ON TENSOR FACTORIZATION FOR REPRESENTATIONS OF FINITE GROUPS

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Abstract. We prove that, given a quasi-primitive complex representation D for a finite group G, the possible ways of decomposing D as an inner tensor product of two projective representations of G are parametrized in terms of the group structure of G. More explicitly, we construct a bijection between the set of such decompositions and a particular interval in the lattice of normal subgroups of G.

Introduction

Throughout the whole paper, all the groups generically denoted by G (or H) are meant to be finite, and all the representations will be finite dimensional representations over the complex field, although $\mathbb{C}$ can be safely replaced with any algebraically closed field of characteristic zero.

It is well known that, given an irreducible representation D for a group G, a good understanding of D is achieved if it is possible to find (and describe) a subgroup H and a representation T for it, such that D is *induced* by T from H. An effective method for recognizing such a pair is provided by Clifford's Theorem (11.1 in [**1**]), iterated applications of which yield a pair (H, T) such that D is induced by T from H, and T is a *quasi-primitive* representation of H (recall that an irreducible representation of a finite group is called quasi-primitive if its restriction to any normal subgroup has pairwise equivalent irreducible constituents). In view of that, understanding the structure of quasi-primitive representations appears as a crucial issue in Representation Theory.

Now, let D be a faithful quasi-primitive representation for G. Although such a representation can still be induced from a proper subgroup of G (but a result due to T. R. Berger excludes this possibility if G is solvable (see [**7**], 11.33)), there seems to be no general method to exploit further the additive structure of D, and it appears natural to investigate D from the point of view of its 'multiplicative' structure. In particular, our aim in this paper is to control and parametrize, in terms of the group structure of G, all the possible ways of decomposing D as an inner tensor product of two projective representations of G. With the further hypothesis that the restriction of D to the Fitting subgroup F of G is irreducible, the main result of this paper (Corollary 2.8, which follows from the more general Theorem 2.6) shows essentially that *there is an explicit bijection between the set*

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of two-factors decompositions[1] *of D and the set of normal subgroups of G lying between the centre Z and F.* The assumption of faithfulness for D was added only for ease of statements in this Introduction, but the hypothesis of irreducibility on the restriction $D{\downarrow_F}$ is serious: it excludes from consideration all (nonabelian) groups in which the Fitting subgroup is central (in particular, it excludes all nonabelian simple groups, for which the problem has an entirely different character that is beyond the present discussion). Of course, when F is central, one cannot expect it to contain much information about factorizations of representations which map it to scalars. If F is noncentral and $D{\downarrow_F}$ is reducible, a well known result (stated here as Lemma 2.1) gives a tensor factorization for D such that one factor represents F irreducibly and the other just by scalars.

Since Corollary 2.8 describes how *the projective-equivalence class* of D factorizes, it can be conveniently translated into the language of characters, and this is done in Theorem 3.2. Finally, the example in Section 4 shows that the hypothesis of irreducibility for $D{\downarrow_F}$ can not be dropped (nor weakened along one particular line) even if D is assumed primitive instead of quasi-primitive.

Tensor factorization of quasi-primitive representations has been extensively studied, from the point of view of characters, in [**3**] (which in turn generalizes some results from [**4**] and [**8**]); in that paper the authors prove that, given a quasi-primitive character χ of a group G, there exists an essentially unique 'admissible set of prime characters' which provides a factorization for χ. It is worth stressing that the aim of such a result is different from ours, as the problem of describing *all* factorizations is not considered; indeed, in some cases (even when the group is solvable) a 'prime character' could be factorized, but no factorization is yelded by those methods. At any rate, taking in account that a prime character fits the hypotheses of Theorem 3.2 (at least in a solvable context), the present approach reveals some interactions with previous works on this subject.

1. Projective representations

In this section we recall some basic definitions and results concerning projective (complex) representations, and we establish some notation and conventions.

Let G be a group, n a positive integer, and P a map from G to $GL(n,\mathbb{C})$. If there exists a map α, from $G\times G$ to $\mathbb{C}^\times$, such that $P(g_1)P(g_2)=\alpha(g_1,g_2)P(g_1g_2)$ holds for all g_1, g_2 in G, then P is called a projective representation of degree n for G. The map α is the *factor set* of P, and it is uniquely determined by P. It is clear that any representation is a projective representation; sometimes, for the sake of emphasis, a representation in the classical sense will be called (following [**9**]) a *genuine* representation.

If P_1 and P_2 are projective representations of degree n for G, then we say that they are equivalent if there exist a matrix A in $GL(n,\mathbb{C})$, and a map $\beta : G\to\mathbb{C}^\times$, such that $P_2(g)=\beta(g)A^{-1}P_1(g)A$ for all g in G. This defines an equivalence

[1] The scope of the word 'explicit', as well as the precise meaning of 'two-factors decomposition' (what is factorized is indeed the projective-equivalence class of D) will be clarified in the body of the paper.

relation on the set of projective representations of G, and we shall denote by $[P]$ the class of a projective representation P modulo this equivalence relation.

