ASYMPTOTIC ANALYSIS FOR VANISHING ACCELERATION IN A THERMOVISCOELASTIC SYSTEM

ELENA BONETTI AND GIOVANNA BONFANTI

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We have investigated a dynamic thermoviscoelastic system (2003), establishing existence and uniqueness results for a related initial and boundary values problem. The aim of the present paper is to study the asymptotic behavior of the solution to the above problem as the power of the acceleration forces goes to zero. In particular, well-posedness and regularity results for the limit (quasistatic) problem are recovered.

1. Introduction

A three-dimensional model for thermoviscoelastic phenomena proposed by Frémond has been investigated in [2] from the point of view of existence and uniqueness of solutions. In the present paper, we examine the asymptotic behavior of the solution to the system considered in [2], as the power of the acceleration forces vanishes. As we are mainly interested in the analytical investigation of the model, we just recall the evolution equations and refer to [2, 9], and the references therein, for a detailed thermomechanical derivation and justification.

Thus, we consider a viscoelastic body (see, e.g., [7]) located in a smooth bounded domain $\Omega \subset \mathbb{R}^3$ during a finite time interval $[0,T]$, $T>0$. Its thermomechanical evolution is described by the variables θ (absolute temperature) and u (small displacement). The latter is considered as a scalar quantity, in regard of simplicity. The equations of the model are the universal balance laws of the thermomechanics, that is, the energy balance and the momentum balance; the constitutive relations are derived from a free energy functional and a pseudo-potential of dissipation (see [9]). In particular, the energy balance law (considering the Fourier heat flux law) leads to the equation

$$c_s\partial_t\theta - k_0\Delta\theta = \theta\nabla\partial_t u\cdot\mathbf{a} + \mu\left|\nabla\partial_t u\right|^2 + r \quad \text{a.e. in } \Omega\times]0,T[, \tag{1.1}$$

and the momentum balance law (accounting for the accelerations) takes the form

$$\varepsilon\partial_{tt}u - \mu\Delta\partial_t u - \nu\Delta u = \nabla\theta\cdot\mathbf{a} + f \quad \text{a.e. in } \Omega\times]0,T[, \tag{1.2}$$

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where $c_s > 0$ denotes the heat capacity of the system, $k_0 > 0$ denotes the thermal conductivity, $\varepsilon > 0$ is proportional to the mass density, $\mu > 0$ and $\nu > 0$ are respectively a dissipation and an interaction coefficient, r stands for an external heat source, and f stands for an external volume force. Moreover, $\mathbf{a}$ is a three-dimensional constant vector related to the thermal expansion coefficient (we will assume, e.g., $\mathbf{a} = (1,1,1)$). Finally, ∂_t denotes $\partial/\partial t$, ∂_{tt} indicates $\partial^2/\partial t^2$, Δ is the Laplacian, and ∇ is the gradient (in space variables).

For the sake of simplicity, we will take $r = f = 0$ and we will get rid of the physical constants. However, we point out that our results apply also to the more general case of nonzero applied volume force f and exterior heat source r, provided f and r are taken with sufficient regularity.

Next, we associate with (1.1)-(1.2) an appropriate set of initial and boundary conditions

$$\begin{gathered}\theta(\cdot,0) = \theta_0, \quad u(\cdot,0) = u_0, \quad \partial_t u(\cdot,0) = u_1, \quad \text{a.e. in } \Omega, \\ \partial_{\mathbf{n}}\theta = 0 \quad \text{a.e. in } \partial\Omega\times]0,T[, \\ u = 0 \quad \text{a.e. in } \partial\Omega\times]0,T[,\end{gathered} \tag{1.3}$$

where $\partial_{\mathbf{n}}$ stands for $\partial/\partial\mathbf{n}$, $\mathbf{n}$ being the outward unit normal vector to the boundary $\partial\Omega$.

We remark that in (1.1) we deal with the complete energy balance equation, as we retain all the mechanically induced heat sources at the right-hand side, coming from the power of interior forces written for the actual velocities, even if they are higher-order nonlinearities. Note that, from the analytical point of view, the presence of quadratic terms in (1.1) involving strain rate introduces strong technical difficulties. Thus, it seems very hard to obtain global well-posedness results for the above system. Similar difficulties arise in the investigations carried out, for example, in [2, 3, 4, 5].

The aim of the present paper is to study the asymptotic behavior of the solution to (1.1), (1.2), and (1.3), as ε vanishes in (1.2). In [2], we provide a local-in-time existence result, as well global uniqueness of sufficiently regular solutions. Nonetheless, such results cannot be considered as the starting point of our asymptotic analysis because the lifetime of the solution depends on the parameter ε and it can vanish when ε tends to 0. Thus, we derive here an alternative proof which provides existence and uniqueness of solution to (1.1), (1.2), and (1.3) in a time interval independent of ε. Next, by a careful derivation of suitable a priori estimates, we prove convergence and regularity results. Concerning applications, we point out that our asymptotic analysis implies, in particular, the local well posedness for a quasistatic viscoelastic problem (see, e.g., [11] for related results) as a limit of the dynamical one, as the mass coefficient in the inertial term tends to zero.

The rest of the present work is as follows. The next Section is devoted to the notation, the assumptions, and the statements of the main results. Section 3 contains the proof of the existence and uniqueness result given by Theorem 2.2. Section 4 is concerned with the proofs of our first convergence theorem (Theorem 2.3) and of the uniqueness result for the limit problem (Theorem 2.4). Finally, Section 5 is devoted to the proof of the regularity result given by Theorem 2.5 (see also Remarks 2.6-2.7 and Proposition 2.8). Throughout the paper some comments and remarks are given.

2. Statement of the problem and main results

We start by fixing some notation. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$. We set $Q_t := \Omega\times]0,t[$ for $t \in]0,T[$ and $Q := \Omega\times]0,T[$. Then, we introduce the notation

$$H := L^2(\Omega), \qquad V := H^1(\Omega), \qquad W := \{u \in H^2(\Omega), \text{ such that } \partial_{\mathbf{n}} u = 0 \text{ on } \partial\Omega\}, \tag{2.1}$$

and identify H with its dual space H', so that $W \hookrightarrow V \hookrightarrow H \hookrightarrow V' \hookrightarrow W'$ with dense and compact embeddings. We use the same symbol for the norm of a space of scalar functions and the norm of the space of corresponding vector-valued functions. For instance, $\|\cdot\|_V$ means the norm of both V and V^3. Besides, let the symbol $\|\cdot\|$ indicate the norm of H (or H^3). Henceforth, we denote by $\langle\cdot,\cdot\rangle$ the duality pairing between V' and V, by $(\cdot,\cdot)$ the scalar product in H, and by $((\cdot,\cdot))$ the scalar product in V. Then, the associated Riesz isomorphism $J : V \to V'$ and the scalar product in V', denoted by $((\cdot,\cdot))_*$, can be specified by

$$\langle Jv_1, v_2\rangle := ((v_1,v_2)), \quad ((u_1,u_2))_* := \langle u_1, J^{-1}u_2\rangle \quad \text{for } v_i \in V,\ u_i \in V',\ i = 1,2. \tag{2.2}$$

For the reader's convenience, we recall the Poincaré inequality

$$\exists C > 0 \text{ such that } \|v\| \le C\|\nabla v\| \quad \forall v \in H_0^1(\Omega), \tag{2.3}$$

as well as the following inequality (which can be obtained, e.g., applying the Green formula, the Hölder inequality, and (2.3)):

$$\exists C' > 0 \text{ such that } \|\nabla v\| \le C'\|\Delta v\| \quad \forall v \in H^2(\Omega) \cap H_0^1(\Omega). \tag{2.4}$$

We start by considering Problem 2.1 below, that is, the system resulting from (1.1), (1.2), and (1.3), by getting rid of some physical constants and taking external sources and forces equal to zero.

