

Discrete time approximation of a COGARCH(p,q) model and its estimation (Preliminary Version)

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Abstract

In this paper, we construct a sequence of discrete time stochastic processes that converges in probability and in the Skorokhod metric to a COGARCH(p,q) model. The result is useful for the estimation of the continuous model defined for irregularly spaced time series data. The estimation procedure is based on the maximization of a pseudo log-likelihood function and is implemented in the `yuima` package.

1 Introduction

The COGARCH(1,1) model has been introduced by Klüppelberg et al. (2004) as a continuous time counterpart of the GARCH(1,1) process. The continuous time model preserves the main features of the GARCH model since the same underlying noise drives the variance and the return processes. For the COGARCH(1,1) case, different methods for its estimation have been proposed. For instance, Haug et al. (2007) develop a procedure based on the matching of theoretical and empirical moments. Maller et al. (2008) use an approximation scheme for obtaining estimates of parameters through the maximization of a pseudo-loglikelihood function while Müller (2010) develop a Markov Chain Monte Carlo estimation procedure based on the same approximation scheme.

The COGARCH(1,1) model has been generalized to the higher order case by Chadraa (2009) and Brockwell et al. (2006). Based on our knowledge, this is the only estimation method for higher order models and it is based on the matching of empirical and theoretical moments.

In this paper, we construct a sequence of discrete time stochastic processes that converges in probability and in the Skorokhod metric to a COGARCH(p,q) model. Our results generalize the approach in Maller et al. (2008) for building a sequence of discrete time stochastic processes based on a GARCH(1,1) model that converges in the Skorokhod metric to its continuous counterpart, i.e COGARCH(1,1) model.

Results derived for a COGARCH(p,q) model in Chadraa (2009) are used in this paper for extending the estimation procedure based on the maximization of the pseudo log-likelihood function. This estimation method is then implemented in the `yuima` package available on CRAN (See Brouste et al., 2014; Brouste and Iacus, 2013; Iacus and Mercuri, 2015; Iacus et al., 2015, for more details on `yuima` package).

The outline of the paper is as follows. In Section 2 we review some useful properties needed in in Section 3 where we introduce a discrete version of our process and prove the convergence to the COGARCH(p,q) model using the Skorokhod metric.

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2 Preliminaries

In this section we review useful results for obtaining a sequence of discrete time processes that converges in Skorokhod distance (Billingsley, 1968, see for example) to a COGARCH(p,q) model.

Definition 1. The sequence of random vectors Q_n is uniformly convergent in probability to Q if and only if:

$$\sup_{\theta \in \Theta} \|Q_{n,\theta} - Q_\theta\| \xrightarrow{P} 0, \quad (1)$$

where $\|\cdot\|$ is the Euclidean norm.

Conjecture 2. The definition holds also for any vector norm $\|\cdot\|_A$ induced by an invertible matrix A , i.e. $\|x\|_A = \|Ax\|$ where A is a non singular matrix.

Definition 3. Let $\|\cdot\|$ be a norm on \mathcal{R}^n , we introduce induced norm $\|\cdot\|_M$ as a function from $\mathcal{R}^{n \times n}$ to \mathcal{R}_+ defined as:

$$\|A\|_M := \sup_{\|x\| \neq 0} \frac{\|Ax\|}{\|x\|} = \sup_{\|z\|=1} \|Az\|$$

where $A \in \mathcal{R}^{q \times q}$.

Theorem 4. The induced norm $\|\cdot\|_M$ satisfies the following properties (see Desoer and Vidyasagar, 1975):

- 1) $\|Ax\| \leq \|A\|_M \|x\|$
 - 2) $\|\alpha A\|_M \leq |\alpha| \|A\|_M$
 - 3) $\|A + B\|_M \leq \|A\|_M + \|B\|_M$
 - 4) $\|AB\|_M \leq \|A\|_M \|B\|_M$,
- where $A \in \mathcal{R}^{q \times q}$, $B \in \mathcal{R}^{q \times q}$ and α is a scalar.

We have also that any induced vector norm satisfies the following inequality: (see Serre, 2002):

$$\left\| \frac{e^{At} - I}{t} - A \right\|_M \leq \frac{e^{\|A\|_M t} - 1 - \|At\|_M}{|t|}, \quad t \in \mathcal{R}. \quad (2)$$

Definition 5. Let $\|\cdot\|_M$ be the induced vector norm by the norm $\|\cdot\|$ defined on \mathcal{R}^n , the logarithmic norm $\mu(A)$ (see Strom, 1975, for its properties) is defined as:

$$\mu(A) := \lim_{t \rightarrow 0^+} \frac{\|I + At\|_M - 1}{t}.$$

Theorem 6. For the logarithmic norm the following inequalities hold:

$$\|e^{At}\|_M \leq e^{\mu(A)t} \leq e^{\|A\|_M t}$$

Let a_n and b_n be sequences of non-negative numbers for $n = 1, \dots, N$. Define as a linear recursive equation the sequence y_n :

$$y_n = a_n y_{n-1} + b_n$$

with initial condition $y_0 = c$ where c is a scalar.