Let P be a projective representation of degree n for G, and V an n-dimensional vector space over $\mathbb{C}$. We say that P is irreducible if the only subspaces of V invariant under the action of the set of matrices $P(G)$ are the zero space and V itself.

Remark 1.1. If $\bar{P}$ is the composite of a projective representation P of degree n for G with the natural homomorphism π which maps $GL(n,\mathbb{C})$ onto $PGL(n,\mathbb{C})$, then $\bar{P}$ is a homomorphism from G to $PGL(n,\mathbb{C})$. Conversely, if $\tau : G \to PGL(n,\mathbb{C})$ is a homomorphism, a projective representation of degree n for G arises in a natural way (as soon as we choose a transversal for $Z(GL(n,\mathbb{C}))$ in the full preimage under π of $\tau(G)$; such a choice is however not relevant up to equivalence).

It is also useful to introduce a concept of equivalence for homomorphisms to projective general linear groups: let τ_1 and τ_2 be homomorphisms of G to $PGL(n,\mathbb{C})$; we say that τ_1 and τ_2 are equivalent if there exists A in $GL(n,\mathbb{C})$ such that $\tau_2(g) = \tau_1(g)^{\pi(A)}$ for all g in G (and, when we write the symbol '$\simeq$' between two such homomorphisms, we refer to this kind of equivalence). If we choose now two projective representations P_1 and P_2 of G with $\bar{P}_1 = \tau_1$ and $\bar{P}_2 = \tau_2$, it is clear that τ_1 and τ_2 are equivalent if and only if P_1 and P_2 are so.

Projective representations play a fundamental role in the present context because it is possible to construct inner tensor products with them, and such a product may yield a genuine representation: if P_1, P_2 are projective representations for the group G, with factor sets α_1, α_2 and degrees n, m respectively, then the map $P_1 \otimes P_2 : G \to GL(nm,\mathbb{C})$ defined by $(P_1 \otimes P_2)(g) := P_1(g) \otimes P_2(g)$ for all g in G, is a projective representation of G whose factor set is the pointwise product of α_1 and α_2 (the symbol '$\otimes$' between two matrices denotes the usual Kronecker product); this projective representation is called the inner tensor product of P_1 and P_2.

We can now introduce some more notation.

Definition 1.2. Let G be a group, and P_1, P_2 projective representations of G; we denote by $\bar{P}_1 \otimes \bar{P}_2$ the homomorphism $\overline{P_1 \otimes P_2}$.

To avoid any confusion arising from the fact that two different concepts of equivalence are floating around for genuine representations (depending on whether they are regarded as genuine or as projective representations), we shall emphasize the distinction, when needed, saying that two representations are *genuine-equivalent* if they are equivalent in the classical sense, whereas we shall call them *projective-equivalent* if they are equivalent only (in principle) as projective representations. It is clear that two genuine representations D_1 and D_2 of G are projective-equivalent if and only if there exists a 1-dimensional representation λ of G such that D_1 and $\lambda \otimes D_2$ are genuine-equivalent.

2. Factorization of representations

The main result of the paper, which is 2.8, follows as a corollary of Theorem 2.6. Before proving it we need to prepare the setting with some lemmas; the first of them is a well known result, and here it is only stated (see [**6**], 21.1(a) and 21.2 for a proof). As the last remark about notation, in what follows we shall denote by I_e (e being a positive integer) the trivial complex representation of degree e, as well as the e-dimensional identity matrix over $\mathbb{C}$.

Lemma 2.1. *Let G be a group, N a normal subgroup of G, T an irreducible representation of N, and D an irreducible representation of G such that $D{\downarrow_N} = I_e \otimes T$. Then there exist projective representations P_1 and P_2 of G such that*

(a) $D(g) = P_1(g) \otimes P_2(g)$ *for all g in G,*
(b) $P_1(x) = I_e$ *and* $P_2(x) = T(x)$ *for all x in N,*
(c) $T(x)^{P_2(g)} = T(x^g)$ *for all x in N and g in G.*

Observe that, if T' is a representation of N which is genuine-equivalent to T, and P, P' are projective representations of G which satisfy condition (c) of 2.1 for T and T' respectively, then (by Schur's Lemma) P and P' are projective-equivalent.

Assume now that the group G is a central product of the subgroups A and B, that is, $G = AB$ with $[A, B] = 1$. In this case G can be identified with a quotient of the (external) direct product $A \times B$ and, by inflation, each representation of G may be viewed as a representation of $A \times B$. In particular, by 3.7.1 of [**5**], each irreducible representation of G may be viewed as an outer tensor product $R \# S$ of some irreducible representation R of A and some irreducible representation S of B, such that $R{\downarrow_{A\cap B}} = \xi \otimes I_{\deg R}$ and $S{\downarrow_{A\cap B}} = \xi \otimes I_{\deg S}$ for a suitable 1-dimensional representation ξ of $A \cap B$ (recall that $R \# S$ maps, by definition, an element (a, b) of $A \times B$ to the matrix $R(a) \otimes S(b)$). Conversely, if R and S are irreducible representations of A and B satisfying this condition for some ξ, then $R \# S$ may be viewed as an (irreducible) representation of G.

