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Supersymmetric solutions of SU(2)-Fayet–Iliopoulos-gauged $\mathcal{N}=2$, $d=4$ supergravity

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Abstract

We explore the construction of supersymmetric solutions of theories of $\mathcal{N}=2, d=4$ supergravity with a SU(2) gauging and SU(2) Fayet–Iliopoulos terms. In these theories an SU(2) isometry subgroup of the Special-Kähler manifold is gauged together with a SU(2) R-symmetry subgroup. We construct several solutions of the $\overline{\mathbb{CP}}^3$ quadratic model directly in four dimensions and of the ST[2, 6] model by dimensional reduction of the solutions found by Cariglia and Mac Conamhna in $\mathcal{N}=(1,0), d=6$ supergravity with the same kind of gauging. In the $\overline{\mathbb{CP}}^3$ model, we construct an $\mathrm{AdS}_2 \times \mathrm{S}^2$ solution which is only $1/8$ BPS and an $\mathbb{R} \times \mathbb{H}^3$ solutions that also preserves 1 of the 8 possible supersymmetries. We show how to use dimensional reduction as in the ungauged case to obtain $\mathbb{R}^n \times \mathrm{S}^m$ and also $\mathrm{AdS}_n \times \mathrm{S}^m$-type solutions (with different radii) in 5- and 4-dimensions from the 6-dimensional $\mathrm{AdS}_3 \times \mathrm{S}^3$ solution.

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0. Introduction

The study of supersymmetric solution of supergravity theories has been one of the most fruitful areas of research in this field over the last few years providing, for instance, backgrounds for string theory with clear spacetime interpretation such as black holes, rings, or branes, their near-horizon geometries, *pp*-waves etc. on which the strings can be quantized consistently. Thus, these solutions have provided the earliest connections between gravity solutions and 2-dimensional conformal field theories (the superstring worldsheet theories) whose states can be counted using standard techniques, paving the way for more general correspondences.

The supersymmetric solutions of many (classes of) supergravity theories have been classified/characterized by now, and, therefore, the independent variables that enter in their fields and the equations that they must obey are well known. However, the explicit construction of these solutions can still be a difficult problem when the equations that need to be solved are non-linear as it is often the case in gauged supergravities, specially with non-Abelian Yang–Mills fields. In this paper we are going to deal with this problem in the context of $\mathcal{N}=2, d=4$ gauged supergravities.

$\mathcal{N}=2, d=4$ supergravities admit several kinds of gaugings[1]:

1. One can just gauge a non-Abelian subgroup of the isometry group of the Special Kähler manifold of the complex scalars from the vector multiplets.[2] This is the simplest possibility: it does not involve the hypermultiplets and trying to gauge an Abelian isometry only would have no effect since all the terms that would have to be added (proportional, for instance, to the Killing vector) vanish identically. In absence of hypermultiplets, these theories have been called in Refs. [4,5] $\mathcal{N}=2, d=4$ Super-Einstein–Yang–Mills (SEYM) because they are the simplest $\mathcal{N}=2$ supersymmetrization of the Einstein–Yang–Mills theories.
2. One can gauge a general subgroup of the isometry group of the Quaternionic Kähler manifold of the scalars in the hypermultiplets.[3] Since this requires coupling to a set of gauge vector fields transforming in the adjoint of the gauge group and the available vectors come in supermultiplets that also contain scalars in a Special Kähler manifold, the gauge group must also be a subgroup of the isometry group of the Special Kähler manifold and must necessarily act on the hypermultiplets and vector multiplets simultaneously. It must act in the adjoint representation on the latter.
   This case can be considered an extension of the previous one in which the hypermultiplets are not mere spectators anymore. There is, however, a very important difference: Abelian gaugings are non-trivial in this setting in the Quaternionic Kähler sector.
3. In absence of hypermultiplets, one can gauge the complete SU(2) factor of the R-symmetry group (U(2)) or just a U(1) subgroup of that SU(2) factor[4] by introducing what would be *constant triholomorphic momentum maps* if there were hypermultiplets. These constants are usually called, respectively, SU(2) or U(1) Fayet–Iliopoulos (FI) terms and the theories obtained are called SU(2)- or U(1)-FI-gauged $\mathcal{N}=2, d=4$ supergravities, respectively.

[1] See, for instance, Refs. [1–3] for a general review on these theories with references to the original literature.
[2] Only isometries that respect the complete Special Geometry structure are global symmetries of the theory and can be gauged.
[3] Only isometries that respect the Quaternionic Kähler structure are global symmetries of the theory and can be gauged.
[4] The U(1) factor cannot be gauged.

The SU(2)-FI-gauged $\mathcal{N}=2, d=4$ theories can be seen as deformations of the $\mathcal{N}=2, d=4$ SEYM theories in which the SU(2) factor of the R-symmetry group is gauged simultaneously with an SU(2) subgroup of the isometry group of the Special Kähler manifold. Gauging the latter is necessary for gauging the SU(2) factor of the R-symmetry group because the global symmetry being gauged has to act on the gauge fields in the adjoint representation and, for the gauging to respect supersymmetry, it must act on the complete vector supermultiplets, including the scalars and this action must, then, be an isometry of the metric.

Our goal in this paper is to search for timelike supersymmetric solutions of this last class of gauged supergravities: SU(2)-FI-gauged $\mathcal{N}=2, d=4$ supergravities with no hypermultiplets.

The timelike supersymmetric solutions of the most general $\mathcal{N}=2, d=4$ supergravities (that is: with the most general matter content and the most general gauging) were classified/characterized in Ref. [6], building on previous results about the supersymmetric solutions of the general $\mathcal{N}=2, d=4$ ungauged theories with vector multiplets and hypermultiplets [7–9], the U(1)-FI-gauged $\mathcal{N}=2, d=4$ theories with no hypermultiplets [10–13] and on the $\mathcal{N}=2, d=4$ SEYM theories, [4,5].

Many solutions of the ungauged, U(1)-FI-gauged and SU(2) SEYM theories have been constructed in the literature but, so far, no supersymmetric solution of SU(2)-FI-gauged theories is explicitly known. This is due to the complexity of the theories and of the equations that need to be solved to construct supersymmetric solutions. Therefore, our very first task will be to describe carefully the structure of SU(2)-FI-gauged $\mathcal{N}=2, d=4$ supergravities with no hypermultiplets (Section 1) and the second will be to spell out in detail the characterization of the timelike supersymmetric solutions of these theories found in Ref. [6] (Section 2), showing that, according to the results of Ref. [14], none of them will be maximally supersymmetric. We will, then (Section 3), consider the simplest theory that admits an SU(2) gauging, the so-called $\overline{\mathbb{CP}}^3$ model, and we will perform the gauging with FI terms, constructing explicitly the scalar potential.

In Section 4 we setup and try to solve by using different methods and ansatzs the equations that the elementary building blocks of supersymmetric solutions must satisfy in the $\overline{\mathbb{CP}}^3$ model. We present 3 different solutions. Finally, in Section 5 we try a different approach which is only valid for ST$[2,n]$ models with $n \geq 6$: the authors of Ref. [15] constructed several timelike supersymmetric solutions of an SU(2)-FI-gauged $\mathcal{N}=(1,0), d=6$ theory and, by dimensional reduction, we can obtain solutions of the corresponding SU(2)-FI-gauged $\mathcal{N}=1, d=5$ and $\mathcal{N}=2, d=4$ theories.[5] Unfortunately, most of the solutions we obtain in this way do not have an good asymptotically behavior (flat, AdS,..) nor they are in general free of naked singularities. The exception is the $\mathrm{AdS}_3 \times \mathrm{S}^2$ solution which can be obtained from the $\mathrm{AdS}_3 \times \mathrm{S}^3$ one in 6 dimensions. There are other possibilities to obtain solutions of the same type in 5 and 4 dimensions that we explain in detail. Section 6 contains our conclusions and directions for future work.

1. SU(2)-FI-gauged $\mathcal{N}=2$, $d=4$ supergravity

In this section we are going to review quickly the kind of theories we will be dealing with. For more details, the reader is referred to Refs. [1,6,3], whose conventions we follow here. More information on the construction of these theories can be found in Ref. [2].

[5] The solutions of SU(2)-FI-gauged $\mathcal{N}=1, d=5$ have not been received much attention, either, and, to the best of our knowledge, none have been presented in the literature up to this moment.

We are considering theories of $\mathcal{N}=2, d=4$ supergravity, where the supergravity multiplet contains the metric $g_{\mu\nu}$ and the graviphoton vector field $A^{0}{}_{\mu}$ plus two gravitini $\psi_{I\,\mu}$, $I, J, \ldots = 1, 2$, coupled to n vector multiplets, each of them consisting of a complex scalar Z^{i} and a vector field $A^{i}{}_{\mu}$ plus two gaugini $\lambda^{i\,I}$, $i = 1, \cdots, n$. All the vector fields are combined into $A^{\Lambda}{}_{\mu}$, $\Lambda, \Sigma, \ldots = 0, 1, \cdots, n$. The complex scalar parametrize a Special-Kähler manifold. The Special-Kähler structure, which determines the Kähler potential $\mathcal{K}$ (and, hence, the Kähler metric $\mathcal{G}_{ij^*} = \partial_i \partial_{j^*}\mathcal{K}$ of the scalar σ-model) and the *period matrix* $\mathcal{N}_{\Lambda\Sigma}(Z, Z^*)$ that describes the coupling of the scalars to the vector field strengths (kinetic matrices), is completely determined by the canonical covariantly-holomorphic symplectic section[6] $\mathcal{V} = \begin{pmatrix} \mathcal{L}^{\Lambda} \\ \mathcal{M}_{\Lambda} \end{pmatrix}$ or by a prepotential $\mathcal{F}$. These two objects determine completely the ungauged theory.

The global symmetries of a theory of $\mathcal{N}=2, d=4$ supergravity coupled to vector supermultiplets are the holomorphic isometries of the Kähler metric that also preserve the rest of the Special-Kähler structure[7] and the R-symmetry group U(2) which only acts on the indices I, J, K of the fermion fields in the fundamental representation. When the group of isometries that are also global symmetries of the theory includes a non-Abelian subgroup[8] which acts in the adjoint representation on a subset of the vector supermultiplets, one can gauge it: if the holomorphic isometries are global symmetries of the theory, there are holomorphic Killing vectors $k_{\Lambda}(Z)$ and associated symplectic generators of the gauge group $\mathcal{T}_{\Lambda}$ satisfying the same Lie algebra

$$[k_{\Lambda}, k_{\Sigma}] = -f_{\Lambda\Sigma}{}^{\Omega} k_{\Omega}\,, \qquad [\mathcal{T}_{\Lambda}, \mathcal{T}_{\Sigma}] = +f_{\Lambda\Sigma}{}^{\Omega}\mathcal{T}_{\Omega}\,, \tag{1.2}$$

where the $f_{\Lambda\Sigma}{}^{\Omega}$ are the structure constants.[9] To gauge the theory, the scalar and vector field strengths are modified in the standard way to make them covariant under the local transformations[10]:

$$\mathfrak{D}_{\mu} Z^{i} = \partial_{\mu} Z^{i} + g A^{\Lambda}{}_{\mu} k_{\Lambda}{}^{i}\,, \tag{1.3}$$

$$F^{\Lambda}{}_{\mu\nu} = 2\partial_{[\mu} A^{\Lambda}{}_{\nu]} + g f_{\Sigma\Omega}{}^{\Lambda} A^{\Sigma}{}_{[\mu} A^{\Omega}{}_{\nu]}\,. \tag{1.4}$$

Here g is the gauge coupling constant. Furthermore, supersymmetry requires the addition of a scalar potential which turns out to be non-negative. The result is the minimal $\mathcal{N}=2$ supersymmetrization of the bosonic Einstein–Yang–Mills theory for that gauge group. These theories were called $\mathcal{N}=2$ Super-Einstein–Yang–Mills (SEYM) theories and their timelike supersymmetric solutions were characterized in Ref. [5] and studied in Refs. [4,16–18].

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6 We will also use

$$\Omega \equiv e^{-\mathcal{K}/2}\mathcal{V} \equiv \begin{pmatrix} \mathcal{X}^{\Lambda} \\ \mathcal{X}_{\Lambda} \end{pmatrix}. \tag{1.1}$$

7 In particular, they must act as transformations of the symplectic group $\mathrm{Sp}(2n+2, \mathbb{R})$ on the symplectic section and, as a consequence, on the period matrix.

8 Abelian subgroups of isometries cannot be gauged in the context of $\mathcal{N}=2, d=4$ theories of supergravity coupled to vector supermultiplets.