Theorem 7. If we have that $a_n \geq 1$ and $b_n \geq 0$, the sequence y_n is non decreasing with

$$y_N = \max_{n=1, \dots, N} y_n$$

and

$$y_N = \left[\prod_{k=0}^{N-1} a_{N-k} \right] y_0 + b_N + \sum_{j=1}^{N-1} \left[\prod_{h=1}^j a_{N+1-h} \right] b_{N-j}.$$

3 Main Result

We recall the definition of a COGAR $\text{CH}(p,q)$ process, introduced in Brockwell et al. (2006), based on the following equations:

$$\begin{aligned} dG_t &= \sqrt{V_t} dL_t \\ V_t &= \alpha_0 + \mathbf{a}^\top Y_{t-} \\ dY_t &= BY_{t-} dt + e (\alpha_0 + \mathbf{a}^\top Y_{t-}) d[L, L]^d \end{aligned} \quad (3)$$

where $B \in \mathcal{R}^{q \times q}$ is matrix of the form:

$$B = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -b_q & -b_{q-1} & \dots & \dots & -b_1 \end{bmatrix}$$

and \mathbf{a} and \mathbf{e} are vectors defined as:

$$\begin{aligned} \mathbf{a} &= [a_1, \dots, a_p, a_{p+1}, \dots, a_q]^\top \\ \mathbf{e} &= [0, \dots, 0, 1]^\top \end{aligned}$$

for $a_{p+1} = \dots = a_q = 0$. As remarked in Brockwell et al. (2006) the state process Y_t in a COGAR $\text{CH}(p,q)$ model is:

$$Y_t = J_{s,t} Y_s + K_{s,t} \quad s \leq t$$

where $J_{s,t} \in \mathcal{R}^{q \times q}$ is a random matrix and $K_{s,t} \in \mathcal{R}^{q \times 1}$ is a random vector.

In particular, if the driven noise is a Compound Poisson the matrices and vectors in the state process have an analytical form. Let N be the number of jumps of a Compound Poisson in the interval $[0, t]$. Define τ_N as the time of the last jump in this interval interval and $Z_N := \Delta L_{\tau_N}^2 = (L_{\tau_N} - L_{\tau_N-})^2$ the square of the jump at time τ_N . The process Y_t can be rewritten as follows:

$$Y_t = e^{B(t-\tau_N)} Y_{\tau_N} \quad t \in [\tau_N, \tau_{N+1})$$

where Y_{τ_N} is the state process at jump time τ_N , i.e. the last jump of size less or equal to t , defined as:

$$Y_{\tau_N} = C_N Y_{\tau_{N-1}} + D_N \quad (4)$$

where the random coefficients C_N and D_N in (4) are respectively:

$$\begin{aligned} C_N &= (I + Z_N \mathbf{e} \mathbf{a}^\top) e^{B \Delta \tau_N} \\ D_N &= \alpha_0 Z_N \mathbf{e}. \end{aligned} \quad (5)$$

As in Maller et al. (2008), we construct a sequence of discrete processes that converges to the COGAR $\text{CH}(p,q)$ model in (3) by means of the Skorokhod distance (see Billingsley, 1968, for more details). For each $n \geq 0$ we consider a sequence of natural numbers N_n such that $\lim_{n \rightarrow +\infty} N_n = +\infty$ and we obtain a partition of the interval $[0, T]$ defined as:

$$0 = t_{0,n} \leq t_{1,n} \leq \dots \leq t_{N_n,n} = T. \quad (6)$$

The mesh of this partition is:

$$\Delta t_n := \max_{i=1, \dots, N_n} \Delta t_{i,n} \xrightarrow{n \rightarrow +\infty} 0.$$

Using the partition in (6), we introduce the process $G_{i,n}$ as follows:

$$G_{i,n} = G_{i-1,n} + \sqrt{V_{i-1,n} \Delta t_{i,n}} \epsilon_{i,n} \quad (7)$$

where the innovations $\epsilon_{i,n}$ are constructed using the first jump approximation method developed in Szimayer and Maller (2007) that we review here quickly.

Let m_n be a strict positive sequence of real numbers satisfying the conditions:

$$\begin{aligned} m_n &\leq 1 \quad \forall n \geq 0, \\ \lim_{n \rightarrow +\infty} m_n &= 0. \end{aligned}$$

We require the Lévy measure Π to satisfy following property:

$$\lim_{n \rightarrow +\infty} \Delta t_n \bar{\Pi}^2(m_n) = 0$$

where $\bar{\Pi}(x) := \int_{|y|>x} \Pi(dx)$.

We define the stopping time process:

$$\tau_{i,n} := \inf \{t \in [t_{i-1,n}, t_{i,n}) : |\Delta L_t| > m_n\} \quad (8)$$

and construct a sequence of independent random variables $(\mathbf{1}_{\tau_{i,n} < +\infty} \Delta L_{\tau_{i,n}})_{i=1, \dots, N_n}$ with density:

$$f(x) = \frac{\Pi(dx)}{\bar{\Pi}(m_n)} \left(1 - e^{-\Delta t_{i,n} \bar{\Pi}(m_n)}\right).$$

We introduce the innovations $\epsilon_{i,n}$ defined as:

$$\epsilon_{i,n} = \frac{\mathbf{1}_{\tau_{i,n} < +\infty} \Delta L_{\tau_{i,n}} - v_{i,n}}{\eta_{i,n}} \quad (9)$$

where $v_{i,n}$ and $\eta_{i,n}$ are respectively the mean and the variance of $\epsilon_{i,n}$. The variance process V_t in (3) is approximated by the process $V_{i,n}$ as:

$$V_{i,n} = \alpha_0 + \mathbf{a}^\top Y_{i,n} \quad (10)$$

where $Y_{i,n}$ is given by:

$$Y_{i,n} = C_{i,n} Y_{i-1,n} + D_{i,n}, \quad (11)$$

with coefficients:

$$\begin{aligned} C_{i,n} &= (I + \epsilon_{i,n}^2 \Delta t_{i,n} \mathbf{e} \mathbf{a}^\top) e^{B \Delta t_{i,n}} \\ D_{i,n} &= \alpha_0 \epsilon_{i,n}^2 \Delta t_{i,n} \mathbf{e}. \end{aligned} \quad (12)$$

The couple $(G_{i,n}, V_{i,n})$ converges to the couple (G_t, V_t) in the Skorokhod distance. The Skorokhod distance between two processes U, V defined on $D^d[0, T]$, i.e. space of càdlàg \mathcal{R}^d stochastic processes on $[0, T]$, is

$$\rho(U, V) := \inf_{\lambda \in \Lambda} \left\{ \sup_{0 \leq t \leq T} \|U_t - V_{\lambda(t)}\| + \sup_{0 \leq t \leq T} |\lambda(t) - t| \right\}$$

where Λ is a set of increasing continuous functions with $\lambda(0) = 0$ and $\lambda(T) = T$.

