CARTAN, SCHOUTEN AND THE SEARCH FOR CONNECTION

ABSTRACT. In this paper we provide an analysis, both historical and mathematical, of two joint papers on the theory of connections by Élie Cartan and Jan Arnoldus Schouten that were published in 1926. These papers were the result of a fertile collaboration between the two eminent geometers that flourished in the two-year period 1925-1926. We describe the birth and the development of their scientific relationship especially in the light of unpublished sources that, on the one hand, offer valuable insight into their common research interests and, on the other hand, provide a vivid picture of Cartan's and Schouten's different technical choices. While the first part of this work is preeminently of a historical character, the second part offers a modern mathematical treatment of some contents of the two contributions.

Contents

1. Introduction

In 1926 the Dutch Academy of Sciences in Amsterdam published two short memoirs, [Cartan and Schouten, 1926b] and [Cartan and Schouten, 1926a], on the theory of connections. They were joint papers by Élie Cartan and Jan Arnoldus Schouten, two of the most eminent geometers of the first half of the last century. The papers dealt with distinct but also closely related issues.

The first one introduced three different connections (now known as Cartan connections) on so–called group manifolds (in German, Schouten referred to them as *Gruppenmannigfaltigkeit*), i.e. on Lie groups. Two of these connections define an absolute parallelism of vectors, that is

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a parallelism that, contrary to the notion introduced by Levi–Civita in 1917, is independent of the path chosen to connect any two points of the manifold. On the contrary, the third connection corresponds to a parallelism that is bounded to the path chosen and in the case of simple and semi–simple Lie groups coincides with the Levi–Civita parallelism of the underlying (pseudo–)Riemannian structure.

The second paper approached a difficult and most fascinating classification problem[1] consisting in determining all Riemannian manifolds that admit an absolute parallelism (i.e. a flat connection) that is consistent with the canonical (Levi–Civita) parallelism induced by the metric tensor.

In many respects these two papers stood out among other contributions to the theory of connections of the same period. In the first place, the collaboration with Schouten represented a noteworthy exception to the relative isolation of Cartan's research that had been following a peculiar and most original path since the early 1900's. In the second place, Cartan's and Schouten's joint works inaugurated in a certain a new approach to the theory of Lie groups. The latter became to be regarded as abstract manifolds, thus allowing the emergence of a geometrical interpretation of classical results of Lie's theory. Eventually, the problems tackled in these papers greatly stimulated further geometrical investigations, especially in the realm of the theory of symmetric spaces, a theory that Cartan was led to develop precisely in the course of his collaboration with Schouten.

In view of these reasons, it seems useful to provide an analysis, both historical and mathematical, of these two papers. Our analyis will be greatly enriched by extensive recourse to unpublished sources kept in the Archives of the Academy of Sciences in Paris (AASP) and in Schouten's Nachlass at the Amsterdam Mathematical Centrum (AMC). The manuscript material which is relevant for our discussion amounts to 43 letters (AASP) + 120 letters (comprehending drafts by Schouten, AMC). The period covered by the correspondence runs from March 1924 up to June 1946. In view of our purposes, we will concentrate exclusively upon letters ranging from March 1924 to the late 1926.[2]

These unpublished sources enable us to get an adequate understanding of how their collaboration orinated, by offering in addition precious insight into the heuristics of the discovery process. Finally, they provide a vivid picture of Cartan's and Schouten's quite diverging methods by testifying to the difficulty and the embarrassment that each of the two authors sometimes experienced in understanding the techniques of the other.

The present paper is divided into two parts. The first one (Sec.s 2-4) is mainly of a historical character. There the scientific contributions of Cartan and Schouten prior to their collaboration are sketchily described. Special emphasis is paid both in underlining their common interest in the dawning theory of connections and in highlighting their different technical approach. Section 3 and 4 offer an analysis of the contents of [Cartan and Schouten, 1926b] and [Cartan and Schouten, 1926a] respectively, in the light of their scientific correspondence, which is here for the first time partly published. The second part of the paper is devoted to a modern mathematical treatment of some outcomes of their collaboration, namely to canonical connections on Lie groups and skew–symmetric torsion tensors.

[1]A precise formulation of the problem is given in section (4) of this paper. In this respect, see also [Wolf, 1972].
[2]It might be useful to recall that an online database has recently been made available to study Cartan's *Nachlass*, his correspondance not being included. See the website http://eliecartanpapers.ahp-numerique.fr/.

2. A common background but different techniques

After the publication of Einstein's fundamental papers on the foundation of General Relativity, the whole realm of differential geometry experienced a period of intense and widespread development. A first manifestation of this phenomenon took place in 1917 when Tullio Levi–Civita published a memoir [Levi-Civita, 1917] in which he provided a geometrical interpretation of Christoffel symbols in terms of the notion of parallelism of vectors. From that moment on, the interest of geometers was devoted to investigating possible extensions of the notion of parallelism to manifolds of a more general type than those of a Riemannian kind. Accordingly, the local character of the space ceased to be exclusively Euclidean allowing the case of more general settings such as affine and projective ones. Incidentally, it should also be borne in mind that since the beginning of such attempts at generalization there had been always an intimate connection with the developments in theoretical physics aimed at achieving extensions of relativity theory.

According to Struik,[3] geometrical investigations dealing with the theory of connections in the period 1917-1930's can be grouped into three different directions. As he put it, in the first place there was the approach, pursued among others by Jan Arnoldus Schouten, which consisted in finding generalizations of Levi–Civita's parallel transport. Besides it, there were the method of the so–called Princeton school (L. P. Eisenhart was its leading exponent) that focused its attention mainly on the study of "paths", i.e. curves with constant directions. Finally, there was the peculiar approach pursued by Élie Cartan who, relying upon the methods of exterior differential calculus, developed an ambitious research programm that aimed at achieving a unification between the conceptual scheme of Klein's *Erlanger Programm* and that of Riemannian geometry.

While the first two directions could be described as mainly analytical, since they relied upon techniques of Ricci calculus, Cartan's method was characterized by a personal geometrical stance that distanced him from the prevalent analytical trend, polemically dubbed by him as an "orgy of indices."[4]

In spite of different choices concerning the techniques to be employed, in the early 1920's Cartan and Schouten shared a common interest into the generalization program of the notion of parallelism to geometrical settings more general than the Riemannian one. This emerges most vividly in a letter (one of the very first and indeed the first one that is extant) that Schouten wrote to Cartan on 3rd March 1924.

> Sir and very honorable Collegue!
> Your notes in the C. R.[5] 174 (1922) p. 437, 593, 734, 857, 1104 On the generalization of the ideas of Riemannian spaces interest me very much since they are closely related to my own research on parallel displacement (Übertragungslehre Math. Zeitschrift 13 (1922) p. 56, 15 (1922) p. 168), of which I had the honour of sending you some offprints. Nonetheless it would be easier for me to draw an exact comparison of my research with yours if I had at my disposal a more extensive publication of your works. You would do me a great favour if you write me whether such a publication already exists. I also ask you, if you can, to send

[3]See [Struik, 1934, pp. 1-4].
[4]See [Cartan, 1946], in particular the preface to the first edition (1928) of this book.
[5]*Comptes Rendus.*

> me some offprints of these articles. I believe that a mutual comparison between our results could bear nice fruits.

It was precisely their common commitment in the research programme aiming at providing an extension of Riemannian geometry that favoured the beginning of their scientific correspondence and, two years later, triggered their intense, though brief, scientific collaboration.

As Schouten's above remark concerning the need of a mutual comparison between their results seems to suggest, Schouten and Cartan's correspondence represented also an important opportunity to compare highly different techniques that were, in a way, in great need of reconciliation.[6]

Indeed, their letters quite often reflect their common difficulty in understanding the point of view of each other and the effort made in overcoming the obstacle represented by the usage of unfamiliar methods. In particular, being quite aware of the peculiarity of his techniques, Cartan frequently emphasized that the results obtained by him, though equivalent to those communicated by Schouten, greatly differed as far as the form of such results was concerned. Cartan's constant recourse to moving frame techniques required, as it were, justification and patient explanation.

In this respect, a letter from Cartan to Schouten dated 16th June 1924 is quite enlightening. Cartan was here referring to a previous letter from Schouten (unfortunately lost) in which very likely Schouten had commented on Cartan's ubiquitous usage of differential forms (Pfaffian expressions), which represented the crucial technical tool of Cartan's moving frame approach. His (Cartan's) response was as follows:

> Let me say something about what you call "my symbolics." In actual fact, I believe that I have no symbolics, but this is a matter of words. Starting from my works on Pfaffian systems, I was led to employ a notation that essentially consists in designating a Pfaffian expression by means of one letter and in attributing the fundamental role not to the differentials of variables but to certain linear combination of the differentials that thus assume a kind of privileged role. Naturally, I found confirmation that my procedure was fertile in the fact that in this way I could create a theory of the structure of transformation groups that was valid both for infinite and finite groups. Subsequently, I applied this procedure to differential geometry with even more certainty since, in connection with my theory of Pfaffian systems, it provides me with the properties of geometrical entities and with their degree of generality. Of course, this does not mean that one cannot arrive at the same results by means of the Ricci calculus, especially in the generalized and completed form that you gave to it; a combination of the two (such as, e.g. the one provided by Mr. Lagrange[7]) might not be without interest. Evidently my method, which was created in view of a certain kind of questions, might produce only inconvenience for other matters.

[6]An interesting episode is recounted in a couple of letters dating back to October 1926. A doubt concerning the appropriate sign of the components of the torsion corresponding to a particular connection had emerged; both Cartan and Schouten felt the need to verify the statements of the other by translating them into their own language, thus revealing the difficulty encountered in employing the technical tools of the other.

[7]Cartan was here referring to [Lagrange, 1923]. Indeed René Lagrange had investigated the possibility of attaining a synthesis between the techniques of the absolute differential calculus and Cartan's techniques based upon the notion of exterior differential forms. See, in this respect, Lagrange's thesis discussed in June 1926 in front of a jury composed by E. Borel, E. Vessiot and Cartan himself.

With respect to Cartan's last remarks, it is interesting to emphasize the balanced and measured tone of his assessment, probably a form of courtesy towards Schouten's work. Indeed, in other circumstances, Cartan did not spare harsh criticisms towards the usage of Ricci's calculus techniques that, as he put it, somehow obscured the geometrical content of some matter.

Before providing an analysis of the results of Cartan's and Schouten's collaboration, it seems worthwhile to discuss briefly their scientific backgrounds in order to appreciate affinities and divergences. We start with Cartan and then concentrate upon Schouten's contributions in the realm of general affine connections.

2.1. **Cartan's theory of generalized spaces.** Starting from the early 1910's, Cartan's research interests experienced a radical shift which oriented his investigations from mainly algebraic and analytical subjects towards geometry and mathematical physics. Such a shift was the consequence of his highly peculiar approach to Lie's theory of infinite continuous groups which allowed him to uncover in a way the geometrical significance of certain differential forms (later to be known as *Maurer–Cartan's forms*) which were first introduced for analytical purposes only. The result was a generalization of Darboux' method of moving frames to homogeneous spaces other than Euclidean ones; Cartan presented it for the first time in [Cartan, 1910].

The advent of General Relativity in 1915-1916 and the subsequent need to attain a better comprehension of the geometrical meaning of the algorithms of Ricci's calculus triggered a wide and profound process of reflection upon differential geometry and, more specifically, upon what we would nowadays call theory of connections. The publication of [Levi-Civita, 1917], which represented the starting point of this process, was soon followed by a conspicuous amount of investigations by many mathematicians including Weyl,[8] König, Schouten and Cartan himself.

Indeed, Cartan soon discovered that his method of moving frames could further be generalized to embrace more general geometrical settings, such as Riemannian spaces and generalizations thereof.[9] The first step in this direction was taken in a long and somehow obscure paper [Cartan, 1922a], whose draft was already completed in 1921. There Cartan provided a proof of the uniqueness of the Einstein tensor G_{ik}, under appropriate conditions.[10] On this very occasion, Cartan provided his own reinterpretation of Riemannian curvature and Levi–Civita parallelism in terms of moving frames. By considering a generalization of the structure equations of the Euclidean group, he could interpret a Riemannian manifold as a deformed Euclidean space. His starting point was the structure equations of the Euclidean group $\mathbf{ASO}(n)$ (written for the general n–dimensional case):

$$d\omega^i = \sum_{j=1}^{n} \omega^j \wedge \omega_j^i \tag{2.1}$$

$$d\omega_j^i = \sum_{k=1}^{n} \omega_j^k \wedge \omega_k^i, \tag{2.2}$$

with $i, j = 1, \ldots, n$ and $\omega_j^i + \omega_i^j = 0$.

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[8]For a general account of Weyl's work on the theory of connections and its application to unified field theories, see [Scholz (eds.), 2001].

[9]For a discussion of generalized spaces modelled upon projective and conformal spaces, see [Nabonnand, 2009].

[10]These conditions are linearity of G_{ik} with respect to the second order derivatives of the metric tensor and its conservation, i.e. its divergencelessness.

