SHIMURA CURVES IN THE PRYM LOCUS

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Abstract. We study Shimura curves of PEL type in A_g generically contained in the Prym locus. We study both the unramified Prym locus, obtained using étale double covers, and the ramified Prym locus, corresponding to double covers ramified at two points. In both cases we consider the family of all double covers compatible with a fixed group action on the base curve. We restrict to the case where the family is 1-dimensional and the quotient of the base curve by the group is $\mathbb{P}^1$. We give a simple criterion for the image of these families under the Prym map to be a Shimura curve. Using computer algebra we check all the examples gotten in this way up to genus 28. We obtain 44 Shimura curves generically contained in the unramified Prym locus and 9 families generically contained in the ramified Prym locus. Most of these curves are not generically contained in the Jacobian locus.

1. Introduction

Denote by R_g the scheme of isomorphism classes $[C,\eta]$, where C is a smooth projective curve of genus g and $\eta \in \mathrm{Pic}^0(C)$ is such that $\eta^2 = \mathcal{O}_C$ and $\eta \neq \mathcal{O}_C$. A point $[C,\eta]$ corresponds to an étale double cover $h : \tilde{C} \longrightarrow C$. The norm map $\mathrm{Nm} : \mathrm{Pic}^0(\tilde{C}) \longrightarrow \mathrm{Pic}^0(C)$ is defined by $\mathrm{Nm}(\sum_i a_i p_i) = \sum_i a_i h(p_i)$. The Prym variety associated to $[C,\eta]$ is the connected component containing 0 of $\ker \mathrm{Nm}$. It is a principally polarized abelian variety of dimension $g-1$, denoted by $P(C,\eta)$ or equivalently $P(\tilde{C},C)$. This defines the Prym map

$$\mathscr{P} : \mathsf{R}_g \longrightarrow \mathsf{A}_{g-1}, \qquad \mathscr{P}([C,\eta]) := [P(C,\eta)].$$

where A_{g-1} is the moduli space of principally polarized abelian varieties of dimension $g-1$. We recall that the Prym map is generically an embedding for $g \geq 7$ [24], [30] and it is generically finite for $g \geq 6$. The Prym map is never injective and it has positive dimensional fibres [18], [40], [42].

Analogously one can consider the moduli space parametrising ramified double coverings and the corresponding Prym varieties. We will only consider the case in which the Prym variety is principally polarised, that is when the map is ramified at two distinct points.

So let $\mathsf{R}_{g,[2]}$ denote the scheme parametrizing triples $[C,\eta,B]$ up to isomorphism, where C is a genus g curve, η a line bundle on C of degree 1, and B a reduced divisor in the linear system $|\eta^2|$ corresponding to a $2:1$ covering $\pi : \tilde{C} \to C$ ramified over B. The *Prym map* is the morphism

$$\mathscr{P} : \mathsf{R}_{g,[2]} \to \mathsf{A}_g$$

which associates to $[C,\eta,B]$ the Prym variety $P(\tilde{C},C)$ of π. It is generically finite for $g \geq 5$ and generically injective for $g \geq 6$ (see [35]).

Denote by

$$j : \mathsf{M}_g \longrightarrow \mathsf{A}_g, \qquad j([C]) := [J(C)].$$

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the Torelli map and by $\overline{j(\mathsf{M}_g)}$ the Torelli locus. The work of Beauville [4] on admissible covers shows that one has the following inclusions

$$\overline{j(\mathsf{M}_g)} \subset \overline{\mathscr{P}(\mathsf{R}_{g,[2]})} \subset \overline{\mathscr{P}(\mathsf{R}_{g+1})}. \tag{1.1}$$

(See also [21] and in the ramified case [35] and also sections 3 and 4 below).

On A_g , viewed as orbifold there is a natural variation of Hodge structure whose fiber at a point A is $H^1(A,\mathbb{Q})$. The Hodge loci for this variation of Hodge structure are called *special* or *Shimura subvarieties* of A_g. A conjecture by Coleman and Oort [43] says that for large genus there should not exist special or Shimura subvarieties of A_g *generically contained* in the Torelli locus, i.e. contained in $\overline{j(\mathsf{M}_g)}$ and intersecting $j(\mathsf{M}_g)$. See [38] for more information, [29, 17, 14, 32, 33] for some results towards the conjecture and [16, 37, 22, 23, 27, 28] for counterexamples to the conjecture in low genera.

Recall that Shimura subvarieties of A_g are totally geodesic with respect to the orbifold metric induced on A_g from the symmetric metric on the Siegel space $\mathfrak{H}_g$. The conjecture is coherent with the fact that the Torelli locus is very curved, and a possible approach to the conjecture is via the study of the second fundamental form of the Torelli map ([15],[14]). The geometry of R_g has many analogies with the geometry of M_g and it has been extensively investigated (see [20] for a nice survey). Moreover, the second fundamental form of the Prym map $\mathscr{P}:\mathsf{R}_g \longrightarrow \mathsf{A}_{g-1}$ has a very similar structure and similar properties as the one of the Torelli map [13].

In view of these similarities and of the inclusions (1.1) it is natural to ask a question analogous to the one of Coleman and Oort for the Prym loci $\overline{\mathscr{P}(\mathsf{R}_{g+1})}$ and $\overline{\mathscr{P}(\mathsf{R}_{g,[2]})}$, namely the following:

Question. *Do there exist special subvarieties of A_g that are generically contained in the Prym loci $\mathscr{P}(\mathsf{R}_{g+1})$ and $\mathscr{P}(\mathsf{R}_{g,[2]})$ for g sufficiently high?*

For low genera ($g \leq 7$) there do exist Shimura subvarieties of A_g contained in the Torelli locus. These have all been constructed as families of Jacobians of Galois coverings of $\mathbb{P}^1$ and of genus one curves ([16], [47], [37], [38], [22], [23]) [27], [28]). All these families of curves C satisfy the sufficient condition that $\dim(S^2H^0(K_C))^G = \dim H^0(2K_C)^G$, where G is the Galois group of the covering (see [22] Theorem 3.9). This condition ensures that the multiplication map $m:(S^2H^0(K_C))^G \to H^0(2K_C)^G$ is an isomorphism. Notice that the multiplication map is the codifferential of the Torelli map. As a first attempt to see the similarity between the Torelli and Prym loci from this point of view, in this paper we construct Shimura curves contained in the Prym loci that satisfy an analogous sufficient condition.

We consider a one-dimensional family of curves $\{\tilde{C}_t\}_{t\in\mathbb{C}-\{0,1\}}$ admitting an action of a group of automorphisms $\tilde{G}$ containing a central involution σ and such that the quotient $\tilde{C}_t/\tilde{G} \cong \mathbb{P}^1$, the covering $\psi_t:\tilde{C}_t \to \tilde{C}_t/\tilde{G}$ is branched at 4 points and the double covering $\tilde{C}_t \to \tilde{C}_t/\langle\sigma\rangle =: C_t$ is either étale or ramified at two distinct points. We give a condition which ensures that the family of the Prym varieties $P(\tilde{C}_t, C_t)$ of the 2:1 coverings yields a Shimura curve. The condition is that the multiplication map $m:(S^2H^0(K_{C_t}\otimes\eta))^{\tilde{G}} \to H^0(2K_{C_t}\otimes 2\eta)^{\tilde{G}}$ is an isomorphism. The multiplication map is the codifferential of the Prym map.

Since the covering ψ_t is branched at 4 points, $\dim(H^0(2K_{C_t}\otimes 2\eta)^{\tilde{G}}) = 1$, so our first requirement is that $\dim((S^2H^0(K_{C_t}\otimes\eta))^{\tilde{G}}) = 1$ (condition (A) of section 3 and section 4).

Unlike the Torelli map, the Prym map has positive dimensional fibers, therefore condition (A) is not enough to ensure that multiplication map m is an isomorphism, or equivalently that m is not zero (condition (B) of section 3 and section 4).

We notice that if $(S^2H^0(K_{C_t}\otimes\eta))^{\tilde{G}}$ is generated by a decomposable tensor (condition (B1) of sections 3 and 4) the multiplication map cannot be zero, hence condition (B) is satisfied.

This happens in particular when the group $\tilde{G}$ is abelian, hence in this case it is enough to verify condition (A) to have a Shimura curve. When $\tilde{G}$ is not abelian we study the geometry of some of these families satisfying condition (A) and we prove that the families of Pryms are not constant, hence condition (B) is satisfied.

As in the Torelli case, all the examples we found up to now are in low dimension, namely in A_g with $g \leq 12$. All the examples where the group is abelian are in A_g with $g \leq 10$. In the ramified case they are all in dimension $g \leq 8$. We also notice that the last example we find satisfying conditions (A) and (B1) are in dimension $g = 10$. To prove that the remaining examples satisfying (A) yield Shimura curves we need ad hoc arguments. On the whole, the number of examples satisfying condition (A) decreases as the dimension grows. This suggests that, as in the Torelli case, one could expect that for high dimension there should not exist Shimura curves contained in the Prym locus constructed in this way.

Let us explain explicitly how we construct these families in the case of unramified double coverings.

A Galois covering $\tilde{C} \to \mathbb{P}^1$ is determined by the Galois group $\tilde{G}$, an epimorphism $\tilde{\theta} : \Gamma_r \to \tilde{G}$ and the branch points $t_1, ..., t_r \in \mathbb{P}^1$ (see section 3 for the notation). We will choose $r = 4$. We also fix a central involution $\sigma \in \tilde{G}$ that does not lie in $\bigcup_{i=1}^r \langle \tilde{\theta}(\gamma_i) \rangle$. Denote by $G = \tilde{G}/\langle\sigma\rangle$. Fixing the Prym datum $(\tilde{G}, \tilde{\theta}, \sigma)$, setting $\{t_1, t_2, t_3\} = \{0, 1, \infty\}$ and letting the point $t_4 = t$ vary we get a one dimensional family of curves and coverings

$$\begin{tikzcd}[column sep=small]
\tilde{C}_t \arrow[rr, "\pi_t"] \arrow[dr] & & C_t = \tilde{C}_t/\langle\sigma\rangle \arrow[dl] \\
& \mathbb{P}^1 \cong \tilde{C}_t/\tilde{G} \cong C_t/G &
\end{tikzcd}$$

and correspondingly a family $\mathsf{R}(\tilde{G}, \tilde{\theta}, \sigma) \subset \mathsf{R}_g$.

Let $\pi : \tilde{C} \to C$ be an element of the family and let $\eta \in Pic^0(C)$ be the 2-torsion element yielding the étale double covering π. Set $V = H^0(\tilde{C}, K_{\tilde{C}})$, and let $V = V_+ \oplus V_-$ be the eigenspace decomposition for the action of σ. The summand V_+ is isomorphic as a G-representation to $H^0(C, K_C)$, while V_- is isomorphic to $H^0(C, K_C \otimes \eta)$. Set $W = H^0(\tilde{C}, 2K_{\tilde{C}})$ and let $W = W_+ \oplus W_-$ be the eigenspace decomposition for the action of σ. We have $W_+ \cong H^0(C, 2K_C)$ and $W_- \cong H^0(C, 2K_C \otimes \eta)$. Consider the multiplication map $m : S^2V \longrightarrow W$. It is the codifferential of the Torelli map $\tilde{j} : \mathsf{M}_{\tilde{g}} \to \mathsf{A}_{\tilde{g}}$ at $[\tilde{C}] \in \mathsf{M}_{\tilde{g}}$. The multiplication map is $\tilde{G}$-equivariant and we have the following isomorphisms

$$(S^2V)^{\tilde{G}} = (S^2V_+)^G \oplus (S^2V_-)^G, \ W^{\tilde{G}} = W_+^G.$$

Therefore m maps $(S^2V)^{\tilde{G}}$ to W_+^G. We are interested in the restriction:

$$m : (S^2V_-)^G \longrightarrow W_+^G. \tag{1.2}$$

By the above discussion this is just the multiplication map $(S^2H^0(C, K_C \otimes \eta))^G \longrightarrow H^0(C, 2K_C)^G$.

Theorem 1.1. *(see Theorem 3.2) Let $(\tilde{G}, \tilde{\theta}, \sigma)$ be a Prym datum. If the map m in* (1.2) *is an isomorphism, then the closure of $\mathscr{P}(\mathsf{R}(\tilde{G}, \tilde{\theta}, \sigma))$ in A_{g-1} is a special subvariety generically contained in the Prym locus.*

In a similar way one can construct families of Pryms in the ramified case and the analogous sufficient condition to ensure that the family yields a Shimura subvariety of A_g (see Theorem 4.2). To produce sistematically these Shimura families we used `MAGMA` [34]. Our script is available at: `http://www.dima.unige.it/~penegini/publications`. Using this script one can in principle determine all the families satisfying condition (A) and (B1) both in the unramified and in the ramified case for every $\tilde{g} = g(\tilde{C})$.

Notice that in the unramified case $\tilde{g} = 2g - 1$, while in the ramified case $\tilde{g} = 2g$, where $g = g(C) = g(\tilde{C}/\langle\sigma\rangle)$. As we have already observed, if $\tilde{G}$ is abelian, condition (B) is automatically satisfied, hence we get a Shimura curve. In the non abelian case we analysed some of the families satisfying condition (A) and we proved that they also yield a Shimura curve. Summarising we have the following theorem.

Theorem 1.2. *In the unramified case, for $\tilde{g} = 2g - 1 \leq 27$ we obtain 41 families satisfying condition* (B1) *(28 are abelian, 13 non-abelian). We obtain three more non-abelian families* *satisfying condition* (B), *namely families 40, 43, 44 of Table 1. So in the unramified case we have found 44 families of Pryms yielding Shimura curves of* A_{g-1} *for* $g \leq 13$.

In the ramified case, for $\tilde{g} = 2g \leq 28$, *we found 9 Shimura families all with* $\tilde{g} \leq 16$. *Of these 9 families 6 satisfy condition* (B1). *Two other families do not satisfy condition* (B1), *but they satisfy condition* (B). *So in the ramified case we found 8 families of Pryms yielding Shimura curves of* A_g *with* $g \leq 8$. *See Table 1.*

The plan of the paper is the following:

In section 2 we recall the definition of special or Shimura subvarieties of A_g and we briefly summarise some of the results of section 3 of [22].

In section 3 we explain the construction of the families of Pryms in the unramified case and we prove Theorem 1.1.

In section 4 we do the analogous construction in the ramified case and we prove the analogous result (Theorem 4.2).

Next we describe a sample of the examples.

All the unramified abelian examples are in A_k with $k \leq 10$. In section 5 we describe the only 7 unramified abelian examples yielding a Shimura curve generically contained in the Prym locus for $k \geq 6$, hence for which the closure of the Prym locus is not all A_k. There are two examples also for $k = 8$, and one example for $k = 10$. Up to now there are no known examples of Shimura varieties generically contained in the Torelli locus in A_k for $k \geq 8$. We also show that the familes in A_8 are not families of Jacobians. Next we describe three unramified non-abelian examples that don't satisfy condition (B1). Hence we prove by ad hoc methods that they do indeed produce Shimura curves generically contained in the Prym locus in A_9 and A_{12} and and we describe their geometry.

In section 6 we describe the examples found in the ramified case. One of the non-abelian examples gives a Shimura curve generically contained in the ramified Prym locus in A_8 and we show that it is not in the Torelli locus.

In the appendix we describe the script and we give the table of the examples.

