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Non-Abelian black string solutions of $\mathcal{N} = (2,0), d = 6$ supergravity

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Abstract: We show that, when compactified on a circle, $\mathcal{N} = (2,0), d = 6$ supergravity coupled to 1 tensor multiplet and n_V vector multiplets is dual to $\mathcal{N} = (2,0), d = 6$ supergravity coupled to just $n_T = n_V + 1$ tensor multiplets and no vector multiplets. Both theories reduce to the same models of $\mathcal{N} = 2, d = 5$ supergravity coupled to $n_{V5} = n_V + 2$ vector fields. We derive Buscher rules that relate solutions of these theories (and of the theory that one obtains by dualizing the 3-form field strength) admitting an isometry. Since the relations between the fields of $\mathcal{N} = 2, d = 5$ supergravity and those of the 6-dimensional theories are the same with or without gaugings, we construct supersymmetric non-Abelian solutions of the 6-dimensional gauged theories by uplifting the recently found 5-dimensional supersymmetric non-Abelian black-hole solutions. The solutions describe the usual superpositions of strings and waves supplemented by a BPST instanton in the transverse directions, similar to the gauge dyonic string of Duff, Lü and Pope. One of the solutions obtained interpolates smoothly between two $\mathrm{AdS}_3\times \mathrm{S}^3$ geometries with different radii.

Keywords: String Duality, Black Holes in String Theory, Superstring Vacua, Supergravity Models

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Introduction. The supergravity theories with 8 real supercharges provide a very interesting arena for the construction and study of supersymmetric solutions because they have enough symmetry to be tractable and exhibit interesting properties such as the attractor mechanism of their black-hole and black-string solutions [1–5] but not so much symmetry that only a few models are permitted.[1]

Most of the work on these theories has been devoted to the 4-and 5-dimensional ones for different reasons: for a given matter content many models are possible; they are the effective theories of type II superstrings compactified on Calabi-Yau 3-folds (times a circle in the 4-dimensional case); they have rich geometrical structures known as Special Geometry (Kähler in $d = 4$, real in $d = 5$); they admit supersymmetric black-hole solutions etc. In fact, most of whose supersymmetric solutions have been classified in refs. [9–15] and refs. [16–23] respectively.

[1] A general but deep review of all these theories can be found in ref. [6] and for the 4-dimensional case, only, in ref. [7]. The 4- and 5-dimensional ones are also reviewed in ref. [8], with emphasis on the supersymmetric bosonic solutions.

Much less work has been done in the 6-dimensional theories (often called $\mathcal{N} = (2,0), d = 6$ supergravities because they have chiral fermions), whose structure is not as rich and which are not associated to Calabi-Yau compactifications. The pure supergravity theory, first constructed in ref. [24] by dimensional reduction from 11-dimensional supergravity [25] contains the graviton, gravitino and a 2-form with anti-selfdual 3-form field strength and it does not admit a covariant action, which makes it more complicated to work with. This theory can be coupled to vector multiplets (which have no scalars), tensor multiplets (which have real scalars which always parametrize the same symmetric space $\mathrm{SO}(1, n_T)/\mathrm{SO}(n_T)$ and 2-forms whose 3-form field strengths are selfdual) and hypermultiplets (with scalars that parametrize arbitrary quaternionic-Kähler manifolds). One way to avoid the complications of having to deal with chiral 2-forms[2] is to consider theories with just one tensor multiplet so the two chiral 2-forms of opposite chiralities combine into one unconstrained 2-form. These theories can describe the effective theory of the truncated, toroidally compactified Heterotic String (metric, Kalb-Ramond 2-form and dilaton) and, coupled to vector multiplets and hypermultiplets were constructed in refs. [26–28]. The coupling to an arbitrary number of tensor multiplets was described in ref. [29] and has attracted much less attention because it has not been identified as the effective field theory of some string or M-theory compactification yet and it cannot be gauged, at least in any conventional sense, because it does not have vectors that can be used as gauge fields. The coupling to tensors, vectors and hypermultiplets with some gaugings was described in ref. [30], which is the reference that we are going to use here.

The supersymmetric solutions of most of these theories have not yet been classified either. The only $\mathcal{N} = (2,0), d = 6$ supergravity theories considered have been the pure supergravity theory in refs. [31, 32] and a theory with one tensor multiplet and a triplet of vector multiplets with SU(2) and U(1) gaugings via Fayet-Iliopoulos terms in ref. [33].

In this paper we are going to study the often disregarded $\mathcal{N} = (2,0), d = 6$ supergravity theories that have several tensor multiplets with or without vector multiplets as a preparation to classify their supersymmetric solutions and to study how these solutions are related to the supersymmetric solutions of the $\mathcal{N} = 2, d = 5$ theories by dimensional reduction on a circle [34]. We are also going to use these results to construct new supersymmetric solutions of the $\mathcal{N} = (2,0), d = 6$ supergravity theories in absence of a classification.

Let us explain how we intend to achieve these goals.

In general, the supersymmetric solutions of theories related by dimensional reduction are also related: all the supersymmetric solutions of the lower-dimensional theory can be uplifted to supersymmetric solutions of the higher-dimensional theory while all the supersymmetric solutions of the higher-dimensional theory admitting translational isometries [35][3] can also be reduced along the associated directions to supersymmetric solutions

[2]That is: 2-form potentials with selfdual or anti-selfdual 3-form field strengths.

[3]In the case of toroidal compactification. The general condition is that the Killing spinors of the higher-dimensional solutions can also be understood as spinors of the lower-dimensional theory. This requires the spinors to have a particular dependence (or independence) on the coordinates of the compactification manifold which, in turn, requires the solution to meet certain conditions. In toroidal compactifications the isometries associated to the circles must act without fixed points (be translational isometries). In more

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of the lower-dimensional theories. Thus, one can get new supersymmetric solutions of one of the theories from known supersymmetric solutions of the other one.[4] The basic reason for this correspondence is that the Killing spinor equations of the higher-dimensional theory always give the Killing spinor equations of the lower-dimensional one and, if the latter admit solutions, also do the former. As explained in the footnote, it may not be true the other way around.

Two conditions have to be met in order to apply this simple solution-generating technique:

1. One has to know which theories are related by dimensional reduction.

2. The detailed relation ("dictionary") between the fields of the higher- and lower-dimensional theories must also be known.

In our case it does not seem to be widely known which models of $\mathcal{N} = 2, d = 5$ supergravity are related by dimensional reduction to which models of $\mathcal{N} = (2,0), d = 6$ supergravity theories, actually. Thus, our first task (section 1) will be to perform the dimensional reduction of a general, ungauged, $\mathcal{N} = (2,0), d = 6$ supergravity theory with an arbitrary number of tensor and vector multiplets[5] to $d = 5$ and identify to which model of $\mathcal{N} = 2, d = 5$ supergravity. A careful identification of the 5-dimensional fields will provide us with the dictionary we need to reduce and uplift solutions (section 2).