Let $\varepsilon > 0$ be given. We introduce the families of initial data $\{\theta_{0\varepsilon}\}_{\varepsilon>0}$, $\{u_{0\varepsilon}\}_{\varepsilon>0}$, $\{u_{1\varepsilon}\}_{\varepsilon>0}$ satisfying

$$\theta_{0\varepsilon} \in H, \quad \theta_{0\varepsilon} \ge 0 \quad \text{a.e. in } \Omega, \tag{2.5}$$

$$u_{0\varepsilon} \in H^2(\Omega) \cap H_0^1(\Omega), \qquad u_{1\varepsilon} \in H_0^1(\Omega). \tag{2.6}$$

Problem 2.1. Find $(\theta_\varepsilon, u_\varepsilon)$ *such that*

$$\theta_\varepsilon \in H^1(0,T;V') \cap C^0([0,T];H) \cap L^2(0,T;V), \quad \theta_\varepsilon \ge 0 \text{ a.e. in } Q, \tag{2.7}$$

$$u_\varepsilon \in H^2(0,T;H) \cap W^{1,\infty}(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^2(\Omega)), \tag{2.8}$$

$$\langle \partial_t\theta_\varepsilon, v\rangle + (\nabla\theta_\varepsilon, \nabla v) = (\theta_\varepsilon \nabla\partial_t u_\varepsilon \cdot \mathbf{a} + |\nabla\partial_t u_\varepsilon|^2, v) \quad \forall v \in V \text{ a.e. in }]0,T[, \tag{2.9}$$

$$\varepsilon\partial_{tt}u_\varepsilon - \Delta\partial_t u_\varepsilon - \Delta u_\varepsilon = \nabla\theta_\varepsilon \cdot \mathbf{a} \quad \text{a.e. in } Q, \tag{2.10}$$

$$\theta_\varepsilon(\cdot,0) = \theta_{0\varepsilon} \quad \text{a.e. in } \Omega, \tag{2.11}$$

$$u_\varepsilon(\cdot,0) = u_{0\varepsilon} \quad \text{a.e. in } \Omega, \tag{2.12}$$

$$\partial_t u_\varepsilon(\cdot,0) = u_{1\varepsilon} \quad \text{a.e. in } \Omega. \tag{2.13}$$

In order to perform an asymptotic analysis on Problem 2.1 as ε tends to 0, from now on, we let ε vary, say, in $]0,1[$. As mentioned in the introduction, as a starting point of our asymptotic study, we need here a result (see Theorem 2.2 below) which provides the existence and uniqueness of the local solution to Problem 2.1 during a time interval independent of ε. To obtain such well-posedness result, we suppose the following bounds on the families of the initial data (in agreement with (2.5)-(2.6)):

$$||\theta_{0\varepsilon}|| + ||\Delta u_{0\varepsilon}|| \leq c_1, \tag{2.14}$$

$$||\nabla u_{1\varepsilon}|| \leq c_2, \tag{2.15}$$

for some $c_1, c_2 > 0$ and any $\varepsilon \in]0,1[$.

Now, we state a local well-posedness result for the Problem 2.1 where the lifetime of the solution is independent of ε.

Theorem 2.2. *Let the assumptions (2.5)-(2.6) and (2.14)-(2.15) hold. Then, there exists a final time $\tau \in]0,T]$ such that for any $\varepsilon \in]0,1[$ there exists a unique pair $(\theta_\varepsilon, u_\varepsilon)$ solving Problem 2.1 during the time interval $]0,\tau[$.*

After proving the above existence and uniqueness result, we are allowed to investigate the asymptotic behavior of the pair $(\theta_\varepsilon, u_\varepsilon)$ given by Theorem 2.2, as $\varepsilon \searrow 0$. To this aim, we require, in addition, the following convergence conditions on the sequences of the initial data allowing us to pass to the limit in (2.9), (2.10), (2.11), and (2.12):

$$\theta_{0\varepsilon} \longrightarrow \theta_0 \quad \text{in } H, \tag{2.16}$$

$$u_{0\varepsilon} \longrightarrow u_0 \quad \text{in } H^2(\Omega) \cap H_0^1(\Omega), \tag{2.17}$$

as $\varepsilon \searrow 0$.

Our first asymptotic result reads as follows.

Theorem 2.3. *Let $\theta_{0\varepsilon}$, $u_{0\varepsilon}$, $u_{1\varepsilon}$, θ_0, and u_0 satisfy (2.15), (2.16), and (2.17). Let $(\theta_\varepsilon, u_\varepsilon)$ be given by Theorem 2.2 corresponding to the data $\theta_{0\varepsilon}$, $u_{0\varepsilon}$, $u_{1\varepsilon}$. Then, there exists a pair (θ, u) fulfilling*

$$\begin{gathered} \theta \in H^1(0,\tau;V') \cap C^0([0,\tau];H) \cap L^2(0,\tau;V), \quad \theta \geq 0 \textit{ a.e. in } Q_\tau, \\ u \in W^{1,\infty}(0,\tau;H_0^1(\Omega)) \cap H^1(0,\tau;H^2(\Omega)), \end{gathered} \tag{2.18}$$

such that the strong, weak, or weak convergences listed below hold:*

$$\begin{gathered} \theta_\varepsilon \rightharpoonup \theta \quad \textit{in } H^1(0,\tau;V') \cap L^2(0,\tau;V), \\ \theta_\varepsilon \overset{*}{\rightharpoonup} \theta \quad \textit{in } L^\infty(0,\tau;H), \\ \theta_\varepsilon \longrightarrow \theta \quad \textit{in } C^0([0,\tau];V') \cap L^2(0,\tau;H), \\ u_\varepsilon \rightharpoonup u \quad \textit{in } H^1(0,\tau;H^2(\Omega)), \\ u_\varepsilon \overset{*}{\rightharpoonup} u \quad \textit{in } W^{1,\infty}(0,\tau;H_0^1(\Omega)), \\ u_\varepsilon \longrightarrow u \quad \textit{in } C^0([0,\tau];H_0^1(\Omega)), \\ \varepsilon\partial_{tt}u_\varepsilon \longrightarrow 0 \quad \textit{in } L^2(0,\tau;H), \end{gathered} \tag{2.19}$$

as $\varepsilon \searrow 0$. *In addition, the pair* (θ, u) *solves the limit problem*

$$\langle \partial_t \theta, v \rangle + (\nabla \theta, \nabla v) = \left(\theta \nabla \partial_t u \cdot \mathbf{a} + |\nabla \partial_t u|^2, v \right) \quad \forall v \in V \text{ a.e. in }]0, \tau[, \tag{2.20}$$

$$-\Delta \partial_t u - \Delta u = \nabla \theta \cdot \mathbf{a} \quad \text{a.e. in } Q_\tau, \tag{2.21}$$

$$\theta(\cdot, 0) = \theta_0 \quad \text{a.e. in } \Omega, \tag{2.22}$$

$$u(\cdot, 0) = u_0 \quad \text{a.e. in } \Omega. \tag{2.23}$$

We stress that Theorem 2.3 provides, in particular, a *local* (in time) existence result to the quasistatic problem (2.18), (2.20), (2.21), (2.22), and (2.23), where θ_0 and u_0 are as in the statement below. Actually, the following *global* uniqueness result concerning (2.18), (2.20), (2.21), (2.22), and (2.23) holds.

Theorem 2.4. *Let* $\theta_0 \in H$, $\theta_0 \geq 0$ *a.e. in* Ω *and* $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ *be given. Let* (θ, u) *be a pair of functions satisfying (2.20), (2.21), (2.22), and (2.23) during some interval* $]0, \tau[$ *with regularity given by (2.18). Then, such a pair is unique on the whole interval* $]0, \tau[$.

From now on, (θ, u) denotes the solution to the problem specified by (2.20), (2.21), (2.22), and (2.23) (with regularity (2.18)). By assuming some stronger hypotheses on the sequences of the initial data, we can derive uniform bounds on $(\theta_\varepsilon, u_\varepsilon)$, which yield further regularity for (θ, u) (see Remarks 2.6 and 2.7).