First of all we need the following auxiliary result.

Theorem 8. *Let $N_n(t)$ be a counting process defined as:*

$$N_n(t) := \# \{i \in \mathcal{N} : \tau_{i,n}^* \leq t\}$$

where $t \leq T$, $N_n(0) = 0$, $N_n(T) = N_n$ and $\tau_{i,n}^* = \min \{\tau_{i,n}, t_{i,n}\}$ with $\tau_{i,n}$ and $t_{i,n}$ in (8) and (6) respectively.

Let L_t be a Compound Poisson with finite second moment, the positive process $H_n(t)$ defined as:

$$H_n(t) := \prod_{k=1}^{N_n(t)} C_{k,n}^*$$

where

$$C_{k,n}^* := (1 + \epsilon_{k,n}^2 \Delta t_{k,n} \|\mathbf{ea}^\top\|_M) e^{\|B\|_M \Delta t_{i,n}}$$

converges uniformly in probability on a compact interval $[0, T]$ (hereafter ucp) to the positive process $\tilde{H}_n(t)$ as:

$$\tilde{H}_n(t) := \prod_{k=1}^{N_n(t)} \tilde{C}_{k,n}$$

with

$$\tilde{C}_{k,n} := \left(1 + \mathbf{1}_{\tau_{k,n} < +\infty} \Delta L_{\tau_{k,n}}^2 \|\mathbf{ea}^\top\|_M\right) e^{\|B\|_M \Delta t_{i,n}}$$

i.e.

$$\sup_{t \in [0, T]} \left| H_n(t) - \tilde{H}_n(t) \right| \xrightarrow{p} 0.$$

For each fixed n , $\tilde{H}_n(t)$ is a non decreasing strictly positive process in the compact interval $[0, T]$ such that $\forall t \in [0, T]$:

$$\tilde{H}_n(t) \leq \tilde{H}_n(T) \leq e^{\|B\|_M T + \sum_{0 \leq s \leq T} \ln(1 + \Delta L_s^2 \|\mathbf{ea}^\top\|_M)}$$

Proof. We start from

$$\begin{aligned} \sup_{t \in [0, T]} \left| H_n(t) - \tilde{H}_n(t) \right| &= \sup_{t \in [0, T]} \left| \prod_{k=1}^{N_n(t)} C_{k,n}^* - \prod_{k=1}^{N_n(t)} \tilde{C}_{k,n} \right| \\ &\leq e^{\|B\|_M T} \sup_{t \in [0, T]} \left| \prod_{k=1}^{N_n(t)} (1 + \epsilon_{k,n}^2 \Delta t_{k,n} \|\mathbf{ea}^\top\|_M) - \prod_{k=1}^{N_n(t)} (1 + \mathbf{1}_{\tau_{k,n} < +\infty} \Delta L_{\tau_{k,n}}^2 \|\mathbf{ea}^\top\|_M) \right| \\ &= e^{\|B\|_M T} \sup_{t \in [0, T]} \left| e^{\sum_{k=1}^{N_n(t)} \ln(1 + \epsilon_{k,n}^2 \Delta t_{k,n} \|\mathbf{ea}^\top\|_M)} - e^{\sum_{k=1}^{N_n(t)} \ln(1 + \mathbf{1}_{\tau_{k,n} < +\infty} \Delta L_{\tau_{k,n}}^2 \|\mathbf{ea}^\top\|_M)} \right|. \end{aligned}$$

Observe that

$$\begin{aligned} L_n &:= \sup_{t \in [0, T]} \left| \sum_{k=1}^{N_n(t)} \ln(1 + \epsilon_{k,n}^2 \Delta t_{k,n} \|\mathbf{ea}^\top\|_M) - \sum_{k=1}^{N_n(t)} \ln(1 + \mathbf{1}_{\tau_{k,n} < +\infty} \Delta L_{\tau_{k,n}}^2 \|\mathbf{ea}^\top\|_M) \right| \\ &\leq \sup_{t \in [0, T]} \left| \sum_{k=1}^{N_n(t)} \left(\epsilon_{k,n}^2 \Delta t_{k,n} \|\mathbf{ea}^\top\|_M - \mathbf{1}_{\tau_{k,n} < +\infty} \Delta L_{\tau_{k,n}}^2 \|\mathbf{ea}^\top\|_M \right) \right| \\ &\leq \|\mathbf{ea}^\top\|_M \sup_{t \in [0, T]} \sum_{k=1}^{N_n(t)} \left| \left(\epsilon_{k,n}^2 \Delta t_{k,n} - \mathbf{1}_{\tau_{k,n} < +\infty} \Delta L_{\tau_{k,n}}^2 \right) \right|. \end{aligned}$$

As shown in Maller et al. (2008), we have that

$$\sup_{t \in [0, T]} \sum_{k=1}^{N_n(t)} \left| \left(\epsilon_{k,n}^2 \Delta t_{k,n} - \mathbf{1}_{\tau_{k,n} < +\infty} \Delta L_{\tau_{k,n}}^2 \right) \right| \xrightarrow{p} 0,$$

that implies

$$\sup_{t \in [0, T]} \left| \sum_{k=1}^{N_n(t)} \ln(1 + \epsilon_{k,n}^2 \Delta t_{k,n} \|\mathbf{ea}^\top\|_M) - \sum_{k=1}^{N_n(t)} \ln(1 + \mathbf{1}_{\tau_{k,n} < +\infty} \Delta L_{\tau_{k,n}}^2 \|\mathbf{ea}^\top\|_M) \right| \xrightarrow{p} 0. \quad (13)$$