9 In this notation the generators of the gauge group carry the same indices as the fundamental vector fields Λ. It is understood that the generators, Killing vectors, structure constants etc. vanish in the directions which remain ungauged. This notation is good enough for our purposes. A more precise (and complicated) notation would require the introduction of the embedding tensor to assign each generator of the gauge group to a gauge field.

10 The field strengths of the fermion fields are also modified, but we will not be concerned with them in this work. See Ref. [3] for more details on this point.

Gauging a subgroup of the R-symmetry group seems to be a different choice, and, indeed it is if the subgroup is Abelian ($U(1) \subset SU(2)$ is the only possibility), because, as we mentioned above, Abelian holomorphic isometries cannot be gauged in these theories. The gauging is done via Fayet–Iliopoulos (FI) terms. The supersymmetric solutions of these theories have been classified and studied in Refs. [10,12].

However, when this subgroup is non-Abelian ($SU(2)$ is the only possibility, via FI terms as well) it turns out that choice is not so different, actually: to gauge it we need gauge vector fields transforming in the adjoint representation of the gauge group. This implies that the whole supermultiplets, and, in particular the complex scalars, must transform in the adjoint representation leaving the whole Special-Kähler structure (and, in particular, the Kähler metric) invariant. Thus, if one gauges a $SU(2)$ subgroup of the R-symmetry group one has to gauge at the same time a $SU(2)$ isometry subgroup of the global symmetry group and one can see the resulting theory as a deformation, via FI terms, of a $\mathcal{N}=2$ SEYM theory with a gauge group that includes a $SU(2)$ factor so that, for a subset of the vector indices $\Lambda, \Sigma, \ldots$ that we are going to denote by the indices $x, y, \ldots$, that only take 3 possible values, the structure constants are those of $SU(2)$:

$$f_{xy}{}^{z} = -\varepsilon_{xyz}\,. \tag{1.5}$$

These are the theories we are interested in. Their timelike supersymmetric solutions were classified as part of the general case studied in Ref. [6]. In the examples we will consider there will be no other factors in the gauge group apart from the $SU(2)$ one.

Since the difference between these theories and the $\mathcal{N}=2$ SEYM theories is the action of the gauge group on the fermions, at the bosonic level the only difference one sees is the scalar potential, which contains additional terms and is no longer non-negative. The scalar and vector field strengths still take the form Eqs. (1.3) and (1.4). The bosonic action is given by

$$\begin{aligned} S = \int d^4x\sqrt{|g|}\big[& R + 2\mathcal{G}_{ij^*}\mathfrak{D}_\mu Z^i \mathfrak{D}^\mu Z^{*\,j^*} + 2\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,\mu\nu}F^{\Sigma}{}_{\mu\nu} \\ & -2\Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,\mu\nu}\star F^{\Sigma}{}_{\mu\nu} - \mathbf{V}(Z,Z^*)\big]\,, \end{aligned} \tag{1.6}$$

where the scalar potential $\mathbf{V}(Z,Z^*)$ is given by

$$\begin{aligned} \mathbf{V}(Z,Z^*) = & -\tfrac{1}{4}g^2(\Im\mathrm{m}\mathcal{N})^{-1|\Lambda\Sigma}\mathcal{P}_\Lambda\mathcal{P}_\Sigma \\ & +\tfrac{1}{2}g^2\left(\mathcal{G}^{ij^*}f^{\Lambda}{}_i f^{*\,\Sigma}{}_{j^*} - 3\mathcal{L}^{*\,\Lambda}\mathcal{L}^{\Sigma}\right)\mathsf{P}_\Lambda{}^x\mathsf{P}_\Sigma{}^x\,, \end{aligned} \tag{1.7}$$

where the objects $f^{\Lambda}{}_i$ are the upper components of the Kähler-covariant derivatives of the canonical symplectic section $(\mathcal{D}_i\mathcal{V}^M) = \begin{pmatrix} f^{\Lambda}{}_i \\ h_{\Lambda\,i}\end{pmatrix}$, $\mathcal{P}_\Lambda$ are the holomorphic momentum maps, and the triholomorphic momentum maps $\mathsf{P}_\Lambda{}^x$, $x, y, \ldots = 1,2,3$, are assumed to be of the form

$$\mathsf{P}_\Lambda{}^x = e_\Lambda{}^x\xi\,, \tag{1.8}$$

for $\xi = 0, 1$[11] and constant tensors $e_\Lambda{}^x$ nonzero for Λ in the range of the $SU(2)$ factor satisfying

$$\varepsilon_{xyz}e_\Lambda{}^y e_\Sigma{}^z = f_{\Lambda\Sigma}{}^{\Omega}e_\Omega{}^x\,, \tag{1.9}$$

[11] The role of this unphysical parameter will be to help us set to zero the FI terms, recovering the $\mathcal{N}=2, d=4$ SEYM theories.

or, taking into account Eq. (1.5),

$$\varepsilon_{xy'z'}e_y{}^{y'}e_z{}^{z'} = -\varepsilon_{xyz'}e_{z'}{}^{x}\,. \tag{1.10}$$

With no loss of generality we will choose the simplest solution

$$e_x{}^{x'} = -\delta_x{}^{x'}\,. \tag{1.11}$$

These constant triholomorphic momentum maps give rise to SU(2) FI terms and often we will use that name for them. With this choice, the scalar potential takes the simple form

$$\mathbf{V}(Z,Z^*) = -\tfrac{1}{4}g^2(\Im\mathrm{m}\mathcal{N})^{-1|\Lambda\Sigma}\mathcal{P}_\Lambda\mathcal{P}_\Sigma + \tfrac{1}{2}\xi^2 g^2\left(\mathcal{G}^{ij^*}f^x{}_i f^{*x}{}_{j^*} - 3\mathcal{L}^{*x}\mathcal{L}^x\right). \tag{1.12}$$

Observe that the first term may contain the contribution of other (necessarily non-Abelian) gauge factors apart from the SU(2) one labeled by $x, y, \ldots$. In the examples that we are going to consider we will not include that possibility and, therefore, the sum over indices $\Lambda, \Sigma, \ldots$ will be restricted to a sum over the SU(2) indices $x, y, \ldots$.

There are other differences between these theories and the SEYM ones in the covariant derivatives of all the fermions (which now transform linearly under the gauge group in the $I, J, \ldots$ indices)[12] and in the supersymmetry transformations as well. We will not deal directly with them and, therefore, we will not describe them here, for the sake of simplicity. All this information can be found in the references mentioned at the beginning of this section.

2. Timelike supersymmetric solutions

The timelike supersymmetric solutions of the theories introduced in the previous section have been characterized in Ref. [6], where the most general gauging of these theories was considered. In this section we are going to particularize the results obtained there to the case of the theories we are dealing with, with only SU(2) as gauge group and with the choice of FI terms Eqs. (1.8) and (1.11).

In order to describe the form of these solutions we start by introducing an auxiliary object X with the same Kähler weight as the canonical symplectic section $\mathcal{V}^M$ so that the quotient $\mathcal{V}^M/X$ has vanishing Kähler weight. Then, we define two real symplectic vectors $\mathcal{R}^M, \mathcal{I}^M$

$$\mathcal{V}^M/X = \mathcal{R}^M + i\mathcal{I}^M\,. \tag{2.1}$$

For any model of $\mathcal{N}=2, d=4$ supergravity (or, equivalently, for any canonical symplectic section $\mathcal{V}^M$) the components $\mathcal{R}^M$ can, in principle, be expressed entirely in terms of the components $\mathcal{I}^M$, although, in practice, this can be very hard to do for certain models. This is often referred to as "solving the stabilization equations" or as "solving the Freudenthal duality equations". We will assume that this has been done and, indeed, that will be the case in the models we will study here. Then, the symplectic product $\mathcal{R}_M\mathcal{I}^M = \langle\mathcal{R}\,|\,\mathcal{I}\rangle = \mathcal{R}_\Lambda\mathcal{I}^\Lambda - \mathcal{R}^\Lambda\mathcal{I}_\Lambda$ is a function of the $\mathcal{I}^M$ only that we call the *Hesse potential*

$$W(\mathcal{I}) \equiv \mathcal{R}_M(\mathcal{I})\mathcal{I}^M\,. \tag{2.2}$$

Now we are ready to describe the form of the fields of the timelike supersymmetric solutions:

[12] In absence of FI terms, the gaugini $\lambda^{i\,I}$ transform as the scalars and vector fields in the same supermultiplets, on the $i, j, \ldots$ indices. The rest of the fermions do not transform at all.

1. First of all, their metric can always be written in the conformastationary form[13]

$$ds^2 = e^{2U}(dt + \hat{\omega})^2 - e^{-2U}\gamma_{\underline{mn}}dx^m dx^n \,. \tag{2.3}$$

The elements that enter in this expression are required to have a specific form or satisfy certain equations:

(a) The metric function e^{-2U} is given by the Hesse potential

$$e^{-2U} = W(\mathcal{I}) = \frac{1}{2|X|^2} \,. \tag{2.4}$$

(b) the 3-dimensional metric $\gamma_{\underline{mn}}$ can be expressed in terms of Dreibein $\hat{V}^x$, $x = 1, 2, 3$

$$\gamma_{\underline{mn}} = V^x{}_{\underline{m}} V^y{}_{\underline{n}} \delta_{xy} \,, \tag{2.5}$$

and these must satisfy the equation

$$d\hat{V}^x - \xi g \epsilon^{xyz} \hat{\tilde{A}}^y \wedge \hat{V}^z + \hat{T}^x = 0 \,, \tag{2.6}$$

where $\hat{\tilde{A}}^\Lambda$ is the effective 3-dimensional gauge connection

$$\tilde{A}^\Lambda{}_{\underline{m}} \equiv A^\Lambda{}_{\underline{m}} + \tfrac{1}{\sqrt{2}} e^{2U} \mathcal{R}^\Lambda \omega_{\underline{m}} \,, \tag{2.7}$$

and

$$\hat{T}^x = \tfrac{1}{\sqrt{2}} \xi g \mathcal{I}^y \hat{V}^y \wedge \hat{V}^x \,. \tag{2.8}$$

(c) The 1-form $\hat{\omega}$ satisfies the equation (in tangent 3-dimensional space)

$$(d\hat{\omega})_{xy} = 2\varepsilon_{xyz} \left\{ \mathcal{I}_M \tilde{\mathfrak{D}}_z \mathcal{I}^M + \tfrac{1}{\sqrt{2}} \xi e^{-2U} \mathcal{R}^z \right\} \,, \tag{2.9}$$

where $\tilde{\mathfrak{D}}$ is the covariant derivative w.r.t. the effective 3-dimensional gauge connection:

$$\tilde{\mathfrak{D}}_z \mathcal{I}^x = \partial_z \mathcal{I}^x - g \varepsilon_{yw}{}^x \tilde{A}^y{}_z \mathcal{I}^w \,, \qquad \tilde{\mathfrak{D}}_z \mathcal{I}_x = \partial_z \mathcal{I}_x - g \varepsilon_{xy}{}^w \tilde{A}^y{}_z \mathcal{I}_w \,, \tag{2.10}$$

$$\tilde{\mathfrak{D}}_z \mathcal{I}^M = \partial_z \mathcal{I}^M \,, \quad \text{when} \quad M \neq x \,, \quad \text{(ungauged directions).} \tag{2.11}$$

2. The time-component of the vector fields has been gauge-fixed to

$$A^\Lambda{}_t = -\tfrac{1}{\sqrt{2}} e^{2U} \mathcal{R}^\Lambda \,, \tag{2.12}$$

and the space components $A^\Lambda{}_x$ together with the functions $\mathcal{I}^M$ are determined by the following generalization of the Bogomol'nyi equations written again in tangent 3-dimensional space:

$$\tilde{F}^\Lambda{}_{xy} = -\tfrac{1}{\sqrt{2}} \varepsilon_{xyz} \left\{ \tilde{\mathfrak{D}}_z \mathcal{I}^\Lambda - \sqrt{2} \xi g \left[\mathcal{R}^\Lambda \mathcal{R}^z + \tfrac{1}{4} e^{-2U} (\Im\mathfrak{m}\mathcal{N})^{-1|\Lambda z} \right] \right\} \,, \tag{2.13}$$

and

$$-\tfrac{1}{\sqrt{2}} \varepsilon_{xyz} \tilde{\mathfrak{D}}_x \tilde{F}_{\Lambda\, yz} = \tfrac{1}{2} g \delta_\Lambda{}^x \left[g \left(\mathcal{I}^x \mathcal{I}^y \mathcal{I}_y - \mathcal{I}_x \mathcal{I}^y \mathcal{I}^y \right) - \tfrac{1}{\sqrt{2}} \xi \varepsilon_{xyz} (d\hat{\omega})_{yz} \right] \,, \tag{2.14}$$

where we have *defined*[14]

$$\tilde{F}_{\Lambda\, xy} \equiv -\tfrac{1}{\sqrt{2}} \varepsilon_{xyz} \left\{ \tilde{\mathfrak{D}}_z \mathcal{I}_\Lambda - \sqrt{2} g \xi \left[\mathcal{R}_\Lambda \mathcal{R}^z + \tfrac{1}{4} e^{-2U} \Re\mathfrak{e}\mathcal{N}_{\Lambda\Gamma} (\Im\mathfrak{m}\mathcal{N})^{-1|\Gamma z} \right] \right\} \,. \tag{2.15}$$

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[13] We use hats to denote differential forms.

