Dynamics of an adaptive randomly reinforced urn

Giacomo Aletti*[1], Andrea Ghiglietti†[1], and Anand N. Vidyashankar‡[2]

[1]ADAMSS center, Università degli Studi di Milano, Milan, Italy

[2]Department of Statistics, George Mason University, Fairfax (VA), USA

Abstract

Adaptive randomly reinforced urn (ARRU) is a two-color urn model where the updating process is defined by a sequence of non-negative random vectors $\{(D_{1,n}, D_{2,n}); n \geq 1\}$ and randomly evolving thresholds which utilize accruing statistical information for the updates. Let $m_1 = E[D_{1,n}]$ and $m_2 = E[D_{2,n}]$. In this paper we undertake a detailed study of the dynamics of the ARRU model. First, for the case $m_1 \neq m_2$, we establish L_1 bounds on the increments of the urn proportion, i.e. the proportion of ball colors in the urn, at fixed and increasing times under very weak assumptions on the random threshold sequences. As a consequence, we deduce weak consistency of the evolving urn proportions. Second, under slightly stronger conditions, we establish the strong consistency of the urn proportions for all finite values of m_1 and m_2. Specifically we show that when $m_1 = m_2$, the proportion converges to a non-degenerate random variable. Third, we establish the asymptotic distribution, after appropriate centering and scaling, for the proportion of sampled ball colors and urn proportions for the case $m_1 = m_2$. In the process, we resolve the issue concerning the asymptotic distribution of the proportion of sampled ball colors for a randomly reinforced urn (RRU). To address the technical issues, we establish results on the harmonic moments of the total number of balls in the urn at different times under very weak conditions, which is of independent interest.
Keywords: generalized Pólya urn, reinforced processes, strong and weak consistency, central limit theorems, crossing times, harmonic moments.
MSC Subject Classification: 60F05, 60F15, 97K50.

1 Introduction

In recent years, randomly reinforced urn (RRU) has been investigated in statistical and probability literature as a model for clinical trial design, computer experiments, and in the context of vertex reinforced random walk (see Hu and Rosenberger (2006); Mahmoud (2008); Pemantle and Volkov (1999)). Introduction of accruing information in designing the reinforcement mechanism leads to an adaptive version of an RRU model, which we refer to as an adaptive randomly reinforced urn (ARRU). In this paper, we study the properties concerning the urn proportions and the proportion of sampled ball colors of an ARRU. We now turn to a precise description of the ARRU.

A randomly reinforced urn (RRU) model (see Muliere et al. (2006)) is characterized by a pair $(Y_{1,n}, Y_{2,n})$ of real random variables representing the number of balls of two colors, red and white. The process is described as follows: at time $n = 0$, the process starts with $(y_{1,0}, y_{2,0})$ balls. A ball is drawn at random. If the color is red, the ball is returned to the urn along with the random numbers $D_{1,1}$ of red balls; otherwise, the ball is returned to the urn along with the random numbers $D_{2,1}$ of white balls. Let $Y_{1,1} = y_{1,0} + D_{1,1}$ and $Y_{2,1} = y_{2,0}$ denote the urn composition when the sampled ball is red; similarly, let $Y_{1,1} = y_{1,0}$ and $Y_{2,1} = y_{2,0} + D_{2,1}$ denote the urn composition when

*giacomo.aletti@unimi.it
†andrea.ghiglietti@unimi.it
‡avidyash@gmu.edu

the sampled ball is white. The process is repeated yielding the collection $\{(Y_{1,n}, Y_{2,n}); n \geq 1\}$. The quantities $\{D_{1,n}; n \geq 1\}$ and $\{D_{2,n}; n \geq 1\}$ are independent collections of independent and identically distributed (i.i.d.) non-negative random variables.

The urn model can be also described using its *replacement matrix* D_n, where $[D_{ij,n}]$ indicates the number of balls of color j that are replaced in the urn when a balls of color i is sampled. In the RRU model, since the off-diagonal elements are 0, we simplify the notation $D_{ii,n}$ to $D_{i,n}$. Hence, the RRU model is characterized by the *replacement matrix*

$$\mathbf{D}_n = \begin{bmatrix} D_{1,n} & 0 \\ 0 & D_{2,n} \end{bmatrix}. \tag{1.1}$$

Let $m_1 := \boldsymbol{E}[D_{1,n}]$ and $m_2 := \boldsymbol{E}[D_{2,n}]$.

Since the replacement matrix in (1.1) is diagonal, the RRU model is not a particular case of class of Generalized Pólya Urns (GPU) whose replacement matrix (or an almost sure limit of certain "conditional" replacement matrices) is assumed to be irreducible. For a review on the literature on GPUs, see for instance Athreya and Karlin (1968); Smythe (1996); Bai and Hu (1999, 2005); Zhang et al. (2006); Laruelle and Pagès (2013); Aletti and Ghiglietti (2017).

The asymptotic properties of the urn proportions in an RRU model were investigated by Durham et al. (1998) for binary reinforcements, and extended to the continuous case by Muliere et al. (2006); Aletti et al. (2009). Specifically, they established that

$$Z_n = \frac{Y_{1,n}}{Y_{1,n} + Y_{2,n}} \stackrel{a.s.}{\to} \begin{cases} 1 & \text{if } m_1 > m_2, \\ Z_\infty & \text{if } m_1 = m_2, \\ 0 & \text{if } m_1 < m_2, \end{cases} \tag{1.2}$$

where $\stackrel{a.s.}{\to}$ stands for almost sure convergence and Z_∞ is a random variable supported on $(0,1)$. The rate of convergence and the limit distribution of Z_n when $m_1 \neq m_2$ has been established in May and Flournoy (2009). For the case $m_1 = m_2$, the properties of the distribution of Z_∞ were studied in Durham et al. (1998); Aletti et al. (2009, 2012). Specifically, the distribution of Z_∞ when $D_{1,n}$ and $D_{2,n}$ are Bernoulli random variables with the same success probability, has been established in Durham et al. (1998). In the more general case where $D_{1,n}$ and $D_{2,n}$ have the same expectations, it has been proved in Aletti et al. (2012) that the distribution of Z_∞ is the unique continuous solution of a functional equation satisfying certain boundary conditions. Additionally, it is shown in Aletti et al. (2009) that $\boldsymbol{P}(Z_\infty = x) = 0$ for any $x \in [0,1]$. Denoting $\{(N_{1,n}, N_{2,n}); n \geq 1\}$ the number of balls of red and white colors sampled from the urn, one can deduce from (1.2) that $N_{1,n}/n$ converges to the same limit as Z_n.

Notice that for an RRU, the limit in (1.2) is always 1 or 0 in the case $m_1 \neq m_2$. This asymptotic behavior can be very attractive in applications such as clinical trials, where the response-adaptive designs based on an RRU model achieve the ethical goal of assigning most subjects to a better performing treatment (see Durham et al. (1998); Muliere et al. (2006)). However, from an inferential perspective, it is common to target a specific value $\rho \in (0,1)$ (see Hu and Rosenberger (2006) for applications in clinical trials). To perform clinical experiments with such a goal, a variant of RRU is needed. This was achieved in Aletti et al. (2013), where the modified randomly reinforced urn (MRRU) model was introduced. The MRRU model is an RRU with two fixed thresholds $0 < \rho_2 \leq \rho_1 < 1$, such that: (i) if a white ball is sampled and $Z_n < \rho_2$, no balls are replaced in urn, and (ii) if a red ball is sampled but $Z_n > \rho_1$, no balls are replaced in the urn. Hence, the replacement matrix (1.1) in this case becomes

$$\mathbf{D}_n = \begin{bmatrix} D_{1,n} \cdot \mathbb{1}_{\{Z_{n-1} \leq \rho_1\}} & 0 \\ 0 & D_{2,n} \cdot \mathbb{1}_{\{Z_{n-1} \geq \rho_2\}} \end{bmatrix}.$$

A more precise description of the MRRU model is provided in Section 2 Remark 2.1.

The strong consistency of Z_n in the case $m_1 \neq m_2$ was established in Aletti et al. (2013), i.e. they showed that

$$Z_n \stackrel{a.s.}{\to} \begin{cases} \rho_1 & \text{if } m_1 > m_2, \\ \rho_2 & \text{if } m_1 < m_2. \end{cases}$$

An efficient test based on the MRRU was implemented in Ghiglietti and Paganoni (2016), while a second order result for Z_n (again when $m_1 \neq m_2$), namely the asymptotic distribution of Z_n after appropriate centering and scaling, was derived in Ghiglietti and Paganoni (2014). We emphasize here that the rate of convergence in this case is not the usual $\sqrt{n}$ but n and the limit distribution is not Gaussian.

In applications, especially in clinical trials (see Hu and Rosenberger (2006)), ρ_1 and ρ_2 are unknown and depend on the parameters of the distributions of $D_{1,1}$ and $D_{2,1}$. Let $\mathcal{F}_{n-1}$ be the σ-algebra generated by the information up to time $n-1$ and let $\hat{\rho}_{1,n-1}$ and $\hat{\rho}_{2,n-1}$ be two random variables that are $\mathcal{F}_{n-1}$-measurable. Ghiglietti et al. (2017) proposed an adaptive randomly reinforced urn model that uses accruing information to construct random thresholds $\hat{\rho}_{1,n-1}$ and $\hat{\rho}_{2,n-1}$ which converge a.s. to specified targets ρ_1 and ρ_2. Thus, using the replacement matrix

$$\mathbf{D}_n = \begin{bmatrix} D_{1,n} \cdot \mathbb{1}_{\{Z_{n-1} \leq \hat{\rho}_{1,n-1}\}} & 0 \\ 0 & D_{2,n} \cdot \mathbb{1}_{\{Z_{n-1} \geq \hat{\rho}_{2,n-1}\}} \end{bmatrix},$$

an MRRU becomes an Adaptive Randomly Reinforced Urn (ARRU). It is worth mentioning here that the random thresholds $\hat{\rho}_{1,n-1}$ and $\hat{\rho}_{2,n-1}$ depend on the *adaptive* estimators of the parameters of the distributions of $D_{1,1}$ and $D_{2,1}$.

Ghiglietti et al. (2017) studied the asymptotic properties of an ARRU when $m_1 \neq m_2$ under various conditions on the rate of convergence of adaptive thresholds. Specifically, they established a strong consistency of (i) the proportion of sampled balls of each color and (ii) the urn proportions, under the assumption that the thresholds converge almost surely and that the limits of the thresholds are different from 0 and 1. Furthermore, they also establish the asymptotic normality for the number of sampled ball colors, under an exponential rate of convergence assumption on the adaptive thresholds and an additional condition that the thresholds are updated at exponential times. Additionally, they provided heuristics as to why the proportion of balls of each color in the urn (urn proportions) may not have a limiting Gaussian distribution, without further hypotheses.

In this manuscript, the first significant contribution concerns weak consistency results for the urn proportions when $m_1 \neq m_2$ under the assumption that the threshold sequence $\{\hat{\rho}_{i,n}, n \geq 1\}$ converges in probability to ρ_i, for $i = 1, 2$. The hypothesis that the thresholds converge only in probability (and not a.s.) brings in subtle challenges which necessitate understanding the dynamics of the ARRU model. More precisely, our proofs involve obtaining L_1 bounds on (i) the increments of the distance $\Delta_n = |Z_n - \rho_1|$, *viz.* $\Delta_{n+1} - \Delta_n$ and (ii) the increments at linearly increasing times $\Delta_{n+nc} - \Delta_n$, where $c > 0$. These results are then combined with a judicious choice of c and using comparison arguments with a specifically designed RRU model, weak consistency is established. The above results are presented in Section 4; specifically, Theorem 4.2, Theorem 4.5 , and Theorem 4.6. The proofs of these theorems rely on the estimates concerning the harmonic moments of the total number of balls in the urn. This result, of independent interest, is established in Theorem 4.1 where even the convergence of thresholds is not required.

The second significant contribution of this manuscript concerns the strong consistency of the urn proportions for all values of m_1 and m_2 under the assumption that the thresholds converge almost surely but without any restriction on their limiting values. As a consequence, we obtain the strong consistency of the proportion of sampled ball colors thus completing Corollary 2.1 in Ghiglietti et al. (2017) for the case $m_1 = m_2$. It is important to notice that in the case $m_1 = m_2$, the urn proportion converges to a proper random variable Z_∞ which is different from the case $m_1 \neq m_2$. As a consequence, we obtain the strong consistency of the proportion of sampled balls of both the colors, thus completing Theorem 2.1 in Ghiglietti et al. (2017) for the case $m_1 = m_2$.

The third significant contribution concerns second order results for the urn proportions and the proportion of sampled ball colors for the case $m_1 = m_2$. Specifically, we establish that, the quantities $(Z_n - Z_\infty)$ and $(n^{-1}N_{1,n} - Z_\infty)$ converge stably at the rate $\sqrt{n}$ to a distribution which is a continuous mixture of a centered Gaussian distribution and the distribution of Z_∞. The proof involves decomposing $(n^{-1}N_{1,n} - Z_\infty)$ into a sum of two terms, one involving comparison of $n^{-1}N_{1,n}$ with the cumulative proportion of red balls up to time n and the other involving the deviation of the cumulative proportion up to time n from Z_∞. The second term is then carefully investigated by invoking the Doob's decomposition theorem and some delicate estimates. In the process, we explicitly identify the variance of the conditional Gaussian distribution. This result also resolves a long-standing open problem in the well-investigated RRU model concerning the limiting distribution of the proportion of sampled ball colors for the case $m_1 = m_2$. Additionally, the results also settle the open problem concerning the urn proportions and proportion of sampled ball colors for the MRRU model when $m_1 = m_2$.

The rest of the paper is structured as follows: Section 2 contains the model, assumptions, and main results; Section 3 is concerned with preliminary estimates and results on the urn process. Sections 4 and 5 are concerned with the proofs of the consistency of the urn proportion and Section 6 is concerned with the proof of the limit distribution of the proportion of sampled balls of both the colors.

2 Model Assumptions, Notations, and Main Results

We begin by describing our model precisely. Let $\boldsymbol{\xi_1} = \{\xi_{1,n}; n \geq 1\}$ and $\boldsymbol{\xi_2} = \{\xi_{2,n}; n \geq 1\}$ be two sequences of i.i.d. random variables. Without loss of generality (wlog), assume that the support S of $\xi_{1,n}$ and $\xi_{2,n}$ to be the same. Additionally, let $\boldsymbol{U} = \{U_n; n \geq 1\}$ denote a sequence of i.i.d. uniform random variables in (0,1) independent of $\boldsymbol{\xi_1}$ and $\boldsymbol{\xi_2}$.

Consider an urn containing $y_{1,0} > 0$ red balls and $y_{2,0} > 0$ white balls, and define $y_0 = y_{1,0} + y_{2,0}$ and $z_0 = y_0^{-1} y_{1,0}$. We note here that $y_{1,0}$ and $y_{2,0}$ may not assume integer values. At time $n = 1$, a ball is drawn at random from the urn and its color is observed. Let the random variable X_1 be such that

$$X_1 = \begin{cases} 1 & \text{if the extracted ball is red,} \\ 0 & \text{if the extracted ball is white.} \end{cases}$$

Then one can express $X_1 = \mathbb{1}_{\{U_1 \leq z_0\}}$. Note that X_1 is a Bernoulli random variable with parameter z_0 and is independent of $\boldsymbol{\xi_1}$ and $\boldsymbol{\xi_2}$.

Let $\hat{\rho}_{1,0}$ and $\hat{\rho}_{2,0}$ be two constants in $[0,1]$ and $\hat{\rho}_{1,0} \geq \hat{\rho}_{2,0}$. Let $u : S \to [a,b]$, $0 < a \leq b < \infty$. (We mention here that it is possible to allow $a = 0$. The case $a > 0$ makes calculations transparent and hence we make this simplifying assumption throughout the manuscript. However, we add remarks on how to get rid of this assumption in various estimates that explicitly use $a > 0$). If $X_1 = 1$ and $z_0 \leq \hat{\rho}_{1,0}$, we return the extracted ball to the urn together with $D_{1,1} = u(\xi_{1,1})$ new red balls. While, if $X_1 = 0$ and $z_0 \geq \hat{\rho}_{2,0}$, we return it to the urn together with $D_{2,1} = u(\xi_{2,1})$ new white balls. If $X_1 = 1$ and $z_0 > \hat{\rho}_{1,0}$, or if $X_1 = 0$ and $z_0 < \hat{\rho}_{2,0}$, the urn composition is not modified. To ease notations, we set $w_{1,0} = \mathbb{1}_{\{z_0 \leq \hat{\rho}_{1,0}\}}$ and $w_{2,0} = \mathbb{1}_{\{z_0 \geq \hat{\rho}_{2,0}\}}$. Formally, the extracted ball is always replaced in the urn together with

$$X_1 D_{1,1} w_{1,0} + (1 - X_1) D_{2,1} w_{2,0}$$

new balls of the same color; now, the urn composition becomes

$$\begin{cases} Y_{1,1} = y_{1,0} + X_1 D_{1,1} w_{1,0} \\ \\ Y_{2,1} = y_{2,0} + (1 - X_1) D_{2,1} w_{2,0}. \end{cases}$$

Set $Y_1 = Y_{1,1} + Y_{2,1}$ and $Z_1 = Y_1^{-1} Y_{1,1}$. Now, by iterating the above procedure we define $\hat{\rho}_{1,1}$ and $\hat{\rho}_{2,1}$ to be two random variables, with $\hat{\rho}_{1,1}, \hat{\rho}_{2,1} \in [0,1]$ and $\hat{\rho}_{1,1} \geq \hat{\rho}_{2,1}$ a.s., measurable with respect to the σ-algebra $\mathcal{F}_1 \equiv \sigma(X_1, \tilde{\xi}_1)$, where $\tilde{\xi}_1 = X_1 \xi_{1,1} + (1 - X_1) \xi_{2,1}$. At the end of time n, let $(Y_{1,n}, Y_{2,n})$ denote the urn composition and $Z_n = \frac{Y_{1,n}}{Y_{1,n} + Y_{2,n}}$.

Now to define the model at time $(n+1)$, let $\hat{\rho}_{1,n}$ and $\hat{\rho}_{2,n}$ be two random variables with $\hat{\rho}_{1,n}, \hat{\rho}_{2,n} \in (0,1)$ and $\hat{\rho}_{1,n} \geq \hat{\rho}_{2,n}$ a.s., measurable with respect to a σ-algebra $\mathcal{F}_n$, where,

$$\mathcal{F}_n = \sigma\left(\mathcal{F}_{n-1}, X_n, \tilde{\xi}_n\right),$$

and $\tilde{\xi}_n = X_n \xi_{1,n} + (1 - X_n) \xi_{2,n}$. We will refer to $\hat{\rho}_{j,n}$ $j = 1, 2$ as threshold parameters.

At time $(n+1)$, a ball is extracted and let $X_{n+1} = 1$ if the ball is red and $X_{n+1} = 0$ otherwise. Equivalently, we can define $X_{n+1} = \mathbb{1}_{\{U_{n+1} \leq Z_n\}}$. Then, the ball is returned to the urn together with

$$X_{n+1} D_{1,n+1} W_{1,n} + (1 - X_{n+1}) D_{2,n+1} W_{2,n}$$

balls of the same color, where $D_{1,n+1} = u(\xi_{1,n+1})$, $D_{2,n+1} = u(\xi_{2,n+1})$, $W_{1,n} = \mathbb{1}_{\{Z_n \leq \hat{\rho}_{1,n}\}}$, $W_{2,n} = \mathbb{1}_{\{Z_n \geq \hat{\rho}_{2,n}\}}$ and $Z_{n+1} = Y_{1,n+1}/Y_{n+1}$ for any $n \geq 1$, where

$$\begin{cases} Y_{1,n+1} = y_{1,0} + \sum_{i=1}^{n+1} X_i D_{1,i} W_{1,i-1} \\ \\ Y_{2,n+1} = y_{2,0} + \sum_{i=1}^{n+1} (1 - X_i) D_{2,i} W_{2,i-1} \end{cases} \tag{2.1}$$

and $Y_{n+1} = Y_{1,n+1} + Y_{2,n+1}$. If $X_{n+1} = 1$ and $Z_n > \hat{\rho}_{1,n}$, i.e. $W_{1,n} = 0$, or if $X_{n+1} = 0$ and $Z_n < \hat{\rho}_{2,n}$, i.e. $W_{2,n} = 0$, the urn composition does not change at time $n+1$. Note that condition $\hat{\rho}_{1,n} \geq \hat{\rho}_{2,n}$ a.s., which implies $W_{1,n} + W_{2,n} \geq 1$, ensures that the urn composition can change with positive probability for any $n \geq 1$, since the replacement matrix is never a zero matrix. Since, conditionally on the σ-algebra $\mathcal{F}_n$, X_{n+1} is assumed to be independent of $\boldsymbol{\xi_1}$ and $\boldsymbol{\xi_2}$, it follows that X_{n+1} is Bernoulli distributed with parameter Z_n.

Remark 2.1. Setting $\hat{\rho}_{1,n} = 1$ and $\hat{\rho}_{2,n} = 0$ for any $n \geq 0$, which implies $W_{1,n} = W_{2,n} = 1$, equation (2.1) expresses the dynamics of an RRU, i.e.

$$\begin{cases} Y_{1,n+1} = y_{1,0} + \sum_{i=1}^{n+1} X_i D_{1,i} \\ \\ Y_{2,n+1} = y_{2,0} + \sum_{i=1}^{n+1} (1 - X_i) D_{2,i}. \end{cases}$$

Setting $\hat{\rho}_{1,n} = \rho_1$ and $\hat{\rho}_{2,n} = \rho_2$ for any $n \geq 0$, which implies $W_{1,n} = \mathbb{1}_{\{Z_n \leq \rho_1\}}$ and $W_{2,n} = \mathbb{1}_{\{Z_n \geq \rho_2\}}$, equation (2.1) expresses the dynamics of a MRRU, i.e.

$$\begin{cases} Y_{1,n+1} = y_{1,0} + \sum_{i=1}^{n+1} X_i D_{1,i} \mathbb{1}_{\{Z_n \leq \rho_1\}} \\ \\ Y_{2,n+1} = y_{2,0} + \sum_{i=1}^{n+1} (1 - X_i) D_{2,i} \mathbb{1}_{\{Z_n \geq \rho_2\}}. \end{cases}$$

Notice that in the MRRU ρ_1 and ρ_2 are known, while in the ARRU they are unknown and typically estimated using the data.

Before we state our results, we recall that $m_1 = \boldsymbol{E}[D_{1,1}]$ and $m_2 = \boldsymbol{E}[D_{2,1}]$.

2.1 Weak Consistency of the Urn Proportions

One of the main results of this paper is concerned with the consistency of the urn proportion Z_n when the random thresholds $\hat{\rho}_{1,n}$ and $\hat{\rho}_{2,n}$ converge in probability to some constants in $\rho_1, \rho_2 \in (0,1)$. To obtain this result, we need to assume that the sequences of the thresholds are bounded away from 0 and 1 with exponentially high probability, which is expressed in the following condition: there exist two constants $0 < \rho_{\min} \leq \rho_{\max} < 1$ and $0 < c_\rho < \infty$ such that

$$\boldsymbol{P}\left(\rho_{\min} \leq \hat{\rho}_{2,n} \leq \hat{\rho}_{1,n} \leq \rho_{\max}\right) \;\geq\; 1 - \exp\left(-c_\rho n\right) \tag{2.2}$$

for large n. The result is described below:

Theorem 2.2. *Assume* (2.2) *and there exist two constants $\rho_1, \rho_2 \in (0,1)$, with $\rho_1 \geq \rho_2$, such that*

$$\hat{\rho}_{1,n} \xrightarrow{p} \rho_1 \qquad\qquad \hat{\rho}_{2,n} \xrightarrow{p} \rho_2. \tag{2.3}$$

Then, when $m_1 \neq m_2$,

$$Z_n \xrightarrow{p} \begin{cases} \rho_1 & \text{if } m_1 > m_2, \\ \rho_2 & \text{if } m_1 < m_2. \end{cases} \tag{2.4}$$

We present the proof of Theorem 2.2 in Section 4.

Remark 2.3. The strong consistency of the urn proportion presented in Ghiglietti et al. (2017, Theorem 2.1), i.e. $\hat{\rho}_{1,n} \xrightarrow{a.s.} \rho_1$ implies $Z_n \xrightarrow{a.s.} \rho_1$, may suggest to prove Theorem 2.2 by applying subsequence arguments. Specifically, $Z_n \xrightarrow{p} \rho_1$ in (2.4) implies that for any subsequence $\{n_k; k \geq 1\}$ there exists a further subsequence $\{n_{k_j}; j \geq 1\}$ such that $Z_{n_{k_j}} \xrightarrow{a.s.} \rho_1$. Moreover, assumption $\hat{\rho}_{1,n} \xrightarrow{p} \rho_1$ in (2.3) guarantees the existence of $\{n_{k_j}; j \geq 1\}$ such that $\hat{\rho}_{1,n_{k_j}} \xrightarrow{a.s.} \rho_1$. Nevertheless, the strong consistency result in Ghiglietti et al. (2017, Theorem 2.1) does not prove that $Z_{n_{k_j}} \stackrel{a.s.}{\to} \rho_1$ with the only assumption that $\hat{\rho}_{1,n_{k_j}} \stackrel{a.s.}{\to} \rho_1$, because this condition does not provide any information on the behavior of $\hat{\rho}_{1,i}$ at times $i \notin \{n_{k_j}; j \geq 1\}$. Hence, the convergence of $\hat{\rho}_{1,n_{k_j}}$ would imply the convergence of $Z_{n_{k_j}}$ only if the urn composition was updated exclusively at times $\{n_{k_j}; j \geq 1\}$.