Using result in (13), we have

$$\sup_{t \in [0, T]} \left| e^{\sum_{k=1}^{N_n(t)} \ln(1 + \epsilon_{k,n}^2 \Delta t_{k,n} \|\mathbf{ea}^\top\|_M)} - e^{\sum_{k=1}^{N_n(t)} \ln(1 + \mathbf{1}_{\tau_{k,n} < +\infty} \Delta L_{\tau_{k,n}}^2 \|\mathbf{ea}^\top\|_M)} \right| \xrightarrow{p} 0.$$

$\tilde{H}_n(t)$ is a non decreasing strictly positive process since is a product of terms $\tilde{C}_{k,n} \geq 1$ a.s. and if $s > t$ then $\tilde{H}_n(s)$ has at least the same terms as in $\tilde{H}_n(t)$. Moreover

$$\tilde{H}_n(T) = e^{\|B\|_M T + \sum_{k=1}^{N_n} \ln(1 + \mathbf{1}_{\tau_k, n} < +\infty \Delta L_{\tau_k, n}^2 \|\mathbf{ea}^\top\|_M)} \leq e^{\|B\|_M T + \sum_{0 \leq s \leq T} \ln(1 + \Delta L_s^2 \|\mathbf{ea}^\top\|_M)}$$

since $\Delta L_s^2 = \Delta L_s^2 \mathbf{1}_{|\Delta L_s| \geq m_n} + \Delta L_s^2 \mathbf{1}_{|\Delta L_s| < m_n}$. \square

Remark. We observe, from Theorem 8, that

$$H_n(t) \xrightarrow{ucp} \tilde{H}_n(t) \leq e^{\|B\|_M T + \sum_{0 \leq s \leq T} \ln(1 + \Delta L_s^2 \|\mathbf{ea}^\top\|_M)} \quad (14)$$

on an interval $[0, T]$. Moreover, the term on the right hand side of the inequality in (14) is bounded almost surely on the compact interval $[0, T]$ since L_t is a Compound Poisson process.

The following theorem is established for the Compound Poisson driven noise case.

Theorem 9. Let L_t be a Compound Poisson process with $E(L_1^2) < +\infty$. The Skorokhod distance computed on the processes $(G_t, V_t)_{t \geq 0}$ and their discretized version $(G_{i,n}, V_{i,n})_{i=1, \dots, N_n}$ converges in probability to zero, i.e.:

$$\rho\left((G_{i,n}, V_{i,n})_{i=1, \dots, N_n}, (G_t, V_t)_{t \geq 0}\right) \xrightarrow{P} 0 \text{ as } n \rightarrow +\infty.$$

Proof. The proof follows the same steps as in Maller et al. (2008)

- Approximation procedure for the underlying process.
- Approximation procedure for the variance process.
- Approximation procedure for the COGARCH(p,q) model.
- Convergence of the pair in the Skorokhod distance.

Steps 1, 2, 4 are exactly the same as in Maller et al. (2008). To prove that the discrete variance process $V_{i,n}$ converges upc on a compact time interval to the continuous-time process V_t we first need to show that $Y_{i,n} \xrightarrow{ucp} Y_t$. This result is achieved through intermediate steps illustrated below.

We introduce the counting process $N_n(t)$ defined as:

$$N_n(t) := \#\{i \in \mathcal{N} : \tau_{i,n}^* \leq t\} \quad (15)$$

with $t \leq T$, $N_n(0) = 0$ and $\tau_{i,n}^* = \min\{\tau_{i,n}, t_{i,n}\}$.

$N_n(t)$ increases by 1 in each subinterval $(t_{i-1,n}, t_{i,n}]$, $i = 1, 2, \dots, n$, at the first time the jump is of magnitude greater or equal to m_n or at $t_{i,n}$ if that jump does not occur.

Using the process $N_n(t)$ in (15) we construct the time process $\Gamma_{t,n}$ as:

$$\Gamma_{t,n} = \sum_{i=1}^{N_n(t)} \Delta t_{i,n}. \quad (16)$$

Now we want to show that the piecewise constant process $Y_{t,n} := Y_{i,n}$ with $t \in [t_{i,n}, t_{i+1,n})$ converges in upc to the process $\bar{Y}_{t,n} := e^{B(t-\Gamma_{t,n})} Y_{i,n}$ i.e.:

$$\sup_{0 \leq t \leq T} \|Y_{t,n} - \bar{Y}_{t,n}\| \xrightarrow{P} 0.$$

For each $t \in [0, T]$, we have:

$$\begin{aligned} \|Y_{t,n} - \bar{Y}_{t,n}\| &= \left\| e^{B(t-\Gamma_{t,n})} Y_{i,n} - Y_{i,n} \right\| \\ &\leq \left\| e^{B(t-\Gamma_{t,n})} - I \right\|_M \|Y_{i,n}\| \\ &= \left\| e^{B(t-\Gamma_{t,n})} - I - B(t-\Gamma_{t,n}) + B(t-\Gamma_{t,n}) \right\|_M \|Y_{i,n}\| \\ &\leq \left(\left\| e^{B(t-\Gamma_{t,n})} - I - B(t-\Gamma_{t,n}) \right\|_M + \|B(t-\Gamma_{t,n})\| \right) \|Y_{i,n}\| \end{aligned}$$

using the inequality in (2), we get:

$$\begin{aligned}\|Y_{t,n} - \bar{Y}_{t,n}\| &\leq \left(e^{\|B(t-\Gamma_{t,n})\|_M} - 1 \right) \|Y_{i,n}\| \\ &\leq \left(e^{\|B\|_M \Delta t_n} - 1 \right) \|Y_{i,n}\|\end{aligned}\quad (17)$$

Since by construction $Y_{t,n} = Y_{i,n}$ with $t \in [t_{i,n}, t_{i+1,n})$ and $Y_{t,n}$ has càdlàg paths, it follows that $\sup_{t \in [0, T]} \|Y_{t,n}\|$ is almost surely finite and

$$\sup_{t \in [0, T]} \|Y_{t,n} - \bar{Y}_{t,n}\| \leq \left(e^{\|B\|_M \Delta t_n} - 1 \right) \sup_{t \in [0, T]} \|Y_{t,n}\| \xrightarrow{P} 0$$

as $n \rightarrow +\infty$.