[14] There are no dual 1-forms A_Λ in this formulation of the gauged theory.

3. Finally, the scalars are given by

$$Z^i = \frac{\mathcal{R}^i + i\mathcal{I}^i}{\mathcal{R}^0 + i\mathcal{I}^0}\,. \tag{2.16}$$

2.1. Maximally supersymmetric vacua

Before we start looking for explicit examples of supersymmetric solutions, it is worth discussing the possible existence of maximally supersymmetric solutions. According to the results of Ref. [14] the supersymmetric solutions of these theories, if any, must be of the same kind as those of the corresponding ungauged theories: in absence of electromagnetic fluxes, Minkowski spacetime M_4 or anti-de Sitter spacetime AdS_4 and, in presence of fluxes, Bertotti–Robinson spacetimes $\mathrm{AdS}_2 \times \mathrm{S}^2$ [19,20] or Kowalski-Glikman homogeneous *pp*-wave spacetimes KG_4 [21]. Furthermore, maximally supersymmetric solutions in gauged supergravities are characterized by the vanishing of all the fermion shifts and of the R-symmetry connection [14].

For the $\mathcal{N}=2, d=4$ the different possibilities were analyzed in detail in Ref. [22]. The maximally supersymmetric solutions with zero curvature (M_4, $\mathrm{AdS}_2 \times \mathrm{S}^2$ and KG_4) must have identically vanishing triholomorphic momentum maps $\mathsf{P}_\Lambda{}^x = 0$, which is not possible in the case we are considering. The remaining possibility is the only maximally supersymmetric solution with negative curvature: AdS_4. The following conditions have to be satisfied in this case:

$$\mathsf{P}_\Lambda{}^x \mathsf{P}_\Sigma{}^{*x} \mathcal{L}^\Lambda \mathcal{L}^{*\Sigma} \neq 0\,, \tag{2.17}$$

$$k_\Lambda{}^i \mathcal{L}^{*\Lambda} = 0\,, \tag{2.18}$$

$$\mathsf{P}_\Lambda{}^x f^\Lambda{}_i = 0\,, \tag{2.19}$$

$$\varepsilon^{xyz} \mathsf{P}_\Lambda{}^y \mathsf{P}_\Sigma{}^{*z} \mathcal{L}^\Lambda \mathcal{L}^{*\Sigma} = 0\,. \tag{2.20}$$

With our choice of FI terms (1.8),(1.11) these conditions take the form

$$\mathcal{L}^x \mathcal{L}^{*x} \neq 0\,, \tag{2.21}$$

$$k_x{}^i \mathcal{L}^{*x} = 0\,, \tag{2.22}$$

$$f^x{}_i = 0\,, \tag{2.23}$$

$$\varepsilon^{xyz} \mathcal{L}^y \mathcal{L}^{*z} = 0\,. \tag{2.24}$$

Using the choice of coordinates $Z^i = \mathcal{X}^i/\mathcal{X}^0$ and the gauge $\mathcal{X}^0 = 1$, it is not difficult to see, from the definition $f^\Lambda{}_i = e^{\frac{\mathcal{K}}{2}} \mathcal{D}_i \mathcal{X}^\Lambda$ that it is not possible to satisfy all the Eqs. (2.23) at the same time.

We conclude that these theories do not admit maximally supersymmetric vacua.

3. The SU(2) gauging of the $\overline{\mathbb{CP}}^3$ model

In order to search for explicit examples of supersymmetric solutions we must specify the model of $\mathcal{N}=2$, $d=4$ supergravity we work with. The simplest example that admits an SU(2) gauging is the $\overline{\mathbb{CP}}^3$ model. Here we quickly review it. This model has 3 vector multiplets and the quadratic prepotential

$$\mathcal{F} = -\tfrac{i}{4}\eta_{\Lambda\Sigma}\mathcal{X}^\Lambda \mathcal{X}^\Sigma\,, \qquad (\eta_{\Lambda\Sigma}) = \mathrm{diag}(+---)\,. \tag{3.1}$$

We can define the 3 complex scalars, which parametrize a U(1, 3)/(U(1) × U(3)) coset space, by

$$Z^i \equiv \mathcal{X}^i/\mathcal{X}^0\,. \tag{3.2}$$

Adding to these $Z^0 \equiv 1$, it is advantageous to use Z^Λ and Z_Λ

$$(Z^\Lambda) \equiv \left(\mathcal{X}^\Lambda/\mathcal{X}^0\right) = (1, Z^i)\,, \qquad (Z_\Lambda) \equiv (\eta_{\Lambda\Sigma}Z^\Sigma) = (1, Z_i) = (1, -Z^i)\,. \tag{3.3}$$

The Kähler potential, the Kähler metric (which is the standard Bergman metric for the symmetric space U(1, 3)/(U(1) × U(3)) [23]) and its inverse in the $\mathcal{X}^0 = 1$ gauge are given by

$$\mathcal{K} = -\log(Z^{*\Lambda}Z_\Lambda)\,, \quad \mathcal{G}_{ij^*} = e^{\mathcal{K}}\left(\delta_{ij^*} + e^{\mathcal{K}}Z^*_i Z_{j^*}\right), \quad \mathcal{G}^{ij^*} = e^{-\mathcal{K}}\left(\delta^{ij^*} - Z^i Z^{*\,j^*}\right), \tag{3.4}$$

which implies that the complex scalars are constrained to the region

$$0 \leq \sum_i |Z^i|^2 < 1\,. \tag{3.5}$$

The covariantly holomorphic symplectic section $\mathcal{V}^M$, its Kähler-covariant derivative $\mathcal{U}_i = \mathcal{D}_i\mathcal{V}$ and the period matrix are given by

$$\begin{aligned}
\mathcal{V} &= e^{\mathcal{K}/2}\begin{pmatrix} Z^\Lambda \\ -\frac{i}{2}Z_\Lambda \end{pmatrix}, \qquad \mathcal{U}_i = e^{\mathcal{K}/2}\begin{pmatrix} -e^{\mathcal{K}}Z^*_i Z^\Lambda + \delta_i{}^\Lambda \\ \frac{i}{2}(e^{\mathcal{K}}Z^*_i Z_\Lambda - \eta_{i\Lambda}) \end{pmatrix}, \\
\mathcal{N}_{\Lambda\Sigma} &= \tfrac{i}{2}\left[\eta_{\Lambda\Sigma} - 2\frac{Z_\Lambda Z_\Sigma}{Z^\Gamma Z_\Gamma}\right].
\end{aligned} \tag{3.6}$$

For later use we also quote

$$\begin{aligned}
\Im\mathfrak{m}\,\mathcal{N}_{\Lambda\Sigma} &= \tfrac{1}{2}\left[\eta_{\Lambda\Sigma} - \left(\frac{Z_\Lambda Z_\Sigma}{Z^\Gamma Z_\Gamma} + \text{c.c}\right)\right], \\
(\Im\mathfrak{m}\,\mathcal{N})^{-1|\Lambda\Sigma} &= 2\left[\eta^{\Lambda\Sigma} - \left(\frac{Z^\Lambda Z^{*\,\Sigma}}{Z^\Gamma Z^*_\Gamma} + \text{c.c}\right)\right],
\end{aligned} \tag{3.7}$$

and the Hesse potential

$$\mathsf{W}(\mathcal{I}) = \tfrac{1}{2}\eta_{\Lambda\Sigma}\mathcal{I}^\Lambda\mathcal{I}^\Sigma + 2\eta^{\Lambda\Sigma}\mathcal{I}_\Lambda\mathcal{I}_\Sigma\,. \tag{3.8}$$

Since the scalars parametrize the symmetric space U(1, 3)/(U(1) × U(3)), the metric (and, indeed, the whole model) is invariant under global U(1, 3) = U(1) × SU(1, 3) transformations. We are interested in the SU(1, 3) subgroup whose SO(3) subgroup we are going to gauge.

The special coordinates $\mathcal{X}^\Lambda$ transform in the fundamental representation of SU(1, 3):

$$\mathcal{X}'^{\,\Lambda} = \Lambda^\Lambda{}_\Sigma\mathcal{X}^\Sigma\,, \qquad \Lambda^{*\,\Gamma}{}_\Lambda\,\eta_{\Gamma\Delta}\,\Lambda^\Delta{}_\Sigma = \eta_{\Lambda\Sigma}\,, \tag{3.9}$$

and, according to their definition, the complex scalars transform non-linearly, as

$$Z'^{\,\Lambda} = \frac{\Lambda^\Lambda{}_\Sigma Z^\Sigma}{\Lambda^0{}_\Sigma Z^\Sigma}\,, \qquad Z'_\Lambda = \frac{\Lambda_\Lambda{}^\Sigma Z_\Sigma}{\Lambda^0{}_\Sigma Z^\Sigma}\,, \quad \text{where } \Lambda_\Lambda{}^\Sigma \equiv \eta_{\Lambda\Gamma}\Lambda^\Gamma{}_\Omega\eta^{\Omega\Sigma}\,. \tag{3.10}$$

We will use the metric $\eta_{\Lambda\Gamma}$ and its inverse to lower and raise the indices of the SU(1, 3) transformations $\Lambda^\Lambda{}_\Sigma$.

These transformations leave the Kähler potential invariant up to Kähler transformations $\mathcal{K}' = \mathcal{K} + f + f^*$ with

$$f(Z) = \log\left(\Lambda^0{}_\Sigma Z^\Sigma\right), \tag{3.11}$$

which implies the exact invariance of the Kähler metric.

The symplectic section $\mathcal{V}^N$ is also left invariant by the combined action of the symplectic transformation that gives the embedding of the group SU(1, 3) in the symplectic group Sp(8, $\mathbb{R}$)

$$(S^M{}_N) = \begin{pmatrix} \Re\mathfrak{e}\,\Lambda^\Lambda{}_\Sigma & -2\Im\mathfrak{m}\,\Lambda^{\Lambda\Sigma} \\ \frac{1}{2}\Im\mathfrak{m}\,\Lambda_{\Lambda\Sigma} & \Re\mathfrak{e}\,\Lambda_\Lambda{}^\Sigma \end{pmatrix}, \tag{3.12}$$

and a Kähler transformation with the parameter $f(Z)$ given in Eq. (3.11). This proves the invariance of the whole model of $\mathcal{N} = 2, d = 4$ supergravity.