2.2 Strong Consistency of the Urn Proportions

The following theorem states the consistency of the urn proportion Z_n for all finite values of m_1 and m_2, when the random thresholds $\hat{\rho}_{1,n}$ and $\hat{\rho}_{2,n}$ converge with probability one.

Theorem 2.4. *Assume there exist two constants $\rho_1, \rho_2 \in [0,1]$, with $\rho_1 \geq \rho_2$, such that*

$$\hat{\rho}_{1,n} \stackrel{a.s.}{\to} \rho_1 \qquad\qquad \hat{\rho}_{2,n} \stackrel{a.s.}{\to} \rho_2. \tag{2.5}$$

Then,

$$Z_n \stackrel{a.s.}{\to} \begin{cases} \rho_1 & \text{if } m_1 > m_2, \\ Z_\infty & \text{if } m_1 = m_2, \\ \rho_2 & \text{if } m_1 < m_2, \end{cases} \tag{2.6}$$

where Z_∞ is a random variable such that $\boldsymbol{P}(Z_\infty \in [\rho_2, \rho_1]) = 1$.

Remark 2.5. As an immediate corollary of the above theorem, it can be seen that the proportion of sampled ball colors, $n^{-1}N_{1,n}$, converges to the same limit as in (2.6).

We present the proof of Theorem 2.4 in Section 5. When the limit of the urn proportion is different from 1 or 0, the following convergence result holds.

Lemma 2.6. *Assume* (2.5) *with $\rho_1 > \rho_2$. Then, on the set $\{\lim_{n\to\infty} Z_n \neq \{0,1\}\}$,*

$$\frac{Y_n}{n} \stackrel{a.s.}{\to} \min\{m_1, m_2\}.$$

The above lemma can be applied for the RRU model only when $m_1 = m_2$. For the case $m_1 \neq m_2$ in an RRU model, May and Flournoy (2009) showed that $\frac{Y_n}{n} \stackrel{a.s.}{\to} \max\{m_1; m_2\}$. In the case $m_1 = m_2$, we are able to establish that the limiting proportion Z_∞ has no point mass of positive probability within the open interval (ρ_2, ρ_1). This is stated in the following lemma.

Lemma 2.7. *Assume* (2.5) *with $\rho_1 > \rho_2$ and $m_1 = m_2$. Then, for any $x \in (\rho_2, \rho_1)$, we have $\boldsymbol{P}(Z_\infty = x) = 0$.*

Point masses of positive probability are possible at values ρ_1 and ρ_2.

2.3 Asymptotic Distribution of the Sampled Ball Colors

The second order asymptotic results concerning the proportion of sampled ball colors involve the concept of stable convergence (see Hall and Heyde (1980)). Formally, let $\{\mathcal{X}_n; n \geq 1\}$ be a random sequence on a probability space $(\Omega, \mathcal{F}, \boldsymbol{P})$; we say that $\mathcal{X}_n \stackrel{d}{\to} \mathcal{X}$ (stably) if, for every point x of continuity for the cumulative distribution function of $\mathcal{X}$ and for every event $E \in \mathcal{F}$,

$$\lim_{n\to\infty} \boldsymbol{P}\left(\mathcal{X}_n \leq x, E \right) = \boldsymbol{P}\left(\mathcal{X} \leq x, E \right).$$

We now present the asymptotic distribution for the proportion of sampled ball colors in an RRU model. Let us denote by $N_{1n} := \sum_{i=1}^n X_i$ and $N_{2n} := \sum_{i=1}^n (1 - X_i) = n - N_{1n}$ the number of red and white balls, respectively, sampled form the urn up to time n. Moreover, let $\sigma_1^2 := \boldsymbol{Var}[D_{1,1}]$ and $\sigma_2^2 := \boldsymbol{Var}[D_{2,1}]$. The result is stated in the following Theorem:

Theorem 2.8. *Consider an RRU model and assume $m_1 = m_2 = m$. Then,*

$$\sqrt{n}\left(\frac{N_{1n}}{n} - Z_\infty\right) \stackrel{d}{\to} \mathcal{N}(0, \Sigma), \qquad (\textit{stably})$$

where

$$\Sigma := \left(1 + \frac{2\bar{\Sigma}}{m^2}\right) Z_\infty(1 - Z_\infty), \qquad \bar{\Sigma} := (1 - Z_\infty)\sigma_1^2 + Z_\infty\sigma_2^2. \tag{2.7}$$

Remark 2.9. In the special case of binary reinforcements with the same mean, i.e. $D_{1,n} \sim Be(p)$ and $D_{2,n} \sim Be(p)$ with $p \in (0,1]$, Theorem 2.8 expresses a Central Limit Theorem with stable convergence for the RRU model studied in Durham et al. (1998), in which Σ in (2.7) reduces to $\big(1 + 2\big(\frac{1-p}{p}\big)\big) Z_\infty (1 - Z_\infty)$. Then, combining Theorem 2.8 with the exact distribution of Z_∞ established in Durham et al. (1998), it is possible to obtain an analytic expression of the asymptotic distribution of $\sqrt{n}\,\big(\frac{N_{1n}}{n} - Z_\infty\big)$ in this special case. A similar calculation also holds for more general binary schemes considered in Aletti et al. (2012, Section 6.2).

It is known that when $D_{1,n} = D_{2,n} = 1$ for any $n \geq 1$, the random variable Z_∞ is Beta-distributed with parameters $(y_{1,0}, y_{2,0})$ (see e.g. Athreya and Karlin (1968)). Furthermore, in this case Theorem 2.8 provides a CLT with stable convergence for the standard Pòlya's urn, in which $\Sigma = Z_\infty(1 - Z_\infty)$. Now, combining Theorem 2.8 with the Beta-distribution, it is possible to obtain an analytic expression of the asymptotic distribution of $\sqrt{n}\,\big(\frac{N_{1n}}{n} - Z_\infty\big)$ for the Pòlya urn case.

We now present the asymptotic distribution for the proportion of sampled ball colors in an ARRU model. This result can be derived using Theorem 2.8 on the set of trajectories that do not cross the thresholds $\hat{\rho}_{1,n}$ and $\hat{\rho}_{2,n}$ infinitely often, and hence $\{Z_\infty \neq \{\rho_2, \rho_1\}\}$. To this end, we introduce a sequence of random sets $\{A_n; n \geq 1\}$ such that $A_n \in \mathcal{F}_n$ and $A_n \subset A_{n+1}$ for any $n \geq 1$, and $\cup_{n \geq 1} A_n = (\rho_2, \rho_1)$. In particular, we fix $0 < \alpha < 1/2$ and we define A_n as follows:

$$A_n \;:=\; \big(\; \rho_2 + C Y_n^{-\alpha} \;,\; \rho_1 - C Y_n^{-\alpha} \;\big)\,, \tag{2.8}$$

where $0 < C < \infty$ is a positive constant. The choice of $\{A_n; n \geq 1\}$ in (2.8) allows us to apply the estimates of Lemma 3.4 in the proof of the limit distribution, in order to obtain the equivalence: $\{Z_n \in A_n, ev.\} = \{Z_\infty \in (\rho_2, \rho_1)\}$ a.s., where *ev.* stands for *eventually*, which means *for all but a finite number of terms.* The limit distribution for the ARRU model is expressed in the following result.

Theorem 2.10. *Assume* (2.5) *with* $\rho_1 > \rho_2$ *and* $m_1 = m_2 = m$. *Then,*

$$\underline{\lim}_n \{Z_n \in A_n\} \;=\; \overline{\lim}_n \{Z_n \in A_n\} \;=\; \{Z_\infty \in (\rho_2, \rho_1)\},$$

and, on the sequence of sets $(\{Z_n \in A_n\}, n \geq 1)$, *we have*

$$\sqrt{n}\left(\frac{N_{1n}}{n} - Z_\infty\right) \;\xrightarrow{d}\; \mathcal{N}(0, \Sigma), \qquad (stably)$$

where, as in (2.7),

$$\Sigma \;:=\; \left(1 + \frac{2\bar{\Sigma}}{m^2}\right) Z_\infty (1 - Z_\infty), \qquad \bar{\Sigma} \;:=\; (1 - Z_\infty)\sigma_1^2 + Z_\infty \sigma_2^2.$$

It is worth noticing that the limiting distribution obtained in Theorem 2.8 and Theorem 2.10 is not Gaussian but a mixture distribution.

As a corollary of the methods of proof of Theorem 2.8 and Theorem 2.10 one can obtain the asymptotic distribution of $\sqrt{n}(Z_n - Z_\infty)$. We state this result without proof.

Theorem 2.11. *Assume* (2.5) *with* $\rho_1 > \rho_2$ *and* $m_1 = m_2 = m$. *Then, conditionally on* $\mathcal{F}_n$, *on the sequence of sets* $(\{Z_n \in A_n\}, n \geq 1)$, *we have*

$$\sqrt{n}\,(Z_n - Z_\infty) \;\xrightarrow{d}\; \mathcal{N}(0, \Sigma_Z), \qquad (stably)$$

where

$$\Sigma_Z \;:=\; \left(1 + \frac{\bar{\Sigma}}{m^2}\right) Z_\infty (1 - Z_\infty), \qquad \bar{\Sigma} \;:=\; (1 - Z_\infty)\sigma_1^2 + Z_\infty \sigma_2^2.$$

3 Preliminary Results

In this section, we present some preliminary estimates that are required to understand the dynamics of the ARRU model and to prove the main results of the paper. Most of the proofs of the results gathered by the literature are omitted, since the original proofs hold for all values of m_1 and m_2.

We start by presenting some basic properties of the ARRU dynamics. Specifically, we provide a useful expression of the excepted increments $(Z_{n+1} - Z_n)$ conditioned on $\mathcal{F}_n$, which is required to prove the consistency result and in particular in the proofs of Theorem 4.2 in Section 4 and of Theorem 2.8 in Section 6. Moreover, we show that the number of balls of both the colors sampled from the urn, namely $N_{1,n}$ and $N_{2,n}$, and the total number of balls in the urn Y_n, increase to infinity almost surely. To do that, we establish a lower bound for the increments of the process Y_n. These results are established in Ghiglietti et al. (2017, Lemma 4.1 and 4.2), and we state the proof below for completeness.

Lemma 3.1. *We have the following results:*

(a) For any $n \geq 0$,

$$\boldsymbol{E}\left[Z_{n+1} - Z_n | \mathcal{F}_n\right] \; = \; Z_n(1 - Z_n)B_n,$$

where

$$B_n \; := \; \boldsymbol{E}\left[\frac{D_{1,n+1}W_{1,n}}{Y_n + D_{1,n+1}W_{1,n}} - \frac{D_{2,n+1}W_{2,n}}{Y_n + D_{2,n+1}W_{2,n}} | \mathcal{F}_n\right]; \tag{3.1}$$

(b) for any $n \geq 0$, *we have that*

$$\boldsymbol{E}\left[Y_{n+1} - Y_n | \mathcal{F}_n\right] \; \geq \; \min\{m_1, m_2\} \cdot \left(\frac{\min\{y_{1,0}; y_{2,0}\}}{y_0 + bn}\right);$$

(c)

$$Y_n \stackrel{a.s.}{\to} \infty;$$

(d)

$$\min\{N_{1,n}; N_{2,n}\} \stackrel{a.s.}{\to} \infty.$$

Proof. The proof of result (a) is based on a modification of the proof in Muliere et al. (2006, Theorem 2). First, note that, by definition

$$Z_{n+1} = X_{n+1} \frac{Y_{1,n} + D_{1,n+1}W_{1,n}}{Y_n + D_{1,n+1}W_{1,n}} + (1 - X_{n+1}) \frac{Y_{1,n}}{Y_n + D_{2,n+1}W_{2,n}}$$

and since X_{n+1} is conditionally on $\mathcal{F}_n$ independent of $D_{1,n+1}$ and $D_{2,n+1}$, we can get that

$$\begin{aligned}
\boldsymbol{E}[Z_{n+1}|\mathcal{F}_n] \quad &= Z_n \boldsymbol{E}\left[\frac{Y_{1,n} + D_{1,n+1}W_{1,n}}{Y_n + D_{1,n+1}W_{1,n}} | \mathcal{F}_n\right] + (1 - Z_n)\boldsymbol{E}\left[\frac{Y_{1,n}}{Y_n + D_{2,n+1}W_{2,n}} | \mathcal{F}_n\right] \\
&= Z_n \boldsymbol{E}\left[\frac{Y_{1,n} + D_{1,n+1}W_{1,n}}{Y_n + D_{1,n+1}W_{1,n}} + \frac{Y_{2,n}}{Y_n + D_{2,n+1}W_{2,n}} | \mathcal{F}_n\right]
\end{aligned}$$

Analogously, we have that

$$\boldsymbol{E}[1 - Z_{n+1}|\mathcal{F}_n] \; = \; (1 - Z_n)\boldsymbol{E}\left[\frac{Y_{2,n} + D_{2,n+1}W_{2,n}}{Y_n + D_{2,n+1}W_{2,n}} + \frac{Y_{1,n}}{Y_n + D_{1,n+1}W_{1,n}} | \mathcal{F}_n\right].$$

Therefore,

$$\begin{aligned}
&\boldsymbol{E}[Z_{n+1} - Z_n|\mathcal{F}_n] \; = \; \boldsymbol{E}[(1 - Z_n)Z_{n+1} - Z_n(1 - Z_{n+1})|\mathcal{F}_n] \\
&= Z_n(1 - Z_n)\boldsymbol{E}\left[\frac{Y_{1,n} + D_{1,n+1}W_{1,n}}{Y_n + D_{1,n+1}W_{1,n}} + \frac{Y_{2,n}}{Y_n + D_{2,n+1}W_{2,n}}\right. \\
&\quad \left. - \frac{Y_{2,n} + D_{2,n+1}W_{2,n}}{Y_n + D_{2,n+1}W_{2,n}} - \frac{Y_{1,n}}{Y_n + D_{1,n+1}W_{1,n}} | \mathcal{F}_n\right] \\
&= Z_n(1 - Z_n)\boldsymbol{E}\left[\frac{D_{1,n+1}W_{1,n}}{Y_n + D_{1,n+1}W_{1,n}} - \frac{D_{2,n+1}W_{2,n}}{Y_n + D_{2,n+1}W_{2,n}} | \mathcal{F}_n\right].
\end{aligned}$$

This concludes the proof of result (a).

Concerning the proof of result (b), first note that

$$Y_{n+1} - Y_n \; = \; X_{n+1}D_{1,n+1}W_{1,n} + (1 - X_{n+1})\,D_{2,n+1}W_{2,n}.$$

Since X_{n+1} and $D_{1,n+1}$ are conditionally independent given $\mathcal{F}_n$, and $W_{1,n}$ is $\mathcal{F}_n$-measurable, we have that

$$\begin{aligned}
\boldsymbol{E}\left[Y_{n+1} - Y_n|\mathcal{F}_n\right] \quad &= \left(m_1 Z_n W_{1,n} + m_2\,(1 - Z_n)\,W_{2,n}\right) \\
&\geq \; \min\{m_1, m_2\} \cdot \left(Z_n W_{1,n} + (1 - Z_n)\,W_{2,n}\right).
\end{aligned}$$

We recall that the variables $W_{1,n}$ and $W_{2,n}$ can only take the values 0 and 1, and by construction we have that $(W_{1,n} + W_{2,n}) \geq 1$ for any $n \geq 0$; then, we can give a further lower bound

$$\boldsymbol{E}\left[Y_{n+1} - Y_n | \mathcal{F}_n\right] \;\geq\; \min\{m_1, m_2\} \cdot \left(\min\left\{Z_n; 1 - Z_n\right\}\right).$$

Finally, the result follows by noting that

$$\min\left\{Z_n; 1 - Z_n\right\} \;=\; \frac{\min\left\{Y_{1,n}; Y_{2,n}\right\}}{Y_n} \;\geq\; \frac{\min\{y_{1,0}; y_{2,0}\}}{y_0 + bn}.$$

This concludes the proof of result (b).

We now focus on the proof of result (c). First, notice that $Y_n = \sum_{i=0}^{n-1}(Y_{i+1} - Y_i) + y_0$. Then, by Chen (1978, Theorem 1), it is sufficient to show that

$$\boldsymbol{P}\Big(\sum_{i=0}^{\infty} \boldsymbol{E}\left[Y_{i+1} - Y_i | \mathcal{F}_i\right] = \infty\Big) = 1.$$

To this end, we will now use the lower bound given by result (b), so obtaining

$$\sum_{i=0}^{n} \boldsymbol{E}\left[Y_{i+1} - Y_i | \mathcal{F}_i\right] \;\geq\; \min\{m_1, m_2\} \left(\sum_{i=0}^{n} \frac{\min\{y_{1,0}; y_{2,0}\}}{y_0 + bi}\right) \to \infty.$$

Hence, we have that $Y_n \overset{a.s.}{\to} \infty$.

Finally, we present the proof of result (d). We will show that $N_{1,n} \overset{a.s.}{\to} \infty$, since the proof for $N_{2,n}$ is analogous. Since $N_{1,n} = \sum_{i=1}^{n} X_i$, by Chen (1978, Theorem 1), it is sufficient to show that

$$\boldsymbol{P}\Big(\sum_{i=1}^{\infty} \boldsymbol{E}\left[X_i | \mathcal{F}_{i-1}\right] = \infty\Big) = 1.$$

Now,

$$\sum_{i=1}^{n} \boldsymbol{E}\left[X_i | \mathcal{F}_{i-1}\right] \;=\; \sum_{i=1}^{n} Z_{i-1} \;\geq\; \sum_{i=1}^{n} \frac{y_{1,0}}{y_0 + (i-1)\, b} \to \infty.$$

Hence, we have that $N_{1,n} \overset{a.s.}{\to} \infty$. $\square$

The following result is needed in the proof of Theorem 2.4 and it is taken from Aletti et al. (2013, Theorem 2.1). This result provides multiple equivalent ways to establish the almost sure convergence of a general real valued process in [0,1], that we will apply to the process $\{Z_n; n \geq 0\}$ of the urn proportion in an ARRU model. For this result, we let d (down) and u (up) be two real numbers such that $0 < d < u < 1$, and we consider two sequences of times $t_j(d,u)$ and $\tau_j(d,u)$ defined as follows: for each $j \geq 0$, $t_j(d,u)$ represents the time of the first up-cross of u after $\tau_{j-1}(d,u)$, and $\tau_j(d,u)$ represents the time of the first down-cross of d after t_j. Note that $t_j(d,u)$ and $\tau_j(d,u)$ are stopping times, since the events $\{t_j(d,u) = k\}$ and $\{\tau_j(d,u) = k\}$ depend on $\{Z_n; n \leq k\}$, which are measurable with respect to $\mathcal{G}_k = \sigma\left(Z_n : n \leq k\right)$.

Lemma 3.2 (Aletti et al. (2013, Theorem 2.1))**.** *Let $\tau_{-1}(d,u) = -1$ and define for every $j \geq 0$ two stopping times*

$$\begin{aligned} t_j(d,u) &= \begin{cases} \inf\{n > \tau_{j-1}(d,u) : Z_n > u\} & \text{if } \{n > \tau_j(d,u) : Z_n > u\} \neq \emptyset; \\ +\infty & \text{otherwise.} \end{cases} \\ \tau_j(d,u) &= \begin{cases} \inf\{n > t_j(d,u) : Z_n < d\} & \text{if } \{n > t_{j-1}(d,u) : Z_n < d\} \neq \emptyset; \\ +\infty & \text{otherwise.} \end{cases} \end{aligned} \tag{3.2}$$

Then, the following three events are a.s. equivalent

(a) Z_n converges a.s.;

(b) for any $0 < d < u < 1$,

$$\lim_{j \to \infty} \boldsymbol{P}\left(t_j(d,u) < \infty\right) = 0;$$

(c) for any $0 < d < u < 1$,

$$\sum_{j\geq 1} \boldsymbol{P}\left(t_{j+1}(d,u) = \infty | t_j(d,u) < \infty\right) = \infty;$$

using the convention that $\boldsymbol{P}\left(t_{j+1}(d,u) = \infty | t_j(d,u) < \infty\right) = 1$ *when* $\boldsymbol{P}\left(t_j(d,u) = \infty\right) = 1$.

We now present a lemma that provides lower bounds for the total number of balls in the urn at the times of up-crossings, Y_{t_j}. The lemma gets used in the proof of Theorem 2.4, where conditioning on a fixed number of up-crossing ensures to have at least a number of balls Y_n determined by the lower bounds of this lemma. This result has been taken by Aletti et al. (2013, Lemma 2.1) and the proof is omitted since the adaptive thresholds and the values of m_1 and m_2 do not play any role during up-crossings. Hence, the proof reported in Aletti et al. (2013, Lemma 2.1) carries over to our model, with D_n replaced by Y_n.

Lemma 3.3 (Aletti et al. (2013, Lemma 2.1))**.** *For any* $0 < d < u < 1$, *we have that*

$$Y_{t_j(d,u)} \geq \left(\frac{u\,(1-d)}{d\,(1-u)}\right) Y_{t_{j-1}(d,u)} \geq \ldots \geq \left(\frac{u\,(1-d)}{d\,(1-u)}\right)^j Y_{t_0(d,u)}.$$

The following lemma provides a uniform bound for the generalized Pólya urn with same reinforcement means, which is needed in the proof of Theorem 2.4.

Lemma 3.4 (Aletti et al. (2013, Lemma 3.2))**.** *Consider an RRU with* $m_1 = m_2$*. If* $Y_0 \geq 2b$, *then*

$$\boldsymbol{P}\left(\sup_{n\geq 1} |Z_n - Z_0| \geq h\right) \leq \frac{b}{Y_0}\left(\frac{4}{h^2} + \frac{2}{h}\right)$$

for every $h > 0$.

Finally, we present an auxiliary result that provides an upper bound on the increments of the urn process Z_n, by imposing a condition on the total number of balls in the urn Y_n.

Lemma 3.5 (Ghiglietti et al. (2017, Lemma 3.1))**.** *For any* $\epsilon \in (0,1)$, *we have that*

$$\left\{ Y_n > b\left(\frac{1-\epsilon}{\epsilon}\right) \right\} \quad \subseteq \quad \{ \, |Z_{n+1} - Z_n| < \epsilon \, \}.$$

4 Proof of Weak Consistency and Related Results

In this section, we prove the weak consistency for the urn proportion of the ARRU model, which is established in Theorem 2.2. This proof requires some probabilistic results concerning the ARRU model, which have been gathered in different subsections. The proof of the weak consistency based on these results is then provided in Subsection 4.4.

Let us start by describing the general structure of the proof. The weak consistency is proved by showing that the process $\{\Delta_n; n \geq 1\}$, defined as

$$\Delta_n := |\rho_1 - Z_n|, \quad \forall\, n \geq 0, \tag{4.1}$$

converges to zero in probability. To prove this, we want to exploit the fact that, unless Δ_n is arbitrarily close to zero, the conditional expected increments of Δ_n are negative. This result is obtained in Subsection 4.2 by studying the conditional expected increments of Z_n. Hence, to show that Δ_n is asymptotically close to zero, we need to investigate the expected increments of the process $\{\Delta_n; n \geq 1\}$. Since the increments of Δ_n are at the same order of Y_n^{-1}, we first determine how fast the total number of balls in the urn, Y_n, increases to infinity. This is addressed in Theorem 4.1, where we show that the total number of balls in the ARRU model increases linearly with the number of extractions from the urn. For this reason, the increments of Δ_n are of the order of n^{-1}; hence, we consider differences of Δ_n evaluated at linearly increasing times, i.e. $G(n,c) := (\Delta_{n+nc} - \Delta_n)$, such that the L_1 bounds obtained for such differences do not vanish as n goes to infinity. More specifically, we provide a negative upper bound for the expected differences $G(n,c)$, which is not negligible unless Δ_n is asymptotically close to zero. Formally, for any $\delta > 0$, we show that for some $0 < C < \infty$

$$\boldsymbol{E}\left[G(n, s_\delta)\right] \;\leq\; -C\boldsymbol{P}(Q(\delta,n)) \;+\; o(1), \tag{4.2}$$

where $0 < s_\delta < \infty$ is an appropriate constant and $Q(\delta, n) := \{\Delta_n > \delta\}$. To obtain (4.2), we prove that the expected differences $G(n, s_\delta)$ are: (i) negative for moderate values of Δ_n (see Theorem 4.5) and (ii) negligible for small values of Δ_n (see Theorem 4.6). These results are derived using comparison arguments with specific auxiliary urn models. Finally, in Subsection 4.4 we use (4.2) and other preliminary results to establish the weak consistency.

4.1 Harmonic Moments of Y_n

In this subsection, we establish that the total number of balls in the ARRU model increases linearly with the number of extractions from the urn. Moreover, this result ensures uniform bounds for the harmonic moments of the total number of balls.