The identification of the 5-dimensional models leads to a surprise: there are two different families of models of $\mathcal{N} = (2,0), d = 6$ supergravity related to the same family of models of $\mathcal{N} = 2, d = 5$ supergravity: the family of models with 1 tensor multiplet and n_V vector multiplets (that we are going to call $\mathcal{N} = 2A, d = 6$ theories)[6] and the family of models with only $n_T = n_V + 1$ tensor multiplets (that we are going to call $\mathcal{N} = 2B, d = 6$ theories) give exactly the same family of models of $\mathcal{N} = 2, d = 5$ supergravity coupled to $n_{V5} = n_V + 2$ vector multiplets with a symmetric tensor C_{IJK} with non-vanishing components $C_{0\,r+1\,s+1} = \frac{1}{3!}\eta_{rs}$ with $r, s = 0, \cdots, n_V + 1$ and $(\eta_{rs}) = (+ - \cdots -)$.

This situation is analogous to what happens when we dimensionally reduce the two maximal 10-dimensional supergravities, $\mathcal{N} = 2A$ and $\mathcal{N} = 2B$, on a circle and we find the same 9-dimensional maximal supergravity[7] [36]. In that case, this coincidence is interpreted as a manifestation at the effective field theory level of the T-duality existing between the two type II superstrings [37–39]. The relation between the fields of the two 10-dimensional

general cases the conditions have not been studied. Observe that this possible problem only arised in the dimensional reduction and never in the oxidation because, by assuming the lower-dimensional solution to be supersymmetric we are assuming the problem has not arisen in the reduction and the lower-dimensional solution has been obtained from a supersymmetric higher-dimensional solution. From a more general perspective: dimensional reduction can break symmetries but dimensional oxidation can never do that.

[4]Of course, the same can be done with non-supersymmetric solutions.

[5]The hypermultiplets do not couple to the vector and tensor multiplets and, clearly, their reduction leads to 5-dimensional hypermultiplets with exactly the same quaternionic-Kähler geometry.

[6]These are the theories related to the toroidal compactification and truncation of the Heterotic String. We also consider the 6-dimensional theories obtained by dualizing the 3-form field strength, related to the compactification of the type IIA superstring on K3. We call them $\mathcal{N} = 2A^*, d = 6$ theories.

[7]It is unique.

supergravities and those of the 9-dimensional one leads to a direct relation between the 10-dimensional fields of the two theories: the type II generalization of the Buscher T-duality rules [40–42] that transform a solution of one of the 10-dimensional theories admitting one isometry into another solution of the other theory (also admitting one isometry) [36].

In the present case it is not clear which is the superstring theory associated to the $\mathcal{N} = 2B, d = 6$ theories (if any), but the relation we have found leads to a new generalization of the Buscher rules transforming 6-dimensional solutions of these theories admitting one isometry (section 3).

In section 4, we are going to exploit the results of section 2 to construct new supersymmetric solutions of the 6-dimensional theories we are discussing ($\mathcal{N} = 2A, 2A^*, 2B$) by uplifting solutions of the $\mathcal{N} = 2, d = 5$ theories they all reduced to. We are going to add a new twist to this story, though. The relations between the fields of two ungauged supergravity theories related by standard dimensional reduction do not change if we gauge both of them in the same way. Thus, we can use the uplifting formulae of section 2 to uplift supersymmetric solutions of the same models of $\mathcal{N} = 2, d = 5$ supergravity but, now, with non-Abelian gaugings.

The supersymmetric solutions of general models of gauged $\mathcal{N} = 2, d = 5$ supergravity were classified in refs. [22, 23], but the construction of explicit examples in the theories with non-Abelian gaugings has only been successfully completed recently in refs. [46, 47]. The method used was essentially the same we are going to use here: the uplifting of solutions of the 4-dimensional non-Abelian gauged theories which are better understood [14, 48–52]. We are just going to consider the simplest solution in ref. [46] to illustrate the procedure, but this will be enough to produce interesting 6-dimensional solutions.

The uplifting of non-Abelian solutions to the $\mathcal{N} = 2A, 2A^*$ theories is well justified, but, what is the justification for the $\mathcal{N} = 2B$ case if these theories cannot be gauged? We believe that a gauged $\mathcal{N} = 2B, d = 6$ theory can be defined at least when the theory is compactified on a circle. The situation is analogous to that of several coincident M5-branes which, at least when wrapped on a circle, must be described by a non-Abelian theory of chiral 2-forms. We do not know how to write such a theory, but at the massless level, we know it is effectively described by a standard non-Abelian theory of vector fields in one dimension less (the theory of coincident D4-branes). We do not know how to describe the non-Abelian $\mathcal{N} = 2B, d = 6$ supergravity theory, which only has chiral 2-forms, but we know that, when compactified on a circle, at the massless level, the theory is described by a standard gauged theory of $\mathcal{N} = 2, d = 5$ supergravity with 1-forms as gauge fields. It is in this limited sense that the non-Abelian solutions of $\mathcal{N} = 2B, d = 6$ supergravity that we are going to construct should be interpreted.

Finally, section 5 contains our conclusions and directions for future work.

1 From six to five dimensions

In this section we are going to consider the dimensional reduction of general theories of ungauged $\mathcal{N} = (2, 0), d = 6$ supergravity coupled to n_T tensor multiplets and n_V vector multiplets to five dimensions. We first review the bosonic sector of the theory explaining our conventions.[8] As usual, we denote the 6-dimensional objects with hats. In particular, $\hat{\mu}, \hat{\nu}, \ldots = 0, \cdots, 5$ and $\hat{a}, \hat{b}, \ldots = 0, \cdots, 5$ are, respectively, 6-dimensional world and tangent space indices. Our metric has mostly minus signature.

The bosonic fields of the n_V vector multiplets, labeled by $i, j, \ldots = 1, \cdots, n_V$, are just the 1-form fields $\hat{A}^i = \hat{A}^i{}_{\hat{\mu}} d\hat{x}^{\hat{\mu}}$. Their 2-form field strengths $\hat{F}^i = \frac{1}{2}\hat{F}^i{}_{\hat{\mu}\hat{\nu}} d\hat{x}^{\hat{\mu}} \wedge d\hat{x}^{\hat{\nu}}$ are defined as

$$\hat{F}^i \equiv d\hat{A}^i \quad \Leftrightarrow \quad \hat{F}^i{}_{\hat{\mu}\hat{\nu}} \equiv 2\partial_{[\hat{\mu}} \hat{A}^i{}_{\hat{\nu}]} \,, \tag{1.1}$$

and are invariant under the gauge transformations

$$\delta \hat{A}^i = d\hat{\Lambda}^i \,, \tag{1.2}$$

for arbitrary 0-forms $\hat{\Lambda}^i$.

