

# A NEW NONLOCAL NONLINEAR DIFFUSION EQUATION FOR IMAGE DENOISING AND DATA ANALYSIS

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**ABSTRACT.** In this paper we introduce and study a new feature-preserving nonlinear anisotropic diffusion for denoising signals. The proposed partial differential equation is based on a novel diffusivity coefficient that uses a nonlocal automatically detected parameter related to the local bounded variation and the local oscillating pattern of the noisy input signal. We provide a mathematical analysis of the existence of the solution of our nonlinear and nonlocal diffusion equation in the two dimensional case (images processing). Finally, we propose a numerical scheme with some numerical experiments which demonstrate the effectiveness of the new method.

## 1. INTRODUCTION

Nonlinear partial differential equations (PDEs) can be used in the analysis and processing of digital images or image sequences, for example to filter out the noise, to produce higher quality image, to extract features and shapes (see e.g. [2, 4, 3, 17, 18] and the References herein). Perhaps, the main application of PDEs based methods in this field is smoothing and restoration of images. From the mathematical point of view, the input (grayscale) image can be modelled by a real function  $u_0(x)$ ,  $u_0 : \Omega \rightarrow \mathbb{R}$ , where  $\Omega \subset \mathbb{R}^d$ , represents the spatial domain. Typically this domain  $\Omega$  is rectangular and  $d = 1, 2, 3$ . The function  $u_0$  is considered as an initial data for a suitable evolution equation with some kind of boundary conditions (usually homogeneous Neumann boundary conditions).

The simplest PDE method for smoothing images is to apply a linear diffusion process, the starting point is the simple observation that the so called Gauss function, with  $\sigma > 0$  and where  $|\cdot|$  is the Euclidean norm,

$$G_\sigma(x) = \frac{1}{(2\pi\sigma^2)^{d/2}} e^{-|x|^2/(2\sigma^2)}$$

is related to the fundamental solution of the linear diffusion (heat) equation. Then, it has been possible to reinterpret the classical smoothing operation of the convolution of an image with  $G_\sigma$ , with a given standard deviation  $\sigma$ , by solving the linear diffusion equation for a corresponding time  $t = \sigma^2/2$  with initial condition given by the original image. For example, when  $d = 2$ , it is a classic result that for any bounded, continuous, and integrable  $u_0(x)$ ,  $x \in \mathbb{R}^2$ , the linear diffusion equation on the whole space (here  $\Delta$

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denotes the Laplacian operator),

$$\frac{\partial u}{\partial t} = \Delta u, \quad u(x, 0) = u_0(x)$$

possesses the following solution

$$u(x, t) = \begin{cases} u_0(x), & t = 0 \\ (G_{\sqrt{2t}} * u_0)(x), & t > 0 \end{cases}$$

where the convolution product  $(g * f)(x)$  between the function  $f$  and  $g$ , is defined as

$$(g * f)(x) = \int_{\mathbb{R}^2} g(x - y)f(y)dy.$$

We point out that for different time (variance)  $t$  we obtain different levels of smoothing: this defines a *scale-space* for the image [12, 19]. That is, we get copies of the image at different scales. Note, of course, that any scale  $t$  can be obtained from a scale  $\tau$ , where  $\tau < t$ , as well as from the original images, this is usually denoted as the causality criteria for scale-spaces [2]. The solution of the above linear diffusion equation is unique, provided we restrict ourselves to functions satisfying some suitable growth conditions. Moreover, it depends continuously on the initial image  $u_0$ , and it fulfils the maximum/minimum principle

$$\inf_{x \in \mathbb{R}^2} u_0(x) \leq u(x, t) \leq \sup_{x \in \mathbb{R}^2} u_0(x) \text{ on } \mathbb{R}^2 \times [0, \infty).$$

For application in image processing we also need to consider appropriate boundary conditions: usually homogeneous Neumann conditions are used.

The flow produced by the linear diffusion equation is also denoted as isotropic diffusion, as it is diffusing the information equally in all directions. Then, the gray values of the initial image will spread, and, in the end, a uniform image, equal to the average of the initial gray values, is obtained. Although this property is good for local reducing noise (averaging is optimal for additive noise), this filtering operation also destroys the image content, that is, the boundaries of the objects and the subregions present in the image (the *edges*). This means that the Gaussian smoothing does not only smooth noise, but also blurs important features and it makes them harder to identify. Furthermore, linear diffusion filtering dislocates edges when moving from finer to coarser scales (see e.g. [19]). So structures which are identified at a coarse scale do not give the right location and have to be traced back to the original image. Moreover, some smoothing properties of Gaussian scale-space do not carry over from the one-dimensional case to higher dimensions: it is generally not true that the number of local extrema, which are related to edges, is non-increasing. As suggested by Hummel [9] the linear diffusion is not the only PDE that can be used to enhance an image and that, in order to keep the scale-space property, we need only to make sure that the corresponding flow holds the maximum principle. Many approaches have been taken in the literature to implement this idea replacing the linear equation with a nonlinear PDE that does not diffuse the image in a uniform way: these flows are normally denoted as anisotropic diffusion. In particular, the diffusion coefficient is locally adapted, becoming negligible as object boundaries are approached. Noise is efficiently removed and object contours are strongly enhanced [18]. There is a vast literature concerning nonlinear anisotropic diffusions with application to image processing which date back to the seminal paper

by Perona and Malik, who, in [15], consider a discrete version of the following equation

$$\begin{cases} \frac{\partial u}{\partial t} - \nabla \cdot (g(|\nabla u|)\nabla u) = 0, & \text{in } \Omega_T = (0, T) \times \Omega, \\ u(x, 0) = u_0(x) & \text{on } \Omega \\ \frac{\partial u}{\partial \vec{n}}(x, t) = 0, & \text{on } \Gamma \times (0, T), \end{cases}$$

where  $\Gamma = \partial\Omega$ , the image domain  $\Omega \subset \mathbb{R}^2$  is an open regular set (typically a rectangle),  $\vec{n}$  denotes the unit outer normal to its boundary  $\Gamma$ , and  $u(x, t)$  denotes the (scalar) image analysed at time (scale)  $t$  and point  $x$ . The initial condition  $u_0(x)$ ,  $u_0$  is, as in the linear case, the original image. In order to reduce smoothing at edges, the diffusivity  $g$  is chosen as a decreasing function of the “edge detector”  $|\nabla u|$  (for a vector  $V = (V_1, V_2) \in \mathbb{R}^2$ ,  $|V|^2 = V_1^2 + V_2^2$ ). A typical choice is,

$$g(s) = \frac{1}{1 + (s/\lambda)^2}, \quad s \geq 0, \lambda > 0.$$

Catté, Lions, Morel and Coll [6] showed that the continuous Perona-Malik model is ill posed, then very close pictures can produce divergent solutions and therefore very different edges. This is caused by the fact that the diffusivity  $g$  leads to flux  $s \cdot g(s)$  decreasing for some  $s$  and the scheme may work locally like the inverse heat equation which is known to be ill posed. This possible misbehaviour surely represents a severe drawback of the Perona-Malik model when applied to data effected by noise. However, discrete implementations work as a regularization factor by introducing implicit diffusion into the model, and the filter is usually observed to be stable (with staircasing effect as the only observable instability). Then, in the continuous settings, a new model has been proposed [6] with the only modification of replacing the gradient  $\nabla u$  in the diffusivity by its spatial regularizations  $(G_\sigma * \nabla u)$ , which are obtained by smoothing the argument by a convolution with a  $C^\infty$  kernel  $G_\sigma$ . Typically  $G_\sigma$  is a Gaussian function and  $\sigma$  determines the scale beyond which regularization occurs. The equation will now diffuse if and only if the gradient is estimated to be small. We point out that the spatial regularization lead to processes where the solution converges to a constant steady state. Then, in order to get nontrivial results, we have to specify a stopping time  $T$ . Sometimes it is attempted to circumvent this task by adding an additional reaction term which keeps the steady state solution close to the original image  $u_0$ , for example

$$\frac{\partial u}{\partial t} - \nabla \cdot (g(|\nabla G_\sigma * u|)\nabla u) = f(u_0 - u),$$

where  $f$  is a Lipschitz continuous, non decreasing function,  $f(0) = 0$ . During the last years, many other nonlinear parabolic equations have been proposed as an image analysis model. The common theme in this proliferation of models is the following, one attempts to fix one intrinsic diffusion direction and tunes the diffusion using the size of the gradient or the value of an estimate of the gradient. A few of the proposed models are even systems of PDEs, for example there exist reaction diffusion systems which have been applied to image restoration and which are connected to Perona-Malik idea or based on Turing's pattern formation model [18].