The next step is to show the convergence upc of $\bar{Y}_{t,n}$ to $\tilde{Y}_{t,n}$ where the last process is defined as:

$$\tilde{Y}_{t,n} = e^{B(t-\Gamma_{t,n})} \tilde{Y}_{i,n} \quad (18)$$

with:

$$\tilde{Y}_{i,n} = \tilde{C}_{i,n} \tilde{Y}_{i-1,n} + \tilde{D}_{i,n} \quad (19)$$

where the random matrix $\tilde{C}_{i,n}$ and the random vector $\tilde{D}_{i,n}$ are respectively:

$$\begin{aligned}\tilde{C}_{i,n} &= \left(I + (\mathbf{1}_{\tau_{i,n} < +\infty} \Delta L_{\tau_{i,n}})^2 \mathbf{e} \mathbf{a}^\top \right) e^{B \Delta t_{i,n}} \\ \tilde{D}_{i,n} &= \alpha_0 (\mathbf{1}_{\tau_{i,n} < +\infty} \Delta L_{\tau_{i,n}})^2 \mathbf{e}.\end{aligned}\quad (20)$$

We consider

$$\sup_{t \in [0, T]} \|\tilde{Y}_{t,n} - \bar{Y}_{t,n}\| \leq e^{\|B\|_M \Delta t_n} \sup_{i=1, \dots, N_n} \|\tilde{Y}_{i,n} - Y_{i,n}\| \quad (21)$$

and observe that, for $i = 1, \dots, N_n$, we have:

$$\|\tilde{Y}_{i,n} - Y_{i,n}\| \leq \|\tilde{C}_{i,n} \tilde{Y}_{i-1,n} - C_{i,n} Y_{i-1,n}\| + \|\tilde{D}_{i,n} - D_{i,n}\|. \quad (22)$$

We analyze the second term in (22) and get:

$$\begin{aligned}\|\tilde{D}_{i,n} - D_{i,n}\| &= \|\alpha_0 (\mathbf{1}_{\tau_{i,n} < +\infty} \Delta L_{\tau_{i,n}})^2 \mathbf{e} - \alpha_0 \epsilon_{i,n}^2 \Delta t_{i,n} \mathbf{e}\| \\ &\leq |\alpha_0| \left| (\mathbf{1}_{\tau_{i,n} < +\infty} \Delta L_{\tau_{i,n}})^2 - \epsilon_{i,n}^2 \Delta t_{i,n} \right|.\end{aligned}\quad (23)$$

The first term in (22) can be bounded by adding and subtracting the quantity $C_{i,n} \tilde{Y}_{i-1,n}$:

$$\begin{aligned}\|\tilde{C}_{i,n} \tilde{Y}_{i-1,n} - C_{i,n} Y_{i-1,n}\| &= \|\tilde{C}_{i,n} \tilde{Y}_{i-1,n} - C_{i,n} \tilde{Y}_{i-1,n} + C_{i,n} \tilde{Y}_{i-1,n} - C_{i,n} Y_{i-1,n}\| \\ &\leq \|\tilde{C}_{i,n} - C_{i,n}\|_M \|\tilde{Y}_{i-1,n}\| + \|C_{i,n}\|_M \|\tilde{Y}_{i-1,n} - Y_{i-1,n}\| \\ &\leq \left[(\mathbf{1}_{\tau_{i,n} < +\infty} \Delta L_{\tau_{i,n}})^2 - \epsilon_{i,n}^2 \Delta t_{i,n} \right] \mathbf{e} \mathbf{a}^\top e^{B \Delta t_{i,n}} \Big|_M \|\tilde{Y}_{i-1,n}\| \\ &\quad + \|C_{i,n}\|_M \|\tilde{Y}_{i-1,n} - Y_{i-1,n}\| \\ &\leq \left| (\mathbf{1}_{\tau_{i,n} < +\infty} \Delta L_{\tau_{i,n}})^2 - \epsilon_{i,n}^2 \Delta t_{i,n} \right| \|\mathbf{e} \mathbf{a}^\top\|_M e^{\|B\|_M \Delta t_{i,n}} \|\tilde{Y}_{i-1,n}\| \\ &\quad + \|C_{i,n}\|_M \|\tilde{Y}_{i-1,n} - Y_{i-1,n}\|\end{aligned}\quad (24)$$

Substituting (24) and (23) into (22) we have:

$$\begin{aligned}\|\tilde{Y}_{i,n} - Y_{i,n}\| &\leq \|C_{i,n}\|_M \|\tilde{Y}_{i-1,n} - Y_{i-1,n}\| \\ &\quad + \left| (\mathbf{1}_{\tau_{i,n} < +\infty} \Delta L_{\tau_{i,n}})^2 - \epsilon_{i,n}^2 \Delta t_{i,n} \right| \left(|\alpha_0| + \|\mathbf{e} \mathbf{a}^\top\|_M e^{\|B\|_M \Delta t_{i,n}} \|\tilde{Y}_{i-1,n}\| \right)\end{aligned}\quad (25)$$

Since a.s.:

$$\|C_{i,n}\|_M \leq (1 + \epsilon_{i,n}^2 \Delta t_{i,n} \|\mathbf{e} \mathbf{a}^\top\|_M) e^{\|B\|_M \Delta t_{i,n}} := C_{i,n}^* \geq 1 \quad (26)$$

and defining

$$K_{i-1,n} := |\alpha_0| + \|\mathbf{ea}^\top\|_M e^{\|B\|_M \Delta t_{i,n}} \|\tilde{Y}_{i-1,n}\| \quad (27)$$

we have:

$$\begin{aligned} \|\tilde{Y}_{i,n} - Y_{i,n}\| &\leq C_{i,n}^* \|\tilde{Y}_{i-1,n} - Y_{i-1,n}\| \\ &\quad + \left| (\mathbf{1}_{\tau_{i,n} < +\infty} \Delta L_{\tau_{i,n}})^2 - \epsilon_{i,n}^2 \Delta t_{i,n} \right| K_{i-1,n}. \end{aligned} \quad (28)$$