The bosonic fields of the supergravity multiplet are the Sechsbein $\hat{e}^{\hat{a}}{}_{\hat{\mu}}$, and a 2-form potential $\hat{B}^0 = \frac{1}{2}\hat{B}^0{}_{\hat{\mu}\hat{\nu}} d\hat{x}^{\hat{\mu}} \wedge d\hat{x}^{\hat{\nu}}$ which satisfies an anti-selfduality constraint whose explicit form depends on the couplings to the matter fields and will be given shortly.

The bosonic fields of the n_T tensor multiplets, labeled by $\alpha, \beta, \ldots = 1, \cdots, n_T$, are the 2-form potentials $\hat{B}^\alpha{}_{\hat{\mu}\hat{\nu}}$ satisfying selfduality constraints whose explicit form will also be given shortly, and the real scalars $\varphi^{\underline{\alpha}}$. These fields can be seen as coordinates in the coset space $\mathrm{SO}(1, n_T)/\mathrm{SO}(n_T)$. It is convenient to use as coset representative the $\mathrm{SO}(1, n_T)$ matrix $\hat{L}_r{}^s$, $r, s, \ldots = 0, 1, \cdots, n_T$ and it is customary to use the following notation: $\hat{L}_r{}^s = (\hat{L}_r, \hat{L}_r{}^\alpha)$ (that is, $\hat{L}_r \equiv \hat{L}_r{}^0$). Then, by definition, these functions satisfy

$$\eta_{rs} = \eta_{tu} \hat{L}_r{}^t \hat{L}_s{}^u = \hat{L}_r \hat{L}_s - \hat{L}_r{}^\alpha \hat{L}_s{}^\alpha \,, \qquad \eta_{rs} = \mathrm{diag}(+, -, -, \cdots, -) \,. \tag{1.3}$$

Using η_{rs} to raise and lower indices we find

$$\hat{L}^r \hat{L}^s \eta_{rs} = \hat{L}^r \hat{L}_r = 1 \,. \tag{1.4}$$

Finally, we define the symmetric $\mathrm{SO}(1, n_T)$ matrix

$$\mathcal{M}_{rs} \equiv \delta_{tu} \hat{L}_r{}^t \hat{L}_s{}^u = 2\hat{L}_r \hat{L}_s - \eta_{rs} \,. \tag{1.5}$$

An $\mathrm{SO}(1, n_T)$-symmetric σ-model for the scalars $\varphi^{\underline{\alpha}}$ can be constructed as usual:

$$\hat{L}_s{}^r \partial_{\hat{a}} \hat{L}^s{}_t \hat{L}_u{}^t \partial^{\hat{a}} \hat{L}^u{}_r = -\partial_{\hat{a}} \hat{L}^r \partial^{\hat{a}} \hat{L}_r \,, \tag{1.6}$$

where we have used the above properties of the coset representative. A simple parametrization of the functions $\hat{L}^r$ in terms of the physical scalars is provided by

$$\hat{L}^0 = \left(1 - \varphi^{\underline{\beta}} \varphi^{\underline{\beta}}\right)^{-1/2} \,, \qquad \hat{L}^\alpha = \varphi^{\underline{\alpha}} \left(1 - \varphi^{\underline{\beta}} \varphi^{\underline{\beta}}\right)^{-1/2} \,, \qquad \Rightarrow \qquad \varphi^{\underline{\alpha}} = \hat{L}^\alpha / \hat{L}^0. \tag{1.7}$$

The matter and supergravity 2-forms are combined into a single $\mathrm{SO}(1, n_T)$ vector $(\hat{B}^r) = (\hat{B}^0, \hat{B}^\alpha)$, with 3-form field strengths $\hat{H}^r = \frac{1}{3!}\hat{H}^r{}_{\hat{\mu}\hat{\nu}\hat{\rho}}\, d\hat{x}^{\hat{\mu}} \wedge d\hat{x}^{\hat{\nu}} \wedge d\hat{x}^{\hat{\rho}}$ defined by

$$\hat{H}^r = d\hat{B}^r + \frac{1}{2} c^r{}_{ij} \hat{F}^i \wedge \hat{A}^j \quad \Leftrightarrow \quad \hat{H}^r{}_{\hat{\mu}\hat{\nu}\hat{\rho}} = 3\partial_{[\hat{\mu}} \hat{B}^r{}_{\hat{\nu}\hat{\rho}]} + \frac{3}{2} c^r{}_{ij} \hat{F}^i{}_{[\hat{\mu}\hat{\nu}} \hat{A}^i{}_{\hat{\rho}]} \,, \tag{1.8}$$

[8] They are, essentially, those of ref. [30].

where $c^r{}_{ij}$ is an array of constant positive-definite matrices. They are invariant under the gauge transformations

$$\delta \hat{B}^r = d\hat{\chi}^r - \frac{1}{2} c^r{}_{ij} \hat{F}^i \hat{\Lambda}^j , \tag{1.9}$$

for arbitrary 1-forms $\hat{\chi}^r$, and they are constrained to satisfy the (anti-) selfduality constraint

$$\mathcal{M}_{rs} \hat{H}^s = -\eta_{rs} \star \hat{H}^s , \quad \text{where} \quad \eta_{rs} = \mathrm{diag}(+, -, -, \cdots , -) . \tag{1.10}$$

Using this constraint in the Bianchi identity of the 3-form field strengths

$$d\hat{H}^r - \frac{1}{2} c^r{}_{ij} \hat{F}^i \wedge \hat{F}^j = 0 , \tag{1.11}$$

one obtains the equation of motion of the 2-forms:

$$d\left(\mathcal{M}_{rs} \star \hat{H}^s\right) + \frac{1}{2} c_{r\,ij} \hat{F}^i \wedge \hat{F}^j = 0 . \tag{1.12}$$

It is convenient to work with the action of the theory but, in general, these theories do not have a covariant action, due to (anti-) selfduality constraints satisfied by the 3-forms [24]. Nevertheless, sometimes, it is possible to construct *pseudoactions* [53] which give the correct equations of motion of the theory upon use of the (anti-) selfduality constraints in the Euler-Lagrange equations that follow from them. The action of the dimensionally reduced theory can then be derived by following these directions:

1. Dimensionally reduce the pseudoaction and the (anti-) selfduality constraints in the standard way.

2. Poincaré-dualize the highest-rank potentials arising from the (anti-) selfdual potentials in the dimensionally-reduced pseudoaction.