2. Special subvarieties of A_g

2.1. Let $E : \mathbb{Z}^{2g} \times \mathbb{Z}^{2g} \to \mathbb{Z}$ be the alternating form of type $(1, \ldots, 1)$ corresponding to the matrix

$$\begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}.$$

The Siegel upper half-space is defined as follows

$$\mathfrak{H}_g := \{J \in \mathrm{GL}(\mathbb{R}^{2g}) : J^2 = -I, J^*E = E, E(x, Jx) > 0, \ \forall x \neq 0\}.$$

The group $\mathrm{Sp}(2g, \mathbb{Z})$ acts on $\mathfrak{H}_g$ by conjugation and this action is properly discontinuous. Set $\mathsf{A}_g := \mathrm{Sp}(2g, \mathbb{Z}) \backslash \mathfrak{H}_g$. This space has the both the structure of a complex analytic orbifold and the structure of a smooth algebraic stack. Throughout the paper we will work with A_g with the orbifold structure. Denote by A_J the real torus $\Lambda_{\mathbb{R}}/\Lambda$ provided with the complex structure $J \in \mathfrak{H}_g$ and the polarization E. It is a principally polarized abelian variety. On $\mathfrak{H}_g$ there is a natural variation of rational Hodge structure, with local system $\mathfrak{H}_g \times \mathbb{Q}^{2g}$ and corresponding to the Hodge decomposition of $\mathbb{C}^{2g}$ in $\pm i$ eigenspaces for J. This descends to a variation of Hodge structure on A_g in the orbifold or stack sense.

2.2. We refer to §2.3 in [38] for the definition of Hodge loci for a variation of Hodge structure. A *special subvariety* $\mathsf{Z} \subseteq \mathsf{A}_g$ is by definition a Hodge locus of the natural variation of Hodge structure on A_g described above. Special subvarieties contain a dense set of CM points and they are totally geodesic [38, §3.4(b)]. Conversely an algebraic totally geodesic subvariety that contains a CM point is a special subvariety [39] (see [36, Thm. 4.3] for a more general result).

The simplest special subvarieties are the *special subvarieties of PEL type*, whose definition is as follows (see [38, §3.9] for more details). Given $J \in \mathfrak{H}_g$, set

$$\operatorname{End}_{\mathbb{Q}}(A_J) := \{f \in \operatorname{End}(\mathbb{Q}^{2g}) : Jf = fJ\}. \tag{2.1}$$

Fix a point $J_0 \in \mathfrak{H}_g$ and set $D := \operatorname{End}_{\mathbb{Q}}(A_{J_0})$. The *PEL type* special subvariety $\mathsf{Z}(D)$ is defined as the image in A_g of the connected component of the set $\{J \in \mathfrak{H}_g : D \subseteq \operatorname{End}_{\mathbb{Q}}(A_J)\}$ that contains J_0. By definition $\mathsf{Z}(D)$ is irreducible.

If $G \subseteq \operatorname{Sp}(2g, \mathbb{Z})$ is a finite subgroup, denote by $\mathfrak{H}_g^G$ the set of points of $\mathfrak{H}_g$ that are fixed by G. Set

$$D_G := \{f \in \operatorname{End}_{\mathbb{Q}}(\mathbb{Q}^{2g}) : Jf = fJ, \ \forall J \in \mathfrak{H}_g^G\}. \tag{2.2}$$

In the following statement we summarize what is needed in the rest of the paper regarding special subvarieties. See [22, §3] for the proofs.

Theorem 2.3. *The subset $\mathfrak{H}_g^G$ is a connected complex submanifold of $\mathfrak{H}_g$. The image of $\mathfrak{H}_g^G$ in A_g coincides with the PEL subvariety $\mathsf{Z}(D_G)$. If $J \in \mathfrak{H}_g^G$, then* $\dim \mathsf{Z}(D_G) = \dim(S^2\mathbb{R}^{2g})^G$ *where $\mathbb{R}^{2g}$ is endowed with the complex structure J.*

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3. Special subvarieties in the unramified Prym locus

In this section we explain how to construct Shimura subvarieties generically contained in the Prym locus, that is contained in $\overline{\mathscr{P}(\mathsf{R}_g)}$ and intersecting $\mathscr{P}(\mathsf{R}_g)$. Recall that one has $\overline{j(\mathsf{M}_{g-1})} \subset \overline{\mathscr{P}(\mathsf{R}_g)}$. In fact it is known already from the work of Wirtinger [49] (see [4] for a modern proof) that Jacobians appear as limits of Pryms. The fiber of the extended Prym map over a generic Jacobian has been studied in detail in [19] and [30]. It is therefore natural to extend the search for Shimura subvarieties contained in the Torelli locus to the case of the Prym locus and to ask whether such Shimura subvarieties exist in high dimension.

For any integer $r \geq 3$ let Γ_r denote the group with presentation $\Gamma_r = \langle \gamma_1, \ldots, \gamma_r | \gamma_1 \cdots \gamma_r = 1\rangle$. A *datum* is a pair (G, θ) where G is a finite group and $\theta : \Gamma_r \longrightarrow G$ is an epimorphism. We will only be concerned with the case $r = 4$. If a datum (G, θ) is fixed, we set $\mathbf{m} := (m_1, \ldots, m_r)$ where m_i is the order of $(\theta(\gamma_i))$. We sometimes stress the importance of the vector $\mathbf{m}$ denoting a datum by $(\mathbf{m}, G, \theta)$. (In fact this is important in the `MAGMA` script, which starts out by computing the possible vectors $\mathbf{m}$ that satisfy the Riemann-Hurwitz formula. So in the computation the vector $\mathbf{m}$ really comes before (G, θ).)

Denote by $\mathsf{T}_{0,r}$ the Teichmüller space in genus 0 and with $r \geq 4$ marked points. The definition of $\mathsf{T}_{0,r}$ is as follows. Fix $r + 1$ distinct points $p_0, \ldots, p_r$ on S^2. For simplicity set $P = (p_1, \ldots, p_r)$. Consider triples of the form $(C, x, [f])$ where C is a curve of genus 0, $x = (x_1, \ldots, x_r)$ is an r-tuple of distinct points in C and $[f]$ is an isotopy class of orientation preserving homeomorphisms $f : (C, x) \to (S^2, P)$. Two such triples $(C, x, [f])$ and $(C', x', [f'])$ are equivalent if there is a biholomorphism $\varphi : C \to C'$ such that $\varphi(x_i) = x'_i$ for any i and $[f] = [f' \circ \varphi]$. The Teichmüller space $\mathsf{T}_{0,r}$ is the set of all equivalence classes, see e.g. [2, Chap. 15] for more details. Since C has genus 0 we can assume that $C = \mathbb{P}^1$. Using the point $p_0 \in S^2 - P$ as base point we can fix an isomorphism $\Gamma_r \cong \pi_1(S^2 - P, p_0)$.

If a datum (G, θ) and a point $t = [\mathbb{P}^1, x, [f]] \in \mathsf{T}_{0,r}$ are fixed, we get an epimorphism $\pi_1(\mathbb{P}^1 - x, f^{-1}(p_0)) \cong \Gamma_r \to G$ and thus a covering $C_t \to \mathbb{P}^1 = C_t/G$ branched over x with monodromy given by this epimorphism. The curve C_t is equipped with an isotopy class of homeomorphisms to a fixed branched cover Σ of S^2. Thus we have a map $\mathsf{T}_{0,r} \to \mathsf{T}_g \cong \mathsf{T}(\Sigma)$ to the Teichmüller space of Σ. The group G embeds in the mapping class group of Σ, denoted Mod_g. This embedding depends on θ and we denote by $G_\theta \subset \operatorname{Mod}_g$ its image. It turns out that the image of $\mathsf{T}_{0,r}$ in T_g is exactly the set of fixed points $\mathsf{T}_g^{G_\theta}$ of the group G_θ. We denote this set by $\mathsf{T}(G,\theta)$. It is a complex submanifold of T_g. The image of $\mathsf{T}(G,\theta)$ in the moduli space M_g is a $(r-3)$-dimensional algebraic subvariety that we denote by $\mathsf{M}(G,\theta)$. See e.g. [26, 8, 9] and [7, Thm. 2.1] for more details.

In the discussion above the choice of the base point p_0 is irrelevant. On the other hand the choice of the isomorphism $\Gamma_r \cong \pi_1(S^2 - P, p_0)$ does matter. To describe this we introduce the braid group:

$$\mathbf{B_r} := \langle \tau_1, \dots, \tau_r | \, \tau_i\tau_j = \tau_j\tau_i \text{ for } |i-j| \geq 2, \, \tau_{i+1}\tau_i\tau_{i+1} = \tau_i\tau_{i+1}\tau_i \rangle$$

There is a morphism $\varphi : \mathbf{B_r} \to \operatorname{Aut}(\Gamma_r)$ defined as follows:

$$\begin{gathered} \varphi(\tau_i)(\gamma_i) = \gamma_{i+1}, \quad \varphi(\tau_i)(\gamma_{i+1}) = \gamma_{i+1}^{-1}\gamma_i\gamma_{i+1}, \\ \varphi(\tau_i)(\gamma_j) = \gamma_j \quad \text{for } j \neq i, i+1. \end{gathered}$$

From this we get an action of $\mathbf{B_r}$ on the set of data: $\tau \cdot (\mathbf{m}, G, \theta) := (\tau(\mathbf{m}), G, \theta \circ \varphi(\tau^{-1}))$, where $\tau(\mathbf{m})$ is the permutation of $\mathbf{m}$ induced by τ. Also the group $\operatorname{Aut}(G)$ acts on the set of data by $\alpha \cdot (\mathbf{m}, G, \theta) := (\mathbf{m}, G, \alpha \circ \theta)$. The orbits of the $\mathbf{B_r} \times \operatorname{Aut}(G)$–action are called *Hurwitz equivalence classes* and elements in the same orbit are said to be related by a *Hurwitz move*. Data in the same orbit give rise to distinct submanifolds of T_g which project to the same subvariety of M_g. So the submanifold $\mathsf{T}(G,\theta)$ is not well-defined, but the subvariety $\mathsf{M}(G,\theta)$ is well-defined. For more details see [46, 8, 6].

Definition 3.1. *A* Prym datum *is triple $\Xi = (\tilde{G}, \tilde{\theta}, \sigma)$, where $\tilde{G}$ is a finite group, $\tilde{\theta} : \Gamma_r \to \tilde{G}$ is an epimorphism and $\sigma \in Z(\tilde{G})$ is an element of order 2, that does not lie in $\bigcup_{i=1}^r \langle \tilde{\theta}(\gamma_i) \rangle$. (Here $Z(\tilde{G})$ denotes the centre of $\tilde{G}$.)*

Set $G := \tilde{G}/\langle\sigma\rangle$ and denote by $\theta : \Gamma_r \to G$ the composition of $\tilde{\theta}$ with the projection $\tilde{G} \to G$. A Prym datum gives rise to two submanifolds of Teichmüller spaces, namely $\mathsf{T}(G,\theta) \subset \mathsf{T}_g$ and $\mathsf{T}(\tilde{G}, \tilde{\theta}) \subset \mathsf{T}_{\tilde{g}}$. Both are isomorphic to $\mathsf{T}_{0,r}$ as explained above. For any $t \in \mathsf{T}_{0,r}$ we have a diagram

$$\begin{tikzcd} \tilde{C}_t \arrow[rr, "\pi_t"] \arrow[dr] & & C_t = \tilde{C}_t/\langle\sigma\rangle. \arrow[dl] \\ & \mathbb{P}^1 & \end{tikzcd}$$

Here $\tilde{C}_t \to \mathbb{P}^1$ is the $\tilde{G}$-covering corresponding to $t \in \mathsf{T}_{0,r}$ and to the datum $(\tilde{G}, \tilde{\theta})$. The quotient map $\pi_t : \tilde{C}_t \to \tilde{C}_t/\langle\sigma\rangle$ is an étale double cover. In fact the elements of $\tilde{G}$ that have fixed points belong to some conjugate of some $\langle\tilde{\theta}(\gamma_i)\rangle$. Since σ is central the definition ensures that it acts freely on $\tilde{C}_t$. Finally it is easy to check that $C_t \longrightarrow \mathbb{P}^1$ is the G-covering corresponding to $t \in \mathsf{T}_{0,r}$ and to the datum (G,θ). Denote by η_t the element of $\operatorname{Pic}^0(C_t)$, corresponding to the covering π_t, i.e. such that $(\pi_t)_*(\mathcal{O}_{\tilde{C}_t}) = \mathcal{O}_{C_t} \oplus \eta_t$. Associating to $t \in \mathsf{T}_{0,r}$ the class of the pair (C_t, η_t) we get a map $\mathsf{T}_{0,r} \longrightarrow \mathsf{R}_g$. This map has discrete fibres. We denote by $\mathsf{R}(\Xi)$ its image. Hence $\dim \mathsf{R}(\Xi) = r - 3$. The following diagram (where $\widetilde{j}$ and j denote the Torelli morphisms) summarizes the construction.

$$\begin{tikzcd} \mathsf{T}_{0,r} \arrow[r, "\cong"] \arrow[d, "\cong"'] & \mathsf{T}(\tilde{G},\tilde{\theta}) \arrow[rr] \arrow[dl, "\cong"] \arrow[dr] & & \mathsf{M}_{\tilde{g}} \arrow[r, "\widetilde{j}"] & \mathsf{A}_{\tilde{g}} \\ \mathsf{T}(G,\theta) \arrow[dr] & & \mathsf{R}(\Xi) \arrow[ur] \arrow[r, "\mathscr{P}"] \arrow[dl] & \mathsf{A}_{g-1} & \\ & \mathsf{M}_g \arrow[r, "j"] & \mathsf{A}_g & & \end{tikzcd} \tag{3.1}$$

Given a Prym datum $\Xi = (\tilde{G}, \tilde{\theta}, \sigma)$ fix an element $\tilde{C}_t$ of the family $\mathsf{T}(\tilde{G}, \tilde{\theta})$ with corresponding étale covering $\pi_t : \tilde{C}_t \longrightarrow C_t$. For simplicty we drop the index t. Set

$$V := H^0(\tilde{C}, K_{\tilde{C}}),$$

and let $V = V_+ \oplus V_-$ be the eigenspace decomposition for the action of σ. The factor V_+ is isomorphic as a G-representation to $H^0(C, K_C)$, while V_- is isomorphic to $H^0(C, K_C \otimes \eta)$. Set

$$W := H^0(\tilde{C}, 2K_{\tilde{C}}),$$

and let $W = W_+ \oplus W_-$ be the eigenspace decomposition for the action of σ. We have $W_+ \cong H^0(C, 2K_C)$ and $W_- \cong H^0(C, 2K_C \otimes \eta)$ as G-representations. The multiplication map

$$m : S^2 V \longrightarrow W$$

is $\tilde{G}$-equivariant and is the codifferential of the Torelli map $\tilde{j} : \mathsf{M}_{\tilde{g}} \to \mathsf{A}_{\tilde{g}}$ at $[\tilde{C}] \in \mathsf{M}_{\tilde{g}}$. We have the following isomorphisms

$$(S^2 V)^{\tilde{G}} = (S^2 V_+)^G \oplus (S^2 V_-)^G, \qquad W^{\tilde{G}} = W_+^G.$$

Therefore m maps $(S^2 V)^{\tilde{G}}$ to W_+^G. We are interested in the restriction of m to $(S^2 V_-)^G$ that for simplicity we denote by the same symbol:

$$m : (S^2 V_{-,t})^G \longrightarrow W_{+,t}^G. \tag{3.2}$$

By the above discussion this is just the multiplication map

$$(S^2 H^0(C, K_C \otimes \eta))^G \longrightarrow H^0(C, 2K_C)^G.$$

Theorem 3.2. *Let $\Xi = (\tilde{G}, \tilde{\theta}, \sigma)$ be a Prym datum. If there is $t \in \mathsf{T}_{0,r}$ such that the map m in* (3.2) *is an isomorphism, then the closure of $\mathscr{P}(\mathsf{R}(\Xi))$ in A_{g-1} is a special subvariety of dimension $r-3$.*