Before presenting the main result, we introduce some notation. For any $0 < c \leq C < \infty$ and for all $n \geq 0$, let $F_n(c, C) \in \mathcal{F}_n$ be the set defined as follows

$$F_n(c, C) := \{y_0 + cn \leq Y_n \leq y_0 + Cn\}.$$

Here, we show that, for some c and C, $\boldsymbol{P}(F_n(c, C))$ converges to one with a sub-exponential rate (a sequence of constants a_n is said to converge at a sub-exponential rate to 1 (resp. 0) if $a_n \geq 1-\exp(-cn^p)$ (resp. $a_n \leq \exp(-cn^p)$) for some $0 < p < 1$), which implies $\boldsymbol{P}(F_n^c(c, C), i.o.) = 0$. Moreover, this result provides uniform bounds for the moments of n/Y_n. The following theorem makes this result precise.

Theorem 4.1. *Under assumption* (2.2)*, there exist two constants* $0 < z_{\min} < \rho_{\min}$ *and* $\rho_{\max} < z_{\max} < 1$ *such that, for some* $\epsilon_z > 0$*, depending on* z_{min} *and* z_{max}*, we have for large* n *that*

$$\boldsymbol{P}\left(z_{\min} \leq Z_n \leq z_{\max}\right) \geq 1 - \exp(-\epsilon_z \sqrt{n}). \tag{4.3}$$

Moreover, there exist two constants $0 < c_1 < C_1 < \infty$ *such that, for some* $\epsilon_y > 0$*, depending on* y_0*, we have for large* n *that*

$$F_n(c_1, C_1) \equiv \boldsymbol{P}(y_0 + c_1 n \leq Y_n \leq y_0 + C_1 n) \;\geq\; 1 - \exp(-\epsilon_y \sqrt{n}). \tag{4.4}$$

As a consequence, for any $j \geq 1$

$$\sup_{n\geq 0}\left\{\boldsymbol{E}\left[\left(\frac{n}{Y_n}\right)^j\right]\right\} \;<\; \infty. \tag{4.5}$$

Proof. This proof has a general structure similar to the proof in Ghiglietti et al. (2017, Theorem 3.1).

Let $c_{\min} := \min\{\rho_{\min}; 1 - \rho_{\max}\}$ and $p_0 := (\frac{y_0}{y_0+b})c_{\min} < c_{\min}$. Fix an arbitrary $0 < c < p_0$ and consider the following sets

$$\begin{aligned} A_{d,n} &:= \bigcup_{n/2\leq i\leq n} \{Z_i < c\}\,, \\ A_{c,n} &:= \bigcap_{n/2\leq i\leq n} \{c < Z_i < 1 - c\}\,, \\ A_{u,n} &:= \bigcup_{n/2\leq i\leq n} \{Z_i > 1 - c\}\,. \end{aligned}$$

Note that by choosing $0 < z_{\min} \leq c$ and $1 - c \leq z_{\max} < 1$ we have $A_{c,n} \subset \{z_{\min} \leq Z_n \leq z_{\max}\}$. Then, since $A_{d,n} \cup A_{c,n} \cup A_{u,n} = \Omega$, result (4.3) can be established by proving that $\boldsymbol{P}(A_{d,n}) + \boldsymbol{P}(A_{u,n}) \leq \exp(-\epsilon_z \sqrt{n})$ for large n. We will focus on the set $A_{d,n}$, since the arguments to deal with $A_{u,n}$ are analogous. First, define $\epsilon > 0$ and, for any $n \geq 1$, the following sets:

$$\begin{aligned} A_{1,n} &:= \left\{ \sup_{i\geq\epsilon\sqrt{n}} \{\hat{\rho}_{1,i}\} > 1 - c_{\min} \right\}, \\ A_{2,n} &:= \left\{ \inf_{i\geq\epsilon\sqrt{n}} \{\hat{\rho}_{2,i}\} < c_{\min} \right\}, \\ A_{3,n} &:= \left\{ \inf_{i\geq\epsilon\sqrt{n}} \{\min\{1 - \hat{\rho}_{1,i}; \hat{\rho}_{2,i}\}\} \geq c_{\min} \right\}. \end{aligned}$$

Note that $A_{1,n} \cup A_{2,n} \cup A_{3,n} = \Omega$, and hence we have that

$$\boldsymbol{P}(A_{d,n}) \;\leq\; \boldsymbol{P}\left(A_{1,n}\right) \;+\; \boldsymbol{P}\left(A_{2,n}\right) \;+\; \boldsymbol{P}\left(A_{3,n} \cap A_{d,n}\right).$$

First, we prove that $\boldsymbol{P}(A_{1,n})$ and $\boldsymbol{P}(A_{2,n})$ converge to zero with a sub-exponential rate. Consider the term $\boldsymbol{P}(A_{1,n})$. From the definition of $A_{1,n}$, we obtain

$$\boldsymbol{P}(A_{1,n}) = \boldsymbol{P}\left(\bigcup_{i\geq\epsilon\sqrt{n}}\{\hat{\rho}_{1,i} > 1-c_{\min}\}\right) \leq \sum_{i\geq\epsilon\sqrt{n}} \boldsymbol{P}(\hat{\rho}_{1,i} > 1-c_{\min}).$$

Since, $(1-c_{\min}) \geq \rho_{\max}$, from (2.2) we have that for large i

$$\boldsymbol{P}(\hat{\rho}_{1,i} > 1-c_{\min}) \leq \exp(-c_\rho i),$$

with $0 < c_\rho < \infty$. Hence, we have that

$$\begin{aligned}\boldsymbol{P}(A_{1,n}) &\leq \sum_{i\geq\epsilon\sqrt{n}} \boldsymbol{P}(\hat{\rho}_{1,i} > 1-c_{\min}) \\ &\leq \sum_{i\geq\epsilon\sqrt{n}} \exp(-c_\rho i) \\ &\leq \exp\left(-\epsilon_z\sqrt{n}\right),\end{aligned}$$

for some constat $\epsilon_z > 0$. Similar arguments can be applied to prove $\boldsymbol{P}(A_{2,n}) \to 0$ with a sub-exponential rate.

Finally, we need to show that $\boldsymbol{P}(A_{3,n} \cap A_{d,n})$ converges to zero with a sub-exponential rate. First, define the set $\tilde{A}_{d,n}$ as follows:

$$\tilde{A}_{d,n} := \bigcap_{\epsilon\sqrt{n}\leq i\leq n/2} \{Z_i < c\}.$$

We now show that, since $0 < c < p_0 = c_{\min}(\frac{y_0}{b+y_0})$, it follows that on the set $A_{3,n}$ we have $\{Z_i \geq c\} \subset \{Z_{i+1} \geq c\}$ for any $i \geq \epsilon\sqrt{n}$, and hence

$$(A_{3,n} \cap A_{d,n}) \subset \left(A_{3,n} \cap \tilde{A}_{d,n}\right), \tag{4.6}$$

for any $n \geq 1$. First, note that on the set $A_{3,n}$, $\{\hat{\rho}_{2,i} \geq c_{\min}\}$ for any $i \geq \epsilon\sqrt{n}$. Hence for any $0 < c < p_0 < c_{\min}$, since $\{Z_i \geq c\} = \{c \leq Z_i < c_{\min}\} \cup \{Z_i \geq c_{\min}\}$, we have

(1) if $\{c \leq Z_i < c_{\min}\}$ we have $W_{2,i} = 0$, which implies $Z_{i+1} \geq Z_i$ and so $Z_{i+1} \geq c$;

(2) if $\{Z_i \geq c_{\min}\}$, the event $\{Z_{i+1} \leq Z_i\}$ is possible but using $X_{i+1} \geq 0$, $W_{2,i} \leq 1$, $D_{2,i+1} \leq b$ and $Y_i \geq y_0$ we obtain

$$Z_{i+1} = \frac{Z_iY_i + X_{i+1}D_{1,i+1}W_{1,i}}{Y_i + X_{i+1}D_{1,i+1}W_{1,i} + (1-X_{i+1})D_{2,i+1}W_{2,i}} \geq \frac{Z_iy_0}{y_0+b} \geq c_{\min}\frac{y_0}{y_0+b} = p_0 > c.$$

This guarantees that (4.6) holds for any $n \geq 1$.

We now show that $\boldsymbol{P}(A_{3,n} \cap \tilde{A}_{d,n})$ converges to zero with a sub-exponential rate. To this end, first note that on the set $A_{3,n}$, we have $\hat{\rho}_{2,i} \geq c_{\min}$ for any $i = \epsilon\sqrt{n}, .., n/2$; moreover, on the set $\tilde{A}_{d,n}$, we have $Z_i < p_0$ for any $i = \epsilon\sqrt{n}, .., n/2$. These considerations imply that

$$W_{2,i} = 0 \text{ and } W_{1,i} = 1 \text{ for any } i = \epsilon\sqrt{n}, .., n/2, \text{ on the set } A_{3,n} \cap \tilde{A}_{d,n}. \tag{4.7}$$

Hence, we can write

$$Z_{n/2} = \frac{Y_{1,\epsilon\sqrt{n}} + \sum_{i=\epsilon\sqrt{n}}^{n/2} X_iD_{1,i}}{Y_{\epsilon\sqrt{n}} + \sum_{i=\epsilon\sqrt{n}}^{n/2} X_iD_{1,i}} \geq \frac{a\sum_{i=\epsilon\sqrt{n}}^{n/2} X_i}{(y_0 + b\epsilon\sqrt{n}) + a\sum_{i=\epsilon\sqrt{n}}^{n/2} X_i}, \tag{4.8}$$

where the inequality follows since $0 \leq Y_{1,\epsilon\sqrt{n}} < Y_{\epsilon\sqrt{n}} \leq y_0 + b\epsilon\sqrt{n}$, $D_{1,i} \geq a$ a.s. for any $i \geq 1$ and the function $\frac{c+x}{C+x}$ is increasing for $x > 0$ and $c < C$. Now, define for any $n \geq 1$ the set $A_{4,n}$ as follows:

$$A_{4,n} := \left\{ \sum_{i=\epsilon\sqrt{n}}^{n/2} X_i > \frac{p_0}{a(1-p_0)}\left(y_0 + b\epsilon\sqrt{n}\right) \right\},$$

and consider the set $A_{3,n} \cap \tilde{A}_{d,n} \cap A_{4,n}$. On the set $A_{3,n} \cap \tilde{A}_{d,n}$ we can use the definition of $A_{4,n}$ in (4.8), so obtaining

$$\left(A_{3,n} \cap \tilde{A}_{d,n} \cap A_{4,n}\right) \subset \left(\{Z_{n/2} > p_0\} \cap \tilde{A}_{d,n}\right).$$

However, $\{Z_{n/2} > p_0\} \cap \tilde{A}_{d,n} = \emptyset$. Hence, $\boldsymbol{P}(A_{3,n} \cap \tilde{A}_{d,n} \cap A_{4,n}) = 0$ and it is sufficient to show that $\boldsymbol{P}\big(A_{3,n} \cap \tilde{A}_{d,n} \cap A_{4,n}^C\big)$ converges to zero with a sub-exponential rate.

To this end, by (4.7), note that on the set $A_{3,n} \cap \tilde{A}_{d,n}$ we have $Z_{i+1} \geq Z_i$ for any $i = \epsilon\sqrt{n}, .., n/2$. Hence, on the set $A_{3,n} \cap \tilde{A}_{d,n}$, $\{X_i, i = \epsilon\sqrt{n}, .., n/2\}$ are conditionally Bernoulli with parameter $p_i \geq Z_{\epsilon\sqrt{n}}$ a.s. Now, let us denote with $\{\varrho_{i,n}; i = 1, .., n/2 - \epsilon\sqrt{n}\}$ a sequence of i.i.d. Bernoulli random variable with parameter $z_{0,n}$, defined as

$$z_{0,n} \ := \ \frac{y_{1,0}}{y_0 + b\epsilon\sqrt{n}} \ \leq \ Z_{\epsilon\sqrt{n}} \quad a.s.;$$

it follows that

$$\boldsymbol{P}\left(A_{3,n} \cap \tilde{A}_{d,n} \cap A_{4,n}^C\right) \leq \boldsymbol{P}\left(\sum_{i=1}^{n/2-\epsilon\sqrt{n}} \varrho_{i,n} \ \leq \ \frac{p_0}{a(1-p_0)}\left(y_0 + b\epsilon\sqrt{n}\right)\right). \tag{4.9}$$

Finally, we use the following Chernoff's upper bound for i.i.d. random variables in $[0,1]$ (see Dembo and Zeitouni (1998))

$$\boldsymbol{P}\left(S_n \leq c_0 \cdot \boldsymbol{E}[S_n]\right) \ \leq \ \exp\left(-\frac{(1-c_0)^2}{2} \cdot \boldsymbol{E}[S_n]\right), \tag{4.10}$$

with $c_0 \in (0,1)$ and $S_n = \sum_i^n X_i$. In our case, we have that RHS of (4.9) can be written as $\boldsymbol{P}\left(S_n \leq c_n \cdot \boldsymbol{E}[S_n]\right)$, where $S_n = \sum_{i=1}^{n/2-\epsilon\sqrt{n}} \varrho_{i,n}$,

$$\boldsymbol{E}\left[S_n\right] = \left(\frac{n}{2} - \epsilon\sqrt{n}\right)\frac{y_{1,0}}{(y_0 + b\epsilon\sqrt{n})} \quad \text{and} \quad c_n = \frac{p_0}{a(1-p_0)}\frac{(y_0 + b\epsilon\sqrt{n})^2}{y_{1,0}(n/2 - \epsilon\sqrt{n})};$$

since $\epsilon > 0$ can be chosen arbitrary small, we can define an integer $n_0 \geq 1$ and a constant $c_0 \in (0,1)$ such that $c_n < c_0$ for any $n \geq n_0$, so that

$$\boldsymbol{P}\left(S_n \leq c_n \cdot \boldsymbol{E}[S_n]\right) \ \leq \ \boldsymbol{P}\left(S_n \leq c_0 \cdot \boldsymbol{E}[S_n]\right).$$

Hence, by using (4.10), for any $n \geq n_0$ we have that

$$\boldsymbol{P}\left(A_{3,n} \cap A_{4,n}^C\right) \ \leq \ \exp\left(-\frac{(1-c_0)^2}{2} \cdot \boldsymbol{E}[S_n]\right),$$

which converges to zero with a sub-exponential rate since

$$\boldsymbol{E}[S_n] \ = \ \frac{y_{1,0}(n/2 - \epsilon\sqrt{n})}{y_0 + b\epsilon\sqrt{n}} \ \sim \ \frac{n}{\sqrt{n}} \ = \ \sqrt{n}.$$

This concludes the proof of (4.3).

Now, we prove (4.4). Since the reinforcements are a.s. bounded, i.e. $|D_{j,n}| < b$ for any $n \geq 1$ and $j = 1, 2$, we trivially have that $\boldsymbol{P}\left(Y_n \geq y_0 + nb\right) = 0$. Thus, we will show that $\boldsymbol{P}\left(Y_n - y_0 \leq c_1 n\right)$ converges to zero with a sub-exponential rate. Moreover, since in the proof of (4.3) we established that for any $0 < c < p_0$ there exists ϵ_z such that $\boldsymbol{P}\left(A_{c,n}\right) \geq 1 - \exp(-\epsilon_z\sqrt{n})$, then to prove (4.4) we can focus on the probability $\boldsymbol{P}\left(\{Y_n - y_0 \leq c_1 n\} \cap A_{c,n}\right)$.

First, consider the following relation on the increments of the total number of balls

$$Y_i - Y_{i-1} = D_{1,i}X_iW_{1,i-1} + D_{2,i}\left(1 - X_i\right)W_{2,i-1} \geq a\left[X_iW_{1,i-1} + \left(1 - X_i\right)W_{2,i-1}\right]$$

Then, note that, on the set $A_{c,n}$, the random variables

$$X_iW_{1,i-1} + \left(1 - X_i\right)W_{2,i-1}, \quad i = n/2, .., n$$

are, conditionally on the σ-algebra $\mathcal{F}_{i-1}$, Bernoulli with parameter greater than or equal to c. Hence, if we introduce $\{B_i; i \geq 1\}$ a sequence of i.i.d. Bernoulli random variables with parameter c, using that Y_n is increasing we have

$$\begin{aligned}
\boldsymbol{P}\left(\{Y_n - y_0 \leq c_1 n\} \cap A_{c,n}\right) \ &\leq \boldsymbol{P}\left(\{Y_n - Y_{n/2} \leq c_1 n\} \cap A_{c,n}\right) \\
&\leq \boldsymbol{P}\left(\left\{a\sum_{i=n/2}^{n} B_i \leq c_1 n\right\} \cap A_{c,n}\right) \\
&\leq \boldsymbol{P}\left(\sum_{i=n/2}^{n} B_i \leq \frac{c_1}{a} n\right).
\end{aligned} \tag{4.11}$$

Now, we want to use the Chernoff's bound for i.i.d. random variables in $[0,1]$ expressed in (4.10), with $S_n = \sum_{i=n/2}^{n} B_i$. In our case, we have $\boldsymbol{E}[S_n] = nc/2$ and so $c_0 = 2c_1/(ac)$. Hence, by choosing c_1 small enough we can obtain $c_0 < 1$ which let us apply Chernoff's bound. This implies (4.4).

Finally, we get the harmonic moments as follows

$$\begin{aligned}\boldsymbol{E}\left[\left(\frac{n}{Y_n}\right)^j\right] &= \boldsymbol{E}\left[\left(\frac{n}{Y_n}\right)^j \mathbb{1}_{F_n(c_1,C_1)}\right] + \boldsymbol{E}\left[\left(\frac{n}{Y_n}\right)^j \mathbb{1}_{F_n^c(c_1,C_1)}\right] \\ &\le \boldsymbol{E}\left[\left(\frac{n}{y_0 + c_1 n}\right)^j \mathbb{1}_{F_n(c_1,C_1)}\right] + \left(\frac{n}{y_0}\right)^j \boldsymbol{E}\left[\mathbb{1}_{F_n^c(c_1,C_1)}\right] \\ &\le c_1^{-j} + y_0^{-j} n^j \exp(-\epsilon_y \sqrt{n}). \quad \square\end{aligned}$$

We notice here that the above proof also works for the case $a = 0$ by directly working with $\sum_{i=1}^{n} X_i D_{1,i}$ and using general large deviation estimates in (4.9) and (4.11).

4.2 L_1 Bound for the Increments of Δ_n

To ease notation in the rest of paper, we will refer to $F_n(c_1, C_1)$ as

$$F_n := \{y_0 + c_1 n \le Y_n \le y_0 + C_1 n\}, \tag{4.12}$$

where $0 < c_1 < C_1 < \infty$ are the constants determined in Theorem 4.1 to obtain (4.4). Also, for any $\varepsilon > 0$, let $R(\varepsilon, n) := \{|\hat{\rho}_{1,n} - \rho_1| < \varepsilon\}$ and $Q(\varepsilon, n) := \{\Delta_n > \varepsilon\}$, where we recall from (4.1) that $\Delta_n = |\rho_1 - Z_n|$. The following result provides an upper bound on the increments of Δ_n.

Theorem 4.2. *Let $m_1 > m_2$ and assume (2.2) and (2.3). For any $\varepsilon > 0$, there exists $0 < c_2 < \infty$ and a sequence of random variables $\{\psi_n; n \ge 0\}$ with $\boldsymbol{E}[|\psi_n|] = o(n^{-1})$, such that*

$$\boldsymbol{E}\left[G(n, n^{-1})\mathbb{1}_{Q(\varepsilon,n)}|\mathcal{F}_n\right] \;\le\; -n^{-1} \cdot c_2 \mathbb{1}_{Q(\varepsilon,n)} \;+\; \psi_n, \tag{4.13}$$

where we recall $G(n, n^{-1}) = (\Delta_{n+1} - \Delta_n)$.

The behavior and the sign of the excepted increments of the urn proportion $G(n, n^{-1})$ required to prove Theorem 4.2 depend on the position of Z_n respect to ρ_1. For this reason, we study separately the cases when Z_n is above or below ρ_1. Formally, we define

$$Q^-(\varepsilon, n) := \{Z_n < \rho_1 - \varepsilon\}, \qquad Q^+(\varepsilon, n) := \{Z_n > \rho_1 + \varepsilon\}, \tag{4.14}$$

so that $Q(\varepsilon, n) = Q^+(\varepsilon, n) \cup Q^-(\varepsilon, n)$. Specifically, we present Lemma 4.3 and Lemma 4.4 that provide bounds for the expected increments $G(n, n^{-1})$ on the sets $Q^-(\varepsilon, n)$ and $Q^+(\varepsilon, n)$, respectively. The proof of Theorem 4.2 is presented after the proofs of Lemma 4.3 and Lemma 4.4.

Lemma 4.3. *Let $A_n \in \mathcal{F}_n$ be such that $A_n \subset Q^-(\varepsilon, n)$. Then, we have that*

$$\boldsymbol{E}\left[(Z_{n+1} - Z_n)\,\mathbb{1}_{A_n}\right] \;\ge\; n^{-1} \cdot c_2 \boldsymbol{P}(A_n) \;-\; o(n^{-1}). \tag{4.15}$$

Proof. Let $I_n := \boldsymbol{E}\left[(Z_{n+1} - Z_n)\,\mathbb{1}_{A_n}\right]$ and, since $A_n \in \mathcal{F}_n$, we can use result (a) of Lemma 3.1 obtaining

$$I_n \;=\; \boldsymbol{E}\left[\boldsymbol{E}\left[Z_{n+1} - Z_n|\mathcal{F}_n\right]\mathbb{1}_{A_n}\right] \;=\; \boldsymbol{E}\left[Z_n(1 - Z_n)B_n \mathbb{1}_{A_n}\right], \tag{4.16}$$

where we recall that B_n is defined in (3.1) as follows

$$B_n \;:=\; \boldsymbol{E}\left[\frac{D_{1,n+1}W_{1,n}}{Y_n + D_{1,n+1}W_{1,n}} - \frac{D_{2,n+1}W_{2,n}}{Y_n + D_{2,n+1}W_{2,n}}|\mathcal{F}_n\right].$$

Now, note the following relation

$$\{Z_n \le \hat{\rho}_{1,n}\} \supset \left(Q^-(\varepsilon, n) \cap R(\varepsilon, n)\right)$$

where $R(\varepsilon, n) = \{|\hat{\rho}_{1,n} - \rho_1| < \varepsilon\}$. Since $A_n \subset Q^-(\varepsilon, n)$, on the set A_n the previous relation becomes $\{Z_n \leq \hat{\rho}_{1,n}\} \supset R(\varepsilon, n)$, which implies $W_{1,n} \geq \mathbb{1}_{R(\varepsilon,n)}$. Combining this argument with $W_{2,n} \leq 1$, we obtain on the set A_n the following inequality

$$B_n \;\geq\; \boldsymbol{E}\left[\left(\frac{D_{1,n+1}\mathbb{1}_{R(\varepsilon,n)}}{Y_n + D_{1,n+1}\mathbb{1}_{R(\varepsilon,n)}} - \frac{D_{2,n+1}}{Y_n + D_{2,n+1}}\right)|\mathcal{F}_n\right].$$

Then, by using $D_{2,n+1} \geq 0$ and $D_{1,n+1}\mathbb{1}_{R(\varepsilon,n)} \leq b$ a.s., we obtain that, on the set A_n,

$$B_n \;\geq\; \boldsymbol{E}\left[\left(\frac{D_{1,n+1}\mathbb{1}_{R(\varepsilon,n)}}{Y_n + b} - \frac{D_{2,n+1}}{Y_n}\right)|\mathcal{F}_n\right] \;=\; E_{1n} \;-\; E_{2n},$$

where

$$E_{1n} := \frac{m_1\mathbb{1}_{R(\varepsilon,n)} - m_2}{Y_n + b}, \text{ and } E_{2n} := \frac{m_2 b}{Y_n(Y_n + b)}.$$

First, note that

$$\boldsymbol{E}\left[Z_n(1-Z_n)E_{2n}\mathbb{1}_{A_n}\right] \;\leq\; \boldsymbol{E}\left[E_{2n}\right] \;\leq\; m_2 b \sup_{k\geq 1}\boldsymbol{E}\left[\left(\frac{k}{Y_k}\right)^2\right] n^{-2}.$$

Now, using (4.5) it follows that

$$\boldsymbol{E}\left[Z_n(1-Z_n)E_{2n}\mathbb{1}_{A_n}\right] \;=\; O(n^{-2}).$$

Thus, from (4.16) we have

$$I_n \;\geq\; \boldsymbol{E}\left[Z_n(1-Z_n)E_{1n}\mathbb{1}_{A_n}\right] - o(n^{-1}). \tag{4.17}$$

Now, consider the set F_n defined in (4.12) as

$$F_n \;=\; \{\; c_1 n \leq Y_n - y_0 \leq C_1 n \;\},$$

where we recall that, by (4.4) in Theorem 4.1, $\boldsymbol{P}(F_n^c) \leq \exp\left(-\epsilon_y\sqrt{n}\right)$. Moreover, let $\mathbb{1}_{A_n} = J_{1n} + J_{2n}$, where $J_{1n} := \mathbb{1}_{A_n \cap F_n}$ and $J_{2n} := \mathbb{1}_{A_n \cap F_n^c}$. Thus, concerning J_{2n} we have that

$$|\boldsymbol{E}\left[Z_n(1-Z_n)E_{1n}J_{2n}\right]| \leq \max_{n\geq 0}\{|E_{1n}|\}\boldsymbol{P}\left(F_n^c\right) = o(n^{-1}),$$

since $\max_{n\geq 0}\{|E_{1n}|\} \leq b/y_0$ a.s. Thus, returning to (4.17) we have that

$$I_n \;\geq\; \boldsymbol{E}\left[Z_n(1-Z_n)E_{1n}J_{1n}\right] - o(n^{-1}). \tag{4.18}$$

