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A remark on generalized complete intersections

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Abstract

We observe that an interesting method to produce non-complete intersection subvarieties, the generalized complete intersections from L. Anderson and coworkers, can be understood and made explicit by using standard Cech cohomology machinery. We include a worked example of a generalized complete intersection Calabi–Yau threefold.

0. Introduction

Calabi–Yau varieties, in particular those of dimension three, are of great interest in string theory. Since there are not many general results yet on their classification, but see [14], the explicit construction of CY threefolds is a quite important enterprise. For example, Kreuzer and Starke classified the toric fourfolds which have CY threefolds as (anticanonical) hypersurfaces [11], [3]. Besides generalizations to complete intersection CYs in certain ambient toric varieties, like products of projective spaces, there are various other examples of CY threefolds constructed with more sophisticated algebro-geometrical methods. Recent examples include [9], [7], [10].

In the recent paper [1], L. Anderson, F. Apruzzi, X. Gao, J. Gray and S-J. Lee found a very nice method to construct many more CY threefolds. The basic idea is to take a hypersurface Y in an ambient variety P and to consider hypersurfaces X in Y. These hypersurfaces need not be complete intersections in P, that is, there need not exist two sections of two line bundles on P

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whose common zero locus is X. There are various generalizations of this method, but we will stick to this basis case. As in [1], we refer to these varieties as generalized complete intersections (gCIs).

A particularly interesting and accessible case that was found and studied by Anderson and coworkers is when the ambient variety is a product of two varieties, one of which is $\mathbf{P}^1$, so $P = P_2 \times \mathbf{P}^1$. The variety P_2 they consider is a product of projective spaces, but this is not essential, one could consider any toric variety or even more general cases. The factor $\mathbf{P}^1$ is important since there are line bundles on $\mathbf{P}^1$ with non-trivial first cohomology group and this is essential to find generalized complete intersections. We review this construction in Section 1.1.

We provide a proposition, proven with standard Cech cohomology methods, that allows one, under a certain hypothesis, to find three equations (more precisely, three sections of three line bundles on P) that define X. In Section 2 we work out a detailed example, with explicit equations, of a CY threefold which was already considered in [1]. The explicit example X has an automorphism of order two and the quotient of X by the involution provides, after desingularization, another CY threefold. More generally, we think that among the gCIs found in [1] one could find more examples of CY threefolds with non-trivial automorphisms. It might be hard though to implement a systematic search as was done in [6] for complete intersection CY threefolds in products of projective spaces. We did not find new CY threefolds with small Hodge numbers (see [5] for an update on these), but the gCICY seem to be a promising class of CYs to search for these. The recent paper [4] by Berglund and Hübsch provides further techniques to deal with gCICYs whereas [2] explores string theoretical aspects of gCICYs.

1. The construction of generalized complete intersections

1.1. The general setting

Let P_2 be a projective variety of dimension n and let $P := P_2 \times \mathbf{P}^1$. We denote the projections to the factors of P by π_1, π_2 respectively. For a coherent sheaf $\mathcal{F}$ on P_2 and an integer d we define a coherent sheaf on P by:

$$\mathcal{F}[d] := \pi_1^*\mathcal{F} \otimes \pi_2^*\mathcal{O}_{\mathbf{P}^1}(d) .$$

The Künneth formula gives

$$H^r(P, \mathcal{F}[d]) = \oplus_{p+q=r} H^p(P_2, \mathcal{F}) \otimes H^q(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(d)) .$$

Recall that the only non-zero cohomology of $\mathcal{O}_{\mathbf{P}^1}(d)$ is: $h^0(\mathcal{O}_{\mathbf{P}^1}(d)) = h^1(\mathcal{O}_{\mathbf{P}^1}(-2-d)) = d+1$ for $d \geq 0$ and a basis for $H^0(\mathcal{O}_{\mathbf{P}^1}(d))$ is given by the monomials $z_0^i z_1^{d-i}$, $i = 0, \dots, d$, where $(z_0 : z_1)$ are the homogeneous coordinates on $\mathbf{P}^1$.

Let L be a line bundle on P_2 and assume that $L[d]$, for some $d \geq 1$, has a non-trivial global section F. Using the Künneth formula, we can write $F = \sum_i f_i z_0^i z_1^{d-i}$ for certain sections $f_i \in H^0(P_2, L)$. Let $Y = (F)$ be the zero locus of F in P. We assume that Y is a (reduced, irreducible) variety, although this will not be essential in this section.