Proof. Over $\mathsf{T}_{0,r}$ we have the families $\tilde{C}_t$, C_t, $\pi_t : \tilde{C}_t \to C_t$ and (C_t, η_t) as in diagram (3.1). The lattice $H_1(\tilde{C}_t, \mathbb{Z})$ is independent of $t \in \mathsf{T}_{0,r}$. Set $\Lambda := H_1(\tilde{C}_t, \mathbb{Z})_-$. Call Q the intersection form on $H_1(\tilde{C}_t, \mathbb{Z})$, i.e. the principal polarization on the Jacobian of $\tilde{C}$. Also Q is independent of t. Set

$$E := (1/2) \cdot Q|_\Lambda.$$

E is an integral symplectic form on Λ. Let $\mathfrak{H}_{g-1}$ be the Siegel upper half-space that parametrizes complex structures on $\Lambda \otimes \mathbb{R} = H_1(\tilde{C}_t, \mathbb{R})_-$ that are compatible with E. For $t \in \mathsf{T}_{0,r}$ we have $H^1(\tilde{C}_t, \mathbb{C}) = V_t \oplus \overline{V_t}$ with $V_t = H^0(\tilde{C}_t, K_{\tilde{C}_t})$ and also $H^1(\tilde{C}_t, \mathbb{C})_- = V_{-,t} \oplus \overline{V_{-,t}}$. Dualizing we get the decomposition

$$H_1(\tilde{C}_t, \mathbb{C})_- = V_{-,t}^* \oplus \overline{V_{-,t}^*}.$$

This decomposition corresponds to a complex structure J_t on $H_1(\tilde{C}_t, \mathbb{R})_-$, that is compatible with E and therefore represents a point of $\mathfrak{H}_{g-1}$, that we denote by $f(t)$. We have thus defined a map $f : \mathsf{T}_{0,r} \to \mathfrak{H}_{g-1}$. The point is that the following diagram commutes:

$$\begin{array}{ccc}
\mathsf{T}_{0,r} & \xrightarrow{f} & \mathfrak{H}_{g-1} \\
\downarrow & & \downarrow \\
\mathsf{R}(\Xi) \subset \mathsf{R}_g & \xrightarrow{\mathscr{P}} & \mathsf{A}_{g-1}.
\end{array}$$

To check this it is enough to recall that

$$P(C_t, \eta_t) = V_{-,t}^* / \Lambda,$$

(see e.g. [1, p. 295ff] or [5, p. 374ff]). Since $\tilde{G}$ preserves Q, G preserves E, so G maps into $\mathrm{Sp}(\Lambda, E)$. Denote by G' the image of G in $\mathrm{Sp}(\Lambda, E)$. The complex structure J_t is G-invariant, i.e. $f(t) = J_t \in \mathfrak{H}_{g-1}^{G'}$. Hence by Theorem 2.3 $P(C_t, \eta_t)$ lies in the PEL special subvariety

$\mathsf{Z}(D_{G'})$. Therefore $\mathscr{P}(\mathsf{R}(\Xi)) \subset \mathsf{Z}(D_{G'})$. Since $f(\mathsf{T}_{0,r}) \subset \mathfrak{H}_{g-1}^{G'}$ we can consider f as a map $f : \mathsf{T}_{0,r} \to \mathfrak{H}_{g-1}^{G'}$. Recall that

$$\Omega^1_{f(t)}\mathfrak{H}_{g-1}^{G'} \cong (S^2 H^0(C_t, K_{C_t} \otimes \eta_t))^G = S^2 V_{-,t},$$
$$\Omega^1_t \mathsf{T}_{0,r} \cong \Omega^1_{[C_t]}\mathsf{T}(G,\theta) \cong H^0(C_t, 2K_{C_t})^G = W^G_{t,+}.$$

The codifferential is simply the multiplication map (see [3] Prop. 7.5)

$$m = (df_t)^* : (S^2 V_{-,t})^G \longrightarrow W^G_{t,+}.$$

This follows from the fact that the codifferential of the Torelli map restricted to $\mathsf{T}_{0,r}$ is the full multiplication map $S^2 V \to W$. By assumption there is a point $t \in \mathsf{T}_{0,r}$ such that the map m is an isomorphism at t. This implies first of all that $\dim(S^2 V_{-,t})^G = \dim W^G_{t,+} = r - 3$. Moreover f is an immersion at point t, hence its image has dimension $r - 3$. As the vertical arrows in (3.1) are discrete maps, both $\mathscr{P}(\mathsf{R}(\Xi))$ and $\mathsf{Z}(D_{G'})$ have dimension $r - 3$. Since $\mathsf{R}(\Xi) \subset \mathsf{Z}(D_{G'})$ and $\mathsf{Z}(D_{G'})$ is irreducible we conclude that $\overline{\mathsf{R}(\Xi)} = \mathsf{Z}(D_{G'})$ as desired. □

The Shimura subvarieties constructed using Theorem 3.2 are *generically contained in the Prym locus*, i.e. they intersect the Prym locus and are contained in its closure.

We wish to apply Theorem 3.2 to construct examples of 1-dimensional special subvarieties (i.e. *Shimura curves*) in A_{g-1}. So from now on we assume $r = 4$.

In the case $r = 4$ the sufficient condition in Theorem 3.2 (namely that m be an isomorphism) can be split in two parts:

(A) $$\dim(S^2 V_-)^{\tilde{G}} = 1.$$

(B) $$m : (S^2 V_-)^{\tilde{G}} \longrightarrow W_+^G \text{ is not identically } 0.$$

Once (A) is known, a sufficient condition ensuring (B) is the following

(B1) $$(S^2 V_-)^{\tilde{G}} \text{ is generated by a decomposable tensor.}$$

In fact if $(S^2 V_-)^{\tilde{G}}$ is generated by $s_1 \otimes s_2$ with $s_i \in V_-$, then $m(s_1 \otimes s_2) = s_1 \cdot s_2$ which cannot vanish identically.

Remark 3.3. We claim if (A) holds, then (B1) is equivalent to the fact that $(S^2 V_-)^{\tilde{G}} = W_1 \otimes W_2$ with W_i 1-dimensional representations. In one direction this is obvious. In the opposite direction, assume that (A) and (B1) hold. Let $V_- = W_1 \oplus \cdots \oplus W_k$ be a decomposition in irreducible representations. Then

$$(S^2 V_-)^{\tilde{G}} = \bigoplus_{i=1}^k (S^2 W_i)^{\tilde{G}} \oplus \bigoplus_{i<j} (W_i \otimes W_j)^{\tilde{G}}.$$

Since $(S^2 V_-)^{\tilde{G}}$ is 1-dimensional, there are two cases: either $(S^2 V_-)^{\tilde{G}} = (S^2 W_i)^{\tilde{G}}$ for some i or $(S^2 V_-)^{\tilde{G}} = (W_i \otimes W_j)^{\tilde{G}}$ for some i and some j. We treat the first case, the other being identical. Let $t \in (S^2 V_-)^{\tilde{G}} = (S^2 W_i)^{\tilde{G}}$ be a generator. By Schur lemma this represents an isomorphism $t : W_i^* \to W_i$. If $d = \dim W_i$, then t has rank d. By (B1) t is decomposable hence $d = 1$, therefore $(S^2 V_-)^{\tilde{G}} = W_i \otimes W_i$.

4. Special subvarieties in the ramified Prym locus

In this section we would like to repeat the construction of the previous section in the case in which the double covering $\pi_t : \tilde{C}_t \to C_t$ is ramified at two points. This is the only other case in which the associated Prym variety is principally polarised [40, 5].

Let C be a curve, η a line bundle on C of degree 1 and B a reduced divisor in the linear system $|\eta^2|$, i.e. $B = p + q$ with $p \neq q$. From this data one gets a double cover $\pi : \tilde{C} \to C$ ramified over B. The *Prym variety* $P(\tilde{C}, C)$ of π is defined as the kernel of the norm map, which in this case is connected. As in the unramified case, the polarization of $J(\tilde{C})$ restricts to the double of a principal polarization E on $P(\tilde{C}, C)$. We will always consider $P(\tilde{C}, C)$ with the principal polarizaztion E. In the case at hand it has dimension g.

Let $\mathsf{R}_{g,[2]}$ denote the scheme parametrizing triples $[C, \eta, B]$ up to isomorphism; the *Prym map* is the morphism

$$\mathscr{P} : \mathsf{R}_{g,[2]} \to \mathsf{A}_g$$

which associates to $[C, \eta, B]$ the Prym variety $P(\tilde{C}, C)$ of π.

We recall that we have the following inclusions $\overline{j(\mathsf{M}_g)} \subset \overline{\mathscr{P}(\mathsf{R}_{g,[2]})} \subset \overline{\mathscr{P}(\mathsf{R}_{g+1})}$. Roughly the inclusion $\overline{\mathscr{P}(\mathsf{R}_{g,[2]})} \subset \overline{\mathscr{P}(\mathsf{R}_{g+1})}$ can be seen as follows: given a double covering of a smooth curve of genus g ramified at two points, we obtain an admissible Beauville covering gluing the two branch points and the corresponding ramification points (see [21] p.763).

The inclusion $\overline{j(\mathsf{M}_g)} \subset \overline{\mathscr{P}(\mathsf{R}_{g,[2]})}$ can be seen as follows: take a smooth genus g curve C. Consider the 2-pointed 1-nodal curve $X = C \cup \mathbb{P}^1$ where C and $\mathbb{P}^1$ meet transversally at a point x and let p, q the two marked points in $\mathbb{P}^1$. Consider the admissible ramified double cover $\tilde{X}$ of X costructed as follows. Take the double cover $f : \mathbb{P}^1 \to \mathbb{P}^1$ ramified in p, q and denote by $\{p_1, p_2\} = f^{-1}(x) \subset \mathbb{P}^1$. Take two copies C_1, C_2 of C, and glue these curves with $\mathbb{P}^1$ identifying the points $x \in C_i$ with p_i. Clearly the Prym $P(\tilde{X}, X)$ is the Jacobian of C.

Thus it is again natural to extend the search for Shimura varieties in the Torelli locus to the ramified Prym locus and the question about the existence of such Shimura subvarieties in high dimension.

Definition 4.1. *A* ramified Prym datum *is triple $\Xi = (\tilde{G}, \tilde{\theta}, \sigma)$, where $\tilde{G}$ is a finite group, $\tilde{\theta} : \Gamma_r \to \tilde{G}$ is an epimorphism and $\sigma \in Z(\tilde{G})$ is an element of order 2, that satisfies one of the following two conditions:*

(1) *there is one and only one index i such that $\sigma \in \langle \tilde{\theta}(\gamma_i) \rangle$ and $m_i = |\tilde{G}|/2$;*

(2) *there are exactly two indices i, j such that $\sigma \in \langle \tilde{\theta}(\gamma_i) \rangle$, $\sigma \in \langle \tilde{\theta}(\gamma_j) \rangle$ and $m_j = m_i = |\tilde{G}|$.*

($Z(\tilde{G})$ denotes the centre of $\tilde{G}$.)

We set $G := \tilde{G}/\langle \sigma \rangle$ and we denote by $\theta : \Gamma_r \to G$ the composition of $\tilde{\theta}$ with the projection $\tilde{G} \to G$. The ramified Prym datum gives rise to two submanifolds of Teichmüller spaces, namely $\mathsf{T}(G, \theta) \subset \mathsf{T}_g$ and $\mathsf{T}(\tilde{G}, \tilde{\theta}) \subset \mathsf{T}_{\tilde{g}}$. Both are isomorphic to $\mathsf{T}_{0,r}$. For any $t \in \mathsf{T}_{0,r}$ we have a diagram

$$\begin{tikzcd}
\tilde{C}_t \arrow[rr, "\pi_t"] \arrow[dr, "\psi"'] & & C_t = \tilde{C}_t/\langle \sigma \rangle . \arrow[dl] \\
& \mathbb{P}^1 &
\end{tikzcd}$$

Here $\tilde{C}_t \to \mathbb{P}^1$ is the $\tilde{G}$-covering corresponding to $t \in \mathsf{T}_{0,r}$ and to the datum $(\tilde{G}, \tilde{\theta})$, while $C_t \to \mathbb{P}^1$ is the G-covering corresponding to (G, θ). The quotient map $\pi_t : \tilde{C}_t \to \tilde{C}_t/\langle \sigma \rangle$ has exactly two ramification points. To check this let $\{t_1, \ldots, t_4\}$ be the critical values of ψ. If Ξ satisfies condition (1) in Definition 4.1, the two critical points of π_t belong to the fibre $\psi^{-1}(t_i)$ and thus $m_i = |\tilde{G}|/2$. If Ξ satisfies condition (2) one critical point of π_t is in $\psi^{-1}(t_i)$ and the other is in $\psi^{-1}(t_j)$ and thus $m_i = m_j = |\tilde{G}|$. Note that $\tilde{g} = 2g$.

Denote by η_t the element of $\operatorname{Pic}^0(C_t)$, corresponding to the covering π_t, so that $(\pi_t)_*(\mathcal{O}_{\tilde{C}_t}) = \mathcal{O}_{C_t} \oplus \eta_t^{-1}$. Let $B_t \in |\eta_t^2|$ be the branch divisor of π_t. Associating to $t \in \mathsf{T}_{0,r}$ the class of the triple (C_t, η_t, B_t) we get a map with discrete fibres $\mathsf{T}_{0,r} \longrightarrow \mathsf{R}_{g,[2]}$. Its image, denoted $\mathsf{R}_{[2]}(\Xi)$, is $(r-3)$-dimensional. The following diagram summarizes the construction.

$$\begin{tikzcd}
\mathsf{T}_{0,r} \arrow[r, "\cong"] \arrow[d, "\cong"'] & \mathsf{T}(\tilde{G}, \tilde{\theta}) \arrow[rrr] \arrow[dl, "\cong"] \arrow[drr] & & & \mathsf{M}_{\tilde{g}} \arrow[r, "\tilde{j}"] & \mathsf{A}_{\tilde{g}} \\
\mathsf{T}(G, \theta) \arrow[drr] & & & \mathsf{R}_{[2]}(\Xi) \arrow[ur] \arrow[r, "\mathscr{P}"] \arrow[dl] & \mathsf{A}_g & \\
& & \mathsf{M}_g \arrow[r, "j"] & \mathsf{A}_g & &
\end{tikzcd} \tag{4.1}$$

Given a ramified Prym datum $(\tilde{G},\tilde{\theta},\sigma)$ and a covering $\pi:\tilde{C}\longrightarrow C$ of the family, we have the eigenspace decomposition for σ just as in unramified case: $V:=H^0(\tilde{C},K_{\tilde{C}})=V_+\oplus V_-$. This time $V_+\cong H^0(C,K_C)$ and $V_-\cong H^0(C,K_C\otimes\eta)$ as G-modules. Similarly $W:=H^0(\tilde{C},2K_{\tilde{C}})=W_+\oplus W_-$, $W_+\cong H^0(2K_C\otimes\eta^2)=H^0(2K_C+B)$ and $W_-\cong H^0(C,2K_C\otimes\eta)$. The multiplication map $m:S^2V\longrightarrow W$ is the codifferential of the Torelli map $\tilde{j}:\mathsf{M}_{\tilde{g}}\to\mathsf{A}_{\tilde{g}}$ at $[\tilde{C}]\in\mathsf{M}_{\tilde{g}}$. It is $\tilde{G}$-equivariant. We have the following isomorphisms

$$(S^2V)^{\tilde{G}}=(S^2V_+)^G\oplus(S^2V_-)^G,$$
$$W^{\tilde{G}}=W_+^G.$$

Therefore m maps $(S^2V)^{\tilde{G}}$ to W_+^G. We are interested in the restriction of m to $(S^2V_-)^G$ that for simplicity we denote by the same symbol:

$$m:(S^2V_-)^G\longrightarrow W_+^G. \tag{4.2}$$

By the above discussion this is just the multiplication map

$$(S^2H^0(C,K_C\otimes\eta))^G\longrightarrow H^0(C,2K_C\otimes\eta^2)^G\cong H^0(2K_C)^G\cong H^0(2K_{\tilde{C}})^{\tilde{G}}.$$