The 15 generators $T_m{}^\Lambda{}_\Sigma$ of $\mathfrak{su}(1, 3)$, defined by

$$\Lambda^\Lambda{}_\Sigma \sim \delta^\Lambda{}_\Sigma + \alpha^m\, T_m{}^\Lambda{}_\Sigma, \tag{3.13}$$

are traceless and such that $T_{m\,\Lambda\Sigma} \equiv \eta_{\Lambda\Gamma}\, T_m{}^\Gamma{}_\Sigma$ is anti-Hermitian. Then, the corresponding $\mathfrak{sp}(1, 3)$ generators, whose exponentiation gives the matrix Eq. (3.12), are given by

$$(\mathcal{T}_m{}^M{}_N) = \begin{pmatrix} \Re\mathfrak{e}T_m{}^\Lambda{}_\Sigma & -2\Im\mathfrak{m}T_m{}^{\Lambda\Sigma} \\ \frac{1}{2}\Im\mathfrak{m}T_{m\,\Lambda\Sigma} & \Re\mathfrak{e}T_{m\,\Lambda}{}^\Sigma \end{pmatrix}. \tag{3.14}$$

The holomorphic Killing vectors that generate the transformations of the scalars Eqs. (3.10) can be written in the form

$$Z'^\Lambda = Z^\Lambda + \alpha^m k_m{}^\Lambda(Z), \qquad k_m{}^\Lambda(Z) = T_m{}^\Lambda{}_\Sigma\, Z^\Sigma - T_m{}^0{}_\Omega\, Z^\Omega Z^\Lambda, \tag{3.15}$$

which allows us to show easily that, if the matrices T_m have the commutation relations $[T_m, T_n] = f_{mn}{}^p\, T_p$, where $f_{mn}{}^p$ are the $\mathfrak{su}(1, 3)$ structure constants, then the commutation relations of the symplectic generators and the Lie brackets of the holomorphic Killing vectors are given by

$$[\mathcal{T}_m, \mathcal{T}_n] = f_{mn}{}^p\, \mathcal{T}_p, \qquad [k_m, k_n] = -f_{mn}{}^p\, k_p. \tag{3.16}$$

The holomorphic functions $\lambda_m(Z)$ defined through

$$\mathcal{L}_{K_m}\mathcal{K} = \lambda_m + \lambda_m^*, \quad \text{where} \quad K_m = k_m(Z) + k_m^*(Z^*), \tag{3.17}$$

are given by

$$\lambda_m = T_m{}^0{}_\Sigma Z^\Sigma, \tag{3.18}$$

and the holomorphic momentum maps $\mathcal{P}_m$, defined through the relation

$$i\mathcal{P}_m = k_m{}^i\partial_i\mathcal{K} - \lambda_m, \tag{3.19}$$

are given by

$$\mathcal{P}_m = ie^{\mathcal{K}}\eta_{\Lambda\Omega}T_m{}^\Lambda{}_\Sigma Z^\Sigma Z^{*\,\Omega}. \tag{3.20}$$

The SU(2) subgroup that we are going to gauge acts in the adjoint representation on the special coordinates $\mathcal{X}^i$ and on the physical scalars Z^i, leaving exactly invariant $\mathcal{X}^0$, the prepotential and the Kähler potential (so $f = \lambda = 0$). We are going to use the indices $x, y, z, \cdots = 1, 2, 3$ to denote the scalars of the gauged directions, instead of $i, j, \cdots$. Thus, the vector fields A^Λ split into A^0

and A^x, the physical scalars are Z^x, the non-vanishing structure constants and the generators are[15]

$$f_{xy}{}^z = -\varepsilon_{xy}{}^z\,, \qquad T_x{}^y{}_z = \varepsilon_x{}^y{}_z\,, \qquad (\mathcal{T}_x{}^M{}_N) = \begin{pmatrix} \varepsilon_x{}^y{}_z & 0 \\ 0 & \varepsilon_{xy}{}^z \end{pmatrix}, \tag{3.21}$$

and the holomorphic momentum maps and Killing vectors are given by

$$\mathcal{P}_x = ie^{\mathcal{K}}\varepsilon_{xyz}Z^yZ^{*z}\,, \qquad k_x{}^y = \varepsilon_x{}^y{}_zZ^z\,, \tag{3.22}$$

and the SU(2) FI terms are given by Eqs. (1.8) and (1.11). Then, the gauge-covariant derivatives, vector field strengths and scalar potential of the model Eqs. (1.3)–(1.12) take the form

$$\mathfrak{D}_\mu Z^x = \partial_\mu Z^x - g\varepsilon^x{}_{yz}A^y{}_\mu Z^z\,, \tag{3.23}$$

$$F^0{}_{\mu\nu} = 2\partial_{[\mu}A^0{}_{\nu]} \tag{3.24}$$

$$F^x{}_{\mu\nu} = 2\partial_{[\mu}A^x{}_{\nu]} - g\varepsilon^x{}_{yz}A^y{}_{[\mu}A^z{}_{\nu]}\,, \tag{3.25}$$

$$\mathbf{V}(Z,Z^*) = 2g^2e^{2\mathcal{K}}(\mathfrak{Re}Z^x\mathfrak{Re}Z^x)(\mathfrak{Im}Z^y\mathfrak{Im}Z^y)\sin^2\alpha + \tfrac{1}{2}g^2\xi^2\left(5-2e^{\mathcal{K}}\right), \tag{3.26}$$

where α is the angle between the 3-vectors $\mathfrak{Re}Z^x$ and $\mathfrak{Im}Z^y$. Observe that the first term in the potential is non-negative but also bounded above due to Eq. (3.5):

$$0 \leq 2g^2(\mathfrak{Re}Z^x\mathfrak{Re}Z^x)(\mathfrak{Im}Z^y\mathfrak{Im}Z^y)\sin^2\alpha \leq 2g^2\,, \tag{3.27}$$

but the second, which is associated to the FI terms, is unbounded below ($e^{\mathcal{K}} \in (1,\infty)$):

$$-\infty \leq \tfrac{1}{2}g^2\xi^2\left(5-e^{\mathcal{K}}\right) \leq 2g^2\,. \tag{3.28}$$

We have explored the minima of this potential and we have found that there is a minimum when all the scalar fields vanish, when one of them vanishes, when two of them are equal or when two of them are real, but the potential is not negative for any of these minima and, therefore, we have not been able to find any (necessarily non-maximally supersymmetric) AdS_4 vacuum in this theory.

As we have already mentioned, the choice of this specific model is due to its simplicity; in particular, its Freudenthal duality equations can easily be solved:

$$\mathcal{R}_\Lambda = \tfrac{1}{2}\eta_{\Lambda\Sigma}\mathcal{I}^\Sigma\,, \qquad \mathcal{R}^\Lambda = -2\eta^{\Lambda\Sigma}\mathcal{I}_\Sigma\,. \tag{3.29}$$

4. Timelike supersymmetric solutions of the SU(2) gauged $\overline{\mathbb{CP}}^3$ model

We just have to adapt the equations of the general recipe reviewed in Section 2 to the gauged model described in the previous section. In particular, we use the imaginary part of period matrix Eqs. (3.7) expressed in terms of the real symplectic vectors $\mathcal{R}^M$ and $\mathcal{I}^M$ and the solution of the Freudenthal duality equations (3.29) to eliminate $\mathcal{R}^M$ from the equations. We are also going to impose

$$\mathcal{I}_\Lambda = 0\,, \tag{4.1}$$

(so that $\mathcal{R}^\Lambda = 0$) in order to simplify the equations. In particular, with this choice, the form ω is closed, and we set it to zero. The equations that remain to be solved are

15 The indices $x, y, \cdots$ are raised and lowered with δ^{xy}, δ_{xy} and, therefore, their actual position is immaterial.

$$F^0{}_{xy} = -\tfrac{1}{\sqrt{2}}\varepsilon_{xyz}\left\{\partial_z\mathcal{I}^0 + \tfrac{1}{\sqrt{2}}g\xi\mathcal{I}^0\mathcal{I}^z\right\}, \tag{4.2}$$

$$F^z{}_{xy} = -\tfrac{1}{\sqrt{2}}\varepsilon_{xyw}\left\{\mathfrak{D}_w\mathcal{I}^z + \tfrac{1}{\sqrt{2}}g\xi\left[e^{-2U}\delta^{zw} + \mathcal{I}^w\mathcal{I}^z\right]\right\}, \tag{4.3}$$

$$\mathfrak{D}_\xi\hat{V}^x = -\tfrac{1}{\sqrt{2}}g\xi\mathcal{I}^y\hat{V}^y\wedge\hat{V}^x, \tag{4.4}$$

where

$$\mathfrak{D}_\xi\hat{V}^x \equiv d\hat{V}^x - g\xi\varepsilon^x{}_{yz}\hat{A}^y\wedge\hat{V}^z. \tag{4.5}$$

For $\xi=1$, $\mathfrak{D}_\xi\hat{V}^x = \mathfrak{D}\hat{V}^x$ and for $\xi=0$, (when the FI terms vanish) $\mathfrak{D}_\xi\hat{V}^x = d\hat{V}^x$ and the last equation would be solved by choosing coordinates $\hat{V}^x = dx^x$.

The integrability condition of the last equation can be obtained by acting with $\mathfrak{D}$ on both sides and using the Ricci identity ($\xi \neq 0$)

$$\mathfrak{D}\mathfrak{D}_\xi\hat{V}^x = -g\xi\varepsilon^{xyz}\hat{F}^y\wedge\hat{V}^z. \tag{4.6}$$

We find, up to the overall factor $g\xi$

$$F^y{}_{xy} + \tfrac{1}{\sqrt{2}}\varepsilon_{xyz}\mathfrak{D}_z\mathcal{I}^y = 0, \tag{4.7}$$

which is satisfied if Eq. (4.3) holds.

4.1. Hedgehog ansatz

It is natural to start by looking for spherically-symmetric solutions. We can adopt the *hedgehog ansatz* for the gauge field $A^x{}_{\underline{m}}$ and the corresponding "Higgs field" Φ^x[16]:

$$-\tfrac{1}{\sqrt{2}}\mathcal{I}^x = \Phi^x(r) = -x^x f(r), \qquad A^x{}_{\underline{m}} = \varepsilon^x{}_{\underline{mn}}x^n h(r). \tag{4.8}$$

We can also assume that the 3-dimensional metric $\gamma_{\underline{mn}}$ is conformally flat and choose Dreibeins of the form

$$V^x{}_{\underline{m}} = \delta^x{}_{\underline{m}}V(r). \tag{4.9}$$

We can also safely assume that

$$-\tfrac{1}{\sqrt{2}}\mathcal{I}^0 = \Phi^0(r). \tag{4.10}$$

The ansatz for the Abelian vector field $A^0{}_{\underline{m}}$ cannot be spherically symmetric: we know that the potential of the Dirac monopole is not spherically symmetric even though the field strength is. If the unit vector s^m indicates the direction of the Dirac string, the Dirac monopole potential can be written in the form

$$A^0{}_{\underline{m}} = \tfrac{1}{2}p\varepsilon_{mnp}\frac{s^n x^p}{r}k(w), \quad \text{where} \quad w \equiv \frac{s^m x^m}{r}, \quad \text{and} \quad k(w) = (1-w)^{-1}. \tag{4.11}$$

We can make the following ansatz in this case:

$$A^0{}_{\underline{m}} = \varepsilon_{mnp}\frac{s^n x^p}{r^2}k(r,w), \tag{4.12}$$

[16] The signs have been chosen so that the equations originally obtained by Protogenov in Ref. [24] coincide with those studied and used in Refs. [16,17,25,18].

so the function k can have additional dependence on r (not through w).

Substituting this ansatz into Eqs. (4.2)–(4.4) we get the following differential equations:

$$V^{-1}[2h + rh'] - f[1 + gr^2 h] - \tfrac{1}{2} g\xi V \left[(\Phi^0)^2 - r^2 f^2\right] = 0\,, \tag{4.13}$$

$$V^{-1}[rh' - gr^2 h^2] - gr^2 hf + rf' + g\xi V r^2 f^2 = 0\,, \tag{4.14}$$

$$(V^{-1})' + g\xi r[hV^{-1} - f] = 0\,, \tag{4.15}$$

$$x^m \partial_{\underline{m}} k = 0\,, \tag{4.16}$$

$$\Phi^{0\,\prime} + V^{-1} s^m \left(\frac{\partial_{\underline{m}} k}{r} - \frac{2x^m k}{r^3}\right) + g\xi r V \Phi^0 f = 0\,, \tag{4.17}$$

where primes indicate differentiation with respect to r, which is the only argument of the functions Φ^0, f, h, V.

Eq. (4.16) above implies that k is a function of w only and we are left with

$$\partial_{\underline{m}} k = k' \left(\frac{s^m}{r} - \frac{w x^m}{r^2}\right)\,, \tag{4.18}$$

and

$$s^m \left(\partial_{\underline{m}} k - \frac{2x^m k}{r^2}\right) = \frac{1}{r}\frac{d}{dw}[(1 - w^2)k]\,. \tag{4.19}$$

This is the only term in Eq. (4.17) that depends on s^m and that dependence must disappear because the corresponding equation is spherically symmetric. Therefore, we must require that

$$\frac{d}{dw}[(1 - w^2)k] = C\,, \tag{4.20}$$

for some constant C. This equation can be integrated to give

$$k = \frac{Cw + D}{1 - w^2}\,, \tag{4.21}$$

for some other integration constant D. The standard form of the Dirac monopole is recovered when we choose $C = D = p/2$. Then, Eq. (4.17) becomes

$$\Phi^{0\,\prime} + C\frac{V^{-1}}{r^2} + g\xi r \Phi^0 f = 0\,, \tag{4.22}$$

and we are left with a non-autonomous system of 4 ordinary differential equations for 4 variables f, h, V, Φ^0 that generalizes Protogenov's [24].