Theorem 2.5. *Let* $\theta_{0\varepsilon}$, $u_{0\varepsilon}$, θ_0, *and* u_0 *satisfy (2.16)-(2.17). Suppose moreover*

$$u_{1\varepsilon} \in H^2(\Omega) \cap H_0^1(\Omega), \tag{2.24}$$

$$||\theta_{0\varepsilon}||_V + \varepsilon^{-1/2} ||\nabla \theta_{0\varepsilon} \cdot \mathbf{a} + \Delta u_{1\varepsilon} + \Delta u_{0\varepsilon}|| \leq c_3, \tag{2.25}$$

for some $c_3 > 0$ *and any* $\varepsilon \in]0, 1[$. *Then, there exists a positive constant c such that*

$$\begin{gathered} ||\theta_\varepsilon||_{H^1(0,\tau;H) \cap L^\infty(0,\tau;V) \cap L^2(0,\tau;W)} \leq c, \\ \varepsilon^{1/2} ||\partial_{tt} u_\varepsilon||_{L^\infty(0,\tau;H)} + ||u_\varepsilon||_{H^2(0,\tau;H_0^1(\Omega)) \cap W^{1,\infty}(0,\tau;H^2(\Omega))} \leq c, \end{gathered} \tag{2.26}$$

for any sufficiently small $\varepsilon > 0$.

Remark 2.6. In view of Theorems 2.3 and 2.4, from (2.26), we deduce the further convergences

$$\theta_\varepsilon \overset{*}{\rightharpoonup} \theta \quad \text{in } H^1(0, \tau; H) \cap L^\infty(0, \tau; V) \cap L^2(0, \tau; W), \tag{2.27}$$

$$u_\varepsilon \overset{*}{\rightharpoonup} u \quad \text{in } H^2(0, \tau; H_0^1(\Omega)) \cap W^{1,\infty}(0, \tau; H^2(\Omega)), \tag{2.28}$$

$$\varepsilon \partial_{tt} u_\varepsilon \longrightarrow 0 \quad \text{in } L^\infty(0, \tau; H). \tag{2.29}$$

Moreover, from (2.27), (2.28), and (2.29), using a classical compactness argument (see [10, page 58]) and the generalized Ascoli theorem (see [12, Corollary 4]), we can deduce the following strong convergences:

$$\theta_\varepsilon \longrightarrow \theta \quad \text{in } C^0([0, \tau]; H) \cap L^2(0, \tau; V), \tag{2.30}$$

$$u_\varepsilon \longrightarrow u \quad \text{in } C^1([0, \tau]; H_0^1(\Omega)). \tag{2.31}$$

Thanks to the above convergences, the pair (θ, u) fulfilling

$$\theta \in H^1(0,\tau;H) \cap C^0([0,\tau];V) \cap L^2(0,\tau;W), \quad \theta \geq 0,$$
$$u \in H^2(0,\tau;H_0^1(\Omega)) \cap W^{1,\infty}(0,\tau;H^2(\Omega)) \tag{2.32}$$

is a solution to (2.20), (2.21), (2.22), (2.23), and (2.20) is now satisfied pointwise almost everywhere in Q_τ, due to the better regularity of θ.

Actually, we can show the further strong convergence

$$u_\varepsilon \longrightarrow u \quad \text{in } H^1(0,\tau;H^2(\Omega)). \tag{2.33}$$

To do this, it is enough to multiply the difference between (2.10) and (2.21) by $-\Delta\partial_t(u_\varepsilon - u)$; to integrate over Q_τ; to take (2.17), (2.29), and (2.30) into account (see (4.13) below for a similar derivation).

Remark 2.7. We point out that, in order to pass to the limit in Problem 2.1 in the more regular framework prescribed by Theorem 2.5, we have required assumption (2.25) as natural compatibility condition on the approximating initial data. Indeed, (2.25) implies (cf. also (2.14), (2.15), (2.16), (2.17), and (2.24))

$$\|\nabla\theta_{0\varepsilon} \cdot \mathbf{a} + \Delta u_{0\varepsilon} + \Delta u_{1\varepsilon}\| \longrightarrow 0, \tag{2.34}$$

as $\varepsilon \searrow 0$. From a mechanical point of view, this corresponds to having for the limit problem that the divergence of the stress is 0 at time $t = 0$, as no volume exterior forces are applied.

In view of Theorem 2.5 (cf. Remark 2.6), we have in addition the following regularity result for the limit problem.

Proposition 2.8. *Let $\theta_0 \in V$, with $\theta_0 \geq 0$ a.e. in Ω, and $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ be given. Then, there exists a unique pair of functions (θ, u), enjoying the regularity specified by (2.32), satisfying (2.21), (2.22), (2.23), and*

$$\partial_t\theta - \Delta\theta = \theta\nabla\partial_t u \cdot \mathbf{a} + |\nabla\partial_t u|^2 \quad \text{a.e. in } Q_\tau. \tag{2.35}$$

3. Proof of Theorem 2.2

We aim here to apply the Schauder fixed point theorem to a suitable operator $\mathcal{T}$ constructed as follows.

For $R > 0$, we consider $Y(\tau, R)$ the closed ball of $H^1(0,\tau;W_0^{1,4}(\Omega))$ with center 0 and radius R, that is,

$$Y(\tau,R) = \{v \in H^1(0,\tau;W_0^{1,4}(\Omega)) \text{ such that } \|v\|_{H^1(0,\tau;W_0^{1,4}(\Omega))} \leq R\}, \tag{3.1}$$

where $\tau \in]0,T]$ will be determined later in such a way that $\mathcal{T} : Y(\tau,R) \to Y(\tau,R)$ is a compact and continuous operator.

We consider the following auxiliary problems whose well-posedness is guaranteed by standard arguments. Thus, for the sake of brevity, we omit any details (cf., e.g., [1, 8]).

Problem 3.1. Given $\hat{u} \in Y(\tau,R)$, find $\overline{\theta}$ such that

$$\overline{\theta} \in [W^{1,1}(0,\tau;H) + H^1(0,\tau;V')] \cap C^0([0,\tau];H) \cap L^2(0,\tau;V), \tag{3.2}$$

$$\langle \partial_t \overline{\theta}, v\rangle + (\nabla\overline{\theta}, \nabla v) = \left(\overline{\theta}\nabla\partial_t\hat{u}\cdot \mathbf{a} + |\nabla\partial_t\hat{u}|^2, v\right) \quad \forall v \in V \text{ a.e. in }]0,\tau[, \tag{3.3}$$

$$\overline{\theta}(\cdot,0) = \theta_{0\varepsilon} \quad \text{a.e. in } \Omega. \tag{3.4}$$

Now, given such $\overline{\theta}$, let $\overline{u}$ be the unique solution of the following problem.

Problem 3.2. Given $\overline{\theta} \in L^2(0,\tau;V)$, find $\overline{u}$ such that

$$\overline{u} \in H^2(0,\tau;H) \cap W^{1,\infty}(0,\tau;H_0^1(\Omega)) \cap H^1(0,\tau;H^2(\Omega)), \tag{3.5}$$

$$\varepsilon\partial_{tt}\overline{u} - \Delta\partial_t\overline{u} - \Delta\overline{u} = \nabla\overline{\theta}\cdot\mathbf{a} \quad \text{a.e. in } Q_\tau, \tag{3.6}$$

$$\overline{u}(\cdot,0) = u_{0\varepsilon} \quad \text{a.e. in } \Omega, \tag{3.7}$$

$$\partial_t\overline{u}(\cdot,0) = u_{1\varepsilon} \quad \text{a.e. in } \Omega. \tag{3.8}$$

We have actually introduced a mapping $\mathcal{T}$ such that $\mathcal{T}(\hat{u}) = \overline{u}$. Our aim is to show that, at least for small times, the Schauder theorem applies to the map $\mathcal{T}$ from $Y(\tau,R)$ into itself. To do this, we start by deriving some a priori bounds on $\overline{\theta}$ and $\overline{u}$. In the sequel, c will denote any positive constant possibly dependent on data of the problem but not on ε. Of course, c may vary from line to line.