In this paper we proposed a new anisotropic diffusion equation introducing a nonlocal diffusive coefficient that takes into account of the “monotonicity” and the oscillating pattern of the image. In other words, a high modulus of the gradient may lead to a small diffusion if the function is, for instance, locally monotone.

**1.1. A motivating 1D model.** At present, the best view of the activity of a neural circuit is provided by multiple-electrode extracellular recording technologies, which allow us to simultaneously measure spike trains from up to a few hundred neurons in one or more brain areas during each trial. While the resulting data provide an extensive picture of neural spiking, their use in characterizing the fine timescale dynamics of a neural circuit is complicated by at least two factors. First, extracellularly captured action potentials provide only an occasional view of the process from which they are generated, forcing us to interpolate the evolution of the circuit between the spikes. Second, the circuit activity may evolve quite differently on different trials that are otherwise experimentally identical. Experimental measurements are noisy. For neural recordings, the noise may arise from a multitude of sources, both intrinsic and extrinsic to the nervous system. Operationally, supposing that recorded data are composed of two parts, signal of interest and other processes unrelated to the experimental conditions, it is a challenge to preserve the essential signal features, such as suitable structures related to the neuronal activity, during the smoothing process. In Figure 1 we show an

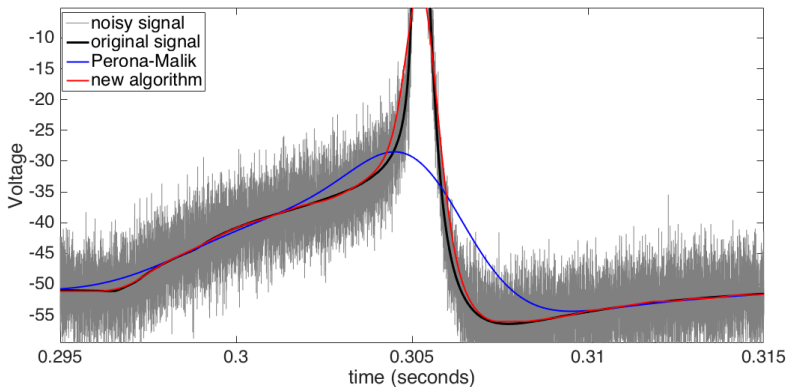


FIGURE 1. 5s simulated membrane potential signals of a network of 20 neurons randomly connected (firing rate =  $5Hz$ ). In black, original signal of a neuron. White gaussian noise was added to the signals (gray). Blue: signal after denoising (Perona-Malik). Red: signal after denoising (new algorithm). Note that the event occurred between  $t = 0.296$  and  $t = 0.297$  is completely removed after Perona-Malik smoothing

example of a noised signal of a neuron, where a white gaussian noise has been superposed to the original signal. The method proposed in this paper is compared with the classical Perona-Malik algorithm. We point out that the diffusivity in the Perona-Malik model, or similar approaches, depends locally on the modulus of the gradient of the function. Instead, we introduce a nonlocal diffusive coefficient that takes into account of the “monotonicity” of the signal. In other words, a high modulus of the gradient may lead to a small diffusion if the function is also locally monotone. Motivated by this fact, we have developed the new approach presented in this paper. More precisely, the diffusion coefficient in a point  $x$  is based on the behavior of the function  $f$  in a interval  $x + Q = (x - q, x + q)$ , where  $Q = (-q, +q)$ , see Figure 2. Analytically, we compute the ratio between the variation  $|f(x + q) - f(x - q)|$  and the total variation  $\int_Q |\nabla f(s + x)| ds$  of the function in  $x + Q$ . A ratio close to 1 will imply a tiny noise in the signal, while a

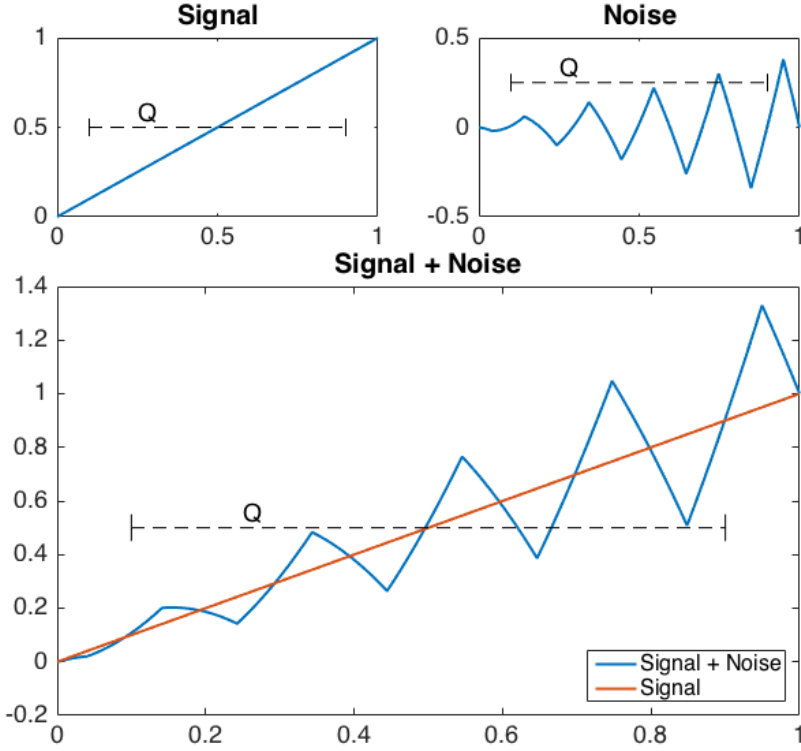


FIGURE 2. Example of a nonlocal signal. The signal and the noise have the same total variation and the same modulus of the gradient. The global increment observed on  $Q$  is instead very different

ratio close to 0 is related to a highly noised signal. As shown in Figure 2, a pure signal and a noised one may have the same total variation and the same modulus of gradient. Therefore, Perona-Malik like methods (and total variation based methods) treat the signals in the same way.

More precisely, for the one dimensional spatial case, let  $u : [a, b] \rightarrow \mathbb{R}$  a real function defined on a bounded interval  $[a, b]$ , and a subinterval  $[c, d] \subset [a, b]$ . We define the *local variation*  $LV_{[c,d]}(u)$  of  $u$  on the interval  $[c, d]$  the value

$$LV_{[c,d]}(u) = |u(d) - u(c)|.$$

We also define the *total local variation*  $TV_{[c,d]}(u)$  of  $u$  on the interval  $[c, d]$  as follows

$$TV_{[c,d]}(u) = \sup_{\mathcal{P}} \sum_{i=0}^{n_P-1} |u(x_{i+1}) - u(x_i)|$$

where  $\mathcal{P} = \{P = \{x_0, \dots, x_{n_P}\} \mid P \text{ is a partition of } [a, b]\}$  is the set of all possible finite partition of the interval  $[c, d]$ . It is easy to prove that if the function  $u$  is a monotone function on the interval  $[c, d]$ , then  $LV_{[c,d]}(u) = TV_{[c,d]}(u)$ . While, if the function  $u$  is not monotone,  $LV_{[c,d]}(u) < TV_{[c,d]}(u)$ . For the 1D signal, as the membrane potential of a neuron, where the independent variable has the dimension of a time, it is convenient

to select instead of a symmetric window  $Q$  an asymmetric interval of a given length  $\delta$ . Let  $\varepsilon \in \mathbb{R}^+$  “small” number, and let  $\delta \in \mathbb{R}^+$  a positive number, we define the ratio,

$$(1.1) \quad R_{\delta,u} = \frac{LV_{[x,x+\delta]}(u)}{\varepsilon + TV_{[x,x+\delta]}(u)}$$

If the parameter  $\delta$  is chosen appropriately we can distinguish between oscillations caused by noise and by electrophysiological stimuli (in the following EPSP) contained in a range of amplitude  $\delta$ . In the case of the membrane potential of a neuron the oscillations due to the noise and to EPSP occur on different time scales: it is possible to choose a value  $\delta$  such that in a range of amplitude  $\delta$  there is at least a full oscillation due to noise, but not to a complete EPSP. Then, there is an oscillation, the signal is not monotone and it is expected that the ratio  $R_{\delta,u}$  is much less than one because  $LV_{[x,x+\delta]}(u) \ll TV_{[x,x+\delta]}(u)$ . While, if in the same time interval there is an EPSP, the ratio  $R_{\delta,u}$  becomes close to one.