To define a codimension two subvariety of P, we consider another line bundle M on P_2. The Künneth formula shows that $M[-e]$ has no global sections if $e \geq 1$. But upon restricting to Y, the vector space $H^0(Y, M[-e]_{|Y})$ could still be non-trivial. In fact, from the exact sequence

$$0 \longrightarrow (L^{-1} \otimes M)[-d-e] \xrightarrow{F} M[-e] \longrightarrow M[-e]_{|Y} \longrightarrow 0 \qquad (1)$$

we deduce the exact sequence

$$0 \longrightarrow H^0(Y, M[-e]_{|Y}) \xrightarrow{d^0} H^1(P, (L^{-1} \otimes M)[-d-e]) \xrightarrow{F_1} H^1(P, M[-e]) \qquad (2)$$

thus $H^0(Y, M[-e]_{|Y}) \cong \ker(F_1)$, where we denote by F_1 the map induced by multiplication by F on the first cohomology groups. Since now

$$H^1(P, (L^{-1} \otimes M)[-d-e]) \cong H^0(P_2, L^{-1} \otimes M) \otimes H^1(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(-d-e)) \qquad (d+e \geq 2)$$

the domain of F_1 is non-trivial if and only if $h^0(P_2, L^{-1} \otimes M) \neq 0$. So for suitable choices of line bundles on P_2 we might find interesting, non-complete intersection, codimension two subvarieties of P in this way. In the proof of Proposition 1.4 we explain how to compute F_1.

1.2. Example

Let $P_2 = \mathbf{P}^n$, $L = \mathcal{O}_{\mathbf{P}^n}(k)$, $M = \mathcal{O}_{\mathbf{P}^n}(k + l)$ with $l \geq 0$, let $d \geq 1$ and $e = 1$. Then $h^0(\mathbf{P}^n, L^{-1} \otimes M) = h^0(\mathcal{O}_{\mathbf{P}^n}(l)) \neq 0$ so $h^1(P, (L^{-1} \otimes M)[-d-e]) \neq 0$, but $h^1(P, M[-e]) = 0$ since $\mathcal{O}_{\mathbf{P}^1}(-1)$ has no cohomology. Thus $H^0(Y, M[-e]_{|Y}) \cong H^1(P, (L^{-1} \otimes M)[-d-e])$ is indeed non-trivial.

1.3. Generalized complete intersections

Given a variety $Y \subset P$ that is the zero locus of $F \in H^0(L[d])$ as in Section 1.1, and given a global section $\tau \in H^0(M[-e]_{|Y})$, its zero locus $X := (\tau) \subset Y$ is called a generalized complete intersection.

The scheme X may not be defined by two global sections σ_1, σ_2 of line bundles L_1, L_2 on P. However in certain cases we can find three sections of line bundles on P which define X:

1.4. Proposition

Let $F \in H^0(P, L[d])$, let $Y = (F)$, let $\tau \in H^0(Y, M[-e]_{|Y})$ with $d, e \geq 1$ be as above and assume that $H^1(P_2, L^{-1} \otimes M) = 0$.

Then there are two global sections $G, H \in H^0(P, M[d-1])$ such that the generalized complete intersection subscheme X of P defined by τ in Y can also be defined as

$$X = \{x \in P : F(x) = G(x) = H(x) = 0\}$$

(the equality is of schemes). Moreover, there is a global section $A \in H^0(P, (L^{-1} \otimes M)[d + e - 2])$ such that $AF = z_1^{d+e-1}G + z_0^{d+e-1}H$, so that on the open subset of $P_2 \times \mathbf{P}^1$ where $z_0 \neq 0$ the subscheme X of P is defined by the two equations $F = G = 0$.

Proof. We use Cech cohomology to make the isomorphism $H^0(Y, M[-e]_{|Y}) \cong \ker(F_1)$, see exact sequence (2), explicit. Let $U_i \subset \mathbf{P}^1$ be the open subset where $z_i \neq 0$. For a coherent sheaf $\mathcal{G}$ on $\mathbf{P}^1$ we have the exact sequence

$$0 \longrightarrow H^0(\mathbf{P}^1, \mathcal{G}) \longrightarrow \mathcal{G}(U_0) \oplus \mathcal{G}(U_1) \xrightarrow{\delta} \mathcal{G}(U_0 \cap U_1) \longrightarrow H^1(\mathbf{P}^1, \mathcal{G}) \longrightarrow 0\,,$$

where $\delta(t_0, t_1) = t_0 - t_1$. The cohomology groups we consider are computed with the Künneth formula. Note that after tensoring this exact sequence by a vector space W, we obtain that $W \otimes H^0(\mathbf{P}^1, \mathcal{G}) = \ker(1_W \otimes \delta)$ and $W \otimes H^1(\mathbf{P}^1, \mathcal{G}) = \text{coker}(1_W \otimes \delta)$.