Moreover, we recall here the Young inequality which will be useful in the sequel:

$$ab \le \delta a^p + \frac{1}{q}(\delta p)^{-q/p} b^q, \tag{3.9}$$

for all $a,b \in \mathbb{R}^+$, $\delta > 0$ and $p > 1$, $q < \infty$ such that $1/p + 1/q = 1$.

Now, in order to obtain a priori bounds on $\overline{\theta}$, we choose $v = \overline{\theta}$ in (3.3) and we integrate from 0 to t, with $0 < t < \tau$. Owing to the Hölder inequality and the continuous embedding $V \hookrightarrow L^4(\Omega)$, we have

$$\begin{aligned}
&\frac{1}{2}||\overline{\theta}(t)||^2 + ||\nabla\overline{\theta}||^2_{L^2(0,t;L^2(\Omega)^3)} \\
&\quad\le \frac{1}{2}||\theta_{0\varepsilon}||^2 + c\int_0^t \left(||\nabla\partial_t\hat{u}(s)||_{L^4(\Omega)}||\overline{\theta}(s)||_{L^4(\Omega)} + ||\nabla\partial_t\hat{u}(s)||^2_{L^4(\Omega)}\right)||\overline{\theta}(s)||ds \\
&\quad\le \frac{1}{2}||\theta_{0\varepsilon}||^2 + c\int_0^t \left(||\nabla\partial_t\hat{u}(s)||_{L^4(\Omega)}||\overline{\theta}(s)||_V + ||\nabla\partial_t\hat{u}(s)||^2_{L^4(\Omega)}\right)||\overline{\theta}(s)||ds.
\end{aligned} \tag{3.10}$$

Next, in order to recover the full V-norm of $\overline{\theta}$ in the left-hand side, we add to (3.10) $||\overline{\theta}||^2_{L^2(0,t;H)}$. Then, we use (3.9) and we get

$$\begin{aligned}
\frac{1}{2}||\overline{\theta}(t)||^2 + ||\overline{\theta}||^2_{L^2(0,t;V)} &\le \frac{1}{2}||\theta_{0\varepsilon}||^2 + \frac{1}{2}||\overline{\theta}||^2_{L^2(0,t;V)} \\
&\quad + c\int_0^t \left(||\nabla\partial_t\hat{u}(s)||^2_{L^4(\Omega)} + 1\right)||\overline{\theta}(s)||^2 ds \\
&\quad + c\int_0^t ||\nabla\partial_t\hat{u}(s)||^2_{L^4(\Omega)}||\overline{\theta}(s)||ds.
\end{aligned} \tag{3.11}$$

Recalling the definition of $Y(\tau,R)$ and (2.14), we apply to (3.11) a generalized version of the Gronwall lemma introduced in [1] and we deduce that there exists a positive constant c_4 depending on T, Ω, c_1, and R such that

$$||\overline{\theta}||_{L^\infty(0,\tau;H)\cap L^2(0,\tau;V)} \leq c_4. \tag{3.12}$$

Next, in order to obtain a priori bounds on $\overline{u}$, we multiply (3.6) by $-\Delta\partial_t\overline{u}$ and we integrate over Q_t, with $0 < t < \tau$. Applying the Hölder inequality, we have

$$\begin{aligned}&\frac{\varepsilon}{2}||\nabla\partial_t\overline{u}(t)||^2 + ||\Delta\partial_t\overline{u}||^2_{L^2(0,t;H)} + \frac{1}{2}||\Delta\overline{u}(t)||^2\\ &\quad\leq \frac{\varepsilon}{2}||\nabla u_{1\varepsilon}||^2 + \frac{1}{2}||\Delta u_{0\varepsilon}||^2 + c||\nabla\overline{\theta}||_{L^2(0,t;L^2(\Omega)^3)}||\Delta\partial_t\overline{u}_\varepsilon||_{L^2(0,t;H)}.\end{aligned} \tag{3.13}$$

In account of (2.14), (2.15), and (3.12), using (3.9), we deduce

$$||\overline{u}||_{H^1(0,\tau;H^2(\Omega)\cap H^1_0(\Omega))} \leq c_5, \tag{3.14}$$

$$\varepsilon^{1/2}||u_\varepsilon||_{L^\infty(0,\tau;H^1_0(\Omega))} \leq c_5, \tag{3.15}$$

for some positive constant c_5 with the same dependence of c_4 and depending in addition on c_2.

Next, proceeding formally, we multiply (3.6) by $\partial_{tt}\overline{u}$ and we integrate over Q_t (the procedure can be made rigorous using, e.g., the tools in [6, appendix]). We have

$$\varepsilon||\partial_{tt}\overline{u}||^2_{L^2(0,t;H)} + \frac{1}{2}||\nabla\partial_t\overline{u}(t)||^2 \leq \frac{1}{2}||\nabla u_{1\varepsilon}||^2 + \int\int_{Q_t}(\Delta\overline{u} + \nabla\overline{\theta}\cdot\mathbf{a})\partial_{tt}\overline{u}. \tag{3.16}$$

We estimate the right-hand side in (3.16) integrating by parts (in time and in space). Using also (3.9), we can write

$$\begin{aligned}\int\int_{Q_t}\Delta\overline{u}\partial_{tt}\overline{u} &= -\int\int_{Q_t}\Delta\partial_t\overline{u}\partial_t\overline{u} + \int_\Omega\Delta\overline{u}(t)\partial_t\overline{u}(t) - \int_\Omega\Delta u_{0\varepsilon}u_{1\varepsilon}\\ &\leq ||\nabla\partial_t\overline{u}||^2_{L^2(0,t;H)} + \frac{1}{8}||\nabla\partial_t\overline{u}(t)||^2 + c||\nabla\overline{u}(t)||^2 + \frac{1}{2}||\Delta u_{0\varepsilon}||^2 + \frac{1}{2}||u_{1\varepsilon}||^2\\ &\leq \frac{1}{8}||\nabla\partial_t\overline{u}(t)||^2 + c\Big(||\nabla\partial_t\overline{u}||^2_{L^2(0,t;H)} + ||\Delta u_{0\varepsilon}||^2 + ||u_{1\varepsilon}||^2\Big).\end{aligned} \tag{3.17}$$

Similarly arguing, we have

$$\int\int_{Q_t}\nabla\overline{\theta}\cdot\mathbf{a}\,\partial_{tt}\overline{u} = \int_0^t\langle\partial_t\overline{\theta},\nabla\partial_t\overline{u}\cdot\mathbf{a}\rangle - \int_\Omega\overline{\theta}(t)\nabla\partial_t\overline{u}(t)\cdot\mathbf{a} + \int_\Omega\theta_{0\varepsilon}\nabla u_{1\varepsilon}\cdot\mathbf{a}. \tag{3.18}$$