Now, consider the further decomposition $J_{1n} = J_{11n} + J_{12n}$, where $J_{11n} := \mathbb{1}_{A_n \cap F_n \cap \{E_{1n} \geq 0\}}$ and $J_{12n} := \mathbb{1}_{A_n \cap F_n \cap \{E_{1n} < 0\}}$. Thus, concerning J_{12n} we have that

$$\boldsymbol{E}\left[Z_n(1-Z_n)E_{1n}J_{12n}\right] \;\geq\; -\left(\frac{m_2}{y_0 + c_1(n+1)}\right)\boldsymbol{P}\left(A_n \cap \{E_{1n} < 0\}\right);$$

moreover, since $\boldsymbol{P}(Z_n < z_{\min})$ and $\boldsymbol{P}(Z_n > z_{\max})$ converge to zero with a sub-exponential rate by (4.3), in Theorem 4.1, we obtain

$$\boldsymbol{E}\left[Z_n(1-Z_n)E_{1n}J_{11n}\right] \;\geq\; \left(\frac{z_{\min}\left(1 - z_{\max}\right)\left(m_1 - m_2\right)}{y_0 + C_1(n+1)}\right)\boldsymbol{P}\left(A_n \cap \{E_{1n} > 0\}\right) \;-\; o(n^{-1})$$

Therefore, from (4.18) we have

$$I_n \;\geq\; n^{-1}c_2\boldsymbol{P}\left(A_n\right) \;-\; O(n^{-1})\boldsymbol{P}\left(E_{1n} < 0\right) \;-\; o(n^{-1}),$$

where $0 < c_2 < \infty$ is an appropriate constant. Hence, since from $m_1 > m_2$ we have $\{E_{1n} < 0\} \equiv R^c(\varepsilon, n)$, result (4.15) is obtained by establishing $\boldsymbol{P}\left(E_{1n} < 0\right) \to 0$. To this end, note that

$$\boldsymbol{P}\left(E_{1n} < 0\right) \;=\; 1 - \boldsymbol{P}\left(R(\varepsilon, n)\right) \;\to\; 0,$$

where $\boldsymbol{P}\left(R(\varepsilon, n)\right) \to 1$ follows from $\hat{\rho}_1 \xrightarrow{p} \rho_1$, which is stated in (2.3) since $m_1 > m_2$. □

Let us recall that from (4.14) $Q^+(\varepsilon, n) = \{Z_n > \rho_1 + \varepsilon\}$. We have the following result

Lemma 4.4. *Let $A_n \in \mathcal{F}_n$ be such that $A_n \subset Q^+(\varepsilon, n)$. Then, we have that*

$$\boldsymbol{E}\left[(Z_{n+1} - Z_n)\mathbb{1}_{A_n}\right] \;\leq\; -n^{-1} \cdot c_2\boldsymbol{P}(A_n) \;+\; o(n^{-1}). \tag{4.19}$$

Proof. The proof of this Lemma is obtained by following analogous arguments of the proof of Lemma 4.3. In fact, we can first apply result (a) of Lemma 3.1, then note that

$$\{Z_n \leq \hat{\rho}_{1,n}\} \subset \big(Q^{+c}(\varepsilon, n) \cup R^c(\varepsilon, n)\big),$$

and

$$\{Z_n \geq \hat{\rho}_{2,n}\} \supset \big(Q^{+}(\varepsilon, n) \cap R(\varepsilon, n)\big),$$

where we recall that $R(\varepsilon, n) := \{|\hat{\rho}_{1,n} - \rho_1| < \varepsilon\}$. Hence, since $A_n \subset Q^+(\varepsilon, n)$, on the set A_n we have that $W_{1,n} \leq \mathbb{1}_{R^c(\varepsilon,n)}$ and $W_{2,n} \geq \mathbb{1}_{R(\varepsilon,n)}$, which lead to the following inequality

$$B_n \;\leq\; \boldsymbol{E}\left[\left(\frac{D_{1,n+1}\mathbb{1}_{R^c(\varepsilon,n)}}{Y_n + D_{1,n+1}\mathbb{1}_{R^c(\varepsilon,n)}} - \frac{D_{2,n+1}\mathbb{1}_{R(\varepsilon,n)}}{Y_n + D_{2,n+1}\mathbb{1}_{R(\varepsilon,n)}}\right)|\mathcal{F}_n\right].$$

Then, by applying some standard calculations, we obtain that, on the set A_n^+,

$$\begin{aligned} B_n \quad &\leq \boldsymbol{E}\left[\left(\frac{D_{1,n+1}\mathbb{1}_{R^c(\varepsilon,n)}}{Y_n} - \frac{D_{2,n+1}\mathbb{1}_{R(\varepsilon,n)}}{Y_n + b\mathbb{1}_{R(\varepsilon,n)}}\right)|\mathcal{F}_n\right] \\ &= \frac{m_1\mathbb{1}_{R^c(\varepsilon,n)}}{Y_n} - \frac{m_2\mathbb{1}_{R(\varepsilon,n)}}{Y_n + b\mathbb{1}_{R(\varepsilon,n)}} \\ &= \frac{m_1\mathbb{1}_{R^c(\varepsilon,n)} - m_2\mathbb{1}_{R(\varepsilon,n)}}{Y_n + b\mathbb{1}_{R(\varepsilon,n)}}. \end{aligned}$$

Now, we can go through the same previous calculations using the results of Theorem 4.1 and $\boldsymbol{P}\left(R^c(\varepsilon, n)\right) \to 0$, in order to prove (4.19). □

Proof of Theorem 4.2. First, note that establishing (4.13) is equivalent to proving that for any $\mathcal{A}_n \in \mathcal{F}_n$ and letting $A_n := \mathcal{A}_n \cap Q(\varepsilon, n)$:

$$\boldsymbol{E}\left[G(n, n^{-1})\mathbb{1}_{A_n}\right] \;\leq\; -n^{-1} \cdot c_2\boldsymbol{P}(A_n) \;+\; o(n^{-1}),$$

where we recall that $G(n, n^{-1}) = (\Delta_{n+1} - \Delta_n)$. Hence, consider $A_n^+ := A_n \cap Q^+(\varepsilon, n)$ and $A_n^- := A_n \cap Q^-(\varepsilon, n)$. Since $A_n^+ \cap A_n^- = \emptyset$ and $A_n^+ \cup A_n^- = A_n$, we have the following decomposition

$$\boldsymbol{E}\left[G(n, n^{-1})\mathbb{1}_{A_n}\right] \;=\; I_n^+ \;-\; I_n^-,$$

where

$$I_n^+ \;:=\; \boldsymbol{E}\left[(Z_{n+1} - Z_n)\,\mathbb{1}_{A_n^+}\right], \qquad I_n^- \;:=\; \boldsymbol{E}\left[(Z_{n+1} - Z_n)\,\mathbb{1}_{A_n^-}\right].$$

By applying Lemmas 4.3 and 4.4 to I_n^- and I_n^+, respectively, we obtain

$$\begin{cases} I_n^- \;\geq\; n^{-1} \cdot c_2\boldsymbol{P}(A_n^-) \;-\; o(n^{-1}), \\ I_n^+ \;\leq\; -n^{-1} \cdot c_2\boldsymbol{P}(A_n^+) \;+\; o(n^{-1}). \end{cases}$$

This concludes the proof. □

4.3 L_1 Bound for Δ_n at Linearly Increasing Times

In this subsection, we provide an upper bound for the increments of Δ_n evaluated at linearly increasing times, i.e. $G(n, c) = (\Delta_{n+nc} - \Delta_n)$ and $c > 0$, where we recall from (4.1) that $\Delta_n = |\rho_1 - Z_n|$. To this end, we claim that, for any fixed $\delta > 0$, there exists a value $c > 0$ such that for all $n \geq 0$

$$\boldsymbol{P}\left(\;\{|Z_{n+ns_\delta} - Z_n| > \delta/2\} \;\cap\; F_n\;\right) \;=\; 0,$$

where we recall from (4.12) that $F_n := \{y_0 + c_1 n \leq Y_n \leq y_0 + C_1 n\}$. We will denote by s_δ one of these values of c.

We can compute precisely the range of values admissible for s_δ: on the set F_n, we obtain

$$|Z_{n+nc} - Z_n| \leq b\sum_{i=n}^{n+nc}\frac{1}{Y_i} \leq \frac{b}{c_1}\sum_{i=n}^{n+nc}\frac{1}{i} \leq \frac{b}{c_1}\log\left(1 + c\right),$$

where we recall that b is the maximum value of the urn reinforcements, i.e. $D_{1,n}, D_{2,n} \leq b$ a.s. for any $n \geq 1$. Then, imposing $|Z_{n+nc} - Z_n| < \delta/2$, we obtain

$$s_\delta \;\in\; \left(\,0\;,\;\exp\left(\frac{c_1}{2b}\delta\right) - 1\,\right). \tag{4.20}$$

This ensures that $\boldsymbol{P}\left(\{|Z_{n+ns_\delta} - Z_n| > \delta/2\} \cap F_n\right) = 0$ for all $n \geq 0$.

The next theorem provides an L_1 upper bound for the difference $G(n, s_\delta) = (\Delta_{n+ns_\delta} - \Delta_n)$ on the set $Q(\delta, n) = \{\Delta_n > \delta\}$. An L_1 upper bound on the set $Q^c(\delta, n)$ is presented in Theorem 4.6.

Theorem 4.5. *Let $m_1 > m_2$, (2.2) and (2.3). Then, for any $\delta > 0$ there exists a constant $0 < C < \infty$ such that*

$$\boldsymbol{E}\left[G(n, s_\delta)\mathbb{1}_{Q(\delta,n)}\right] \;\leq\; -C\boldsymbol{P}(Q(\delta, n)) \;+\; o(1).$$

Proof. First, note that using (4.4) in Theorem 4.1, we have

$$\left|\boldsymbol{E}\left[G(n, s_\delta)\mathbb{1}_{Q(\delta,n)\cap F_n^c}\right]\right| \;\leq\; \boldsymbol{P}(F_n^c) \;\to\; 0.$$

Hence, define

$$G_n \;:=\; \boldsymbol{E}\left[G(n, s_\delta)\mathbb{1}_{Q(\delta,n)\cap F_n}\right],$$

and consider the following expression

$$G_n \;=\; \sum_{i=n}^{n+ns_\delta-1} \boldsymbol{E}\left[G(i, i^{-1})\mathbb{1}_{Q(\delta,n)\cap F_n}\right], \tag{4.21}$$

where we recall that $G(i, i^{-1}) = (\Delta_{i+1} - \Delta_i)$. From the definition of s_δ in (4.20), on the set F_n we have that for all $i \in \{n, .., n + ns_\delta\}$

$$Q(\delta, n) \;\subset\; Q(\delta/2, i),$$

where we recall that $Q(\delta, n) = \{\Delta_n > \delta\}$ and $Q(\delta/2, i) = \{\Delta_i > \delta/2\}$. Hence, by applying Theorem 4.2 to each term of the sum in (4.21), since $Q(\delta, n) \cap F_n \in \mathcal{F}_i$ for all $i \in \{n, .., n + ns_\delta\}$, we obtain

$$\begin{aligned}
\boldsymbol{E}\left[G(i, i^{-1})\mathbb{1}_{Q(\delta,n)\cap F_n}\right] &= \boldsymbol{E}\left[\boldsymbol{E}\left[G(i, i^{-1})\mathbb{1}_{Q(\delta/2,i)}|\mathcal{F}_i\right]\mathbb{1}_{Q(\delta,n)\cap F_n}\right] \\
&\leq \boldsymbol{E}\left[\left(-i^{-1}\cdot c_2\mathbb{1}_{Q(\delta/2,i)} \;+\; \psi_i\right)\mathbb{1}_{Q(\delta,n)\cap F_n}\right] \\
&= -\,i^{-1}\cdot c_2\boldsymbol{P}\left(Q(\delta, n) \cap F_n\right) \;+\; \boldsymbol{E}\left[\psi_i\mathbb{1}_{Q(\delta,n)\cap F_n}\right].
\end{aligned}$$

Now, note that from (4.4) in Theorem 4.1 we have that $\boldsymbol{P}\left(Q(\delta, n) \cap F_n\right) = \boldsymbol{P}\left(Q(\delta, n)\right) - o(i^{-1})$; moreover, from Theorem 4.2, $|\boldsymbol{E}\left[\psi_i\mathbb{1}_{Q(\delta,n)\cap F_n}\right]| \leq \boldsymbol{E}\left[|\psi_i|\right] = o(i^{-1})$. Thus, from (4.21) we have that

$$\begin{aligned}
G_n \;&\leq\; -\sum_{i=n}^{n+ns_\delta-1} i^{-1}\cdot c_2\boldsymbol{P}(Q(\delta, n)) \;+\; \sum_{i=n}^{n+ns_\delta-1} o\left(i^{-1}\right) \\
&\leq\; -\log\left(1 + s_\delta\right)\cdot c_2\boldsymbol{P}(Q(\delta, n)) \;+\; o(1).
\end{aligned}$$

The result follows after calling $C := c_2 \log\left(1 + s_\delta\right)$. □

Now, we show that the expected difference $G(n, s_\delta)$ is asymptotically non-positive on the set $Q^c(\delta, n)$, for any $\delta > 0$, where we recall that $G(n, s_\delta) = (\Delta_{n+ns_\delta} - \Delta_n)$, $Q(\delta, n) = \{\Delta_n > \delta\}$ and $\Delta_n = |\rho_1 - Z_n|$. The result is stated precisely in the following theorem.

Theorem 4.6. *Let $m_1 > m_2$, (2.2) and (2.3). Then, for any $\delta > 0$,*

$$\overline{\lim}_n\boldsymbol{E}\left[G(n, s_\delta)\mathbb{1}_{Q^c(\delta,n)}\right] \;\leq\; 0. \tag{4.22}$$

To prove Theorem 4.6, we need to compare the ARRU model with two new urn models: $\{\tilde{Z}_n^+; n \geq 1\}$ and $\{\tilde{Z}_n^-; n \geq 1\}$. The dynamics of these processes is based on a sequence of random times $\{t_n; n \geq 1\}$ which describes relation between the process $\{\Delta_n; n \geq 1\}$ and an arbitrary fixed value $\nu > 0$. Specifically, fix $\nu > 0$ and, for any $n \geq 0$, define the set

$$\mathcal{T}_n \;:=\; \{0 \leq k \leq n \;:\; Q^c(\nu, n-k)\},$$

where we recall $Q^c(\nu, n-k) = \{\Delta_{n-k} \leq \nu\}$. Let $\{t_n; n \geq 1\}$ be the sequence of random times defined as

$$t_n = \begin{cases} \inf\{\mathcal{T}_n\} & \text{if } \mathcal{T}_n \neq \emptyset; \\ \infty & \text{otherwise.} \end{cases} \tag{4.23}$$

The time $(n - t_n)$ indicates the last time up to n the urn proportion is in the interval $(\rho_1 - \nu, \rho_1 + \nu)$.

First, let us describe the urn model $\{\tilde{Z}_n^-; n \geq 1\}$. Let $\tilde{I}^- = 1$, $\tilde{y}_0 \in (0, y_0)$ and $\tilde{z}_0^- \in (0, \rho_1 - \nu)$. The process $\{\tilde{Z}_n^-; n \geq 1\}$, $\tilde{Z}_n^- = \tilde{Y}_{1,n}/(\tilde{Y}_{1,n} + \tilde{Y}_{2,n})$, evolves as follows: if $t_{n-1} = 0$, i.e. $\Delta_{n-1} \leq \nu$, or $t_{n-1} = \infty$, then $\tilde{X}_n = \mathbb{1}_{\{U_n < \tilde{z}_0^-\}}$ and

$$\begin{cases} \tilde{Y}_{1,n} = \tilde{z}_0^- \cdot \tilde{y}_0 + \tilde{X}_n D_{1,n} \tilde{I}^-, \\ \tilde{Y}_{2,n} = \left(1 - \tilde{z}_0^-\right) \cdot \tilde{y}_0 + \left(1 - \tilde{X}_n\right) D_{2,n}; \end{cases} \tag{4.24}$$

if $t_{n-1} = k \geq 1$, i.e. $\Delta_{n-1} > \nu$, then $\tilde{X}_n = \mathbb{1}_{\{U_n < \tilde{Z}_{n-1}\}}$ and

$$\begin{cases} \tilde{Y}_{1,n} = \tilde{Y}_{1,n-1} + \tilde{X}_n D_{1,n} \tilde{I}^-, \\ \tilde{Y}_{2,n} = \tilde{Y}_{2,n-1} + \left(1 - \tilde{X}_n\right) D_{2,n} \end{cases} \tag{4.25}$$

then, $\tilde{Y}_n := \tilde{Y}_{1,n} + \tilde{Y}_{2,n}$ and $\tilde{Z}_n := \tilde{Y}_{1,n}/\tilde{Y}_n$. The urn model is well defined since t_{n-1} is $\mathcal{F}_{n-1}$-measurable.

Analogously, the urn model $\{\tilde{Z}_n^+; n \geq 1\}$, $\tilde{Z}_n^+ = \tilde{Y}_{1,n}/(\tilde{Y}_{1,n} + \tilde{Y}_{2,n})$, is defined by the same equations (4.24) and (4.25), with $\tilde{I}^-$ and $\tilde{z}_0^-$ are replaced by $\tilde{I}^+ = 0$ and $\tilde{z}_0^+ \in (\rho_1 + \nu, 1)$, respectively.

In the next lemma, we state an important relation among the processes $\{\tilde{Z}_n^-; n \geq 1\}$, $\{\tilde{Z}_n^+; n \geq 1\}$ and the urn proportion of the ARRU model $\{Z_n; n \geq 1\}$. This result is needed in the proof of Theorem 4.6. To ease calculations, let $h > 0$ and fix the initial proportions $\tilde{z}_0^-$ and $\tilde{z}_0^+$ as follows:

$$\rho_1 - \tilde{z}_0^- = \tilde{z}_0^+ - \rho_1 = \nu + h. \tag{4.26}$$

Let $M_n := \sum_{i=n}^{n+ns_\delta} \mathbb{1}_{R^c(\nu,n)}$ and, for any $\epsilon > 0$ define the set

$$\mathcal{M}_n^\epsilon := \{M_n < ns_\delta \epsilon\}, \tag{4.27}$$

where we recall that $R(\nu, n) = \{|\hat{\rho}_{1,n} - \rho_1| \leq \nu\}$, s_δ is such that $\boldsymbol{P}\left(\{|G(n, s_\delta)| > \delta/2\} \cap F_n\right) = 0$, with $F_n = \{y_0 + c_1 n \leq Y_n \leq y_0 + C_1 n\}$ from (4.12). Moreover, for any $n \geq 1$ and $k \in \{n, .., n + ns_\delta\}$ let us define the set

$$E(n, k) := \cup_{j=n}^k Q^c(\nu, j) \equiv \{\, \exists j \in \{n, .., k\} : \{\Delta_j \leq \nu\}\,\}. \tag{4.28}$$

We also introduce the following notation: $\tilde{\Delta}_l^- := |\rho_1 - \tilde{Z}_l^-|$, $\tilde{\Delta}_l^+ := |\rho_1 - \tilde{Z}_l^+|$ and $\tilde{\Delta}_l^* := \max\left\{\tilde{\Delta}_l^-, \tilde{\Delta}_l^+\right\}$. Thus, we have the following result:

Lemma 4.7. *Let $m_1 > m_2$, (2.2) and (2.3). Fix $n \geq 1$, $\tilde{y}_0 \in (0, y_0 + c_1 n)$, $\tilde{z}_0^-$ and $\tilde{z}_0^+$ as in (4.26). Consider the set $\mathcal{M}_n^\epsilon$ as defined in (4.27) with*

$$0 < \epsilon < \frac{c_1 h}{b s_\delta}. \tag{4.29}$$

Then, for any $n \geq 1$ and $l_n \in \{n+1, .., n + ns_\delta\}$, on the set $\mathcal{M}_n^\epsilon \cap F_n$ we have that

$$E(n, l_n) \subset Q^c(\tilde{\Delta}_l^*, l) \ a.s., \tag{4.30}$$

for all $l \in \{l_n + 1, .., n + ns_\delta\}$.

Proof. First, fix $l \in \{l_n + 1, .., n + ns_\delta\}$ and note that, from the definition of $\{t_n; n \geq 1\}$ in (4.23) and $E(n,k)$ in (4.28), we always have

$$\{t_{l-1} = \infty\} \cap E(n, l_n) = \emptyset.$$

Hence, we never consider in this proof the set $\{t_{l-1} = \infty\}$.

Then, consider the set $\{t_{l-1}=0\}$ and note that, from the definition of t_n in (4.23), $\{t_{l-1}=0\}\equiv Q^c(\nu,l-1)$, which implies that, on the set $\{t_{l-1}=0\}\cap\{X_l=0\}$,

$$Z_l \;\geq\; \frac{(\rho_1-\nu)Y_{l-1}}{Y_{l-1}+D_{2,l}W_{2,l-1}} \;\geq\; \frac{\tilde{z}_0^-\tilde{y}_0}{\tilde{y}_0+D_{2,l}} \;=\; \tilde{Z}_l^- \quad a.s., \tag{4.31}$$

and, on the set $\{t_{l-1}=0\}\cap\{X_l=1\}$,

$$Z_l \;\leq\; \frac{(\rho_1+\nu)Y_{l-1}+D_{1,l}W_{1,l-1}}{Y_{l-1}+D_{1,l}W_{1,l-1}} \;\leq\; \frac{\tilde{z}_0^+\tilde{y}_0+D_{1,l}}{\tilde{y}_0+D_{1,l}} \;=\; \tilde{Z}_l^+ \quad a.s. \tag{4.32}$$

From (4.31) and (4.32) we have $\tilde{Z}_l^-\leq Z_l\leq\tilde{Z}_l^+$ a.s., that ensures that (4.30) is verified whenever $\{t_{l-1}=0\}$.

To prove (4.30) on the set $\{1\leq t_{l-1}<\infty\}$, we will show that, defining $\tilde{A}_l^-:=\left\{\tilde{Z}_l^-\leq Z_l\right\}$, $\tilde{A}_l^+:=\left\{Z_l\leq\tilde{Z}_l^+\right\}$ and $B:=\mathcal{M}_n^\epsilon\cap F_n\cap\{1\leq t_{l-1}<\infty\}$,

$$\begin{aligned}\big(B\cap Q^-(\nu,l-t_{l-1})\big) &\subseteq \big(\tilde{A}_l^-\cap Q^-(\nu,l-t_{l-1})\big),\\ \big(B\cap Q^+(\nu,l-t_{l-1})\big) &\subseteq \big(\tilde{A}_l^+\cap Q^+(\nu,l-t_{l-1})\big).\end{aligned} \tag{4.33}$$

Moreover, from the definition of $\{t_n;n\geq 1\}$ in (4.23), on the set $\{1\leq t_{l-1}<\infty\}$, we note that

$$\{X_{l-t_{l-1}}=0\} \;\equiv\; \{Z_{l-t_{l-1}}\leq\rho_1-\nu\}=Q^-(\nu,l-t_{l-1}).$$

$$\{X_{l-t_{l-1}}=1\} \;\equiv\; \{Z_{l-t_{l-1}}\geq\rho_1+\nu\}=Q^+(\nu,l-t_{l-1}),$$

Hence, showing (4.33) is equivalent to establish the following

$$\begin{aligned}\big(B\cap\{X_{l-t_{l-1}}=0\}\big) &\subseteq \big(\tilde{A}_l^-\cap\{X_{l-t_{l-1}}=0\}\big),\\ \big(B\cap\{X_{l-t_{l-1}}=1\}\big) &\subseteq \big(\tilde{A}_l^+\cap\{X_{l-t_{l-1}}=1\}\big).\end{aligned} \tag{4.34}$$

To this end, we will prove by induction on $j\in\{l-t_{l-1}+1,..,l\}$ the following results:

$$\big(B\cap\{X_{l-t_{l-1}}=0\}\big) \;\subseteq\; \Big(\Big(\bigcap_{i=l-t_{l-1}}^{j}\tilde{A}_i^-\Big)\cap\{X_{l-t_{l-1}}=0\}\Big), \tag{4.35}$$

$$\big(B\cap\{X_{l-t_{l-1}}=1\}\big) \;\subseteq\; \Big(\Big(\bigcap_{i=l-t_{l-1}}^{j}\tilde{A}_i^+\Big)\cap\{X_{l-t_{l-1}}=1\}\Big). \tag{4.36}$$

First, note that by (4.31) it follows that condition (4.35) is verified for $j=l-t_{l-1}$. Hence, the result is achieved by establishing (4.35) for $j\in\{l-t_{l-1}+1,..,l\}$, assuming that (4.35) holds for $(j-1)$.