Lemma 2.2. *Let the group G be a central product of the subgroups A and B. Then the following properties hold:*

(a) *if R is an irreducible genuine representation for one of the central factors, say A, then there exists a unique homomorphism ϱ, from G to $PGL(\deg R, \mathbb{C})$, such that $\varrho{\downarrow_A} = \bar{R}$ and $B \leq \ker \varrho$,*
(b) *let R and S be irreducible genuine representations, for A and B respectively, such that $R \# S$ is a representation for G. If ϱ and σ are homomorphisms as in* (a) *for R and S, then we have $\overline{R \# S} = \varrho \otimes \sigma$.*

Proof of (a). For each element g of G, consider an element a in A and an element b in B such that $g = ab$, and define $\varrho(g)$ to be $\bar{R}(a)$. It is routine to check that ϱ is well defined and satisfies the required conditions. ∎

Proof of (b). Let X and Y be projective representations of G such that $\bar{X} = \varrho$ and $\bar{Y} = \sigma$ (see Remark 1.1); following Definition 1.2, $\varrho \otimes \sigma$ is the homomorphism $\overline{X \otimes Y}$. Now, given an element g in G, let a and b be elements of A and B respectively, such that $g = ab$; since we get $\bar{X}(g) = \varrho(g) = \bar{R}(a)$, and $\bar{Y}(g) = \sigma(g) = \bar{S}(b)$, claim (b) follows simply by applying the definitions. ∎

It will be convenient to have a temporary name for a class of groups which will play an important role in our proofs (but will not appear in our conclusions).

Definition 2.3. Let F be a finite group; we say that F is a *good group* if $Z(F)$ is cyclic and $F/Z(F)$ is abelian of squarefree exponent.

The reason these groups are so important here is the following lemma; for a proof, see 1.4 of [**3**].

Lemma 2.4. *Let G be a group with Fitting subgroup F and centre Z. If G has a faithful quasi-primitive representation, then F/Z is an abelian group of squarefree exponent.*

The relevant properties of good groups will be outlined next.

Lemma 2.5. *Let F be a good group, and let Z denote its centre; then the following properties hold:*

(a) *if K is a subgroup of F such that $Z(K) = Z$, then F is the (central) product of K and $C_F(K)$;*

(b) *if P is an irreducible projective representation of F with $Z \leq \ker \bar{P}$, then we have $(\deg P)^2 = |F : \ker \bar{P}|$;*

(c) *if D is a faithful irreducible representation of F, and $\bar{D} \simeq \bar{P}_1 \otimes \bar{P}_2$ where P_1 and P_2 are projective representations of F (here equivalence is in the sense of Remark* 1.1*), then we have $F = \ker \bar{P}_1 \cdot \ker \bar{P}_2$;*

(d) *with the same assumptions as in* (c)*, if K is the kernel of $\bar{P}_1$, then $Z(K)$ coincides with Z; moreover, denoting by L the kernel of $\bar{P}_2$, we have $L = C_F(K)$;*

(e) *with the same assumptions as in* (c)*, there exist genuine representations R and S, of $K := \ker \bar{P}_1$ and $L := \ker \bar{P}_2 = C_F(K)$ respectively, such that $R \# S$ is a representation of F which is genuine-equivalent to D. Moreover, we have $\bar{P}_1 = \sigma$ and $\bar{P}_2 = \varrho$, where ϱ and σ are the homomorphisms linked to R and S which were constructed in* 2.2(a)*, so that we have $\bar{P}_1{\downarrow_L} = \bar{S}$ and $\bar{P}_2{\downarrow_K} = \bar{R}$. This implies that $P_1{\downarrow_L}$ is a genuine representation of L up to multiplying it by a suitable map from L to $\mathbb{C}^\times$, and P_2 behaves similarly with respect to K.*

Proof of (a). Let Q be a (nontrivial) Sylow subgroup of F, say for the prime q. We claim first that $Z(Q) = Z(K \cap Q)$. To see this, note that $Z(Q) \leq Z$ because F is nilpotent; as $Z \leq K$, this proves that $Z(Q)$ is contained in $K \cap Q$, and then of course $Z(Q) \leq Z(K \cap Q)$. Conversely, $Z(K \cap Q)$ lies in $Z(K) = Z$ because K is nilpotent and $K \cap Q$ is the Sylow q-subgroup of K, and hence $Z(K \cap Q) \leq Z(Q)$.