As in the Perona-Malik model, we adapt the diffusive coefficient by using the above ratio  $R_{\delta,u}$ . For small values of the latter we have to reduce the noise, while for values close to 1, the upper bound of  $R_{\delta,u}$ , we have to preserve the signal variation (as the edges in the image). The resulting non-local equation is the following,

$$(1.2) \quad \frac{\partial u}{\partial t} - \frac{\partial}{\partial x} \left( g \left( \frac{LV_{[x,x+\delta]}(u)}{\varepsilon + TV_{[x,x+\delta]}(u)} \right) \frac{\partial u}{\partial x} \right) = 0,$$

where the function  $g$  has the same properties as in the Perona-Malik model and  $\delta > 0$ . If  $u$  is a differentiable function and  $u'$  is integrable, the total variation can be written as,

$$TV_{[x,x+\delta]}(u) = \int_x^{x+\delta} |u'(s)| ds,$$

and the non linear diffusion equation (1.2) can be state as

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x} \left( g \left( \frac{\left| \int_x^{x+\delta} u'(s) ds \right|}{\varepsilon + \int_x^{x+\delta} |u'(s)| ds} \right) \frac{\partial u}{\partial x} \right) = 0.$$

for a function  $u(x,t)$ ,  $x \in (a,b)$ ,  $t > 0$ . As initial condition we take the original signal  $u_0$  but with some regularization obtain with a standard smoothing filter, e.g. a Gaussian filter, and we assume homogeneous Neumann condition at the boundary, that is  $\partial u / \partial x = 0$  for  $x = a, b$  and  $t > 0$ .

**1.2. The multidimensional case.** In order to apply the new model to a multidimensional signal, in particular in the two-dimensional case (a gray level digital image), we have to generalize the ratio  $R_{\delta,u}$ , see (1.1). Let  $A \subseteq \mathbb{R}^d$  and  $u : \Omega \rightarrow \mathbb{R}$  an integrable function smooth function, the total variation  $TV(u)$  (or  $BV$  seminorm), can be computed as [20]

$$TV(u) = \int_A |\nabla u| dx$$

where  $\nabla u$  is the gradient of  $u$ . Here, we consider the anisotropic total variation,

$$TV_a(u) = \int_A |\nabla u|_1 dx$$

considering the  $l^1$  norm,  $|v|_1 = |v_1| + |v_2| + \dots + |v_d|$ , instead of the Euclidean norm. The usual total variation  $TV(u)$  is invariant to rotation of the domain, but anisotropic  $TV_a$  is not. However, the latter allows for other approaches that do not apply with the usual  $TV$ , for example the graph-cuts algorithm [7]. Moreover the  $TV_a(u)$  has the

advantage of making the total variation satisfies the coarea formula [8], which allows us to interpret  $TV_a(u)$  as the cumulated length of the level lines of the function  $u$ .

For the local variation term, the numerator of the ratio  $R_{\delta,u}$ , we have to compute the variation of the function  $u$  in a region  $A$  by taking into account the flux of  $u$  at the boundary  $\partial A$  of the same set  $A$ . Following the definition of the  $BV$ -seminorm [20], and the choice we propose the definition,

$$LV_A(u) = \sup\left\{\int_A \nabla u(x) \nabla h(x) dx, |\nabla h|_1 \leq 1, h \text{ harmonic on } A\right\}.$$

In the above definition, due to the properties of the test function  $h$ , we have

$$\int_A \nabla u \nabla h dx = \int_{\partial A} u \nabla h \cdot \vec{n}_A ds - \int_A u \operatorname{div}(\nabla h) dx$$

where  $\operatorname{div}$  is the divergence operator,  $\vec{n}_A$  denotes the unit outer normal to  $\partial A$ , and the last integral is equal to zero because  $h$  is harmonic. Then

$$\int_A \nabla u \nabla h dx = \int_{\partial A} u \nabla h \cdot \vec{n}_A ds,$$

and the supremum for the  $LV_A(u)$  is taken considering all the possible orientations of the vector  $\nabla h$  with respect to  $\vec{n}_A$ . Returning to the one-dimensional case, for  $A = [c, d]$ , we obtain,

$$LV_A(u) = \sup\{(u(c) - u(d)), (-u(c) + u(d))\} = |u(d) - u(c)|.$$

The remainder of this paper is organized as follows. In Section 2, we provide the mathematical analysis of the new non-linear and non-local diffusion equation in the two dimensional spatial case. We show in particular the existence of a solution for the model by using a suitable semidiscrete scheme under reasonable hypotheses for applications in image processing. In Section 3, we build an explicit, in time, numerical scheme for the new model coupling a finite element method based on bilinear element  $\mathcal{Q}_1$ , a finite difference approximation for the numerical gradients, and a decomposition with respect to a suitable set of eigenfunctions. In Section 3 we also show some numerical experiments.

**Notation.** In the following,  $\Omega \subset \mathbb{R}^2$  denotes a open bounded domain with Lipschitz continuous boundary  $\Gamma = \partial\Omega$ , and  $\Omega_T = \Omega \times (0, T)$ , with  $T > 0$ . We denote by  $H^k(\Omega)$ ,  $k$  is a positive integer, the Sobolev space of all function  $u$  defined in  $\Omega$  such that  $u$  and its distributional derivatives of order  $k$  all belong to  $L^2(\Omega)$ . Let  $D^s$  the distributional derivatives,  $H^k$  is a Hilbert space for the norm,

$$\|u\|_k = \|u\|_{H^k} = \left( \sum_{|s| \leq k} \int_{\Omega} |D^s u(x)|^2 dx \right)^{1/2}, \quad \|u\|_0 = \|u\|_{L^2}.$$

Let  $V = H^1$ ,  $V^*$  stands for its dual space. We denote by  $L^p(0, T; H^k(\Omega))$  the set of all functions  $u$ , such that, for almost every  $t$  in  $(0, T)$ ,  $u(t)$  belong to  $H^k(\Omega)$ ,  $L^p(0, T; H^k(\Omega))$  is a normed space for the norm

$$\|u\|_{L^p(0, T; H^k(\Omega))} = \left( \int_0^T \|u\|_k^p dt \right)^{1/p}$$

$p \geq 1$  and  $k$  a positive integer. We denote by  $(\cdot, \cdot)$ , the scalar product in  $L^2(\Omega)$ .

## 2. ANALYSIS OF THE NEW NONLOCAL AND NONLINEAR EQUATION, 2D CASE

In this section we will consider the two-dimensional spatial case and we will prove the existence of a variational solution of the corresponding non-local diffusion equation. From the discussion in the subsections (1.1)-(1.2), given  $U \in L^2(0, T; V)$  and  $Q = (-q_1, +q_1) \times (-q_2, +q_2)$  (the local window), we can define the ratio coefficient  $R$  as the function

$$R_{Q,U}(x, t) = \begin{cases} \frac{\sup\{\int_Q \nabla U(x+y, t) \nabla h(y) dy, |\nabla h|_1 \leq 1, h \text{ harmonic on } Q\}}{\int_{x+Q} |\nabla U(y, t)|_1 dy}, & \text{if } \int_{x+Q} |\nabla U(y, t)|_1 dy > 0; \\ 0, & \text{otherwise;} \end{cases}$$

where  $|\cdot|_1$  is the  $l^1$  norm in  $\mathbb{R}^2$ . It is easy to verify that the function  $R_{Q,U}(x, t)$  is measurable, and  $0 \leq R_{Q,U} \leq 1$ . Moreover, note that  $\int_{x+Q} |\nabla U(y, t)|_1 dy$  is continuous in  $x$  since  $U \in L^2(0, T; V)$ .

Let  $g : [0, +\infty) \rightarrow \mathbb{R}$  be a Lipschitz continuous nonincreasing function such that  $g(0) = 1$ ,  $g(s) > 0$ ,  $\forall s \geq 0$ ,  $g(1) = \epsilon > 0$ . It follows that  $1 \geq g(R_{Q,U}(x, t)) \geq \epsilon$ .

Let  $Q$  be the window that is used in the definition of the diffusive coefficient  $R_{Q,u}$ . We assume that

$$\text{(Assumption 1)} \quad \inf_{x \in \Omega} \frac{|\Omega \cap \{x + Q/3\}|}{|\{x + Q/3\}|} = q_\Omega > 0,$$

where if  $A$  is a measurable set, let  $|A|$  be the Lebesgue measure of  $A$ .