The right hand side in (28) is a linear recursive equation with random coefficients and condition (26) implies that:

$$\begin{aligned} \sup_{i=1,\dots,N_n} \|\tilde{Y}_{i,n} - Y_{i,n}\| &\leq \left[\prod_{i=0}^{N_n-1} C_{N_n-i,n}^* \right] \|\tilde{Y}_{0,n} - Y_{0,n}\| + \left| (\mathbf{1}_{\tau_{N_n,n} < +\infty} \Delta L_{\tau_{N_n,n}})^2 - \epsilon_{N_n,n}^2 \Delta t_{N_n,n} \right| K_{N_n-1,n} \\ &\quad + \sum_{i=1}^{N_n-1} \left[\prod_{h=1}^i C_{N_n+1-h,n}^* \right] \left| (\mathbf{1}_{\tau_{N_n-i,n} < +\infty} \Delta L_{\tau_{N_n-i,n}})^2 - \epsilon_{N_n-i,n}^2 \Delta t_{N_n-i,n} \right| K_{N_n-1-i,n}. \end{aligned} \quad (29)$$

The term:

$$\left[\prod_{i=0}^{N_n-1} C_{N_n-i,n}^* \right] \|\tilde{Y}_{0,n} - Y_{0,n}\| \geq 0 \quad n \geq 1$$

with

$$E \left[\left(\prod_{i=0}^{N_n-1} C_{N_n-i,n}^* \right) \|\tilde{Y}_{0,n} - Y_{0,n}\| \right] = E \left[\left(\prod_{i=0}^{N_n-1} C_{N_n-i,n}^* \right) \right] \|\tilde{Y}_{0,n} - Y_{0,n}\|$$

since $\tilde{Y}_{0,n} = Y_{0,n}$ we have:

$$E \left[\left(\prod_{i=0}^{N_n-1} C_{N_n-i,n}^* \right) \|\tilde{Y}_{0,n} - Y_{0,n}\| \right] = 0 \Rightarrow \left(\prod_{i=0}^{N_n-1} C_{N_n-i,n}^* \right) \|\tilde{Y}_{0,n} - Y_{0,n}\| = 0 \text{ a.s.} \quad (30)$$

Condition (29) becomes:

$$\begin{aligned} \sup_{i=1,\dots,N_n} \|\tilde{Y}_{i,n} - Y_{i,n}\| &\leq \left| (\mathbf{1}_{\tau_{N_n,n} < +\infty} \Delta L_{\tau_{N_n,n}})^2 - \epsilon_{N_n,n}^2 \Delta t_{N_n,n} \right| K_{N_n-1,n} \\ &\quad + \sum_{i=1}^{N_n-1} \left[\prod_{h=1}^i C_{N_n+1-h,n}^* \right] \left| (\mathbf{1}_{\tau_{N_n-i,n} < +\infty} \Delta L_{\tau_{N_n-i,n}})^2 - \epsilon_{N_n-i,n}^2 \Delta t_{N_n-i,n} \right| K_{N_n-1-i,n}. \end{aligned} \quad (31)$$

Defining:

$$\begin{aligned} Q_n &:= \left| (\mathbf{1}_{\tau_{N_n,n} < +\infty} \Delta L_{\tau_{N_n,n}})^2 - \epsilon_{N_n,n}^2 \Delta t_{N_n,n} \right| K_{N_n-1,n} \\ &\quad + \sum_{i=1}^{N_n-1} \left[\prod_{h=1}^i C_{N_n+1-h,n}^* \right] \left| (\mathbf{1}_{\tau_{N_n-i,n} < +\infty} \Delta L_{\tau_{N_n-i,n}})^2 - \epsilon_{N_n-i,n}^2 \Delta t_{N_n-i,n} \right| K_{N_n-1-i,n}. \end{aligned}$$

we observe that Q_n can be bounded. Indeed, $\forall i = 1, \dots, N_n$:

$$\prod_{h=1}^i C_{N_n+1-h,n}^* \leq \prod_{h=1}^{N_n} C_{N_n+1-h,n}^*$$

and, from Theorem 8, the quantity $\prod_{h=1}^{N_n} C_{N_n+1-h,n}^*$ converges in probability to a non negative r.v. that is a.s. bounded by:

$$e^{\|B\|_M T + \sum_{0 \leq s \leq T} \ln(1 + \Delta L_s^2 \|\mathbf{ea}^\top\|_M)}.$$

Even $\sup_{i=1,\dots,N_n} K_{i,n}$ is bounded a.s. $\forall n$. Consequently we have:

$$Q_n \leq \left[\prod_{h=1}^{N_n} C_{N_n+1-h,n}^* \right] \left[\sup_{i=1,\dots,N_n} K_{i,n} \right] \sum_{i=1}^{N_n} \left| (\mathbf{1}_{\tau_{i,n} < +\infty} \Delta L_{\tau_{i,n}})^2 - \epsilon_{i,n}^2 \Delta t_{i,n} \right| \quad (32)$$

Since $\lim_{n \rightarrow +\infty} \sup_{i=1, \dots, N_n} K_{i,n} = M < +\infty$ a.s. and, as shown in Maller et al. (2008),

$$\sup_{t \in [0, T]} \sum_{i=1}^{N_n(t)} \left| \left(\mathbf{1}_{\tau_{i,n} < +\infty} \Delta L_{\tau_{i,n}} \right)^2 - \epsilon_{i,n}^2 \Delta t_{i,n} \right| \xrightarrow{P} 0$$

as $n \rightarrow +\infty$, then $Q_n \xrightarrow{P} 0$ that implies $\bar{Y}_{t,n} \xrightarrow{ucp} \tilde{Y}_{t,n}$.