Cartan supposed that a measure of a deviation from the Euclidean space could be provided by introducing additional terms in 2.2 that express the fact that the differential system ceases to be completely integrable. Cartan thus obtained:

$$d\omega_j^i = \sum_{k=1}^{n} \omega_i^k \wedge \omega_k^j + \Omega_j^i, \qquad i,j = 1,\dots,n. \tag{2.3}$$

Later (for example in [Cartan, 1923]) he referred to the 2–forms Ω_j^i as the curvature form associated to the connection defined by the Pfaffian forms ω^i, ω_j^i, $i,j = 1,\dots$. Indeed, as he explicitly observed in [Cartan, 1922a, p. 154], Ω_j^j can be expressed in terms of the coefficients of the Riemannian curvature as follows:

$$\Omega_j^i = \sum_{k<l} A_{jkl}^i \omega^k \wedge \omega^l, \qquad i,j = 1,\dots,n; \tag{2.4}$$

where, he said, "the quantities A_{jkl}^i are the *coefficients of the Riemannian curvature*".[11] In this respect one should be a little careful in identifying A_{jkl}^i with the components of the Riemann tensor, in the usual sense. To understand this point, let us consider once again the forms ω_i and ω_j^i in the *homogeneous* Euclidean case. As we said above, such forms can be interpreted (and Cartan did so for sure) as differential forms defined upon the group $\mathbf{ASO}(n)$. Now since the transformations of this group are in bijective correspondence with the set of direct orthonormal frames, one should think of ω^i, ω_j^i as differential 1−forms defined upon this manifold. Accordingly, in the case of a Riemannian manifold M we might regard the forms ω^i, ω_j^i as being defined upon the *bundle of orthonormal frames* $O(M)$ associated to M. After all, this interpretation is supported by passages such as [Cartan, 1922a, p. 151], where Cartan explicitly remarked that the Pfaffian forms ω^i, ω_j^i depend upon additional $\frac{n(n-1)}{2}$ parameters, beyond the coordinates of the manifold M. As a consequence of this, Cartan's ω^i's would be identified with the components (with respect to the natural basis of $\mathbb{R}^n$) of the so-called canonical form.[12] Furthermore, we can interpret the 1−forms ω_j^i as the connection forms associated to the Levi–Civita connection on the principal bundle of orthonormal frames. Thus, in order to reobtain the components of the classical Riemann tensor, one should replace the forms ω_i, ω_{ij} with their pull–backs to M via the choice of a frame $X : M \to O(M)$. By defining $\bar{\omega}^i = X^*\omega^i$, $\bar{\omega}_j^i = X^*\omega_j^i$ and $\bar{\Omega}_j^i = X^*\Omega_j^i$, we deduce the components of the Riemannian tensor (with respect to the frame X) as:

$$\bar{\Omega}_j^i = \sum_{k<l} A_{jkl}^i \bar{\omega}^k \wedge \bar{\omega}^l, \qquad i,j = 1,\dots,n. \tag{2.5}$$

In the light of these observation, we can understand how Cartan reformulated the conditions for Levi-Civita parallelism as given in [Cartan, 1922a, p. 152]:

$$d\xi^i + \sum_k \xi^k \omega_k^i = 0, \qquad i = 1,\dots,n, \tag{2.6}$$

where ξ^i, $i = 1,\dots,n$ indicate the components of a tangent vector field with respect to a coordinate base, say $\left(\frac{\partial}{\partial u^1},\dots,\frac{\partial}{\partial u^n}\right)$. Indeed, if we interpret ω_k^i as $\bar{\omega}_k^i = X^*\omega_k^i$, where $X : M \to O(M)$ is the coordinate frame $X = \left(\frac{\partial}{\partial u^1},\dots,\frac{\partial}{\partial u^n}\right)$, then $\bar{\omega}_k^i = \Gamma_{jk}^i \bar{\omega}^j$, where Γ_{jk}^i are the Christoffel symbols of the Levi–Civita parallelism and $\bar{\omega}^j = du^j$.

[11] [Cartan, 1922a, p. 154], emphasis in the original.
[12] See [Kobayashi and Nomizu, 1963, III, §7].

Although anachronistic, this interpretation offers to modern readers a viable tool to appreciate Cartan's peculiar techniques. Needless to say, Cartan's treatment remained somehow informal and unrigorous. It was with the work of Charles Ehresmann, one of Cartan's students, that a first attempt towards axiomatization was made by introducing the notion of fiber manifold (*espace fibré*).

Interestingly, Cartan emphasized his deliberate choice of avoiding the tools of Ricci's calculus[13] in favor of what he regarded as the more geometrical method of moving frames. Incidentally, it should be mentioned that such a choice was fostered also by recourse to the theory of equivalence which he had been developing since 1902 in terms of exterior differential forms and which provided him with a very powerful instrument to approach the uniqueness proof.

In that same year, in [Cartan, 1922c] a similar generalization involving the translational components ω^k was conveyed. Cartan introduced a sort of translational curvature which he named "torsion". According to his own admission, Cartan was driven to such a notion through the work of the brothers Cosserat on generalized continua. Starting from 1896, the Cosserats had pursed a vast research programm aiming at providing a completely new formulation of continuum mechanics in the light of Darboux's method of moving frames[14]. A crucial innovation of their approach was represented by the possibility of conceiving elastic media that, contrary to the assumptions of classical elasticity theory, were capable of carrying internal torque too. As Cartan explicitly observed in [Cartan, 1922c, p. 595], a Cosserat medium with constant pressure and torque offered a mechanical representation of a 3-dimensional space endowed with non-vanishing torsion[15] and vanishing curvature. In the light of terminology employed by Cartan in [Cartan and Schouten, 1926b], one can say that this represented Cartan's first published example of space endowed with absolute parallelism.

Already in [Cartan, 1922c] Cartan associated the notion of torsion to non-complete integrability of the Pfaffian equations involving the (infinitesimal) translational components. Nonetheless, it was in [Cartan, 1922b] that he provided its analytical expression as follows:

$$\Omega^i = d\omega^i - \sum_{k=1}^{n} \omega^k \wedge \omega_k^i = \sum_{k<l} T^i_{kl}\omega^k \wedge \omega^l, \quad i = 1, \dots, n, \tag{2.7}$$

where the coefficients T^i_{kl} coincide (modulo pullbacks to the base manifold via moving frame) with the components of the torsion tensor. Curvature and torsion was thus regarded by Cartan as complementary terms which provide a measure of the deformation of the Euclidean space. The first one involves rotational components, while the second one involves translational components.

Further generalizations were attained in a series of papers[16] dealing with the geometrical foundations of relativity theory. Therein a careful analysis of manifolds endowed with torsion was carried out too. Interestingly, [Cartan, 1923, chap. IV] was devoted to investigating the relationship between the Levi-Civita connection (i.e. the unique Euclidean connection without torsion, in Cartan's language) and connections with torsion. In particular, Cartan was interested in finding conditions that guarantee the preservation of the geodesics associated to the two

[13] [Cartan, 1922a, p. 143].

[14] See [Cosserat and Cosserat, 1896] and [Cosserat and Cosserat, 1909].

[15] In this respect, see [Lazar and Hehl, 2010] which provides a detailed and modern discussion of Cartan's assertion.

[16] [Cartan, 1923], [Cartan, 1924b], [Cartan, 1925].

distinct connections.[17] In this respect he proved that one such condition was provided by the requirement that "the translation associated to an arbitrary plane element of the manifold be normal to the element itself" [Cartan, 1923, §66]. By this Cartan meant that the tangent vector $T(X,Y)$ associated to the plane generated by $X, Y \in T_xM$ is normal (with respect to the metric g) for every $x \in M$; more explicitly $g(T(X,Y),X) = g(T(X,Y),Y) = 0\ \forall X, Y \in T_xM$.

Because of evident dimensional reasons, this condition, which is actually both necessary and sufficient for preservation of geodesics, is never fulfilled when the dimension of the manifold is equal to 2. In order to explain this phenomenon, Cartan conveyed an example that is both very simple and instructive. He considered[18] the two dimensional sphere with a flat connection given by the following prescription: two vectors originating from two points $p, q \in S^2$, say w and v, are parallel if they form the same angle with the meridians passing through p and q respectively. The parallelism so defined is an absolute parallelism in the sense that it is independent of the path chosen to go from p to q; although this property is evident from the adopted definition of parallelism, one can verify it straightforwardly by computing, as Cartan did, the curvature of the connection which is identically equal to zero. On the other hand, the space carries a torsion which can easily be computed by means of the structure equations (2.7). To this end, Cartan introduced the usual parametrization of (a portion of) the sphere in terms of colatitude θ and longitude ϕ, with $0 < \theta < \pi$ and $0 < \phi < 2\pi$ and defined a system of moving frames by choosing unitary (with respect to the standard metric $d\theta^2 + \sin^2\theta d\phi^2$) vectors $e_1 = \frac{1}{\sin\theta}\partial_\phi$ and $e_2 = \partial_\theta$. The dual 1–forms ω_1 and ω_2 are given by $\omega_1 = \sin\theta d\phi$ and $\omega_2 = d\theta$. As a consequence of the adopted prescription one has $de_i = 0$, $i = 1, 2$ (or, in modern terms, $\nabla_{e_i} e_j = 0$, $i, j = 1, 2$) and thus

$$\Omega_1 = \cot\theta\omega_1 \wedge \omega_2, \qquad \Omega_2 = 0.$$

At this point Cartan remarked that the torsion could be represented by a vector tangent at each point to the parallel through it and equal, in length, to $\cot\theta$. Furthermore, he observed, the geodesics corresponding to this connection are loxodromics, which in general do not coincide with the shortest paths, the only exception being represented by meridians and the equator. This was presented by Cartan as a special case of a more general result according to which, in dimension $n = 2$, the only manifolds with geodesics coinciding with shortest paths (i.e. with the geodesics of the Levi–Civita connection) are those manifolds with vanishing torsion.[19] This example deserves our attention especially in view of the fact Cartan employed it, already in 1922, to explain to Einstein the concept of "Fernparallelismus" (i.e. absolute parallelism) on the occasion of Einstein's visit to Paris.[20]

As already established in [Cartan, 1923, §66], manifolds with dimension $n \geq 3$ exhibit a different behavior. Cartan dealt with this topic in a more elementary way in the overview article [Cartan, 1924a] where he outlined the recent developments of the notion of space following the advent of General Relativity and Levi–Civita's geometrical interpretation of Christoffel symbols. His aim was to show that, in contrast with the 2–dimensional case, there exist Euclidean connections with torsion whose geodesics coincide with the geodesics of the Levi–Civita connection.

[17] By *geodesic* (or *straight line*) on a manifold with an affine connection Cartan meant a curve such that its tangent vector is transported parallel to itself along the curve. See [Cartan, 1923, §62].
[18] [Cartan, 1923, p. 408-409].
[19] [Cartan, 1923, p. 408].
[20] See, in this respect, [Cartan et al., 1979, pp. 4-9].

This was the context in which, after recalling the 2–sphere example that we have just mentioned, Cartan discussed for the first time Clifford parallelism in elliptic 3–space.[21] His starting point in [Cartan, 1924a] consisted in providing the Cayley–Klein classical model for the elliptic space. Following a long standing tradition, Cartan defined Clifford's parallelism by having recourse to the double system of generators of the fundamental quadric and observed that "the space endowed with the Euclidean connection induced by Clifford's parallelism of either first or second kind is without curvature" and thus, by employing a denomination which Cartan did not use at this moment, it is a space with absolute parallelism.

Interestingly, to the projective Cayley–Klein model Cartan preferred a kinematical model which allowed a "curious" representation of both Clifford's and Levi–Civita's parallelisms. Despite the fact that at this point he did not mention it explicitly, Cartan was there alluding to the possibility if interpreting the elliptic space as the group manifold of rotations in Euclidean space. Let us read the relevant passage *in extenso*:

> Consider, in the ordinary space, a rigid body S moving about a fixed point O; there are three degrees of freedom. The set of all positions thus constitutes a 3–dimensional manifold (*variété*) or a *pseudo–space*, whose every element or *pseudo–point* corresponds to a position of the body. We call *pseudo–straight–lines* the set of all pseudo–points corresponding to those positions of S that can be deduced from a given position through rotation about a given axis. We call *pseudo–distance* between two pseudo–points of this pseudo–straight–line the semi–angle of the rotation which brings to coincidence the two corresponding positions of the rigid body. Thence, one can see that pseudo–straight–lines are closed and of finite length equal to π. Every pseudo–straight-line AM can be represented by the position A of the rigid body and by the axis OR of the rotation that gives the other positions M corresponding to the other pseudo–points of the pseudo–straight–line. Pseudo–distance AM is given by $\sqrt{x^2+y^2+z^2}$, where x, y, z indicate the rectangular coordinates of the end point of vector $\overrightarrow{OR}$, having length equal to the semi–angle of the rotation which brings A to coincide with M. From this, it follows that the angle between two pseudo–straight–lines drawn through A is equal to the angle between the corresponding axes OR.[22]

On the basis of these positions, Cartan defined Clifford parallelism of the two kinds (*espèces*) as follows.

> Two pseudo–straight–lines are Clifford's parallels of the first kind if their axis OR is the same; while two pseudo–pseudo–lines, drawn through two pseudo–points A and B, are Clifford's parallel's of the second kind if their corresponding rotation axes OR and OS are placed in the same way with respect to the positions of the rigid body which correspond to A and B.[23]

Cartan computed torsion and curvature of the connection induced by the two Clifford parallelisms by means of a standard technique of his which consisted in finding translational and rotational components of the displacement to which a moving trihedron is subjected when it is parallel transported along a closed path. The elliptic space endowed with Clifford's parallelism

[21]For a general historical introduction to this fascinating phenomenon see [Cogliati, 2015].
[22][Cartan, 1924a, p. 306].
[23][Cartan, 1924a, p. 306].

was thus interpreted as a generalized space with vanishing curvature and non–vanishing torsion and, consequently, as a space with absolute parallelism. As Cartan claimed (without proof), the geodesics of the two Clifford's connections coincide with those of the standard Levi–Civita connection; thus, confirmation was provided that connections with torsion can exhibit the same geodesics as those corresponding to the Levi–Civita connection.

Furthermore, since elliptic space was susceptible of being regarded as the group–manifold of rotations, Cartan quite naturally was led to confront the problem of extending this result to any Lie group and of investigating the geometry of group–manifolds from a more general standpoint.

2.2. **Schouten and his *Ricci's Calculus*.** As aptly observed [Nijenhuis, 1972], the name of Jan Arnoldus Schouten is naturally associated with tensors. Indeed, since his first steps in advanced mathematical research, his attention was devoted to providing a certain degree of systematization to the inhomogeneous corpus of techniques of the so–called vector analysis. Still at the beginning of the 20th century, this discipline was afflicted by the existence of a multitude of formalisms (e.g. those of Gibbs, Grassmann, Hamilton etc.) that rendered further developments somehow difficult. Schouten's doctoral dissertation aimed precisely at remedying this unsatisfying state of affairs. Inspired by the unification power of Klein's *Erlanger Programm* that regarded geometry as the study of invariants under the action of a group, Schouten intended to carry out an analogous programme for the classification of vectorial and tensorial quantities that was based upon their behaviour with respect to the action of a group. The results of his attempts materialized in his doctoral dissertation that was published as a book under the title *Grundlagen der Vektor- und Affinoranalysis*, with a short preface written by Felix Klein. As Nijenhuis remarked, Schouten's dissertation seems to have exerted little influence. Schouten himself was dissatisfied with his notation, which he regarded as too complicate.