The smoothing process of the image  $u_I$  is obtained by the solution  $u(x, t)$  of the following non-linear, non-local diffusion equation,

$$\begin{aligned} (2.3) \quad & \frac{\partial u}{\partial t} - \operatorname{div}(g(R_{Q,u}(x, t)) \nabla u) = 0, \quad \text{in } \Omega_T; \\ & \frac{\partial u}{\partial \vec{n}} = 0, \quad \text{on } \Gamma \times (0, T); \\ & u(x, 0) = u_0(x) \in V; \end{aligned}$$

with homogeneous Neumann boundary conditions for the normal derivative  $\frac{\partial u}{\partial \vec{n}}$ , and initial data  $u_0 \in V$  which is a smoother version of the original image  $u_I$ .

**Remark 2.1.** *We point out that the initial data is more regular with respect to classical parabolic theory but we need to ensure the well-posedness of the diffusion coefficients  $R_{Q,u}$ . In the numerical approximation of the equation (2.3) we obtain a suitable initial data from the original signal  $u_I$  by using a convolutional operator with a Gaussian filter.*

**2.1. Rothe method and a priori estimates.** In order to prove the existence of a solution  $u \in L^2(0, T; V) \cap C^0(0, T; L^2)$  we consider the so called Rothe-type approximation [10] of (2.3) which consists in using time discretization to approximate the evolution problem by a sequence of an elliptic one. To show the convergence of such a process, a common approach is to follow the following steps:

- (1) for each approximate problem, prove the existence of a solution, and derive a-priori estimates satisfied by any solution;
- (2) then use compactness arguments to show (up to the extraction of a subsequence) the existence of a limit;
- (3) Finally, prove that the previous limit satisfies the original problem.

Let  $0 = t_0 < t_1 < \dots < t_N = T$  denote the time discretization with  $t_{i+1} = t_i + \tau$ , where  $\tau$  is the time step. Let  $u_i$  be the solution of linear equation,

$$(2.4) \quad \frac{u_i - u_{i-1}}{\tau} - \operatorname{div}(g(R_{Q,u_{i-1}}(x, t)) \nabla u_i) = 0,$$



with  $u_0 = u(x, 0)$ , and homogeneous Neumann boundary conditions. Let  $\delta u_i = (u_i - u_{i-1})/\tau$  the backward difference at time  $t_i$ , we understand the solution of (2.4) in the variational sense, i.e., we look for  $u_i \in V$ , for  $i = 1, \dots, N$  satisfies the identity

$$(2.5) \quad (\delta u_i, v) + (g(R_{Q, u_{i-1}}) \nabla u_i, \nabla v) = 0, \quad \forall v \in V,$$

where  $u_0 \in V$  is given. By introducing the bilinear form  $a_{\tau, w}$ , on  $V \times V$ ,

$$a_{\tau, w} = (u, v) + \tau a_w(u, v), \quad a_w(u, v) = (g(R_{Q, w}) \nabla u, \nabla v),$$

for a given  $w \in V$ , we can rewrite the previous identity as,

$$(2.6) \quad a_{\tau, u_{i-1}}(u_i, v) = (u_{i-1}, v), \quad \forall v \in V.$$

The term  $a_w(u, v)$  in (Referencesatau-form) is weakly coercive, i.e., there exist two constants  $c_1 > 0$  and  $c_2 > 0$  such that

$$(2.7) \quad a_w(u, u) + c_2 \|u\|_0^2 \geq c_1 \|u\|_1^2, \quad \forall u \in V.$$

Furthermore, the form  $a_{\tau, w}$  is continuous and it verifies

$$a_{\tau, w}(u, u) \geq \tau c_1 \|u\|_1^2 + (1 - \tau c_2) \|u\|_0^2,$$

then it is  $V$ -elliptic if  $\tau c_2 \leq 1$ . Under this coercivity condition, the existence and the uniqueness of  $u_i \in V$ ,  $i = 1, \dots, N$  from (2.5) is guaranteed by the Lax-Milgram Theorem [16]. Now, we introduce the so-called Rothe function

$$u^{(N)}(t) = u_{i-1} + (t - t_{i-1})\delta u_i, \quad \text{for } t_{i-1} \leq t \leq t_i, \quad i = 1, \dots, N$$

which we consider as a linear piecewise approximation of the problem (2.3). Together with  $u^{(N)}$  we consider the step function

$$\bar{u}^{(N)}(t) = u_i, \quad \text{for } t_{i-1} \leq t \leq t_i, \quad i = 1, \dots, N$$

with  $\bar{u}^{(N)}(0) = u_0$ . In the following  $C$  denotes the generic positive constant.

**Lemma 2.2.** *Let  $u_i$ ,  $i = 1, \dots, N$ , be the solution of problem (2.6), then*

$$(2.8) \quad \max_{1 \leq i \leq N} \|u_i\|_0 \leq C,$$

hold uniformly for  $N$ , and  $\bar{u}^{(N)}(t)$ ,  $u^{(N)}(t) \in L^\infty(0, T; L^2)$ .

*Proof.* First we test (2.5) at time  $t_{k+1}$  by  $v = \tau u_{k+1}$  and sum over  $k = 0, \dots, p \geq 1$ , we obtain (let us define  $a_{u_k}(u, v) = a_k(u, v)$ ),

$$\sum_{k=0}^p (u_{k+1} - u_k, u_{k+1}) + \tau \sum_{k=0}^p a_k(u_{k+1}, u_{k+1}) = 0, \quad p = 1, \dots, (n-1).$$

By using the identity  $2(u - v, u) = (u - v, u - v) + (u, u) - (v, v)$ , we have

$$\sum_{k=0}^p \|u_{k+1} - u_k\|_0^2 + \|u_{p+1}\|_0^2 - \|u_0\|_0^2 + 2\tau \sum_{k=0}^p a_k(u_{k+1}, u_{k+1}) = 0,$$

Then, from (2.7),

$$\sum_{k=0}^p \|u_{k+1} - u_k\|_0^2 + \|u_{p+1}\|_0^2 - \|u_0\|_0^2 + 2\tau c_1 \sum_{k=0}^p \|u_{k+1}\|_1^2 - 2\tau c_2 \sum_{k=0}^p \|u_{k+1}\|_0^2 \leq 0,$$

which can be rewritten as follows

$$\sum_{k=0}^p \|u_{k+1} - u_k\|_0^2 + \|u_{p+1}\|_0^2 + 2\tau c_1 \sum_{k=0}^p \|u_{k+1}\|_1^2 \leq 2\tau c_2 \sum_{k=0}^p \|u_{k+1}\|_0^2 + \|u_0\|_0^2.$$

Let us define

$$s_p = \sum_{k=0}^{p-1} \|u_{k+1} - u_k\|_0^2 + \frac{1}{2} \|u_{p+1}\|_0^2 + 2\tau c_1 \sum_{k=0}^{p-1} \|u_{k+1}\|_1^2,$$

It is easy to verify that

$$s_{p+1} \leq C_0 + 2\tau c_1 \sum_{k=0}^p \|u_k\|_1^2,$$

where the constant  $C_0$  depends only on the initial datum  $u_0$ . By the inequality  $2s_p \geq \|u_p\|_0^2$ , we obtain

$$s_{p+1} \leq C_0 + 4\tau c_1 \sum_{k=0}^p s_k.$$

Applying the discrete Gronwall lemma we have the following inequalities,

$$s_p \leq C_0 (1 + 4\tau c_1)^{p-1} \leq C_0 (1 + 4\tau c_1)^N.$$

The function  $\tau \rightarrow (1 + 4\tau c_1)^{T/\tau}$  is bounded for  $\tau \in (0, +\infty)$ , then, for a suitable constant  $\bar{C}$ ,

$$(2.9) \quad \max_p \frac{\|u_p\|_0^2}{2} \leq \max_p s_p \leq C_0 \bar{C} = C,$$

which leads to the a-priori estimates in (2.8).  $\square$

**Lemma 2.3.** *The estimates*

$$(2.10) \quad \tau \sum_{k=1}^N \|\nabla u_k\|_0^2 \leq C, \quad \sum_{k=1}^N \|u_k - u_{k-1}\|_0^2 \leq C$$

hold uniformly with respect to  $N$ .

*Proof.* The estimates followed by the upper bound obtained in the Lemma 2.2, see (2.9),

$$\sum_{k=0}^p \|u_{k+1} - u_k\|_0^2 + \frac{1}{2} \|u_{p+1}\|_0^2 + 2\tau c_1 \sum_{k=0}^p \|u_{k+1}\|_1^2 \leq C,$$

and from the properties of the bilinear form  $a_k(u, v)$ .  $\square$

**2.2. Compactness and passage to the limit.** We remember that  $\Omega_T = \Omega \times (0, T)$ , the estimates (2.8,2.10) will lead the equicontinuity of the Rothe approximation  $u^{(N)}$ , together the step function  $\bar{u}(N)$ .