To this end, consider

$$Z_j=\frac{Z_{l-t_{l-1}-1}Y_{l-t_{l-1}-1}+\sum_{i=l-t_{l-1}}^{j}X_iD_{1,i}W_{1,i-1}}{Y_{l-t_{l-1}-1}+\sum_{i=l-t_l}^{j}X_iD_{1,i}W_{1,i-1}+\sum_{i=l-t_{l-1}}^{j}(1-X_i)\,D_{2,i}W_{2,i-1}}.$$

Now, note that by (4.35) we have $X_i=\mathbb{1}_{\{U_i<Z_{i-1}\}}\geq\mathbb{1}_{\{U_i<\tilde{Z}_{i-1}^-\}}=\tilde{X}_i^-$ for any $(i-1)=l-t_{l-1},...,(j-1)$, and since $Z_{l-t_{l-1}-1}\geq\rho_1-\nu\geq\tilde{z}_0^-$ we also have that $X_{l-t_{l-1}}\geq\tilde{X}_{l-t_{l-1}}^-$. Moreover, since $Y_{l-t_{l-1}-1}\geq\tilde{y}_0$ and $X_{l-t_{l-1}}=0$, it follows that

$$Z_j \;\geq\; \frac{(\rho_1-\nu)\tilde{y}_0+\sum_{i=l-t_{l-1}+1}^{j}\tilde{X}_i^-D_{1,i}W_{1,i-1}}{\tilde{y}_0+\sum_{i=l-t_l+1}^{j}\tilde{X}_i^-D_{1,i}W_{1,i-1}+\sum_{i=l-t_{l-1}}^{j}\left(1-\tilde{X}_i^-\right)D_{2,i}W_{2,i-1}}.$$

Note that, letting n_0 such that $\boldsymbol{P}(R(\nu,n_0))>\eta>0$, for any $n\geq n_0$ we have the following relation

$$\{Z_n\leq\hat{\rho}_{1,n}\}\supset\big(Q^-(\nu,n)\cap R(\nu,n)\big),$$

where we recall that $R(\nu,n)=\{|\hat{\rho}_{1,n}-\rho_1|<\nu\}$ and $Q^-(\nu,n)=\{Z_n<\rho_1-\nu\}$. Hence, by definition of t_{l-1} in (4.23), we have $Q^-(\nu,i)$ for any $i=l-t_{l-1},...,j-1$, and $\{Z_i\leq\hat{\rho}_{1,i}\}\supset R(\nu,i)$, which implies $W_{1,i}\geq\mathbb{1}_{R(\nu,i)}$. Combining this argument with $W_{2,i}\leq 1$, we have that

$$Z_j \;\geq\; \frac{(\rho_1-\nu)\tilde{y}_0+\sum_{i=l-t_{l-1}+1}^{j}\tilde{X}_i^-D_{1,i}\mathbb{1}_{R(\nu,i-1)}}{\tilde{y}_0+\sum_{i=l-t_{l-1}+1}^{j}\tilde{X}_i^-D_{1,i}\mathbb{1}_{R(\nu,i-1)}+\sum_{i=l-t_{l-1}}^{j}\left(1-\tilde{X}_i^-\right)D_{2,i}}.$$

In addition, on the set $\mathcal{M}_n^{\epsilon}$ we have that

$$\sum_{i=l-t_{l-1}+1}^{j} \tilde{X}_i^- D_{1,i} \mathbb{1}_{R(\nu,i-1)} \geq \sum_{i=l-t_{l-1}+1}^{j} \tilde{X}_i^- D_{1,i} - bM_n$$
$$\geq \sum_{i=l-t_{l-1}+1}^{j} \tilde{X}_i^- D_{1,i} - nbs_\delta\epsilon.$$

Moreover, condition (4.29) ensures that

$$(\rho_1 - \nu)\tilde{y}_0 - nbs_\delta\epsilon \ \geq\ \tilde{z}_0^- \tilde{y}_0,$$

which implies $\tilde{A}_l^- = \{Z_j \geq \tilde{Z}_j^-\}$.

Analogous arguments can be followed to establish (4.36) for any $j \in \{l - t_{l-1} + 1, .., l\}$. Finally, combining (4.35) and (4.36), we obtain (4.34). This concludes the proof. □

In the next lemma, we show an important result required in the proof of Theorem 4.6, concerning the probability that $\tilde{Z}_n$ exceeds an arbitrary threshold $l > 0$. This result is obtained by using comparison arguments between the process $\{\tilde{\Delta}_n^*; n \geq 1\}$ and the urn proportion of an RRU model, where we recall that $\tilde{\Delta}_n^* = \max\{\tilde{\Delta}_l^-, \tilde{\Delta}_l^+\}$, $\tilde{\Delta}_l^- := |\rho_1 - \tilde{Z}_l^-|$ and $\tilde{\Delta}_l^+ := |\rho_1 - \tilde{Z}_l^+|$. The result is the following,

Lemma 4.8. *Let $m_1 > m_2$, and*

$$\tilde{T}_n \ :=\ \{k_n < t_n < \infty\}, \qquad H_n \ :=\ \left\{\tilde{\Delta}_n^* > \nu\right\}, \tag{4.37}$$

where $\{k_n; n \geq 1\}$ is a deterministic sequence such that $k_n \to \infty$. Fix $0 < \tilde{y}_0 < \infty$ and define $\tilde{z}_0^-$ and $\tilde{z}_0^+$ as in (4.26)*. Then,*

$$\lim_{n\to\infty} \boldsymbol{P}\left(H_n \cup \tilde{T}_n\right) = 0. \tag{4.38}$$

Proof. Since $H_n = H_n^- \cup H_n^+$ where

$$H_n^- := \left\{\tilde{Z}_n^- < \rho_1 - \nu\right\}, \qquad \text{and} \qquad H_n^+ := \left\{\tilde{Z}_n^+ > \rho_1 + \nu\right\},$$

equation (4.38) is established by proving

$$\lim_{n\to\infty} \boldsymbol{P}\left(H_n^- \cup \tilde{T}_n\right) + \boldsymbol{P}\left(H_n^+ \cup \tilde{T}_n\right) = 0.$$

We will show that $\boldsymbol{P}\left(H_n^- \cup \tilde{T}_n\right) \to 0$, since the proof of $\boldsymbol{P}\left(H_n^+ \cup \tilde{T}_n\right) \to 0$ is analogous.

First, we recall that t_n, defined in (4.23), satisfies that $Q^c(\nu, n - t_n) = \{\Delta_{n-t_n} \leq \nu\}$ and when $t_n > 0$, $Q(\nu, i) = \{\Delta_i > \nu\}$ for any $n - t_n < i \leq n$. Hence, on the set $\tilde{T}_n$ the process $\tilde{Z}_i^-$ evolves at times $n - t_n < i \leq n$ as described in (4.25), yielding $\tilde{X}_i = \mathbb{1}_{\{U_i < \tilde{Z}_{i-1}^-\}}$ and

$$\begin{cases} \tilde{Y}_{1,n}^- \ =\ \tilde{z}_0^- \tilde{y}_0 \ +\ \sum_{i=n-t_n+1}^{n} \tilde{X}_i D_{1,i}, \\ \\ \tilde{Y}_{2,n}^- \ =\ (1 - \tilde{z}_0^-)\tilde{y}_0 \ +\ \sum_{i=n-t_n+1}^{n} \left(1 - \tilde{X}_i\right) D_{2,i}. \end{cases} \tag{4.39}$$

Now, consider an RRU model $\{Z_j^R; j \geq 1\}$ with initial composition $(\tilde{y}_{1,0}^R, \tilde{y}_{2,0}^R) = (\tilde{z}_0^- \tilde{y}_0, (1 - \tilde{z}_0^-)\tilde{y}_0)$; the reinforcements are defined as $D_{1,j}^R = D_{1,n-t_n+j}$ and $D_{2,j}^R = D_{2,n-t_n+j}$ for any $i \geq 1$ a.s.; the sampling process is modeled by $X_j^R := \mathbb{1}_{\{U_j^R < Z_{j-1}^R\}}$ and $U_j^R = U_{n-t_n+j}$ a.s., Hence, the composition of the RRU model at time $j \geq 1$ can be expressed as follows:

$$\begin{aligned} Y_{1,j}^R &= \tilde{y}_{1,0}^R \ +\ \sum_{i=1}^{j} X_{n-t_n+i} D_{1,n-t_n+i} \\ &= \tilde{z}_0^- \tilde{y}_0^- \ +\ \sum_{i=n-t_n+1}^{n-t_n+j} X_i D_{1,i}, \\ Y_{2,j}^R &= \tilde{y}_{2,0}^R \ +\ \sum_{i=1}^{j} (1 - X_{n-t_n+i})\, D_{2,n-t_n+i} \\ &= (1 - \tilde{z}_0^-)\tilde{y}_0 \ +\ \sum_{i=n-t_n+1}^{n-t_n+j} (1 - X_i)\, D_{2,i}. \end{aligned} \tag{4.40}$$

Hence, combining (4.39) and (4.40) with $j = t_n$, we have that on the set $\tilde{T}_n$

$$(\tilde{Y}_{1,n}^-, \tilde{Y}_{2,n}^-) = (Y_{1,t_n}^R, Y_{2,t_n}^R).$$

Now, from the asymptotic behavior of the RRU studied in Muliere et al. (2006, Theorem 8) we have that (since $m_1 > m_2$) $\boldsymbol{P}(\lim_{n\to\infty} Z_n^R = 1) = 1$. Thus, on the set $\tilde{T}_n$ we have $\{\lim_{n\to\infty} Z_n^R = 1\}$, which implies $\boldsymbol{P}\left(H_n^- \cup \tilde{T}_n\right) \to 0$. This concludes the proof. □

Proof of Theorem 4.6. First, consider the set $F_n = \{y_0 + c_1 n \leq Y_n \leq y_0 + C_1 n\}$ defined in (4.12) and by using (4.4) in Theorem 4.1 we have

$$\overline{\lim}_n \boldsymbol{P}(F_n^c) \ = \ 0.$$

Hence, since $|G(n, s_\delta)| \leq \max\{Z_{n+ns_\delta}; Z_n\} < 1$ a.s., to prove (4.22) it is enough to show that for any $0 < h < 1/2$

$$\boldsymbol{E}\left[G_{n,s_\delta}\mathbb{1}_{Q^c(\delta,n)\cap F_n}\right] \ \leq \ h \ + \ o(1), \tag{4.41}$$

where we recall that $G(n, s_\delta) = (\Delta_{n+ns_\delta} - \Delta_n)$ and $Q(\delta, n) = \{\Delta_n > \delta\}$. Now, define $H := [\delta/h]$ and note that

$$[0, \delta] \ \subset \ [0, (H+1)h] \ = \ \cup_{i=0}^H [ih, (i+1)h];$$

then, calling

$$\bar{Q}((i+1)h, n) := Q^c((i+1)h, n) \setminus Q^c(ih, n) = \{ih < \Delta_n < (i+1)h\},$$

(where for any two sets A and B, $A \setminus B = A \cap B^c$), we have $Q^c(\delta, n) = \cup_{i=0}^H \bar{Q}((i+1)h, n)$ and hence the left-hand side of (4.41) can be written as

$$\boldsymbol{E}\left[G(n, s_\delta)\mathbb{1}_{Q^c(\delta,n)\cap F_n}\right] = \sum_{i=0}^{H} \boldsymbol{E}\left[G(n, s_\delta)\mathbb{1}_{\bar{Q}((i+1)h,n)\cap F_n}\right];$$

thus, result (4.41) can be achieved by establishing the following

$$\boldsymbol{E}\left[G(n, s_\delta)\mathbb{1}_{\bar{Q}((i+1)h,n)\cap F_n}\right] \ \leq \ h \cdot \boldsymbol{P}\left(\bar{Q}((i+1)h, n)\right) \ + \ o(1), \tag{4.42}$$

for any $i \in \{1, .., H\}$. Now, fix $i \in \{0, .., H\}$, call $\nu := (i+1)h$ and consider the set $\mathcal{M}_n^\epsilon := \{M_n < ns_\delta\epsilon\}$ defined in (4.27), where we recall that $M_n = \sum_{i=n}^{n+ns_\delta} \mathbb{1}_{R^c(\nu,n)}$. The left-hand side of (4.42) can be so decomposed $\boldsymbol{E}\left[G(n, s_\delta)\mathbb{1}_{\bar{Q}(\nu,n)\cap F_n}\right] = \mathcal{G}_{1n} + \mathcal{G}_{2n}$, where

$$\mathcal{G}_{1n} := \boldsymbol{E}\left[G(n, s_\delta)\mathbb{1}_{\bar{Q}(\nu,n)\cap F_n\cap\mathcal{M}_n^\epsilon}\right], \text{ and } \mathcal{G}_{2n} := \boldsymbol{E}\left[G(n, s_\delta)\mathbb{1}_{\bar{Q}(\nu,n)\cap F_n\cap\mathcal{M}_n^{\epsilon c}}\right].$$

Since $\boldsymbol{P}(R(\nu, n)) \to 1$ from (2.3), and by using Markov's inequality we have that

$$\boldsymbol{P}(\mathcal{M}_n^{\epsilon c}) \ \leq \ \epsilon^{-1}\frac{1}{ns_\delta}\sum_{i=n}^{n+ns_\delta} \boldsymbol{P}\left(R^c(\nu, n)\right) \ \to \ 0;$$

thus, since $|G(n, s_\delta)| \leq \max\{Z_{n+ns_\delta}; Z_n\} < 1$ a.s., we have $\mathcal{G}_{2n} \to 0$ and hence result (4.42) can be achieved by establishing the following

$$\mathcal{G}_{1n} \ = \ \boldsymbol{E}\left[G(n, s_\delta)\mathbb{1}_{\bar{Q}(\nu,n)\cap F_n\cap\mathcal{M}_n^\epsilon}\right] \ \leq \ h \cdot \boldsymbol{P}\left(\bar{Q}(\nu, n)\right) \ + \ o(1), \tag{4.43}$$

where we recall that $\bar{Q}(\nu, n) = \{\nu - h < \Delta_n < \nu\}$.

Now, following the same arguments used to determine s_δ in (4.20), we can fix a value s_h such that

$$\boldsymbol{P}\left(\ \{|G(n, s_h)| > h/2\} \ \cap \ F_n \ \right) \ = \ 0,$$

where we recall that $G(n, s_h) = (\Delta_{n+ns_h} - \Delta_n)$. Analogously to (4.20), the range of values admissible for s_h is

$$s_h \ \in \ \left(\ 0 \ , \ \exp\left(\frac{c_1}{2b}h\right) - 1 \ \right),$$

where we recall that $c_1 > 0$ is a constant introduce in (4.12) to define F_n.

Now, consider the random time t_j defined in (4.23) as the smallest time k such that $Q^c(\nu, n-k)$ occurs, i.e. $n - t_n$ indicates the last time up to n the urn proportion is in the interval $(\rho_1 - \nu, \rho_1 + \nu)$. Then, call $\tau_n := t_{n+ns_\delta}$ and note that, since $\bar{Q}(\nu, n) \subset Q^c(\nu, n)$ by definition of $\bar{Q}(\nu, n)$, we have that

$$\boldsymbol{P}\left(\tau_n \leq ns_\delta \mid \bar{Q}(\nu, n)\right) = 1.$$

Hence, define $S_H := [s_\delta/s_h]$ and, assuming wlog that $s_\delta = S_H s_h h + 1$, on the set $\bar{Q}(\nu, n)$, consider the partition $\{0, .., ns_\delta\} = \cup_{k=0}^{S_H} \mathcal{T}_k^n$, where $\mathcal{T}_k^n := \{nks_h, .., n(k+1)s_h\}$; thus, the left-hand side of (4.43) can be decomposed as $\mathcal{G}_{1n} = \sum_{k=0}^{S_H} T_k^n$, where for any $k \in \{0, .., S_H\}$

$$T_k^n := \boldsymbol{E}\left[G(n, s_\delta)\mathbb{1}_{\bar{Q}(\nu,n)\cap F_n \cap \mathcal{M}_n^\epsilon \cap \{\tau_n \in \mathcal{T}_k^n\}}\right]. \tag{4.44}$$

Hence, equation (4.43) can be achieved by establishing the following

$$T_k^n \leq h \cdot \boldsymbol{P}\left(\bar{Q}(\nu, n) \cap \{\tau_n \in \mathcal{T}_k^n\}\right) + o(1), \quad \forall k \in \{0, .., S_H\}. \tag{4.45}$$

First, consider $k = 0$ in (4.45). From the definition of τ_n, we have

$$\{\tau_n \in \mathcal{T}_0^n\} \subset Q^c(\nu + h, n + ns_\delta), \tag{4.46}$$

where we recall that $Q^c(\nu + h, n + ns_\delta) = \{\Delta_{n+ns_\delta} < \nu + h\}$. Hence, using (4.46) in (4.44), it is immediate to obtain (4.45).

For $k \in \{1, .., S_H\}$ in (4.45), from the definition of τ_n and $E_{n,k}$ in (4.28), we have that

$$\{\tau_n \in \mathcal{T}_k^n\} \subset E(n, n + n(s_\delta - ks_h)),$$

where we recall $E(n, k) = \cup_{j=n}^{k} Q^c(\nu, j)$. Hence, we can use Lemma 4.7 with $l_n = n + n(s_\delta - ks_h)$, to obtain, on the set $\mathcal{M}_n^\epsilon \cap F_n$, for any $j \in \{n + n(s_\delta - ks_h) + 1, .., n + ns_\delta\}$

$$Q^c(\nu, n + n(s_\delta - ks_h)) \subset Q^c(\tilde{\Delta}_j^*, j) \quad a.s., \tag{4.47}$$

where we recall that $Q^c(\nu, j) = \{\Delta_j < \nu\}$ and $Q^c(\tilde{\Delta}_j^*, j) = \{\Delta_j < \tilde{\Delta}_j^*\}$, $\tilde{\Delta}_j^* = \max\left\{\tilde{\Delta}_j^-, \tilde{\Delta}_j^+\right\}$, $\tilde{\Delta}_j^- = |\rho_1 - \tilde{Z}_j^-|$ and $\tilde{\Delta}_j^+ = |\rho_1 - \tilde{Z}_j^+|$. In particular, by using (4.47) and since $\bar{Q}(\nu, n) \subset Q(\nu - h, n) = \{\Delta_n > \nu - h\}$, from (4.44) we obtain

$$T_k^n \leq \boldsymbol{E}\left[(\tilde{\Delta}_{n+ns_\delta}^* - \nu + h)\mathbb{1}_{\bar{Q}(\nu,n)\cap F_n \cap \mathcal{M}_n^\epsilon \cap \{\tau_n \in \mathcal{T}_k^n\}}\right]. \tag{4.48}$$

Note that, from the definition of τ_n and $\mathcal{T}_k^n$, we have

$$\{\tau_n \in \mathcal{T}_k^n\} \subset \{nks_h < t_{n+ns_\delta} < n(k+1)s_h\}.$$

Hence, we can apply Lemma 4.8 with $k_{n+ns_\delta} = nks_h$, $\tilde{T}_j := \left\{\tilde{\Delta}_j^* > \nu\right\}$ and $H_j := \{k_j < t_j < \infty\}$ as defined in (4.37), so obtaining

$$\boldsymbol{E}\left[(\tilde{\Delta}_{n+ns_\delta}^* - \nu)^+ \mathbb{1}_{\{\tau_n \in \mathcal{T}_k^n\}}\right] \leq \boldsymbol{P}\left(H_{n+ns_\delta} \cup \tilde{T}_{n+ns_\delta}\right) \to 0.$$

Hence, applying these results to (4.48), we obtain

$$T_k^n \leq h \cdot \boldsymbol{P}\left(\bar{Q}(\nu, n) \cap \{\tau_n \in \mathcal{T}_k^n\}\right) + o(1),$$

that corresponds to (4.45). This concludes the proof. □

4.4 Proof of Weak Consistency

Proof of Theorem 2.2. The result is established by proving that, for any $l > 0$ and any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\boldsymbol{P}\left(Q(l, n)\right) < \epsilon, \tag{4.49}$$

for any $n \geq n_0$, where we recall that $Q(l, n) = \{\Delta_n > l\}$ and $\Delta_n = |\rho_1 - Z_n|$. To this end, fix $0 < \epsilon' < \frac{l\epsilon}{3}$ and $0 < \delta < \epsilon'$ to define the conditions

$$\mathcal{A}_n := \{\boldsymbol{P}\left(Q(\delta, n)\right) < \epsilon'\}, \qquad \mathcal{B}_n := \{\boldsymbol{E}[\Delta_n] < 2\epsilon'\}.$$

It is immediate to see that $\mathcal{B}_n$ implies (4.49). Thus, (4.49) can be established by proving that

(a) for any $N \geq 1$ there exists $n_0 \geq N$ such that $\mathcal{A}_{n_0}$ occurs;

(b) there exists $n_0 \geq 1$ such that for any $n \geq n_0$ $\mathcal{A}_n \subset \mathcal{B}_k$ for all $k \in \{n+1, .., n(1+s_\delta)\}$;

(c) there exists $n_0 \geq 1$ such that for any $n \geq n_0$ $\mathcal{B}_n \subset \mathcal{B}_k$ for all $k \in \{n(1+s_\delta), .., (n+1)(1+s_\delta)\}$.

For part (a), we will show that there cannot exist $N \geq 1$ such that

$$\mathcal{A}_n^c \;:=\; \{\boldsymbol{P}\left(Q(\delta, n)\right) \geq \epsilon'\}, \tag{4.50}$$

occurs for all $n \geq N$. First, we combine Theorem 4.5 and Theorem 4.6 to obtain

$$\boldsymbol{E}\left[G(n, s_\delta)\right] \;\leq\; -C\left(\boldsymbol{P}(Q(\delta, n)) \;-\; \frac{\epsilon'}{2}\right), \tag{4.51}$$

with $0 < C < \infty$, where we recall that $G(n, s_\delta) = (\Delta_{n+ns_\delta} - \Delta_n)$. Now, if (4.50) holds, then there exists a subsequence $\{k_n; n \geq 1\}$ such that, $k_1 = N$ and $k_n = k_{n-1}(1+s_\delta)$ for all $n \geq 2$, and by (4.51)

$$\boldsymbol{E}\left[\Delta_{k_n}\right] = \sum_{i=1}^{n} \boldsymbol{E}\left[G(k_{i-1}, s_\delta)\right] \leq -\sum_{i=1}^{n} C\frac{\epsilon'}{2} \;=\; -\infty,$$

where $G(k_{i-1}, s_\delta) = (\Delta_{k_i} - \Delta_{k_{i-1}})$, which is a contradiction and hence part (a) holds. For part (b), consider the time n at which $\mathcal{A}_n$ occurs. Fix $k \in \{n+1, .., n+ns_\delta\}$ and note that $\boldsymbol{E}[\Delta_k] \;\leq\; J_{1n} \;+\; J_{2n,k}$ where

$$J_{1n} \;:=\; \boldsymbol{E}[\Delta_n], \quad \text{and} \quad J_{2n,k} \;:=\; \boldsymbol{E}[|\Delta_k - \Delta_n|].$$

From definition of s_δ in (4.20) we have

$$\begin{aligned} J_{2n,k} \quad &\leq \boldsymbol{E}[|\Delta_k - \Delta_n| \mathbb{1}_{F_n}] \;+\; \boldsymbol{E}[|\Delta_k - \Delta_n| \mathbb{1}_{F_n^c}] \\ &\leq \delta \;+\; \boldsymbol{P}(F_n^c), \end{aligned}$$

and using $\boldsymbol{P}(F_n^c) \to 0$ from (4.4) in Theorem 4.1 we have that $\lim_{n\to\infty} J_{2n,k} \leq \delta$. Thus, there exists $n_0 \geq 1$ such that $J_{2n,k} < 2\delta$ for any $n \geq n_0$. Then, note that $J_{1n} \;=\; J_{3n} \;+\; J_{4n}$ where

$$J_{3n} \;:=\; \boldsymbol{E}[\Delta_n \mathbb{1}_{Q^c(\delta,n)}], \quad \text{and} \quad J_{4n} \;:=\; \boldsymbol{E}[\Delta_n \mathbb{1}_{Q(\delta,n)}].$$

Notice that $J_{3n} \leq \delta \boldsymbol{P}(Q^c(\delta, n)) < \delta$ and $J_{4n} \leq \boldsymbol{P}(Q(\delta, n)) < \epsilon'$, and hence we have $J_{1n} < \delta + \epsilon'$. Thus, combining J_{1n} and J_{2n}, since $\delta < \epsilon'/3$, we obtain for any $n \geq n_0$

$$\boldsymbol{E}[\Delta_k] \;\leq\; J_{1n} \;+\; J_{2n,k} \;<\delta \;+\; \epsilon' \;+\; 2\delta \;<\; 2\epsilon',$$

that implies (b). For part (c), for any $k \in \{n(1+s_\delta), .., (n+1)(1+s_\delta)\}$ consider

$$\boldsymbol{E}[|\Delta_k - \Delta_{n+ns_\delta}|] \;\leq\; \boldsymbol{E}[|\Delta_k - \Delta_{n+ns_\delta}| \mathbb{1}_{F_n}] \;+\; \boldsymbol{E}[|\Delta_k - \Delta_{n+ns_\delta}| \mathbb{1}_{F_n^c}].$$

First, note that $\boldsymbol{P}(F_n^c) \to 0$ from (4.4) in Theorem 4.1. Then, since $|k-(n+ns_\delta)| \leq (1+s_\delta)$ and $|Z_{n+1} - Z_n| < b/Y_n$ a.s., we have that

$$\boldsymbol{P}\left(\; \left\{ |Z_k - Z_{n+ns_\delta}| > \left(\frac{b}{y_0 + c_1 n}\right)(1+s_\delta) \right\} \;\cap\; F_n \;\right) \;=\; 0.$$

Thus, for any $k \in \{n(1+s_\delta), .., (n+1)(1+s_\delta)\}$ we have

$$\boldsymbol{E}[|\Delta_k - \Delta_{n+ns_\delta}|] \;\leq\; \left(\frac{b(1+s_\delta)}{y_0 + c_1 n}\right) \;+\; \boldsymbol{P}(F_n^c) \;\to\; 0. \tag{4.52}$$

Now, since $\mathcal{B}_n \subset \mathcal{A}_n \cup \mathcal{C}_n$, where $\mathcal{C}_n = (\mathcal{B}_n \cap \mathcal{A}_n^c)$, part (c) is established by proving that there exists $n_0 \geq 1$ such that, for any $n \geq n_0$,

(c1) $\mathcal{A}_n \subset \mathcal{B}_k$ for all $k \in \{n(1+s_\delta), .., (n+1)(1+s_\delta)\}$;

(c2) $\mathcal{C}_n \subset \mathcal{B}_k$ for all $k \in \{n(1+s_\delta), .., (n+1)(1+s_\delta)\}$;

For part (c1), we can follow the same arguments of part (b), except for $J_{2n,k}$ since here $k \in \{n(1+s_\delta),..,(n+1)(1+s_\delta)\}$ and hence

$$\begin{aligned} J_{2n,k} \quad &\leq \boldsymbol{E}[|\Delta_k - \Delta_n|\mathbb{1}_{F_n}] \ + \ \boldsymbol{E}[|\Delta_k - \Delta_n|\mathbb{1}_{F_n^c}] \\ &\leq \boldsymbol{E}[|\Delta_k - \Delta_{n+ns_\delta}|\mathbb{1}_{F_n}] \ + \ \boldsymbol{E}[|\Delta_{n+ns_\delta} - \Delta_n|\mathbb{1}_{F_n}] \ + \ \boldsymbol{P}(F_n^c) \\ &\leq \boldsymbol{E}[|\Delta_k - \Delta_{n+ns_\delta}|\mathbb{1}_{F_n}] \ + \ \delta \ + \ \boldsymbol{P}(F_n^c); \end{aligned}$$

However, by using (4.52), we still have $\lim_{n\to\infty} J_{2n,k} \leq \delta$ and so, analogously to part (b), there exists $n_0 \geq 1$ such that $J_{n2} < 2\delta$ for any $n \geq n_0$. Since J_{1n} does not depend on k, (c1) follows. For part (c2), we combine (4.51) and $\mathcal{A}_n^c$ to obtain

$$\boldsymbol{E}\left[G(n, s_\delta)\right] \ \leq \ -C\frac{\epsilon'}{2} \tag{4.53}$$

where we recall that $G(n, s_\delta) = (\Delta_{n+ns_\delta} - \Delta_n)$. Moreover, by (4.52) there exists $n_0 \geq 1$ such that $\boldsymbol{E}[|\Delta_k - \Delta_{n+ns_\delta}|] \leq C\frac{\epsilon'}{2}$ for any $n \geq n_0$. Hence, (c2) follows by combining (4.52), (4.53) and $\mathcal{B}_n$ as follows:

$$\boldsymbol{E}[\Delta_k] \ \leq \ \boldsymbol{E}[|\Delta_k - \Delta_{n+ns_\delta}|] \ + \ \boldsymbol{E}[G(n, s_\delta)] \ + \ \boldsymbol{E}[\Delta_n] \ = \ 2\epsilon'. \quad \square$$

Remark 4.9. It is possible to present a modification of the current arguments along the traditional probabilistic lines. We chose to present the above alternative logical argument.