It follows that $(K \cap Q) \cap C_Q(K \cap Q) = Z(Q)$, whence the product of $(K \cap Q)/Z(Q)$ and $C_Q(K \cap Q)/Z(Q)$ is a direct product. We want to show next that this is all of

$Q/Z(Q)$. Suppose that $|(K\cap Q)/Z(Q)| = q^n$ so $(K\cap Q)/Z(Q)$ is an n-dimensional vector space over $GF(q)$, and choose a basis $\{Z(Q)x_1, \ldots, Z(Q)x_n\}$ for it. We have $C_Q(K\cap Q) = \bigcap_{i=1}^n C_Q(x_i)$. Using that F/Z is abelian of squarefree exponent, it is easy to see that each map $\alpha_i : Q \to Z$ defined by $\alpha_i(x) = [x_i, x]$ is a homomorphism whose kernel is $C_Q(x_i)$ and whose image has exponent dividing q. Since Z is cyclic, we conclude that $|Q : C_Q(x_i)| \leq q$, whence $|Q/C_Q(K\cap Q)| \leq q^n$. Thus the dimension of $(K\cap Q)/Z(Q) \times C_Q(K\cap Q)/Z(Q)$ (as a vector space over $GF(q)$) is at least the dimension of $Q/Z(Q)$, and our claim follows.

Finally, let $\{Q_1, \ldots, Q_h\}$ be the set of (nontrivial) Sylow subgroups of F: we have

$$F = (K\cap Q_1)C_{Q_1}(K\cap Q_1)\cdots(K\cap Q_h)C_{Q_h}(K\cap Q_h) = KC_F(K),$$

as desired. ∎

Proof of (b). Since P is an irreducible projective representation such that $\ker \bar{P}$ contains Z, we have that $\bar{P}(F)$ is an irreducible abelian subgroup of $PGL(\deg P, \mathbb{C})$ (which is of course isomorphic to $F/\ker \bar{P}$); if M is the preimage of $\bar{P}(F)$ in $GL(\deg P, \mathbb{C})$ under the natural homomorphism, we have $Z(GL(\deg P, \mathbb{C})) \leq Z(M)$ but, since M is irreducible, equality holds. Moreover, M is nilpotent of class 2, so that $(\deg P)^2 = |M/Z(M)| = |\bar{P}(F)|$ (this is not hard to prove; see for example [**2**], 4.3) and our claim follows. ∎

Proof of (c). Since D is faithful, we have $\ker \bar{P}_1 \cap \ker \bar{P}_2 = Z$ (this is easily seen, as the Kronecker product of two matrices is a scalar matrix if and only if the factors are scalar matrices), so that $\ker \bar{P}_1/Z \cdot \ker \bar{P}_2/Z = \ker \bar{P}_1/Z \times \ker \bar{P}_2/Z$; this is a subgroup of the abelian group with squarefree exponent F/Z, hence it suffices to show that $l(\ker \bar{P}_1/Z) + l(\ker \bar{P}_2/Z) = l(F/Z)$, where $l(G)$ denotes the composition length of a given group G.

Since we have $l(\ker \bar{P}_i/Z) = l(F/Z) - l(F/\ker \bar{P}_i)$, what we want to show is $l(F/Z) = l(F/\ker \bar{P}_1) + l(F/\ker \bar{P}_2)$. Let H be the set $\{(\bar{P}_1(x), \bar{P}_2(x)) : x \in F\}$, which is indeed a subgroup of the (external) direct product $\bar{P}_1(F) \times \bar{P}_2(F)$, and let $\varphi : H \to PGL(\deg D, \mathbb{C})$ be the map defined by $\varphi((\bar{P}_1(x), \bar{P}_2(x))) := \overline{P_1(x) \otimes P_2(x)}$. It is easily seen that φ is a monomorphism, and therefore $H \simeq \varphi(H) \simeq \bar{D}(F)$ holds; but now, by part (b), we have

$$|\bar{D}(F)| = (\deg D)^2 = (\deg P_1)^2(\deg P_2)^2 = |\bar{P}_1(F)||\bar{P}_2(F)|,$$

whence $\bar{D}(F) \simeq \bar{P}_1(F) \times \bar{P}_2(F)$, and the claim is proved (as $F/Z \simeq \bar{D}(F)$ and $F/\ker \bar{P}_i \simeq \bar{P}_i(F)$). ∎

Proof of (d). By part (c) we have $F = KL$. Let us now prove that $[K, L] = 1$. Denoting by x an element of K and by y an element of L, there exist A in $GL(\deg D, \mathbb{C})$ and λ, μ in $\mathbb{C}^\times$ such that

$$A^{-1}D(x)A = \lambda I_{\deg P_1} \otimes P_2(x) \quad \text{and} \quad A^{-1}D(y)A = P_1(y) \otimes \mu I_{\deg P_2};$$

it is now clear that $[D(K), D(L)] = 1$, and the faithfulness of D yields what we wanted. Of course now we have $Z(K) = Z$ and, since the conditions $F = KL$, $L \leq C_F(K)$ and $K \cap L = K \cap C_F(K) = Z$ hold, we conclude that $L = C_F(K)$. ■