**Lemma 2.4.** *There exists  $u \in L^2(0, T; V)$  with  $\partial u / \partial t \in L^2(0, T; V^*)$  such that (in the sense of subsequences)*

$$\begin{aligned} u^{(N)} &\rightharpoonup u, & \bar{u}^{(N)} &\rightharpoonup u & \text{in } L^2(\Omega_T); \\ \bar{u}^{(N)} &\rightharpoonup u & & & \text{in } L^2(0, T; V); \\ \frac{\partial u^{(N)}}{\partial t} &\rightharpoonup \frac{\partial u}{\partial t} & & & \text{in } L^2(0, T; V^*). \end{aligned}$$

*Proof.* Let  $s \in (0, T)$ , we consider the time translate variation of the approximation  $\bar{u}^{(N)}(t)$ ,

$$J_s = \int_0^{T-s} \|\bar{u}^{(N)}(t+s) - \bar{u}^{(N)}(t)\|_0^2 dt.$$

There exists an integer  $k$  such that  $k\tau \leq s \leq (k+1)\tau$ , then, by the definition of the step function  $\bar{u}^{(N)}$ ,

$$J_s = \tau \sum_{l=0}^{N-k} \|u_{l+k} - u_l\|_0^2.$$

From the Lemma 2.2 and 2.3 it follows that

$$J_s \leq Ck\tau,$$

for a suitable constant  $C$ , and the time translate estimate,

$$(2.11) \quad \int_0^{T-s} \|\bar{u}^{(N)}(t+s) - \bar{u}^{(N)}(t)\|_0^2 dt \leq C(s+\tau).$$

By using again the estimates (2.8), (2.10), and the definition of the step approximation  $\bar{u}^{(N)}$ , it is easy to show that

$$(2.12) \quad \int_0^T \|\bar{u}^{(N)}(t)\|_1^2 dt \leq C_u.$$

Given a vector  $\xi \in \mathbb{R}^2$ , let  $\Omega_\xi = \{x \in \Omega : x + \xi \in \Omega\}$  and  $\Omega_{\xi,T} = \Omega_\xi \times (0, T)$ . From the inequality (2.12) we have the following space translate estimate,

$$(2.13) \quad \int_{\Omega_{\xi,T}} \|\bar{u}^{(N)}(x+\xi, t) - \bar{u}^{(N)}(x, t)\|_0^2 dx dt \leq C_\xi |\xi|^2, \quad \text{for } |\xi| \text{ sufficiently small.}$$

Due to the time and translate estimates, respectively (2.11) and (2.13), the set  $\{\bar{u}^{(N)}\}_N$  is compact in  $L^2(\Omega_T)$  because of Kolmogorov's relative compactness criterion [5, 13]. The, we can conclude  $\bar{u}^{(N)} \rightarrow u$  in  $L^2(\Omega_T)$  (and also pointwise in  $\Omega_T$ ), and  $u \in L^2(0, T; V)$ . From the definition of  $\bar{u}^{(N)}$  and  $u^{(N)}$ , and from Lemma 2.3 it follows the estimate

$$\int_0^T \|\bar{u}^{(N)}(t) - u^{(N)}(t)\|_0^2 dt \leq \frac{C_d}{N},$$

which holds uniformly with respect to  $N$ . Then  $u^{(N)} \rightarrow u$  in  $L^2(\Omega_T)$ .

Observing that  $\partial u^{(N)}/\partial t = (u_i - u_{i-1})/\tau$ , for  $t \in (t_{i-1}, t_i)$ , we can compute

$$\left\| \frac{\partial u^{(N)}}{\partial t} \right\|_* = \sup_{v \in V, \|v\| \leq 1} |((u_i - u_{i-1})/\tau, v)|.$$

Then, the following estimate holds uniformly for  $N$ ,

$$\int_0^T \left\| \frac{\partial u^{(N)}}{\partial t} \right\|_*^2 dt \leq C_*,$$

and we can deduce the weak convergence of the time derivative of the Rothe approximations  $u^{(N)}$ .  $\square$

**Lemma 2.5.** *With the notation of Lemma 2.4,*

$$u^{(N)} \rightarrow u, \quad \bar{u}^{(N)} \rightarrow u \quad \text{in } L^2(0, T; V);$$

*Proof.* Now we shall prove the  $L^2(0, T; V)$  convergence of  $\bar{u}^{(N)}$  to  $u$  (which belongs to the space  $L^2(0, T; V)$ ). So, let us test (2.5) by  $v = \bar{u}^{(N)} - u$  and integrate it over the time interval  $(0, T)$  by using the partition from the subinterval  $(t_{i-1}, t_i)$ ,

$$(2.14) \quad \sum_i^N \int_{t_{i-1}}^{t_i} [((u_i - u_{i-1})/\tau, u_i - u) + (g(R_{Q, u_{i-1}}) \nabla u_i, \nabla u_i - \nabla u)] dt = 0.$$

We recall that  $1 \geq g(R_{Q,u_{i-1}}) \geq g(1) = \epsilon$ , then we obtain

$$\epsilon \int_0^T \|\nabla \bar{u}^{(N)} - \nabla u\|_0^2 dt \leq \sum_i^N \int_{t_{i-1}}^{t_i} (g(R_{Q,u_{i-1}}) \nabla u_i - \nabla u, \nabla u_i - \nabla u) dt.$$

From (2.14), and the above inequality, we have

$$\begin{aligned} \epsilon \int_0^T \|\nabla \bar{u}^{(N)} - \nabla u\|_0^2 dt + \sum_i^N \int_{t_{i-1}}^{t_i} (g(R_{Q,u_{i-1}}) \nabla u_i, \nabla u_i - \nabla u) dt \\ \leq \sum_i^N \int_{t_{i-1}}^{t_i} ((u_i - u_{i-1})/\tau, u - u_i) dt. \end{aligned}$$

Now, from the Lemma 2.2 and the Lemma 2.3, and the convergence  $\bar{u}^{(N)} \rightarrow u$  in  $L^2(\Omega_T)$ , and the weak convergence  $\partial \bar{u}^{(N)}/\partial t \rightharpoonup \partial u/\partial t$  in the space  $L^2(0, T; V^*)$ , we deduce that exists a vanishing sequence  $\{C_N\}$ ,  $C_N \in \mathbb{R}$ ,  $\lim_{N \rightarrow \infty} C_N = 0$  such that

$$\epsilon \int_0^T |\nabla \bar{u}^{(N)} - \nabla u|^2 dt \leq C_N,$$

which implies  $\bar{u}^{(N)} \rightarrow u$  in  $L^2(0, T; V)$ . To prove the convergence  $u^{(N)} \rightarrow u$  in  $L^2(0, T; V)$  it is possible to consider a time average approximation starting from the values  $u_i$  and proceeding as in [11].  $\square$

**2.3. Existence of a variational solution.** In order to prove that the limit  $u$  is a variational solution of (2.3) we have to consider the property of the stability of the kernel  $g(R_{Q,u})$  with respect a variation in the space  $V$ . First, we introduce some results from the measure theory.

**Lemma 2.6** (Vitali covering). *Let  $\cup_{i=1}^n x_i + Q/3$  be a finite cover of a set  $\tilde{\Omega} \subseteq \Omega$ . Then there exists a finite sub-cover  $\cup_{j=1}^m \{x_j + Q\}$  such that  $\{x_j + Q/3, j = 1, \dots, m\}$  are disjoint.*

**Corollary 2.7.** *Let  $\tilde{\Omega} \subseteq \Omega$ . Then there exists  $x_0, \dots, x_{N_0} \in \tilde{\Omega}$  such that  $\tilde{\Omega} \subseteq \cup_{j=1}^{N_0} \{x_j + Q\}$  and  $N_0 \leq \frac{3^2 |\Omega|}{|Q| q_\Omega}$ .*

*Proof.* Denote by  $\hat{\Omega}$  the closure of  $\tilde{\Omega}$  in  $\Omega$ . Take the open cover of  $\hat{\Omega}$  made by  $\mathcal{C} = \{x + Q/3, x \in \tilde{\Omega}\}$ . Since  $\bar{\Omega}$  is compact, then  $\hat{\Omega}$  is compact and hence there exists a finite cover of  $\hat{\Omega}$  made by  $\{\cup_{i=1}^n \{x_i + Q/3\}\}$ . By the Vitali covering Lemma, there exists a finite sub-cover  $\cup_{j=1}^{N_0} \{x_j + Q\}$  such that  $\{x_j + Q/3, j = 1, \dots, N_0\}$  are disjoint. Moreover,

$$|\Omega| \geq |\Omega \cap (\cup_{j=1}^{N_0} \{x_j + Q/3\})| = \sum_{j=1}^{N_0} |\Omega \cap \{x_j + Q/3\}| \geq q_\Omega N_0 \frac{|Q|}{3^2},$$

the last inequality being a consequence of (Assumption 1), and since  $|\{x_j + Q/3\}| = |Q/3| = |Q|/3^2$ .  $\square$

The following result shows the stability of the kernel  $g(R_{Q,u})$  when the limiting function is not locally constant.