For an affine open subset $V \subset \mathbf{P}^1$, the cohomology of the exact sequence (1) on $P_2 \times V$ gives the exact sequence, where we extend $M[-e]_{|Y}$ by zero to $P_2 \times V$,

$$H^0(P_2 \times V, M[-e]) \longrightarrow H^0(P_2 \times V, M[-e]_{|Y}) \longrightarrow H^1(P_2 \times V, (L^{-1} \otimes M)[-d-e]) .$$

The Künneth formula, combined with the assumption $H^1(P_2, L^{-1} \otimes M) = 0$ and the fact that $H^1(V, \mathcal{F}) = 0$ for any coherent sheaf $\mathcal{F}$ since V is affine, implies that the last group is zero.

Taking $V = U_0, U_1$, the exact sequence (1) on $P_2 \times V$ thus gives two exact sequences whose sum (term by term) is

$$\begin{gathered} 0 \longrightarrow \oplus_{i=0}^1 H^0(L^{-1} \otimes M) \otimes (\mathcal{O}_{\mathbf{P}^1}(-d-e)(U_i)) \stackrel{F}{\longrightarrow} \\ \oplus_{i=0}^1 H^0(M) \otimes (\mathcal{O}_{\mathbf{P}^1}(-e)(U_i)) \longrightarrow \oplus_{i=0}^1 (M[-e]_{|Y})(P_2 \times U_i) \longrightarrow 0 . \end{gathered} \tag{3}$$

Similarly taking $V = U_0 \cap U_1$ one has the exact sequence:

$$\begin{gathered} 0 \longrightarrow H^0(L^{-1} \otimes M) \otimes (\mathcal{O}_{\mathbf{P}^1}(-d-e)(U_0 \cap U_1)) \stackrel{F}{\longrightarrow} \\ H^0(M) \otimes (\mathcal{O}_{\mathbf{P}^1}(-e)(U_0 \cap U_1)) \longrightarrow (M[-e]_{|Y})(P_2 \times (U_0 \cap U_1)) \longrightarrow 0 . \end{gathered} \tag{4}$$

Next we use the Cech boundary map δ to map sequence (3) to sequence (4) and we obtain a commutative diagram with three complexes as columns. The first two columns are Cech complexes for the covering $\{U_i\}_{i=0,1}$ of $\mathbf{P}^1$, their cohomology groups are respectively

$$\begin{aligned} H^0(L^{-1} \otimes M) \otimes H^q(\mathcal{O}_{\mathbf{P}^1}(-d-e)) &\cong H^q(P, (L^{-1} \otimes M)[-d-e]), \\ H^0(M) \otimes H^q(\mathcal{O}_{\mathbf{P}^1}(-e)) &\cong H^q(P, M[-e]), \quad (q = 0, 1) . \end{aligned}$$

The zero-th cohomology group of the last column is $H^0(Y, M[-e]_{|Y})$. So we conclude that the map F_1 can be computed with the long exact cohomology sequence associated to this diagram.

We observe, but will not use, that the Künneth formula implies that $H^2(P, (L^{-1} \otimes M)[-d-e]) = 0$ and thus the cohomology sequence of (1) gives a six term exact sequence with the zero-th and first cohomology groups. The first 5 terms are the same as those of the long exact sequence associated to the diagram, so we conclude that the first cohomology group of the last column is $H^1(Y, M[-e]_{|Y})$.

Given $\tau \in H^0(Y, M[-e]_{|Y})$, let $q := d^0(\tau) \in \ker(F_1)$. Since the first row (3) of the complex is exact, the section τ is locally given by restricting sections $\tau_i \in M[-e](P_2 \times U_i)$ to Y. By the snake lemma, they satisfy $\tau_0 - \tau_1 = Fq$ on $P_2 \times (U_0 \cap U_1)$, in particular $\tau_0 = \tau_1$ on $Y \cap (P_2 \times (U_0 \cap U_1))$ since $F = 0$ on Y.