Theorem 4.2. *Let $\Xi=(\tilde{G},\tilde{\theta},\sigma)$ be a ramified Prym datum. If for some $t\in\mathsf{T}_{0,r}$ the map m in (4.2) is an isomorphism, then the closure of $\mathscr{P}(\mathsf{R}_{[2]}\Xi)$ in A_g is a special subvariety.*

Proof. Over $\mathsf{T}_{0,r}$ we have the families $\tilde{C}_t$, C_t, η_t, B_t. The lattice $H_1(\tilde{C}_t,\mathbb{Z})$ the intersection form Q on $H_1(\tilde{C}_t,\mathbb{Z})$ and the the sublattice $\Lambda:=H_1(\tilde{C}_t,\mathbb{Z})_-$ are independent of t. Moreover $E:=(1/2)\cdot Q|_\Lambda$ is an integer-valued form on Λ. Let $\mathfrak{H}_g$ be the Siegel upper half-space parametrizing complex structures on $\Lambda\otimes\mathbb{R}=H_1(\tilde{C}_t,\mathbb{R})_-$ that are compatible with E. For any $t\in\mathsf{T}_{0,r}$ we have a decomposition $H^1(\tilde{C}_t,\mathbb{C})_-=V_{-,t}\oplus\overline{V_{-,t}}$. Dualizing we get a decomposition $H_1(\tilde{C}_t,\mathbb{C})_-=V^*_{-,t}\oplus\overline{V^*_{-,t}}$ that corresponds to a complex structure J_t on $H_1(\tilde{C}_t,\mathbb{R})_-$. J_t is compatible with E and therefore represents a point of $\mathfrak{H}_g$, that we denote by $f(t)$. We have thus defined a map $f:\mathsf{T}_{0,r}\to\mathfrak{H}_g$ that fits in following diagram:

$$\begin{array}{ccc} \mathsf{T}_{0,r} & \xrightarrow{\quad f\quad} & \mathfrak{H}_g \\ \downarrow & & \downarrow \\ \mathsf{R}_{[2]}(\Xi)\subset\mathsf{R}_{g,[2]} & \xrightarrow{\ \mathscr{P}\ } & \mathsf{A}_g. \end{array} \tag{4.3}$$

The diagram commutes since also in this case

$$P(\tilde{C}_t,C_t)=V^*_{-,t}/\Lambda,$$

(see e.g. [1, p. 295ff] or [5, p. 374ff]). Since $\tilde{G}$ preserves Q, G preserves E, so G maps into $\mathrm{Sp}(\Lambda,E)$. Denote by G' the image of G in $\mathrm{Sp}(\Lambda,E)$. The complex structure J_t is G-invariant, i.e. $f(t)=J_t\in\mathfrak{H}_g^{G'}$. Hence by Theorem 2.3 $P(\tilde{C}_t,C_t)$ lies in the PEL special subvariety $\mathsf{Z}(D_{G'})$. Therefore $\mathscr{P}(\mathsf{R}(\Xi))\subset\mathsf{Z}(D_{G'})$. Since $f(\mathsf{T}_{0,r})\subset\mathfrak{H}_g^{G'}$ we can consider f as a map $f:\mathsf{T}_{0,r}\to\mathfrak{H}_g^{G'}$. Recall that

$$\Omega^1_{f(t)}\mathfrak{H}_g^{G'}\cong(S^2H^0(C_t,K_{C_t}\otimes\eta_t))^G=S^2V_{-,t},$$
$$\Omega^1_t\mathsf{T}_{0,r}=\Omega^1_{[C_t]}\mathsf{T}(G,\theta)=H^0(C_t,2K_{C_t})^G\cong H^0(C_t,2K_{C_t}\otimes\eta_t^2)^G=W^G_{t,+}.$$

The codifferential is simply the multiplication map

$$m=(df_t)^*:(S^2V_{-,t})^G\longrightarrow W^G_{t,+}$$

(see [41] Prop. 3.1, or [31]). By assumption there is some $t\in\mathsf{T}_{0,r}$ such that the map m is an isomorphism at t. This implies first of all that $\dim(S^2V_{-,t})^G=\dim W^G_{t,+}=r-3$. Moreover f is an immersion at t, hence its image has dimension $r-3$. As the vertical arrows in (4.3) are discrete maps, both $\mathscr{P}(\mathsf{R}(\Xi))$ and $\mathsf{Z}(D_{G'})$ have dimension $r-3$. Since $\mathsf{R}(\Xi)\subset\mathsf{Z}(D_{G'})$ and $\mathsf{Z}(D_{G'})$ is irreducible we conclude that $\overline{\mathsf{R}(\Xi)}=\mathsf{Z}(D_{G'})$ as desired. $\square$

The special subvarieties in A_g constructed using Theorem 4.2 are generically contained in the ramified Prym locus.

We wish to use Theorem 4.2 to construct special curves. So we set $r = 4$. Just as in the unramified case we can then split the hypothesis of the Theorem in two conditions:

$$\text{(A)} \qquad \dim(S^2V_-)^{\tilde{G}} = 1.$$

$$\text{(B)} \qquad m : (S^2V_-)^{\tilde{G}} \longrightarrow W_+^{G} \text{ is not identically } 0.$$

Again once (A) is true, a sufficient condition ensuring (B) is the following

$$\text{(B1)} \qquad (S^2V_-)^{\tilde{G}} \text{ is generated by a decomposable tensor.}$$

5. Examples in the Prym locus

In this section we discuss several examples of Shimura curves in the Prym locus obtained using theorem 3.2 and the scripts. Although we do not study in detail all the examples gotten in this way (which are listed in Tables 1 and 2) we give several informations for various of them. In particular for each example we recall the genera of $\tilde{C}$ and C, the group $\tilde{G}$ with a presentation and the monodromy, i.e. the epimorphism $\tilde{\theta}$. With these data it is possible to compute everything of the family, at least in principle, and such presentation for all the examples of Tables 1 and 2 be found in the lists on-line (see Appendix).

Before describing the examples, let us recall the description of two Shimura families of Jacobians constructed in [37], namely family (3) and (4) in Table 1 in [37]. These two families will show up frequently in the following discussions.

As observed in [22] (see Table 1 and Table 2 in [22]), these two families have extra automorphisms: the group D_6 for (3) and D_4 for (4), in fact (3)=(30) and (4)=(29) in the enumeration of [22]. For every non-central element a of order 2 in D_6 and for any curve $\tilde{C}_t$ in (3), the quotient $\tilde{C}_t/\langle a\rangle$ is an elliptic curve E_t. One easily shows that $J(\tilde{C}_t)$ is isogenous to $E_t \times E_t$.

The same happens for (4) taking E_t to be the quotient by a non-central element of order 2 in D_4. Therefore these two families of Jacobians are both isogenous to the product of the same elliptic curve $E \times E$ which moves.

We also notice that family (3) has the equation: $\tilde{C}_t : v^2 = (u^3 + 1)(u^3 + t)$, while family (4) has the equation: $\tilde{C}_t : v^2 = u(u^2 - 1)(u^2 - t)$.

We notice that many of the examples give rise to decomposable Pryms. It would be interesting to study this aspect more in detail. For related questions in the case of Jacobians see e.g. [44].

Remark 5.1. *Notice that if one of the families of Pryms we constructed satisfying* (A) *and* (B) *is a family of Jacobians, it must satisfy condition* $(*)$ *of Theorem 3.9 in* [22]*. Hence if the dimension of the Pryms is* ≤ 9*, they yield a Shimura curve that must appear in Table 2 of* [22].

Lemma 5.2. *Let* $(\tilde{G}, \tilde{\theta})$ *be a datum. Assume that for any* $t \in \mathsf{T}_{0,r}$ *there is a* $\tilde{G}$*-invariant rational Hodge substructure* $W_t \subset H^1(P(\tilde{C}_t, C_t), \mathbb{C})$*. If* $(S^2 W_t^{1,0})^{\tilde{G}} = \{0\}$*, then the abelian variety corresponding up to isogeny to* W_t *does not depend on* t*.*

Proof. It is enough to observe that the period matrix of the abelian variety corresponding up to isogeny to W_t lies in $\mathfrak{H}_k^{\tilde{G}}$, where $k = \dim W_t^{1,0}$, and that $\mathfrak{H}_k^{\tilde{G}}$ is a point by the assumption. □

There are only 28 abelian examples, all in A_k with $k \leq 10$. Recall that if the group is abelian both conditions (A) and (B1) are satisfied. Theorem 3.2 tells us that these families of Pryms yield special subvarieties of A_k. We give here a descriptions of the 7 examples with $k \geq 6$, for which the closure of the Prym locus is not all A_k.

5.1. **The unramified abelian examples in A_6 and in A_7.** Note that for $k = 6, 7$, in the abelian examples we always have $\tilde{G} = \mathbb{Z}/2 \times \mathbb{Z}/n$ and for these examples we give explicit equations describing $\tilde{C}_t$ and C_t as n-coverings of $\mathbb{P}^1$, via the quotient by $\mathbb{Z}/n$.

In the following ζ_n denotes a primitive n-th root of unity.

We denote by ρ_n^i the character of $\langle g \rangle = \mathbb{Z}/n$ mapping g to ζ_n^i, while $W_{\zeta_n^i}$ denotes the irreducible representation of $\langle g \rangle$ corresponding to this character, i.e. mapping g to ζ_n^i. Since $\langle g \rangle \hookrightarrow \tilde{G} \to G = \tilde{G}/\langle \sigma \rangle$ is an isomorphism, we consider V_- as a representation of $\langle g \rangle$.

Example 30.
$\tilde{g} = 13$, $g = 7$,
$\tilde{G} = G(16, 5) = \mathbb{Z}/2 \times \mathbb{Z}/8 = \langle g_2, g_1 \mid g_2^2 = 1,\ g_1^8 = 1,\ g_1 g_2 = g_2 g_1 \rangle$, $\sigma = g_1^4 g_2$.
$\tilde{\theta}(\gamma_1) = g_1, \quad \tilde{\theta}(\gamma_2) = g_1^3, \quad \tilde{\theta}(\gamma_3) = g_1 g_2, \quad \tilde{\theta}(\gamma_4) = g_1^3 g_2$.
$\tilde{C}_t: \quad y^8 = u^2(u^2-1)^7(u^2-t)^5 \quad \pi : \tilde{C} \to \mathbb{P}^1, \quad \pi(u, y) = u.$
$g_2 : (u, y) = (-u, -y), \quad g_1(u, y) = (u, -\zeta_8 y) = (u, \zeta_8^5 y) \quad \sigma(u, y) = (-u, y).$
$C_t: \quad y^8 = x(x-1)^7(x-t^2)^5 \quad (x, y) = (u^2, y).$
$V_- = W_{\zeta_8^2} \oplus 2W_{\zeta_8^5} \oplus W_{\zeta_8^6} \oplus 2W_{\zeta_8^7} \qquad (S^2 V_-)^{\tilde{G}} \cong W_{\zeta_8^2} \otimes W_{\zeta_8^6}.$

Here $P(\tilde{C}_t, C_t)$ is not isogenous to a Jacobian, since Table 2 of [22] does not contain families of genus 6 curves with an action of $\mathbb{Z}/8$. $P(\tilde{C}_t, C_t)$ is isogenous to the product of a fixed CM abelian 4-fold T' with a (Shimura) family of abelian surfaces with an action of $\mathbb{Z}/4$. Geometrically set $D_1 := \tilde{C}/\langle g_1^4 \rangle$, $D_2 := \tilde{C}/\langle g_2 \rangle$, $B := \tilde{C}/\langle g_2, g_1^4 \rangle$. Then $g(D_2) = 7, g(D_1) = 5, g(B) = 3$, $P(\tilde{C}, C) \sim P(D_2, B) \times P(D_1, B)$, where $T' = P(D_2, B)$, while $P(D_1, B)$ is a Shimura family of abelian surfaces with an action of $\mathbb{Z}/4$.

Example 31.
$\tilde{g} = 13$, $g = 7$,
$\tilde{G} = G(20, 5) = \mathbb{Z}/2 \times \mathbb{Z}/10 = \langle g_1, g_2, g_3 \mid g_1^2 = 1, g_2^2 = 1, g_3^5 = 1 \rangle$, $\sigma = g_1 g_2$,
$\tilde{\theta}(\gamma_1) = g_2, \quad \tilde{\theta}(\gamma_2) = g_2 g_3, \quad \tilde{\theta}(\gamma_3) = g_1 g_3^2, \quad \tilde{\theta}(\gamma_4) = g_1 g_3^2$.
$\tilde{C}_t: \quad z^{10} = (u^2-1)(u^2-t), \quad \pi : \tilde{C} \to \mathbb{P}^1, \quad \pi(u, y) = u$
$g_2(u, z) = (-u, z), \quad g_1(u, z) = (u, \zeta_{10}^2 z), \quad g_3(u, z) = (u, \zeta_{10}^5 z) \quad \sigma(u, z) = (-u, -z).$
$C_t: \quad y^{10} = x^5(x-1)(x-t), \quad (x, y) := (u^2, u^{-1}z).$
$V_- = W_{\zeta_{10}} \oplus W_{\zeta_{10}^2} \oplus 2W_{\zeta_{10}^4} \oplus W_{\zeta_{10}^7} \oplus W_{\zeta_{10}^8} \qquad (S^2 V_-)^{\tilde{G}} \cong W_{\zeta_{10}^2} \otimes W_{\zeta_{10}^8}.$

$P(\tilde{C}, C)$ is isogenous to $T \times A''$, where T is a fixed CM abelian surface and A'' is a moving abelian 4-fold. Geometrically, set $D_1 := \tilde{C}/\langle g_1 \rangle$, $D_2 := \tilde{C}/\langle g_2 \rangle$, $F := \tilde{C}/\langle g_1, g_2 \rangle$. Then $g(D_1) = 6$, $g(D_2) = 4$, $g(F) = 2$ and $T = P(D_2, F)$, $A'' = P(D_1, F)$. Notice that A'' is not isogenous to any Shimura family of Jacobians, since Table 2 of [22] does not contain any family of Jacobians of genus 4 curves admitting an action of $\mathbb{Z}/10$.