**Lemma 2.8.** *The function  $R_{Q,U}(x, t)$  is continuous at  $U \in V$  on the set*

$$\left\{ (x, t) : \int_{x+Q} |\nabla U(y, t)|_1 dy > 0 \right\}.$$

*Proof.* Denote by

$$N_U(x, t) = \sup \left\{ \int_Q \nabla U(x + y, t) \nabla h(y) dy, |\nabla h|_1 \leq 1, h \text{ harmonic in } Q \right\},$$

$$D_U(x, t) = \int_{x+Q} |\nabla U(y, t)|_1 dy,$$

and by

$$|\nabla u|_1 = \left| \frac{\partial u}{\partial x_1} \right| + \left| \frac{\partial u}{\partial x_2} \right|, \quad |\nabla u|_2 = \sqrt{\left( \frac{\partial u}{\partial x_1} \right)^2 + \left( \frac{\partial u}{\partial x_2} \right)^2},$$

the seminorm  $|\cdot|_1$  and, respectively,  $|\cdot|_2$ , then  $\frac{|\nabla u|_1}{\sqrt{2}} \leq |\nabla u|_2 \leq |\nabla u|_1$ . Let  $U_n \rightarrow U$  in  $L^2(0, T; V)$ . By definition, for a.e.  $t \in (0, T)$ ,

$$\int_{\Omega} |\nabla U - \nabla U_n|_2^2 dy \rightarrow 0,$$

and hence

$$(2.15) \quad \frac{1}{2 \cdot |\Omega|} \left( \int_{\Omega} |\nabla U - \nabla U_n|_1 dy \right)^2 \leq \int_{\Omega} \left( \frac{|\nabla U - \nabla U_n|_1}{\sqrt{2}} \right)^2 dy \leq \int_{\Omega} |\nabla U - \nabla U_n|_2^2 dy \rightarrow 0.$$

As a direct consequence,

$$(2.16) \quad D_{U_n}(x, t) \rightarrow D_U(x, t) > 0, \quad \text{for a.e. } (x, t): \int_{x+Q} |\nabla U(y, t)|_1 dy > 0.$$

For what concerns  $N_U$ , if  $|\nabla h|_1 \leq 1$  then we get  $|\nabla h|_{\infty} \leq 1$ . Then

$$\begin{aligned} & \left| \int_Q \nabla U_1(x + y, t) \nabla h(y) dy - \int_Q \nabla U_2(x + y, t) \nabla h(y) dy \right| \\ & \leq \int_Q |\nabla U_1(x + y, t) - \nabla U_2(x + y, t) \nabla h(y)| dy \\ & \leq \|\nabla U_1(x + \cdot, t) - \nabla U_2(x + \cdot, t)\|_1 \|\nabla h\|_{\infty} \\ & \leq \int_{x+Q} |\nabla U_1(y, t) - \nabla U_2(y, t)|_1 dy \end{aligned}$$

By (2.15),  $N_{U_n}(x, t) \rightarrow N_U(x, t)$ , which concludes the proof together with (2.16).  $\square$

**Lemma 2.9.** *Let  $u_n \rightarrow u$  in  $V$ . For any  $w \in C^1(\Omega; \mathbb{R})$  and  $t \in (0, T)$ ,*

$$\lim_{n \rightarrow \infty} \left( \int_{\Omega} g(R_{Q, u_n}) \nabla u_n \nabla w dx - \int_{\Omega} g(R_{Q, u}) \nabla u \nabla w dx \right) = 0.$$

*Proof.* Note that

$$(2.17) \quad \left| \int_{\Omega} g(R_{Q, u_n}) \nabla u_n \nabla w dx - \int_{\Omega} g(R_{Q, u}) \nabla u \nabla w dx \right|$$

$$\leq \int_{\Omega} |g(R_{Q, u_n})| |(\nabla u_n - \nabla u) \nabla w| dx$$

$$+ \int_{\Omega} |g(R_{Q, u_n}) - g(R_{Q, u})| |\nabla u \nabla w| dx.$$

For what concerns the RHS of (2.17), the first term vanishes as  $n$  goes to infinity since  $g$  is bounded. For the second term of the RHS of (2.17), we get

$$(2.18) \quad \int_{\Omega} |g(R_{Q,u_n}) - g(R_{Q,u})| |\nabla u \nabla w| dx = \int_{\tilde{\Omega}^\epsilon} |g(R_{Q,u_n}) - g(R_{Q,u})| |\nabla u \nabla w| dx \\ + \int_{\Omega \setminus \tilde{\Omega}^\epsilon} |g(R_{Q,u_n}) - g(R_{Q,u})| |\nabla u \nabla w| dx$$

where,

$$\tilde{\Omega}^\epsilon = \left\{ x : \int_{x+Q} |\nabla u|_1 dy < \frac{\epsilon |Q| q_\Omega}{g(0) K 3^2 |\Omega|} \leq \frac{\epsilon}{g(0) K N_0} \right\},$$

$K$  is such that  $\max(|\frac{\partial w}{\partial x_1}|, |\frac{\partial w}{\partial x_2}|) \leq K$ , and  $N_0$  is defined in Corollary 2.7. The first term of the RHS of (2.18) is uniformly bounded (in  $n$ ) by  $\epsilon$ : by Corollary 2.7

$$\int_{\tilde{\Omega}^\epsilon} |g(R_{Q,u_n}) - g(R_{Q,u})| |\nabla u \nabla w| dy \leq g(0) K \int_{\tilde{\Omega}^\epsilon} |\nabla u|_1 dy \\ \leq g(0) K \int_{\cup_{j=1}^{N_0} \{x_j+Q\}} |\nabla u|_1 dy \\ \leq g(0) K \sum_{j=1}^{N_0} \int_{x_j+Q} |\nabla u|_1 dy \\ \leq g(0) K N_0 \int_{x_j+Q} |\nabla u|_1 dy \\ \leq \epsilon.$$

The second term of the RHS of (2.18) vanishes as a consequence of the Dominated Convergence Theorem. In fact,  $|g(R_{Q,u_n}) - g(R_{Q,u})| |\nabla u \nabla w| \leq g(0) |\nabla u \nabla w| \in L^1$ ; and  $g(R_{Q,u_n}) \rightarrow g(R_{Q,u})$  on  $\Omega \setminus \tilde{\Omega}^\epsilon$  by Lemma 2.8.  $\square$

Now we prove that the limit  $u$  is a variational solution of (2.3).

**Lemma 2.10** (Existence). *For any  $u_0 \in V$ , there exists  $u \in L^2(0, T; V)$  with  $\frac{\partial u}{\partial t} \in L^2(0, T; V^*)$  such that  $u(x, 0) = u_0(x)$  on  $\Omega$ ,  $\frac{\partial u}{\partial n} = 0$  on  $\Gamma \times (0, T)$  and*

$$\int_{\Omega} \frac{\partial u}{\partial t} w dx = \int_{\Omega} \operatorname{div}(g(R_{Q,u}) \nabla u) w dx, \quad \forall w \in C_0^1(\Omega).$$

*Proof.* By Lemma 2.4 and Lemma 2.5, there exists a sequence  $u^{(N)}$  such that

$$u^{(N)} \rightarrow u \text{ in } L^2(0, T; V), \quad \frac{\partial u^{(N)}}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \text{ in } L^2(0, T; V^*).$$

Let  $\phi \in C_c^\infty(0, T)$  be a real-valued test function and  $w \in C_0^1(\Omega)$ . Taking  $v(x, t) = \phi(t)w(x)$  as a test function and integrating the result with respect to  $t$ , we find that

$$\int_0^T \left( \int_{\Omega} \frac{\partial u^{(N)}}{\partial t} v(x, t) dx + \int_{\Omega} g(R_{Q,u_N}) \nabla u^{(N)} \nabla v(x, t) dx \right) dt = 0.$$