Remark 4.10. An anonymous referee raised the issue of relaxing the hypothesis concerning the boundedness of $u(\cdot)$. While such condition has been used in several estimates, we notice that it is not required in the proof of the Theorem 4.1. In this case, under weak additional conditions on the tails of $u(\xi_{1,1})$ and $u(\xi_{2,1})$ one can modify the arguments to obtain an analogous version of Theorem 4.1. The challenge however is to establish the comparison arguments between various urns without this hypothesis. This seems to be a challenging task at this moment even though the authors believe that the results should hold without the boundedness assumption. It is worth pointing out that even for the MRRU model, the limit theorems without the boundedness condition are not known.

5 Proof of Strong Consistency

In this section, we provide the proof of the strong consistency of the urn proportion Z_n for any values of m_1 and m_2, when the random thresholds $\hat{\rho}_{1,n}$ and $\hat{\rho}_{2,n}$ converge with probability one.

Proof of Theorem 2.4. We divide the proof into three steps:

(a) $\boldsymbol{P}\left(\ \rho_2 \ \leq \ \underline{\lim}_n Z_n \leq \overline{\lim}_n Z_n \ \leq \ \rho_1 \ \right) \ = \ 1,$

(b)

$$\begin{cases} \boldsymbol{P}(\overline{\lim}_n Z_n \geq \rho_1) = 1 & \text{if } m_1 > m_2, \\ \boldsymbol{P}(\underline{\lim}_n Z_n \leq \rho_2) = 1 & \text{if } m_1 < m_2. \end{cases}$$

(c) $\boldsymbol{P}\left(\ \lim_n Z_n \ \textit{exists} \ \right) \ = \ 1.$

For part (a), firstly note that, when $\rho_1 = 1$ and $\rho_2 = 0$, result (a) is trivially true, hence consider $0 < \rho_2 \leq \rho_1 < 1$. We show that $\boldsymbol{P}(\overline{\lim}_n Z_n \leq \rho_1) = 1$, since the proof of $\boldsymbol{P}(\underline{\lim}_n Z_n \geq \rho_2) = 1$ is completely analogous. To this end, we show that there cannot exist $\epsilon > 0$ and $\rho' > \rho_1$ such that

$$\boldsymbol{P}\left(\overline{\lim}_n Z_n > \rho_1'\right) \ \geq \ \epsilon \ > \ 0. \tag{5.1}$$

We prove this by contradiction using a comparison argument with an RRU model. The proof involves last exit time arguments. Now, suppose (5.1) holds and let $A_1 := \{\overline{\lim}_n Z_n > \rho_1'\}$. Let

$$R_1 := \left\{ \ k \geq 0 \ : \ \hat{\rho}_{1,k} \geq \frac{\rho_1' + \rho_1}{2} \ \right\},$$

and denote the last time the process $\{\hat{\rho}_{1,n}; n \geq 1\}$ is above $(\rho_1' + \rho_1)/2$ by

$$t_{\frac{\rho_1'+\rho_1}{2}} = \begin{cases} \sup\{R_1\} & \text{if } R_1 \neq \emptyset; \\ 0 & \text{otherwise.} \end{cases}$$

Since $\hat{\rho}_{1,n} \overset{a.s.}{\to} \rho_1$ by (2.5), then we have that $\boldsymbol{P}\left(t_{\frac{\rho_1'+\rho_1}{2}} < \infty\right) = 1$. Hence, there exists $n_\epsilon \in \mathbb{N}$ such that

$$\boldsymbol{P}\left(t_{\frac{\rho_1'+\rho_1}{2}} > n_\epsilon\right) \;\leq\; \frac{\epsilon}{2}. \tag{5.2}$$

Setting $B_1 := \left\{t_{\frac{\rho_1'+\rho_1}{2}} > n_\epsilon\right\}$ and using (5.2), it follows that

$$\epsilon \;\leq\; \boldsymbol{P}(A_1) \;\leq\; \epsilon/2 \;+\; \boldsymbol{P}(A_1 \cap B_1^c).$$

Now, we show that $\boldsymbol{P}(A_1 \cap B_1^c) = 0$. Setting

$$C_1 \;=\; \left\{ \underline{\lim}_n Z_n < \frac{\rho_1' + \rho_1}{2} \right\},$$

we decompose $\boldsymbol{P}(A_1 \cap B_1^c)$ as follows:

$$\boldsymbol{P}(A_1 \cap B_1^c) \;\leq\; \boldsymbol{P}(E_1) \;+\; \boldsymbol{P}(E_2),$$

where $E_1 = A_1 \cap B_1^c \cap C_1$ and $E_2 = A_1 \cap B_1^c \cap C_1^c$.

Consider the term $\boldsymbol{P}(E_2)$. Note that on the set C_1^c, we have $\left\{\underline{\lim}_n Z_n \geq \frac{\rho_1'+\rho_1}{2}\right\}$ and on the set B_1^c we have $\{\hat{\rho}_{1,n} \leq \frac{\rho_1'+\rho_1}{2}\}$ for any $n \geq n_\epsilon$. Hence, since $(B_1^c \cap C_1^c) \supset E_2$, on the set E_2 we have that $W_{1,n} = \mathbb{1}_{\{Z_n \leq \hat{\rho}_{1,n}\}} \overset{a.s.}{\to} 0$. Then, letting $\tau_W := \sup\{k \geq 1 : W_{1,k} = 1\}$ we have $\boldsymbol{P}(E_2 \cap \{\tau_W < \infty\}) = \boldsymbol{P}(E_2)$ and, on the set E_2, for any $n \geq \tau_W$ the ARRU model can be written as follows:

$$\begin{cases} Y_{1,n+1} = Y_{1,\tau_W} \\ \\ Y_{2,n+1} = Y_{2,\tau_W} + \sum_{i=\tau_W}^{n+1} (1 - X_i)\, D_{2,i}, \end{cases}$$

where $W_{1,i-1} = 0$ for any $i \geq \tau_W$, and $W_{2,i-1} = 1$ because $W_{2,i-1} + W_{2,i-1} \geq 1$ by construction. Now, consider an RRU model $\{Z_i^R; i \geq 1\}$ with initial composition $(Y_{1,0}^R, Y_{2,0}^R) = (Y_{1,\tau_W}, Y_{2,\tau_W})$ a.s.; the reinforcements are defined as $D_{1,i}^R = 0$ and $D_{2,i}^R = D_{2,\tau_W+i}$ for any $i \geq 1$ a.s.; the drawing process is modeled by $X_{i+1}^R := \mathbb{1}_{\{U_{i+1}^R < Z_i^R\}}$ and $U_i^R = U_{\tau_W+i}$ a.s., where $\{U_n; n \geq 1\}$ is the sequence such that $X_{n+1} = \mathbb{1}_{\{U_n < Z_n\}}$ for any $n \geq 1$. Formally, this RRU model can be described for any $n \geq 1$ as follows:

$$\begin{cases} Y_{1,n+1}^R = Y_{1,0}^R = Y_{1,\tau_W} \\ \\ Y_{2,n+1}^R = Y_{2,0}^R + \sum_{i=0}^{n+1} \left(1 - X_i^R\right) D_{2,i}^R = Y_{2,\tau_W} + \sum_{i=\tau_W}^{n+\tau_W+1} (1 - X_i)\, D_{2,i}. \end{cases}$$

Hence, on the set E_2 we have that for any $n \geq \tau_W$

$$(Y_{1,n}, Y_{2,n}) = (Y_{1,n-\tau_W}^R, Y_{2,n-\tau_W}^R).$$

Since from Muliere et al. (2006, Theorem 8) $\boldsymbol{P}(\overline{\lim}_n Z_n^R = 0) = 1$, on the set E_2 we have that $\{\overline{\lim}_n Z_n = 0\}$. This is incompatible with the set A_1 which includes E_2. Hence $\boldsymbol{P}(E_2) = 0$.

We now turn to the proof that $\boldsymbol{P}(E_1) = 0$. To this end, let

$$\tau_\epsilon \;:=\; \inf\left\{k \geq n_\epsilon \;:\; \left\{Z_k < \frac{\rho_1' + \rho_1}{2}\right\} \cap \left\{Y_k > \frac{b}{(\rho_1' - \rho_1)/2}\right\}\right\}$$

and note that, since by result (c) of Lemma 3.1 $Y_n \overset{a.s.}{\to} \infty$, $\boldsymbol{P}(C_1 \cap \{\tau_\epsilon < \infty\}) = \boldsymbol{P}(C_1)$. Moreover, on the set B_1^c we have that $\{\hat{\rho}_{1,n} \leq \frac{\rho_1'+\rho_1}{2}\}$ for any $n \geq n_\epsilon$. We now show by induction that on the set $B_1^c \cap C_1$ we have $\{Z_n < \rho_1' \;\forall n \geq \tau_\epsilon\}$. By definition we have $Z_{\tau_\epsilon} < \frac{\rho_1'+\rho_1}{2}$, and by Lemma 3.5 this implies $Z_{\tau_\epsilon+1} < \rho_1'$; now, consider an arbitrary $n > \tau_\epsilon$; if $Z_n < \frac{\rho_1'+\rho_1}{2}$, then by Lemma 3.5 we have $Z_{n+1} < \rho_1'$; if $\frac{\rho_1'+\rho_1}{2} < Z_n < \rho_1'$ we have

$W_{1,n} = 0$ and so $Z_{n+1} \leq Z_n < \rho_1'$. Hence, since $(B_1^c \cap C_1) \subset E_1$, on the set E_1 we have $\{Z_n < \rho_1' \ \forall n \geq \tau_\epsilon\}$. This is incompatible with the set A_1 which also includes E_1. Hence $\boldsymbol{P}(E_1) = 0$. Combining all together we have $\epsilon \ \leq \ \epsilon/2 + \boldsymbol{P}(E_1) + \boldsymbol{P}(E_2) \ = \ \epsilon/2$, which is impossible. Thus, we conclude that $\boldsymbol{P}(A_1^c) = \boldsymbol{P}(\overline{\lim}_n Z_n \leq \rho_1) = 1$.

For part (b), wlog we assume $m_1 > m_2$ to show that $\boldsymbol{P}(\overline{\lim}_n Z_n \geq \rho_1) = 1$, since the proof of $\boldsymbol{P}(\underline{\lim}_n Z_n \leq \rho_2) = 1$ when $m_1 < m_2$ follows the same arguments. To this end, we now show that there cannot exist $\epsilon > 0$ and $\rho' < \rho_1$ such that

$$\boldsymbol{P}\left(\overline{\lim}_n Z_n < \rho_1'\right) \ \geq \ \epsilon \ > \ 0. \tag{5.3}$$

We prove this by contradiction, using a comparison argument with an RRU model. Now suppose (5.3) holds and let $A_2 := \{\overline{\lim}_n Z_n < \rho_1'\}$. Let

$$R_2 := \left\{ \ k \geq 0 \ : \ \hat{\rho}_{1,k} < \frac{\rho_1' + \rho_1}{2} \ \right\},$$

and define the last time the process $\{\hat{\rho}_{1,n}; n \geq 1\}$ is less than $(\rho_1' + \rho_1)/2$ by

$$\tau_{\frac{\rho_1'+\rho_1}{2}} = \begin{cases} \sup\{R_2\} & \text{if } R_2 \neq \emptyset; \\ 0 & \text{otherwise.} \end{cases}$$

Since $\hat{\rho}_{1,n} \stackrel{a.s.}{\to} \rho_1$, then we have that $\boldsymbol{P}\left(\tau_{\frac{\rho_1'+\rho_1}{2}} < \infty\right) = 1$. Hence, there exists $n_\epsilon \in \mathbb{N}$ such that

$$\boldsymbol{P}\left(\tau_{\frac{\rho_1'+\rho_1}{2}} > n_\epsilon\right) \ \leq \ \frac{\epsilon}{2}. \tag{5.4}$$

Setting $B_2 := \left\{\tau_{\frac{\rho_1'+\rho_1}{2}} > n_\epsilon\right\}$ and using (5.4), it follows that

$$\epsilon \ \leq \ \boldsymbol{P}(A_2) \ \leq \ \epsilon/2 \ + \ \boldsymbol{P}(A_2 \cap B_2^c).$$

Let $E_3 := A_2 \cap B_2^c$. We now show that $\boldsymbol{P}(E_3)$=0. On the set A_2, we have $\left\{\overline{\lim}_n Z_n \leq \rho_1'\right\}$ and on the set B_2^c, we have $\{\hat{\rho}_{1,n} \geq \frac{\rho_1'+\rho_1}{2}\}$ for any $n \geq n_\epsilon$. Hence, on the set E_3 we have that $W_{1,n} = \mathbb{1}_{\{Z_n \leq \hat{\rho}_{1,n}\}} \stackrel{a.s.}{\to} 1$. Then, letting $\tau_W := \sup\{k \geq 1 : W_{1,n} = 0\}$ we have $\boldsymbol{P}(E_3 \cap \{\tau_W < \infty\}) = \boldsymbol{P}(E_3)$. Now, analogously to the proof of $\boldsymbol{P}(E_2) = 0$, we can use comparison arguments with an RRU model to show that on the set E_3 we have $\{\overline{\lim}_n Z_n = 1\}$. This is incompatible with the set A_2, which also includes E_3. Hence $\boldsymbol{P}(E_3) = 0$. Combining all together we have $\epsilon \ \leq \ \epsilon/2 + \boldsymbol{P}(E_3) \ = \ \epsilon/2$, which is impossible. Thus, we conclude that the event $A_2^c = \{\overline{\lim}_n Z_n \geq \rho_1\}$ occurs with probability one.

For part (c), note that, combining (a) and (b), we have shown that

$$\begin{cases} \boldsymbol{P}(\overline{\lim}_n Z_n = \rho_1) = 1 & \text{if } m_1 > m_2, \\ \boldsymbol{P}(\rho_2 \leq \underline{\lim}_n Z_n \leq \overline{\lim}_n Z_n \leq \rho_1) = 1 & \text{if } m_1 = m_2, \\ \boldsymbol{P}(\underline{\lim}_n Z_n = \rho_2) = 1 & \text{if } m_1 < m_2. \end{cases}$$

Therefore, if the process $\{Z_n; n \geq 1\}$ converges almost surely, we obtain (2.6). Wlog, assume $m_1 \geq m_2$, since the proof of the case $m_1 \leq m_2$ is completely analogous.

First, let d, u, γ and ρ_1' $(d < u < \gamma < \rho_1' < \rho_1)$ be four constants in $(0,1)$. Let $\{\tau_j(d,u); j \geq 1\}$ and $\{t_j(d,u); j \geq 1\}$ be the sequences of random variables defined in (3.2). Since d and u are fixed in this proof, we sometimes denote $\tau_j(d,u)$ by τ_j and $t_j(d,u)$ by t_j. It is easy to see that τ_n and t_n are stopping times with respect to $\{\mathcal{F}_n; n \geq 1\}$.

Recall that, by Lemma 3.2, we have that for every $0 < d < u < 1$

$$\begin{aligned} Z_n \text{ converges a.s.} \quad & \Leftrightarrow \boldsymbol{P}(t_n(d,u) < \infty) \to 0, \\ & \Leftrightarrow \sum_{n=1}^{\infty} \boldsymbol{P}(t_{n+1}(d,u) = \infty | t_n(d,u) < \infty) = \infty. \end{aligned}$$

Now, to prove that Z_n converges a.s., it is sufficient to show that

$$\boldsymbol{P}(t_n(d,u) < \infty) \to 0,$$

for all $0 < d < u < 1$. Suppose Z_n does not converges a.s.. This implies that $\boldsymbol{P}(t_n < \infty) \downarrow \phi_1 > 0$, since $\boldsymbol{P}(t_n < \infty)$ is a non-increasing sequence. We will show that for large j there exists a constant $\phi < 1$ dependent on ϕ_1, such that

$$\boldsymbol{P}(t_{j+1} < \infty | t_j < \infty) \leq \phi. \tag{5.5}$$

This result implies that $\sum_n \boldsymbol{P}(t_{n+1} = \infty | t_n < \infty) = \infty$, establishing by Lemma 3.2 that $\boldsymbol{P}(t_n < \infty)$ converges to zero as n goes to infinity, which is a contradiction.

Consider the term $\boldsymbol{P}(t_{i+1} < \infty | t_i < \infty)$. First, let us denote by $\tau_{\rho_1'}$ the last time the process $\hat{\rho}_{1,n}$ is below ρ_1', i.e.

$$\tau_{\rho_1'} = \begin{cases} \sup\{n \geq 1 : \hat{\rho}_{1,n} \leq \rho_1'\} & \text{if } \{n \geq 1 : \hat{\rho}_{1,n} \leq \rho_1'\} \neq \emptyset; \\ 0 & \text{otherwise.} \end{cases}$$

Since $\hat{\rho}_{1,n} \overset{a.s.}{\to} \rho_1$, we have that $\boldsymbol{P}\left(\tau_{\rho_1'} < \infty\right) = 1$. Hence, for any $\epsilon \in \left(0, \frac{1}{2}\right)$ there exists $n_\epsilon \in \mathbb{N}$ such that

$$\frac{1}{\phi_1} \boldsymbol{P}\left(\tau_{\rho_1'} > n_\epsilon\right) \leq \epsilon. \tag{5.6}$$

By denoting $\boldsymbol{P_i}(\cdot) = \boldsymbol{P}(\cdot | t_i < \infty)$ and using $t_i \leq \tau_i \leq t_{i+1}$ we obtain

$$\boldsymbol{P}(t_{i+1} < \infty | t_i < \infty) \leq \boldsymbol{P_i}(\tau_i < \infty).$$

Hence

$$\boldsymbol{P_i}(\tau_i < \infty) \leq \boldsymbol{P_i}\left(\{\tau_i < \infty\} \cap \{\tau_{\rho_1'} \leq n_\epsilon\}\right) + \boldsymbol{P_i}\left(\tau_{\rho_1'} > n_\epsilon\right). \tag{5.7}$$

We start with the second term in (5.7). Note that

$$\boldsymbol{P_i}\left(\tau_{\rho_1'} > n_\epsilon\right) \leq \frac{\boldsymbol{P}\left(\tau_{\rho_1'} > n_\epsilon\right)}{\boldsymbol{P}(t_i < \infty)} \leq \frac{\boldsymbol{P}\left(\tau_{\rho_1'} > n_\epsilon\right)}{\phi_1} \leq \epsilon,$$

where the last inequality follows from (5.6).

Now, consider the first term in (5.7). Since the probability is conditioned on the set $\{t_i < \infty\}$, in what follows we will consider the urn process at times n after the stopping time t_i. Since we want to show (5.5) for large i, we can choose an integer $i \geq n_\epsilon$ and

$$i > \log_{\frac{u(1-d)}{d(1-u)}} \left(\frac{b}{Y_0(\gamma - u)}\right),$$

so that

(i) $t_i \geq i \geq n_\epsilon$ a.s.;

(ii) from Lemma 3.3, we have that $Y_{\tau_i} > b/(\gamma - u)$ a.s.

These two properties imply respectively that, on the set $\{n \geq t_i\}$

(i) $\hat{\rho}_{1,n} \geq \rho_1'$, since from $\{\tau_{\rho_1'} \leq n_\epsilon\}$ we have that $n \geq \tau_{\rho_1'}$;

(ii) $Z_{t_i} \in (u, \gamma)$, since $Z_{t_i - 1} \leq u$ and $Z_{t_i} > u$ and from Lemma 3.5 we have that $|Z_n - Z_{n-1}| < (\gamma - u)$.

Now, let us define two sequences of stopping times $\{t_n^*; n \geq 1\}$ and $\{\tau_n^*; n \geq 1\}$, where t_n^* represents the first time after τ_{n-1}^* the process $Z_{t_i + n}$ up-crosses ρ_1', while τ_n^* represents the first time after t_n^* the process Z_{t_i+n} down-crosses γ. Formally, let $\tau_0^* = 0$ and define for every $j \geq 1$ two stopping times

$$t_j^* = \begin{cases} \inf\{n > \tau_{j-1}^* : Z_{t_i+n} > \rho_1'\} & \text{if } \{n > \tau_j^* : Z_{t_i+n} > \rho_1'\} \neq \emptyset; \\ +\infty & \text{otherwise.} \end{cases}$$

$$\tau_j^* = \begin{cases} \inf\{n > t_j^* : Z_{t_i+n} \leq \gamma\} & \text{if } \{n > t_{j-1}^* : Z_{t_i+n} \leq \gamma\} \neq \emptyset; \\ +\infty & \text{otherwise.} \end{cases}$$

Note that, since $Z_{t_i + \tau_j^* - 1} \geq \gamma$ and $Z_{t_i + \tau_j^*} < \gamma$, from (ii) we have that $Z_{t_i + \tau_j^*} \in (u, \gamma)$.

For any $j \geq 0$, let $\{\tilde{Z}_n^j; n \geq 1\}$ be an RRU model defined as follows:

(1) $\left(\tilde{Y}^j_{1,0}, \tilde{Y}^j_{2,0}\right) = \left(Y_{1,t_i+\tau^*_j}, Y_{1,t_i+\tau^*_j}\frac{u+d}{2-u-d}\right)$ a.s., which implies that $\tilde{Z}^j_0 = \frac{u+d}{2}$;

(2) the drawing process is modeled by $\tilde{X}^j_{n+1} = \mathbb{1}_{\{\tilde{U}^j_{n+1} < \tilde{Z}^j_n\}}$, where $\tilde{U}^j_{n+1} = U_{t_i+\tau^*_j+n+1}$ a.s. and U_n is such that $X_n = \mathbb{1}_{\{U_n < Z_{n-1}\}}$;

(3) the reinforcements are defined as $\tilde{D}^j_{2,n+1} = D_{2,t_i+\tau^*_j+n+1} + (m_1 - m_2)$, $\tilde{D}^j_{1,n+1} = D_{1,t_i+\tau^*_j+n+1}$ a.s.; this means $\boldsymbol{E}[\tilde{D}^j_{1,n}] = \boldsymbol{E}[\tilde{D}^j_{2,n}]$ for any $n \geq 1$;

(4) the urn process evolves as an RRU model, i.e. for any $n \geq 0$

$$\begin{cases} \tilde{Y}^j_{1,n+1} = \tilde{Y}^j_{1,n} + \tilde{X}^j_{n+1}\tilde{D}^j_{1,n+1}, \\ \tilde{Y}^j_{2,n+1} = \tilde{Y}^j_{2,n} + \left(1 - \tilde{X}^j_{n+1}\right)\tilde{D}^j_{2,n+1}, \\ \tilde{Y}^j_{n+1} = \tilde{Y}^j_{1,n+1} + \tilde{Y}^j_{2,n+1}, \\ \tilde{Z}^j_{n+1} = \frac{\tilde{Y}^j_{1,n+1}}{\tilde{Y}^j_{n+1}}. \end{cases}$$

We will compare the process $\{\tilde{Z}^j_n; n \geq 1\}$ with the ARRU process $\{Z_{t_i+n}; n \geq 1\}$. Note that at time n, we have defined only the processes $\tilde{Z}^j$ such that $\tau^*_j < n$.