3. Identify the resulting potentials with the lowest-rank potentials arising from the (anti-) selfdual potentials. This identification should be completely equivalent to the use of the dimensionally reduced (anti-) selfduality constraint in the action.

A well-known example of this procedure is the dimensional reduction to $d = 9$ of the $\mathcal{N} = 2B, d = 10$ supergravity theory [54–56] carried out in ref. [57]: in this case there is a RR 4-form potential $\hat{C}^{(4)}$ whose 5-form field strength $\hat{G}^{(5)}$ is self-dual $\hat{G}^{(5)} = \star_{10} \hat{G}^{(5)}$ and the equations of motion can be derived from the pseudoaction constructed in ref. [53] by imposing a selfduality constraint. The dimensional reduction of the 4-form potential $\hat{C}^{(4)}$ gives rise to a 4- and a 3-form $C^{(4)}, C^{(3)}$ potentials whose 5- and 4-form field strengths $G^{(5)}$ and $G^{(4)}$ are related by the dimensionally reduced selfduality constraint $G^{(5)} \sim \star G^{(4)}$. Following the above recipe, in ref. [57] the pseudoaction and selfduality constraint were reduced to $d = 9$ first. Then, the 9-dimensional 4-form potential $C^{(4)}$ was Poincaré-dualized into a 9-dimensional 3-form potential $\tilde{C}^{(3)}$ in the pseudoaction. At this point the theory has two different 3-form potentials $\tilde{C}^{(3)}$ and $C^{(3)}$ and the selfduality constraint takes the form $\tilde{G}^{(4)} = G^{(4)}$ indicating that the two 3-forms are one and the same $\tilde{C}^{(3)} = C^{(3)}$. Making this identification in the pseudoaction gives the correct 9-dimensional action.

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In the case at hands, the bosonic equations of motion (in particular, eq. (1.12)) can be found by varying the pseudoaction

$$\hat{S} = \int d^6\hat{x}\sqrt{|\hat{g}|}\Big\{\hat{R} - \partial_{\hat{a}}\hat{L}^r\partial^{\hat{a}}\hat{L}_r + \frac{1}{3}\mathcal{M}_{rs}\hat{H}^r{}_{\hat{a}\hat{b}\hat{c}}\hat{H}^{s\,\hat{a}\hat{b}\hat{c}} - \hat{L}_r c^r{}_{ij}\hat{F}^i{}_{\hat{a}\hat{b}}\hat{F}^{j\,\hat{a}\hat{b}} - \frac{1}{4}c_{r\,ij}\hat{\epsilon}^{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}\hat{f}}\hat{B}^r{}_{\hat{a}\hat{b}}\hat{F}^i{}_{\hat{c}\hat{d}}\hat{F}^j{}_{\hat{e}\hat{f}}\Big\} . \tag{1.13}$$

and imposing on the resulting Euler-Lagrange equations the (anti-) selfduality conditions eqs. (1.10). However, due to the Chern-Simons term, this action is gauge invariant if and only if the following condition holds [58]

$$\eta_{rs}c^r{}_{i(j}c^s{}_{kl)} = 0 , \tag{1.14}$$

and we will assume this condition to hold through our work. Only then one gets consistent five-dimensional theories.

1.1 Reduction of the fields

Having described the bosonic sector of the theories we want to study, we are now ready to reduce them to $d = 5$.

We are going to follow the standard procedure proposed in ref. [59] with the particular conventions of ref. [8]. Thus, we assume that none of the fields depends explicitly on the compact coordinate, that we will call z, we split the world and tangent-space indices as follows

$$\hat{\mu} = \mu, \underline{z} , \qquad \hat{a} = a, z , \tag{1.15}$$

and we decompose the components of the Sechsbein basis (which we choose to be upper-triangular) $\hat{e}^{\hat{a}}{}_{\hat{\mu}}$ into those of a Fünfbein $e^a{}_\mu$, a (Kaluza-Klein (KK)) vector A_μ and a KK scalar k as follows:

$$\left(\hat{e}^{\hat{a}}{}_{\hat{\mu}}\right) = \begin{pmatrix} e^a{}_\mu & kA_\mu \\ 0 & k \end{pmatrix} , \qquad \left(\hat{e}_{\hat{a}}{}^{\hat{\mu}}\right) = \begin{pmatrix} e_a{}^\mu & -A_a \\ 0 & k^{-1} \end{pmatrix} , \tag{1.16}$$

where $A_a = e_a{}^\mu A_\mu$.

The scalars are the same z-independent functions in both dimensions. In particular, $\hat{L}_r = L_r$.

The vector fields $\hat{A}^i$ decompose into vector fields A^i and scalar fields l^i as follows:

$$\hat{A}^i{}_a \equiv A^i{}_a \quad \Leftrightarrow \quad \hat{A}^i{}_\mu = A^i{}_\mu + l^i A_\mu , \tag{1.17}$$

$$\hat{A}^i{}_z \equiv k^{-1}l^i \Leftrightarrow \hat{A}^i{}_{\underline{z}} = l^i . \tag{1.18}$$

This leads to the following decomposition of the vector field strengths:

$$\hat{F}^i{}_{ab} = \mathcal{F}^i{}_{ab} = F^i{}_{ab} + l^i F_{ab} , \tag{1.19}$$

$$\hat{F}^i{}_{az} = k^{-1}\partial_a l^i , \tag{1.20}$$

where F^i and F are the 5-dimensional field strengths

$$F^i \equiv dA^i , \qquad F \equiv dA . \tag{1.21}$$

Each 2-form $\hat{B}^r$ produces a 2- and 1-form in five dimensions (B^r and A^r respectively). They will be related by the (anti-) selfduality constraints. It turns out that the following definitions give potentials with good gauge transformation properties:

$$\hat{B}^r{}_{\mu\underline{z}} \equiv A^r{}_\mu + \frac{1}{2}c^r{}_{ij}l^i A^j{}_\mu\,, \tag{1.22}$$

$$\hat{B}^r{}_{\mu\nu} \equiv B^r{}_{\mu\nu} - A_{[\mu}A^r{}_{\nu]} - c^r{}_{ij}A_{[\mu}A^i{}_{\nu]}l^j\,. \tag{1.23}$$