The next step is to try to rewrite this system as an autonomous system by a change of variables. For the Protogenov system this is explained in Ref. [16]. Actually, the same change of variables works here. Defining

$$gr^2 \equiv e^{2\eta}\,, \qquad 1 + gr^2 h \equiv N\,, \qquad gr^2 f \equiv I\,, \qquad gr^2(\Phi^0)^2 \equiv K^2\,, \qquad C' = g^{1/2} C\,, \tag{4.23}$$

and combining the differential equations we arrive at the autonomous system

$$\partial_\eta N = V\left[IN - \tfrac{1}{2}\xi V I^2 + \tfrac{1}{2} g\xi V K^2\right]\,, \tag{4.24}$$

$$\partial_\eta I = (N^2 - 1)V^{-1} + I - \tfrac{1}{2}\xi V I^2 - \tfrac{1}{2} g\xi V K^2\,, \tag{4.25}$$

$$\partial_\eta V^{-1} = -\xi(N - 1)V^{-1} + \xi I\,, \tag{4.26}$$

$$\partial_\eta K = K - C' V^{-1} - \xi V K I\,. \tag{4.27}$$

When $\xi = 0$, the third equation is solved by $V =$ constant and, setting that constant to 1, the first two equations become those of the Protogenov system and involve only two variables: N and I. When $\xi = 1$ the four equations are coupled in a non-trivial way and we have to make additional assumptions in order to simplify the system and find solutions.

Observe that there are no solutions with vanishing scalars, that is, with $I = 0$: setting $I = 0$ in Eqs. (4.24) and (4.25) and combining them to eliminate K we obtain a differential equation that only involves N and can be integrated to give $N = -\tanh\eta + \alpha$ where α is some integration constant. Then, Eq. (4.25) cannot be satisfied for any real V or K.

A further change of variables, $\mathfrak{I} = VI$ and $\mathfrak{K} = VK$, allows us to rewrite the system in a simpler way:

$$\partial_\eta N = N\mathfrak{I} - \tfrac{1}{2}\mathfrak{I}^2 + \tfrac{1}{2}g\mathfrak{K}^2\,, \tag{4.28}$$

$$\partial_\eta \mathfrak{I} = N^2 - 1 + N\mathfrak{I} - \tfrac{3}{2}\mathfrak{I}^2 - \tfrac{1}{2}g\mathfrak{K}^2\,, \tag{4.29}$$

$$\partial_\eta \mathfrak{K} = \mathfrak{K}N - C' - 2\mathfrak{K}\mathfrak{I}\,, \tag{4.30}$$

$$\partial_\eta \log V = N - \mathfrak{I} - 1\,. \tag{4.31}$$

This system admits a solution in which N, $\mathfrak{I}$ and $\mathfrak{K}$ are constants: the first three equations are algebraic and the fourth is trivial to solve. This allows us to obtain the first solution of this theory.

4.1.1. *Solution 1:* $\mathrm{AdS}_2 \times \mathrm{S}^2$

With no loss of generality we can assume $\mathfrak{I}$ to be positive, and the solution, dependent on two constants $\mathfrak{I}$, v is given by:

$$\begin{aligned}
C' &= \pm\sqrt{\tfrac{\mathfrak{I}}{g}}\left(3\mathfrak{I} + \sqrt{3\mathfrak{I}^2+1}\right)\left(3\mathfrak{I} + 2\sqrt{3\mathfrak{I}^2+1}\right)^{\frac{1}{2}}\,,\\
N &= -\mathfrak{I} - \sqrt{3\mathfrak{I}^2+1}\,,\\
\mathfrak{K} &= \mp\sqrt{g}\left(3\mathfrak{I}^2 + 2\mathfrak{I}\sqrt{3\mathfrak{I}^2+1}\right)^{\frac{1}{2}}\,,\\
V &= v g^{-\mathfrak{I}-\frac{1}{2}-\frac{1}{2}\sqrt{3\mathfrak{I}^2+1}}\, r^{-2\mathfrak{I}-1-\sqrt{3\mathfrak{I}^2+1}}\,.
\end{aligned} \tag{4.32}$$

The physical fields are then given by:

$$\begin{aligned}
ds^2 &= \frac{v^2}{2\mathfrak{I}} g^{-2\mathfrak{I}+1-\sqrt{3\mathfrak{I}^2+1}}\left(\mathfrak{I} + \sqrt{3\mathfrak{I}^2+1}\right)^{-1} r^{-4\mathfrak{I}-2\sqrt{3\mathfrak{I}^2+1}} dt^2\\
&\quad - 2\mathfrak{I}\left(\mathfrak{I} + \sqrt{3\mathfrak{I}^2+1}\right)\frac{1}{g^2 r^2}\left(dr^2 + r^2 d\Omega^2_{(2)}\right)\,,\\
Z^x &= \pm\frac{x^x}{gr}\mathfrak{I}\left(3\mathfrak{I}^2 + 2\mathfrak{I}\sqrt{3\mathfrak{I}^2+1}\right)^{\frac{1}{2}}\,,\\
\Phi^0 &= \frac{1}{v} g^{\mathfrak{I}+\frac{1}{2}+\frac{1}{2}\sqrt{3\mathfrak{I}^2+1}}\left(3\mathfrak{I}^2 + 2\mathfrak{I}\sqrt{3\mathfrak{I}^2+1}\right)^{\frac{1}{2}} r^{2\mathfrak{I}+\sqrt{3\mathfrak{I}^2+1}}\,,\\
A^x{}_{\underline{m}} &= \varepsilon^x{}_{mn}\frac{x^n}{gr^2}\left(-\mathfrak{I} - 1 - \sqrt{3\mathfrak{I}^2+1}\right)\,.
\end{aligned} \tag{4.33}$$

This metric turns out to be that of $\mathrm{AdS}_2 \times \mathrm{S}^2$ (with different radii), independently of the value of $\mathfrak{I}$, as can be seen performing the following change of variables,

$$\begin{aligned}\rho &= r^{-2\Im-\sqrt{3\Im^2+1}}\,,\\ \tau &= v\left(7\Im^2+1+4\Im\sqrt{3\Im^2+1}\right)^{-1} g^{-2\Im-1-\sqrt{3\Im^2+1}}\,t\,,\end{aligned} \tag{4.34}$$

which leads to:

$$\begin{aligned} ds^2 &= \frac{1}{2\Im}\frac{7\Im^2+1+4\Im\sqrt{3\Im^2+1}}{\Im+\sqrt{3\Im^2+1}}g^2\rho^2 d\tau^2 - 2\Im\frac{\Im+\sqrt{3\Im^2+1}}{7\Im^2+1+4\Im\sqrt{3\Im^2+1}}g^{-2}\frac{d\rho^2}{\rho^2}\\ &\quad -2\Im\left(\Im+\sqrt{3\Im^2+1}\right)g^{-2}d\Omega^2_{(2)}\,,\\ Z^i &= \pm\frac{x^i}{g}\Im\left(3\Im^2+2\Im\sqrt{3\Im^2+1}\right)^{\frac{1}{2}}\rho^{\frac{1}{2\Im+\sqrt{3\Im^2+1}}}\,,\\ \Phi^0 &= \frac{1}{v\rho}g^{\Im+\frac{1}{2}+\frac{1}{2}\sqrt{3\Im^2+1}}\left(3\Im^2+2\Im\sqrt{3\Im^2+1}\right)^{\frac{1}{2}}\,,\\ A^x{}_{\underline{m}} &= \varepsilon^x{}_{mn}\frac{x^n}{g}\left(-\Im-1-\sqrt{3\Im^2+1}\right)\rho^{\frac{2}{2\Im+\sqrt{3\Im^2+1}}}\,.\end{aligned} \tag{4.35}$$

The potential (3.26) assumes in this situation a constant value, which can be negative for certain values of the parameter $\Im$:

$$\mathbf{V}<0 \quad\Leftrightarrow\quad \Im^2\left(3\Im^2+2\Im\sqrt{3\Im^2+1}\right) < g^2 < \tfrac{5}{3}\Im^2\left(3\Im^2+2\Im\sqrt{3\Im^2+1}\right)\,. \tag{4.36}$$

By construction this solution is supersymmetric. In order to determine which fraction of the total supersymmetry it preserves (the minimal amount is $\frac{1}{8}$) we take advantage of the analysis performed in Ref. [6]: the gaugini Killing Spinor Equation is solved imposing three projection operators, each of which projects out half of the components of the Killing spinor. However, if some gaugini's shifts

$$W^{ix} = g\mathcal{G}^{ij^*} f^{*\Lambda}{}_{j^*}\mathsf{P}_\Lambda{}^x\,, \tag{4.37}$$

vanish identically for the configuration we are examining, the corresponding projector does not need to be imposed, and the supersymmetry preserved can be larger. From Eqs. (3.4) and (3.6) we get, for the model we are dealing with,

$$W^{ix}=0 \quad\Leftrightarrow\quad Z^i Z^{*x} - \tfrac{1}{2}\delta^{ix} = 0\,, \tag{4.38}$$

which can never be satisfied for the solution we are presenting, where $Z^x \propto x^x$. This solution, therefore, is only $\frac{1}{8}$-BPS.

4.2. *Another ansatz*

In order to generalize the ansatz we made in Section 4.1 we are going to relax Eq. (4.9): it will have the same form

$$V^x{}_{\underline{m}} = \delta^x{}_{\underline{m}} V\,, \tag{4.39}$$

but now we will allow V to be an arbitrary (that is: not necessarily spherically-symmetric) function of the coordinates $x^{\underline{m}}$.

With this choice, Eq. (4.4) can be solved by

$$A^x{}_{\underline{m}} = \varepsilon^x{}_{\underline{mn}} h^n\,, \tag{4.40}$$

$$\partial_{\underline{m}} V = gV\left(h^m + V\Phi^m\right) \tag{4.41}$$

for some triplet of arbitrary functions h^m that, in particular, can vanish identically. We consider first this possibility.

4.2.1. Solution 2

Let us consider the ansatz (4.40), (4.41) making some further assumptions: $h^m = 0$ and all the functions involved depend on a single direction, say x^1, so that

$$A^x{}_{\underline{m}} = 0\,, \qquad \partial_{\underline{1}} V^{-1} = -g\Phi^1\,, \qquad \Phi^2 = \Phi^3 = 0\,. \tag{4.42}$$

This ansatz is adequate to find domain-wall-type solutions.

Under these assumptions, Eq. (4.2) implies that the only non-trivial component of $F^0{}_{mn}$ is $F^0{}_{23}$. However, since, by assumption, the components $A^0{}_{\underline{2},\underline{3}}$ are functions of x^1 only, they must be constants and the purely spatial components of the field strength $F^0{}_{\underline{mn}}$ must vanish identically.

The equations in (4.2) and (4.3) that remain to be solved are

$$\partial_{\underline{1}} V^{-1} = -g\Phi^1\,, \tag{4.43}$$

$$\partial_{\underline{1}} \Phi^1 = \tfrac{1}{2} g V \left[(\Phi^0)^2 + (\Phi^1)^2\right], \tag{4.44}$$

$$\partial_{\underline{1}} \Phi^0 = g\Phi^0\Phi^1 V\,, \tag{4.45}$$

and can be rewritten in this form

$$\partial_{V^{-1}} \Phi^0 = -\Phi^0 V\,, \tag{4.46}$$

$$\partial_{V^{-1}} \Phi^1 = -\tfrac{1}{2}\frac{V}{\Phi^1}\left[(\Phi^0)^2 + (\Phi^1)^2\right], \tag{4.47}$$

$$\partial_{\underline{1}} V^{-1} = -g\Phi^1\,, \tag{4.48}$$

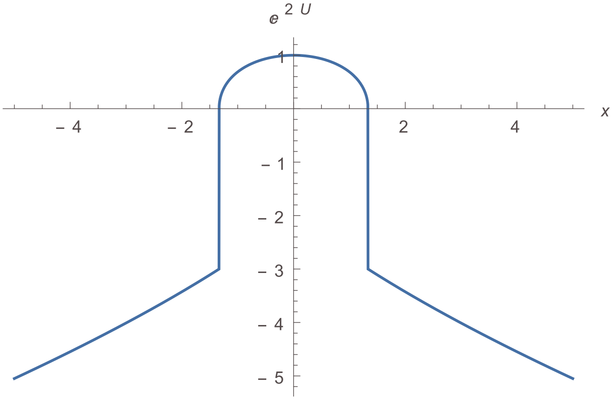
that can be immediately integrated, giving

$$\begin{aligned}
\Phi^0 &= p^0 V\,,\\
\Phi^1 &= \pm\sqrt{\left(p^0\right)^2 V^2 + p^1 V}\,,\\
V &= -2^{\frac{5}{3}}(p^0)^2(p^1)^2 \Bigg\{(p^1)^3\left[16(p^0)^2 - 9(p^1)^4\left(-gx^1+v\right)^2\right]^2\\
&\quad + 3\sqrt{(p^1)^{10}\left(-gx^1+v\right)^2\left[-16(p^0)^2 + 9(p^1)^4\left(-gx^1+v\right)^2\right]^3}\Bigg\}^{-\frac{1}{3}}\\
&\quad - 2^{\frac{1}{3}}\Bigg\{(p^1)^3\left[16(p^0)^2 - 9(p^1)^4\left(-gx^1+v\right)^2\right]^2\\
&\quad + 3\sqrt{(p^1)^{10}\left(-gx^1+v\right)^2\left[-16(p^0)^2 + 9(p^1)^4\left(-gx^1+v\right)^2\right]^3}\Bigg\}^{\frac{1}{3}}\\
&\quad \left[16(p^0)^2 - 9(p^1)^4\left(-gx^1+v\right)^2\right]^{-1}
\end{aligned} \tag{4.49}$$

where p^0, p^1 and v are integration constants. The metric function for these solutions is $e^{-2U} = (\Phi^0)^2 - (\Phi^1)^2 = -p^1 V(x^1)$ and the complete metric has the form

$$ds^2 = -\frac{1}{p^1 V}dt^2 + p^1 V^3[(dx^1)^2 + (dx^2)^2 + (dx^3)^2]\,. \tag{4.50}$$

We must set $p^0 \neq 0$ because, otherwise, $\Phi^0 = 0$ and the metric function would always be negative and we must require $p^1 V < 0$ so $e^{-2U} > 0$. The profile of e^{-2U} changes dramatically with the integration constants and it is not easy to find physically meaningful solutions. One of the few simple examples that we have found corresponds to the choice, $p^0 = -1$, $p^1 = 1$, $v = 0$, (if $g = 1$) for which $e^{-2U}(x^1)$ is positive in an interval of the real line (see the figure where we have represented the inverse, e^{2U}).