After substituting $\partial_t\overline{\theta}$ from (3.3), applying the Hölder inequality and (3.9), we get

$$\begin{aligned}\int\int_{Q_t} \nabla\overline{\theta}\cdot\mathbf{a}\,\partial_{tt}\overline{u} = &-\int\int_{Q_t}\nabla\overline{\theta}\cdot\nabla(\nabla\partial_t\overline{u}\cdot\mathbf{a})\\ &+\int\int_{Q_t}\left(\overline{\theta}\nabla\partial_t\hat{u}\cdot\mathbf{a}+|\nabla\partial_t\hat{u}|^2\right)\nabla\partial_t\overline{u}\cdot\mathbf{a}\\ &-\int_\Omega\overline{\theta}(t)\nabla\partial_t\overline{u}(t)\cdot\mathbf{a}+\int_\Omega\theta_{0\varepsilon}\nabla u_{1\varepsilon}\cdot\mathbf{a}\le c\|\Delta\partial_t\overline{u}\|^2_{L^2(0,t;H)}\\ &+c\|\nabla\overline{\theta}\|^2_{L^2(0,t;L^2(\Omega)^3)}\\ &+c\int_0^t\left\|\overline{\theta}(s)\nabla\partial_t\hat{u}(s)\cdot\mathbf{a}+|\nabla\partial_t\hat{u}(s)|^2\right\|\,\|\nabla\partial_t\overline{u}(s)\|ds\\ &+c\|\overline{\theta}(t)\|^2+\frac{1}{8}\|\nabla\partial_t\overline{u}(t)\|^2+c\left(\|\theta_{0\varepsilon}\|^2+\|\nabla u_{1\varepsilon}\|^2\right),\end{aligned}\tag{3.19}$$

for some positive constant c. In account of (2.14), (2.15), (3.12), (3.14), (3.17), and (3.19), we can write (3.16) as follows:

$$\begin{aligned}&\varepsilon\|\partial_{tt}\overline{u}\|^2_{L^2(0,t;H)}+\frac{1}{4}\|\nabla\partial_t\overline{u}(t)\|^2\\ &\quad\le c+c\int_0^t\left\|\overline{\theta}(s)\nabla\partial_t\hat{u}(s)\cdot\mathbf{a}+|\nabla\partial_t\hat{u}(s)|^2\right\|\,\|\nabla\partial_t\overline{u}(s)\|ds.\end{aligned}\tag{3.20}$$

Then, we recall that, due to (3.1) and (3.12), $\|\overline{\theta}\nabla\partial_t\hat{u}\cdot\mathbf{a}+|\nabla\partial_t\hat{u}|^2\|$ belongs to $L^1(0,\tau)$ and the Gronwall lemma enables us to deduce

$$\|\overline{u}\|_{W^{1,\infty}(0,\tau;H_0^1(\Omega))}\le c_6,\tag{3.21}$$

$$\varepsilon^{1/2}\|\partial_{tt}u_\varepsilon\|_{L^2(0,\tau;H)}\le c_6,\tag{3.22}$$

for some positive constant c_6 with the same dependence of the previous constants.

Now our aim is to find τ (independent of ε) such that $\mathcal{T}$ maps $Y(\tau,R)$ into itself. Using standard interpolation inequalities for L^p-norms, from (3.14) and (3.21), we can deduce

$$\|\overline{u}\|_{W^{1,8/3}(0,\tau;W_0^{1,4}(\Omega))}\le c_7,\tag{3.23}$$

for some positive constant c_7.

By the Hölder inequality we have

$$\|\overline{u}\|_{H^1(0,\tau;W_0^{1,4}(\Omega))}\le c\tau^{1/8}\|\overline{u}\|_{W^{1,8/3}(0,\tau;W_0^{1,4}(\Omega))}.\tag{3.24}$$

Thus, to ensure that $\overline{u}\in Y(\tau,R)$, that is

$$\|\overline{u}\|_{H^1(0,\tau;W_0^{1,4}(\Omega))}\le\tau^{1/8}c_8\le R,\tag{3.25}$$

it is enough to choose $\tau\in(0,T]$ such that, for example, $\tau\le R^8/c_8^8$. Note that τ can be chosen independently of ε.

Moreover, the above arguments (cf. (3.14), (3.21), and (3.22)) lead to

$$\|\overline{u}\|_{H^2(0,\tau;H)\cap W^{1,\infty}(0,\tau;H_0^1(\Omega))\cap H^1(0,\tau;H^2(\Omega))}\le k,\tag{3.26}$$

for a positive constant k independent of the choice of $\widehat{u} \in Y(\tau, R)$ (depending on ε and the previous constants), which ensures that $\mathcal{T}$ is a compact operator. Hence, to achieve the proof of the Schauder theorem, it remains to show that $\mathcal{T}$ is continuous with respect to the natural strong topology induced in $Y(\tau, R)$ by $H^1(0,\tau; W_0^{1,4}(\Omega))$. To this aim, we consider a sequence $\{\widehat{u}_n\}_{n\in\mathbb{N}} \subset Y(\tau, R)$ such that

$$\widehat{u}_n \longrightarrow \widehat{u} \quad \text{strongly in } H^1(0,\tau; W_0^{1,4}(\Omega)), \tag{3.27}$$

as $n \to +\infty$.

Denote now by $\overline{\theta}_n$ (resp., $\overline{\theta}$) the solution to Problem 3.1 corresponding to the datum $\widehat{u}_n$ (resp., $\widehat{u}$) and by $\overline{u}_n = \mathcal{T}(\widehat{u}_n)$ (resp., $\overline{u} = \mathcal{T}(\widehat{u})$) the solution to Problem 3.2 corresponding to $\overline{\theta}_n$ (resp., $\overline{\theta}$). Let finally $\widetilde{\theta} = \overline{\theta}_n - \overline{\theta}$ and $\tilde{u} = \overline{u}_n - \overline{u}$. We consider the difference between the corresponding equations (3.3), we test by $\widetilde{\theta}$, and we integrate over $(0,t)$. Arguing as in the derivation of (3.10), we get

$$\begin{aligned}
&\frac{1}{2}\|\widetilde{\theta}(t)\|^2 + \|\nabla\widetilde{\theta}\|^2_{L^2(0,t;L^2(\Omega)^3)} \\
&\quad \le c'\int_0^t \|\overline{\theta}_n(s)\|_V \|\nabla\partial_t\widehat{u}_n(s) - \nabla\partial_t\widehat{u}(s)\|_{L^4(\Omega)}\|\widetilde{\theta}(s)\|ds \\
&\qquad + c'\int_0^t \Big(\|\widetilde{\theta}(s)\|_V\|\nabla\partial_t\widehat{u}(s)\|_{L^4(\Omega)} + \Big\| |\nabla\partial_t\widehat{u}_n(s)|^2 - |\nabla\partial_t\widehat{u}(s)|^2\Big\|\Big)\|\widetilde{\theta}(s)\|ds,
\end{aligned} \tag{3.28}$$

for some positive constant c' independent of n. Applying (3.9), we have

$$\begin{aligned}
\frac{1}{2}\|\widetilde{\theta}(t)\|^2 + \|\widetilde{\theta}\|^2_{L^2(0,t;V)} &\le \frac{1}{2}\|\widetilde{\theta}\|^2_{L^2(0,t;V)} + \|\nabla\partial_t\widehat{u}_n - \nabla\partial_t\widehat{u}\|^2_{L^2(0,t;L^4(\Omega)^3)} \\
&\quad + c''\int_0^t \Big(\|\overline{\theta}_n(s)\|_V^2 + \|\nabla\partial_t\widehat{u}(s)\|^2_{L^4(\Omega)} + 1\Big)\|\widetilde{\theta}(s)\|^2 ds \\
&\quad + c''\int_0^t \Big\| |\nabla\partial_t\widehat{u}_n(s)|^2 - |\nabla\partial_t\widehat{u}(s)|^2\Big\|\|\widetilde{\theta}(s)\|ds,
\end{aligned} \tag{3.29}$$

for some positive constant c'' independent of n. Then, applying the Gronwall lemma to (3.29), in account of (3.12) and (3.27), we infer

$$\|\widetilde{\theta}\|_{L^\infty(0,\tau;H)\cap L^2(0,\tau;V)} \longrightarrow 0, \tag{3.30}$$

as $n \to +\infty$.

Now, we consider the difference between the corresponding equations (3.6), we multiply by $-\Delta\partial_t\tilde{u}$, and we integrate over Q_t (cf. (3.13)). Mainly exploiting (3.30), we readily deduce

$$\|\tilde{u}\|_{H^1(0,\tau;H^2(\Omega)\cap H_0^1(\Omega))} \longrightarrow 0 \tag{3.31}$$

which implies

$$\|\tilde{u}\|_{H^1(0,\tau;W_0^{1,4}(\Omega))} \longrightarrow 0, \tag{3.32}$$

as $n \to +\infty$.