The 3-form field strengths $\hat{H}^r$ decompose as follows:

$$\hat{H}^r{}_{abc} \equiv H^r{}_{abc}\,, \tag{1.24}$$

$$\hat{H}^r{}_{abz} \equiv k^{-1}\mathcal{F}^r{}_{ab} \equiv k^{-1}\left[F^r + c^r{}_{ij}l^i F^j + \frac{1}{2}c^r{}_{ij}l^i l^j F\right]\,, \tag{1.25}$$

where

$$H^r = dB^r - \frac{1}{2}F\wedge A^r - \frac{1}{2}F^r\wedge A + \frac{1}{2}c^r{}_{ij}F^i\wedge A^j\,, \tag{1.26}$$

$$F^r = dA^r\,. \tag{1.27}$$

This completely fixes the reduction of fields and field strengths. Plugging these decompositions in the pseudoaction eq. (1.13) together with the decomposition of the Levi-Civita symbol

$$\hat{\epsilon}^{abcdez} \equiv \epsilon^{abcde}\,, \tag{1.28}$$

we get in a straightforward manner the 5-dimensional pseudoaction

$$\begin{aligned} S = \int d^5x\sqrt{|g|}k\Bigg\{ & R - \frac{1}{4}k^2F^2 - \partial_\mu L^r\partial^\mu L_r + 2k^{-2}L_r c^r{}_{ij}\partial_\mu l^i\partial^\mu l^j \\ & + \frac{1}{3}\mathcal{M}_{rs}H^r H^s - k^{-2}\mathcal{M}_{rs}\mathcal{F}^r\mathcal{F}^s - L_r c^r{}_{ij}\mathcal{F}^i\mathcal{F}^j \\ & + \frac{k^{-1}\epsilon}{6\sqrt{|g|}}c_{r\,ij}\left[H^r\left(\mathcal{F}^i l^j - 2\partial l^i A^j\right) - 3\mathcal{F}^r\mathcal{F}^i A^j\right]\Bigg\}\,, \end{aligned} \tag{1.29}$$

where the indices are assumed to be contracted in the obvious way: $\mathcal{F}^r\mathcal{F}^s \equiv \mathcal{F}^r{}_{\mu\nu}\mathcal{F}^{s\,\mu\nu}$, $\epsilon H^r c_{r\,ij}(\mathcal{F}^i l^j - 2\partial l^i A^j) = \epsilon^{\mu\nu\rho\kappa\sigma}H^r{}_{\mu\nu\rho}c_{r\,ij}(\mathcal{F}^i{}_{\kappa\sigma}l^j - 2\partial_{[\kappa}l^i A^j{}_{\sigma]})$, etc.

Finally, we make a rescaling of the metric in order to express the action in the "Einstein frame" metric $g_{E\,\mu\nu}$ (minimal coupling to Ricci scalar) in the following way:

$$g_{\mu\nu} = k^{-2/3}g_{E\,\mu\nu}\,, \tag{1.30}$$

and redefine the KK scalar k in order to give it a kinetic term with standard normalization

$$k = e^{\sqrt{3/8}\phi}\,. \tag{1.31}$$

The result, up to total derivatives, is the pseudoaction

$$S = \int d^5x\sqrt{|g_E|}\Bigg\{ R_E + \frac{1}{2}(\partial\phi)^2 - \partial_\mu L^r\partial^\mu L_r + 2e^{-\sqrt{3/2}\phi}L_r c^r{}_{ij}\partial_\mu l^i\partial^\mu l^j - \frac{1}{4}e^{\sqrt{8/3}\phi}F^2$$

$$- e^{-\sqrt{2/3}\phi}\mathcal{M}_{rs}\mathcal{F}^r\mathcal{F}^s - L_r c^r{}_{ij}e^{\phi/\sqrt{6}}\mathcal{F}^i\mathcal{F}^j + \frac{1}{3}e^{\sqrt{2/3}\phi}\mathcal{M}_{rs}H^rH^s$$

$$+ \frac{\epsilon}{6\sqrt{|g_E|}}c_{r\,ij}\left[H^r\left(\mathcal{F}^i l^j - 2\partial l^i A^j\right) - 3\mathcal{F}^r\mathcal{F}^i A^j\right]\Bigg\}. \tag{1.32}$$

The reduction of the (anti-) selfduality constraints eqs. (1.10) offers no problems and becomes a duality relation between the 2- and 1-form potentials B^r, A^r

$$\mathcal{M}_{rs}H^s = -e^{-\sqrt{2/3}\phi}\eta_{rs}\star\mathcal{F}^s\,. \tag{1.33}$$

The equations of motion of the 5-dimensional theory can be obtained by varying the above pseudoaction and imposing the duality constraints. However, in order to identify the 5-dimensional theories obtained with models of $\mathcal{N}=2, d=5$ supergravity coupled to vector multiplets it is convenient to eliminate this constraint. We carry out this task next.

1.2 Dualization

Following the procedure outlined at the beginning of this section, we are going to Poincaré dualize the 2-forms B^r into 1-forms $\tilde{A}_r$. First, we are going replace the 2-forms B^r by their 3-form field strengths H^r as variables of the pseudoaction eq. (1.32). This is possible because the pseudoaction only depends on the 2-forms through their field strengths. However, we have to add a Lagrange-multiplier term to enforce the Bianchi identities of the H^r, which have the form

$$4\partial_{[\mu}H^r{}_{\nu\rho\sigma]} + 6F^r{}_{[\mu\nu}F_{\rho\sigma]} - 3c^r{}_{ij}F^i{}_{[\mu\nu}F^j{}_{\rho\sigma]} = 0\,. \tag{1.34}$$

The Lagrange-multiplier term to be added to the pseudoaction to enforce the Bianchi identity is (again, with the indices contracted in the obvious way)

$$\frac{\epsilon}{\sqrt{|g_E|}}\tilde{A}_r\left(\partial H^r + \frac{3}{2}F^rF - \frac{3}{4}c^r{}_{ij}F^iF^j\right)\,, \tag{1.35}$$

where the Lagrange multiplier is the 1-form field $\tilde{A}_r$.

Adding this term to the pseudoaction and integrating it by parts we get

$$S = \int d^5x\sqrt{|g_E|}\Bigg\{ R_E + \frac{1}{2}(\partial\phi)^2 - \partial_\mu L^r\partial^\mu L_r + 2e^{-\sqrt{3/2}\phi}L_r c^r{}_{ij}\partial_\mu l^i\partial^\mu l^j - \frac{1}{4}e^{\sqrt{8/3}\phi}F^2$$

$$- e^{-\sqrt{2/3}\phi}\mathcal{M}_{rs}\mathcal{F}^r\mathcal{F}^s - L_r c^r{}_{ij}e^{\phi/\sqrt{6}}\mathcal{F}^i\mathcal{F}^j + \frac{1}{3}e^{\sqrt{2/3}\phi}\mathcal{M}_{rs}H^rH^s$$

$$+ \frac{\epsilon}{6\sqrt{|g_E|}}\Big[c_{r\,ij}H^r(\mathcal{F}^i l^j - 2\partial l^i A^j) - 3c_{r\,ij}\mathcal{F}^r\mathcal{F}^i A^j$$

$$+3\tilde{F}_r(H^r + \frac{3}{2}FA^r + \frac{3}{2}F^rA - \frac{3}{2}c^r{}_{ij}F^iA^j)\Big]\Bigg\}, \tag{1.36}$$

where

$$\tilde{F}_r \equiv d\tilde{A}_r\,. \tag{1.37}$$

Since in this pseudoaction H^r is an independent field, we can compute its field equation, which will relate it to $\tilde{F}_r$. It is given by

$$\mathcal{M}_{rs}H^s = -\frac{1}{2}e^{-\sqrt{2/3}\phi}\star\left[c_{r\,ij}\left(\mathcal{F}^i l^j - 2\partial l^i A^j\right) + 3\tilde{F}_r\right], \tag{1.38}$$