The images of the $z_0^{-j} z_1^{-d-e+j} \in \mathcal{O}_{\mathbf{P}^1}(-d-e)(U_0 \cap U_1)$, $j = 1, \dots, d+e-1$, form a basis of $H^1(\mathbf{P}^1, \mathcal{O}(-d-e))$. A cohomology class $q \in H^1(P, (L^{-1} \otimes M)[-d-e]) \cong H^0(P_2, L^{-1} \otimes M) \otimes H^1(\mathbf{P}^1, \mathcal{O}(-d-e))$ can thus be represented by $q = \sum_j q_j z_0^{-j} z_1^{-d-e+j}$ with $q_i \in H^0(P_2, L^{-1} \otimes M)$. Let $F = \sum_i f_i z_0^i z_1^{d-i}$, where $f_i \in H^0(P_2, L)$, then Fq is homogeneous of degree $d - (d+e) = -e$ and it is a sum of terms $r_k z_0^k z_1^{-e-k}$ with $r_k \in H^0(P_2, M)$. Writing

$$\begin{aligned} Fq &= \sum_{k=-d-e+1}^{d-1} r_k z_0^k z_1^{-e-k} \\ &= \left(\sum_{k=-d-e+1}^{-e} r_k z_0^k z_1^{-e-k} \right) + \left(\sum_{k=-e+1}^{-1} r_k z_0^k z_1^{-e-k} \right) + \left(\sum_{k=0}^{d-1} r_k z_0^k z_1^{-e-k} \right), \end{aligned}$$

the first summand lies in $M[-e](P_2 \times U_0)$ (where $z_0 \neq 0$) and the last summand lies in $M[-e](P_2 \times U_1)$, we denote these summands by τ_0 and $-\tau_1$ respectively. The middle summand has monomials $z_0^a z_1^b$ with both $a, b < 0$. Thus Fq represents a class in $q' \in H^1(P, M[-e])$, which is the same as the class represented by the middle summand. By definition, one has $q' = F_1(q)$ and thus $q \in \ker(F_1)$ when all coefficients r_k, $k = -e+1, \dots, -1$, are zero.

Since $q \in \ker(F_1)$ this middle summand is zero, so that $Fq = \tau_0 - \tau_1$ as desired. Now we define $G := z_0^{d+e-1}\tau_0$ and $H := -z_1^{d+e-1}\tau_1$ so that all their monomials $z_0^a z_1^b$ have $a, b \geq 0$ and $a+b = d-1$, thus both $G, H \in H^0(P, M[d-1])$. Then $(z_0 z_1)^{d+e-1} Fq = z_1^{d+e-1} G + z_0^{d+e-1} H$ and with $A := (z_0 z_1)^{d+e-1} q \in H^0(P, (L^{-1} \otimes M)[d+e-2])$ we find the desired relation. □

1.5. Example

With the choices of P_2, L, M as in Example 1.2, and if X is a smooth variety (of dimension $n-1$), then $H^1(P_2, L^{-1} \otimes M) = H^1(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(l)) = 0$, for any l, if $n > 1$. The adjunction formula implies that X has trivial canonical bundle if we choose $l = n+1-2k$ and $d = 3$. In that case $P = \mathbf{P}^n \times \mathbf{P}^1$ and F is homogeneous of bidegree $(k, 3)$ whereas G, H have bidegree $(n+1-k, 2)$.

1.6. A fibration on X

Given X as in the proposition, the projection $\pi_2 : P_2 \times \mathbf{P}^1 \to \mathbf{P}^1$ restricts to X to give a fibration denoted by $\pi_2 : X \to \mathbf{P}^1$. For a point $p = (z_0 : z_1) \in \mathbf{P}^1$, we denote by $F_p \in H^0(P_2, L)$, $H_p \in H^0(P_2, M)$ the restrictions of F and H to the fiber X_p. The equation $AF = z_1^{d+e-1} G + z_0^{d+e-1} H$ shows that if $z_1 \neq 0$ then F_p and H_p define the fiber X_p, which is thus a complete intersection in P_2.