Independently of Levi-Civita, in 1918, while investigating the operation of differentiating tensors (or affinors as he called them by following Franz Jung[24]), Schouten discovered the notion of parallelism in Riemannian geometry by introducing what he called geodesically moving reference systems. With respect to Levi-Civita's, Schouten's treatment had the advantage of avoiding any recourse to embedding into Euclidean space, nonetheless Levi–Civita's article[25] soon established itself as the standard reference on the subject.[26]

Over the following years, Schouten immersed himself in the task of providing a systematization of the techniques of the "absolute differential calculus" with special emphasis upon its application to the theory of connections. A noteworthy example of such commitment is offered by [Schouten, 1922]. Interestingly, in the introduction Schouten conveyed some general remarks concerning the latter developments of differential geometry. He started out by observing that recent contributions by Hessenberg, Weyl, and König had finally made it clear that the fundamental tensor actually fulfilled a double role. As he put it, firstly there is the metric in the strict sense that fixes the length of tangent vectors; secondly, there is the notion of displacement (or connection), fixed as well by the metric tensor through the Christoffel symbols, which allows to define an invariant differentiation. The adequate assessment of this fundamental distinction, Schouten continued, allowed an extension of the notion of connection that could thus embrace a

[24]See in this respect [Reich, 1994, Chap. 4].
[25][Levi-Civita, 1917].
[26]See in this respect the most interesting article [Struik, 1989] that contains personal recollections too. See also [Cogliati, 2016].

wide range of new possibilities. In spite of these advances, there was still room for complaining about the lack of a systematic treatment of the subject. In this respect, he wrote:

> However, some further steps are still to be taken in order to achieve a complete overall view of the different possibilities of the extended differential geometry. Firstly, to my knowledge the most general "linear" connection (*Zusammenhang*) has not yet been defined and consequently a complete classification of possible different cases is still missing. Secondly, Weyl is the only one who has characterized the connection chosen by him in an invariant manner, by means of a vector field and a tensor field of second order; while, as for the other already known more general connections, such a characterization, which should be regarded as essential, has not been provided yet. Thirdly, the curvature-magnitudes of the fourth degree (*Krümmungsgrößen vierten Grades*) corresponding to the different connections, which generalize the Riemann–Christoffel four index symbols, have not been investigated yet in their full generality and in their reciprocal relationships.[27]

In his attempt to remedy this state of affairs, Schouten conveyed an axiomatic treatment of the general notion of linear connection in terms of the corresponding covariant differential δ. To this aim he required the following conditions to be satisfied: i) a given quantity[28] (*Größe*) and its covariant differential are quantities of the same kind; ii) the covariant differential of a quantity is linear in the differential of the coordinates, i.e. $\delta\Phi = \nabla_\mu \Phi dx^\mu$; iii) the operator δ is linear; iv) it follows the Leibniz rule, i.e. $\delta(\Phi\Psi) = (\delta\Phi)\Psi + \Phi(\delta\Psi)$ and v) the differential of a scalar quantity reduces to ordinary differential, i.e. $\nabla_\mu\Phi = \partial_\mu\Phi$.

From all these postulates the following formulas for covariant differentiation were easily derived:

$$\nabla_\mu v^\nu = \frac{\partial v^\nu}{\partial x^\mu} + \Gamma^\nu_{\lambda\mu} v^\lambda, \qquad \nabla_\mu v_\lambda = \frac{\partial v_\lambda}{\partial x^\mu} +' \Gamma^\nu_{\lambda\mu} v_\nu, \tag{2.8}$$

It should be observed that Schouten did not require any relation between the coefficients $\Gamma^\nu_{\lambda\mu}$ and $'\Gamma^\nu_{\lambda\mu}$, contrary to the Levi-Civita connection where $\Gamma^\nu_{\lambda\mu} = -'\Gamma^\nu_{\lambda\mu}$. Thus, according to him, a linear connection was defined by the assignment of $2n^3$ coefficients a priori and independent.

Schouten also provided a classification (he listed 18 types of different connections) that was based upon tensor fields such as the torsion tensor $S_{\lambda\nu}^{\cdot\cdot\,\mu} := \frac{1}{2}(\Gamma^\mu_{\lambda\nu} - \Gamma^\mu_{\nu\lambda})$,[29] the non–metricity tensor, $Q_\mu^{\cdot\,\lambda\nu} := \nabla_\mu g^{\lambda\nu}$ and the tensor $C_{\mu\lambda}^{\cdot\cdot\,\nu} := \Gamma^\nu_{\mu\lambda} +' \Gamma^\nu_{\mu\lambda}$. In this framework, the Levi–Civita connection can be characterized as that connection for which all these tensors vanish simultaneously.

Over the following years, non–symmetric connections, i.e. connections endowed with non–vanishing torsion, attracted Schouten's attention especially in view of physical applications; indeed, they offered a promising approach towards a unified field theory of gravitation and electromagnetism.[30] From a purely mathematical standpoint, the notion of non–symmetric connection soon found applications in the realm of group theory. Indeed, it was Luther Eisenhart

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[27][Schouten, 1922, p. 57].

[28]By this Schouten meant an arbitrary tensor, covariant, controvariant or mixed. See in this respect also [Schouten, 1924].

[29]As for the notation employed by Schouten to denote the components of a tensor, see [Schouten, 1924, pp. 3-4], namely the section *Bezeichung der gemischten Größen*.

[30]See [Schouten, 1923].

who found out that the infinitesimal generators of a simply–transitive continuous group (that can be viewed as one of the parametric groups associated to a continuous group of transformations in Lie's sense) serve to define unsymmetric connections. In particular Eisenhart succeeded in computing the torsion (he did not use this denomination) and expressing it in terms of the structure constants of the group.[31] More precisely, if

$$X_i = \sum_{\mu=1}^{n} \lambda_i^{\mu} \frac{\partial}{\partial x^{\mu}}, \qquad n = 1, \dots, n,$$

denote n generators of a simply transitive group, then Eisenhart proved that the symbols

$$\Gamma^{\mu}_{\nu\rho} = \sum_{j=1}^{n} \lambda_j^{\mu} \frac{\partial \Lambda^j_{\rho}}{\partial x^{\nu}},$$

where Λ^j_ρ are defined by $\sum_{\rho=1}^n \Lambda^j_\rho \lambda^\rho_i = \delta^j_i$, define a linear connection with vanishing curvature and such that the Γ's are unsymmetric. Indeed,

$$\Gamma^{\mu}_{\nu\rho} - \Gamma^{\mu}_{\rho\nu} = -c^i_{jk} \lambda^{\mu}_i \Lambda^j_\nu \Lambda^k_\rho, \tag{2.9}$$

where c^i_{jk} are the classical Lie's structure constants, $[X_j, X_k] = c^i_{jk} X_i$.

3. The geometry of Lie groups

The collaboration between Cartan and Schouten was triggered by a letter (16th December 1925) from Schouten to Cartan to which Schouten attached the draft of a paper on projective and conformal curvature of non–symmetric parallel displacements, to be published later on as [Schouten, 1927]. What Schouten wrote in the introduction to this memoir attracted Cartan's attention and thus stimulated the desire of the latter to know more details about Schouten's recent research.

Indeed, in the introduction to [Schouten, 1927] Schouten hinted at the fruitfulness of the notion of non–symmetric connections both in geometry and physics and mentioned the important case of connections in continuous groups. He wrote:

> Non–symmetric connections (*Übertragungen*) occur in many places and seem predestined to play a more important role both in future geometrical and maybe also physical investigations. Eisenhart [Schouten referred to [Eisenhart, 1925]] sketchily proved that there is a non–symmetric connection that is associated to any simply transitive continuous group. Starting from Cartan's and Weyl's investigations one can show in general that the adjoint group induces a non–symmetric connection in the manifold of the transformations of every continuous groups. This connection leaves $S_{\lambda\mu}^{\cdot\cdot\,\nu}$ [i.e. the torsion tensor] invariant and its curvature is null. If the group is semi–simple then there is an invariant non–degenerate tensor of degree two and thus the connection is metric.

Schouten was here referring to the very recent contribution [Eisenhart, 1925]. On 14th January 1926, having had the time to read carefully Schouten's paper, these sketchy remarks included, very interestingly Cartan explained to Schouten how he was driven to study the same topic, albeit with a different motivation. While Schouten's researches seem to have been triggered by

[31]See [Eisenhart, 1925, p.249].

Eisenhart's paper and his general interest in the theory of connections, Cartan was mainly motivated by his attempt to provide a generalization of the group theoretic treatment of Clifford's parallelism that he had recently provided. He wrote:

> What you say at the outset [in [Schouten, 1927]] about the theory of continuous groups interests me very much. I have read Einsenhart's article, which I did not know, but I did not understand the end very well. I would like to ask you whether you published something (or you intend to publish something) concerning the "Übertragung" induced by the adjoint group of a continuous group. I was driven to the same question by a recent paper by Mr. Enea Bortolotti on Clifford's parallelism "Parallelismo assoluto e vincolato negli S_3 a curvatura costante ed estensione alle V_3 qualunque," Venezia 1925. Therein he alluded to a paper that I probably sent to you on recent generalizations of the notion of space (Bulletin de math. t. 48, 1924, 294-320); in it I precisely dealt with Clifford's absolute parallelism from the point of view of group theory as a "nichtsymmetrische Uebertragung" with vanishing curvature in the space of orthogonal transformations in three variables. It is this point of view that can be extended to any r–parameter group; in the manifold of transformations there exist two absolute parallelisms (i.e. with vanishing curvature). Analytically, if one denotes with S_x the transformation with parameters x, two infinitesimally close vectors $(x, x + dx)$ and $(y, y + dy)$ are equipollent of the first kind if one has
>
> $$S_{x+dx}S_x^{-1} = S_{y+dy}S_y^{-1};$$
>
> they are equipollent of the second kind if one has:
>
> $$S_{x+dx}^{-1}S_x = S_{y+dy}^{-1}S_y;$$
>
> In both cases one has a manifold with an affine connection characterized by vanishing curvature and whose geodesics are the same in the two cases, i.e. the 1–parameter groups of the total group. [...]
>
> Analytically, if $\omega_1, \ldots, \omega_r$ are the infinitesimal components of the infinitesimal transformation $S_{x+dx}S_x^{-1}$, the structure equations of the space defined by the affine connection of the first kind are
>
> $$\omega_s' = \sum c_{iks}[\omega_i\omega_k],$$
>
> with $\omega_i^j = 0$ (c_{iks} being the Lie constants).
>
> The second connection involves the parameters $\varpi_1, \ldots, \varpi_r$ of the infinitesimal transformation $S_{x+dx}^{-1}S_x$ (see my paper "sur la structure des groupes des transformations et la théorie du trièdre mobile", Bull. Sc. Math. (2), 34, 1910).

Cartan continued by commenting upon the possibility of endowing a given Riemannian (or pseudo–Riemannian) manifold with absolute parallelisms that are consistent with the canonical Levi–Civita connection.

> As you pointed out, if the group is semisimple, to each of the affine connections without curvature that were previously defined, the space is a Riemannian manifold with the same ds^2 in the two cases; moreover, the geodesics previously considered are still geodesics (in the sense of Riemann). Thus, one can set out

> to find all Riemannian spaces in which one can define an *absolute* parallelism (corresponding to a linear Uebertragung) for which the geodesics join the property of being autoparallel. Among them, there are those spaces that correspond to semisimple groups according to what was previously said. In all these spaces Ricci principal directions are undetermined; furthermore, the Riemannian curvature of a facet is equal to the square of the semi–torsion that each absolute parallelism confers on the facet (the torsions being equal and opposite for the two parallelisms);[32] finally, the vector displaced by Levi–Civita parallelism is the bisector of the angle between the positions of the vector that are obtained following the absolute parallelisms.
>
> Are there other solutions to this problem? I have not decided yet, in one direction or the other. There are no other solutions for $n \leq 6$; if there are any, then the corresponding Riemannian spaces admits a continuous family of absolute parallelisms joining the indicated property alongside with a transitive group of rigid movements with more than n parameters. These Riemannian spaces, if they exist, are never with constant curvature.

On January 18th Schouten communicated to Cartan the recent results which he had obtained in the realm of the geometry of Lie groups. In particular, Schouten had shown that every Lie group, regarded as a differential manifold, could be endowed with three different connections; two of them without curvature and with non–vanishing torsion (termed $(\pm)$–connections), the third with vanishing torsion and non–vanishing curvature (termed (0)–connection) which in the case of semi–simple groups turned out to be a Riemannian connection (i.e. the Levi–Civita connection univocally associated to the Riemannian metric).

In moderately modern terms, the results alluded by Cartan and Schouten can be sketched as follows. We start from the case of the $(+)$–connection. To this end we need to recall the notion of parametric groups.[33] Consider an r–parameter continuous group of transformations, $G_{n,r}$ acting on variables $\vec{x} \in \mathbb{R}^n$. Its elements are transformations of type $\vec{x}' = f(\vec{x}, \vec{a})$. As a consequence of the definition of group, such transformations induce analogous transformations on the space of parameters (a subset of $\mathbb{R}^r$ or of $\mathbb{C}^r$), $\vec{c} = \phi(a; b)$, where ϕ's are such that $f(\vec{x}; \phi(\vec{a}; \vec{b})) = f(f(\vec{a}; \vec{x}); \vec{b})$. Now, consider the transformations $\vec{c} = \phi(\vec{a}; \vec{b})$. They define two groups of transformations (called parameter groups). If $\vec{a}$ is regarded as a parameter while $\vec{b}$ is thought of as a variable which is transformed through ϕ, we obtain a group of left transformations ($\vec{b} \mapsto \vec{c} = \vec{a} \cdot \vec{b}$). This group is called *first* parameter group. Similarly, if we look at $\vec{b}$ as a parameter we obtain a group of right transformations ($\vec{a} \mapsto \vec{c} = \vec{a} \cdot \vec{b}$) which is called *second* parameter group. The first parameter group, as any other continuous group, can (at least locally) be generated by infinitesimal transformations (vector fields), i.e. linear differential operator of type $A_i = \sum_{\alpha=1}^{r} A_i^{\alpha}(a) \dfrac{\partial}{\partial a^{\alpha}}$, $i = 1, \ldots, r$. Such vector fields can be thought of as obtained from

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[32]In modern terms, that means that $K(X,Y) = \dfrac{1}{4} g(T(X,Y), T(X,Y))$, where K indicates the sectional curvature, T the torsion tensor (regarded as a (1,2)–tensor) and g the metric tensor. It is intended that X, Y are tangent orthonormal vectors in $x \in M$ i.e., $g(X,X) = g(Y,Y) = 1$ and $g(X,Y) = 0$. In this respect, see [Agricola and Friedrich, 2010, Prop. 2.1].

[33]Our presentation is purely local. It should be observed that starting from the second half of the 1920's, mathematicians started to appreciate the difference between local and global aspects of the theory of Lie groups; in this respect, see for example [Chorlay, 2015].

a parallel displacement of r given vectors, say $A_i^\alpha(a^0)$, according to the connection defined by

$$\overset{+}{\Gamma}{}^\alpha_{\beta\delta} = -\sum_{k=1}^{r} A_k^\alpha \frac{\partial A_\beta^k}{\partial a^\delta},$$

where $A_\alpha^k A_k^\beta = \delta_\alpha^\beta$. As it would be easy to check,[34] the corresponding parallel displacement (Π) coincides with the tangent map of a right–translation; more explicitly, if h, k are points in G (now regarded as a Lie group in the modern sense) then a vector v in T_hG is mapped into w (the parallel vector to v) in T_kG by $(dR_{h^{-1}k})_h$. The connection so obtained defines an absolute parallelism upon G, i.e. a parallelism which is independent from the path followed. In the language of exterior forms, to which Cartan made recourse, the connection is defined by the (right–invariant) Maurer–Cartan forms ϖ_i, $i = 1, \ldots, r$ associated to G_r.