Example 32.
$\tilde{g} = 13$, $g = 7$,
$\tilde{G} = G(24, 9) = \mathbb{Z}/2 \times \mathbb{Z}/12 = \langle g_1, g_2, g_3 \mid g_1^4 = 1, g_2^2 = 1, g_3^3 = 1 \rangle$, $\sigma = g_1^2 g_2$.
$\tilde{\theta}(\gamma_1) = g_2, \quad \tilde{\theta}(\gamma_2) = g_1, \quad \tilde{\theta}(\gamma_3) = g_3 g_1^2, \quad \tilde{\theta}(\gamma_4) = g_1 g_2 g_3^2$.
$\tilde{C}_t: \quad z^{12} = (u^2-1)^3(u^2-t)^2, \quad \pi : \tilde{C} \to \mathbb{P}^1, \quad \pi(u, y) = u.$
$g_2(u, z) = (-u, z), \quad g_1(u, z) = (u, \zeta_{12}^3 z), \quad g_3(u, z) = (u, \zeta_{12}^4 z), \ \sigma(u, z) = (-u, -z).$
$C_t: \quad y^{12} = x^6(x-1)^3(x-t)^2, \quad (x, y) := (u^2, u^{-1}z).$
$V_- = W_{\zeta_{12}^2} \oplus W_{\zeta_{12}^3} \oplus W_{\zeta_{12}^4} \oplus W_{\zeta_{12}^5} \oplus W_{\zeta_{12}^{10}} \oplus W_{\zeta_{12}^{11}} \qquad (S^2 V_-)^{\tilde{G}} \cong \zeta_{12}^2 \otimes \zeta_{12}^{10}.$

Here $P(\tilde{C}, C)$ is isogenous to the product a fixed CM abelian 4-fold T'' with the Shimura family (3) of [37]. Set $D_1 := \tilde{C}/\langle g_1^2 \rangle$, $D_2 := \tilde{C}/\langle g_2 \rangle$, $F_1 := \tilde{C}/\langle g_1 g_2 \rangle$, $E_\rho = \tilde{C}/\langle g_1 \rangle$, $E_i = \tilde{C}/\langle g_2, g_3 \rangle$ (these are the two CM elliptic curves), $F := \tilde{C}/\langle g_1^2, g_2 \rangle$. Then $g(D_1) = g(D_2) = 4$, $g(F) = 1$, $g(F_1) = 2$, $P(\tilde{C}, C) \sim P(D_1, F) \times P(D_2, F)$ and $P(D_1, F) \sim J(F_1) \times E_\rho$ and $J(F_1)$ is the family (3) of [37]. Moreover, $P(D_2, F) \sim Y \times E_i$, where Y is a CM abelian surface, so $T'' = Y \times E_\rho \times E_i$.

Example 35.
$\tilde{g}=15$, $g=8$,
$\tilde{G}=G(24,9)=\mathbb{Z}/2\times\mathbb{Z}/12=\langle g_1,g_2,g_3 \mid g_1^4=1, g_2^2=1, g_3^3=1\rangle$, $\sigma=g_1^2g_2$.
$\tilde{\theta}(\gamma_1)=g_1^2,\quad \tilde{\theta}(\gamma_2)=g_2g_3,\quad \tilde{\theta}(\gamma_3)=g_1g_3,\quad \tilde{\theta}(\gamma_4)=g_1g_2g_3.$
$\tilde{C}_t:\quad z^{12}=u^8(u^2-1)^6(u^2-t)^7,\quad \pi:\tilde{C}\to\mathbb{P}^1,\quad \pi(u,y)=u.$
$g_2(u,z)=(-u,z),\quad g_1(u,z)=(u,\zeta_{12}^3z),\quad g_3(u,z)=(u,\zeta_{12}^4z)\quad \sigma(u,z)=(-u,-z).$
$C_t:\quad y^{12}=x^{10}(x-1)^6(x-t)^7,\quad (x,y):=(u^2,u^{-1}z).$
$V_-=2W_{\zeta_{12}}\oplus W_{\zeta_{12}^2}\oplus W_{\zeta_{12}^3}\oplus W_{\zeta_{12}^5}\oplus W_{\zeta_{12}^7}\oplus W_{\zeta_{12}^8}\qquad (S^2V_-)^{\tilde{G}}\cong W_{\zeta_{12}^5}\otimes W_{\zeta_{12}^7}.$

Here $P(\tilde{C},C)\sim T'''\times E_\rho\times E_i\times E_\rho$, where T''' is a moving abelian fourfold not isogenous to a Jacobian, since it carries an action of $\mathbb{Z}/2\times\mathbb{Z}/12$ and in Table 2 of [22] there does not exist any family of Jacobians of genus 4 curves with an action of $\mathbb{Z}/12$. More geometrically, set $E:=\tilde{C}/\langle g_1,g_2\rangle$, $D_2:=\tilde{C}/\langle g_2\rangle$, $F:=\tilde{C}/\langle g_1^2,g_2\rangle$, $F_1:=\tilde{C}/\langle g_1g_2\rangle$, $F_2:=\tilde{C}/\langle g_1\rangle$, $E_i\cong\tilde{C}/\langle g_2,g_3\rangle$ (in fact it carries the action of $\mathbb{Z}/4\cong\langle g_1\rangle$). Then $g(D_1)=4$, $g(D_2)=7$, $g(F)=g(F_1)=g(F_2)=2$, $P(\tilde{C},C)\sim P(F_1,E)\times P(F_2,E)\times P(D_2,F)$ and $P(F_1,E)\sim P(F_2,E)\sim E_\rho$, $P(D_2,F)\sim E_i\times T'''$.

5.2. **The unramified abelian examples in A_8.** We describe now the two only examples with $\tilde{G}$ abelian, yielding a Shimura curve generically contained in the Prym locus in A_8. We notice that up to now there are no known examples of Shimura varieties generically contained in the Torelli locus for $g\geq 8$. On the other hand, by Remark 5.1 these families are not families of Jacobians since Table 2 in [22] contains no example at all in genus 8.

Example 36.
$\tilde{g}=17$, $g=9$.
$\tilde{G}=G(24,9)=\mathbb{Z}/2\times\mathbb{Z}/12=\langle g_1,g_2,g_3 \mid g_1^4=1, g_2^2=1, g_3^3=1\rangle$, $\sigma=g_1^2g_2$.
$\tilde{\theta}(\gamma_1)=g_1,\quad \tilde{\theta}(\gamma_2)=g_1g_2,\quad \tilde{\theta}(\gamma_3)=g_1^3g_3,\quad \tilde{\theta}(\gamma_4)=g_1g_2g_3^2.$
$V_-=W_{\zeta_{12}^3}\oplus W_{\zeta_{12}^9}\oplus W_{\zeta_{12}^4}\oplus 2W_{\zeta_{12}^7}\oplus 2W_{\zeta_{12}^{11}}\oplus W_{\zeta_{12}^2}\qquad (S^2V_-)^{\tilde{G}}\cong W_{\zeta_{12}^3}\otimes W_{\zeta_{12}^9}.$

We have $P(\tilde{C},C)\sim P(D,E)\times A$, where A is a fixed CM abelian 5-fold and $D=\tilde{C}/\langle g_2,g_3\rangle$, $E=\tilde{C}/\langle g_2,g_3,\sigma\rangle$, $g(D)=3$, $g(E)=1$ and $H^{1,0}(P(D,E))\cong W_{\zeta_{12}^3}\oplus W_{\zeta_{12}^9}$.

Example 37.
$\tilde{g}=17$, $g=9$.
$\tilde{G}=G(32,21)=\mathbb{Z}/4\times\mathbb{Z}/4\times\mathbb{Z}/2\cong\langle g_1\rangle\times\langle g_2\rangle\times\langle g_3\rangle$,
where $o(g_1)=o(g_2)=4$, $o(g_3)=2$, $\sigma=g_2^2g_3$.
$\tilde{\theta}(\gamma_1)=g_2$, $\tilde{\theta}(\gamma_2)=g_2g_3$, $\tilde{\theta}(\gamma_3)=g_1$, $\tilde{\theta}(\gamma_4)=g_1^3g_2^2g_3$.
$V_-=W_{0,1,0}\oplus W_{1,0,1}\oplus 2W_{1,1,0}\oplus W_{1,2,1}\oplus W_{1,3,0}\oplus W_{2,1,0}\oplus W_{3,1,0}$,
$(S^2V_-)^{\tilde{G}}\cong W_{1,3,0}\otimes W_{3,1,0}$,
where W_{a_1,a_2,a_3} is the irreducible representation of the group $\tilde{G}$ corresponding to the character ρ_{a_1,a_2,a_3} mapping g_i to $\zeta_{k_i}^{a_i}$, for i from 1 to 3 ($k_1=k_2=4$, $k_3=2$).

Since $\tilde{G}$ is abelian both conditions (A) and (B1) are satisfied. Theorem 3.2 tells us that this family of Pryms yields a special subvariety of A_8. Set $E_1:=\tilde{C}/\langle g_1\rangle$, $E_2:=\tilde{C}/\langle g_2\rangle$, $E_3:=\tilde{C}/\langle g_2g_3\rangle$, $E_4:=\tilde{C}/\langle g_1g_2^2g_3\rangle$. These are all elliptic curves with a $\mathbb{Z}/4$-action, hence isomorphic to E_i. We have $H^0(E_1,K_{E_1})\cong W_{0,1,0}$, $H^0(E_2,K_{E_2})\cong W_{1,0,1}$, $H^0(E_3,K_{E_3})\cong W_{1,2,1}$, $H^0(E_4,K_{E_4})\cong W_{2,1,0}$. There is a diagram of coverings

$$
\begin{tikzcd}
 & \tilde{C} \arrow[dl] \arrow[d] \arrow[dr] & \\
C_1=\tilde{C}/M \arrow[dr, "\pi_1"'] & & C_2=\tilde{C}/N \arrow[dl, "\pi_2"] \\
 & F=\tilde{C}/H &
\end{tikzcd}
\tag{5.1}
$$

where $M = \langle g_3, g_1g_2\rangle$, $N = \langle g_3, g_1g_2^3\rangle$ and $H = \langle g_3, g_1g_2, g_1g_2^3\rangle$. We have $g(C_1) = g(C_2) = 3$, $g(F) = 1$, and $H^{1,0}(P(C_1,F)) \cong W_{3,1,0} \oplus W_{1,3,0}$, $H^{1,0}(P(C_2,F)) \cong 2W_{1,1,0}$. Hence

$$P(\tilde{C},C) \sim E_1 \times E_2 \times E_3 \times E_4 \times P(C_1,F) \times P(C_2,F) = 4E_i \times P(C_1,F) \times P(C_2,F)$$

Since $(S^2(V_-))^{\tilde{G}} \cong S^2H^{1,0}(P(C_1,F))$, by Lemma 5.2, $P(\tilde{C},C)$ is isogenous to the product of a fixed CM abelian variety $A = 4E_i \times P(C_2,F)$ admitting an action of $\mathbb{Z}/4$, with the Shimura family of abelian surfaces $P(C_1,F)$ having an action of $\tilde{G}/M \cong \mathbb{Z}/4$ and moving in $\mathsf{A}_2(\Theta)$, where $\mathsf{A}_2(\Theta)$ is the moduli space of abelian surfaces with a given type of polarisation Θ.

5.3. **The unramified abelian example in A_{10}.** We now describe the only abelian unramified example in A_{10}.

Example 41.
$\tilde{g} = 21$, $g = 11$.
$\tilde{G} = G(32,3) = \mathbb{Z}/4 \times \mathbb{Z}/8 \cong \langle g_2\rangle \times \langle g_1\rangle$, where $o(g_1) = 8$, $o(g_2) = 4$, $\sigma = g_2^2g_1^4$.
$\tilde{\theta}(\gamma_1) = g_2$, $\tilde{\theta}(\gamma_2) = g_2g_1^4$, $\tilde{\theta}(\gamma_3) = g_1$, $\tilde{\theta}(\gamma_4) = g_1^3g_2^2$.
$V_- = W_{0,1} \oplus 2W_{2,1} \oplus W_{2,3} \oplus W_{4,1} \oplus W_{5,0} \oplus W_{5,2} \oplus W_{6,1} \oplus W_{7,0} \oplus W_{7,2}$,
$(S^2V_-)^{\tilde{G}} \cong W_{2,3} \otimes W_{6,1}$,
where W_{a_1,a_2} is the irreducible representation of the group $\tilde{G}$ corresponding to the character ρ_{a_1,a_2} mapping g_1 to $\zeta_8^{a_1}$, and g_2 to $\zeta_4^{a_2}$.

Since $\tilde{G}$ is abelian both conditions (A) and (B1) are satisfied. Theorem 3.2 tells us that this family of Pryms yields a special subvariety of A_{10}. Set $F = \tilde{C}/\langle g_1\rangle$, $D = \tilde{C}/\langle g_2\rangle$, $Z = \tilde{C}/\langle g_2g_1^4\rangle$, $X = \tilde{C}/\langle g_1^7g_2\rangle$, $E = \tilde{C}/\langle g_1g_2,\sigma\rangle$, $L = \tilde{C}/\langle g_1g_2^2\rangle$. We have $g(F) = g(E) = g(L) = 1$, $g(D) = g(Z) = 2$, $g(X) = 3$,

$$P(\tilde{C},C) \sim F \times L \times J(D) \times J(Z) \times P(X,E) \times P(Y,E),$$

where $H^0(F,K_F) = W_{0,1}$, $H^0(L,K_L) = W_{4,1}$, $H^0(D,K_D) = W_{7,0} \oplus W_{5,0}$, $H^0(Z,K_Z) = W_{5,2} \oplus W_{7,2}$, $H^{1,0}(P(X,E)) = 2W_{2,1}$, $H^{1,0}(P(Y,E)) = W_{2,3} \oplus W_{6,1}$. Since $(S^2(V_-))^{\tilde{G}} \cong S^2H^{1,0}(P(Y,E))$, by Lemma 5.2, $P(\tilde{C},C)$ is isogenous to the product of a fixed CM abelian variety $F \times L \times J(D) \times J(Z) \times P(X,E)$ with the Shimura family of abelian surfaces $P(Y,E)$.

5.4. **Non abelian examples.** In this section we describe three non-abelian examples satisfying condition (A), but not (B1). We prove by ad hoc arguments that condition (B) holds. Notice that these three examples are examples of Shimura curves generically contained in the Prym locus in A_g, with $g = 9$ or $g = 12$. Moreover by Remark 5.1, Example 38 is not a family of Jacobians.

Example 40.
$\tilde{g} = 19$, $g = 10$
$\tilde{G} = G(108,28) = ((\mathbb{Z}/3 \times \mathbb{Z}/3) \rtimes \mathbb{Z}/3) \rtimes (\mathbb{Z}/2 \times \mathbb{Z}/2) \cong$
$\cong (((\langle g_4\rangle \times \langle g_5\rangle) \rtimes \langle g_3\rangle) \rtimes (\langle g_1\rangle \times \langle g_2\rangle)$,
where $o(g_4) = o(g_5) = o(g_3) = 3$, $o(g_1) = o(g_2) = 2$,
$Z(\tilde{G}) = \langle g_5, g_2\rangle \cong \mathbb{Z}/3 \times \mathbb{Z}/2$,
$g_3^{-1}g_4g_3 = g_4g_5$, $g_1^{-1}g_3g_1 = g_3^{-1}$, $g_1^{-1}g_4g_1 = g_4^{-1}$, $\sigma = g_2$.
$\tilde{\theta}(\gamma_1) = g_1$, $\tilde{\theta}(\gamma_2) = g_1g_4$, $\tilde{\theta}(\gamma_3) = g_1g_2g_3$, $\tilde{\theta}(\gamma_4) = g_1g_2g_3g_4^2$.
$V_- = V_{15} \oplus V_{16} \oplus V_{20}$ (the notation is the one used by `MAGMA`).
$\dim(V_{15}) = \dim(V_{16}) = \dim(V_{20}) = 3$.
$\dim(S^2(V_-))^{\tilde{G}} = \dim(V_{15} \otimes V_{20})^{\tilde{G}} = 1$, hence condition (A) is satisfied.