1.7. Example

This example illustrates that X, as in Proposition 1.4, might be reducible, even if $h^0(Y, M[-e]_{|Y})$ is rather large. The example is taken from [1, Table 4], third item (with $i = 2$) where it is in fact observed that no smooth varieties arise in that case. We take

$$P_2 := \mathbf{P}^2 \times \mathbf{P}^1 \times \mathbf{P}^1 , \qquad L := \mathcal{O}(0,1,1), \quad M := \mathcal{O}(3,1,1) , \qquad d = 4,\ e = -2 .$$

Notice that $H^1(P_2, L^{-1} \otimes M) = H^1(\mathbf{P}^2 \times \mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}(3,0,0)) = 0$ by the Künneth formula, so we can, but will not, apply Proposition 1.4. Since $h^1((L^{-1} \otimes M)[-d-e]) = h^1(\mathcal{O}(3,0,0)[-6]) = 10 \cdot 1 \cdot 1 \cdot 5 = 50$ and $h^1(M[-e]) = 10 \cdot 2 \cdot 2 \cdot 1 = 40$, we find $h^0(M[-e]_{|Y}) \geq 10$. We will show that, for general Y, $h^0(M[-e]_{|Y}) = 10$ but that all sections of $M[-e]_{|Y}$ define reducible subvarieties of Y.

Due to the first zero in $L = \mathcal{O}(0,1,1)$, the variety Y is a product, $Y = \mathbf{P}^2 \times S \subset P$, with $S \subset (\mathbf{P}^1)^3$ the surface defined by a section of $\mathcal{O}(1,1,4)$. Then we have $h^0(M[-e]_{|Y}) = h^0(\mathbf{P}^2 \times S, \pi_1^*\mathcal{O}_{\mathbf{P}^2}(3) \otimes \pi_2^*\mathcal{O}_S(1,1,-2))$ and using the Künneth formula we find $h^0(M[-e]_{|Y}) = h^0(\mathcal{O}_{\mathbf{P}^2}(3)) h^0(\mathcal{O}_S(1,1,-2)) = 10 h^0(\mathcal{O}_S(1,1,-2))$. The exact sequence

$$0 \longrightarrow \mathcal{O}_{(\mathbf{P}^1)^3}(0,0,-6) \stackrel{f}{\longrightarrow} \mathcal{O}_{(\mathbf{P}^1)^3}(1,1,-2) \longrightarrow \mathcal{O}_S(1,1,-2) \longrightarrow 0 ,$$

where f is the equation of S, shows that (with f_1 the map induced by f on H^1):

$$h^0(\mathcal{O}_S(1,1,-2)) = \dim\ker\Big(f_1 : H^1(\mathcal{O}_{(\mathbf{P}^1)^3}(0,0,-6)) \to H^1(\mathcal{O}_{(\mathbf{P}^1)^3}(1,1,-2))\Big) .$$

Since these spaces have dimensions $1 \cdot 1 \cdot 5 = 5$ and $2 \cdot 2 \cdot 1 = 4$ respectively, one expects $h^0(\mathcal{O}_S(1,1,-2)) = 1$. In that case any section $\tau \in H^0(M[-e]_{|Y})$ would be the product $\tau = gs$ with $g \in H^0(\mathcal{O}_{\mathbf{P}^2}(3))$ and $s \in H^0(\mathcal{O}_S(1,1,-2))$ the unique (up to scalar multiple) section, hence X would be reducible.

To see that indeed $h^0(\mathcal{O}_S(1,1,-2)) = 1$ for a general equation f, take a smooth (genus one) curve C of bidegree $(2,2)$ in $\mathbf{P}^1 \times \mathbf{P}^1$ and choose eight distinct points on C which are not cut out by another curve of bidegree $(2,2)$. As curves of bidegree $(1,4)$ depend on $2 \cdot 5 = 10$ parameters, we can find two polynomials g_0, g_1 of bidegree $(1,4)$ such that $g_0 = g_1 = 0$ consists of these eight points on C. Take $f = x_0 g_0 + x_1 g_1$ with $(x_0 : x_1) \in \mathbf{P}^1$, the first copy of $\mathbf{P}^1$ in $(\mathbf{P}^1)^3$, and the g_i on the last two copies of $\mathbf{P}^1$. The surface $S \subset (\mathbf{P}^1)^3$ defined by f is thus the blow up of $\mathbf{P}^1 \times \mathbf{P}^1$ in the eight points where $g_0 = g_1 = 0$. The adjunction formula shows that the line bundle $\mathcal{O}_S(1,1,-2)$ is the anticanonical bundle of S. The effective anticanonical divisors are the strict transforms of bidegree $(2,2)$-curves on passing through these eight points. Hence the strict transform of C in S will be the unique effective anticanonical divisor on S and therefore $h^0(\mathcal{O}_S(1,1,-2)) = 1$.