In a completely similar way, one can introduce the (−)-connection, $\overset{-}{\Gamma}{}^\alpha_{\beta\delta}$ associated to left–invariant vector fields and one–forms.

As for the Levi–Civita connection that Schouten dubbed (0)-connection, the corresponding Christoffel symbols turned out to be expressed as the arithmetic mean of the connections $\overset{+}{\Gamma}{}^\alpha_{\beta\delta}, \overset{-}{\Gamma}{}^\alpha_{\beta\delta}$, i.e.:

$$\overset{0}{\Gamma}{}^\alpha_{\beta\delta} = \frac{1}{2}\left(\overset{+}{\Gamma}{}^\alpha_{\beta\delta} + \overset{-}{\Gamma}{}^\alpha_{\beta\delta}\right)$$

Indeed, in the case of a simple group G_r, the connection $\overset{0}{\Gamma}{}^\alpha_{\beta\delta}$ is the one canonically associated with the Riemannian structure defined upon G_r that is, in turn, defined by the Killing–Cartan form. More explicitly, the coefficients $\overset{0}{\Gamma}{}^\alpha_{\beta\delta}$ are the Christoffel symbols that correspond to the (pseudo–)Riemannian metric induced by the Killing form $g_{ij} = \sum_{a,b} c_{ia}^b c_{jb}^a$.[35]

In accordance to Cartan's remark in the above quoted letter, it can be easily proved through direct calculation that the Riemannian space associated to $\overset{0}{\Gamma}{}^\alpha_{\beta\delta}$ is an Einstein space (i.e. the Ricci tensor is proportional to the metric). This is what Cartan meant by the old–fashioned expression "Ricci's principal directions are undetermined".[36]

Schouten and Cartan's results on the geometry of Lie groups were soon published in a short memoir [Cartan and Schouten, 1926b] that appeared in the Proceedings of the Academy of Sciences of Amsterdam. It should be remarked that Schouten was the main responsible for putting in writing their common results in [Cartan and Schouten, 1926b]. As a consequence of this, the choice in the exposition was mainly oriented towards the language of absolute differential calculus with no recourse, or even mention, to Maurer–Cartan's forms and structure equations. Over the following years, both Cartan and Schouten took up the subject again by offering more extensive treatments that followed their own personal taste as far as techniques and language are concerned.

Interestingly, Schouten decided to devote a lecture course on the geometry of Lie groups at Leiden University in 1926-1927, that, according to his own remarks, could be regarded as propaedeutical (*als Vorbereitung*) to the treatment of the Killing–Cartan classification theory of Lie algebras.

[34]See section 5.2 of this paper.
[35]The coefficients of the metric are defined as $g_{\alpha\beta} = c \cdot g_{ij} A_\alpha^i A_\beta^j$. See [Cartan and Schouten, 1926b, pp. 809-810]. It should be observed that the non–degeneracy of g_{ij} is a consequence of Cartan's semisimplicity criterion.
[36]In this respect, see [Eisenhart, 1966, §34].

On his part Cartan, already in spring 1926, set out to compose a paper[37] aiming at providing an extended and more detailed treatment of the results contained in his joint memoir with Schouten. As he explained in a letter to Schouten himself (April 1926), Cartan wanted to emphasize in it the possibility of furnishing a geometrical interpretation of classical results of the theory of continuous groups (such as, the Jacobi identity that can be geometrically expressed in terms of the conservation of the torsion tensor associated to the connections $(\pm)$). Furthermore, as Cartan went on to explain, one motivation consisted in investigating the properties of those Riemannian spaces whose Riemann curvature tensor is parallel, $\nabla R = 0$ and to provide a classification thereof. Indeed, as the correspondence with Schouten clearly shows, the Spring 1926 was for Cartan a period of extraordinary creative fervor. Not only did he devote himself to extending some of the results on the geometry of Lie groups, but he at the same time also managed to attain a complete classification of irreducible (locally) symmetric spaces.

4. Riemannian spaces with absolute parallelism

The study of the geometry of Lie groups, and especially of the Levi-Civita connection associated to them, represented an important stimulus for Cartan's interest in the study of what he had dubbed *Levy's Problem* (after Harry Levy[38]), i.e. the determination of all Riemannian spaces such that $\nabla R = 0$. Furthermore, Cartan's attention towards this unexplored field was motivated as well by the, somehow unexpected, solution to the following problem to which Cartan had alluded in the above quoted letter to Schouten, dated 14th January 1926:

Problem 1. *To determine all Riemannian manifolds that admit an* absolute *parallelism consistent with the parallelism induced by the Levi-Civita connection. By requiring that the two parallelisms be consistent, one means that the following properties are fulfilled: i) the (Levi-Civita's) geodesics of the Riemannian manifold are autoparallel curves with respect to the connection induced by the absolute parallelism; ii) this connection (i.e. the one stemming from the absolute parallelism) is metric, i.e. preserves lengths and angles.*[39]

Indeed, as it soon turned out, those Riemannian spaces that are solution to this problem are necessarily symmetric in the sense that the covariant derivative of the Riemann tensor is zero.[40] In this respect, it is interesting to read the sketchy remarks that Cartan inserted in the introduction to [Cartan, 1926] where we can find as well further confirmation of the heuristic role played by classical Clifford's parallelism:

> I was led to pose myself this problem [i.e. the classification problem of irreducible symmetric Riemannian manifolds] in connection with another problem, which I have investigated in collaboration with Mr. J. A. Schouten, whose solution has appeared in a note presented to the Royal Academy of Sciences in Amsterdam. This problem consisted in studying all possible generalizations of the well known Clifford's parallelism in 3–dimensional elliptic space. More precisely, the problem was to find all Riemannian spaces in which there exists an absolute parallelism for geodesics with the property that the angle between two geodesics is preserved when one draws through an arbitrary point two parallel lines. All these manifolds

[37] See [Cartan, 1927].
[38] In this respect, see e.g. [Akivis and Rosenfeld, 2011, p. 186].
[39] This is a modern version of Cartan's formulation. See also section 5.4.
[40] See section 5.6 for a modern proof of this fundamental property.

> are $\mathcal{E}$–spaces; but, in actual fact they only represent a small portion of those. If we restrict ourselves to the irreducible case, then, beyond the manifolds that are representative of simple groups, there is only one relevant case represented by the 7–dimensional elliptic space.

Before providing a description of the tortuous path that finally led Cartan and Schouten to discover the solution to *problem* (1), it seems interesting to examine in some detail the different formulations of the problem offered by the two mathematicians.

Let us start with Schouten. In a letter to Cartan dated 18th January 1926, after expressing great interest in the problem posed by Cartan, he offered the following technical formulation based upon transformation of the connection coefficients. The basic formula was the following one:

$$'\Gamma^{\nu}_{\kappa\mu} = \Gamma^{\nu}_{\kappa\mu} + 2\delta^{\nu}_{\kappa} p_{\mu},$$

where p_μ are the components of an arbitrary vector. More specifically, if $\Gamma^{\nu}_{\kappa\mu}$ are the Christoffel symbols of the Levi–Civita connection then one has

$$'\Gamma^{\nu}_{\kappa\mu} = \Gamma^{\nu}_{\kappa\mu} + p_\kappa \delta^{\nu}_{\mu} + p_\mu \delta^{\nu}_{\kappa} + S_{\mu\nu}^{\cdot\,\cdot\,\kappa}, \tag{4.1}$$

where $S_{\mu\nu}^{\cdot\,\cdot\,\kappa}$ designate the components of the torsion tensor corresponding to the connection defined by $'\Gamma^{\nu}_{\kappa\mu}$. Although Schouten did not mention it at this stage of their collaboration, the requirement that the connection $'\Gamma^{\nu}_{\kappa\mu}$ be metric imposes further restriction upon the form of the transformation (4.1). Indeed, as it was remarked at the outset [Cartan and Schouten, 1926a], the condition

$$\nabla g_{\mu\nu} = 0 \tag{4.2}$$

implies $p_k = 0$ and that the tensor $S_{\lambda\mu\rho} = S_{\lambda\mu}^{\cdot\,\cdot\,\nu} g_{\nu\rho}$ is antisymmetric in the exchange of every pair of indices and, thus, it is a *trivector*.[41] Finally, the condition that the parallelism induced by $'\Gamma^{\nu}_{\kappa\mu}$ is absolute can be expressed by the requirement that the corresponding curvature tensor is identically equal to zero.

As for Cartan's approach, it is once again through his correspondence with Schouten that we can appreciate his peculiar formulation of the problem. Generally speaking, one can say that his approach was characterized by a distinct geometrical stance. Not surprisingly, his starting point was represented by the structural equations for Euclidean connection (in Cartan's language) with vanishing curvature. In this special case, the structural equations with respect to a parallel coframe ω^i, $i = 1, \ldots, n$, are easily seen to be reduced to

$$\omega^i_j = 0, \qquad d\omega^i = \sum_{j<k=1}^{n} T^i_{jk} \theta^j \wedge \theta^k. \tag{4.3}$$

Cartan explained very clearly his approach in a letter to Schouten dated 20th January 1926. He wrote to his Dutch colleague:

> Here is the core of the general problem that I pointed out to you and for which you have been so kind to show some interest. One has to find, in the most general way, r independent Pfaffian expressions $\omega_1, \ldots, \omega_r$ such that
>
> $$\omega_i' = \sum c_{\alpha\beta i} \left[\omega_\alpha \omega_\beta\right], \tag{4.4}$$

[41] For a modern proof of this property in the language of moving frames, see section 5.4, in particular equation 5.28.

> the coefficients $c_{\alpha\beta\gamma}$ (sic!) being the components of a system of trivectors:
>
> (4.5) $$c_{\alpha\beta\gamma} = c_{\beta\gamma\alpha} = c_{\gamma\alpha\beta} = -c_{\alpha\gamma\beta} = -c_{\beta\alpha\gamma} = -c_{\gamma\beta\alpha}.$$
>
> If the $c_{\alpha\beta\gamma}$ are constant (but without the symmetry condition (4.5)), these satisfy Lie's relations (Jacobi's identities) that one obtains by writing that the trilinear covariants of the righthand side of equations (4.4) are zero. The question is then the following: is it possible that the coefficients $c_{\alpha\beta\gamma}$ are not constant? It would appear, at first sight, that they are not constant, however I am inclined to think that they are actually constant: nonetheless, I am still far away from being sure and I would be interested in knowing it if you discover something relevant to the problem.

Although Cartan did not mention it explicitly, we can understand the origin of the symmetry condition (4.5)) in the light of what Cartan had observed in [Cartan, 1923, §66]. Indeed, the requirement that the torsion associated to a plane element be orthogonal to the plane itself is easily seen to be equivalent to the condition (4.5)), when the coefficients $c_{\alpha\beta\gamma}$ are regarded as the components of the torsion tensor.

Following Cartan's suggestion, over the following weeks Schouten set out to investigate the conjecture to problem (1) according to which, beyond the case represented by semi–simple (compact) Lie groups, there are no other Riemannian manifolds that exhibit the properties required. Indeed, as clearly emerges from an undated draft composed very likely a few days after 20th January 1926 (soon after receiving Cartan's above quoted letter), Schouten communicated to him his hope that through attentive usage of tensorial identities the existence of Riemannian manifolds other than Lie groups and still satisfying the requirements of problem (1) could be excluded. Indeed on 19th February 1926 Schouten wrote to Cartan that he obtained a proof of a theorem according to which there are no other solutions to problem (1) except those represented by semisimple Lie groups. Two days later, Schouten sent to Cartan the details of the demonstration along with a proposal for a joint publication. He wrote:

> I plan to write a note of about 20-30 pages for the Royal Academy of Amsterdam, containing i) an overview of the three different connections of simple and semisimple group manifolds; ii) the demonstration of the theorem that has been proved.
>
> Nonetheless, it would be disagreeable to publish these two results on my own. You found the different connections of group manifolds together with me; as for the theorem, I believe that the divination of a theorem so beautiful and general is at least as worthy as providing a demonstration.
>
> Thus, I propose you to do me the honor to publish this note as a joint work with me.

On 26th February, Cartan responded to Schouten by accepting the proposal and also confirming the correctness of Schouten's proof that he had achieved through moving frame techniques.

To the great surprise of both of them, the validity of the theorem excluding any other case of Riemannian manifold turned out to be defective. Indeed, only a few days later (1st March) Cartan was able to exhibit a counterexample that undermined their beliefs on the subject. A letter of Cartan provides a vivid description of the state of despair caused by the discovery of the counterexample together with the subsequent relief inspired by the detection of a mistake in the preceding proof.

> Dear and honorable Colleague,
> Here are some news. I have found a simple example that invalidates the theorem under dispute; I had abandoned this example which I had already considered a month ago, since it led to a manifold of constant curvature, an eventuality that I had demonstrated to be impossible, since such a space could not admit a pseudoparallelism with the required properties.
>
> Upon resuming this example, I have been conducted to a possible pseudoparallelism in the 7–dimensional elliptic space; in actual fact, I have proved the existence of ∞^7 pseudoparallelisms in this space. However, at the same I did not find any mistakes in the proof of the impossibility of such pseudoparallelisms in space with costant curvature! Thus, for 24 hours I had been in the distressing position of conciliating two irreconcilable theorems, both of them having been proved. I have discovered that my demonstration was defective since in an equation where there were to sums with two summation indices, I mistakenly changed the summation indices in a sum, a most natural thing to do!

The counterexample exhibited by Cartan, as he went on to explain in his letter to Schouten, was an almost straightforward generalization of Clifford's parallelism to the 7–dimensional elliptic space. The possibility of such generalization was due to the existence of the (non–associative) division algebra of the octonions of Graves and Cayley. In Cartan's words:

> It should be remarked that the preceding definition [i.e. the definition of parallelism in S^7] is an immediate generalization of Clifford's parallelism in the elliptic 3–space; in the latter case, one has just to replace the octonions with the quaternions.

Clearly, the exhibition of a counterexample did not resolve the classification problem at all. There was still the possibility of the existence of Riemannian manifolds, other than semi–simple Lie groups and S^7, that were solutions to problem (1). In this respect, possibly because of Frobenius' theorem on the classification of division algebras, Cartan was inclined to think that, besides S^3, S^7 was the only elliptic space admitting this type of parallelisms. He even ventured to say that S^7 could represent the only exception. He wrote in the above quoted letter to Schouten:

> It is not impossible that the 7–dimensional elliptic space is the only manifold that represents an exception to the theorem under examination. Can one provide a general proof thereof [...]? This is the question!