This equation can be used to eliminate completely H^r from the pseudoaction and from the duality relation eq. (1.33). After this operation, the 2-forms B^r have disappeared from both, having been replaced by the dual 1-forms $\tilde{A}_r$. We only write explicitly the constraint after this replacement (and some massaging):

$$\tilde{F}_r = \frac{2}{3}\Big(\eta_{rs}F^s + c_{r\,ij}\partial\left(l^i A^j\right)\Big)\,, \tag{1.39}$$

which implies the following algebraic relation between potentials

$$\tilde{A}_r = \frac{2}{3}\eta_{rs}A^s + \frac{1}{3}c_{r\,ij}l^i A^j\,, \tag{1.40}$$

that we can use in the pseudoaction to eliminate completely $\tilde{A}_r$. After this operation the 1-forms A^r are the only fields remaining from the reduction of the 2-forms B^r. Furthermore, there are no constraints to be imposed and the pseudoaction is the standard action

$$\begin{aligned} S = \int d^5x\sqrt{|g_E|}\Bigg\{ & R_E + \frac{1}{2}(\partial\phi)^2 - \partial_\mu L^r\partial^\mu L_r + 2e^{-\sqrt{3/2}\phi}L_r c^r{}_{ij}\partial_\mu l^i\partial^\mu l^j \\ & - \frac{1}{4}e^{\sqrt{8/3}\phi}F^2 - 2e^{-\sqrt{2/3}\phi}\mathcal{M}_{rs}\mathcal{F}^r\mathcal{F}^s - L_r c^r{}_{ij}e^{\phi/\sqrt{6}}\mathcal{F}^i\mathcal{F}^j \\ & + \frac{\epsilon}{\sqrt{|g_E|}}\Big(\eta_{rs}F^rF^sA - c_{r\,ij}F^iF^jA^r\Big)\Bigg\}\,. \end{aligned} \tag{1.41}$$

1.3 Identification with five-dimensional supergravity

The next step is to identify the previous theory as a model of $\mathcal{N}=1, d=5$ supergravity coupled to n_{V5} vector multiplets. These theories[9] contain $n_{V5}+1$ 1-form fields A^I, $I,J,\ldots = 0,1,\cdots,n_{V5}$ and n_{V5} scalars ϕ^x, $x,y,\ldots = 1,\cdots,n_{V5}$, and their interactions (in fact, the whole theory) are determined by the constant and completely symmetric tensor C_{IJK}. In particular, the scalar manifold is the n_{V5}-dimensional hypersurface in $\mathbb{R}^{n_{V5}+1}$ defined by the cubic equation

$$C_{IJK}h^I(\phi)h^J(\phi)h^K(\phi) = 1, \tag{1.42}$$

the kinetic matrix of the vector fields $a_{IJ}(\phi)$ is given by

$$a_{IJ} = -2C_{IJK}h^K + 3h_Ih_J\,, \tag{1.43}$$

[9]We use the conventions of refs. [60] and [21].

where the $h_I(\phi)$ are defined by

$$h_I \equiv C_{IJK} h^J h^K \,, \tag{1.44}$$

and the σ-model metric $g_{xy}(\phi)$ is given by

$$g_{xy} \equiv 3 a_{IJ} \frac{\partial h^I}{\partial \phi^x} \frac{\partial h^J}{\partial \phi^y} = -2 C_{IJK} \frac{\partial h^I}{\partial \phi^x} \frac{\partial h^J}{\partial \phi^y} h^K \,. \tag{1.45}$$

The action is given by

$$S = \int d^5x \sqrt{|g|} \left\{ R + \frac{1}{2} g_{xy} \partial_\mu \phi^x \partial^\mu \phi^y - \frac{1}{4} a_{IJ} F^I F^J + \frac{\epsilon}{12\sqrt{3}\sqrt{|g|}} C_{IJK} F^I F^J A^K \right\} . \tag{1.46}$$

In order to identify the models corresponding to the theories we have obtained by dimensional reduction, we start by rescaling the vector fields

$$A \to \frac{1}{\sqrt{12}} A \,, \qquad A^r \to \frac{1}{\sqrt{12}} A^r \,, \qquad A^i \to \frac{1}{\sqrt{12}} A^i \,, \tag{1.47}$$

so that the action becomes

$$\begin{aligned} S = \int d^5x \sqrt{|g_E|} \Bigg\{ & R_E + \frac{1}{2} (\partial\phi)^2 - \partial_\mu L^r \partial^\mu L_r + 2 e^{-\sqrt{3/2}\phi} L_r c^r{}_{ij} \partial_\mu l^i \partial^\mu l^j \\ & - \frac{1}{48} e^{\sqrt{8/3}\phi} F^2 - \frac{1}{12} L_r c^r{}_{ij} e^{\phi/\sqrt{6}} (F^i{}_{\mu\nu} + l^i F_{\mu\nu})(F^j{}_{\mu\nu} + l^j F_{\mu\nu}) \\ & - \frac{1}{6} e^{-\sqrt{2/3}\phi} \mathcal{M}_{rs} \left(F^r{}_{\mu\nu} + c^r{}_{ij} l^i F^j{}_{\mu\nu} + \frac{1}{2} c^r{}_{ij} l^i l^j F_{\mu\nu} \right) \\ & \times \left(F^s{}_{\mu\nu} + c^s{}_{ij} l^i F^j{}_{\mu\nu} + \frac{1}{2} c^s{}_{ij} l^i l^j F_{\mu\nu} \right) \\ & + \frac{\epsilon}{12\sqrt{3}\sqrt{|g_E|}} \left(\frac{1}{2} \eta_{rs} F^r F^s A - \frac{1}{2} c_{r\,ij} F^i F^j A^r \right) \Bigg\} . \end{aligned} \tag{1.48}$$

Comparing this theory with eq. (1.46) we first see that $n_{V5} = n_T + n_V + 1$ (there is a total of $n_T + n_V + 2$ 1-forms). We can decompose the 5-dimensional index I as $I = 0, r+1, i+n_T+1$ where the indices take the values $r = 0, \ldots, n_T$, $i = 1, \ldots, n_V$ and identify

$$A^0 = A \,, \qquad A^{I=r+1} = A^r \,, \qquad A^{I=i+n_T+1} = A^i \,, \tag{1.49}$$

where the fields in the l.h.s.'s are those of eq. (1.46) and the fields in the r.h.s.'s are those of eq. (1.48).