We will prove, by induction, that on the set $\{\tau_{\rho'_1} \leq n_\epsilon\}$, for any $j \in \mathbb{N}$ and for any $n \leq t^*_{j+1} - \tau^*_j$

$$\tilde{Z}^j_n < Z_{t_i+\tau^*_j+n}, \qquad \tilde{Y}^j_{2,n} \geq Y_{2,t_i+\tau^*_j+n}, \qquad \tilde{Y}^j_{1,n} < Y_{1,t_i+\tau^*_j+n}. \tag{5.8}$$

In other words, we will show, provided that $t_i > \tau_{\rho'_1}$, that for each $j \geq 1$ the process $\tilde{Z}^j_n$ is always dominated by the original process $Z_{t_i+\tau^*_j+n}$, as long as $Z_{t_i+\tau^*_j+n}$ is dominated by ρ'_1 (i.e. for $n \leq t^*_{j+1} - \tau^*_j$). By construction we have that

$$\tilde{Z}^j_0 = \frac{d+u}{2} < u < Z_{t_i+\tau^*_j}, \qquad \tilde{Y}^j_{1,0} = Y_{1,t_i+\tau^*_j}$$

which immediately implies $\tilde{Y}^j_{2,0} > Y_{2,t_i+\tau^*_j}$. To this end, we assume (5.8) by induction hypothesis. First, we will show that $\tilde{Y}^j_{2,n+1} > Y_{2,t_i+\tau^*_j+n+1}$. Since from (5.8) $\tilde{Z}^j_n < Z_{t_i+\tau^*_j+n}$ for $n \leq t^*_{j+1} - \tau^*_j$, by construction we obtain that

$$\tilde{X}^j_{n+1} = \mathbb{1}_{\{\tilde{U}^j_n < \tilde{Z}^j_n\}} \leq \mathbb{1}_{\{U_{t_i+\tau^*_j+n+1} < Z_{t_i+\tau^*_j+n}\}} = X_{t_i+\tau^*_j+n+1}.$$

As a consequence, since $W_n \leq 1$ for any $n \geq 1$, we have that

$$\begin{aligned} \left(Y_{2,t_i+\tau^*_j+n+1} - Y_{2,t_i+\tau^*_j+n}\right) &= \left(1 - X_{t_i+\tau^*_j+n+1}\right) D_{2,t_i+\tau^*_j+n+1} W_{2,t_i+\tau^*_j+n} \\ &\leq (1 - \tilde{X}^j_{n+1})\tilde{D}^j_{2,n+1} \\ &= \left(\tilde{Y}^j_{2,n+1} - \tilde{Y}^j_{2,n}\right), \end{aligned}$$

which, using hypothesis (5.8), implies $\tilde{Y}^j_{2,n+1} > Y_{2,t_i+\tau^*_j+n+1}$. Similarly, we now show that $\tilde{Y}^j_{1,n+1} \leq Y_{1,t_i+\tau^*_j+n+1}$. We have

$$\left(Y_{1,t_i+\tau^*_j+n+1} - Y_{1,t_i+\tau^*_j+n}\right) = X_{t_i+\tau^*_j+n+1} D_{1,t_i+\tau^*_j+n+1} W_{1,t_i+\tau^*_j+n}.$$

From (i) we have that, as long as Z remains below ρ'_1, Z is also above the process $\hat{\rho}_{1,n}$. Since we consider the behavior of $Z_{t_i+\tau^*_j+n}$ when it is below ρ'_1, i.e. $n \leq \tau^*_{j+1} - t^*_j$, we have that $W_{1,t_i+\tau^*_j+n} = 1$. Thus,

$$\left(Y_{1,t_i+\tau^*_j+n+1} - Y_{1,t_i+\tau^*_j+n}\right) \geq \tilde{X}^j_{n+1}\tilde{D}^j_{1,n+1} = \left(\tilde{Y}^j_{1,n+1} - \tilde{Y}^j_{1,n}\right),$$

which using hypothesis (5.8) implies $\tilde{Y}^j_{1,n+1} \leq Y_{1,t_i+\tau^*_j+n+1}$. Thus, we have shown that, on the set $\{\tau_{\rho'_1} \leq n_\epsilon\}$, for any $n \leq t^*_{j+1} - \tau^*_j$, $\tilde{Z}^j_{n+1} < Z_{t_i+\tau^*_j+n+1}$, $\tilde{Y}^j_{1,n+1} \leq Y_{1,t_i+\tau^*_j+n+1}$ and $\tilde{Y}^j_{2,n+1} > Y_{2,t_i+\tau^*_j+n+1}$ hold.

Now, for any $j \geq 1$, let T_j be the stopping time for $\tilde{Z}^j_n$ to exit from (d, u), i.e.:

$$T_j = \begin{cases} \inf\{R_3\} & \text{if } R_3 \neq \emptyset; \\ +\infty & \text{otherwise,} \end{cases}$$

where $R_3 := \{n \geq 1 : \tilde{Z}_n^j \leq d \text{ or } \tilde{Z}_n^j \geq u\}$. Note that, on the set $\{\tau_{\rho_1'} \leq n_\epsilon\}$,

$$\{\tau_i < \infty\} = \left\{\inf_{n\geq 1}\{Z_{t_i+n}\} < d\right\} \subset \left(\cup_{j:\tau_j^* \leq n}\left\{\inf_{n\geq 1}\left\{\tilde{Z}^j_{n-\tau_j^*}\right\} < d\right\}\right)$$
$$\subset \left(\cup_{j=0}^{\infty}\{T_j < \infty\}\right).$$

Hence, by denoting $\boldsymbol{P_i}(\cdot) = \boldsymbol{P}(\cdot|t_i < \infty)$ and $\boldsymbol{E_i}[\cdot] = \boldsymbol{E}[\cdot|t_i < \infty]$, we have that

$$\begin{aligned}\boldsymbol{P_i}\left(\{\tau_i < \infty\} \cap \{\tau_{\rho_1'} \leq n_\epsilon\}\right) &\leq \boldsymbol{P_i}\left(\{\cup_{j=0}^{\infty}\{T_j < \infty\}\} \cap \{\tau_{\rho_1'} \leq n_\epsilon\}\right)\\ &\leq \sum_{j=0}^{\infty}\boldsymbol{P_i}\left(\{T_j < \infty\} \cap \{\tau_{\rho_1'} \leq n_\epsilon\}\right),\end{aligned}$$

and, by setting $h = \frac{u-d}{2}$, each term of the series is less or equal than

$$\boldsymbol{P_i}\left(\left\{\sup_{n\geq 1}|\tilde{Z}_n^j - \tilde{Z}_0^j| \geq h\right\} \cap \{\tau_{\rho_1'} \leq n_\epsilon\}\right) \leq \boldsymbol{P_i}\left(\sup_{n\geq 1}|\tilde{Z}_n^j - \tilde{Z}_0^j| \geq h\right).$$

Note that $\{\tilde{Z}_n^j; n \geq 1\}$ is the proportion of red balls in an RRU model with same reinforcement means. Then, by using Lemma 3.4 we obtain

$$\begin{aligned}\boldsymbol{P_i}\left(\sup_{n\geq 1}|\tilde{Z}_n^j - \tilde{Z}_0^j| \geq h\right) &= \boldsymbol{E_i}\left[\boldsymbol{P}\left(\sup_{n\geq 1}|\tilde{Z}_n^j - \tilde{Z}_0^j| \geq h \middle| \mathcal{F}_{\tau_i + t_j^*}\right)\right]\\ &\leq \boldsymbol{E_i}\left[\frac{b}{Y_{t_j^*}}\right]\left(\frac{4}{h^2} + \frac{2}{h}\right).\end{aligned}$$

Moreover, by using Lemma 3.3, the right hand side can be expressed as

$$\boldsymbol{E_i}\left[\frac{b}{Y_{t_i}}\right]\left(\frac{\rho_1'(1-\gamma)}{\gamma(1-\rho_1')}\right)^j\left(\frac{4}{h^2} + \frac{2}{h}\right).$$

Since by result (c) of Lemma 3.1 Y_n converges a.s. to infinity, and since $\tau_i \to \infty$ a.s. because $\tau_i \geq i$, we have that $\boldsymbol{E_i}\left[Y_{t_i}^{-1}\right]$ tends to zero as i increases. As a consequence, we can choose an integer i large enough such that

$$\boldsymbol{E_i}\left[\frac{b}{Y_{t_i}}\right]\left(\frac{4}{h^2} + \frac{2}{h}\right)\left(\frac{1-\rho_1'}{1-\rho_1'/\gamma}\right) < \frac{1}{2},$$

which by setting $\phi = 1/2 + \epsilon$ implies (5.5), i.e.

$$\boldsymbol{P}(t_{i+1} < \infty | t_i < \infty) \leq \phi < 1.$$

This concludes the proof. □

Proof of Lemma 2.6. We divide the proof in two parts:

(i) $m_1 \neq m_2$ and $0 < \rho_2 < \rho_1 < 1$;

(ii) $m_1 = m_2$ and $0 \leq \rho_2 < \rho_1 \leq 1$, on the set $\{Z_\infty \neq \{0, 1\}\}$;

For part (i), assume $m_1 > m_2$, since the proof in the case $m_1 < m_2$ follows the same arguments. In this case $\min\{m_1, m_2\} = m_2$ and, by using Theorem 2.4, we have $Z_n \overset{a.s.}{\to} \rho_1$; thus, since $\hat{\rho}_{2,n} \overset{a.s.}{\to} \rho_2$ and $\rho_1 > \rho_2$, denoting by $\tau \in \mathbb{N}$ the last time Z_n crosses $\hat{\rho}_{2,n}$, i.e. $\tau := \sup\{k \geq 1, Z_k < \hat{\rho}_{2,k}\}$, we have that $\boldsymbol{P}(\tau < \infty) = 1$. Then, since $\{\tau \leq n\} \subset \{W_{2,k} = 1, \forall k \geq n\}$, we use the following decomposition, on the set $\{\tau \leq n\}$,

$$\frac{Y_{2,n}}{n} = \frac{1}{n}\sum_{i=1}^{n}(1 - X_i)D_{2,i}W_{2,i-1} = \mathcal{W}_{0,n} + \mathcal{W}_{1,n},$$

where

$$\begin{aligned}\mathcal{W}_{0,n} &:= \frac{1}{n}\sum_{i=1}^{\tau}(1 - X_i)D_{2,i}(W_{2,i-1} - 1),\\ \mathcal{W}_{1,n} &:= \frac{1}{n}\sum_{i=\tau}^{n}(1 - X_i)D_{2,i}.\end{aligned}$$

Since $\boldsymbol{P}(\tau < \infty) = 1$, we have $\mathcal{W}_{0,n} \overset{a.s.}{\to} 0$, while since

$$\boldsymbol{E}[(1 - X_i)D_{2,i}|\mathcal{F}_{i-1}] = (1 - Z_{i-1})m_2 \overset{a.s.}{\to} (1 - Z_\infty)m_2,$$

we have that $\mathcal{W}_{1,n} \stackrel{a.s.}{\to} (1-Z_\infty)m_2$. Finally, since $Y_n = (1-Z_n)^{-1}Y_{2,n}$, we have $\frac{Y_n}{n} \stackrel{a.s.}{\to} m_2 = \min\{m_1, m_2\}$.

For part (ii), since $m_1 = m_2 = m$, by using Theorem 2.4 we have $Z_n \stackrel{a.s.}{\to} Z_\infty \in [\rho_2, \rho_1]$; then, on the set $\{Z_\infty \in (0,1)\}$, we can follow the arguments of part (i), so obtaining

$$\frac{Y_{2,n}}{n} \stackrel{a.s.}{\to} (1-Z_\infty)m, \qquad \frac{Y_{1,n}}{n} \stackrel{a.s.}{\to} Z_\infty m.$$

Thus, $\frac{Y_n}{n} = \frac{Y_{1,n}}{n} + \frac{Y_{2,n}}{n} \stackrel{a.s.}{\to} m$. □

The proof of Lemma 2.7 is based on comparison arguments between the ARRU and RRU model. Specifically, for any $n_0 \geq 1$, we consider an RRU process $\{\tilde{Z}_k(n_0); k \geq 0\}$ coupled with the ARRU process $\{Z_{n_0+k}; k \geq 0\}$ as follows: the initial composition is $(\tilde{Y}_{1,0}(n_0), \tilde{Y}_{2,0}(n_0)) = (Y_{1,n_0}, Y_{2,n_0})$ and for any $k \geq 1$

$$\begin{cases} \tilde{Y}_{1,k}(n_0) = \tilde{Y}_{1,k-1}(n_0) + \tilde{X}_{1,k}(n_0) D_{1,k} \\ \tilde{Y}_{2,k}(n_0) = \tilde{Y}_{2,k-1}(n_0) + \left(1 - \tilde{X}_k(n_0)\right) D_{2,k}, \end{cases} \tag{5.9}$$

where $\tilde{X}_k(n_0) = \mathbb{1}_{\{U_k \leq \tilde{Z}_{k-1}(n_0)\}}$. The relation between $\tilde{Z}_k(n_0)$ and Z_{n_0+k} required in the proof of Lemma 2.7 is expressed in the following result.

Lemma 5.1. *For any $n_0, n_1 \geq 1$, we have that*

$$\left(\cap_{k=1}^{n_1} \{\hat{\rho}_{2,n_0+k} \ \leq \ Z_{n_0+k} \ \leq \ \hat{\rho}_{1,n_0+k} \ \}\right) \subset \left(\cap_{k=1}^{n_1} \{ \ Z_{n_0+k} = \tilde{Z}_k(n_0) \ \}\right). \tag{5.10}$$

Proof. First, consider the dynamics of the RRU process $\{\tilde{Z}_k(n_0); k \geq 0\}$ expressed in (5.9) and the dynamics of the ARRU process $\{Z_{n_0+k}; k \geq 0\}$ expressed as follows:

$$\begin{cases} Y_{1,n_0+k} = Y_{1,n_0+k-1} + X_{1,n_0+k} D_{1,n_0+k} W_{1,n_0+k-1} \\ Y_{2,n_0+k} = Y_{2,n_0+k-1} + (1 - X_{1,n_0+k}) D_{2,n_0+k} W_{2,n_0+k-1}, \end{cases} \tag{5.11}$$

where $X_{n_0+k} = \mathbb{1}_{\{U_k \leq Z_{n_0+k-1}\}}$. Hence, (5.10) follows by noticing that for any $1 \leq k \leq n_1$

$$\{\hat{\rho}_{2,n_0+k} \ \leq \ Z_{n_0+k} \ \leq \ \hat{\rho}_{1,n_0+k} \ \} \subset \{W_{1,n_0+k-1} = W_{2,n_0+k-1} = 1\}. \quad \square$$

Proof of Lemma 2.7. The proof is structured as follows: we assume there exist $x \in (\rho_2, \rho_1)$ and $p > 0$ such that $\boldsymbol{P}(Z_\infty = x) = p$ and we show that this assumption leads to a contradiction. To this end, fix $\epsilon > 0$ such that $\rho_2 < x - \epsilon < x + \epsilon < \rho_1$ and denote by $\tau \in \mathbb{N}$ the last time Z_n exceeds $I_\epsilon := (x - \epsilon, x + \epsilon)$: formally,

$$\tau = \begin{cases} \sup\{k > 1 : Z_k \notin I_\epsilon, \} & \text{if } \{k > 1 : Z_k \notin I_\epsilon\} \neq \emptyset; \\ -\infty & \text{otherwise.} \end{cases}$$

Since $\{Z_\infty = x\} \subset \{\tau < \infty\}$ and by (2.5) $\hat{\rho}_{j,n} \stackrel{a.s.}{\to} \rho_j \notin I_\epsilon$, $j \in \{1,2\}$, there exists an integer $k_0 \in \mathbb{N}$ such that,

$$\boldsymbol{P}\left(\ \{\hat{\rho}_{j,n} \notin I_\epsilon, \forall n \geq k_0\} \cap \{\tau \leq k_0\} \cap \{Z_\infty = x\} \ \right) \ \geq \ \frac{p}{2}. \tag{5.12}$$

Now, by using Lemma 5.1, we have that

$$\left(\{\hat{\rho}_{j,n} \notin I_\epsilon, \forall n \geq k_0\} \cap \{\tau \leq k_0\}\right) \ \subset \ \left\{Z_{k_0+n} = \tilde{Z}_n(k_0), \forall n \geq k_0\right\},$$

and hence (5.12) is equivalent to

$$\boldsymbol{P}\left(\ \{\hat{\rho}_{j,n} \notin I_\epsilon, \forall n \geq k_0\} \cap \{\tau \leq k_0\} \cap \{\tilde{Z}_\infty(k_0) = x\} \ \right) \ \geq \ \frac{p}{2}.$$

Finally, the contradiction follows by noticing that by Aletti et al. (2009, Theorem 2), for an RRU model, we have $\boldsymbol{P}(\tilde{Z}_\infty(k_0) = x) = 0$. □

Remark 5.2. As described in Remark 4.10, the boundedness of $u(\cdot)$ plays a critical role in the proofs. Additionally, when $m_1 = m_2$ the weak law of large numbers (even for a bounded $u(\cdot)$) is unclear. Here, the behavior of the thresholds $\hat{\rho}_{1,n}$ and $\hat{\rho}_{2,n}$ is much more erratic and linking this behavior with the tail conditions of $u(\xi_{1,1})$ and $u(\xi_{2,1})$ remains a challenge.

6 Proofs of Limit Distribution of the Proportion of Sampled Ball Colors

We start by presenting the limit distribution of the proportion of sampled ball colors for the RRU model.

Proof of Theorem 2.8. Note that

$$\sqrt{n}\left(\frac{N_{1n}}{n} - Z_\infty\right) \;=\; T_{1n} \;+\; T_{2n},$$

where

$$T_{1n} := n^{-1/2}\left(N_{1n} - \sum_{i=1}^{n} Z_{i-1}\right), \qquad T_{2n} := n^{-1/2}\sum_{i=1}^{n}\left(Z_{i-1} - Z_\infty\right).$$

Now, calling $\Delta Z_j = Z_j - Z_{j-1}$ and $(j \wedge n) := \min\{j, n\}$, we have that

$$\begin{aligned} T_{2n} &= n^{-1/2}\sum_{i=1}^{n}\sum_{j=i}^{\infty}(-\Delta Z_j) \;=\; -n^{-1/2}\sum_{j=1}^{\infty}\sum_{i=1}^{j\wedge n}\Delta Z_j \\ &= -\,n^{-1/2}\sum_{j=1}^{\infty}(j \wedge n)\Delta Z_j \;=\; -(T_{3n} \;+\; T_{4n}), \end{aligned}$$

where, since $(j \wedge n) = n$ for all $j \geq n+1$, we have

$$T_{3n} := n^{-1/2}\sum_{j=1}^{n} j\Delta Z_j, \qquad T_{4n} := n^{1/2}(Z_\infty - Z_n).$$

Now, by using the Doob's decomposition $\Delta Z_j = \Delta M_j + \Delta A_j$ (see Durrett (2010)), where $\boldsymbol{E}[\Delta M_j|\mathcal{F}_{j-1}] = 0$ and $A_j \in \mathcal{F}_{j-1}$, we have $T_{3n} \;=\; T_{5n} \;+\; T_{6n}$, where

$$T_{5n} := n^{-1/2}\sum_{j=1}^{n} j\Delta M_j, \qquad T_{6n} := n^{-1/2}\sum_{j=1}^{n} j\Delta A_j.$$

Then, recalling that

$$\sqrt{n}\left(\frac{N_{1n}}{n} - Z_\infty\right) \;=\; T_{1n} \;-\; T_{4n} \;-\; T_{5n} \;-\; T_{6n},$$

the limit distribution is established by proving the following results:

(a) $T_{4n}|\mathcal{F}_n \xrightarrow{d} \mathcal{N}(0, \Sigma_a)$ (stably), where $\Sigma_a = Z_\infty(1 - Z_\infty)(1 + \frac{\bar{\Sigma}}{m^2})$;

(b) $T_{6n} \xrightarrow{p} 0$;

(c) $(T_{1n} - T_{5n}) \xrightarrow{d} \mathcal{N}(0, \Sigma_c)$ (stably), where $\Sigma_c = Z_\infty(1 - Z_\infty)\frac{\bar{\Sigma}}{m^2}$;

(d) $T_{4n} + (T_{1n} - T_{5n}) \xrightarrow{d} \mathcal{N}(0, \Sigma_a + \Sigma_c)$ (stably).

Part (a) follows from Aletti et al. (2009, Theorem 1), Crimaldi et al. (2007); Crimaldi (2009).

For part (b), by using result (a) of Lemma 3.1, for any $j \geq 0$, we have that

$$\Delta A_j \;=\; \boldsymbol{E}\left[\Delta Z_j|\mathcal{F}_{j-1}\right] \;=\; Z_{j-1}(1 - Z_{j-1})B_{j-1},$$

with $W_{1,j-1} = W_{2,j-1} = 1$ (since for any $j \geq 1$, the process is an RRU model). By using Aletti et al. (2009, Lemma 2), we have $|B_{j-1}| < c_1 Y_{j-1}^{-2}$ a.s. for some constant $c_1 > 0$, and hence

$$T_{6n} \;\leq\; n^{-1/2}\sum_{j=1}^{n} j|\Delta A_j| \;\leq\; c_1 n^{-1/2}\sum_{j=1}^{n} jY_{j-1}^{-2};$$

in addition, by using Aletti et al. (2009, Lemma 3), we have $\boldsymbol{E}[Y_{j-1}^{-2}] \leq c_2(j-1)^{-2}$ for some constant $c_2 > 0$ and hence

$$\boldsymbol{E}[T_{6n}] \;\leq\; c_1 c_2 n^{-1/2}\sum_{j=1}^{n} j(j-1)^{-2} \;=\; O\left(n^{-1/2}\log(n)\right).$$

Thus, (b) follows

For part (c), let $T_{1n} - T_{5n} = \sum_{j=1}^{n}\Delta S_{jn}$ where

$$\Delta S_{jn} \;:=\; n^{-1/2}(X_j - Z_{j-1} - j\Delta M_j).$$

Since $(T_{1n} - T_{5n})$ is a martingale with respect to the filtration $\{\mathcal{F}_n; n \geq 1\}$, we apply the Martingale CLT (MCLT) after establishing the following conditions (see Hall and Heyde (1980, Theorem 3.2)):

(i) $\max_{1\leq j\leq n}|\Delta S_{jn}| \xrightarrow{p} 0$;

(ii) $\sup_{n\geq 1}\boldsymbol{E}[\max_{1\leq j\leq n}(\Delta S_{jn})^2]<\infty$;

(iii) $\sum_{j=1}^n \boldsymbol{E}[(\Delta S_{jn})^2|\mathcal{F}_{j-1}] \xrightarrow{p} \Sigma_c$.

For part (i), since $|X_j - Z_{j-1}|\leq 1$ a.s. and $\Delta M_j = (\Delta Z_j - \Delta A_j)$, we have that

$$|\Delta S_{jn}| \ \leq\ n^{-1/2}(|X_j - Z_{j-1}| + |j\Delta M_j|) \ \leq\ n^{-1/2}(1+|j(\Delta Z_j - \Delta A_j)|).$$

Now, since $|\Delta Z_j| < bY_{j-1}^{-1}$ and $|\Delta A_j| < c_1 Y_{j-1}^{-2}$ a.s. by Aletti et al. (2009, Lemma 2), we have

$$|\Delta S_{jn}| \ \leq\ n^{-1/2}(1+bjY_{j-1}^{-1}+c_1 jY_{j-1}^{-2}) \ \ a.s.$$

Since by Lemma 2.6 $(jY_j^{-1}) \xrightarrow{a.s.} m^{-1}$, we have $\sup_{j\geq1}(jY_j^{-1})<\infty$ a.s., and thus $|\Delta S_{jn}| \xrightarrow{a.s.} 0$.