The search for such a general proof was completed in the short span of a few weeks. Both Schouten and Cartan succeeded in demonstrating that the solutions to problem (1) belonged to two sharply distinct types. On one hand, there was the case of semisimple Lie groups; on the other, the exception of the 7–dimensional elliptic case.

In order to understand how this distinction emerged, we now follow the treatment offered in [Cartan and Schouten, 1926b] by adding some comments. It should be stressed that the first of part of [Cartan and Schouten, 1926b] was written by Schouten alone. This is reflected in the tensorial character of the treatment. It is unclear how Cartan obtained the proof of this result. Nonetheless, as will be seen, the characterization of the second case, that corresponding to S^7, was Cartan's merit. His methods, relying upon holonomy group techniques, turned out to be the most appropriate ones for this aim.

The starting point of the classification was represented by equation (4.1) that in view of the metricity condition (4.2) can be written as follows:

$$'\Gamma^{\nu}_{\kappa\mu} = \Gamma^{\nu}_{\kappa\mu} + S_{\mu\nu}^{\cdot\cdot\,\kappa}, \tag{4.6}$$

$S_{\lambda\mu\kappa} = S_{\lambda\mu}^{\cdot\cdot\,\rho} g_{\rho\kappa}$ being a trivector. By computing the Riemann curvature tensor for the connection $'\Gamma^{\nu}_{\kappa\mu}$ and imposing that this is identically equal to zero (in view of the flatness condition), one obtains the following formula that expresses the Riemann tensor of the Levi-Civita connection[42] in terms of the torsion tensor of $'\Gamma$:

$$R_{\omega\mu\lambda\nu} = \frac{1}{3} S_{\alpha\mu\lambda} S^{\alpha}_{\cdot\,\nu\omega} - \frac{2}{3} S_{\alpha\lambda\nu} S^{\alpha}_{\cdot\,\mu\omega} + \frac{1}{3} S_{\alpha\nu\mu} S^{\alpha}_{\cdot\,\lambda\omega}. \tag{4.7}$$

It can be shown through direct computation that the Riemannian tensor $R_{\omega\mu\lambda\nu}$ is covariantly constant and thus that the Riemannian manifold is (locally) symmetric. Evidently this implies that the Ricci tensor is covariantly constant as well: $\nabla R_{\nu\mu\lambda}^{\quad\;\nu}{}_{\cdot} = 0$. At this point Schouten and Cartan introduced a technical requirement that was functional to the subsequent analysis, namely the irreducibility of the manifold. Their treatment was in this respect not completely clear. Nonetheless, it seems that the main aim of such hypothesis was represented by the applicability of a theorem due to Eisenhart concerning the existence of symmetric tensors of the second order whose covariant derivative is zero. More precisely, Eisenhart had proved that "a necessary and sufficient condition that a Riemann space admit a symmetric covariant tensor of the second order $A_{\mu\nu}$ other than [...] the metric $g_{\mu\nu}$, such that its first covariant derivative is zero, is that the metric g be reducible to a sum of forms $g_1 \oplus g_2 \oplus \ldots \oplus g_s$." As a consequence of this theorem together with the irreducibility requirement it follows that the manifold is Einstein, i.e., in Cartan's and Schouten's language, the Ricci principal directions are undetermined: $R_{\mu\lambda} = c g_{\mu\lambda}$. In a similar way, since the symmetric[43] tensor $B_{\lambda\mu} := g_{\lambda\alpha\beta} S_{\mu}^{\cdot\,\alpha\beta}$, where $g_{\lambda\alpha\beta} := S_{\gamma\lambda}^{\cdot\cdot\,\sigma} S_{\sigma\alpha}^{\cdot\cdot\,\rho} S_{\rho\beta}^{\cdot\cdot\,\gamma}$, is covariantly constant, that is $\nabla B_{\lambda\mu} = 0$, Cartan and Schouten could easily deduce that $B_{\lambda\mu}$ is proportional to the metric tensor $g_{\lambda\mu}$, $B_{\lambda\mu} = A g_{\lambda\mu}$. By choosing A equal to $-c\rho$, they obtained:

$$g_{\lambda\mu\nu} = \rho S_{\lambda\mu\nu}. \tag{4.8}$$

Now, applying the second Bianchi identity to the Riemann curvature of the Levi–Civita connection in the form $\nabla_{[\xi} K_{\omega\mu]\lambda\nu} = 0$,[44] one has:

$$c S_{\lambda\alpha[\mu} S^{\alpha}_{\cdot\,\omega\xi]} = -2 g_{\lambda\alpha[\mu} S^{\alpha}_{\cdot\,\omega\xi]}. \tag{4.9}$$

The substitution of (4.8) into (4.9) allowed Cartan and Schouten to single out two distinct cases. Indeed, one obtains:

$$(c + 2\rho)\, S_{\lambda\alpha[\mu} S^{\alpha}_{\cdot\,\omega\xi]} = 0. \tag{4.10}$$

[42] As for Schouten's definition of the components of the Riemannian tensor, see e.g. [Schouten, 1924, §12-14].

[43] Its symmetry can be proved via a direct computation by exploiting the complete antisymmetry of the trivector $S_{\alpha\beta\gamma}$.

[44] As for the squared bracket notation, see [Schouten, 1924, p. 4], namely the section *Bezeichnung der Alternation und der Mischung*. Consider for example the tensor $v_{\lambda\mu}$, then $v_{[\lambda\mu]} = \frac{1}{2}(v_{\lambda\mu} - v_{\mu\lambda})$.

There are two possibilities: either $c = -2\rho$ or $S_{\lambda\alpha[\mu}S^{\alpha}_{\cdot\,\omega\xi]} = 0$. In the latter case, the geometry of simple Lie groups is recovered since the condition $S_{\lambda\alpha[\mu}S^{\alpha}_{\cdot\,\omega\xi]} = 0$ is equivalent to the conservation of the torsion and thus that the functions c's in Cartan's structural equations (i.e. the components of the torsion tensor with respect to a parallel frame) are constant. Furthermore, it can be proved that the conservation of the torsion tensor implies $c = +2\rho$.

The case $c = -2\rho$ represents the above discussed exception. It was Cartan's merit alone the discovery that S^7 is the only possible case (beyond those represented by Lie groups).[45] His approach was based upon consideration of the holonomy group of a manifold along with classification results form the theory of simple Lie algebras. We will not go into this part of the theory at all. Nonetheless, it seems worthwhile to provide a discussion of the much weaker result according to which the 3–dimensional and the 7–dimensional elliptic spaces are the only manifolds with constant curvature that are solutions to problem (1). We follow in this respect the draft of a letter written by Schouten on 1st March 1926. Starting from the above made positions, i.e. (4.7) and (4.8), Schouten derived the following relation:

$$R_{\omega\mu\lambda\nu}S^{\lambda\nu}_{\cdot\,\cdot\,\xi} = -\frac{2}{3}\,(c+\rho)\,S_{\omega\mu\xi}. \tag{4.11}$$

Since the space is supposed to have constant Riemannian curvature, $R_{\lambda\mu\omega\nu}$ can be expressed as

$$R_{\omega\mu\lambda\nu} = -\frac{2c}{n-1}g_{[\lambda[\omega}g_{\mu]\nu]}. \tag{4.12}$$

By substituing (4.12) in (4.11), after some tedious calculations and manipulations with indices, one obtains:

$$-\frac{2c}{n-1} = -\frac{2}{3}\,(c+\rho)\,. \tag{4.13}$$

Now, the only possible values for n are, for $c = 2\rho$, $n = 3$ (in this case, one recovers the 3–dimensional elliptic space) and, for $c = -\rho$, $n = 7$ which gives S^7 as a possible solution to problem (1).

5. A modern description of the problem

The purpose of this section is that of offering the reader a modern treatment of some technical aspects of the collaboration. In particular, we want to discuss in modern terms the three canonical connections that can be attributed to Lie groups and to provide a deduction of some properties that characterize the connections relevant to the solution of problem (1), namely the 3–form character of the torsion tensor (a property that Cartan and Schouten referred to with the denomination "trivector") and the property according to which the Riemann curvature tensor, associated to Levi–Civita connection, is covariantly constant.

5.1. **Basics on the method of moving frames.** In this section we follow closely the presentation in [Alías et al., 2016]. Let (M, g) be a Riemannian manifold of dimension m with metric g. Let $p \in M$ and let (U, φ) be a *local chart* such that $p \in U$. Denote by $x^1, \ldots, x^m$, $m = \dim M$ the coordinate functions on U. Then, at any $q \in U$ we have

$$g = g_{ij}\,dx^i \otimes dx^j \tag{5.1}$$

[45] It should be noted that the proof of this result provided by Cartan is somehow obscure. It is extremely difficult to follow every single detail. Mathematicians, such as J. Wolf and I. Agricola, expressed some doubts on the correctness of Cartan's reasoning. See e.g. [Wolf, 1972].

where dx^i denotes the differential of the function x^i and g_{ij} are the (local) components of the metric defined by $g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$. Applying in q the Gram-Schmidt orthonormalization process we can find linear combinations of the 1-forms dx^i which we will call θ^i for $i = 1, \ldots, m$. Then (5.1) takes the form

$$g = \delta_{ij}\theta^i \otimes \theta^j, \tag{5.2}$$

where δ_{ij} is the Kronecker symbol. Since, as q varies in U, the previous process gives rise to coefficients that are C^∞ functions of q, the set of 1-forms $\left\{\theta^i\right\}$ defines an orthonormal system on U for the metric g, i.e. a (local) *orthonormal coframe.* It is usual to write

$$g = \sum_{i=1}^{m}(\theta^i)^2,$$

instead of (5.2). We also define the (local) *dual orthonormal frame* $\{e_i\}$, for $i = 1, \ldots, m$, as the set of vector fields on U satisfying

$$\theta^j(e_i) = \delta_i^j, \tag{5.3}$$

where δ_i^j is the Kronecker symbol. We have the following result.

Proposition 5.1. *Let $\{\theta^i\}$ be a local orthonormal coframe defined on the open set $U \subset M$; then on U there exist unique 1-forms $\left\{{}^g\theta_j^i\right\}$, for $i, j = 1 \ldots, m$, such that*

$$d\theta^i = -\,{}^g\theta_j^i \wedge \theta^j \tag{5.4}$$

and

$${}^g\theta_j^i + {}^g\theta_i^j = 0 \tag{5.5}$$

The forms ${}^g\theta_j^i$ are called the *Levi-Civita connections forms* associated to the orthonormal coframe $\{\theta^i\}$, while equation (5.4) is called the *first structure equation.*

Starting from the Levi-Civita connection forms, we can define a *covariant derivative* ∇^g on every tensor bundle. Let $\{e_i\}, \{\theta^i\}$ be an orthonormal frame and its dual coframe on the open set U. The *connection* ∇^g *induced by the Levi-Civita connection forms* is defined by

$$\nabla^g e_i = {}^g\theta_i^j \otimes e_j, \tag{5.6}$$

and, for every $X, Y \in \mathfrak{X}(U)$, $f \in C^\infty(U)$, by the rules

$$\nabla^g(X + Y) = \nabla^g X + \nabla^g Y, \qquad \nabla^g(fX) = df \otimes X + f\nabla X; \tag{5.7}$$

the *dual connection*, still denoted with ∇^g, is given by the formula

$$\nabla^g\theta^i = -\,{}^g\theta_j^i \otimes \theta^j$$

(which follows imposing the condition $\nabla^g\theta^i(e_j) + \theta^i(\nabla^g e_j) = \nabla^g\big(\theta^i(e_j)\big) = d\big(\theta^i(e_j)\big) = 0$.

The *curvature forms* $\{{}^g\Theta_j^i\}$ are associated to the orthonormal coframe $\{\theta^i\}$ through the *second structure equation*

$$d\,{}^g\theta_j^i = -\,{}^g\theta_k^i \wedge {}^g\theta_j^k + {}^g\Theta_j^i. \tag{5.8}$$

Because of (5.5) it follows immediately that

$${}^g\Theta_j^i + {}^g\Theta_i^j = 0. \tag{5.9}$$

Using the basis $\{\theta^i \wedge \theta^j\}$, for $1 \leq i < j \leq m$, of the space of skew-symmetric 2-forms $\Lambda^2(U)$ on the open set U, we may write

$$ {}^g\Theta^i_j = \frac{1}{2}\, {}^gR^i_{jkt}\theta^k \wedge \theta^t \tag{5.10}$$

for some coefficients ${}^gR^i_{jkt} \in C^\infty(U)$ satisfying

$$ {}^gR^i_{jkt} + {}^gR^i_{jtk} = 0. \tag{5.11}$$

These are the coefficients of the $(1,3)$–version of the *Riemann curvature tensor* which we denote by ${}^gR$. More precisely, in this local orthonormal frame we have

$$ {}^gR^i_{jkt} = {}^g\Theta^i_j(e_k, e_t) = (d\,{}^g\theta^i_j + {}^g\theta^i_s \wedge {}^g\theta^s_j)(e_k, e_t) = g({}^gR(e_k, e_t)e_j, e_i), \tag{5.12}$$

so that its components are

$$ {}^gR = {}^gR^i_{jkt}\theta^k \otimes \theta^t \otimes \theta^j \otimes e_i. $$

Note that (5.9) implies

$$ {}^gR^i_{jkt} + {}^gR^j_{ikt} = 0. \tag{5.13}$$

The $(0,4)$–version of R is defined by ${}^g\mathrm{Riem}(X,Y,Z,W) = g({}^gR(Z,W)Y,X)$, so that its local coefficients ${}^gR_{ijkt}$ satisfy

$$ {}^gR_{ijkt} = {}^g\mathrm{Riem}(e_i, e_j, e_k, e_t) = g({}^gR(e_k, e_t)e_j, e_i) = {}^gR^i_{jkt} \tag{5.14}$$

and thus in the local orthonormal frame

$$ {}^g\mathrm{Riem} = {}^gR_{ijkt}\theta^i \otimes \theta^j \otimes \theta^k \otimes \theta^t. \tag{5.15}$$

We recall that the Riemann curvature tensor also has the simmetry

$$ {}^gR^i_{jkt} = {}^gR^k_{tij} \qquad \text{(equivalently: } {}^gR_{ijkt} = {}^gR_{ktij}). \tag{5.16}$$

For the proof and for further details, we refer to [Alías et al., 2016].

The *Ricci tensor* is the symmetric $(0,2)$–tensor obtained from (5.15) by tracing either with respect to i and k or, equivalently, with respect to j and t. Thus

$$ {}^g\mathrm{Ric} = {}^gR_{ij}\theta^i \otimes \theta^j $$

with

$$ {}^gR_{ij} = {}^gR_{itjt} = {}^gR_{titj}. $$

5.2. **Connections on Lie groups.** We now provide a modern treatment of the three canonical connections on Lie groups that Schouten and Cartan introduced in [Cartan and Schouten, 1926b]. To this aim, we will closely follow [Postnikov, 2001, Chap. 6].