We conclude that $\mathcal{T}$ has a fixed point in $Y(\tau,R)$, that is, there exists a pair $(\theta_\varepsilon,u_\varepsilon)$ (hereafter we make explicit the dependence of the solution on ε, by using a subscript ε) solving Problem 2.1 during the time interval $]0,\tau[$. Actually, we have to complete the proof of the regularity for θ_ε specified by (2.7). To this aim, after adding θ_ε to both sides of (2.9), we test by $J^{-1}\partial_t\theta_\varepsilon$ and we integrate over $(0,t)$. Using the Hölder inequality, recalling the definition of J and the continuous embedding $V \hookrightarrow L^4(\Omega)$, we obtain

$$\begin{aligned}
&\|\partial_t\theta_\varepsilon\|^2_{L^2(0,t;V')}+\frac{1}{2}\|\theta_\varepsilon(t)\|^2\\
&\quad\le \frac{1}{2}\|\theta_{0\varepsilon}\|^2+c\int_0^t\big(\|\theta_\varepsilon(s)\|+\|\nabla\partial_t u_\varepsilon(s)\|\big)\|\nabla\partial_t u_\varepsilon(s)\|_{L^4(\Omega)}\|J^{-1}\partial_t\theta_\varepsilon(s)\|_V\,ds\\
&\qquad+c\int_0^t\|\theta_\varepsilon(s)\|\,\|J^{-1}\partial_t\theta_\varepsilon(s)\|_V\,ds\le\frac{1}{2}\|\theta_{0\varepsilon}\|^2+\frac{1}{2}\|\partial_t\theta_\varepsilon\|^2_{L^2(0,t;V')}\\
&\qquad+c\Big(\|\nabla\partial_t u_\varepsilon\|^2_{L^\infty(0,\tau;H)}+\|\theta_\varepsilon\|^2_{L^\infty(0,\tau;H)}\Big)\|\nabla\partial_t u_\varepsilon\|^2_{L^2(0,t;L^4(\Omega)^3)}+c\|\theta_\varepsilon\|^2_{L^2(0,\tau;H)}.
\end{aligned}\tag{3.33}$$

Hence, the further regularity specified by (2.7) easily follows.

Remark 3.3. The nonnegativity of θ_ε a.e. in Q_τ can be proved proceeding as in [2], using a maximum principle argument, and exploiting the nonnegativity of the initial datum $\theta_{0\varepsilon}$ (cf. (2.5)). For the sake of brevity, we just sketch the proof and refer to [2] for the details. Note that in the case of nonzero heat source r, one has to prescribe in addition $r \ge 0$ (and analogously for a nonhomogeneous Neumann boundary condition for the temperature). Test (2.9) by $-\theta_\varepsilon^-$ (θ_ε^- is the so-called negative part of the function θ_ε), and integrate over $(0,t)$. Thanks to the nonnegativity of $\theta_{0\varepsilon}$, standard arguments show

$$\|\theta_\varepsilon^-(t)\|^2+\|\theta_\varepsilon^-\|^2_{L^2(0,t;V)}\le c\Big(\|\theta^-\|^2_{L^2(0,t;H)}+\int_0^t\|\nabla\partial_t u\|^2_{L^4(\Omega)}\|\theta_\varepsilon^-\|^2\Big),\tag{3.34}$$

from which the Gronwall lemma implies

$$\|\theta_\varepsilon^-\|^2_{L^\infty(0,\tau;H)\cap L^2(0,\tau;V)}\le 0,\tag{3.35}$$

that is, $\theta_\varepsilon^-=0$ a.e. in Q_τ, from which it follows $\theta_\varepsilon\ge 0$ a.e. in Q_τ.

Now, in order to complete the proof of Theorem 2.2, we have to show that the solution to Problem 2.1 is unique. This fact follows applying [2, Theorem 2.2]. Nonetheless, we prove here directly the uniqueness result, providing an alternative proof. We stress that such a result holds in any interval where the solution exists. We proceed by contradiction. Let $(\theta_{\varepsilon1},u_{\varepsilon1})$ and $(\theta_{\varepsilon2},u_{\varepsilon2})$ be two solutions to Problem 2.1 during some interval $]0,\tau[$, $\tau\in]0,T]$. Set $\widetilde{\theta}_\varepsilon=\theta_{\varepsilon1}-\theta_{\varepsilon2}$ and $\tilde{u}_\varepsilon=u_{\varepsilon1}-u_{\varepsilon2}$. We consider the difference between the corresponding equations (2.10), we multiply it by $\partial_t\tilde{u}_\varepsilon$, and we integrate over Q_t. We get

$$\begin{aligned}
&\frac{\varepsilon}{2}\|\partial_t\tilde{u}_\varepsilon(t)\|^2+\|\nabla\partial_t\tilde{u}_\varepsilon\|^2_{L^2(0,t;L^2(\Omega)^3)}+\frac{1}{2}\|\nabla\tilde{u}_\varepsilon(t)\|^2\\
&\quad\le\frac{1}{2}\|\nabla\partial_t\tilde{u}_\varepsilon\|^2_{L^2(0,t;L^2(\Omega)^3)}+\frac{|\mathbf{a}|^2}{2}\|\widetilde{\theta}_\varepsilon\|^2_{L^2(0,t;H)}.
\end{aligned}\tag{3.36}$$

Next, we consider the difference between the corresponding equations (2.9), we add (in both sides) $\tilde{\theta}_\varepsilon$, we test by $J^{-1}\widetilde{\theta}_\varepsilon$, and we integrate from 0 to t; we obtain

$$\frac{1}{2}||\widetilde{\theta}_\varepsilon(t)||^2_{V'} + ||\widetilde{\theta}_\varepsilon||^2_{L^2(0,t;H)} \le ||\widetilde{\theta}_\varepsilon||^2_{L^2(0,t;V')} + \sum_{i=1}^{3} |I_i(t)|, \tag{3.37}$$

where the integrals $I_i(t)$, $i = 1,2,3$, are specified as follows:

$$\begin{aligned} I_1(t) &= \int_0^t (\mathbf{a}\cdot\nabla\partial_t u_{\varepsilon 1}\widetilde{\theta}_\varepsilon, J^{-1}\widetilde{\theta}_\varepsilon),\\ I_2(t) &= \int_0^t (\theta_{\varepsilon 2}\,\mathbf{a}\cdot\nabla\partial_t\tilde{u}_\varepsilon, J^{-1}\widetilde{\theta}_\varepsilon),\\ I_3(t) &= \int_0^t ((\nabla\partial_t u_{\varepsilon 1} + \nabla\partial_t u_{\varepsilon 2})\cdot\nabla\partial_t\tilde{u}_\varepsilon, J^{-1}\widetilde{\theta}_\varepsilon). \end{aligned} \tag{3.38}$$

We handle $I_1(t)$ using again the Hölder inequality, (3.9) and the Sobolev embeddings. Recalling also the definition of J, we get

$$\begin{aligned} |I_1(t)| &\le c\int_0^t ||\nabla\partial_t u_{\varepsilon 1}(s)||_{L^4(\Omega)}||\tilde{\theta}_\varepsilon(s)||\,||J^{-1}\widetilde{\theta}_\varepsilon(s)||_{L^4(\Omega)}ds\\ &\le c\int_0^t ||\partial_t u_{\varepsilon 1}(s)||_{H^2(\Omega)}||\widetilde{\theta}_\varepsilon(s)||\,||\tilde{\theta}_\varepsilon(s)||_{V'}ds\\ &\le \frac{1}{4}||\widetilde{\theta}_\varepsilon||^2_{L^2(0,t;H)} + c\int_0^t ||\partial_t u_{\varepsilon 1}(s)||^2_{H^2(\Omega)}||\tilde{\theta}_\varepsilon(s)||^2_{V'}ds. \end{aligned} \tag{3.39}$$

By analogous arguments, for any $\delta > 0$, we can write

$$\begin{aligned} &|I_2(t)| + |I_3(t)|\\ &\quad\le \delta||\nabla\partial_t\tilde{u}_\varepsilon||^2_{L^2(0,t;L^2(\Omega)^3)}\\ &\qquad + c(\delta)\int_0^t \Big(||\theta_{\varepsilon 2}(s)||^2_V + ||\partial_t u_{\varepsilon 1}(s)||^2_{H^2(\Omega)} + ||\partial_t u_{\varepsilon 2}(s)||^2_{H^2(\Omega)}\Big)||\tilde{\theta}_\varepsilon(s)||^2_{V'}ds, \end{aligned} \tag{3.40}$$

where $c(\delta)$ is deduced applying (3.9).