At the boundary of that region e^{-2U} and V blow up, and so does the scalar potential, which in this case is given by

$$\mathbf{V} = \tfrac{1}{2} g^2 \left(5 - \frac{2(p^0)^2 V}{p^1} \right) . \tag{4.51}$$

On the other hand, the condition $W^{ix} = 0$ cannot be satisfied for any x, meaning that the solution is $\frac{1}{8}$-BPS.

4.2.2. Solution 3

If, in the context of the ansatz Eqs. (4.40), (4.41), we still assume that all the functions involved depend only on x^1 but we do not assume the vanishing of h^m, the non-trivial components of Eq. (4.4) take the form

$$\partial_{\underline{1}} V = g V \left(h^1 - V \Phi^1 \right) , \tag{4.52}$$

$$h^{2,3} = -V \Phi^{2,3} , \tag{4.53}$$

those of Eq. (4.3) take the form

$$\partial_{\underline{1}} A^0{}_{\underline{2}} = -g V \Phi^0 \Phi^3 , \tag{4.54}$$

$$\partial_{\underline{1}} A^0{}_{\underline{3}} = g V \Phi^0 \Phi^2 , \tag{4.55}$$

$$\partial_{\underline{1}} \Phi^0 = g V \Phi^0 \Phi^1 , \tag{4.56}$$

and, finally, those of Eq. (4.2) take the form

$$\partial_{\underline{1}} \Phi^{2,3} = g h^1 \Phi^{2,3} , \tag{4.57}$$

$$\Phi^2 \Phi^3 = 0 , \tag{4.58}$$

$$\partial_{\underline{1}} h^1 = -gVh^1\Phi^1 + \tfrac{1}{2}gV^2\left[(\Phi^0)^2 - (\Phi^1)^2 - (\Phi^2)^2 + (\Phi^3)^2\right], \tag{4.59}$$

$$\partial_{\underline{1}} h^1 = -gVh^1\Phi^1 + \tfrac{1}{2}gV^2\left[(\Phi^0)^2 - (\Phi^1)^2 + (\Phi^2)^2 - (\Phi^3)^2\right], \tag{4.60}$$

$$\partial_{\underline{1}} \Phi^1 = -\frac{g}{V}(h^1)^2 + 2gV\Phi^2\Phi^3 + \tfrac{1}{2}gV\left[(\Phi^0)^2 + (\Phi^1)^2 - (\Phi^2)^2 - (\Phi^3)^2\right]. \tag{4.61}$$

It is immediate to conclude that

$$\Phi^2 = \Phi^3 = 0\,, \qquad A^0{}_{\underline{2},\underline{3}} = \text{const.}\,, \qquad A^2 = h^1 dx^3\,, \qquad A^3 = -h^1 dx^2\,, \tag{4.62}$$

and the equations that remain to be solved are

$$\partial_{\underline{1}} V = gV\left(h^1 - V\Phi^1\right), \tag{4.63}$$

$$\partial_{\underline{1}} h^1 = -gVh^1\Phi^1 + \tfrac{1}{2}gV^2\left[(\Phi^0)^2 - (\Phi^1)^2\right], \tag{4.64}$$

$$\partial_{\underline{1}} \Phi^0 = gV\Phi^0\Phi^1\,, \tag{4.65}$$

$$\partial_{\underline{1}} \Phi^1 = -\frac{g}{V}(h^1)^2 + \tfrac{1}{2}gV\left[(\Phi^0)^2 + (\Phi^1)^2\right]. \tag{4.66}$$

This system of equations can be simplified by setting $\Phi^1 = 0$; in this way, the resulting equations

$$\Phi^0 = \pm\sqrt{2}\frac{h^1}{V} = \text{const.}\,, \tag{4.67}$$

$$\partial_{\underline{1}} V = gVh^1\,, \tag{4.68}$$

$$\partial_{\underline{1}} h^1 = g(h^1)^2\,, \tag{4.69}$$

are easy to solve, and the solution is determined by the following non-vanishing fields:

$$\Phi^0 = \pm\frac{\sqrt{2}}{b}\,, \tag{4.70}$$

$$A^3{}_{\underline{2}} = -A^2{}_{\underline{3}} = \frac{1}{gx^1}\,, \tag{4.71}$$

$$ds^2 = \frac{2}{b^2}dt^2 - \frac{b^4}{2g^2(x^1)^2}dx^m dx^m\,, \tag{4.72}$$

where b is an integration constant.

The spatial part of the metric is the metric of a 3-dimensional hyperboloid in coordinates analogous to the Poincaré coordinates of AdS_3[17] and, therefore, the complete metric has the

[17] If we define the hyperboloid as the hypersurface

$$(X^1)^2 + (X^2)^2 + (X^3)^2 - (X^4)^2 = -1\,, \tag{4.73}$$

in the $\mathbb{R}^4$ endowed with the metric

$$ds^2 = (dX^1)^2 + (dX^2)^2 + (dX^3)^2 - (dX^4)^2\,, \tag{4.74}$$

then, if we parametrize it with coordinates x^1, x^2, x^3

$$X^1 + X^4 \equiv -\frac{1}{x^1}\,, \qquad X^{2,3} \equiv \frac{x^{2,3}}{x^1}\,, \tag{4.75}$$

geometry of $\mathbb{R} \times \mathbb{H}^3$ and it is supported only by a non-Abelian field whose field strength is related to the volume form of $\mathbb{H}^3$ by

$$F^x{}_{yz} = -g\varepsilon_{\underline{xyz}}\,. \tag{4.77}$$

Usually, p-form field strengths support p- of $(d-p)$-dimensional symmetric spaces. For instance, 2-form field strengths support $\mathrm{AdS}_2 \times \mathrm{S}^2$ solutions in 4 dimensions and $\mathrm{AdS}_2 \times \mathrm{S}^3$ or $\mathrm{AdS}_3 \times \mathrm{S}^2$ solutions in 5 dimensions. In this sense, this solution is exceptional and the exceptionality is related to the rank of the form and to the dimension of the gauge group.

The potential is again equal to a positive constant when this configuration is considered, and the amount of supersymmetry preserved by the solution is $\frac{1}{8}$.

5. Solutions from dimensional reduction

An alternative procedure to construct solutions of a given theory is by dimensional reduction or oxidation of known solutions, provided that there are theories related to the one we are interested in by these mechanisms and that there are known solutions of them which, if they are to be dimensionally reduced, have enough isometries.

$\mathcal{N}=2, d=4$ supergravity theories are directly related by dimensional reduction or oxidation to other supergravity theories with 8 supercharges.[18] These only exist in $d \leq 6$ and, to the best of our knowledge, theories with SU(2) FI gaugings have only been studied in $\mathcal{N}=(1,0), d=6$ supergravity coupled to one tensor multiplet and a triplet of vector multiplets in Ref. [15]. This theory is unique[19] and describes a truncation of the Heterotic String compactified on T^4 that includes the metric $\tilde{g}_{\tilde{\mu}\tilde{\nu}}$, a complete[20] (Kalb–Ramond) 2-form $\tilde{B}_{\tilde{\mu}\tilde{\nu}}$, a real scalar (dilaton) $\tilde{\varphi}$ and the three vector fields $\tilde{A}^A_{\tilde{\mu}}$, $A=1,2,3$. The FI terms induces a simple potential for the dilaton, and the action takes the form [15,26]

$$\tilde{S} = \int d^6\tilde{x}\sqrt{|\tilde{g}|}\left\{\tilde{R} + \tfrac{1}{2}(\partial\tilde{\varphi})^2 + \tfrac{1}{3}e^{\sqrt{2}\tilde{\varphi}}\tilde{H}^2 - e^{\tilde{\varphi}/\sqrt{2}}\tilde{F}^i\tilde{F}^i - \tfrac{3}{2}g_6^2 e^{-\tilde{\varphi}/\sqrt{2}}\right\}, \tag{5.1}$$

where g_6 is the 6-dimensional coupling constant.

The dimensional reduction of $\mathcal{N}=(1,0), d=6$ supergravity theories coupled to tensor and vector multiplets on a circle has been studied and the models of $\mathcal{N}=1, d=5$ supergravity coupled to vector multiplets they give rise to have been determined in Ref. [26]. We can use the results in that paper to dimensionally reduce the 6-dimensional solutions found in Ref. [15] to solutions of SU(2) FI-gauged $\mathcal{N}=1, d=5$ supergravity since the relation between the 6- and 5-dimensional fields of the gauged theories is the same as in the ungauged case, as long as the gauge groups are the same in both theories. These relations are given Appendix A. The

the induced metric is

$$ds^2 = \frac{1}{(x^1)^2}dx^m dx^m\,. \tag{4.76}$$

[18] The relation with theories with different number of supercharges must necessarily involve truncations and constraints on the solutions and we will not consider them here.

[19] As different from $d=4,5$ supergravities with 8 supercharges, in the $d=6$ case, there is only one model for each possible matter content.

[20] That is: not subject to any self- or anti-self-duality (*chirality*) constraints because it is, actually the sum of the 2-form of the supergravity multiplet and the 2-form of the tensor multiplet, which have opposite chiralities.

5-dimensional model obtained in the dimensional reduction is completely characterized by the symmetric tensor $C_{0rs} = \frac{1}{3!}\eta_{rs}$, $r, s = 1, \ldots, 5$. The bosonic fields in this theory are the metric $\hat{g}_{\hat{\mu}\hat{\nu}}$, the 6 gauge fields $\hat{A}^I{}_{\hat{\mu}}$, $I = 0, \cdots, 5$, 5 of which, $\hat{A}^r{}_{\hat{\mu}}$, correspond to 5 vector multiplets,[21] and 5 scalar fields. Due to the reduction procedure, $\hat{A}^{0,1,2}{}_{\hat{\mu}}$ are Abelian fields, while $\hat{A}^{A+2}{}_{\hat{\mu}}$ are the three SU(2) gauge fields in five dimensions. The physical scalars $\hat{\phi}^r$ are encoded in the scalar functions $\hat{h}^I$, constrained by the fundamental relation of Real Special Geometry

$$C_{IJK}\hat{h}^I\hat{h}^J\hat{h}^K = \tfrac{1}{2}\hat{h}^0\eta_{rs}\hat{h}^r\hat{h}^s = 1\,. \tag{5.2}$$