Lie groups are differential manifolds with many special properties; among them, parallelizability is the most relevant one in view of our scope. For an arbitrary n–dimensional manifold M this means that the $C^\infty(M)$-module of differential vector fields $\mathfrak{X}(M)$ is a free module of rank n. For Lie groups, even more it is true in the sense that the following result holds: every base of the Lie algebra $\mathfrak{g}$ of G is also a base of the $C^\infty(G)$–module $\mathfrak{X}(G)$. We now define a connection ∇ to be left invariant if for any two vector fields $X, Y \in \mathfrak{g}$ the field $\nabla_X Y$ is also left invariant, i.e. $\nabla_X Y \in \mathfrak{g}$. Now, define the mapping $\alpha : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ through the relation $\alpha(X,Y) = \nabla_X Y$. By the definition, the mapping α is $\mathbb{R}$–linear and thus it is a multiplication in $\mathfrak{g}$. As a consequence of the above made remarks on parallelizability of Lie groups, we easily see that every left invariant connection on G is univocally determined by the fields $A_{ij} := \alpha(X_i, X_j)$, where $X_i, X_j \in \mathfrak{g}$, $i, j = 1, \ldots, n$. There is indeed a bijective correspondence between left invariant connections on Lie groups and multiplications in the corresponding Lie algebras.

Evidently, the multiplication map α can be decomposed into a symmetric (α') and a skew–symmetric (α'') addendum as follows:

$$\alpha(X, Y) = \frac{\alpha(X, Y) + \alpha(Y, X)}{2} + \frac{\alpha(X, Y) - \alpha(Y, X)}{2}. \tag{5.17}$$

Now we are in the position to introduce the modern definition of Schouten's and Cartan's $(\pm)$, (0)–connections. In order to do that we preliminarily observe that for each vector $A \in T_eG$ there are two, in principle distinct, curves passing through the identity e and having A as tangent vector in e. One of them is the one parameter group subgroup (β_A) corresponding to the left invariant vector field univocally associated to A; the other is the geodesic passing through e and having A as tangent vector in e, (γ_A). We are led to the following definition of Cartan (or canonical) connections on a Lie group:

Definition 5.2. *A left invariant connection ∇ on a Lie group G is said to be a* canonical connection *if for any vector $A \in T_eG$ the corresponding one parameter subgroup and geodesic coincide.*

As a consequence of the fact that β_A is the integral curve of the left invariant vector field $\widetilde{A}$ associated to $A \in \mathfrak{g}$, it can be proved that $\beta_A : t \mapsto \beta_A$ is also a geodesic if, and only if, $\alpha(\widetilde{A}, \widetilde{A}) = 0$. Thus, a left–invariant connection ∇ is a Cartan connection if, and only if, the corresponding multiplication α is skew–symmetric, i.e. $\alpha' = 0$. An easy stipulation for $\alpha(X, Y)$ is as follows:

$$\nabla_X Y = \lambda[X, Y], \qquad X, Y \in \mathfrak{g}. \tag{5.18}$$

Depending on the value that one attributes to λ, namely $\lambda = 1, \lambda = \frac{1}{2}$, or $\lambda = 0$, we obtain the connections $(+), (0)$ and $(-)$, respectively.

Let us now consider in some detail the case of the $(-)$ connection. The condition $\nabla_X Y = 0$ holds for all fields $X \in \mathfrak{X}(G)$ if, and only if, it holds for all $X \in \mathfrak{g}$ (again, this is a consequence of parallelizability). Now, the field $Y = f^i X_i$, with $X_1, \ldots, X_n$ base of $\in \mathfrak{g}$ and $f^i \in C^\infty(G)$, is covariantly constant if, and only if $Xf^i = 0$ for any $i = 1, \ldots, n$ and any $X \in \mathfrak{g}$. But $X_i f = 0$ for $i = 1, \ldots, n$ implies $f = \text{const}$. Thus, $Y \in \mathfrak{g}$. Consequently, the vector fields Y that are covariantly constant with respect to the connection $(-)$ are exactly the left–invariant vector fields. This implies that for any points $p, q \in G$, the parallel translation corresponding to connection $(-)$, $\Pi_{p,q} : T_pG \to T_qG$ is given by $\Pi_{p,q} = dL_{qp^{-1}}$, where L_p is the left translation by $p \in G$. This means that with respect to connection $(-)$, the Lie group G is a space with absolute parallelism.

In a completely similar way, one can prove that the vector fields that are covariantly constant with respect to the connection $(+)$ are exactly the right–invariant vector fields of the Lie group G. Finally, one can easily demonstrate that the connection (0) is characterized by a curvature that is given by $R(X, Y)Z = -\frac{1}{4}\,[[X, Y]\, Z]$ for $X, Y, Z \in \mathfrak{g}$.

5.3. **Types of connection.** Here and in the rest of the section we follow [Agricola and Friedrich, 2010] and [Agricola, 2006], including all the relevant computations. We explicitly note that our definition of the curvature tensor Riem differs from the one employed in the aforementioned works by a minus sign.

Let (M, g) be a m-dimensional Riemannian manifold with metric g and associated Levi-Civita connection ∇^g. If ∇ is another (linear) connection, we recall that the *torsion* of ∇ on M is the $(1, 2)$-tensor field defined by

$$\mathrm{Tor}(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y], \quad X, Y \in \mathfrak{X}(M), \tag{5.19}$$

where $[\ ,\ ]$ is the Lie bracket and $\mathfrak{X}(M)$ is the set of all smooth vector fields on M. Note that Tor^g, the torsion tensor associated to ∇^g, is vanishing by the fundamental theorem of Riemannian geometry. A simple computation shows that the difference between ∇ and ∇^g is a $(1, 2)$-tensor field that we denote with $\mathcal{A}$, i.e.

$$\nabla_X Y = \nabla^g_X Y + \mathcal{A}(X, Y), \quad X, Y \in \mathfrak{X}(M), \tag{5.20}$$

and which, in a very important case, can be related to the torsion Tor, as we shall see in a shortwhile. From now on, when needed, we shall also use the $(0, 3)$-version of Tor and $\mathcal{A}$, respectively T and A, that are defined by

$$A(X, Y, Z) = g(\mathcal{A}(X, Y), Z) \quad \text{and} \quad T(X, Y, Z) = g(\mathrm{Tor}\,(X, Y), Z). \tag{5.21}$$

The vanishing of the symmetric or of the antisymmetric part of $\mathcal{A}$ has direct geometric consequences: indeed, a simple computation shows the validity of the following

Lemma 5.3. *The connection* ∇

- i) *is torsion-free if and only if $\mathcal{A}$ is symmetric;*
- ii) *has the same geodesics of the Levi-Civita connection if and only if $\mathcal{A}$ is skew-symmetric;*
- iii) *is compatible with the metric if and only if $A(X, Y, Z) + A(X, Z, Y) = 0$ for all $X, Y, Z \in \mathfrak{X}(M)$.*

Using the same notation of [Agricola, 2006], if $\mathcal{T}$ is the $\frac{n^2(n-1)}{2}$-dimensional space of all possible torsion tensors,

$$\mathcal{T} = \left\{T \in \otimes^3 TM \mid T(X, Y, Z) = -T(Y, X, Z)\right\} \cong \Lambda^2 TM \otimes TM$$

and $\mathcal{A}^g$ is the space

$$\mathcal{A}^g = TM \otimes \Lambda^2 TM = \left\{A \in \otimes^3 TM \mid A(X, V, W) + A(X, W, V) = 0\right\},$$

then we have that $\dim \mathcal{A}^g = \dim \mathcal{T}$: this is related to the fact that metric connections can be uniquely characterized by their torsion, indeed we have the following (see [Agricola, 2006, Proposition 2.1], [Cartan, 1925], [Tricerri and Vanhecke, 1983], [Salamon, 1989])

Proposition 5.4. *The spaces $\mathcal{T}$ and $\mathcal{A}^g$ are isomorphic as $O(n)$ representations, an equivariant bijection being*

$$T(X, Y, Z) = A(X, Y, Z) - A(Y, X, Z),$$
$$2A(X, Y, Z) = T(X, Y, Z) - T(Y, Z, X) + T(Z, X, Y).$$

For $n \geq 3$, they split under the action of $O(n)$ into the sum of three irreducible representations,

$$\mathcal{T} \cong TM \oplus \Lambda^3 TM \oplus \mathcal{T}'.$$

The last module (denoted $\mathcal{A}'$ if viewed as a subspace of $\mathcal{A}^g$) is equivalent to the Cartan product of representations $TM \otimes \Lambda^2 TM$,

$$\mathcal{T}' = \left\{ T \in \mathcal{T} \mid \overset{X,Y,Z}{\mathfrak{S}} T(X,Y,Z) = 0, \sum_{i=1}^{n} T(e_i, e_i, X) = 0 \,\forall\, X, Y, Z \right\}$$

for any orthonormal frame $e_1, \ldots, e_n$ and where $\overset{X,Y,Z}{\mathfrak{S}}$ denotes a sum over a cyclic permutation of X, Y, Z. For $n = 2$, $\mathcal{T} \cong \mathcal{A}^g \cong \mathbb{R}^2$ is $O(2)$-irreducible.

The eight classes of linear connections are now defined by the possible components of their torsions T in these spaces. Incidentally, it should be noticed that Cartan was the first to provide a classification of the types of torsion tensor in [Cartan, 1925, p. 51].

5.4. **Skew-symmetric torsion.**

Definition 5.5. *The connection ∇ is said to have* (totally) skew-symmetric torsion *if its torsion tensor lies in the second space of the decomposition of Proposition 5.4, i.e. if it is given by a 3-form.*

We have the following

Corollary 5.6. *A connection ∇ on (M, g) is metric and geodesic-preserving if and only if its torsion T lies in $\Lambda^3 TM$. In this case we have $T = 2A$,*

$$\nabla_X Y = \nabla^g_X Y + \frac{1}{2} \operatorname{Tor}(X, Y, \cdot), \tag{5.22}$$

and the ∇-Killing vector fields coincide with the Riemannian Killing vector fields.

Note that equation (5.22) can be written as

$$\nabla Y = \nabla^g Y + \frac{1}{2} \operatorname{Tor}(\cdot, Y). \tag{5.23}$$

5.4.1. *Moving frame for skew-symmetric torsion.* Let $\{\theta^i\}$ be a local orthonormal coframe, with dual frame $\{e_i\}$. We write

$$\operatorname{Tor} = T^k_{ij} \theta^j \otimes \theta^i \otimes e_k, \quad T^k_{ij} = -T^k_{ji}, \tag{5.24}$$

so that, if $X, Y \in \mathfrak{X}(M)$ and $X = X^i e_i$, $Y = Y^j e_j$,

$$\operatorname{Tor}(\cdot\,, Y) = T^k_{ij} Y^i \theta^j \otimes e_k, \quad \operatorname{Tor}(X, Y) = T^k_{ij} X^j Y^i e_k.$$

Remark 5.7. We explicitly note that our convention for the torsion, chosen for computational reasons, differs from that of Cartan for a minus sign (see, for example, equation (2.7)).

Our aim is to find how the Levi-Civita connection forms θ^i_j and the curvature forms Ω^i_j of the connection ∇ are related to the corresponding Levi-Civita connection forms ${}^g\theta^i_j$ and the curvature forms ${}^g\Omega^i_j$ of the Levi-Civita connection ∇^g. To this purpose we note that equation (5.23) implies

$$\nabla e_i = \nabla^g e_i + \frac{1}{2} \operatorname{Tor}(\cdot\,, e_i),$$

that is,

$$\nabla e_i = {}^g\theta^j_i \otimes e_j + \frac{1}{2} T^j_{ik} \theta^k \otimes e_j,$$

which implies, renaming indexes and setting $\nabla e_i = \theta^j_i \otimes e_j$,

$$\theta^i_j = {}^g\theta^i_j + \frac{1}{2} T^i_{jk} \theta^k. \tag{5.25}$$

From (5.25) and (5.4) we immediately deduce the *first structure equation* for the connection ∇, that is

$$d\theta^i = -\left(\theta^i_j - \frac{1}{2}T^i_{jk}\theta^k\right) \wedge \theta^j. \tag{5.26}$$

Recalling the definition (5.21), we note that, writing $T = T_{ijk}\theta^k \otimes \theta^j \otimes \theta^i$, we have

$$T_{ijk} = T(e_k, e_j, e_i) = g\left(\mathrm{Tor}\,(e_k, e_j), e_i\right) = T^i_{jk},$$

thus

$$T^i_{jk} = T_{ijk}, \tag{5.27}$$

which is consistent with the convention in [Alías et al., 2016]. Note also that the skew-symmetry of T implies

$$T_{ijk} = -T_{jik} = T_{jki} = -T_{kji} = -T_{ikj}. \tag{5.28}$$

For the covariant derivative of the torsion with respect to the connection ∇ we use the notation

$$\nabla\,\mathrm{Tor} = {}^{\nabla}T^k_{ij,t}\theta^t \otimes \theta^j \otimes \theta^i \otimes e_k, \quad \nabla T = {}^{\nabla}T_{ijk,t}\theta^t \otimes \theta^k \otimes \theta^j \otimes \theta^i, \tag{5.29}$$

and we explicitly note that, since ∇ is metric by assumption, for every $X, Y, Z, T, W \in \mathfrak{X}(M)$

$$(\nabla_W T)(X, Y, Z) = g\left((\nabla_W\,\mathrm{Tor})(X, Y), Z\right), \tag{5.30}$$

which implies (choosing $X = e_k$, $Y = e_j$ and $Z = e_i$)

$${}^{\nabla}T_{ijk,t} = {}^{\nabla}T^i_{jk,t}. \tag{5.31}$$

We have the following:

Proposition 5.8. *The Riemann curvature tensors of the Levi-Civita connection and of the connection ∇ satisfy the relation*

$${}^{g}R^i_{jkt} = {}^{\nabla}R^i_{jkt} + \frac{1}{2}\left({}^{\nabla}T^i_{jk,t} - {}^{\nabla}T^i_{jt,k}\right) + \frac{1}{2}T^i_{jr}T^r_{kt} + \frac{1}{4}T^i_{rk}T^r_{jt} + \frac{1}{4}T^i_{rt}T^r_{kj}, \tag{5.32}$$

or, equivalently,

$${}^{g}R_{ijkt} = {}^{\nabla}R_{ijkt} + \frac{1}{2}\left({}^{\nabla}T_{ijk,t} - {}^{\nabla}T_{ijt,k}\right) + \frac{1}{2}T_{ijr}T_{rkt} + \frac{1}{4}T_{irk}T_{rjt} + \frac{1}{4}T_{irt}T_{rkj}, \tag{5.33}$$