We have to verify that also condition (B) is satisfied. Let $H := \langle g_1, g_3\rangle \cong S_3$, $K := \langle g_1g_2, g_3\rangle \cong S_3$. By Riemann-Hurwitz $\tilde{C}/H =: E$ has genus 1 and $\tilde{C}/K =: D$ has genus 2. The trace of g_1 on V_{15} is -1. Since g_1 has order 2, we have a decomposition $V_{15} = X_{15} \oplus W_{15}$, where $\dim(X_{15}) = 1$, $\dim(W_{15}) = 2$ and $g_{1|X_{15}} = Id_{X_{15}}$, $g_{1|W_{15}} = -Id_{W_{15}}$. The same happens for $V_{20} = X_{20} \oplus W_{20}$, where $\dim(X_{20}) = 1$, $\dim(W_{20}) = 2$ and $g_{1|X_{20}} = Id_{X_{20}}$, $g_{1|W_{20}} = -Id_{W_{20}}$.

The trace of g_1 on V_{16} is 1, so $V_{16} = X_{16} \oplus W_{16}$, where $\dim(X_{16}) = 2$, $\dim(W_{16}) = 1$ and $g_{1|X_{16}} = Id_{X_{16}}$, $g_{1|W_{16}} = -Id_{W_{16}}$.

Since g_2 acts as $-Id$ on $V_- = V_{15} \oplus V_{16} \oplus V_{20}$ we have $g_1 g_{2|X_j} = -Id_{X_j}$, $g_1 g_{2|W_j} = Id_{W_j}$, for $j = 15, 16, 20$.

The group S_3 has three irreducible representations, Y_1, Y_2, Y_3, where $\dim(Y_i) = 1$, $i = 1, 2$, $\dim(Y_3) = 2$, Y_1 is the trivial one, Y_2 is the one given by the sign. Looking at the action of the subgroups $H \cong S_3$ on V_j, $j = 15, 16, 20$, one sees that $V_{15} \cong Y_2 \oplus Y_3$ and the same happens for V_{20}, since $g_{1|Y_2} = -Id_{Y_2}$, $g_{3|Y_2} = Id_{Y_2}$, $g_{1|Y_3}$ has eigenvalues $1, -1$, $g_{3|Y_3}$ has eigenvalues ζ_3, ζ_3^2. Similarly one sees that $V_{16} \cong Y_1 \oplus Y_3$, hence the fixed point locus of the action of H on V (which must be isomorphic to $H^0(E, K_E)$, hence one dimensional) is contained in $V_{16} \subset V_-$. Therefore $H^0(E, K_E) = Y_1 \subset V_{16} \subset V_-$.

On the other hand, if we look at the action of the subgroup $K \cong S_3$ on V_j, $j = 15, 16, 20$, we clearly have $V_{15} \cong Y_1 \oplus Y_3$ and the same for V_{20}, while $V_{16} \cong Y_2 \oplus Y_3$, hence the fixed point locus of the action of K on V, which we know to be two dimensional, is given by two copies of Y_1, one contained in V_{15} and the other contained in V_{20}. Therefore $H^0(D, K_D) \subset V_{15} \oplus V_{20} \subset V_-$. Hence $P(\tilde{C}, C) \sim J(D) \times E \times T$ for some 6-dimensional abelian variety T. Since $H^0(E, K_E) \subset V_{16}$ and $(S^2(V_-))^{\tilde{G}} = (V_{15} \otimes V_{20})^{\tilde{G}}$, the elliptic curve E does not move by Lemma 5.2. To prove condition (B) we will show that $J(D)$ moves.

Consider the action on $\tilde{C}$ of the subgroup $L := \langle g_1, g_2, g_3 \rangle \cong H \times \mathbb{Z}/2$. By Riemann-Hurwitz $\tilde{C}/L \cong \mathbb{P}^1$ and we have a factorisation

$$\begin{tikzcd} \tilde{C} \arrow[r, "\varphi"] \arrow[dr] & D = \tilde{C}/H \arrow[d, "2:1"', "p_D"] \\ & \mathbb{P}^1 = \tilde{C}/L = D/\langle g_2 \rangle. \end{tikzcd} \tag{5.2}$$

If we prove that the 6 critical values of the hyperelliptic covering p_D move, we are done. Denote by $\psi : \tilde{C} \to \mathbb{P}^1 = \tilde{C}/\tilde{G}$ the original covering and consider the factorisation

$$\begin{tikzcd} \tilde{C} \arrow[r, "\varphi"] \arrow[d, "\psi"] & D = \tilde{C}/H \arrow[dl, "\pi"] \\ \mathbb{P}^1 = \tilde{C}/\tilde{G}. & \end{tikzcd} \tag{5.3}$$

The $18:1$ covering π factors as follows

$$\begin{tikzcd} D \arrow[dr, "p_D"] \arrow[dd, "\pi"] & \\ & D/\langle g_2 \rangle \cong \mathbb{P}^1 \arrow[dl, "\pi'"] \\ \mathbb{P}^1. & \end{tikzcd} \tag{5.4}$$

Denote by $\{P_1, P_2, P_3, P_4\}$ the critical values of ψ and by $\{y_1, y_2, y_3, z_1, z_2, z_3\}$ the critical values of p_D. Looking at the above diagrams, one easily checks that the critical values of p_D all lie in $\pi'^{-1}(P_1) \cup \pi'^{-1}(P_2)$. More precisely:

$\pi'^{-1}(P_1)$ consists of 3 critical values $\{y_1, y_2, y_3\}$ of p_D which are regular for π' and of three critical points of order 2 for π' which are regular values for p_D.

$\pi'^{-1}(P_2)$ consists of 3 critical values $\{z_1, z_2, z_3\}$ of p_D which are regular for π' and of three critical points of order 2 for π' which are regular values for p_D.

$\pi'^{-1}(P_3)$ consists of three regular points and three critical points of order 2 of π' (all regular values for p_D).

$\pi'^{-1}(P_4)$ consists of two critical points of π', one of order 3 and one of order 6 (both regular values for p_D).

To understand better the $9:1$ map π' let us consider this last factorisation

$$\begin{tikzcd} \mathbb{P}^1 = D/\langle g_2\rangle \arrow[d, "\pi'"] \arrow[r, "p_5"] & \mathbb{P}^1 = D/\langle g_2, g_5\rangle \arrow[dl, "\bar\pi"] \\ \mathbb{P}^1 = D/\langle g_2, g_4, g_5\rangle. & \end{tikzcd} \tag{5.5}$$

We have the following: $\bar\pi^*(P_i) = w_i + 2q_i$, for all $i = 1, 2, 3, 4$, and $p_5^{-1}(w_1) = \{y_1, y_2, y_3\}$, $p_5^{-1}(w_2) = \{z_1, z_2, z_3\}$. The critical values of the Galois $3:1$ covering p_5 are w_4 and q_4.

Consider the $3:1$ covering $\bar\pi : \mathbb{P}^1 \to \mathbb{P}^1$. Composing with automorphisms of $\mathbb{P}^1$ in the source and in the target, we can assume that $P_4 = \infty$, $P_3 = 0$, $P_2 = 1$. We denote P_1 by the parameter λ, $w_4 = 0$, $q_4 = \infty$, $w_3 = 1$, and set $q_3 = a$ for simplicity. Hence $\bar\pi(z) = b\frac{(z-1)(z-a)^2}{z}$, where b is nonzero.

Computing the derivative of $\bar\pi$ we see that the other two critical points q_1, q_2 are $\frac{1\pm\sqrt{1+8a}}{4}$. Imposing that 1 and λ are the corresponding critical values, we see that a, w_1, w_2 are all non constant functions on λ. We can assume that $p_5(z) = z^3$, hence $\{y_1, y_2, y_3\} = p_5^{-1}(w_1) = \{z \in \mathbb{P}^1 \mid z^3 = w_1\}$ and $\{z_1, z_2, z_3\} = p_5^{-1}(w_2) = \{z \in \mathbb{P}^1 \mid z^3 = w_2\}$, and since w_1 and w_2 are non-constant functions of λ, the same holds for y_i, z_i, $i = 1, 2, 3$.

This proves that as λ varies, the hyperelliptic covering $p_D : D \to \mathbb{P}^1$ varies, and hence the genus 2 curve D varies, so $J(D)$ varies and hence $P(\tilde C, C) \sim 3E \times 3J(D)$ varies. Therefore condition (B) is satisfied.

Example 43.
$\tilde g = 25$, $g = 13$.
$\tilde G = G(48, 32) = \mathbb{Z}/2 \times SL(2, \mathbb{F}_3) \cong \langle g_1\rangle \times SL(2, \mathbb{F}_3)$, where

$$SL(2,\mathbb{F}_3) = \Big\langle g_2 = \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix} \ g_3 = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \ g_4 = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \ g_5 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \mid g_3^2 = g_4^2 = g_5, \\ g_5^2 = 1, \ g_2^3 = 1, \ g_2^{-1} g_3 g_2 = g_4, \ g_2^{-1} g_4 g_2 = g_3 g_4, g_3^{-1} g_4 g_3 = g_4 g_5 \Big\rangle.$$

$\sigma = g_5$, $Z(\tilde G) = \langle g_1, g_5\rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2$.
$\tilde\theta(\gamma_1) = g_1$, $\tilde\theta(\gamma_2) = g_1 g_2 g_5$, $\tilde\theta(\gamma_3) = g_1 g_2 g_4 g_5$, $\tilde\theta(\gamma_4) = g_1 g_2 g_3 g_4 g_5$.
$V_- = V_7 \oplus 2V_8 \oplus 2V_{11} \oplus V_{12}$.
Here V_i are the irreducible representations as enumerated by `MAGMA`. Note that $\dim(V_7) = \dim(V_8) = \dim(V_{11}) = \dim(V_{12}) = 2$.
$(S^2(V_-))^{\tilde G} = (V_8 \otimes V_8)^{\tilde G} = (\Lambda^2 V_8)^{\tilde G}$ is one dimensional, hence condition (A) is satisfied.
We have to check condition (B).

Consider the commutative diagram:

$$\begin{tikzcd} & \tilde C \arrow[dl] \arrow[d] \arrow[dr] & \\ C'' = \tilde C/\langle g_1 g_5\rangle \arrow[dr, "p"] & C' = \tilde C/\langle g_1\rangle \arrow[d, "q"] & C = \tilde C/\langle g_5\rangle \arrow[dl] \\ & E = \tilde C/\langle g_1, g_5\rangle. & \end{tikzcd} \tag{5.6}$$

The curves C' and C'' have genus 7, while E has genus 1. One can check that $P(\tilde C, C) \sim P(C', E) \times P(C'', E)$, since $H^0(P(C', E), K_{P(C',E)}) \cong V_7 + 2V_{11}$ and $H^0(P(C'', E), K_{P(C'',E)}) \cong 2V_8 + V_{12}$. Since $(S^2(V_-))^{\tilde G} = (V_8 \otimes V_8)^{\tilde G}$, the abelian variety $P(C', E)$ does not move by Lemma 5.2. To prove condition (B) we need to show that $P(C'', E)$ moves.

We have $(S^2(H^0(C'', K_{C''})))^{\tilde{G}} \cong (S^2(2V_8 + V_{12}))^{\tilde{G}} + (S^2V_3)^{\tilde{G}} = (\Lambda^2 V_8)^{\tilde{G}} + (S^2V_3)^{\tilde{G}} = (\Lambda^2 V_8)^{\tilde{G}}$, as one can check. Therefore $(S^2(H^0(C'', K_{C''})))^{\tilde{G}}$ has dimension 1. So the family $C'' \to C''/H = \tilde{C}/\tilde{G}$, where $H = \tilde{G}/\langle g_1 g_5\rangle \cong SL(2, \mathbb{F}_3)$, satisfies condition $(*)$ of [22], i.e. the codifferential of the Torelli map, i.e. the multiplication map $(S^2(H^0(C'', K_{C''})))^H \to H^0(C'', 2K_{C''})^H$, is an isomorphism. So this is the Shimura family (40) of [22]. Since $J(C'') \sim P(C'', E) \times E$ and $J(C'')$ moves, while E is fixed, $P(C'', E)$ necessarily moves. Therefore $P(\tilde{C}, C)$ moves as well and condition (B) is satisfied. Notice that on E there is an action of $\langle g_2\rangle \cong \mathbb{Z}/3$, hence $E = E_\rho$.

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$\tilde{g} = 25$, $g = 13$. $\tilde{G} = G(48, 30) = A_4 \rtimes \mathbb{Z}/4 = A_4 \rtimes \langle g_1\rangle$,
where $A_4 = \langle g_3 = (123), g_4 = (12)(34), g_5 = (13)(24)$,
$g_3^3 = 1,\ g_4^2 = 1,\ g_5^2 = 1,\ g_3^{-1} g_4 g_3 = g_5,\ g_3^{-1} g_5 g_3 = g_4 g_5,\ g_4 g_5 = g_5 g_4$,
$g_1^{-1} g_3 g_1 = g_3^2, g_1^{-1} g_4 g_1 = g_5, g_1^{-1} g_5 g_1 = g_4\rangle$.
$\sigma = g_2 = g_1^2$, $Z(\tilde{G}) = \langle g_2\rangle \cong \mathbb{Z}/2$.
$\tilde{\theta}(\gamma_1) = g_1 g_5,\ \tilde{\theta}(\gamma_2) = g_1 g_4,\ \tilde{\theta}(\gamma_3) = g_1 g_3 g_4,\ \tilde{\theta}(\gamma_4) = g_1 g_3 g_4 g_5$.
$V_- = 2V_3 \oplus 2V_5 \oplus 2V_{10}$, where $\dim(V_3) = 1$, $\dim(V_5) = 2$, $\dim(V_{10}) = 3$.
(Notation of `MAGMA` as above.)
$(S^2(V_-))^{\tilde{G}} = (V_5 \otimes V_5)^{\tilde{G}} = (\Lambda^2 V_5)^{\tilde{G}} = \Lambda^2 V_5$.
So $(S^2(V_-))^{\tilde{G}} = (V_5 \otimes V_5)^{\tilde{G}} = \Lambda^2 V_5$ is 1-dimensional, hence condition (A) is satisfied. We check now condition (B).

Consider the normal subgroup $H := \langle g_4, g_5\rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2 \lhd \tilde{G}$. Set $C' = \tilde{C}/H$. One sees that $\tilde{C} \to C' = \tilde{C}/H$ is a $4:1$ étale covering and C' has genus 7. Moreover $H^0(C', K_{C'}) = 2V_3 + 2V_5 + V_2$.

Set $H' := \langle\sigma\rangle \times A_4$. The quotient $E := \tilde{C}/H'$ is a genus one curve and $H^0(E, K_E) = V_2$. So we have the following commutative diagram:

$$\begin{tikzcd} & \tilde{C} \arrow[dl] \arrow[dr] & \\ C' = \tilde{C}/H \arrow[dr, "p"] & & C = \tilde{C}/\langle\sigma\rangle \arrow[dl, "q"] \\ & E = \tilde{C}/H'. & \end{tikzcd}$$

Hence $P(\tilde{C}, C) \sim P(C', E) \times A$, where A is a fixed abelian 6-fold. Consider the groups $L = H'/H < K = \tilde{G}/H$, $L \cong \langle\sigma\rangle \times (A_4/H) \cong \mathbb{Z}/6$. We have

$$\begin{tikzcd} & C' \arrow[dl, "\varphi"] \arrow[d] \arrow[dr] & \\ D = C'/\langle g_3\rangle \arrow[dr, "\pi"] & & E' = C'/\langle g_2\rangle \arrow[dl, "\bar{q}"] \\ & E. & \end{tikzcd} \tag{5.7}$$

Notice that D has genus 3 and φ is a $3:1$ étale covering. Moreover, $H^0(C', K_{C'})^{\langle g_3\rangle} \cong H^0(K_D) \cong V_2 \oplus 2V_3$, hence $H^{1,0}(P(C', D)) \cong 2V_5$ and $P(\tilde{C}, C) \sim P(C', D) \times A'$, where A' is a fixed abelian variety of dimension 8.