Hence, we add (3.36) multiplied by $1/|\mathbf{a}|^2$ to (3.37), in account of (3.39)-(3.40). Choosing, for example, $\delta = 1/(4|\mathbf{a}|^2)$, we get

$$\begin{aligned} &\frac{1}{2}||\widetilde{\theta}_\varepsilon(t)||^2_{V'} + \frac{1}{4}||\widetilde{\theta}_\varepsilon||^2_{L^2(0,t;H)} + \frac{\varepsilon}{2|\mathbf{a}|^2}||\partial_t\tilde{u}_\varepsilon(t)||^2\\ &\qquad + \frac{1}{4|\mathbf{a}|^2}||\nabla\partial_t\tilde{u}_\varepsilon||^2_{L^2(0,t;L^2(\Omega)^3)} + \frac{1}{2|\mathbf{a}|^2}||\nabla\tilde{u}_\varepsilon(t)||^2\\ &\quad\le c\int_0^t \Big(1 + ||\theta_{\varepsilon 2}(s)||^2_V + ||\partial_t u_{\varepsilon 1}(s)||^2_{H^2(\Omega)} + ||\partial_t u_{\varepsilon 2}(s)||^2_{H^2(\Omega)}\Big)||\tilde{\theta}_\varepsilon(s)||^2_{V'}ds. \end{aligned} \tag{3.41}$$

Owing to the regularity for $\theta_{\varepsilon 2}$, $u_{\varepsilon 1}$, $u_{\varepsilon 2}$ specified by (2.7)-(2.8), we can apply the Gronwall lemma to (3.41) and deduce $\widetilde{\theta}_\varepsilon = \tilde{u}_\varepsilon = 0$ a.e. in Q_τ which concludes the proof of Theorem 2.2.

4. Proof of Theorems 2.3 and 2.4

In the present section, we will study the asymptotic behavior of the solution to Problem 2.1 given by Theorem 2.2, as $\varepsilon \searrow 0$. To this aim, we need some a priori estimates (independent of the parameter ε) concerning the pair $(\theta_\varepsilon, u_\varepsilon)$, holding in the interval $]0,\tau[$. Remark that such asymptotic analysis is meaningful, because τ can be chosen independently of ε. The a priori estimates allowing us to pass to the limit in (2.9)-(2.10) can be established arguing as in the previous section. In fact, the fixed point argument employed in the proof of Theorem 2.2 provides some uniform bounds on the solution according to the regularity specified by (2.7)-(2.8) (recall that $u_\varepsilon \in Y(\tau,R)$, that is $\|u_\varepsilon\|_{H^1(0,\tau;W_0^{1,4}(\Omega))} \le R$, for any $\varepsilon \in]0,1[$). Thus, we can reason as in the derivation of (3.12), (3.14), (3.21), (3.22), and (3.33) and deduce

$$||\theta_\varepsilon||_{H^1(0,\tau;V')\cap L^\infty(0,\tau;H)\cap L^2(0,\tau;V)} \le c, \tag{4.1}$$

$$||u_\varepsilon||_{W^{1,\infty}(0,\tau;H_0^1(\Omega))\cap H^1(0,\tau;H^2(\Omega))} \le c, \tag{4.2}$$

$$\varepsilon^{1/2}||\partial_{tt}u_\varepsilon||_{L^2(0,\tau;H)} \le c \tag{4.3}$$

for some positive constant c independent of ε.

The uniform bounds (4.1), (4.2), and (4.3) allow us to pass to the limit in (2.9)-(2.10). In fact, by well-known weak and weak star compactness results, there exists a pair (θ, u) such that, at least for a subsequence of $\varepsilon \searrow 0$,

$$\theta_\varepsilon \rightharpoonup \theta \quad \text{in } H^1(0,\tau;V') \cap L^2(0,\tau;V), \tag{4.4}$$

$$\theta_\varepsilon \overset{*}{\rightharpoonup} \theta \quad \text{in } L^\infty(0,\tau;H), \tag{4.5}$$

$$u_\varepsilon \rightharpoonup u \quad \text{in } H^1(0,\tau;H^2(\Omega)), \tag{4.6}$$

$$u_\varepsilon \overset{*}{\rightharpoonup} u \quad \text{in } W^{1,\infty}(0,\tau;H_0^1(\Omega)), \tag{4.7}$$

$$\varepsilon\partial_{tt}u_\varepsilon \longrightarrow 0 \quad \text{in } L^2(0,\tau;H). \tag{4.8}$$

In particular, by compactness (see [10, page 58] and [12, Corollary 4]), we deduce the following strong convergences:

$$\theta_\varepsilon \longrightarrow \theta \quad \text{in } C^0([0,\tau];V') \cap L^2(0,\tau;H), \tag{4.9}$$

$$u_\varepsilon \longrightarrow u \quad \text{in } C^0([0,\tau];H_0^1(\Omega)). \tag{4.10}$$

Thanks to the above convergences and also to (2.16)-(2.17), the pair (θ, u) fulfilling

$$\begin{gathered} \theta \in H^1(0,\tau;V') \cap C^0([0,\tau];H) \cap L^2(0,\tau;V), \quad \theta \ge 0 \text{ a.e. in } Q_\tau, \\ u \in W^{1,\infty}(0,\tau;H_0^1(\Omega)) \cap H^1(0,\tau;H^2(\Omega)) \end{gathered} \tag{4.11}$$

satisfies (2.21) (we pass to the limit in (2.10)) in the time interval $]0,\tau[$ along with (2.22)-(2.23). Note that the strong convergence (4.9) guarantees the nonnegativity of θ, once it holds for θ_ε.

Our next goal is to pass to the limit in (2.9). The critical terms are the nonlinearities in the right-hand side. In order to handle them, we will show the further strong convergence

$$u_\varepsilon \longrightarrow u \quad \text{in } H^1(0,\tau;H_0^1(\Omega)). \tag{4.12}$$

To this aim, we consider the difference between (2.10) and (2.21); we multiply it by $\partial_t(u_\varepsilon - u)$ and we integrate over Q_t, with $0 < t < \tau$. After an integration by parts in space, using (2.3) and (3.9), we obtain

$$\begin{aligned} &\|\nabla\partial_t(u_\varepsilon - u)\|^2_{L^2(0,t;L^2(\Omega)^3)} + \frac{1}{2}\|\nabla(u_\varepsilon - u)(t)\|^2 \le \frac{1}{2}\|\nabla u_{0\varepsilon} - \nabla u_0\|^2 \\ &\quad + \frac{1}{2}\|\nabla\partial_t(u_\varepsilon - u)\|^2_{L^2(0,t;L^2(\Omega)^3)} + c\Big(\|\theta_\varepsilon - \theta\|^2_{L^2(0,t;H)} + \varepsilon^2\|\partial_{tt}u_\varepsilon\|^2_{L^2(0,t;H)}\Big). \end{aligned} \tag{4.13}$$

Thus, (4.12) follows from (4.13), due to (2.17), (4.8), and (4.9).

Finally, exploiting the regularity for θ and u specified in (2.18), we can prove Theorem 2.4, arguing as in the proof of the uniqueness part of Theorem 2.2 (cf. (3.36), (3.37), (3.38), (3.39), (3.40), and (3.41)).

5. Proof of Theorem 2.5

Owing to the better regularity assumed on the families of the initial data (cf. (2.24)-(2.25)), we derive here some further bounds (independent of the parameter ε) concerning the pair $(\theta_\varepsilon, u_\varepsilon)$.