Proof of (e). By assumption, there exist an element A in $GL(\deg D, \mathbb{C})$ and a map λ from F to $\mathbb{C}^\times$ such that $\lambda(f)A^{-1}D(f)A = P_1(f) \otimes P_2(f)$ holds for all f in F. Now, for all k in K, we get $\lambda(k)A^{-1}D(k)A = \mu(k)I_{\deg P_1} \otimes P_2(k)$, where μ is a map from K to $\mathbb{C}^\times$; defining $R(k)$ as $\lambda(k)^{-1}\mu(k)P_2(k)$ we get $A^{-1}D(k)A = I_{\deg P_1} \otimes R(k)$, so that R is a genuine representation of K. Similarly, a genuine representation S for L can be defined so that, for all l in L, we have $A^{-1}D(l)A = S(l) \otimes I_{\deg P_2}$. Now, for every element f in F, we can choose k in K and l in L such that $f = kl$, obtaining

$$\begin{aligned} A^{-1}D(f)A &= \big(A^{-1}D(k)A\big)\big(A^{-1}D(l)A\big) \\ &= \big(I_{\deg P_1} \otimes R(k)\big)\big(S(l) \otimes I_{\deg P_2}\big) \\ &= S(l) \otimes R(k) \\ &= (S \# R)(f), \end{aligned}$$

so that D is genuine-equivalent to $R \# S$ (swapping the factors does not change the equivalence type), and both of R and S are irreducible. Finally, recalling that σ is defined by $\sigma(f) := \bar{S}(l)$, and observing that we have $\bar{S}(l) = \bar{P}_1(l) = \bar{P}_1(f)$, we conclude that $\sigma = \bar{P}_1$; in an entirely similar way we also get $\varrho = \bar{P}_2$. ■

We are now in a position to prove the main results of the paper.

Theorem 2.6. *Let H be a group, and let F be a good group such that F is a normal subgroup of H and $C_H(F)$ is contained in F. Let D be a faithful representation of H such that $D{\downarrow_F}$ is irreducible. Then there exists a bijection between the set of all the pairs $([P_1], [P_2])$, where P_1, P_2 are projective representations of H such that $\bar{D} \simeq \bar{P}_1 \otimes \bar{P}_2$, and the set of normal subgroups K of H such that $K \leq F$ and $Z(K) = Z(H)$ hold. In particular, such a bijection can be constructed by mapping $([P_1], [P_2])$ to $K := \ker(\bar{P}_1{\downarrow_F})$ and, K being so defined, we also have $\ker(\bar{P}_2{\downarrow_F}) = C_F(K)$.*

Proof. we shall denote by Z the centre of H; also, we shall denote by $\mathcal{F}_D$ the set of all the pairs $([P_1], [P_2])$, where P_1, P_2 are projective representations of H such that $\bar{D} \simeq \bar{P}_1 \otimes \bar{P}_2$, and by $\mathcal{S}$ the set of normal subgroups K of H such that K lies in F and $Z(K) = Z$. Next we observe that, since D is faithful and its restriction to F is irreducible, we have $Z(F) \leq Z$; but F contains its own centralizer in H, therefore $Z(F)$ coincides with Z.

Now, as the first step in the proof, we shall construct a map α from $\mathcal{F}_D$ to $\mathcal{S}$: consider an element $([P_1], [P_2])$ in $\mathcal{F}_D$ and define $\alpha(([P_1], [P_2]))$ as the kernel of $\bar{P}_1{\downarrow_F}$. Since equivalent projective representations yield homomorphisms (to the relevant projective general linear group) which have the same kernel, the 'value' $\alpha(([P_1], [P_2]))$ does not depend on the choice of representatives for the classes $[P_1]$ and $[P_2]$; moreover, denoting by K the kernel of $\bar{P}_1{\downarrow_F}$, Lemma 2.5(d) tells us that $Z(K) = Z$, and certainly we have $K \leq F$ and $K \trianglelefteq H$. The discussion above shows that α is actually a map from $\mathcal{F}_D$ to $\mathcal{S}$. Also, again by Lemma 2.5(d), we get $C_F(K) = \ker(\bar{P}_2{\downarrow_F})$.