2. An example: a generalized complete intersection Calabi–Yau threefold

2.1. Introduction

We illustrate the use of Proposition 1.4 (and its proof) for the generalized complete intersection Calabi Yau discussed in [1, Section 2.2.2]. We also consider an explicit example which has a non-trivial involution and we compute the Hodge numbers of a desingularization of the quotient threefold which is again a CY.

2.2. The varieties P_2 and Y

We consider the case that $P_2 = \mathbf{P}^4$, we choose the line bundle $L := \mathcal{O}_{\mathbf{P}^4}(2)$ and we let $d = 3$. Then the line bundle $L[d] = \mathcal{O}_P(2,3)$ is very ample on $P = \mathbf{P}^4 \times \mathbf{P}^1$ and thus a general section F will define a smooth fourfold Y of P. To obtain a CY threefold in Y, we consider global sections of the anticanonical bundle of Y. By adjunction, $\omega_Y = (\mathcal{O}_P(-5,-2) \otimes \mathcal{O}_P(2,3))_Y = \mathcal{O}_Y(-3,1)$. Thus we take $M = \mathcal{O}_{\mathbf{P}^4}(3)$ and $e = 1$, so that $M[-e]_{|Y} = \mathcal{O}_Y(3,-1) = \omega_Y^{-1}$. As the H^1 of any line bundle on $\mathbf{P}^4$ is trivial, we can use (the proof of) Proposition 1.4 to find polynomials $G, H \in H^0(P, \mathcal{O}_P(3,2))$ which together with F define a generalized complete intersection X.

As in Example 1.2, we get

$$H^0(\mathcal{O}_Y(3,-1)) \xrightarrow{\cong} H^1(\mathcal{O}_P(1,-4)) .$$

To find explicit elements of $H^0(\mathcal{O}_Y(3,-1))$, we write the defining equation of Y as

$$F = P_0 z_0^3 + P_1 z_0^2 z_1 + P_2 z_0 z_1^2 + P_3 z_1^3 \qquad (\in H^0(P, \mathcal{O}_P(2,3))) ,$$

with $P_i \in H^0(\mathbf{P}^4, \mathcal{O}(2))$ homogeneous polynomials of degree two in $y = (y_0 : \ldots : y_4)$. As $H^1(\mathcal{O}_P(1,-4)) \cong H^0(\mathcal{O}_{\mathbf{P}^4}(1)) \otimes H^1(\mathcal{O}_{\mathbf{P}^1}(-4))$, a basis of this $5 \cdot 3 = 15$ dimensional vector space are the products of one of $y_0, \ldots, y_4$ with one of $z_0^{-3} z_1^{-1}, z_0^{-2} z_1^{-2}, z_0^{-1} z_1^{-3}$. Thus any class $q \in H^1(\mathcal{O}_P(1,-4))$ has a representative

$$q = Q_0 z_0^{-3} z_1^{-1} + Q_1 z_0^{-2} z_1^{-2} + Q_2 z_0^{-1} z_1^{-3} \qquad (\in H^1(\mathcal{O}_P(1,-4))) ,$$

with linear forms $Q_i \in H^0(\mathbf{P}^4, \mathcal{O}(1))$. As in the proof of Proposition 1.4 we must write:

$$Fq = \tau_0 - \tau_1, \qquad G := z_0^3\tau_0, \qquad H := -z_1^3\tau_1\ ,$$

with $\tau_i \in \mathcal{O}_P(3,-1)(\mathbf{P}^4 \times U_i)$. So we find

$$G = z_0^2(P_1Q_0 + P_2Q_1 + P_3Q_2) + z_0z_1(P_2Q_0 + P_3Q_1) + z_1^2P_3Q_0\ ,$$
$$H = z_0^2P_0Q_2 + z_0z_1(P_0Q_1 + P_1Q_2) + z_1^2(P_0Q_0 + P_1Q_1 + P_2Q_2).$$