We take the limit of this equation as  $N \rightarrow \infty$ . Since the function  $t \rightarrow \phi(t)w$  belongs to  $L^2(0, T; V)$ , we have

$$\int_0^T \int_{\Omega} \frac{\partial u^{(N)}}{\partial t} v(x, t) dx dt \longrightarrow \int_0^T \int_{\Omega} \frac{\partial u}{\partial t} v(x, t) dx dt.$$

Moreover, Lemma 2.9 shows that

$$\int_0^T \left( \int_{\Omega} g(R_{Q,u_N}) \nabla u^{(N)} \nabla w \, dx \right) \phi(t) \, dt \rightarrow \int_0^T \left( \int_{\Omega} g(R_{Q,u}) \nabla u \nabla w \, dx \right) \phi(t) \, dt$$

It therefore follows that  $u$  satisfies

$$\int_0^T \phi(t) \left( \int_{\Omega} \frac{\partial u^{(N)}}{\partial t} w \, dx + \int_{\Omega} g(R_{Q,u}) \nabla u \nabla w \, dx \right) dt = 0, \quad \forall \phi \in C_c^\infty(0, T),$$

and hence, for almost every  $t \in (0, T)$ ,

$$\int_{\Omega} \frac{\partial u}{\partial t} w \, dx = \int_{\Omega} \operatorname{div}(g(R_{Q,u}) \nabla u) w \, dx, \quad \forall w \in C_0^1(\Omega).$$

□

### 3. NUMERICAL APPROXIMATION

We introduce here a numerical scheme for the one-dimensional and the two-dimensional case, that is for  $1D$  signal and for the gray level images. We point out that a digital signal/image is usually defined on a uniform subdivision of an interval or a rectangular domain. While for the one-dimensional case we consider a finite differences approach, for the  $2D$  spatial problem we have to couple different numerical approaches in order to estimate the function  $R_{Q,u}$  and the corresponding diffusive coefficient.

**3.1.  $1D$  problem.** We approximate the  $1D$  non linear diffusion equation (1.2), for  $x \in [0, L]$ ,  $t \in [0, T]$ , coupled with homogeneous Neumann conditions for  $x = 0$ ,  $x = L$ , using a finite difference scheme. For convenience, we will use a uniform grid, with grid spacing  $h = L/N$ ,  $N > 1$  integer. If we wish to refer to one of the points in the spatial grid, we shall call the points  $x_i$ ,  $i = 0, \dots, N$ , where  $x_i = ih$ , obtaining the corresponding partition  $0 = x_0 < x_1 < \dots < x_N = L$ . Likewise, we discretize the time domain  $[0, T]$  similarly by place a grid on the temporal axis with grid spacing  $\tau = T/M$  and time points  $t_k = k\tau$ ,  $k = 0, \dots, M$ . We will define  $u_i^k$  to be a function defined at the point  $(ih, k\tau)$ , the function  $u_i^k$  will be our approximation to the solution of the diffusion problem at the same point  $(ih, k\tau)$ .

For simplicity we choose the parameter  $\delta$  (the length of the window for the non local term) as  $\delta = lh$ , where  $l \in \mathbb{N}$ . We approximate the local variation  $LV_{[x_i, x_i+\delta]}(u)$  by  $LV_i = |u_{i+l} - u_i|$ , and we define  $TV_i = \sum_{j=0}^{l-1} |u_{i+j+1} - u_{i+j}|$  as a discretization of the total variation  $TV_{[x_i, x_i+\delta]}(u)$ . We also define the following quantities where the function

$g(s)$  is the edge-stopping,

$$\begin{aligned}
g_i^k &= g\left(\frac{LV_i}{\varepsilon + TV_i}\right) \\
g_{i+\frac{1}{2}}^k &= \frac{g_i^k + g_{i+1}^k}{2} \\
\partial_x u_i^{k+1} &= \frac{u_{i+1}^{k+1} - u_i^{k+1}}{h} \\
\phi_{i+\frac{1}{2}}^k &= g_{i+\frac{1}{2}}^k \partial_x u_i^{k+1} \\
[\partial_x (g \partial_x u)]_i^k &= \frac{\phi_{i+\frac{1}{2}}^k - \phi_{i-\frac{1}{2}}^k}{h} = \\
&= \frac{g_{i-1}^k + g_i^k}{2h^2} u_{i-1}^{k+1} - \frac{g_{i-1}^k + 2g_i^k + g_{i+1}^k}{2h^2} u_i^{k+1} + \frac{g_i^k + g_{i+1}^k}{2h^2} u_{i+1}^{k+1} = \\
&= \beta_i^k u_{i-1}^{k+1} - \alpha_i^k u_i^{k+1} + \gamma_i^k u_{i+1}^{k+1}
\end{aligned}$$

Then we can state our semi-implicit numerical scheme

$$\frac{U^{k+1} - U^k}{\tau} = A(U^k) U^{k+1}$$

where  $U^k$  is the vector of the values  $u_i^k$ , and the matrix  $A(U^k)$  is defined as

$$A(U^k) = \begin{pmatrix} \alpha_1^k & \gamma_1^k & & & & \\ \beta_2^k & \alpha_2^k & \gamma_2^k & & & \\ & \ddots & \ddots & \ddots & & \\ & & \beta_{N-1}^k & \alpha_{N-1}^k & \gamma_{N-1}^k & \\ & & & \beta_N^k & \alpha_N^k & \end{pmatrix}$$

Then at each time step we have to numerically solve the following linear system,

$$(I - \tau A(U^k)) U^{k+1} = U^k$$

Let  $B(U^k)$  the matrix  $(I - \tau A(U^k))$ , it is easy to show that  $B$  is a strictly diagonally dominant matrix, then it is non singular.

**3.2. 1D Test.** In this numerical experiment we consider a recorded calcium imaging data from a 3D cultures of cortical neurons. The sampling rate was 65Hz and the sampling time interval was about 8 seconds. The data was collected in the Department of Neuroscience and Brain Technologies of the Fondazione Istituto Italiano di Tecnologia. In Figure 3 we show a typical trace of the calcium signal together with different smoothed signals at different time  $T$ . In the test we used  $\tau = 0.1$ ,  $\delta = 20$ , and  $h = 1/65$ , the initial data  $U^0$  is obtained by the convolution of the original signal with a Gaussian filter with  $\sigma = 0.01$ .

**3.3. 2D problem.** Without loss of generality we can choose a square domain  $(x, y) \in [1, L] \times [1, L]$ , with  $L$  a positive integer which is the number of pixels for each row (or column) of the image. Moreover, we can fix the grid spacing  $h_x = h_y = 1$  because it represents the distance between two adjacent pixels. Also for the time interval  $[0, T]$  we use an uniform grid with time spacing  $\tau$ , let  $\mathbf{j} = (j_1, j_2)$ , the node  $(i, \mathbf{j})$  corresponds



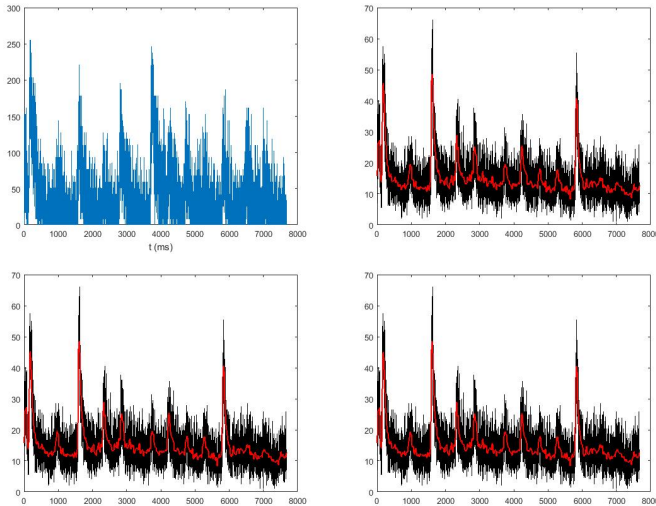


FIGURE 3. Reconstruction with the non linear, non local diffusion equation (1.2) of a Calcium trace. The original signal was obtained in the IIT lab based in Genova (Italy). First line, on the left the original signal, on the right the solution for  $T = 300$ . Second line, on the left the solution for  $T = 400$ , on the right the solution for  $T = 500$ . In all numerical experiments,  $\delta = 20$ ,  $\tau = 0.1$ .