We want to show that $P(C', D)$ moves and hence yields a Shimura curve.

Since the map φ is a 3:1 étale covering, it corresponds to a 3-torsion line bundle η on D and the pairs $[C', D]$ vary in a curve $\mathcal{B}$ in the moduli space $\mathcal{R}'_{3,3}$ parametrising pairs $[C', D]$ where $C' \to D$ is a $3:1$ étale Galois covering of a genus three curve D. Denote by $\mathcal{P} : \mathcal{R}'_{3,3} \to \mathsf{A}_4(\Theta)$ the corresponding Prym map. To conclude we need to show that the differential of the restriction of $\mathcal{P}$ to $\mathcal{B}$ is injective. Notice that the image of $d\mathcal{P}_{[C',D]} : T_{[C',D]}\mathcal{R}'_{3,3} \to T_{P(C',D)}\mathsf{A}_4(\Theta)$ is contained in the $\mathbb{Z}/3$ invariant part of $T_{P(C',D)}\mathsf{A}_4(\Theta)$. Therefore

$$d\mathcal{P}^\vee_{[D,\eta]} : (T^*_{P(C',D)}\mathsf{A}_4(\Theta))^{\mathbb{Z}/3} \cong S^2 H^{1,0}(P(C', D))^{\mathbb{Z}/3} \to T^*_{[D,\eta]}\mathcal{R}'_{3,3} \cong H^0(2K_D).$$

Observe that $H^{1,0}(P(C',D)) \cong H^0(K_D(\eta)) \oplus H^0(K_D(\eta^2))$, hence

$$S^2H^{1,0}(P(C',D))^{\mathbb{Z}/3} \cong H^0(K_D(\eta)) \otimes H^0(K_D(\eta^2))$$

and the codifferential is identified with the multiplication map

$$m : H^0(K_D(\eta)) \otimes H^0(K_D(\eta^2)) \to H^0(2K_D).$$

First of all we prove that m is injective. Observe that injectivity follows from the base point free pencil trick if we show that $|K_D(\eta^2)|$ is base point free. In fact in this case the kernel of m would be $H^0(\eta) = 0$.

Let us now prove that $|K_D(\eta^2)|$ is base point free.

So assume that $|K_D(\eta^2)|$ has a base point $p \in D$. Then $h^0(K_D(\eta^2)(-p)) = h^1(\eta(p)) = 2$, hence $h^0(\eta(p)) = 1$, therefore there exists a point $q \in D$ such that $\eta = \mathcal{O}_D(q-p)$. By the commutativity of diagram (5.7), we know that $\eta = \pi^*(\eta_E)$, where η_E is the 3-torsion line bundle on E corresponding to the $3:1$ étale covering $\bar{q}$. In particular η is invariant by the covering involution ι of π. Hence we have $p - q \equiv \iota(p) - \iota(q)$, equivalently $p + \iota(q) \equiv \iota(p) + q$, which is impossible since D is not hyperelliptic. In fact the family $D \to \mathbb{P}^1$ is the family (4) of [37], which is not hyperelliptic.

Now denote by α the line bundle on E yielding the $2:1$ covering π. We have $K_D = \pi^*(\alpha)$. Via the projection formula, the map m can be identified with the multiplication map

$$m_E : H^0(\alpha \otimes \eta_E) \otimes H^0(\alpha \otimes \eta_E^2) \to H^0(\alpha^2) \subset H^0(\alpha^2) \oplus H^0(\alpha) \cong H^0(2K_D).$$

Notice that $H^0(\alpha^2)$ can be identified with the cotangent space to the bielliptic locus at the point D and the cotangent space $T^*_{[C',D]}\mathcal{B}$ is identified to a 1 dimensional subspace of it via the forgetful map $\mathcal{R}'_{3,3} \to \mathcal{M}_3$. Since $\dim(H^0(\alpha^2)) = 4$ and m is injective, m_E is an isomorphism, hence the differential of the restriction of the Prym map to $\mathcal{B}$ at the point $[C',D]$ is injective. Therefore the family $P(\tilde{C},C)$ moves.

6. Examples in the ramified Prym locus

In this section we briefly describe the examples of families of ramified Pryms satisfying conditions (A) and (B), hence yielding Shimura curves generically contained in the ramified Prym locus.

Example 1.
$\tilde{g} = 4, g = 2, \quad \tilde{G} = \mathbb{Z}/6 = \mathbb{Z}/2 \times \mathbb{Z}/3 = \langle g_1, g_2 \mid g_1^2 = 1,\ g_2^3 = 1,\ g_1g_2 = g_2g_1\rangle$.
$\tilde{\theta}(\gamma_1) = g_2, \quad \tilde{\theta}(\gamma_2) = g_2^2, \quad \tilde{\theta}(\gamma_3) = g_1g_2, \quad \tilde{\theta}(\gamma_4) = g_1g_2^2$.
$\tilde{C}_t: \quad y^3 = u(u^2-1)^2(u^2-t), \quad \pi : \tilde{C}_t \to \mathbb{P}^1, \quad \pi(u,y) = u$.
$\sigma = g_1 : (u,y) \to (-u,-y), g_2 : (u,y) \to (u, \zeta_3 y)$.
$C_t: \quad z^3 = x^2(x-1)^2(x-t) \quad (x,z) = (u^2, yu)$.
Let ζ_3^i denote the character of $\langle g_2\rangle$ mapping g_2 to ζ_3^i. Let $W_{\zeta_3^i}$ be the irreducible representation of $\langle g_2\rangle$ corresponding to the character ζ_3^i.
As a representation of $\langle g_2\rangle$ we have: $V_- = W_{\zeta_3} \oplus W_{\zeta_3^2}, \quad (S^2V_-)^{\tilde{G}} \cong W_{\zeta_3} \otimes W_{\zeta_3^2}$.
In the notation of Magma $V_4 = W_{\zeta_3}, \quad V_6 = W_{\zeta_3^2}$. The orbit of W_{ζ_3} under the action of $Gal(\mathbb{Q}(\zeta_3),\mathbb{Q})$ is clearly $\{W_{\zeta_3}, W_{\zeta_3^2}\}$. The Pryms $P(\tilde{C},C)$ form a 1-dimensional family of abelian surfaces with a $\mathbb{Z}/3$-action. This yields a Shimura curve, hence it is family (3) of [37].

Example 2.
$\tilde{g} = 4, \qquad g = 2$,
$\tilde{G} = D_6 = G(12,4) = \langle g_1, g_2, g_3 \mid g_1^2 = g_2^2 = g_3^3 = 1,\ g_1^{-1}g_3g_1 = g_3^{-1},\ g_1g_2 = g_2g_1,\ g_2g_3 = g_3g_2\rangle$,
$\sigma = g_2$.
$\tilde{\theta}(\gamma_1) = g_1, \quad \tilde{\theta}(\gamma_2) = g_1g_2, \quad \tilde{\theta}(\gamma_3) = g_3, \quad \tilde{\theta}(\gamma_4) = g_2g_3^2$.
We observe that this is the same family as in Example 1, since the family of the curves C is family (3) of [37]. In fact family (3) is equal to family (28) of [22].

In the following two examples we have
$\tilde{g} = 8, \qquad g = 4,$
$\tilde{G} = \mathbb{Z}/2 \times \mathbb{Z}/5 = \langle g_1, g_2 \mid g_1^2 = 1,\ g_2^5 = 1,\ g_1g_2 = g_2g_1 \rangle$, $\sigma = g_1$.
In both cases the family of Pryms is a 1-dimensional family of abelian 4-folds with an action of $\mathbb{Z}/5$, that yields a Shimura curve .

Example 3.
$\tilde{\theta}(\gamma_1) = g_2, \quad \tilde{\theta}(\gamma_2) = g_2^2, \quad \tilde{\theta}(\gamma_3) = g_1g_2, \quad \tilde{\theta}(\gamma_4) = g_1g_2.$
$\tilde{C}_t : \quad y^5 = u^2(u^2-1)^2(u^2-t), \quad \pi : \tilde{C}_t \to \mathbb{P}^1, \quad \pi(u,y) = u.$
$g_1(u,y) = (-u,y), \quad g_2(u,y) = (u, \zeta_5 y).$
$C_t : \quad y^5 = x(x-1)^2(x-t) \quad (x,y) = (u^2, y).$
$V_- = W_{\zeta_5} \oplus 2W_{\zeta_5^3} \oplus W_{\zeta_5^4}.$
$(S^2V_-)^{\tilde{G}} \cong W_{\zeta_5} \otimes W_{\zeta_5^4}.$

Example 4.
$\tilde{\theta}(\gamma_1) = g_2, \quad \tilde{\theta}(\gamma_2) = g_2^2, \quad \tilde{\theta}(\gamma_3) = g_1g_2^3, \quad \tilde{\theta}(\gamma_4) = g_1g_2^4.$
$\tilde{C}_t : \quad y^5 = u(u^2-1)^2(u^2-t), \quad \pi : \tilde{C} \to \mathbb{P}^1, \quad \pi(u,y) = u.$
$g_1 : (u,y) \to (-u,-y), \quad g_2 : (u,y) \to (u, \zeta_5 y).$
$C_t := \quad z^5 = x^3(x-1)^2(x-t) \quad (x,z) = (u^2, yu).$
$V_- = 2W_{\zeta_5} \oplus W_{\zeta_5^2} \oplus W_{\zeta_5^3}.$
$(S^2V_-)^{\tilde{G}} \cong W_{\zeta_5^2} \otimes W_{\zeta_5^3}.$

In the next example the group $\tilde{G}$ is not abelian and condition (B1) is not satisfied. We show with a geometrical argument that that condition (B) holds and therefore we get a Shimura curve in A_4.

Examples 5
$\tilde{g} = 8, g = 4,$
$\tilde{G} = G(24,10) \cong \mathbb{Z}/3 \times D_4 =$
$= \langle g_1, g_2, g_3 \mid g_1^2 = g_2^2 = g_3^3 = 1,\ (g_2g_1)^4 = 1,\ g_3g_i = g_ig_3\ i = 1,2, \rangle \cong$
$\cong \langle g_3 \rangle \times \langle x = g_2g_1, y = g_1 \mid x^4 = y^2 = 1,\ yx = x^{-1}y \rangle$
$\sigma = (g_2g_1)^2$
$\tilde{\theta}(\gamma_1) = g_2, \quad \tilde{\theta}(\gamma_2) = g_1, \quad \tilde{\theta}(\gamma_3) = g_3, \quad \tilde{\theta}(\gamma_4) = g_1g_2g_3^2.$
$V_- = V_{14} \oplus V_{15}$, where $\dim(V_{14}) = \dim(V_{15}) = 2$ (notation of `MAGMA`),
$(S^2(V_-))^{\tilde{G}} = (V_{14} \otimes V_{15})^{\tilde{G}}$, it is one dimensional, hence condition (A) is satisfied. We need to check condition (B). Consider $\langle g_1 \rangle \cong \mathbb{Z}/2$ and set $D = \tilde{C}/\langle g_1 \rangle$. The quotient $\tilde{C} \to D$ is a double cover ramified in 6 points, hence $g(D) = 3$. We have the following commutative diagram:

(6.1)
$$\begin{tikzcd} & \tilde{C} \arrow[dl] \arrow[dr] & \\ D = \tilde{C}/\langle g_1 \rangle \arrow[dr, "p"] & & C = \tilde{C}/\langle \sigma \rangle \arrow[dl, "q"'] \\ & E = \tilde{C}/\langle \sigma, g_1 \rangle. & \end{tikzcd}$$

Here q is a double cover ramified in 6 points and E is an elliptic curve with an action of $\langle g_3 \rangle \cong \mathbb{Z}/3$, hence it is constant. From the above diagram one sees that $P(\tilde{C}, C) \sim P(D,E) \times A$, where A is an abelian surface. To prove that $P(\tilde{C}, C)$ moves, we will show that $P(D,E)$ moves. Since E is fixed, it is equivalent to show that $J(D)$ moves in a one dimensional family. Denote by $\psi : \tilde{C} \to \mathbb{P}^1 = \tilde{C}/\tilde{G}$ our original covering, by P_1, P_2, P_3, P_4 the branch points of ψ and by $\pi : E \to E/\langle g_2, g_3 \rangle \cong \tilde{C}/\tilde{G}$. The branch points of the map π (given by the $\mathbb{Z}/6$-action on E) are P_1, P_3, P_4, hence, since E does not move, the three branch points of the original map ψ, P_1, P_3, P_4 do not move, therefore P_2 must move. The map p has 4 branch points $\{e_1, e_2, e_3, e_4\} \subset E$, where $\pi(e_i) = P_2$ for $i = 1,2,3$, while $\pi(e_4) = P_4$. Since P_2 moves, the three branch points $\{e_1, e_2, e_3\}$ move, hence the covering $p : D \to E$ moves and so do D and $J(D)$. This concludes the argument.

The following two examples both have
$\tilde{g} = 12, \qquad g = 6,$
$\tilde{G} = \mathbb{Z}/2 \times \mathbb{Z}/7 = \langle g_1, g_2 \mid g_1^2 = 1,\ g_2^7 = 1,\ g_1 g_2 = g_2 g_1 \rangle$, $\sigma = g_1$.
In both cases the family $P(\tilde{C}, C)$ is a 1-dimensional family of abelian 6-folds with an action of $\mathbb{Z}/7$, that yields a Shimura curve.