2.3. *The base locus of* $|-K_Y|$

In Section 2.2 we showed how to find the global sections of $\omega_Y^{-1} = \mathcal{O}_Y(3,-1)$ explicitly, locally such a section is given by the polynomials G and H. From the formula for F we see that if $x \in \mathbf{P}^4$ and $P_0(x) = \ldots = P_3(x) = 0$, then the curve $\{x\} \times \mathbf{P}^1$ lies in Y. This curve also lies in the zero loci of G and H, for any choice of $Q_0, Q_1, Q_2 \in H^0(\mathcal{O}_{\mathbf{P}^4}(1))$, hence it lies in the base locus of anticanonical system $|-K_Y|$. Since the four quadrics $P_i = 0$ in $\mathbf{P}^4$ intersect in at least 2^4 points, counted with multiplicity, we see that this base locus is non-empty. Thus we cannot use Bertini's theorem to guarantee that there are smooth CY threefolds $X \subset Y$, but we resort to an explicit example, see below.

2.4. *The CY threefold X*

To obtain an explicit example, we choose

$$P_0 := y_0^2 + y_1^2 + y_2^2 + y_3^2 + y_4^2, \quad P_1 := y_0^2 + y_4^2,$$
$$P_2 := y_1^2 + y_3^2, \quad P_3 := y_0^2 + y_1^2 - y_2^2 - y_3^2 - y_4^2,$$

and

$$Q_0 := y_0, \qquad Q_1 := y_1, \qquad Q_2 := y_2\ .$$

Using a computer algebra system (we used Magma [12]), one can verify that $Y := (F = 0)$ and $X := (F = G = H = 0)$ are smooth varieties in P. The variety X is a Calabi–Yau threefold since it is an anticanonical divisor on Y. In [1, (2.27), (2.28)] one finds that the Hodge numbers of X are $(h^{1,1}(X), h^{2,1}(X)) = (2, 46)$, in particular, $h^2(X) = 2$, $h^3(X) = 94$.

2.5. *Parameters*

The CY threefold X is defined by a section $F \in H^0(P, \mathcal{O}_P(2,3))$ and a section $\tau \in H^0(Y, \mathcal{O}_Y(3,-1))$. The first is a vector space of dimension

$$h^0(P, \mathcal{O}_P(2,3)) = h^0(\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4}(2)) \cdot h^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(3)) = 15 \cdot 4 = 60\ ,$$

whereas the second has dimension 15. The group $GL(5, \mathbf{C}) \times GL(2, \mathbf{C})$ acts on $H^0(\mathcal{O}_P(2,3))$ and has dimension $5^2 + 2^2 = 29$. The subgroup of elements $(\lambda I_5, \mu I_2)$ with $\lambda^2\mu^3 = 1$ acts trivially, so we get $60 - 28 = 32$ parameters for P and next $15 - 1 = 14$ parameters for τ, so we do get $32 + 14 = 46 = h^{2,1}(X)$ parameters for X. So the general deformation of X seems to be again a gCICY of the same type as X. (In [1], just below (2.28), the dependence of X on $\mathcal{P}$, which gives 32 parameters, seems to have been overlooked.)

2.6. A CY quotient

A well-known method to obtain Calabi–Yau threefolds is to consider desingularizations of quotients of such threefolds by finite groups, see for example [6]. In the example above, we see that $X \subset \mathbf{P}^4 \times \mathbf{P}^1$ has a subgroup $(\mathbf{Z}/2\mathbf{Z})^2 \subset Aut(X)$ given by the sign changes of y_3 and y_4. We consider the involution

$$\iota : X \longrightarrow X, \qquad \Big((y_0 : \ldots : y_4), (z_0 : z_1)\Big) \longmapsto \Big((y_0 : y_1 : y_2 : -y_3 : -y_4), (z_0 : z_1)\Big) .$$

Its fixed point locus has two components, one defined by $y_3 = y_4 = 0$ and the other by $y_0 = y_1 = y_2 = 0$ in X. The first is a curve in $\mathbf{P}^2 \times \mathbf{P}^1 \subset P$, which is smooth, irreducible and reduced of genus 8 according to Magma. Similarly, the other component is a genus 2 curve in $\mathbf{P}^1_{(y_3:y_4)} \times \mathbf{P}^1_{(z_0:z_1)} \subset P$. In fact, only $F = 0$ provides a non-trivial equation for this curve since $y_0 = y_1 = y_2 = 0$ implies $Q_0 = Q_1 = Q_2 = 0$ and hence $G = H = 0$ on this $\mathbf{P}^1 \times \mathbf{P}^1$. As $F = 0$ defines a smooth curve of bidegree $(2, 3)$ in $\mathbf{P}^1 \times \mathbf{P}^1$, this curve has genus $(2 - 1)(3 - 1) = 2$.