For part (ii), using the relation $\boldsymbol{E}[S]=\int_0^\infty \boldsymbol{P}(S>t)dt$ that holds for any non negative r.v. S, we obtain

$$\boldsymbol{E}\left[\max_{1\leq j\leq n}(\Delta S_{jn})^2\right] \ \leq\ \sum_{j=1}^n\int_0^\infty \boldsymbol{P}((\Delta S_{jn})^2>t)dt.$$

By applying arguments analogous to part (i), we obtain

$$\begin{aligned} n(\Delta S_{jn})^2 \quad &\leq 2\left[(X_j - Z_{j-1})^2 + (j\Delta M_j)^2\right] \\ &\leq 2\left[1+2\left[(j\Delta Z_j)^2+(j\Delta A_j)^2\right]\right] \\ &\leq 2\left[1+2\left[b^2(jY_{j-1}^{-1})^2 + c_1^2(jY_{j-1}^{-2})^2\right]\right]. \end{aligned}$$

Now, by using Markov's inequality we obtain

$$\begin{aligned} \boldsymbol{P}((\Delta S_{jn})^2>t) \quad &\leq \boldsymbol{P}\left(C\left(\frac{j}{Y_{j-1}}\right)^2 > nt\right) \\ &\leq\ \max\left\{\ 1\ ;\ \left(\frac{C}{nt}\right)^2\boldsymbol{E}\left[\left(\frac{j}{Y_{j-1}}\right)^4\right]\ \right\}. \end{aligned}$$

Now, since by Aletti et al. (2009, Lemma 3) $\sup_{j\geq1}\boldsymbol{E}\left[\left(\frac{j}{Y_{j-1}}\right)^4\right]<\infty$, it follows that there exists a constant C independent of j such that $\int_0^\infty \boldsymbol{P}((\Delta S_{jn})^2>t)\leq Cn^{-2}$ and hence

$$\sup_{n\geq1}\boldsymbol{E}\left[\max_{1\leq j\leq n}(\Delta S_{jn})^2\right] \ \leq\ \sup_{n\geq1}Cn^{-1} \ \leq\ C.$$

For part (iii), since $\Delta M_j = \Delta Z_j - \Delta A_j$, $\Delta A_j\in\mathcal{F}_{j-1}$ and hence $\boldsymbol{E}[\Delta Z_j\Delta A_j|\mathcal{F}_{j-1}]=(\Delta A_j)^2$, we have the following decomposition:

$$\boldsymbol{E}[(\Delta S_{jn})^2|\mathcal{F}_{j-1}] \ =\ \frac{1}{n}\boldsymbol{E}[Q_j^2|\mathcal{F}_{j-1}] \ +\ \frac{2}{n}(j\Delta A_j)^2,$$

where $Q_j := (X_j - Z_{j-1} - j\Delta Z_j)$. Since $|\Delta A_j| < c_1 Y_{j-1}^{-2}$ a.s. and by Lemma 2.6 $(jY_j^{-1}) \xrightarrow{a.s.} m^{-1}$, we have that $(j\Delta A_j)^2 \xrightarrow{a.s.} 0$. Thus, $\frac{2}{n}\sum_{j=1}^n(j\Delta A_j)^2 \xrightarrow{a.s.} 0$ and hence (iii) is obtained by establishing that

$$\sum_{j=1}^n\boldsymbol{E}[(\Delta S_{jn})^2|\mathcal{F}_{j-1}] \ =\ \frac{1}{n}\sum_{j=1}^n\boldsymbol{E}[Q_j^2|\mathcal{F}_{j-1}] \ \xrightarrow{p}\ \Sigma_c.$$

To this end, we will show that $\boldsymbol{E}[Q_j^2|\mathcal{F}_{j-1}] \xrightarrow{a.s.} \Sigma_c$. First, note that, since $X_j\in\{0,1\}$, we can express ΔZ_j as follows

$$\Delta Z_j \ =\ X_j\left((1-Z_{j-1})\frac{D_{1,j}}{Y_{j-1}}\right) \ +\ (1-X_j)\left(-Z_{j-1}\frac{D_{2,j}}{Y_{j-1}}\right).$$

As a consequence, we consider $Q_j^2 = X_jQ_{j,1}^2 + (1-X_j)Q_{j,0}^2$, where, denoting by $M_{j-1}:=Y_{j-1}/j$,

$$\begin{aligned} Q_{j,1} \quad &:= (1-Z_{j-1})\left(1-\frac{D_{1,j}}{M_{j-1}}\right) \ =\ \left(\frac{1-Z_{j-1}}{M_{j-1}}\right)(M_{j-1}-D_{1,j}), \\ Q_{j,0} \quad &:= Z_{j-1}\left(-1+\frac{D_{2,j}}{M_{j-1}}\right) \ =\ \left(\frac{Z_{j-1}}{M_{j-1}}\right)(-M_{j-1}+D_{2,j}). \end{aligned}$$

Then, since $D_{1,j}$, $D_{2,j}$ and X_j are independent conditionally on $\mathcal{F}_{j-1}$ and using

$$\begin{aligned}\boldsymbol{E}[(M_{j-1} - D_{1,j})^2 \,|\mathcal{F}_{j-1}] &= (M_{j-1} - m)^2 \;+\; \sigma_1^2,\\ \boldsymbol{E}[(-M_{j-1} + D_{2,j})^2 \,|\mathcal{F}_{j-1}] &= (M_{j-1} - m)^2 \;+\; \sigma_2^2,\end{aligned}$$

we have that

$$\begin{aligned}\boldsymbol{E}[Q_j^2|\mathcal{F}_{j-1}] &= Z_{j-1}\boldsymbol{E}[Q_{j,1}^2|\mathcal{F}_{j-1}] \;+\; (1 - Z_{j-1})\boldsymbol{E}[Q_{j,0}^2|\mathcal{F}_{j-1}]\\ &= Z_{j-1}\left(\frac{1 - Z_{j-1}}{M_{j-1}}\right)^2 \left[(M_{j-1} - m)^2 + \sigma_1^2\right]\\ &+ (1 - Z_{j-1})\left(\frac{Z_{j-1}}{M_{j-1}}\right)^2 \left[(M_{j-1} - m)^2 + \sigma_2^2\right].\end{aligned}$$

Finally, since by Lemma 2.6 $M_{j-1} \stackrel{a.s.}{\to} m$ and by Theorem 2.4 $Z_{j-1} \stackrel{a.s.}{\to} Z_\infty$, it follows that

$$\sum_{j=1}^{n} \boldsymbol{E}[(\Delta\tilde{S}_{jn})^2|\mathcal{F}_{j-1}] \;\stackrel{a.s.}{\to}\; \Sigma_c \;=\; Z_\infty(1 - Z_\infty)\left(\frac{\bar{\Sigma}}{m^2}\right).$$

For part (d), the result follows by combining part (a), (c), Crimaldi et al. (2007), and Crimaldi (2009) and by noticing that $(T_{1n} - T_5) \in \mathcal{F}_n$. □

We now turn to consider the ARRU model. The limit distribution for an ARRU model can be obtained using Theorem 2.8 on the set of trajectories that do not cross the thresholds $\hat{\rho}_{1,n}$ and $\hat{\rho}_{2,n}$ i.o., and hence $\{Z_\infty \in (\rho_2, \rho_1)\}$. Since this set is not $\mathcal{F}_n$-measurable, we consider a sequence of sets $\{A_n; n \geq 1\}$ such that $\{Z_n \in A_n, ev.\} = \{Z_\infty \in (\rho_2, \rho_1)\}$ a.s. Specifically, we consider the sequence of sets $\{A_n; n \geq 1\}$ defined in (2.8) as follows:

$$A_n \;:=\; \left(\; \rho_2 + CY_n^{-\alpha}\;,\; \rho_1 - CY_n^{-\alpha}\;\right), \tag{6.1}$$

where $0 < C < \infty$ is a positive constant and $0 < \alpha < \frac{1}{2}$. Consider the partition $\Omega = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3$, where

$$\begin{aligned}\mathcal{A}_1 &:= \{Z_k \in A_k, ev.\},\\ \mathcal{A}_2 &:= \{Z_k \in A_k, i.o.\} \cap \{Z_k \notin A_k, i.o.\},\\ \mathcal{A}_3 &:= \{Z_k \notin A_k, ev.\}.\end{aligned} \tag{6.2}$$

The following lemma establishes the relation between $\mathcal{A}_j$, $j \in \{1, 2, 3\}$, and Z_∞.

Lemma 6.1. *Assume $m_1 = m_2 = m$ and (2.5) with $\rho_1 > \rho_2$. Then,*

(a) $\mathcal{A}_1 \;=\; \{Z_\infty \in (\rho_2, \rho_1)\}$ *a.s.;*

(b) $\boldsymbol{P}(\mathcal{A}_2) \;=\; 0$*;*

(c) $\mathcal{A}_3 \;=\; \{Z_\infty \in \{\rho_2, \rho_1\}\}$ *a.s.*

The proof of Lemma 6.1 is based on comparison arguments between the ARRU and an RRU model presented in Lemma 5.1. This relation is possible when only one random threshold modifies the dynamics of the ARRU. For this reason, we fix $\epsilon \in (0, (\rho_1 - \rho_2)/2)$ and we introduce the following times

$$\begin{aligned}T_1 &:= \sup\{\; n \geq 1 \;:\; Z_n > \min\{\hat{\rho}_{1n}; \rho_1 - \epsilon\}\;\},\\ T_2 &:= \sup\{\; n \geq 1 \;:\; Z_n < \max\{\hat{\rho}_{2n}; \rho_2 + \epsilon\}\;\}.\end{aligned}$$

Let $\mathcal{T}_1 := \{T_1 < \infty\}$ and $\mathcal{T}_2 := \{T_2 < \infty\}$. Since $\hat{\rho}_{1n}, \hat{\rho}_{2n}$ and Z_n converge a.s., $\boldsymbol{P}(\mathcal{T}_1 \cup \mathcal{T}_2) = 1$. Then, by comparing the ARRU process with the RRU process defined in (5.9) we have the following result:

Lemma 6.2. *On the set $\mathcal{T}_1$, for any $n_0, k \geq 1$ we have*

$$\{n_0 \geq \mathcal{T}_1\} \;\subset\; \left\{\; \tilde{Z}_k(n_0) \;\leq\; Z_{n_0+k} \;\leq\; \rho_1 - \epsilon \;\right\}. \tag{6.3}$$

Analogously, on the set $\mathcal{T}_2$, for any $n_0, k \geq 1$ we have

$$\{n_0 \geq \mathcal{T}_2\} \;\subset\; \left\{\; \rho_2 + \epsilon \;\leq\; Z_{n_0+k} \;\leq\; \tilde{Z}_k(n_0) \;\right\}. \tag{6.4}$$

Proof. Consider the dynamics of the RRU process $\{\tilde{Z}_k(n_0); k \geq 0\}$ expressed in (5.9) and the dynamics of the ARRU process $\{Z_{n_0+k}; k \geq 0\}$ expressed in (5.11). Then, since $\{n_0 \geq \mathcal{T}_1\} \subset \{W_{1,n_0+k-1} = 1\}$ and $W_{2,n_0+k-1} \leq 1$ we obtain (6.3). Analogously, since $\{n_0 \geq \mathcal{T}_2\} \subset \{W_{2,n_0+k-1} = 1\}$ and $W_{1,n_0+k-1} \leq 1$ we have (6.4). □

Proof of Lemma 6.1. First, let $A := [\rho_2, \rho_1]$, $t_0 = 0$ and define for every $j \geq 1$

$$\tau_j = \begin{cases} \inf\{k > t_{j-1} : Z_k \in A_k\} & \text{if } \{k > t_{j-1} : Z_k \in A_k\} \neq \emptyset; \\ +\infty & \text{otherwise.} \end{cases}$$

$$t_j = \begin{cases} \inf\{k > \tau_j : \tilde{Z}_{k-\tau_j}(\tau_j) \notin A\} & \text{if } \{k > \tau_j : \tilde{Z}_{k-\tau_j}(\tau_j) \notin A\} \neq \emptyset; \\ +\infty & \text{otherwise.} \end{cases}$$

Denoting by T_0 the last finite time in $\{t_j, \tau_j, j \geq 1\}$, we have the following partition $\Omega = S_t \cup S_\infty \cup S_\tau$, where

$$\begin{aligned} S_t &:= \{T_0 \in \{t_j, j \geq 1\}\} = \cap_{k \geq T_0}\{Z_k \notin A_k\}, \\ S_\infty &:= \{T_0 = \infty\}, \\ S_\tau &:= \{T_0 \in \{\tau_j, j \geq 1\}\} = \cap_{k \geq T_0}\{\tilde{Z}_{k-T_0}(T_0) \in (\rho_2, \rho_1)\}. \end{aligned}$$

Thus, we establish the following result:

(i) $\boldsymbol{P}(S_\infty) = 0$,

(ii) $S_\tau \subset \mathcal{A}_1$, and

(iii) $S_\tau \subset \{Z_\infty \in (\rho_2, \rho_1)\}$.

For part (i), this result is obtained by establishing that there exists $i_0 \geq 1$ such that, for any $i \geq i_0$,

$$\boldsymbol{P}(t_i < \infty | \tau_i < \infty) \leq \frac{1}{2}.$$

To see this, we recall that by Lemma 3.4 we have, for any $h \in (0, 1)$,

$$\boldsymbol{P}\left(\sup_{k \geq 1} |\tilde{Z}_k - \tilde{Z}_0| \geq h\right) \leq \frac{b}{Y_0}\left(\frac{4}{h^2} + \frac{2}{h}\right) \leq \frac{6b}{Y_0} h^{-2}.$$

Thus, by using Lemma 3.4 with $h = C(\tilde{Y}_0(\tau_j))^{-\alpha}$ we obtain

$$\begin{aligned} \boldsymbol{P}(t_i < \infty | \tau_i < \infty) &= \boldsymbol{P}\left(\cup_{k \geq 1} \tilde{Z}_k(\tau_i) \notin [\rho_2, \rho_1] \ |\tau_i < \infty\right) \\ &\leq \boldsymbol{P}\left(\sup_{k \geq 1} |\tilde{Z}_k(\tau_j) - \tilde{Z}_0(\tau_j)| > C(\tilde{Y}_0(\tau_j))^{-\alpha} \ |\tau_i < \infty\right) \\ &\leq \boldsymbol{E}\left[\left(\frac{6b}{\tilde{Y}_0(\tau_j)}\right) \left(C(\tilde{Y}_0(\tau_j))^{-\alpha}\right)^{-2} \ |\tau_i < \infty\right] \\ &= \frac{6b}{C^2} \boldsymbol{E}\left[(\tilde{Y}_0(\tau_j))^{2\alpha - 1} \ |\tau_i < \infty\right], \end{aligned}$$

and hence the result follows by recalling that $0 < \alpha < \frac{1}{2}$ and by (c) of Lemma 3.1. For part (ii), by Lemma 6.2, we have that

$$\begin{aligned} (S_\tau \cap \mathcal{T}_1) &\subset \left(\cap_{k \geq T_0} \{\tilde{Z}_{k-T_0}(T_0) \leq Z_k \leq \rho_1 - \epsilon\}\right), \\ (S_\tau \cap \mathcal{T}_2) &\subset \left(\cap_{k \geq T_0} \{\rho_2 + \epsilon \leq Z_k \leq \tilde{Z}_{k-T_0}(T_0)\}\right). \end{aligned}$$

Thus, the result follows by $\boldsymbol{P}(\mathcal{T}_1 \cup \mathcal{T}_2) = 1$ and $\tilde{Z}_{k-T_0}(T_0) \overset{a.s.}{\to} \tilde{Z}_\infty(T_0) \in (\rho_2, \rho_1)$. For part (iii), from part (ii) we have that

$$S_\tau \subset \{\min\{\rho_2 + \epsilon, \tilde{Z}_\infty(T_0)\} \leq Z_\infty \leq \max\{\rho_1 - \epsilon, \tilde{Z}_\infty(T_0)\}\};$$

thus, the result follows by noticing that

$$\Big(\min\{\rho_2 + \epsilon, \tilde{Z}_\infty(T_0)\}, \max\{\rho_1 - \epsilon, \tilde{Z}_\infty(T_0)\} \Big) \subset (\rho_2, \rho_1).$$

Now, to complete the proof of Lemma 6.1, we notice that from (i), (ii) and $\{\mathcal{A}_3 = S_t\}$, it follows that $\boldsymbol{P}(\mathcal{A}_2) = 0$ and $\{S_\tau = \mathcal{A}_1\}$. Then, combining (iii) and $\mathcal{A}_3 \subset \{Z_\infty \in \{\rho_2, \rho_1\}\}$, we obtain the result. □

We now present the proof of the limit distribution of the proportion of sampled ball colors for an ARRU model.

Proof of Theorem 2.10. First, take the sets $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ defined in (6.2). Note that, since $\mathcal{A}_1 = \underline{\lim}_n\{Z_n \in A_n\}$ and $\mathcal{A}_3^c = \overline{\lim}_n\{Z_n \in A_n\}$, by Lemma 6.1 we have

$$\underline{\lim}_n\{Z_n \in A_n\} \ = \ \overline{\lim}_n\{Z_n \in A_n\} \ = \ \{Z_\infty \in (\rho_2, \rho_1)\}.$$

Then, the proof is based on applying Theorem 2.8 to the ARRU model. To this end, consider the decomposition $\{Z_n \in A_n\} = \mathcal{A}_{1n} \cup \mathcal{A}_{2n} \cup \mathcal{A}_{3n}$, where $\mathcal{A}_{jn} = \{Z_n \in A_n\} \cap \mathcal{A}_j$ for any $j \in \{1,2,3\}$. Since by using Lemma 6.1 $\boldsymbol{P}(\mathcal{A}_2) = 0$, we have $\boldsymbol{P}(\mathcal{A}_{2n}) = 0$ for any $n \geq 1$. Moreover, by definition we have that $\boldsymbol{P}(\mathcal{A}_{3n}) \to 0$ and $\boldsymbol{P}(\mathcal{A}_{1n}) \to \boldsymbol{P}(\mathcal{A}_1)$. Thus, calling $\mathcal{N}_n := \sqrt{n}(\frac{N_{1n}}{n} - Z_\infty)$, we have

$$\lim_{n\to\infty} \boldsymbol{P}\left(\ \mathcal{N}_n \leq x\ ,\ \{Z_n \in A_n\}\ \right) \ = \ \lim_{n\to\infty} \boldsymbol{P}\left(\ \mathcal{N}_n \leq x\ ,\ \mathcal{A}_1\ \right),$$

and since by Lemma 6.1 $\mathcal{A}_1 = \{Z_\infty \in (\rho_2, \rho_1)\}$, this is equivalent to

$$\lim_{n\to\infty} \boldsymbol{P}\left(\ \mathcal{N}_n \leq x\ ,\ \{Z_\infty \in (\rho_2, \rho_1)\}\ \right).$$

Now, consider the RRU model $\{\tilde{Z}_k(n_0), k \geq 1\}$ described in (5.9) coupled with the ARRU model $\{Z_{n_0+k}, k \geq 1\}$. By using Lemma 5.1, for any $n_0 \geq 1$, we have

$$\left(\cap_{k=n_0}^\infty \{\hat{\rho}_{2,k} \ \leq \ Z_k \ \leq \ \hat{\rho}_{1,k}\ \}\right) \ \subset \ \left(\cap_{k=1}^\infty \{Z_{n_0+k} = \tilde{Z}_k(n_0)\ \}\right).$$

Hence, on this set the ARRU process Z_{n_0+k} is equivalent to the RRU process $\tilde{Z}_k(n_0)$; thus, we can obtain the limit distribution for the ARRU by applying the limit distribution for the RRU expressed in Theorem 2.8 on the set where the trajectories of the two processes are equivalent. To this end, define

$$T^* \ := \ \sup\{\ k \geq 1\ :\ \{Z_k < \hat{\rho}_{2,k}\} \cup \{Z_k > \hat{\rho}_{1,k}\}\ \},$$

and note that, for any $n_0 \geq 1$,

$$\{T^* \leq n_0\} \ \subset \ \left(\cap_{k=1}^\infty \{Z_{n_0+k} = \tilde{Z}_k(n_0)\ \}\right).$$

Let $\mathcal{S}$ be a r.v. with characteristic function $\boldsymbol{E}[\exp(\frac{1}{2}\Sigma t^2)]$. Thus, by applying Theorem 2.8 we have that, for any $n_0 \geq 1$ and any set $\mathcal{T} \in \mathcal{F}$,

$$\lim_{n\to\infty} \boldsymbol{P}\left(\ \mathcal{N}_n \leq x\ ,\ \mathcal{T} \cap \{T^* \leq n_0\}\ \right) \ = \ \boldsymbol{P}(\ \mathcal{S} \leq x\ ,\ \mathcal{T} \cap \{T^* \leq n_0\}\ ).$$

Now, since $\{Z_\infty \in (\rho_2, \rho_1)\} \subset \{T^* < \infty\}$, we have

$$\lim_{n_0\to\infty} \boldsymbol{P}(\{T^* \leq n_0\} \cap \{Z_\infty \in (\rho_2, \rho_1)\}) \ = \ \boldsymbol{P}(Z_\infty \in (\rho_2, \rho_1)),$$

which implies that

$$\lim_{n\to\infty} \boldsymbol{P}\left(\ \mathcal{N}_n \leq x, \{Z_\infty \in (\rho_2, \rho_1)\}\ \right) \ = \ \boldsymbol{P}(\ \mathcal{S} \leq x, \{Z_\infty \in (\rho_2, \rho_1)\}\ ).$$

This concludes the proof. □

Acknowledgement. The authors would like to thank the anonymous reviewers for their valuable comments and suggestions that improved the presentation of the paper.

References

Aletti, G. and Ghiglietti, A. (2017), "Interacting generalized Friedman's urn systems," *Stochastic Process. Appl.*, Forthcoming paper, doi:10.1016/j.spa.2016.12.003.

Aletti, G., Ghiglietti, A., and Paganoni, A. M. (2013), "Randomly reinforced urn designs with prespecified allocations," *J. Appl. Probab.*, 50, 486–498.

Aletti, G., May, C., and Secchi, P. (2009), "A central limit theorem, and related results, for a two-color randomly reinforced urn," *Adv. Appl. Probab.*, 41, 829–844.

— (2012), "A functional equation whose unknown is $\mathcal{P}([0,1])$ valued," *J. Theoret. Probab.*, 25, 1207–1232.

Athreya, K. B. and Karlin, S. (1968), "Embedding of urn schemes into continuous time Markov branching processes and related limit theorems," *Ann. Math. Statist.*, 39, 1801–1817.

Bai, Z. D. and Hu, F. (1999), "Asymptotic theorems for urn models with nonhomogeneous generating matrices," *Stochastic Process. Appl.*, 80, 87–101.

Bai, Z.-D. and Hu, F. (2005), "Asymptotics in randomized urn models," *Ann. Appl. Probab.*, 15, 914–940.

Chen, L. H. Y. (1978), "A short note on the conditional Borel-Cantelli lemma," *Ann. Probab.*, 6, 699–700.

Crimaldi, I. (2009), "An almost sure conditional convergence result and an application to a generalized Pólya urn," *Int. Math. Forum*, 4, 1139–1156.

Crimaldi, I., Letta, G., and Pratelli, L. (2007), "A strong form of stable convergence," in *Séminaire de Probabilités XL*, Springer, Berlin, vol. 1899 of *Lecture Notes in Math.*, pp. 203–225.

Dembo, A. and Zeitouni, O. (1998), *Large deviations techniques and applications*, vol. 38 of *Applications of Mathematics (New York)*, Springer-Verlag, New York, 2nd ed.

Durham, S. D., Flournoy, N., and Li, W. (1998), "A sequential design for maximizing the probability of a favourable response," *Canad. J. Statist.*, 26, 479–495.

Durrett, R. (2010), *Probability: theory and examples*, vol. 31 of *Cambridge Series in Statistical and Probabilistic Mathematics*, Cambridge University Press, Cambridge, 4th ed.

Ghiglietti, A. and Paganoni, A. M. (2014), "Statistical properties of two-color randomly reinforced urn design targeting fixed allocations," *Electron. J. Stat.*, 8, 708–737.

— (2016), "An urn model to construct an efficient test procedure for response adaptive designs," *Stat. Methods Appl.*, 25, 211–226.

Ghiglietti, A., Vidyashankar, A. N., and Rosenberger, W. F. (2017), "Central limit theorem for an adaptive randomly reinforced urn model," *Ann. Appl. Probab.*, Forthcoming paper.

Hall, P. and Heyde, C. C. (1980), *Martingale limit theory and its application*, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, probability and Mathematical Statistics.

Hu, F. and Rosenberger, W. F. (2006), *The theory of response-adaptive randomization in clinical trials*, Wiley Series in Probability and Statistics, Wiley-Interscience [John Wiley & Sons], Hoboken, NJ.

Laruelle, S. and Pagès, G. (2013), "Randomized urn models revisited using stochastic approximation," *Ann. Appl. Probab.*, 23, 1409–1436.

Mahmoud, H. (2008), *Pólya urn models*, CRC press.

May, C. and Flournoy, N. (2009), "Asymptotics in response-adaptive designs generated by a two-color, randomly reinforced urn," *Ann. Statist.*, 37, 1058–1078.

Muliere, P., Paganoni, A. M., and Secchi, P. (2006), "A randomly reinforced urn," *J. Statist. Plann. Inference*, 136, 1853–1874.

Pemantle, R. and Volkov, S. (1999), "Vertex-reinforced random walk on **Z** has finite range," *Ann. Probab.*, 27, 1368–1388.

Smythe, R. T. (1996), "Central limit theorems for urn models," *Stochastic Process. Appl.*, 65, 115–137.

Zhang, L.-X., Hu, F., and Cheung, S. H. (2006), "Asymptotic theorems of sequential estimation-adjusted urn models," *Ann. Appl. Probab.*, 16, 340–369.