We can also identify the components of the C_{IJK} tensor that characterizes the model of $\mathcal{N} = 2, d = 5$ supergravity

$$C_{0\,r+1\,s+1} = \frac{1}{3!} \eta_{rs} \,, \qquad C_{r+1\,i+n_T+1\,j+n_T+1} = -\frac{1}{3!} c_{r\,ij} \,. \tag{1.50}$$

We will discuss later the properties of these models, picking two particular subfamilies. Now, knowing C_{IJK} and the expected forms of a_{IJ} and g_{xy}, we can identify the scalar fields of eq. (1.48) with the scalar functions h^I and the physical scalars ϕ^x.

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The components of a_{IJ} in eq. (1.48) are

$$
\begin{aligned}
a_{00} &= \frac{1}{12}\left[e^{2\phi/\sqrt{6}} + 2L_r\xi^r e^{-\phi/\sqrt{6}}\right]^2 , \\
a_{0\,r+1} &= \frac{1}{3}\mathcal{M}_{rs}\xi^s e^{-\sqrt{2/3}\phi}, \\
a_{0\,i+n_T+1} &= \frac{1}{3}L_r c^r{}_{ij} l^j e^{-\phi/\sqrt{6}}\left(e^{2\phi/\sqrt{6}} + 2L_s\xi^s e^{-\phi/\sqrt{6}}\right), \\
a_{r+1\,s+1} &= \frac{2}{3}e^{-\sqrt{2/3}\phi}\mathcal{M}_{rs}, \\
a_{r+1\,i+n_T+1} &= \frac{2}{3}e^{-\sqrt{2/3}\phi}\mathcal{M}_{rs}c^s{}_{ij}l^j, \\
a_{i+n_T+1\,j+n_T+1} &= \frac{2}{3}e^{-\sqrt{2/3}\phi}\mathcal{M}_{rs}c^r{}_{ik}c^s{}_{jl}l^k l^l + \frac{1}{3}e^{\phi/\sqrt{6}}L_r c^r{}_{ij},
\end{aligned}
\tag{1.51}
$$

where $\xi^r \equiv c^r{}_{ij}l^i l^j$ and we have made some simplifications by using the properties $L^r L_r = 1$, $\xi^r\xi_r = 0$, $\xi^r c_{r\,ij}l^i = 0$ and $\mathcal{M}_{rs} = 2L_r L_s - \eta_{rs}$. Finally, if we use as physical scalar fields $(\phi^x) = (\phi^1, \cdots, \phi^{n_V+n_T+1}) = (\phi, \varphi^{\underline{\alpha}}, l^i)$, we see from (1.48) that only the diagonal components of g_{xy} are non-vanishing:

$$
\begin{aligned}
g_{11} &= 1 , \\
g_{\underline{\alpha}+1\,\underline{\beta}+1} &= -2\partial_{\underline{\alpha}}L^r\partial_{\underline{\beta}}L_r, \\
g_{i+n_T+1\,j+n_T+1} &= 4e^{-\sqrt{3/2}\phi}L_r c^r{}_{ij} .
\end{aligned}
\tag{1.52}
$$

Comparing these expressions with the formulae eqs. (1.43) and (1.45) for the theories with symmetric tensor given by eq. (1.50) we conclude that the scalar functions h^I are given by

$$
h^0 = 2e^{-2\phi/\sqrt{6}} , \qquad h^r = L^r e^{\phi/\sqrt{6}} + \xi^r e^{-2\phi/\sqrt{6}} , \qquad h^i = -2e^{-2\phi/\sqrt{6}} l^i . \tag{1.53}
$$

For the sake of convenience we also give the h_I:

$$
h_0 = \frac{1}{6}\left(e^{2\phi/\sqrt{6}} + 2\xi_r L^r e^{-\phi/\sqrt{6}}\right) , \qquad h_r = \frac{2}{3}L_r e^{-\phi/\sqrt{6}} , \qquad h_i = \frac{2}{3}e^{-\phi/\sqrt{6}}c_{r\,ij}L^r l^j . \tag{1.54}
$$

We are interested in two particular cases which correspond to models of the same family characterized by the symmetric tensor with non-vanishing components $C_{0ab} = \frac{1}{3!}\eta_{ab}$ with $(\eta_{ab}) = \mathrm{diag}(+-\cdots-)$ and $a, b = 1, \cdots, n$ for some value of n that depends on the model: $n = n_T$ for $n_V = 0$ and $n = n_V + 1$ for $n_T = 1$. These models can be identified with the Riemannian symmetric spaces $\mathrm{SO}(1,1) \times \mathrm{SO}(1,n)/\mathrm{SO}(n)$ by simple inspection of the metric in eqs. (1.52). However, it is not difficult to see that the scalar manifold is, topologically, the symmetric space $\mathrm{SO}(2,n)/\mathrm{SO}(1,n)$, which is that of AdS_{n+1}: this manifold can be identified with the hypersurface

$$
\left(X^0\right)^2 + \left(X^1\right)^2 - \left(X^2\right)^2 - \cdots - \left(X^n\right)^2 = 1 , \tag{1.55}
$$

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in $\mathbb{R}^{n+2}$. Any change of coordinates such that[10]

$$(X^0)^2 + (X^1)^2 = \frac{1}{2}h^0\,(h^1)^2\,, \quad (X^2)^2 = \frac{1}{2}h^0(h^2)^2\,, \;\cdots\; (X^n)^2 = \frac{1}{2}h^0(h^n)^2\,, \tag{1.56}$$

brings the above definition of the hypersurface to the cubic form

$$\frac{1}{2}\eta_{ab}h^0h^ah^b = 1\,, \tag{1.57}$$

that characterizes the models under discussion. It is also easy to see in this cubic form that the conformal transformations of η_{ab} (the group $\mathrm{SO}(2,n)$), compensated by rescalings of h^0, leave invariant the definition.

Although the scalar manifold is the same manifold of AdS_{n+1}, as metric spaces they are totally different because the metric in $\mathbb{R}^{n+2}$ is not the $\mathrm{SO}(2,n)$-symmetric one, but a_{IJ}. Furthermore, observe that only the subgroup $\mathrm{SO}(1,1)\times\mathrm{SO}(1,n)\subset\mathrm{SO}(2,n)$ is linearly realized on the h^I coordinates of the Real Special Geometry.