In particular, the singular locus of the quotient X/ι consists of two curves of A_1-singularities. Since the fixed point locus X^ι consists of two curves, we conclude that locally on X the involution is given by $(t_1, t_2, t_3) \mapsto (-t_1, -t_2, t_3)$ in suitable coordinates. Hence ι acts trivially on the nowhere vanishing holomorphic 3-form on the CY threefold X. Thus the blow up Z of X/ι in the singular locus will again be a CY threefold.

We determine the Hodge numbers of Z. To do so, it is more convenient to consider the blow up $\tilde{X}$ of X in the fixed point locus X^ι. The involution extends to an involution $\tilde{\iota}$ on $\tilde{X}$, the fixed point set of $\tilde{\iota}$ consists of the two exceptional divisors and the quotient $\tilde{X}/\tilde{\iota}$ is the same Z. Moreover, $H^i(Z, \mathbf{Q}) \cong H^i(\tilde{X}, \mathbf{Q})^{\tilde{\iota}}$, the $\tilde{\iota}$-invariant subspace.

Standard results on the blow up of smooth varieties in smooth subvarieties (cf. [13, Thm 7.31]) show that $h^2(\tilde{X}) = h^2(X) + 2 = 4$ (due to the two exceptional divisors over the two fixed curves) and $h^3(\tilde{X}) = h^3(X) + 2 \cdot 8 + 2 \cdot 2 = 114$ (the contribution of the H^1 of the fixed curves to H^3 of the blow up). The Lefschetz fixed point formula for $\tilde{\iota}$ gives

$$\chi(\tilde{X}^{\tilde{\iota}}) = \sum_{i=0}^{6} (-1)^i tr(\tilde{\iota}^* | H^i(\tilde{X}, \mathbf{Q})) .$$

Notice that $\tilde{\iota}^*$ is the identity on H^0, H^2, H^4, H^6, in particular $h^2(Z) = \dim H^2(\tilde{X}, \mathbf{Q})^{\tilde{\iota}} = 4$. The fixed points of $\tilde{\iota}$ are the two exceptional divisors, these are $\mathbf{P}^1$-bundles over the exceptional curves hence

$$2(2 - 2 \cdot 2) + 2(2 - 2 \cdot 8) = 1 - 0 + 4 - t_3 + 4 - 0 + 1 \quad \Longrightarrow \quad t_3 = 42 .$$

If the $+, -$ eigenspaces of $\tilde{\iota}$ on $H^3(\tilde{X}, \mathbf{Q})$ have dimensions a, b respectively, then $a + b = 114$ and $a - b = 42$, thus $a = 78$ and $a = \dim H^3(\tilde{X}, \mathbf{Q})^{\tilde{\iota}} = h^3(Z)$. As Z is a CY threefold it has $h^{3,0}(Z) = 1$ and thus $h^{2,1}(Z) = (78 - 2)/2 = 38$. Other examples of CY threefolds with $(h^{1,1}, h^{2,1}) = (4, 38)$ are already known.

2.7. A (singular) projective model of Z

The fibers of $\pi_2 : X \to \mathbf{P}^1$ are K3 surfaces, complete intersections of a quadric and a cubic hypersurface in $\mathbf{P}^4$. The involution ι on X restricts to a Nikulin involution on each smooth fiber. The quotient of such a fiber by the involution will in general be isomorphic to a K3 surface in

$\mathbf{P}^2 \times \mathbf{P}^1$, defined by an equation of bidegree $(3, 2)$ (see [8, Section 3.3]). Using the same method as in that reference, we found that the rational map

$$\mathbf{P}^4 \times \mathbf{P}^1 \dashrightarrow \mathbf{P}^2 \times \mathbf{P}^1 \times \mathbf{P}^1,$$
$$\Big((y_0 : \ldots : y_4), (z_0 : z_1)\Big) \longmapsto \Big((y_0 : y_1 : y_2), (y_3 : y_4), (z_0 : z_1)\Big)$$

factors over X/ι and the image, defined by an equation of multidegree $(3, 2, 2)$, is birational with Z. Using the explicit equation for the image and Magma, we found that the image has 38 singular points.

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