to the point  $(i\tau, j_1 h_x, j_2 h_y) = (i\tau, j_1, j_2)$ . We denote by  $u_{i,(j_1+1,j_2)}$  the approximation of the solution  $u$  in the node  $(i, \mathbf{j})$ . Each component of the gradient vector and the divergence of a vector will be approximated with central differences formula,

$$\begin{aligned} \nabla u_{i,\mathbf{j}} &= \left( \frac{(u_{i,(j_1+1,j_2)} - u_{i,(j_1-1,j_2)})/2}{(u_{i,(j_1,j_2+1)} - u_{i,(j_1,j_2-1)})/2} \right), \\ \operatorname{div}(v_{i,\mathbf{j}}, w_{i,\mathbf{j}}) &= \frac{v_{i,(j_1+1,j_2)} - v_{i,(j_1-1,j_2)}}{2} + \frac{w_{i,(j_1,j_2+1)} - w_{i,(j_1,j_2-1)}}{2}. \end{aligned}$$

The main effort, given a symmetric rectangular  $Q = (-q_1, +q_1) \times (-q_2, +q_2)$ , is the computation of

$$R_{Q,u_{i-1,\mathbf{j}}} = \frac{\sup\{\int_{x+Q} \nabla u(y,t) \nabla h \, dy, |\nabla h|_1 \leq 1, h \text{ harmonic on } Q\}}{\epsilon + \int_{x+Q} |\nabla u(y,t)|_1 \, dy}.$$

**Step 1:** computation of  $\int_{x+Q} |\nabla u(y,t)|_1 \, dy$ .

- We compute the total variation  $TV_{i,(j_1+0.5,j_2+0.5)}$  in each square by taking the  $Q_1$  finite element approximation  $P(x_1, x_2)$  defined by its values at the corner nodes  $u_{i,(j_1,j_2)}, u_{i,(j_1+1,j_2)}, u_{i,(j_1+1,j_2+1)}, u_{i,(j_1,j_2+1)}$ . We may then exactly compute

$$TV_{i,(j_1+0.5,j_2+0.5)} = \int_0^1 \int_0^1 |\nabla P(x_1, x_2)|_1 \, dx_1 \, dx_2$$

as a function of the corner values;

- with a pre-computed filter, we sum values of the vertices of each of the  $4q_1q_2$  squares in the neighborhood  $Q$  of  $\mathbf{j}$ :

$$TV_{Q,i;\mathbf{j}} = \sum_{k_1=j_1-q_1}^{q_1-1} \sum_{k_2=j_2-q_2}^{q_2-1} TV_{i,(k_1+0.5,k_2+0.5)}.$$

**Step 2:** computation of

$$(3.19) \quad \sup\left\{ \int_{x+Q} \nabla u(y,t) \nabla h \, dy, \quad |\nabla h|_1 \leq 1, h \text{ harmonic on } Q \right\}.$$

Since  $\operatorname{div}(\nabla h) = 0$ , in this case we have

$$\int_{x+Q} \nabla u(y,t) \nabla h \, dy = \int_{x+\partial Q} u(y,t) (\nabla h \cdot \vec{n}) \, ds,$$

and hence (3.19) depends on  $u$  only through its values at  $x+\partial Q$  (as in 1D). To compute

$$\sup\left\{ \int_{x+\partial Q} u(s,t) (\nabla h \cdot \vec{n}) \, ds, \quad |\nabla h|_1 \leq 1, h \text{ harmonic on } Q \right\},$$

we approximate the set  $\{h \text{ harmonic on } Q, |\nabla h|_1 \leq 1\}$  with a spectral decomposition using suitable eigenfunctions in the following way:

- we map  $Q$  on the square  $S = [0,1] \times [0,1]$ ; up to a constant, the following functions form a base for  $\{h \text{ harmonic on } S\}$

$$\begin{aligned} & x, y, xy, \\ & \frac{\sin(k\pi x) \sinh(k\pi(1-y))}{\cosh(\pi k)}, \frac{\sin(k\pi y) \sinh(k\pi x)}{\cosh(\pi k)}, \\ & \frac{\sin(k\pi x) \sinh(k\pi y)}{\cosh(\pi k)}, \frac{\sin(k\pi y) \sinh(k\pi(1-x))}{\cosh(\pi k)}, \end{aligned}$$

- we choose  $M$  and we approximate the base by taking  $k \leq M$ , the first  $M$  modes and  $x, y, xy$ ;
- we compute the approximate base  $\mathcal{H} = \{f_h, h = 1, \dots, 4M+3\}$  on  $Q$ , with a linear transformation of  $x$  e  $y$  that preserves harmonicity;
- we compute the gradient  $\nabla f_h$  of each element of  $\mathcal{H}$ ;
- we solve the linear problem

$$\begin{aligned} \max_{a_1, \dots, a_{4M+3}} \sum_{h=1}^{4M+3} a_h \int_{x+\partial Q} u(s,t) (\nabla f_h \cdot \vec{n}) \, ds \\ \text{subject to } \left| \sum_{h=1}^{4M+3} a_h \nabla f_h \right|_1 \leq 1, \text{ uniformly on } Q, \end{aligned}$$

by noticing that each component of  $\sum_{h=1}^{4M+3} a_h \nabla f_h$  is still an harmonic function, and hence it attains its maximum on  $\partial Q$ . With a very fine mesh  $\{y_1, \dots, y_L\}$  on the boundary  $\partial Q$ , the problem (3.19) is hence well approximated by the linear

problem

$$\begin{aligned} & \max_{a_1, \dots, a_{4M+3}} \sum_{h=1}^{4M+3} a_h \int_{x+\partial Q} u(s, t) (\nabla f_h \cdot \vec{n}) ds \\ & \text{subject to } \sum_{h=1}^{4M+3} a_h \left( \pm \frac{\partial f_h(y_l)}{\partial x} \pm \frac{\partial f_h(y_l)}{\partial y} \right) \leq 1, \\ & \text{for any } l = 1, \dots, L. \end{aligned}$$

This linear problem has  $4M + 3$  unknown and  $4L$  constrains and it is solved independently for each node  $(i, \mathbf{j})$ . We have parallelized this operation, once we have preallocated the quantities involving  $\nabla f_h(y_l)$ ;

- furthermore,  $u(s, t)$  is evaluated with a linear interpolation on the values  $u_{i, \mathbf{j}}$  of the nodes  $\mathbf{j} \in \partial Q$ , since the integral is made on the boundary of an element of  $Q_1$ . Therefore, there exist suitable constants such that

$$\int_{x+\partial Q} u(y, t) (\nabla f_h \cdot \vec{n}) ds = \sum_{\mathbf{j} \in \partial Q} u_{i, \mathbf{j}} H_{h, \mathbf{j}}.$$

Again, we have calculated the quantities  $H_{h, \mathbf{j}}$  at the beginning of the code.



FIGURE 4. Numerical solutions of equation (2.4): the left plot refers to initial value, the right one is obtained after 300 iterations. Resolution: top  $126 \times 126$  pixels, bottom  $512 \times 512$  pixels.  $Q = (-2, +2)^2$ .

In Figure 4 two numerical simulations are performed. In order to avoid the solution of a linear system, as in the  $1D$  case, we used to advance in time the classical forward Euler scheme. We have therefore experimentally found a condition of stability by selecting an

appropriate value for the time spacing  $\tau$ . In both the cases, we have used  $q_1 = q_2 = 2$ ,  $M = 3$ ,  $L = 400$  with Dual-Simplex Algorithm. The details under the magnitude of  $Q$  are treated as noise, and smoothed, while the edges are clearly magnified.

#### 4. CONCLUSION

Image denoising/smoothing is one of basic issues in image processing. It plays a key preliminary step in many computer based vision systems, but also it is a starting point towards more complex tasks. Since image noise removal represents a relevant issue in various image analysis and computer vision problems, it is a challenge to preserve the essential image features, such as edges and other sharp structures during the smoothing process. The feature preserving image noise reduction still represents a challenging image processing task. In this paper we propose a new method based on a non linear and non local diffusion equation. The new approach has already been successfully applied in the analysis of membrane potentials in a neural network [1], and for data analysis of recorded calcium signals in a 3D culture cells [14]: these denoising experiments provided very encouraging results. Here we focused on the mathematical analysis of the model and its numerical approximation. In particular, we provided an existence theorem for the variational solution and a numerical scheme both for the 1D and 2D case.

We observe that the uniqueness of the solution of the novel equation remains an open problem. Also the analysis of the stability and convergence of proposed numerical schemes should be completed. We have already developed some preliminary results that will be reported in a forthcoming paper. In particular, it is possible to show that the semi-implicit numerical scheme satisfies the same discrete scale-space properties as for the Perona-Malik method. Finally, a more complete comparison with other methods has to be done, but this goes beyond the aims of this paper.

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