As the second step we shall show that, given an element K of $\mathcal{S}$, there exists a unique element $([P_1],[P_2])$ in $\mathcal{F}_D$ such that $\ker(\bar{P}_1{\downarrow_F}) = K$; this will prove that α is a bijection. So, let us start from an element K in $\mathcal{S}$; by Lemma 2.5(a) we get $F = KC_F(K)$. If we denote by χ the character afforded by D, we have $\chi{\downarrow_F}= \varphi$ for some φ in $\operatorname{Irr} F$; now K is a normal subgroup of F and, if ϑ is an irreducible constituent of $\varphi{\downarrow_K}$, then the inertia subgroup $I_F(\vartheta)$ is all of F. We conclude that $\chi{\downarrow_K}= e\vartheta$, where e is a positive integer, hence we can assume $D{\downarrow_K}= I_e \otimes T$ where T is an irreducible representation of K affording the character ϑ. We are now in a position to apply Lemma 2.1, which ensures the existence of an element $([P_1],[P_2])$ in $\mathcal{F}_D$ with the properties that K is contained in $\ker(\bar{P}_1{\downarrow_F})$, $\deg P_2 = \deg T$ and $T(x^h) = T(x)^{P_2(h)}$ for all x in K and h in H. We want now to prove that K coincides with $\ker(\bar{P}_1{\downarrow_F})$. Let x be in $\ker(\bar{P}_1{\downarrow_F})$, and let k, c be elements, of K and $C_F(K)$ respectively, such that $x = kc$ (again we are using Lemma 2.5(a)). Since K is contained in $\ker(\bar{P}_1{\downarrow_F})$, c lies in $\ker(\bar{P}_1{\downarrow_F})$ as well, so that we have $D(c) = \mu I_e \otimes P_2(c)$ for some μ in $\mathbb{C}^\times$; moreover, c is in $C_F(K)$, hence $T(y)^{P_2(c)} = T(y^c) = T(y)$ holds for all y in K, so that $P_2(c)$ is a scalar matrix, say $\nu I_{\deg T}$ for some ν in $\mathbb{C}^\times$. We conclude that $D(c)$ is given by the Kronecker product of two scalar matrices, therefore c lies in Z (by the faithfulness of D) and the claim follows.

To complete the second step of the proof, we need to show that K determines *uniquely* an element $([P_1],[P_2])$ of $\mathcal{F}_D$ with the property that $\ker(\bar{P}_1{\downarrow_F}) = K$. For this purpose observe that, since $\bar{D} \simeq \bar{P}_1 \otimes \bar{P}_2$, there exist A in $GL(\deg D, \mathbb{C})$ and a map λ from H to $\mathbb{C}^\times$ such that $A^{-1}D(h)A = (\lambda(h))^{-1}P_1(h)\otimes P_2(h)$ holds for all h in H. Moreover, we get $A^{-1}D(k)A = I_{\deg P_1} \otimes R(k)$ for all k in K, where R is the genuine irreducible representation of K defined in the proof of Lemma 2.5(e). From $D(k^h) = D(k)^{D(h)}$ we obtain now $I_{\deg P_1} \otimes R(k^h) = I_{\deg P_1} \otimes R(k)^{P_2(h)}$, whence $R(k^h) = R(k)^{P_2(h)}$ for all h in H and k in K. Since the genuine-equivalence type of R is uniquely determined by K and by the genuine-equivalence type of D, we conclude that the projective-equivalence type of P_2, that is, $[P_2]$, is uniquely determined by K and by the genuine-equivalence type of D. Similarly, we see that $[P_1]$ is uniquely determined by $\ker(\bar{P}_2{\downarrow_F})$, which is in turn determined by K, being its centralizer in F (Lemma 2.5(d)), and by the genuine-equivalence type of D. ∎

Definition 2.7. Let G be a group, and D a representation of G. We define the subgroups $Z(D)$ and $F(D)$ of G by $Z(D)/\ker D := Z(G/\ker D)$ and $F(D)/\ker D := F(G/\ker D)$.

Corollary 2.8. *Let G be a group, and D a quasi-primitive representation of G such that $D{\downarrow_{F(D)}}$ is irreducible. There is a bijection between the set of all the pairs $([P_1],[P_2])$, where P_1 and P_2 are projective representations of G such that $\bar{D} \simeq \bar{P}_1\otimes\bar{P}_2$, and the interval $[Z(D),F(D)]$ in the lattice of normal subgroups of G. Such a bijection can be constructed by mapping $([P_1],[P_2])$ to $K := \ker(\bar{P}_1{\downarrow_{F(D)}})$*

and, K being so defined, we also have $\ker(\bar{P}_2{\downarrow}_{F(D)}) = \{x \in F(D) : [x, K] \subseteq \ker D\}$.