We observe that, since the driven noise is a Compound Poisson, we have only a finite number of jumps in a compact interval $[0, T]$. We indicate with τ_k the time of the k -th jump. Since the irregular grid becomes finer as n increases and satisfies the following two conditions:

$$\begin{aligned} \Delta t_n &:= \max_{i=1, \dots, N_n} \Delta t_{i,n} \xrightarrow{n \rightarrow +\infty} 0 \\ T &= \sum_{i=1}^{N_n} \Delta t_{i,n}, \end{aligned}$$

then exists n^* such that for $n \geq n^*$, all jump times $\tau_k \in \{t_{0,n}, t_{1,n}, \dots, t_{N_n,n}\}$. The COGARCH(p,q) state process Y_t in (4) can be defined equivalently $\forall n \geq n^*$ as:

$$Y_{t_{i,n}} = C_{t_{i,n}} Y_{t_{i-1,n}} + D_{t_{i,n}} \quad (33)$$

with coefficients $C_{t_{i,n}}$ and $D_{t_{i,n}}$ defined as:

$$\begin{aligned} C_{t_{i,n}} &= \left(I + \Delta L_{t_{i,n}}^2 \mathbf{e} \mathbf{a}^\top \right) e^{B \Delta t_{i,n}} \\ D_{t_{i,n}} &= \alpha_0 \Delta L_{t_{i,n}}^2 \mathbf{e}. \end{aligned}$$

To show the ucp convergence of process $\tilde{Y}_{t,n}$ to Y_t , we start observing that:

$$\begin{aligned} \sup_{t \in [0, T]} \left\| Y_t - \tilde{Y}_{t,n} \right\| &= \sup_{t \in [0, T]} \left\| e^{B(t - \Gamma_{t,n})} \left(Y_{t_{i,n}} - \tilde{Y}_{i,n} \right) \right\| \\ &\leq e^{\|B\|_M T} \sup_{i=1, \dots, N_n} \left\| \left(Y_{t_{i,n}} - \tilde{Y}_{i,n} \right) \right\|. \end{aligned} \quad (34)$$

We work on $\sup_{i=1, \dots, N_n} \left\| \left(Y_{t_{i,n}} - \tilde{Y}_{i,n} \right) \right\|$ and for $i = 1, \dots, N_n$ and for fixed n we have:

$$\left\| \left(Y_{t_{i,n}} - \tilde{Y}_{i,n} \right) \right\| \leq \left\| \left(C_{t_{i,n}} Y_{t_{i-1,n}} - \tilde{C}_{i,n} \tilde{Y}_{i-1,n} \right) \right\| + \left\| D_{t_{i,n}} - \tilde{D}_{i,n} \right\|. \quad (35)$$

The term $\left\| D_{t_{i,n}} - \tilde{D}_{i,n} \right\|$ in (35) is bounded as follows:

$$\left\| D_{t_{i,n}} - \tilde{D}_{i,n} \right\| \leq |\alpha_0| \left| \Delta L_{t_{i,n}}^2 - \mathbf{1}_{\tau_{i,n} < +\infty} \Delta L_{\tau_{i,n}}^2 \right|. \quad (36)$$

Since

$$\Delta L_{t_{i,n}}^2 := \Delta L_{t_{i,n}}^2 \mathbf{1}_{|\Delta L_{t_{i,n}}| \geq m_n} + \Delta L_{t_{i,n}}^2 \mathbf{1}_{|\Delta L_{t_{i,n}}| < m_n}, \quad (37)$$

the inequality in (36) becomes:

$$\begin{aligned} \left\| D_{t_{i,n}} - \tilde{D}_{i,n} \right\| &\leq |\alpha_0| \left| \Delta L_{t_{i,n}}^2 \mathbf{1}_{0 < |\Delta L_{t_{i,n}}| < m_n} \right| \\ &\leq m_n |\alpha_0| \left| \mathbf{1}_{|\Delta L_{t_{i,n}}| > 0} \right|. \end{aligned} \quad (38)$$

Inserting (38) into (35), we have:

$$\left\| \left(Y_{t_{i,n}} - \tilde{Y}_{i,n} \right) \right\| \leq \left\| \left(C_{t_{i,n}} Y_{t_{i-1,n}} - \tilde{C}_{i,n} \tilde{Y}_{i-1,n} \right) \right\| + m_n |\alpha_0| \left| \mathbf{1}_{|\Delta L_{t_{i,n}}| > 0} \right| \quad (39)$$

We add and subtract the term $C_{t_{i,n}} \tilde{Y}_{i-1,n}$ into the quantity $\|C_{t_{i,n}} Y_{t_{i-1,n}} - \tilde{C}_{i,n} \tilde{Y}_{i-1,n}\|$. By exploiting the triangular inequality we obtain:

$$\begin{aligned} \|C_{t_{i,n}} Y_{t_{i-1,n}} - \tilde{C}_{i,n} \tilde{Y}_{i-1,n}\| &\leq \|C_{t_{i,n}} Y_{t_{i-1,n}} - C_{t_{i,n}} \tilde{Y}_{i-1,n}\| + \|C_{t_{i,n}} \tilde{Y}_{i-1,n} - \tilde{C}_{i,n} \tilde{Y}_{i-1,n}\| \\ &\leq \|C_{t_{i,n}}\|_M \|Y_{t_{i-1,n}} - \tilde{Y}_{i-1,n}\| + \|C_{t_{i,n}} - \tilde{C}_{i,n}\|_M \|\tilde{Y}_{i-1,n}\| \\ &\leq \|C_{t_{i,n}}\|_M \|Y_{t_{i-1,n}} - \tilde{Y}_{i-1,n}\| \\ &\quad + \left| \Delta L_{t_{i,n}}^2 - \mathbf{1}_{\tau_{i,n} < +\infty} \Delta L_{\tau_{i,n}}^2 \right| \|\mathbf{ea}^\top\|_M e^{\|B\|_M \Delta t_{i,n}} \|\tilde{Y}_{i-1,n}\|. \end{aligned} \quad (40)$$