A convenient parametrization is $\hat{\phi}^r = \hat{h}^r$ so $\hat{h}^0 = 2/(\phi\eta\phi) \equiv \hat{\phi}^0$, where $\phi\eta\phi \equiv \hat{\phi}^r\eta_{rs}\hat{\phi}^s$. In this parametrization, the last 3 scalars $\hat{\phi}^{A+2}$ transform in the adjoint representation of SU(2) and the action of the theory can be written in the compact form

$$\begin{aligned}\hat{S} = \int d^5\hat{x}\sqrt{\hat{g}}\,\Big\{ &\hat{R} + \tfrac{3}{2}\hat{a}_{IJ}\hat{\mathfrak{D}}_{\hat{\mu}}\hat{\phi}^I\hat{\mathfrak{D}}^{\hat{\mu}}\hat{\phi}^J - \tfrac{1}{4}\hat{a}_{IJ}\hat{F}^{I\,\hat{\mu}\hat{\nu}}\hat{F}^J{}_{\hat{\mu}\hat{\nu}} - 18\,g_5^2\left(\hat{\phi}^0\right)^{-1} \\ &+ \tfrac{1}{24\sqrt{3}}\eta_{rs}\frac{\hat{\varepsilon}^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\alpha}}}{\sqrt{\hat{g}}}\hat{A}^0{}_{\hat{\mu}}\hat{F}^r{}_{\hat{\nu}\hat{\rho}}\hat{F}^s{}_{\hat{\sigma}\hat{\alpha}}\Big\}\,,\end{aligned} \tag{5.3}$$

where

$$\hat{\mathfrak{D}}_{\hat{\mu}}\hat{\phi}^{0,1,2} = \partial_{\hat{\mu}}\hat{\phi}^{0,1,2}\,, \qquad \hat{\mathfrak{D}}_{\hat{\mu}}\hat{\phi}^{A+2} = \partial_{\hat{\mu}}\hat{\phi}^{A+2} - g_5\epsilon^A{}_{BC}\hat{A}^B{}_{\hat{\mu}}\hat{\phi}^{C+2}\,, \tag{5.4}$$

and where the non-vanishing components of the metric a_{IJ} are

$$a_{00} = \tfrac{1}{12}(\phi\eta\phi)\,, \qquad a_{rs} = \frac{-2\eta_{rs}(\phi\eta\phi) + 4\eta_{rr'}\hat{\phi}^{r}\eta_{ss'}\hat{\phi}^{r}}{3(\phi\eta\phi)^2}\,. \tag{5.5}$$

Observe that the 6- and 5-dimensional gauge coupling constants are related by

$$g_6 = \sqrt{12}g_5\,. \tag{5.6}$$

The dimensional reduction of $\mathcal{N} = 1, d = 5$ supergravity on a circle gives cubic models of $\mathcal{N} = 2, d = 4$ supergravity. Therefore, the $\overline{\mathbb{CP}}^3$ model cannot be obtained in this way. The model that actually arises in the dimensional reduction of the above 5-dimensional model is the ST[2, 6] model, which is characterized by the prepotential[22]

$$\mathcal{F} = -\tfrac{1}{3!}\frac{d_{ijk}\mathcal{X}^i\mathcal{X}^j\mathcal{X}^k}{\mathcal{X}^0}\,, \tag{5.7}$$

where $i = 1, 2\cdots, 6$ labels the 6 vector multiplets and where the fully symmetric tensor d_{ijk} has as only non-vanishing components

$$d_{1\alpha\beta} = \eta_{\alpha\beta}\,, \quad \text{where} \quad (\eta_{\alpha\beta}) = \mathrm{diag}(+-\cdots-)\,, \quad \text{and} \quad \alpha, \beta = 2, \cdots, 6\,. \tag{5.8}$$

The 6 complex scalars parametrize the coset space

$$\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{SO}(2)} \times \frac{\mathrm{SO}(2,5)}{\mathrm{SO}(2)\times\mathrm{SO}(5)}\,, \tag{5.9}$$

[21] The reduction of the KR 2-form gives just 2 vector fields.

[22] More details on this theory and, in particular, on its relation with the toroidal compactification of the Heterotic string can be found in Refs. [17,18].

and the group SO(3) acts in the adjoint on the coordinates $\alpha = 4, 5, 6$ that we are going to denote with $A, B, \ldots$ indices. These are the directions which are gauged. With our conventions, the $\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{SO}(2)}$ factor is parametrized by the scalar Z^1 which is often called the axidilaton field since its real and imaginary parts are, respectively, an axion and a dilaton field.

The action of the ST[2, 6] model can be constructed using the standard formulae valid for any cubic model.[23] It has a complicated form that we are not going to use directly and, therefore, we refrain from writing it here. The computation of the scalar potential using the general formula Eq. (1.7) requires the computation of the momentum maps etc., but we can also obtain it by dimensional reduction using the relation between 5- and 4-dimensional fields that can be found, for instance, in Ref. [18]. It takes the extremely simple form

$$\mathbf{V}(Z, Z^*) = -\tfrac{3}{4}\, g_4^2 \,\frac{1}{\Im\mathfrak{m} Z^1}\,, \tag{5.10}$$

(that is: proportional to the exponential of the dilaton field and, therefore, negative definite) where now the 5- and 4-coupling constants are related by

$$g_5 = -\tfrac{1}{\sqrt{24}}\, g_4\,. \tag{5.11}$$

Thus, to summarize this discussion, we can obtain supersymmetric solutions of the above SU(2) FI-gauged supergravities by dimensional reduction of the 6-dimensional supersymmetric solutions constructed in Ref. [15], using the relations in the Appendix. In the rest of this section we are going to do just that for some of those 6-dimensional solutions.

5.1. Solution 1

The first solution of Ref. [15] that we are going to reduce to 4 dimensions is given in Section 6.2.1 of that reference and it is, perhaps, the simplest: it is a generalization of the solution with geometry $\mathbb{M}_4 \times \mathrm{S}^2$ found by Salam in Sezgin in Ref. [27] that has $\mathbb{M}_3 \times \mathrm{S}^3$ metric, a constant dilaton field whose value is proportional to the square of the radius of the S^3 and to the square of the coupling constant, a meronic gauge field and vanishing 2-form. The non-vanishing field are given by

$$\begin{aligned} d\tilde{s}^2 &= dt^2 - dz^2 - dy^2 - a^2 d\Omega^2_{(3)}\,, \\ e^{\frac{\tilde{\varphi}}{\sqrt{2}}} &= \frac{a^2 g_6^2}{2}\,, \\ \tilde{A}^A &= -\frac{1}{2g_6}\sigma^A\,, \end{aligned} \tag{5.12}$$

where the σ^A are the left-invariant Maurer–Cartan 1-forms satisfying $d\sigma^A = \frac{1}{2}\varepsilon^A{}_{BC}\sigma^B \wedge \sigma^C$, $d\Omega^2_{(3)} = \frac{1}{4}\sigma^A\sigma^A$ and a is a constant parameter.

Reducing along the z coordinate using Eqs. (A.1), we get a solution of the 5-dimensional theory with the following non-vanishing fields:

---

[23] See, for instance, Ref. [3].

$$\begin{aligned}
d\hat{s}^2 &= dt^2 - dy^2 - a^2 d\Omega^2_{(3)}\,,\\
\hat{h}^0 &= 6a^2 g_5^2\,,\\
\hat{h}^1 &= 1 + \frac{1}{12a^2 g_5^2}\,,\\
\hat{h}^2 &= 1 - \frac{1}{12a^2 g_5^2}\,,\\
\hat{A}^{A+2} &= -\frac{1}{2g_5}\sigma^A\,.
\end{aligned} \tag{5.13}$$

This solution belongs to the same class as its 6-dimensional parent: it has constant scalars and a meronic gauge field that support an $\mathbb{M}_2 \times \mathrm{S}^3$ geometry.

Reducing further along the y coordinate using Eqs. (A.2), we obtain a 4-dimensional solution of the same kind with non-vanishing fields

$$\begin{aligned}
ds^2 &= dt^2 - a^2 d\Omega^2_{(3)}\,,\\
Z^1 &= \tfrac{i}{4} a^2 g_4^2\,,\\
Z^2 &= i\left(1 + \frac{2}{a^2 g_4^2}\right),\\
Z^3 &= i\left(1 - \frac{2}{a^2 g_4^2}\right),\\
A^{A+3} &= -\frac{1}{2\,g_4}\sigma^A\,.
\end{aligned} \tag{5.14}$$

5.2. Solution 2

The second solution we are going to consider is the *dyomeronic black string* constructed in Section 6.2.2 of Ref. [15], which corresponds to a black string lying along the z direction with electric and magnetic 3-form and a meronic gauge field in the 4-dimensional transverse space. Its non-vanishing fields are given by

$$\begin{aligned}
d\tilde{s}^2 &= \frac{r}{\sqrt{Q_1 + \frac{Q_2}{r^2}}}\left(dt^2 - dz^2\right) - \frac{\sqrt{Q_1 + \frac{Q_2}{r^2}}}{r}\left(dr^2 + a^2 r^2\, d\Omega^2_{(3)}\right),\\
e^{\sqrt{2}\tilde{\varphi}} &= \frac{a^4 g_6^4}{4\left(1-a^2\right)^2} r^2 \left(Q_1 + \frac{Q_2}{r^2}\right),\\
\tilde{A}^i &= -\frac{1-a^2}{2g_6}\sigma^i\,,\\
\tilde{H} &= \frac{1-a^2}{g_6^2}\left[\frac{a}{4} r\,\sigma^1\wedge\sigma^2\wedge\sigma^3 + \frac{2Q_2}{a^2}\frac{1}{r^3\left(Q_1 + \frac{Q_2}{r^2}\right)^2}\, dt\wedge dr\wedge dz\right],
\end{aligned} \tag{5.15}$$

where the parameter a satisfies $a^2 < 1$. This solution is not asymptotically AdS (or some other known vacuum solution) but has a horizon at $r = 0$ and in the near-horizon limit $r \to 0$ the metric is of the form $\mathrm{AdS}_3 \times \mathrm{S}^3$ where the two factors have different radii. Since this limit is equivalent to setting $Q_1 = 0$, the $\mathrm{AdS}_3 \times \mathrm{S}^3$ near-horizon limit is a supersymmetric solution as well.

If we reduce along the z direction, the following 5-dimensional solution is obtained

$$
\begin{aligned}
d\hat{s}^2 &= r^{\frac{4}{3}}\left(Q_1+\frac{Q_2}{r^2}\right)^{-\frac{2}{3}} dt^2 - r^{-\frac{2}{3}}\left(Q_1+\frac{Q_2}{r^2}\right)^{\frac{1}{3}}\left(dr^2+a^2r^2d\Omega^2_{(3)}\right),\\
\hat{h}^0 &= \frac{6a^2g_5^2}{1-a^2}r^{\frac{4}{3}}\left(Q_1+\frac{Q_2}{r^2}\right)^{\frac{1}{3}},\\
\hat{h}^1 &= r^{-\frac{2}{3}}\left(Q_1+\frac{Q_2}{r^2}\right)^{\frac{1}{3}}\left[1+\frac{1-a^2}{12a^2g_5^2\left(Q_1+\frac{Q_2}{r^2}\right)}\right],\\
\hat{h}^2 &= r^{-\frac{2}{3}}\left(Q_1+\frac{Q_2}{r^2}\right)^{\frac{1}{3}}\left[1-\frac{1-a^2}{12a^2g_5^2\left(Q_1+\frac{Q_2}{r^2}\right)}\right],\\
\hat{F}^0 &= 12^2\sqrt{3}\,g_5^2\frac{a^2}{1-a^2}r^{\frac{5}{2}}\left(Q_1+\frac{Q_2}{r^2}\right)^{-\frac{1}{4}}dt\wedge dr,\\
\hat{F}^1 &= -\hat{F}^2 = \frac{1-a^2}{2\sqrt{3}a^2g_5^2}\frac{Q_2}{r^3}\left(Q_1+\frac{Q_2}{r^2}\right)^{-2}dt\wedge dr,\\
\hat{A}^{A+2} &= -\frac{1-a^2}{2g_5}\sigma^A.
\end{aligned}
\tag{5.16}
$$

This solution is singular at $r=0$ and it is not asymptotically AdS (or some other known vacuum solution). If we reduce it again along the coordinate ϕ, defined by $d\Omega^2_{(3)} = \frac{1}{4}\left[(d\phi+\cos\theta\, d\psi)^2+d\theta^2+\sin^2\theta\, d\psi^2\right]$, we get a 4-dimensional solution which we will refrain from writing explicitly because it has the same problems as the 5-dimensional one.