Proof. Denoting by X the kernel of D, consider the quotient group $\hat{G} := G/X$; if Δ is the representation of $\hat{G}$ defined by $\Delta(Xg) := D(g)$ for all Xg in $\hat{G}$, we have that Δ is faithful and $\Delta{\downarrow}_{F(\hat{G})}$ is irreducible, therefore $C_{\hat{G}}(F(\hat{G}))$ is in the centre of $\hat{G}$ and, in particular, it lies in $F(\hat{G})$. This implies $Z(\hat{G}) = Z(F(\hat{G}))$ and, since Δ is quasi-primitive, we conclude that $F(\hat{G})$ is a good group (see 2.4), obviously a normal subgroup of $\hat{G}$. Now we are in a position to apply Theorem 2.6, obtaining that there exists a bijection between the set $\mathcal{F}_\Delta$ of all the pairs $([Q_1], [Q_2])$, where Q_1 and Q_2 are projective representations of $\hat{G}$ such that $\bar{\Delta} \simeq \bar{Q}_1 \otimes \bar{Q}_2$, and the set of normal subgroups $\hat{K}$ of $\hat{G}$ such that $\hat{K} \leq F(\hat{G})$ and $Z(\hat{K}) = Z(\hat{G})$. We also know that, if $([Q_1], [Q_2])$ corresponds to $\hat{K}$ in the relevant bijection, then we have $\hat{K} = \ker(\bar{Q}_1{\downarrow}_{F(\hat{G})})$ and $C_{F(\hat{G})}(\hat{K}) = \ker(\bar{Q}_2{\downarrow}_{F(\hat{G})})$.

Consider now a projective representation P of G such that X is contained in $\ker \bar{P}$; we can choose a projective representation Q of $\hat{G}$ such that $\bar{Q}(Xg) := \bar{P}(g)$ for all Xg in $\hat{G}$ and, associating $[P]$ with $[Q]$, we can easily construct a bijection between $\mathcal{F}_\Delta$ and the set of all the pairs $([P_1], [P_2])$, where P_1 and P_2 are projective representations of G such that $\bar{D} \simeq \bar{P}_1 \otimes \bar{P}_2$. Also, the natural correspondence between normal subgroups of $\hat{G}$ and normal subgroups of G containing X provides, by restriction, a bijection between the set of normal subgroups $\hat{K}$ of $\hat{G}$ such that $\hat{K} \leq F(\hat{G})$ and $Z(\hat{K}) = Z(\hat{G})$, and the interval $[Z(D), F(D)]$ in the lattice of normal subgroups of G; the proof can be now easily completed. ∎

3. Factorization of characters

We give next an interpretation of the discussion above in terms of characters.

Definition 3.1. Let G be a group; we denote by $(\tilde{G}, \pi)$ a Schur covering of G (so that $\tilde{G}$ is a Schur representation group for G; see [**7**], Chapter 11), and by A the kernel of π, which is a central subgroup of $\tilde{G}$; if H is a subgroup of G, we define $\tilde{H}$ as $\pi^{-1}(H)$.

If χ and ψ are irreducible characters of G, we say that they are equivalent (and we write $\chi \simeq \psi$) if there exists λ in $\operatorname{Irr} G$ such that $\lambda(1) = 1$ and $\chi = \lambda\psi$. It is clear that, in this way, an equivalence relation on the set $\operatorname{Irr} G$ is defined; we shall denote by $[\chi]$ the equivalence class of the character χ modulo this equivalence relation.

Finally, we define $Z(\chi)$ and $F(\chi)$ in analogy with Definition 2.7; observe that, if χ_{inf} is the character of $\tilde{G}$ obtained from χ by inflation, we have $\widetilde{F(\chi)} = F(\chi_{\mathrm{inf}})$.

Theorem 3.2. *Let G be a group, and χ a quasi-primitive character of G such that $\chi{\downarrow}_{F(\chi)}$ is irreducible. Then the following properties hold:*

(a) *if N is a normal subgroup of G with $Z(\chi) \leq N \leq F(\chi)$, then there exist characters ϱ_1 and ϱ_2 of $\tilde{G}$ such that $\chi_{\mathrm{inf}} \simeq \varrho_1\varrho_2$ and $Z(\varrho_1{\downarrow}_{\widetilde{F(\chi)}}) = \tilde{N}$;*

(b) *let ϱ_1, ϱ_2, ϱ_3 and ϱ_4 be irreducible characters of G such that $\chi \simeq \varrho_1\varrho_2$ and $\chi \simeq \varrho_3\varrho_4$; if $Z(\varrho_1{\downarrow}_{F(\chi)})$ is the same as $Z(\varrho_3{\downarrow}_{F(\chi)})$, then we have $\varrho_1 \simeq \varrho_3$ and $\varrho_2 \simeq \varrho_4$;*

(c) *there is a bijection between the set of all the pairs* $([\varrho_1],[\varrho_2])$, *where* ϱ_1 *and* ϱ_2 *are characters of* $\tilde{G}$ *such that* $\chi_{\inf} \simeq \varrho_1\varrho_2$, *and the interval* $[Z(\chi), F(\chi)]$ *in the lattice of normal subgroups of* G. *Such a bijection can be constructed by mapping* $([\varrho_1],[\varrho_2])$ *to the subgroup* N *such that* $Z(\varrho_1{\downarrow}_{\widetilde{F(\chi)}}) = \tilde{N}$.