Defining:

$$C_{t_{i,n}}^{**} := \left(1 + \Delta L_{t_{i,n}}^2 \|\mathbf{ea}^\top\|_M\right) e^{\|B\|_M \Delta t_{i,n}} \geq \|C_{t_{i,n}}\|_M,$$

substituting (40) into (39) and using the same arguments as in (37) and (38), we obtain:

$$\begin{aligned} \|Y_{t_{i,n}} - \tilde{Y}_{i,n}\| &\leq C_{t_{i,n}}^{**} \|Y_{t_{i-1,n}} - \tilde{Y}_{i-1,n}\| + m_n |\alpha_0| \left| \mathbf{1}_{|\Delta L_{t_{i,n}}| > 0} \right| \\ &\quad + \left| \Delta L_{t_{i,n}}^2 - \mathbf{1}_{\tau_{i,n} < +\infty} \Delta L_{\tau_{i,n}}^2 \right| \|\mathbf{ea}^\top\|_M e^{\|B\|_M \Delta t_{i,n}} \|\tilde{Y}_{i-1,n}\|. \end{aligned}$$

Using $K_{i,n}$ in (27), we have:

$$\|Y_{t_{i,n}} - \tilde{Y}_{i,n}\| \leq C_{t_{i,n}}^{**} \|Y_{t_{i-1,n}} - \tilde{Y}_{i-1,n}\| + m_n K_{i-1,n} \left| \mathbf{1}_{|\Delta L_{t_{i,n}}| > 0} \right|. \quad (41)$$

We introduce a stochastic recurrence equation on the grid $\{t_{i,n}\}_{i=0,\dots,N_n}$ defined as as:

$$\zeta_{i,n} = C_{t_{i,n}}^{**} \zeta_{i-1,n} + m_n K_{i-1,n} \mathbf{1}_{|\Delta L_{t_{i,n}}| > 0}$$

with initial condition $\zeta_{0,n} := \|Y_{t_{0,n}} - \tilde{Y}_{0,n}\| = 0$ a.s.. Since $\forall i C_{t_{i,n}}^{**} \geq 1$ and $m_n K_{i-1,n} \mathbf{1}_{|\Delta L_{t_{i,n}}| > 0} \geq 0$ a.s., $\zeta_{i,n}$ is a non decreasing process that is an upper bound for $\|Y_{t_{i,n}} - \tilde{Y}_{i,n}\|$ for each fixed i then:

$$\begin{aligned} \sup_{i=1,\dots,N_n} \|Y_{t_{i,n}} - \tilde{Y}_{i,n}\| &\leq \left[\prod_{i=0}^{N_n-1} C_{N_n-i,n}^{**} \right] \|Y_{t_{0,n}} - \tilde{Y}_{0,n}\| \\ &\quad + m_n \left\{ \sum_{i=1}^{N_n-1} \left[\prod_{h=1}^i C_{N_n+i-h,n}^{**} \right] \mathbf{1}_{|\Delta L_{t_{N_n-i,n}}| > 0} K_{N_n-1-i,n} + \mathbf{1}_{|\Delta L_{t_{N_n,n}}| > 0} K_{N_n-1,n} \right\} \end{aligned} \quad (42)$$

The right-hand side in (42) is non-negative as a summation of non-negative terms. We split it into two parts:

$$\begin{aligned} G_n &:= \left[\prod_{i=0}^{N_n-1} C_{N_n-i,n}^{**} \right] \|Y_{t_{0,n}} - \tilde{Y}_{0,n}\| \\ W_n &:= m_n \left\{ \sum_{i=1}^{N_n-1} \left[\prod_{h=1}^i C_{N_n+i-h,n}^{**} \right] \mathbf{1}_{|\Delta L_{t_{N_n-i,n}}| > 0} K_{N_n-1-i,n} + \mathbf{1}_{|\Delta L_{t_{N_n,n}}| > 0} K_{N_n-1,n} \right\}. \end{aligned}$$

Using the same arguments as in (30), we can say that:

$$G_n = 0 \text{ a.s. } \forall n \geq 0$$

and

$$W_n \xrightarrow{n \rightarrow +\infty} 0$$

since for $n \rightarrow +\infty$, the quantity

$$\sum_{i=1}^{N_n-1} \left[\prod_{h=1}^i C_{N_n+i-h,n}^{**} \right] \mathbf{1}_{|\Delta L_{t_{N_n-i,n}}| > 0} K_{N_n-1-i,n} + \mathbf{1}_{|\Delta L_{t_{N_n,n}}| > 0} K_{N_n-1,n}$$

is composed by a finite number of terms and then it finite a.s. for the same arguments in (32). In conclusion we have:

$$\sup_{i=1,\dots,N_n} \left\| Y_{t_{i,n}} - \tilde{Y}_{i,n} \right\| \leq G_n + W_n \xrightarrow{n \rightarrow +\infty} 0$$

that implies

$$Y_{t,n} \xrightarrow{ucp} Y_t \quad (43)$$

where $Y_{t,n}$ is the constant piecewise process associated to the process $Y_{i,n}$ defined in (11). From (43) we obtain the ucp convergence of process $V_{i,n}$ to the COGARCH(p,q) variance process V_t . The remaining part of the proof follows the same steps as in Maller et al. (2008) \square

The result can be generalized to any COGARCH(p,q) model driven by a finite variation Lévy process since, as shown in Brockwell et al. (2006), a COGARCH(p,q) driven by a general Lévy can be approximated by the same COGARCH(p,q) process driven by a Compound Poisson. Then using the triangular inequality, the discrete process $(G_{i,n}, V_{i,n})$ converges in the Skorokhod metric and in probability to any COGARCH(p,q) model.

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