Proof. From equations (5.12), (5.25) and (5.26) we have

$$\begin{aligned}
{}^{g}R(e_k, e_t)e_j &= \left[{}^{g}\theta^i_r \wedge {}^{g}\theta^r_j)(e_k, e_t) + (d\,{}^{g}\theta^i_j)(e_k, e_t)\right]e_i \\
&= \left\{\left[\left(\theta^i_r - \frac{1}{2}T^i_{rs}\theta^s\right) \wedge \left(\theta^r_j - \frac{1}{2}T^r_{jl}\theta^l\right)\right](e_k, e_t) + d\left(\theta^i_j - \frac{1}{2}T^i_{js}\theta^s\right)(e_k, e_t)\right\}e_i \\
&= {}^{\nabla}R(e_k, e_t)e_j + \frac{1}{2}\left\{\left[-T^r_{js}\theta^i_r \wedge \theta^s - T^i_{rs}\theta^s \wedge \theta^r_j + \frac{1}{2}T^i_{rs}T^r_{jl}\theta^s \wedge \theta^l - dT^i_{js} \wedge \theta^s - T^i_{jr}d\theta^r\right](e_k, e_t)\right\}e_i \\
&= {}^{\nabla}R(e_k, e_t)e_j - \frac{1}{2}\left\{\left[{}^{\nabla}T^i_{js,l}\theta^l \wedge \theta^s + \frac{1}{2}\left(T^i_{jr}T^r_{sl} + T^i_{rs}T^r_{jl}\right)\theta^l \wedge \theta^s\right](e_k, e_t)\right\}e_i \\
&= {}^{\nabla}R(e_k, e_t)e_j - \frac{1}{2}\left\{\left({}^{\nabla}T^i_{js,l} + \frac{1}{2}T^i_{jr}T^r_{sl} + \frac{1}{2}T^i_{rs}T^r_{jl}\right)\left(\theta^l \wedge \theta^s\right)(e_k, e_t)\right\}e_i.
\end{aligned}$$

Since $\left(\theta^l \wedge \theta^s\right)(e_k, e_t) = \delta^l_k\delta^s_t - \delta^l_t\delta^s_k$, the previous relation becomes

$${}^{g}R(e_k, e_t)e_j = {}^{\nabla}R(e_k, e_t)e_j$$

which immediately implies (5.32).

Equation (5.33) now follows from (5.32) using (5.14), (5.27) and (5.31). □

Remark 5.9. In global notation, equation (5.33) becomes

$$
\begin{aligned}
{}^gR(X,Y,Z,W) &= {}^\nabla R(X,Y,Z,W) + \frac{1}{2}(\nabla_X T)(Y,Z,W) - \frac{1}{2}(\nabla_Y T)(X,Z,W) \\
&+ \frac{1}{2}g\left(\mathrm{Tor}(X,Y),\mathrm{Tor}(Z,W)\right) + \frac{1}{4}g\left(\mathrm{Tor}(Y,Z),\mathrm{Tor}(X,W)\right) \\
&+ \frac{1}{4}g\left(\mathrm{Tor}(Z,X),\mathrm{Tor}(Y,W)\right) \\
&= {}^\nabla R(X,Y,Z,W) + \frac{1}{2}(\nabla_X T)(Y,Z,W) - \frac{1}{2}(\nabla_Y T)(X,Z,W) \\
&+ \frac{1}{4}\sigma_T(X,Y,Z,W) + \frac{1}{4}g\left(\mathrm{Tor}(X,Y),\mathrm{Tor}(Z,W)\right)
\end{aligned}
\tag{5.34}
$$

for every $X,Y,Z,W \in \mathfrak{X}(M)$, where the quantity σ_T, which is defined as

$$
\sigma_T(X,Y,Z,W) = g\left(\mathrm{Tor}\,(X,Y),\mathrm{Tor}\,(Z,W)\right) + g\left(\mathrm{Tor}\,(Y,Z),\mathrm{Tor}\,(X,W)\right) + g\left(\mathrm{Tor}\,(Z,X),\mathrm{Tor}\,(Y,W)\right),
\tag{5.35}
$$

measures, in a certain sense, the non-degeneracy of the torsion (see [Agricola et al., 2015]).

Remark 5.10. Using the transformation law for the covariant derivative of the torsion, that is

$$
{}^\nabla T^i_{jk,t} = {}^gT^i_{jk,t} - \frac{1}{2}(T_{rjk}T_{rit} + T_{irk}T_{rjt} + T_{ijr}T_{rkt})
\tag{5.36}
$$

(which follows easily from ${}^\nabla T^i_{jk,t}\theta^t = dT_{ijk} - T_{rjk}\theta^r_i - T_{irk}\theta^r_j - T_{ijr}\theta^r_k$ and equation (5.25)), we can write equation (5.33) in the more compact form

$$
{}^\nabla R_{ijkt} = {}^gR_{ijkt} + \frac{1}{2}({}^gT_{ijk,t} - {}^gT_{ijt,k}) + \frac{1}{4}T_{irk}T_{rjt} + \frac{1}{4}T_{irt}T_{rkj}.
\tag{5.37}
$$

Note that we could have obtained the same result starting from the second structure equation (5.8) and using (5.25): this approach leads to the *second structure equation* for the connection ∇, which reads as

$$
d\theta^i_j = -\theta^i_k \wedge \theta^k_j + \Omega^i_j, \qquad \Omega^i_j = \frac{1}{2}\,{}^\nabla R^i_{jkt}\theta^k \wedge \theta^t.
\tag{5.38}
$$

5.5. **Symmetries of the curvature tensor.**

Proposition 5.11. *For every connection and for every $X,Y,Z,W \in \mathfrak{X}(M)$ we have*

$$
R(Z,W)Y = -R(W,Z)Y \Leftrightarrow \mathrm{Riem}\,(X,Y,Z,W) = -\,\mathrm{Riem}\,(X,Y,W,Z),
\tag{5.39}
$$

while for every connection compatible with the metric

$$
g\left(R(Z,W)Y,X\right) = -g\left(R(Z,W)X,Y\right) \Leftrightarrow \mathrm{Riem}\,(X,Y,Z,W) = -\,\mathrm{Riem}\,(Y,X,Z,W).
\tag{5.40}
$$

5.6. **Parallelism of ${}^gR$.** In this section we want to prove, using the moving frame method, the following theorem:

Theorem 5.12. *Let (M,g) be a m-dimensional Riemannian manifold with metric g. If (M,g) admits a flat connection ∇ which is compatible with the metric and with the same geodesics of ${}^g\nabla$, then the Riemann curvature tensor associated to the Levi-Civita connection ${}^gR$ is parallel with respect to the latter, that is,*

$$
{}^g\nabla\,{}^gR = 0.
$$

Proof. Since ∇ is flat equation (5.37) becomes

$$ {}^{g}R_{ijkt} = \frac{1}{2}({}^{g}T_{ijk,t} - {}^{g}T_{ijt,k}) + \frac{1}{4}(T_{rik}T_{rjt} + T_{rit}T_{rkj}). \tag{5.41} $$

Step 1: First we prove that ${}^{\nabla}T_{ijk,t} = \frac{1}{3}\sigma_T(e_k, e_j, e_i, e_t)$, ${}^{g}T_{ijk,t} = -\frac{1}{6}\sigma_T(e_k, e_j, e_i, e_t)$.

By definition (see equation (5.35)) we have

$$ \sigma_T(X, Y, Z, W) = g\left(\mathrm{Tor}\,(X, Y), \mathrm{Tor}\,(Z, W)\right) + g\left(\mathrm{Tor}\,(Y, Z), \mathrm{Tor}\,(X, W)\right) + g\left(\mathrm{Tor}\,(Z, X), \mathrm{Tor}\,(Y, W)\right), $$

thus

$$ \sigma_T(e_k, e_j, e_i, e_t) = T^s_{jk}T^s_{ti} + T^s_{ij}T^s_{tk} + T^s_{ki}T^s_{tj} $$

or, equivalently,

$$ \sigma_T(e_k, e_j, e_i, e_t) = T_{sij}T_{stk} + T_{sjk}T_{sti} + T_{ski}T_{stj}. \tag{5.42} $$

The first Bianchi identity ${}^{g}R_{ijkt} + {}^{g}R_{iktj} + {}^{g}R_{itjk} = 0$ implies, using (5.41), that

$$ \begin{aligned} 0 &= \frac{1}{2}({}^{g}T_{ijk,t} - {}^{g}T_{ijt,k}) + \frac{1}{2}({}^{g}T_{ikt,j} - {}^{g}T_{ikj,t}) + \frac{1}{2}({}^{g}T_{itj,k} - {}^{g}T_{itk,j}) \\ &+ \frac{1}{4}(T_{rik}T_{rjt} + T_{rit}T_{rkj}) + \frac{1}{4}(T_{rit}T_{rkj} + T_{rij}T_{rtk}) + \frac{1}{4}(T_{rik}T_{rij} + T_{rik}T_{rjt}), \end{aligned} $$

that is

$$ {}^{g}T_{ijk,t} + {}^{g}T_{ijk,t} + {}^{g}T_{ijk,t} = -\frac{1}{2}(T_{sij}T_{stk} + T_{sjk}T_{sti} + T_{ski}T_{stj}) = -\frac{1}{2}\sigma_T(e_k, e_j, e_i, e_t). \tag{5.43} $$

Equation (5.41) implies

$$ {}^{g}R_{ktij} = \frac{1}{2}({}^{g}T_{kti,j} - {}^{g}T_{ktj,i}) + \frac{1}{4}(T_{rik}T_{rjt} + T_{rit}T_{rkj}); \tag{5.44} $$

now, using the symmetry (5.16), from (5.41) and (5.44) we deduce

$$ {}^{g}T_{ijk,t} - {}^{g}T_{ijt,k} = {}^{g}T_{kti,j} - {}^{g}T_{ktj,i}. \tag{5.45} $$

From equation (5.45) we get

$$ {}^{g}T_{ikt,j} = {}^{g}T_{ijk,t} + {}^{g}T_{itj,k} + {}^{g}T_{ktj,i} $$

and, permuting indices,

$$ {}^{g}T_{itj,k} = {}^{g}T_{ikt,j} + {}^{g}T_{ijk,t} + {}^{g}T_{tjk,i}; $$

adding the two previous relations gives

$$ 0 = 2({}^{g}T_{ijk,t} + {}^{g}T_{ktj,i}) = 2({}^{g}T_{jki,t} + {}^{g}T_{jkt,i}), $$

that is,

$$ {}^{g}T_{jkt,i} = -{}^{g}T_{jki,t} \tag{5.46} $$

thus ${}^{g}\nabla T$ is a 4-form. This implies

$$ {}^{g}T_{ijk,t} + {}^{g}T_{ijk,t} + {}^{g}T_{ijk,t} = 3\,{}^{g}T_{ijk,t}, $$

thus, from equation (5.43), we deduce

$$ {}^{g}T_{ijk,t} = -\frac{1}{6}(T_{sij}T_{stk} + T_{sjk}T_{sti} + T_{ski}T_{stj}) \\ = \frac{1}{6}(T_{sij}T_{skt} + T_{sjk}T_{sit} + T_{ski}T_{sjt}) \\ = -\frac{1}{6}\sigma_T(e_k, e_j, e_i, e_t). \tag{5.47}$$

On the other hand, from the definition of covariant derivative and (5.25) we have

$$ {}^{\nabla}T_{ijk,t}\theta^t = dT_{ijk} - T_{rjk}\theta^r_i - T_{irk}\theta^r_j - T_{ijr}\theta^r_k \\ = \left(dT_{ijk} - T_{rjk}\,{}^{g}\theta^r_i - T_{irk}\,{}^{g}\theta^r_j - T_{ijr}\,{}^{g}\theta^r_k\right) - \frac{1}{2}(T_{rjk}T_{rit} + T_{irk}T_{rjt} + T_{ijr}T_{rkt})\theta^t \\ = {}^{g}T_{ijk,t}\theta^t + \frac{1}{2}(T_{rij}T_{rtk} + T_{rjk}T_{rti} + T_{rki}T_{rtj})\theta^t,$$

that is

$$ {}^{\nabla}T_{ijk,t} = {}^{g}T_{ijk,t} + \frac{1}{2}(T_{sij}T_{stk} + T_{sjk}T_{sti} + T_{ski}T_{stj})$$

which implies, using (5.47),

$$ {}^{\nabla}T_{ijk,t} = \frac{1}{3}\sigma_T(e_k, e_j, e_i, e_t) \tag{5.48}$$

Step 2: now we want to express ${}^{g}R_{ijkt,r}$ using the components of T.

From (5.41) and (5.46) we have

$$ {}^{g}R_{ijkt} = {}^{g}T_{ijk,t} + \frac{1}{4}(T_{ski}T_{stj} + T_{sti}T_{sjk}) \\ = \frac{1}{6}(T_{sij}T_{skt} + T_{sjk}T_{sit} + T_{ski}T_{sjt}) - \frac{1}{4}(T_{ski}T_{sjt} + T_{sti}T_{skj}) \\ = \frac{1}{6}T_{sij}T_{skt} - \frac{1}{12}(T_{ski}T_{sjt} + T_{sjk}T_{sit}),$$

thus

$$ 6\,{}^{g}R_{ijkt} = T_{sij}T_{skt} - \frac{1}{2}(T_{ski}T_{sjt} + T_{sjk}T_{sit}). \tag{5.49}$$

Now we take the covariant derivative (with respect to the Levi-Civita connection) of the previous relation, deducing

$$ 6\,{}^{g}R_{ijkt,r} = {}^{g}T_{sij,r}T_{skt} + T_{sij}\,{}^{g}T_{skt,r} - \frac{1}{2}({}^{g}T_{ski,r}T_{sjt} + T_{ski}\,{}^{g}T_{sjt,r} + {}^{g}T_{sjk,r}T_{sit} + T_{sjk}\,{}^{g}T_{sit,r}). \tag{5.50}$$

Now we use repeatedly equation (5.47) in (5.50) to get, after some manipulation,

$$ 36\,{}^{g}R_{ijkt,r} = T_{sij}(T_{lsk}T_{ltr} - T_{lst}T_{lkr}) + T_{skt}(T_{lsi}T_{ljr} - T_{lsj}T_{lir}) \\ + \frac{1}{2}[T_{sit}(T_{lsk}T_{ljr} - T_{lsj}T_{lkr}) - T_{sjt}(T_{lsk}T_{lir} - T_{lsi}T_{lkr})] \\ + \frac{1}{2}[T_{skj}(T_{lsi}T_{ltr} - T_{lst}T_{lir}) - T_{ski}(T_{lsj}T_{ltr} - T_{lst}T_{ljr})]. \tag{5.51}$$