Proof of (a). Let D be a representation of G which affords χ; D is quasi-primitive, its restriction to $F(D) = F(\chi)$ is irreducible, and N is a normal subgroup of G with $Z(D) \leq N \leq F(D)$; hence Corollary 2.8 yields that there exist projective representations P_1 and P_2 of G such that $\bar{D} \simeq \bar{P}_1 \otimes \bar{P}_2$ and $\ker(\bar{P}_1{\downarrow}_{F(D)}) = N$. As $(\tilde{G}, \pi)$ is a Schur covering for G, we can find genuine representations D_1 and D_2 of $\tilde{G}$, together with maps ξ_1 and ξ_2 from $\tilde{G}$ to $\mathbb{C}^\times$, such that $\xi_1(x)D_1(x) = P_1(Ax)$ and $\xi_2(x)D_2(x) = P_2(Ax)$ for all x in $\tilde{G}$ (here we are identifying G with $\tilde{G}/A$); it is now easy to see that D, viewed by inflation as a representation of $\tilde{G}$, is projective-equivalent to $D_1 \otimes D_2$. We conclude that, denoting by ϱ_1 and ϱ_2 the characters of $\tilde{G}$ afforded by D_1 and D_2, we get $\chi_{\inf} \simeq \varrho_1\varrho_2$; moreover, it is easily checked that $Z(\varrho_1{\downarrow}_{\widetilde{F(\chi)}})$ coincides with $\tilde{N}$. ∎

Proof of (b). Let D_i be a representation which affords ϱ_i, for i in $\{1,2,3,4\}$; we have $\bar{D} \simeq \bar{D}_1 \otimes \bar{D}_2$ and $\bar{D} \simeq \bar{D}_3 \otimes \bar{D}_4$; moreover,

$$\ker(\bar{D}_1{\downarrow}_{F(D)}) = Z(\varrho_1{\downarrow}_{F(\chi)}) = Z(\varrho_3{\downarrow}_{F(\chi)}) = \ker(\bar{D}_3{\downarrow}_{F(D)})$$

holds. By Corollary 2.8 we conclude that $\bar{D}_1 \simeq \bar{D}_3$ and $\bar{D}_2 \simeq \bar{D}_4$, so that the claim follows. ∎

Proof of (c). This follows at once by the two previous statements. ∎

4. A final remark

Let G be a group, and P, Q, R projective representations of G such that $\bar{P} \otimes \bar{Q} \simeq \bar{P} \otimes \bar{R} =: \bar{D}$; if D happens to be a genuine quasi-primitive representation of G whose restriction to $F(D)$ is irreducible, then it follows from Corollary 2.8 that Q and R are equivalent (and therefore we have, under the right assuptions, a 'cancellation law').

Even this claim fails if we weaken the hypothesis of irreducibility for $D{\downarrow}_{F(D)}$, assuming only, for instance, that the restriction of D to $F^*(D)$ is irreducible (here $F^*(D)$ denotes the preimage, under the natural homomorphism, of the generalized Fitting subgroup of $G/\ker D$). Consider for example $G = A_9$; if we denote by P the 8-dimensional irreducible representation of G, by Q and R the two 21-dimensional irreducible representations of G (which are inequivalent), and by D the 168-dimensional irreducible representation of G, we see that D is quasi-primitive (indeed primitive) and of course irreducible when restricted to $F^*(G)$. Moreover, D is genuine-equivalent to both of $P \otimes Q$ and $P \otimes R$, and therefore we have $\bar{D} \simeq \bar{P} \otimes \bar{Q} \simeq \bar{P} \otimes \bar{R}$. But it is clear that Q and R are not equivalent, even in a projective sense.

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References

[**1**] Charles W. Curtis and Irving Reiner, *Methods of representation theory I* (Wiley, New York, 1981).

[**2**] John D. Dixon, *The structure of linear groups* (Van Nostrand Reinhold, London, 1971).

[**3**] Pamela A. Ferguson and Alexandre Turull, 'Prime characters and factorizations of quasi-primitive characters', *Math. Z.* **190** (1985), 583–604.

[**4**] Dilip Gajendragadkar, 'A characteristic class of characters of finite π-separable groups', *J. Algebra* **59** (1979), 237–259.

[**5**] Daniel Gorenstein, *Finite groups* (Harper & Row, New York, 1968).

[**6**] Bertram Huppert, *Character theory of finite groups* (De Gruyter, Berlin, 1998).

[**7**] I. Martin Isaacs, *Character theory of finite groups* (Academic Press, New York, 1976).

[**8**] I. Martin Isaacs, 'Primitive Characters, Normal Subgroups and M-Groups', *Math. Z.* **177** (1981), 267–284.

[**9**] L. G. Kovács, 'On tensor induction of group representations', *J. Aust. Math. Soc. Ser. A* **49** (1990), 486–501.

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