Of course, we could have used this coordinate ϕ in the reduction from 6 to 5 dimensions. Doing that we get a 5-dimensional solution with the properties similar to those of the 6-dimensional one:

$$
\begin{aligned}
d\hat{s}^2 &= \left(\frac{a}{2}\right)^{\frac{2}{3}} r^{\frac{4}{3}}\left(Q_1+\frac{Q_2}{r^2}\right)^{-\frac{1}{3}}\left(dt^2-dz^2\right) - \left(\frac{a}{2}\right)^{\frac{2}{3}} r^{-\frac{2}{3}}\left(Q_1+\frac{Q_2}{r^2}\right)^{\frac{2}{3}}dr^2\\
&\quad -\left(\frac{a}{2}\right)^{\frac{8}{3}} r^{\frac{4}{3}}\left(Q_1+\frac{Q_2}{r^2}\right)^{\frac{2}{3}}d\Omega^2_{(2)},\\
\hat{h}^0 &= \frac{3\cdot 2^{\frac{1}{3}}a^{\frac{8}{3}}g_5^2}{1-a^2}r^{\frac{4}{3}}\left(Q_1+\frac{Q_2}{r^2}\right)^{\frac{2}{3}},\\
\hat{h}^1 &= \left(\frac{2}{a}\right)^{\frac{4}{3}} r^{-\frac{2}{3}}\left(Q_1+\frac{Q_2}{r^2}\right)^{-\frac{1}{3}}\left[1+\frac{\left(a^2-1\right)\left(a^2-2\right)}{4\cdot 12\,g_5^2}\right],\\
\hat{h}^2 &= \left(\frac{2}{a}\right)^{\frac{4}{3}} r^{-\frac{2}{3}}\left(Q_1+\frac{Q_2}{r^2}\right)^{-\frac{1}{3}}\left[1-\frac{\left(a^2-1\right)\left(a^2-2\right)}{4\cdot 12\,g_5^2}\right],\\
\hat{h}^{A+2} &= -\frac{1-a^2}{2^{\frac{2}{3}}\cdot 3a^{\frac{4}{3}}g_5}r^{-\frac{2}{3}}\left(Q_1+\frac{Q_2}{r^2}\right)^{-\frac{1}{3}}\frac{x^A}{r},\\
\hat{F}^0 &= \frac{3^{\frac{5}{2}}a^6g_5^2}{1-a^2}Q_2\,r^{\frac{3}{2}}\left(Q_1+\frac{Q_2}{r^2}\right)^{-\frac{1}{4}}\cos\theta\, d\theta\wedge d\psi,
\end{aligned}
\tag{5.17}
$$

$$\hat{F}^1 = -\hat{F}^2 = \left[\frac{\left(1-a^2\right)a}{16\sqrt{3}\,g_5^2}\,r - 2\sqrt{3}\right]\sin\theta\, d\theta\wedge d\psi\,,$$

$$\hat{A}^3 = \frac{1-a^2}{2\,g_5}\left(-\sin\psi\, d\theta + \cos\theta\sin\theta\cos\psi\, d\psi\right)\,,$$

$$\hat{A}^4 = \frac{1-a^2}{2\,g_5}\left(\cos\psi\, d\theta + \cos\theta\sin\theta\sin\psi\, d\psi\right)\,,$$

$$\hat{A}^5 = -\frac{1-a^2}{2\,g_5}\cos\theta\left(1+\cos\theta\right)d\psi\,,$$

where we have introduced 3 Cartesian coordinates x^A related to the spherical coordinates r, θ, ψ in the standard way.

This solution is regular in the $r \to 0$ limit, where the metric becomes that of the product $\mathrm{AdS}_3 \times \mathrm{S}^2$ with different radii:

$$d\hat{s}^2 \to \left(\frac{a}{2}\right)^{2/3}\frac{Q_2^{2/3}}{\rho^2}\left(dt^2 - dz^2 - d\rho^2\right) - \left(\frac{a}{2}\right)^{8/3} Q_2^{\frac{2}{3}}\, d\Omega_2^2\,, \tag{5.18}$$

where $\rho \equiv Q_2^{1/2}/r$ but, again, it is not asymptotically AdS.

The $r \to 0$ limit of the complete solution coincides with the solution that one gets by setting $Q_1 = 0$. Thus, there is a globally regular $\mathrm{AdS}_3 \times \mathrm{S}^2$ solution in this theory. It could have been obtained directly by dimensional reduction from the 6-dimensional $\mathrm{AdS}_3 \times \mathrm{S}^3$ solution.

Further reduction along the z coordinate would lead to the same 4-dimensional solution mentioned above. There are, however, other possibilities inspired in the results of Ref. [28], in which the relation between $\mathrm{AdS}_n \times \mathrm{S}^m$ vacua of the 4-, 5- and 6-dimensional theories with 8 supercharges was studied. The main observation is that, just as S^3 can be seen as a U(1) fibration over S^2 and one gets that S^2 by dimensional reduction along that fiber,[24] AdS_3 can be seen as a U(1) fibration over AdS_2 and, by dimensional reduction along that fiber one gets AdS_2. Thus, if instead of using the coordinate z along which the 6-dimensional string lies, one uses the U(1) fiber of the AdS_3 in the $\mathrm{AdS}_3 \times \mathrm{S}^3$ solution, we would have obtained an $\mathrm{AdS}_2 \times \mathrm{S}^3$ solution in 5 dimensions and then an $\mathrm{AdS}_2 \times \mathrm{S}^2$ solution in 4 dimensions.

A more general dimensional reduction is possible: one can rotate the two U(1) fibers of the 6-dimensional solution and dimensionally reduce along one of the rotated fibers. As in the ungauged case studied in Ref. [28] one would get a solution that describes geometry of the near-horizon limit of the BMPV black hole in which the remaining U(1) is non-trivially fibered over $\mathrm{AdS}_2 \times \mathrm{S}^2$. This space is obtained in 4 dimensions after dimensional reduction along the remaining fiber.

The main difference with the ungauged case, apart from the presence of non-trivial SU(2) gauge field, is the difference between the radii of the two factors of these metrics.

Carrying out these alternative dimensional reductions following Ref. [28] is straightforward, albeit quite involved due to the necessity to rewrite the 6-dimensional solution in different coordinates. We leave it for a future publication.

[24] This is what we have done here to go from the $\mathrm{AdS}_3 \times \mathrm{S}^3$ to the $\mathrm{AdS}_3 \times \mathrm{S}^2$ solution.

6. Conclusions

Exploring the space of the supersymmetric solutions of a supergravity theory is one of the most elementary steps one can take to get a more complete understanding of its structure, providing information about the possible vacua and some of the solitonic objects that can exist on it. In this paper we have taken this step for two particular examples (the $\overline{\mathbb{CP}}^3$ and ST[2, 6] models) of a wide class of theories with a class of gaugings that has been overlooked so far: SU(2)-FI-gauged $\mathcal{N}=2, d=4$ supergravity.

Although, as we have shown, no maximally supersymmetric solutions exist in these theories, there are non-maximally-supersymmetric solutions that can be seen as a deformation of the maximally supersymmetric vacua of the ungauged theory, such as the $\mathrm{AdS}_2 \times \mathrm{S}^2$ solutions with different radii. Actually, in the ST[2, 6] model, the $\mathrm{AdS}_2 \times \mathrm{S}^2$ solution must have a higher-dimensional origin analogous to that of the ungauged case [28] and we have indicated the existence of a family of vacua of $\mathcal{N}=1, d=5$ supergravity similar to the near-horizon geometry of the 5-dimensional BMPV black-hole originating in the $\mathrm{AdS}_3 \times \mathrm{S}^3$ solution of $\mathcal{N}=(1,0), d=6$ supergravity with different radii constructed by Cariglia and Mac Conamhna in Ref. [15].

It is likely the existence of deformed versions of the rest of the maximally supersymmetric vacua of $\mathcal{N}=1, d=5$ supergravity (Hpp-waves and Gödel spacetimes [29,30]). It may be possible to obtain them from the above-mentioned solutions by different limiting procedures [31]. On the other hand, it would be interesting to find complete black-hole and black-string solutions whose near-horizon geometries were precisely the $\mathrm{AdS}_m \times \mathrm{S}^n$ solutions we have discussed, but it is not guaranteed that they are always going to exist and their asymptotic behaviour is uncertain.

Apart from these solutions we have found solutions whose geometry is of the form $\mathbb{M}_m \times \mathrm{S}^n$ in 4 and 5 dimensions which descend from a 6-dimensional solution of the same kind and a solution of the $\overline{\mathbb{CP}}^3$ model with $\mathbb{R} \times \mathbb{H}^3$ geometry which deserves further study. Work in this direction is in progress [31].

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Appendix A. Rules for dimensional reduction

A.1. 6 → 5

Following Ref. [26], for the supergravity theories considered in Section 5, if we perform the dimensional reduction along the coordinate z, the 5-dimensional fields can be expressed in terms of the 6-dimensional fields as follows:

$$
\begin{aligned}
\hat{g}_{\hat\mu\hat\nu} &= \tilde{g}_{\hat\mu\hat\nu}\left|\tilde{g}_{\underline{zz}}\right|^{\frac{1}{3}} + \tilde{g}_{\hat\mu\underline{z}}\,\tilde{g}_{\hat\nu\underline{z}}\left|\tilde{g}_{\underline{zz}}\right|^{-\frac{2}{3}}\,,\\
\hat{h}^0 &= e^{\frac{\tilde\varphi}{\sqrt{2}}}\left|\tilde{g}_{\underline{zz}}\right|^{\frac{1}{3}}\,,\\
\hat{h}^1 &= \left|\tilde{g}_{\underline{zz}}\right|^{-\frac{2}{3}}\left(1+\tilde{A}^i{}_{\underline{z}}\tilde{A}^i{}_{\underline{z}}\right) + \tfrac{1}{2}e^{-\frac{\tilde\varphi}{\sqrt{2}}}\left|\tilde{g}_{\underline{zz}}\right|^{\frac{1}{3}}\,,\\
\hat{h}^2 &= \left|\tilde{g}_{\underline{zz}}\right|^{-\frac{2}{3}}\left(1-\tilde{A}^i{}_{\underline{z}}\tilde{A}^i{}_{\underline{z}}\right) - \tfrac{1}{2}e^{-\frac{\tilde\varphi}{\sqrt{2}}}\left|\tilde{g}_{\underline{zz}}\right|^{\frac{1}{3}}\,,\\
\hat{h}^{i+2} &= -2\left|\tilde{g}_{\underline{zz}}\right|^{-\frac{2}{3}}\tilde{A}^i{}_{\underline{z}}\,,\\
\hat{F}^0{}_{\hat a\hat b} &= -4\sqrt{3}\left|\tilde{g}_{\underline{zz}}\right|^{\frac{2}{3}} e^{\sqrt{2}\tilde\varphi}\,\epsilon_{\hat a\hat b\hat c\hat d\hat e}\,\tilde{H}^{\hat c\hat d\hat e}\,,\\
\hat{F}^1{}_{\hat\mu\hat\nu} &= \sqrt{3}\,\tilde{H}_{\hat\mu\hat\nu\underline{z}} + 4\sqrt{3}\,\tilde{A}^i{}_{\underline{z}}\,\tilde{F}^i{}_{\hat\mu\hat\nu} + 2\sqrt{3}\,\partial_{[\hat\mu}\left[\frac{\tilde{g}_{\hat\nu]\underline{z}}}{\tilde{g}_{\underline{zz}}}\left(\tilde{A}^i{}_{\underline{z}}\tilde{A}^i{}_{\underline{z}}+1\right)\right],\\
\hat{F}^2{}_{\hat\mu\hat\nu} &= -\sqrt{3}\,\tilde{H}_{\hat\mu\hat\nu\underline{z}} - 4\sqrt{3}\,\tilde{A}^i{}_{\underline{z}}\,\tilde{F}^i{}_{\hat\mu\hat\nu} - 2\sqrt{3}\,\partial_{[\hat\mu}\left[\frac{\tilde{g}_{\hat\nu]\underline{z}}}{\tilde{g}_{\underline{zz}}}\left(\tilde{A}^i{}_{\underline{z}}\tilde{A}^i{}_{\underline{z}}-1\right)\right],\\
\hat{A}^{i+2}{}_{\hat\mu} &= \sqrt{12}\,\tilde{A}^i{}_{\hat\mu} + 2\sqrt{3}\,\frac{\tilde{g}_{\hat\mu\underline{z}}}{\tilde{g}_{\underline{zz}}}\,\tilde{A}^i{}_{\underline{z}}\,.
\end{aligned}
\tag{A.1}
$$

A.2. 5 → 4

Following Ref. [18], for the supergravity theories considered in Section 5, if we perform the dimensional reduction along the coordinate y, the 4-dimensional fields can be expressed in terms of the 5-dimensional ones as follows:

$$
\begin{aligned}
g_{\mu\nu} &= \left|\hat{g}_{\underline{yy}}\right|^{\frac{1}{2}}\left[\hat{g}_{\mu\nu} - \frac{\hat{g}_{\mu\underline{y}}\,\hat{g}_{\nu\underline{y}}}{\hat{g}_{\underline{yy}}}\right],\\
Z^i &= \tfrac{1}{\sqrt{3}}\hat{A}^{i-1}{}_{\underline{y}} + i\left|\hat{g}_{\underline{yy}}\right|^{\frac{1}{2}}\hat{h}^{i-1}\,,\\
A^0{}_\mu &= \tfrac{1}{2\sqrt{2}}\frac{\hat{g}_{\mu\underline{y}}}{\hat{g}_{\underline{yy}}}\,,\\
A^i{}_\mu &= -\tfrac{1}{2\sqrt{6}}\left[\hat{A}^{i-1}{}_\mu - \hat{A}^{i-1}{}_{\underline{y}}\frac{\hat{g}_{\mu\underline{y}}}{\hat{g}_{\underline{yy}}}\right].
\end{aligned}
\tag{A.2}
$$

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