Now, a long but straightforward computation using the second Bianchi identity

$$ {}^{g}R_{ijkt,r} + {}^{g}R_{ijtr,k} + {}^{g}R_{ijrk,t} = 0 \tag{5.52}$$

and equation (5.51) shows that

$$0 = T_{sij}(T_{lsk}T_{ltr} - T_{lst}T_{lkr} + T_{lsr}T_{lkt}) + T_{skt}(T_{lsi}T_{ljr} - T_{lsj}T_{lir}) + T_{str}(T_{lsi}T_{ljk} - T_{lsj}T_{lik}) + T_{srk}(T_{lsi}T_{ljt} - T_{lsj}T_{lit}); \tag{5.53}$$

equivalently, renaming indexes,

$$T_{sij}(T_{lsk}T_{ltr} - T_{lst}T_{lkr}) + T_{skt}(T_{lsi}T_{ljr} - T_{lsj}T_{lir}) = T_{sij}T_{lsr}T_{ltk} + T_{ski}T_{lsj}T_{ltr} - T_{skj}T_{lsi}T_{ltr} + T_{sit}T_{lsj}T_{lkr} - T_{sjt}T_{lsi}T_{lkr}. \tag{5.54}$$

Inserting (5.54) into (5.51) we get

$$36\,{}^gR_{ijkt,r} = T_{sij}T_{lsr}T_{ltk} + \frac{1}{2}[T_{sit}(T_{lsk}T_{ljr} + T_{lsj}T_{lkr}) - T_{sjt}(T_{lsk}T_{lir} + T_{lsi}T_{lkr})] + \frac{1}{2}[T_{skj}(T_{lsi}T_{ltr} + T_{lst}T_{lir}) - T_{ski}(T_{lsj}T_{ltr} + T_{lst}T_{ljr})]. \tag{5.55}$$

Also,

$$36\,{}^gR_{ktij,r} = T_{skt}T_{lsr}T_{lji} + \frac{1}{2}[T_{skj}(T_{lsi}T_{ltr} + T_{lst}T_{lir}) - T_{stj}(T_{lsi}T_{lkr} + T_{lsk}T_{lir})] + \frac{1}{2}[T_{sik}(T_{lst}T_{ljr} + T_{lsj}T_{ltr}) - T_{sit}(T_{lsk}T_{ljr} + T_{lsj}T_{lkr})]. \tag{5.56}$$

Since, by the symmetry (5.16), ${}^gR_{ijkt,r} = {}^gR_{ktij,r}$, substracting (5.56) from (5.55) we obtain

$$0 = 2T_{sij}T_{lsr}T_{ltk} + [T_{sit}(T_{lsk}T_{ljr} - T_{lsj}T_{lkr}) - T_{sjt}(T_{lsk}T_{lir} - T_{lsi}T_{lkr})] + [T_{skj}(T_{lsi}T_{ltr} - T_{lst}T_{lir}) - T_{ski}(T_{lsj}T_{ltr} - T_{lst}T_{ljr})]$$

which implies, by (5.55), ${}^gR_{ijkt,r} = 0$. □

6. Some final remarks

Our paper represents a first attempt at a detailed study of the rich scientific correspondence between Schouten and Cartan. We limited ourselves to that part of the correspondence that is relevant to the contextualization and the understanding of their joint papers [Cartan and Schouten, 1926b] and [Cartan and Schouten, 1926a]. Nonetheless, to be sure, the richness and wideness of the manuscript legacy concerning the scientific relationship between the two illustrious geometers deserve further attention and historical studies. From a wider perspective, we would like to draw the attention of historians towards the necessity of pursuing a more systematic study of Schouten's Nachlass (Amsterdam Mathematical Centrum) that represents an invaluable source of information about the historical development of differential geometry in the first half of the 20th century. We hope that our contribution may somehow foster the undertaking of this enterprise.

7. Transcription of part of the correspondence

This section provides transcription of some passages of letters between Schouten and Cartan, whose English translation was provided in the corpus of the paper. The transcription closely respect the original orthography and punctuation, thus maintaining in some cases misprints.

Letter from Schouten to Cartan 3rd March 1924

Monsieur et très honoré Collègue!
Vos notes dans les C.R. [Comptes Rendus] 174 (1922) p. 437, 593, 734, 857, 1104 Sur la généralisation de l'idée des éspaces riemanniens m'intéressent profondement parce qu'ils se rattachent à mes propres récherches sur le déplacement parallèle (Übertragungslehre Math. Zeitschrift 13 (1922) p. 56, 15 (1922) p. 168) dont j'avais l'honneur de vous envoyer des tirés à part. Il me sera cependant plus facile de comparer exactement vos recherches aux miennes quand j'aurai une publication plus étendue des vos travaux. Vous me rendriez un grand service en m'écrivant si une telle publication existe déjà. Je vous prie aussi de m'envoyer si possible quelques tirés à part des articles en question. Je crois qu'une comparaison mutuelle de nos résultat puisse rendre de belles fruits.

Letter from Cartan to Schouten 16th June 1924

[...] Permettez moi encore de dire un mot sur ce que vous appellez ma "symbolique". Je crois qu'en réalité je n'ai pas de symbolique mais c'est la surtout une question de mot. J'ai été amené par mes traveaux sur les systèmes de Pfaff à employer une notation qui consiste surtout à désigner par une seule lettre une expression de Pfaff et à faire jouer le rôle fondamental non pas aux différentielles des variables mais à certaines combinaisons linéaires de ces différentielles jouant un rôle plus au moins privilégié. J'ai naturallement été confirmé dans l'idée que ma manière de procéder était féconde par le fait que je pu ainsi créer une théorie de la structure des groupes des transformations valable aussi bien pour les groupes infinis que pour les groupes finis. J'ai ensuite appliqué mon procédé à la Géométrie différentielle avec d'autant plus de sûreté que, en liaison avec ma théorie des systèmes de Pfaff elle me donne simultanément les propriétés des êtres géométriques étudiés et leur degré de généralité. Bien entendu cela ne veut pas dire qu'on ne puisse pas arriver aux mêmes résultats par le calcul de Ricci surtout tel que vous l'avez généralisé et complété; une combinaison des deux (telle au fond que l'a essayé M. Lagrange) ne serait peut–être pas sans intérêt. Il est bien évident aussi que créé pour certains genres de questions, il peut n'offrir que des inconvénients pour d'autres. [...]

Letter from Cartan to Schouten 14th January 1926

Monsieur et cher Collègue,
Ce que vous dites au début de la théorie des groupes continus m'intéresse plus particulièrment. J'ai lu l'article de M. Eisenhart que je ne connaissais pas et donc je n'ai pas du reste très bien compri la fin. Je voudrais vous demander si vous avez publié quelque chose (ou su vous vous proposez de le faire) relativement à l'"Uebertragung" induite par le groupe adjoint d'un groupe continus. J'ai précisement été conduit à cette question par une note récente de M. Enea Bortolotti sur le parallelisme de Clifford, "Parallelismo assoluto e vincolato negli S_3 a curvatura costante ed estensione alle V_3 qualunque", Venezia 1925. Il y fait allusion à un article que j'ai dû vous envoyer sur les récentes généralisations de la notion d'espace (Bulletin de math. t. 48, 1924, 294-320) et où je traitais précisément le parallélisme absolu de Clifford du point de vue de la théorie des groupes comme une "nichtsymmetrische Uebertragung" de courbure nulle

dans l'espace des transformations orthogonales à trois variables. Ce point de vue s'étend de lui même à un groupe quelconque à r paramètres et dans l'espace des transformations il existe deux parallélismes absolus (c'est à dire à courbure nulle). Analytiquement si on désigne par S_x la transformation de paramètres x deux vecteurs infinitement petits $(x, x+dx)$ *et* $(y, y+dy)$ sont equipollents de première espèce si on a

$$S_{x+dx}S_x^{-1} = S_{y+dy}S_y^{-1};$$

ils sont equipollents de seconde espèce si on a:

$$S_{x+dx}^{-1}S_x = S_{y+dy}^{-1}S_y$$

Dans les deux cas on a une variété à connexion affine à courbure nulle et les géodésiques sont les mêmes dans les deux cas, à savoir les groupes à un paramètre du groupe total. [...]

Analytiquement, si $\omega_1, \ldots, \omega_r$ sont le composantes infinitement petits de la transformation infinitésimale $S_{x+dx}S_x^{-1}$, les équations de structure de l'espace défini par la connexion affine de première espèce sont

$$\omega_s' = \sum c_{iks}\,[\omega_i \omega_k]\,,$$

avec $\omega_i^j = 0$ (c_{iks} constants de Lie).

La seconde connexion affine ferait intervenir les paramètres $\varpi_1, \ldots, \varpi_r$ de la transformation infinitesimal $S_{x+dx}^{-1}S_x$ (Voir mon article sur la structure des groupes des transformations et la théorie du trièdre mobile, Bull. Sc. Math. (2), 34, 1910).

Comme vous le faites remarquer si le groupe est semi–simple, avec chacune des connexions affines précédemment definies l'espace est de Riemann avec le même géodésiques au sens de Riemann. On peut alors se proposer de trover tous les espaces de Riemann dans lesquels il est possible de définir un parallélisme absolu pour lequel les géodésiques (?)[46] l'autoparallélisme (correspondant à une lineare Uebertragung). Il y a d'abord ceux qui correspondent aux groupes semi–simple de la manière indiquée plus haut; dans tous ces espaces les directions principals de Ricci sont indéterminées; de plus la courbure riemannienne d'une facette est égale au carré de la demi torsion que chacun des parallelismes absolus confère à la facette (les torsions étant égales et opposées pour les deux parallélismes); enfin un vector déplacé par parallélisme de Levi–Civita se dirige suivant la bissectrice de l'angle formé par les deux positions de ce vecteur transporté suivant les deux parallelismes absolus. En dehors de ces solutions du probleme y en a il d'autres? Je n'ai pas le décidé ni par l'affirmative ni par la négative. Il n'y en a pas pour $n \leq 6$; s'il y en a, les espaces de Riemann correspondants admettent une famille continue de parallélismes absolus jouiants de la propriété indiquée, ainsi qu'un groupe transitif de déplacements rigides à plus de n paramètres. Les espaces de Riemann, s'ils existent, ne sont jamais à corbure constante. [...]

Letter from Cartan to Schouten 20th January 1926

[...] En ce qui concerne le problème général que je vous ai signalé, et auquel vous voulez bien vous intéresser, voici au fond à quoi je le ramène. Il s'agit de trouver de la manière la plus générale possible r expressions de Pfaff indépendantes $\omega_1, \ldots, \omega_r$ telles qu'on a

(7.1)
$$\omega_i' = \sum_{\alpha\beta} c_{\alpha\beta i}\,[\omega_\alpha \omega_\beta]\,.$$

[46] Unreadable.

les coefficients $c_{\alpha\beta\gamma}$ étant les composantes d'un système de trivecteurs:

$$c_{\alpha\beta\gamma} = c_{\beta\gamma\alpha} = c_{\gamma\alpha\beta} = -c_{\alpha\gamma\beta} = -c_{\beta\alpha\gamma} = -c_{\gamma\beta\alpha} \tag{7.2}$$

Si les $c_{\alpha\beta\gamma}$ sont des constantes (mais sans les conditions de symmétrie (7.2)), elles satisfont aux relations de Lie (identités de Jacobi), qu'on obtient en écrivant que les covariantes trilinéaires des second membre de (7.1) sont nuls. La question est alors la suivante: *est il possible que les coefficients $c_{\alpha\beta\gamma}$ ne soient pas tous constants?* Il semblerait à première vue que en général les coefficients ne sont pas constants, mais je serais assez porté à croire qu'ils le sont toujours: cependant c'est loin d'être sûr et je serai très intéressé si vous trouvez quelque chose là–dessus.

La recherche des connexion affines à courbure nulle, en se plaçcant au point de vue précédent, revient à prendre n expressions de Pfaff arbitraires $\omega_1, \omega_2, \ldots, \omega_n$; elles définissent, avec les expressions $\omega_i^j = 0$, l'espace à connexion affine sans courbure le plus général. On a

$$(\omega^i)' = \sum_{\alpha\beta} c^i_{\alpha\beta} \left[\omega^\alpha \omega^\beta\right]$$

Le cas de $c^i_{\alpha\beta}$ constants correspond aux connexions sans courbure associées à un groupe quelconque; pour que ce cas se présente, il faut et il suffit qu'il existe dans l'espace un groupe ponctuel à n paramètres (simplement transitif) changeant tout vecteur en un vecteur parallèle (ou plutôt équipollent). Ce groupe est formé des transformations qui laissent invariante chacune des expressions ω^i. [...]

Letter from Schouten 21st February 1926

[...] J'ai fait le projet d'écrire une note de circa 20 à 30 pages pour l'Académie Royale à Amsterdam, contenant

(1) Un aperçu des trois differentes connexions dans la variété d'un groupe simple ou semi–simple.
(2) La démonstration du théorème trouvé.

Mais il ne m'était pas agréable de publier ces deux pour moi seul. Les connexions différentes des variétés des groupes vous les avez trouvé en même temps que moi, et quant au théorème je crois que diviner un théorème si beau et si général c'est au moins si méritable que faire la démonstration.

Ainsi je vous propose de me faire l'honneur de publier cette note en commun comme publication de nous deux. [...]

Letter from Cartan 1st March 1926

[...] Voici des nouveaux. J'ai trouvé un exemple simple mettant en défaut le théorème en litige, et cet exemple que j'avais été sur le point de mettre sur pied il y a plus d'un mois, je l'avais abandonné, parce qu'il me conduisait à un espace à courbure constante et que d'autre part j'avais démontré qu'un espace à courbure constante ne pouvait admettre du pseudo parallélisme jouissant des propriétés voulues.

En reprenant cet exemple, il m'a suggeré un pseudo parallélisme possible dans l'espace elliptique à 7 dimensions, j'ai démontré effectivement dans cet espace l'existence de ∞^7 pseudo parallélisme. Mais, en même temps je ne trouvais rien à ridire à ma démonstration de l'impossibilité de tels pseudo parallélisme dans un espace à courbure costante! De sorte que pendant 24 heures j'étais dans la situation angoissante de concilier deux théorèmes inconciliables et tous les deux démontrés. Je viens enfin de voir que ma démonstration était fausse et que dans une égalité où il y avait deux sommes à deux indices de sommations, j'avais eu tort d'échanger les indices de sommations dans une de sommes, chose pourtant bien naturelle! [...]

Il est à remarquer que la définiton précedente généralise immédiatement celle du parallélisme de Clifford dans l'espace elliptique à 3 dimensions; dans ce dernier cas il n'y a qu'à remplacer les octaves par les quaternions. [...]

Il ne serait pas impossible que l'espace elliptique à 7 dimensions fût le seul à faire exception au théorème en litige. Peut on faire une démonstration générale [...]? C'est l'inconnue! [...]

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