Example 6. $\tilde{\theta}(\gamma_1) = g_2, \quad \tilde{\theta}(\gamma_2) = g_2^3, \quad \tilde{\theta}(\gamma_3) = g_1 g_2^4, \quad \tilde{\theta}(\gamma_4) = g_1 g_2^6.$
$\tilde{C}_t : \quad y^7 = u(u^2-1)^3(u^2-t), \quad \pi : \tilde{C} \to \mathbb{P}^1, \quad \pi(u, y) = u,$
$g_1 : (u, y) \to (-u, -y), g_2 : (u, y) \to (u, \zeta_7 y),$
$C_t : \quad z^7 = x^4(x-1)^3(x-t) \quad (x, z) = (u^2, yu).$
$V_- = 2W_{\zeta_7} \oplus W_{\zeta_7^2} \oplus 2W_{\zeta_7^3} \oplus W_{\zeta_7^5}.$
$(S^2 V_-)^{\tilde{G}} \cong W_{\zeta_7^2} \otimes W_{\zeta_7^5}.$

Example 7. $\tilde{\theta}(\gamma_1) = g_2, \quad \tilde{\theta}(\gamma_2) = g_2^3, \quad \tilde{\theta}(\gamma_3) = g_1 g_2^5, \quad \tilde{\theta}(\gamma_4) = g_1 g_2^5.$
$\tilde{C}_t : \quad y^7 = u^3(u^2-1)^3(u^2-t), \quad \pi : \tilde{C} \to \mathbb{P}^1, \quad \pi(u, y) = u.$
$g_1 : (u, y) \to (-u, -y), g_2 : (u, y) \to (u, \zeta_7 y).$
$C_t : \quad z^7 = x^5(x-1)^3(x-t) \quad (x, z) = (u^2, yu).$
$V_- = 2W_{\zeta_7} \oplus W_{\zeta_7^3} \oplus W_{\zeta_7^4} \oplus 2W_{\zeta_7^5}.$
$(S^2 V_-)^{\tilde{G}} \cong W_{\zeta_7^3} \otimes W_{\zeta_7^4}.$

Example 8. $\tilde{g} = 14, \qquad g = 7,$
$\tilde{G} = G(18, 2) = \mathbb{Z}/2 \times \mathbb{Z}/9 = \langle g_1, g_2 \mid g_1^2 = 1,\ g_2^9 = 1,\ g_1 g_2 = g_2 g_1 \rangle$, $\sigma = g_1$.
$\tilde{\theta}(\gamma_1) = g_2^3, \quad \tilde{\theta}(\gamma_2) = g_2, \quad \tilde{\theta}(\gamma_3) = g_1 g_2^7, \quad \tilde{\theta}(\gamma_4) = g_1 g_2^7.$
$\tilde{C}_t : \quad y^9 = u^7(u^2-1)^6(u^2-t)^5, \quad \pi : \tilde{C} \to \mathbb{P}^1, \quad (u, y) \to u.$
$g_1 : (u, y) \to (-u, -y), g_2 : (u, y) \to (u, \zeta_9 y).$
$C_t : \quad z^9 = x^8(x-1)^6(x-t)^5 \quad (x, z) = (u^2, yu).$
$V_- = W_{\zeta_9} \oplus 2W_{\zeta_9^2} \oplus 2W_{\zeta_9^4} \oplus W_{\zeta_9^6} \oplus W_{\zeta_9^8}\ .$
$(S^2 V_-)^{\tilde{G}} \cong W_{\zeta_9} \otimes W_{\zeta_9^8}.$

In the next example that satisfies condition (A), the group $\tilde{G}$ is not abelian and condition (B1) does not hold. Hence we show again with a geometrical argument that also condition (B) holds and therefore it gives a Shimura curve generically contained in the ramified Prym locus in A_8. Notice that by Remark 5.1 it is not contained in the Torelli locus.

Example 9. $\tilde{g} = 16, \qquad g = 8,$
$\tilde{G} = G(40, 10) \cong \mathbb{Z}/5 \times D_4 =$
$= \langle g_1, g_2, g_3 \mid g_1^2 = g_2^2 = g_3^5 = 1 (g_2 g_1)^4 = 1,\ g_3 g_i = g_i g_3\ i = 1, 2, \rangle \cong$
$\cong \langle g_3 \rangle \times \langle x = g_2 g_1, y = g_1 \mid x^4 = y^2 = 1,\ yx = x^{-1} y \rangle$
$\sigma = (g_2 g_1)^2.$
$\tilde{\theta}(\gamma_1) = g_2, \quad \tilde{\theta}(\gamma_2) = g_1, \quad \tilde{\theta}(\gamma_3) = g_3, \quad \tilde{\theta}(\gamma_4) = g_1 g_2 g_3^{-1}.$
$V_- = V_{22} \oplus V_{23} \oplus 2V_{24}$, where $\dim(V_{22}) = \dim(V_{23}) = \dim(V_{24}) = 2$ (notation of `MAGMA`).
$(S^2(V_-))^{\tilde{G}} = (V_{22} \otimes V_{23})^{\tilde{G}}$ and it is one dimensional, hence condition (A) is satisfied. We check now condition (B). Consider $\langle g_1 \rangle \cong \mathbb{Z}/2 \subset \tilde{G}$ and denote by $D = \tilde{C}/\langle g_1 \rangle$. One sees that $\tilde{C} \to D$ is a double cover ramified in 10 points, hence $g(D) = 6$. We have the following commutative diagram:

$$
\begin{tikzcd}
 & \tilde{C} \arrow[dl] \arrow[dr] & \\
D = \tilde{C}/\langle g_1 \rangle \arrow[dr, "p"] & & C = \tilde{C}/\langle \sigma \rangle \arrow[dl, "q"'] \\
 & F = \tilde{C}/\langle \sigma, g_1 \rangle. &
\end{tikzcd}
\tag{6.2}
$$

where q is a double cover ramified in 10 points and F is a genus 2 curve with an action of $\langle g_3\rangle \cong \mathbb{Z}/5$. Therefore F is a CM curve for any value of the parameter. It follows that F is constant. From the above diagram one sees that $P(\tilde{C}, C) \sim P(D, F) \times A$, where A is an abelian surface. Therefore, to prove that $P(\tilde{C}, C)$ moves, we will show that $P(D, F)$ moves. Since F is fixed, this is equivalent to show that $J(D)$ moves in a one dimensional family. Denote by $\psi : \tilde{C} \to \mathbb{P}^1 = \tilde{C}/\tilde{G}$ our original covering, by P_1, P_2, P_3, P_4 the branch points of ψ and by $\pi : F \to F/\langle g_2, g_3\rangle \cong \tilde{C}/\tilde{G}$. The branch points of the map π (given by the $\mathbb{Z}/10$-action on F) are P_1, P_3, P_4, hence, since F does not move, the three branch points of the original map ψ, P_1, P_3, P_4 do not move, therefore P_2 must move. The map p has 4 branch points $\{e_1, e_2, e_3, e_4\} \subset F$, where $\pi(e_i) = P_2$ for $i = 1, 2, 3$, while $\pi(e_4) = P_4$. Since P_2 moves, the three branch points $\{e_1, e_2, e_3\}$ move, hence the covering $p : D \to F$ moves and so D (and $J(D)$) moves. This concludes the argument.

Appendix

This appendix gives the relevant information on the script and contains the tables of all the Prym data, which satisfy condition (A). Table 1 is for étale Prym data, while Table 2 is for the ramified Prym data.

To perform the calculations done in this paper we wrote a `GAP4` [25] and a `MAGMA` [34] script, both of them are available at:

`http://www.dima.unige.it/~penegini/publications`

We now describe the `GAP4` program `PrymGenerators_v2.gap`.

The main routine is the function `PossibleGoodPrym`. One fixes a range for the genus of the covering curve $\tilde{C}$ (we used $4 \leq \tilde{g} \leq 30$), a range for the number of branch points of the covering $\tilde{C} \to \mathbb{P}^1$ (we considered only the case of 4 branch points) and the type x of Prym. The latter means the following:: $x = 1$ for étale Prym datum, $x = 2$ for ramified Prym datum satisfying (2) of Definition 4.1, $x = 3$ for ramified Prym datum satisfying (1) of Definition 4.1. Once all these data are fixed the program performs the following calculations.

(1) First it calculates all possible signature types (Group order, $\mathbf{m}$) for the coverings $\tilde{C} \to \mathbb{P}^1$.
(2) After that, the program calculates for each signature type all the Prym data up to Hurwitz equivalence. These are: a group $\tilde{G}$ of a fixed order, all spherical systems of generators (SSG) for $\tilde{G}$ (images of $\tilde{\theta}$) of the fixed type $\mathbf{m}$ up to Hurwitz moves, and an order 2 central element in $\tilde{G}$. Here the script calls some parts of the script given in [45] (in particular the function `NrOfComponents`). We refer to the appendix of [45] for an explanation of the algorithm.

While looking for the Prym data in the unramified case we can forget from the very beginning the cyclic groups thanks to the following lemma.

Lemma 6.1. *If (G, θ) is an unramified Prym datum, then $\tilde{G}$ is not cyclic.*

Proof. Assume by contradiction that $\tilde{G} = \langle x \rangle$ with $o(x) = 2n$ and let $\{x^{n_i} = \tilde{\theta}(\gamma_i)\}_{i=1}^k$ be a set of generators for $\tilde{G}$. There is only one element of order 2 in $\tilde{G}$, namely $\sigma := x^n$. It follows that $\sigma \in \langle a \rangle$ if and only if $o(a)$ is even. Since $\sigma \notin \langle x^{n_i} \rangle$, $o(x^{n_i})$ is odd for any i. On the other hand if $a = x^s$, then $o(a) = 2n/(2n, s)$. Write $n = 2^p q$ and $n_i = 2^{p_i} q_i$ with q and q_i odd. Then $o(x^{n_i}) = 2^{p+1-\min\{p+1, p_i\}} \cdot \frac{q}{(q, q_i)}$. As this number is odd, we have $p_i \geq p + 1$, so n_i is even for any i. Then clearly $[n_i]_{2n}$ cannot generate $\mathbb{Z}/(2n)$, contradiction. $\square$

We used the `GAP4` program because the algorithm for finding inequivalent pairs ($\tilde{G}$, SSG) up to Hurwitz moves is efficient and quite fast. One can find the output of this program at the web page

`http://www.dima.unige.it/~penegini/publications/`

The remaining computations are performed using a `MAGMA` program `PrymMagma_v6`, that we now describe.

(1) The function `GoodExample` calculates the dimension $N_1 := \dim(S^2 V)^{\tilde{G}}$ using the script `PossGruppigFix_v2Hwr` written for the paper [22] (we refer to [22] for explanations). The input for this function are the data previously calculated by `PrymGenerators_v2.gap`.
(2) The function `ProjSSG` constructs an SSG for the group G (for the covering $C \to C/G \cong \mathbb{P}^1$) compatible with the given SSG of $\tilde{G}$.
(3) Afterwards we calculate the dimension $N_2 = \dim(S^2 V_+)^G$, again with the function `GoodExample`.
(4) The function `GoodPrym(N1,N2)` checks condition (A) in the form $N_1 - N_2 = 1$. If the condition is satisfied the program will print `GOOD EXAMPLE`. The resulting lists are Table 1 and 2 here.

(5) Finally the function `IsGoodGood` checks condition (B1).

All the results are available at

`http://www.dima.unige.it/~penegini/publications/`

A brief explanation of the tables.

The tables list all Prym data with $\tilde{g} \leq 28$ satisfying conditions (A) and (B1) up to Hurwitz equivalence. It also contains all the non-abelian examples satisfying (A) (but not (B1)) for which we have verified condition (B). For each datum we list a number that identifies the datum, the genera of $\tilde{C}$ and C, the group $\tilde{G}$ and its `MAGMA SmallGroupId`. The last two columns contain information about conditions (B1) and (B). There is a checkmark for (B1) if and only if (B1) is satisfied. If (B1) is true, then (B) follows. When there is a checkmark for (B), this means that we proved that (B) holds.

In the tables some data are grouped together because they differ only by $\tilde{\theta}$.

We do not give the full presentation of $\tilde{G}$, nor the morphism $\tilde{\theta}$, since that would take too much space. The complete information is of course available at the page above.

The data satisfying (B) yield Shimura curves in the Prym loci.

n	$g(\tilde{C})$	$g(C)$	$\tilde{G}$	`SmallGroupId`	$B1$	B
$1-3$	5	3	$\mathbb{Z}/2 \times \mathbb{Z}/4$	$G(8,2)$	✓	✓
4	5	3	$\mathbb{Z}/2 \times \mathbb{Z}/6$	$G(12,5)$	✓	✓
5	5	3	$(\mathbb{Z}/2 \times \mathbb{Z}/4) \rtimes \mathbb{Z}/2$	$G(16,3)$	✓	✓
$6-8$	5	3	$\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/4$	$G(16,10)$	✓	✓
9	5	3	$\mathbb{Z}/2 \times A_4$	$G(24,13)$	✓	✓
10	7	4	$\mathbb{Z}/2 \times \mathbb{Z}/6$	$G(12,5)$	✓	✓
$11-12$	7	4	$\mathbb{Z}/4 \rtimes \mathbb{Z}/4$	$G(16,4)$	✓	✓
13	7	4	$\mathbb{Z}/2 \times Q_8$	$G(16,12)$	✓	✓
14	9	5	$\mathbb{Z}/4 \times \mathbb{Z}/4$	$G(16,2)$	✓	✓
15	9	5	$\mathbb{Z}/4 \rtimes \mathbb{Z}/4$	$G(16,4)$	✓	✓
$16-20$	9	5	$\mathbb{Z}/2 \times \mathbb{Z}/8$	$G(16,5)$	✓	✓
$21-23$	9	5	$\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/4$	$G(16,10)$	✓	✓
$24-25$	9	5	$\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/6$	$G(24,15)$	✓	✓
26	9	5	$\mathbb{Z}/2 \times ((\mathbb{Z}/2 \times \mathbb{Z}/4) \rtimes \mathbb{Z}/2)$	$G(32,22)$	✓	✓
27	9	5	$\mathbb{Z}/4 \times D_4$	$G(32,25)$	✓	✓
28	11	5	$\mathbb{Z}/2 \times \mathbb{Z}/8$	$G(16,5)$	✓	✓
29	11	6	$\mathbb{Z}/2 \times \mathbb{Z}/12$	$G(24,9)$	✓	✓
30	13	7	$\mathbb{Z}/2 \times \mathbb{Z}/8$	$G(16,5)$	✓	✓
31	13	7	$\mathbb{Z}/2 \times \mathbb{Z}/10$	$G(20,5)$	✓	✓
32	13	7	$\mathbb{Z}/2 \times \mathbb{Z}/12$	$G(24,9)$	✓	✓
33	13	7	$(\mathbb{Z}/2 \times \mathbb{Z}/8) \rtimes \mathbb{Z}/2$	$G(32,9)$	✓	✓
34	13	7	$(\mathbb{Z}/4 \times \mathbb{Z}/4) \rtimes \mathbb{Z}/2$	$G(32,24)$	✓	✓
35	15	8	$\mathbb{Z}/2 \times \mathbb{Z}/12$	$G(24,9)$	✓	✓
36	17	9	$\mathbb{Z}/2 \times \mathbb{Z}/12$	$G(24,9)$	✓	✓

n	$g(\tilde{C})$	$g(C)$	$\tilde{G}$	`SmallGroupId`	$B1$	B
37	17	9	$\mathbb{Z}/2 \times \mathbb{Z}/4 \times \mathbb{Z}/4$	$G(32, 21)$	✓	✓
38	17	9	$(\mathbb{Z}/2 \times \mathbb{Z}/12) \rtimes \mathbb{Z}/2$	$G(48, 14)$	✓	✓
39	17	9	$(\mathbb{Z}/4 \times \mathbb{Z}/4 \times \mathbb{Z}/2) \rtimes \mathbb{Z}/2$	$G(64, 71)$	✓	✓
40	19	10	$(\mathbb{Z}/2 \times \mathbb{Z}/3 \times \mathbb{Z}/3) \rtimes S_3$	$G(108, 28)$		✓
41	21	11	$\mathbb{Z}/4 \times \mathbb{Z}/8$	$G(32, 3)$	✓	✓
42	21	11	$\mathbb{Z}/4 \times D_8$	$G(64, 118)$	✓	✓
43	25	13	$\mathbb{Z}/2 \times SL(2, 3)$	$G(48, 32)$		✓
44	25	13	$A_4 \rtimes \mathbb{Z}/4$	$G(48, 30)$		✓

Table 1

n	$g(\tilde{C})$	$g(C)$	$\tilde{G}$	`SmallGroupId`	$B1$	B
1	4	2	$\mathbb{Z}/6$	$G(6, 2)$	✓	✓
2	4	2	D_6	$G(12, 4)$	✓	✓
$3-4$	8	4	$\mathbb{Z}/10$	$G(10, 2)$	✓	✓
5	8	4	$\mathbb{Z}/3 \times D_4$	$G(24, 10)$		✓
$6-7$	12	6	$\mathbb{Z}/14$	$G(14, 2)$	✓	✓
8	14	7	$\mathbb{Z}/18$	$G(18, 2)$	✓	✓
9	16	8	$\mathbb{Z}/5 \times D_4$	$G(40, 10)$		✓

Table 2

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