First, we test (2.9) by $\partial_t\theta_\varepsilon$ and we integrate over $(0,t)$, with $0 < t < \tau$. Using the Hölder inequality and (3.9), we obtain

$$\begin{aligned} &\|\partial_t\theta_\varepsilon\|^2_{L^2(0,t;H)} + \frac{1}{2}\|\nabla\theta_\varepsilon(t)\|_{L^2(0,t;L^2(\Omega)^3)} \le \frac{1}{2}\|\nabla\theta_{0\varepsilon}\|^2 + \frac{1}{2}\|\partial_t\theta_\varepsilon\|^2_{L^2(0,t;H)} \\ &\quad + c\int_0^t \|\nabla\partial_t u_\varepsilon(s)\|^2_{L^4(\Omega)}\Big(\|\theta_\varepsilon(s)\|^2_{L^4(\Omega)} + \|\nabla\partial_t u_\varepsilon(s)\|^2_{L^4(\Omega)}\Big)ds. \end{aligned} \tag{5.1}$$

Now, we add to (5.1) $\|\theta_\varepsilon(t)\|^2$, recovering the full V-norm of $\theta_\varepsilon(t)$ in the left-hand side. Thus, owing to (2.25), (4.1), and the Sobolev embeddings, we can write (5.1) as follows:

$$\|\partial_t\theta_\varepsilon\|^2_{L^2(0,t;H)} + \|\theta_\varepsilon(t)\|^2_V \le c + c\int_0^t \|\Delta\partial_t u_\varepsilon(s)\|^2\Big(\|\theta_\varepsilon(s)\|^2_V + \|\Delta\partial_t u_\varepsilon(s)\|^2\Big)ds. \tag{5.2}$$

Next, we differentiate (2.10) with respect to time; we multiply formally by $\partial_{tt}u_\varepsilon$, and we integrate over Q_t. We get

$$\begin{aligned} &\frac{\varepsilon}{2}\|\partial_{tt}u_\varepsilon(t)\|^2 + \|\nabla\partial_{tt}u_\varepsilon\|^2_{L^2(0,t;L^2(\Omega)^3)} + \frac{1}{2}\|\nabla\partial_t u_\varepsilon(t)\|^2 \\ &\quad \le \frac{\varepsilon}{2}\|\partial_{tt}u_\varepsilon(0)\|^2 + \frac{1}{2}\|\nabla u_{1\varepsilon}\|^2 + \frac{1}{2}\|\nabla\partial_{tt}u_\varepsilon\|^2_{L^2(0,t;L^2(\Omega)^3)} + k_1\|\partial_t\theta_\varepsilon\|^2_{L^2(0,t;H)}, \end{aligned} \tag{5.3}$$

for some positive constant k_1 independent of ε. To handle the right-hand side of (5.3), we consider (2.10) written for $t = 0$, in account of (2.11), (2.12), and (2.13). Recalling (2.25), we obtain

$$\varepsilon\|\partial_{tt}u_\varepsilon(0)\|^2 = \varepsilon^{-1}\|\nabla\theta_{0\varepsilon}\cdot\mathbf{a}+\Delta u_{1\varepsilon}+\Delta u_{0\varepsilon}\|^2 \le c. \tag{5.4}$$

Moreover, note that (2.17) and (2.25) imply that $\|\Delta u_{1\varepsilon}\|$ is bounded independently of ε. Thus, due to (2.4),

$$\|\nabla u_{1\varepsilon}\| \le c, \tag{5.5}$$

for some $c > 0$ and any sufficiently small $\varepsilon > 0$. Then, in account of (5.4) and (5.5), we infer

$$\varepsilon\|\partial_{tt}u_\varepsilon(t)\|^2 + \|\nabla\partial_{tt}u_\varepsilon\|^2_{L^2(0,t;L^2(\Omega)^3)} + \|\nabla\partial_t u_\varepsilon(t)\|^2 \le c + 2k_1\|\partial_t\theta_\varepsilon\|^2_{L^2(0,t;H)}. \tag{5.6}$$

Next, a comparison in (2.10) gives, for a.a. $t \in]0,\tau[$,

$$\|\Delta\partial_t u_\varepsilon(t)\|^2 \le k_2\left(\varepsilon^2\|\partial_{tt}u_\varepsilon(t)\|^2 + \|\Delta u_\varepsilon(t)\|^2 + \|\theta_\varepsilon(t)\|_V^2\right), \tag{5.7}$$

for some positive constant k_2 independent of ε. Now, we add (5.7) multiplied by $1/(2k_2)$ to (5.2). Taking (4.2) and (5.6) into account, we get

$$\begin{aligned}
&\|\partial_t\theta_\varepsilon\|^2_{L^2(0,t;H)} + \|\theta_\varepsilon(t)\|_V^2 + \frac{1}{2k_2}\|\Delta\partial_t u_\varepsilon(t)\|^2 \\
&\quad\le c + c\int_0^t \|\Delta\partial_t u_\varepsilon(s)\|^2\left(\|\theta_\varepsilon(s)\|_V^2 + \|\Delta\partial_t u_\varepsilon(s)\|^2\right)ds + \frac{1}{2}\|\theta_\varepsilon(t)\|_V^2 + \frac{\varepsilon^2}{2}\|\partial_{tt}u_\varepsilon(t)\|^2 \\
&\quad\le c + c\int_0^t \|\Delta\partial_t u_\varepsilon(s)\|^2\left(\|\theta_\varepsilon(s)\|_V^2 + \|\Delta\partial_t u_\varepsilon(s)\|^2\right)ds + \frac{1}{2}\|\theta_\varepsilon(t)\|_V^2 + \varepsilon k_1\|\partial_t\theta_\varepsilon\|^2_{L^2(0,t;H)}.
\end{aligned} \tag{5.8}$$

Thus, for any sufficiently small ε (e.g., for any $\varepsilon \in]0, \min(1, 1/(2k_1))[$), we get

$$\begin{aligned}
&\|\partial_t\theta_\varepsilon\|^2_{L^2(0,t;H)} + \|\theta_\varepsilon(t)\|_V^2 + \|\Delta\partial_t u_\varepsilon(t)\|^2 \\
&\quad\le c + c\int_0^t \|\Delta\partial_t u_\varepsilon(s)\|^2\left(\|\theta_\varepsilon(s)\|_V^2 + \|\Delta\partial_t u_\varepsilon(s)\|^2\right)ds.
\end{aligned} \tag{5.9}$$

Finally, observing that, because of (4.2), $\|\Delta\partial_t u_\varepsilon\|^2 \in L^1(0,\tau)$, we can apply the Gronwall lemma and deduce the following upper bounds:

$$\|\theta_\varepsilon\|_{H^1(0,\tau;H)\cap L^\infty(0,\tau;V)} \le c, \tag{5.10}$$

$$\|u_\varepsilon\|_{W^{1,\infty}(0,\tau;H^2(\Omega)\cap H_0^1(\Omega))} \le c, \tag{5.11}$$

for any sufficiently small ε. Thanks to (5.10)-(5.11), a comparison in (2.9) gives

$$\|\theta_\varepsilon\|_{L^2(0,\tau;W)} \le c. \tag{5.12}$$

Moreover, due to (5.10), from (5.6), it follows that

$$\begin{gathered} \|u_\varepsilon\|_{H^2(0,\tau;H_0^1(\Omega))} \le c, \\ \varepsilon^{1/2}\|\partial_{tt}u_\varepsilon\|_{L^\infty(0,\tau;H)} \le c, \end{gathered} \tag{5.13}$$

and the proof of Theorem 2.5 is complete.

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Elena Bonetti: Dipartimento di Matematica, Università degli Studi di Pavia, via Ferrata 1, 27100 Pavia, Italy

E-mail address: elena.bonetti@unipv.it

Giovanna Bonfanti: Dipartimento di Matematica, Università degli Studi di Brescia, via Branze 38, 25123 Brescia, Italy

E-mail address: bonfanti@ing.unibs.it