1.3.1 Case $n_V = 0$

If we begin with a six-dimensional theory with an arbitrary number n_T of tensor multiplets and no vector multiplets, we arrive to the model with $n_{V5} = n_T + 1$ characterized by

$$C_{0rs} = \frac{1}{3!}\eta_{rs}\,, \tag{1.58}$$

and with the parametrization

$$h^0 = 2e^{-2\phi^1/\sqrt{6}}\,, \qquad h^r = e^{\phi^1/\sqrt{6}}L^r\,, \tag{1.59}$$

with $L^r = L^r(\phi^2,\cdots,\phi^{n_T+1})$.

The $n_{V5} = n_T + 1$ scalars of these models parametrize the coset $\mathrm{SO}(1,1)\times\mathrm{SO}(1,n_T)/\mathrm{SO}(n_T)$. Upon dimensional reduction one obtains an $ST[2,n_T+1]$ model of $\mathcal{N}=2, d=4$ supergravity coupled to $n_{V4} = n_{V5}+1 = n_T+2$ vector multiplets parametrizing the coset space $\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{SO}(2)}\times\frac{\mathrm{SO}(2,n_T+1)}{\mathrm{SO}(2)\times\mathrm{SO}(n_T+1)}$.

1.3.2 Case $n_T = 1$

Let us start from a six-dimensional theory with $n_T = 1$ and an arbitrary number of vector multiplets n_V and let us choose the coefficients $c_{r\,ij}$ to be

$$c_{0\,ij} = c_{1\,ij} = \delta_{ij}\,, \tag{1.60}$$

which is a particularly simple solution of the constraint eq. (1.14). These theories contain two 2-forms of opposite selfduality that can be combined into a single, unconstrained, 2-form that can be identified with the Kalb-Ramond field, a single scalar that can be identified with the dilaton field and a set of Abelian vector fields. These theories can be obtained by toroidal compactification to 6 dimensions and subsequent truncation of the

[10]These $n+1$ relations need to be suplemented by another one such as, for instance, $X^1/X^0 = h^1$ to have the change of coordinates completely defined.

Heterotic String theory, assuming that the number of Abelian vectors does not exceed 16. We will show later how to rewrite it in the standard form. Now we just want to show that, after dimensional reduction, these theories also belong to the same family as those of the $n_V = 0$ case.

With the above choice of coefficients, the parametrization of $\tilde{h}^i$ is given by[11]

$$\begin{aligned} \tilde{h}^0 &= 2e^{-2\phi/\sqrt{6}}\,, & \tilde{h}^1 &= \tilde{L}^0 e^{\phi/\sqrt{6}} + l^2 e^{-2\phi/\sqrt{6}}\,, \\ \tilde{h}^2 &= \tilde{L}^1 e^{\phi/\sqrt{6}} - l^2 e^{-2\phi/\sqrt{6}}\,, & \tilde{h}^i &= -2e^{-2\phi/\sqrt{6}} l^i\,. \end{aligned} \tag{1.61}$$

These functions satisfy the equation

$$1 = \tilde{C}_{IJK}\tilde{h}^I\tilde{h}^J\tilde{h}^K = \frac{1}{2}\tilde{h}^0\left[\left(\tilde{h}^1\right)^2 - \left(\tilde{h}^2\right)^2\right] - \frac{1}{2}\left(\tilde{h}^1 + \tilde{h}^2\right)\tilde{h}^i\tilde{h}^i\,. \tag{1.62}$$

However, we are free to make linear transformations of the $\tilde{h}^I$ and A^I in order to obtain equivalent theories. In particular, if we perform the transformation $(\tilde{h}^0, \tilde{h}^1, \tilde{h}^2, \tilde{h}^i) \to (h^0, h^r)$, with $r = 1, 2, i + 2$, given by

$$\begin{aligned} \tilde{h}^0 &= h^1 + h^2, \\ \tilde{h}^1 &= \frac{1}{2}\left(h^0 + h^1 - h^2\right), \\ \tilde{h}^2 &= \frac{1}{2}\left(h^0 - h^1 + h^2\right), \\ \tilde{h}^i &= h^{i+2}\,, \end{aligned} \tag{1.63}$$

we find that the new variables satisfy

$$1 = \frac{1}{2}h^0\big((h^1)^2 - (h^2)^2 - h^{i+2}h^{i+2}\big) = \frac{1}{2}h^0 h^r h^s \eta_{rs} \equiv C_{IJK}h^I h^J h^K\,, \tag{1.64}$$

so these models are equivalent to those with $C_{0rs} = \frac{1}{3!}\eta_{rs}$.

We conclude that $\mathcal{N} = (2, 0), d = 6$ supergravity coupled to n_T tensor multiplets gives the same five-dimensional supergravity model as $\mathcal{N} = (2, 0), d = 6$ supergravity coupled to just 1 tensor multiplet and and $n_V = n_T - 1$ vector multiplets. Furthermore, the 5-dimensional theory that one obtains by dimensional reduction of those two 6-dimensional theories can be embedded in Heterotic String theory.

These two 6-dimensional supergravity theories, dimensionally reduced on a circle, are dual in the same sense in which the 10-dimensional $\mathcal{N} = 2A$ and $\mathcal{N} = 2B$ supergravity theories are T-dual [36], a fact related to the T-duality of the type IIA and IIB superstring theories compactified on circles of dual radii [37–39]. Before we can interpret this duality between supergravity theories in the context of superstring theory as a large-small radii or coupling constant duality (for instance) we need to find the dictionary that relates the fields of both 6-dimensional theories. This dictionary will be the analogous of the Buscher rules for T-duality [36, 40–42, 61] and it will allow us to transform any solution of one of these theories admitting one isometry into a solution of the dual theory.

[11] We are going to denote the objects of these theories with tildes.

The initial step to derive this dictionary will be to find out how each solution of the 5-dimensional theory can be oxidized to two different solutions of two different 6-dimensional theories: one which only contains chiral 2-forms and one with a non-chiral 2-form and vector fields.

To simplify the discussions, in what follows we are going to call the 6-dimensional supergravity theories with just one tensor multiplet and n_V vector multiplets and $c_{0\ ij} = c_{1\ ij} = \delta_{ij}$, $\mathcal{N} = 2A$ theories and the dual theories with $n_T = n_V + 1$ tensor multiplets and no vector multiplets, $\mathcal{N} = 2B$ theories.

Now we will focus on the 5-dimensional theories with $n_{V5} = n_V + 2$ vector multiplets which have these two possible 6-dimensional origins.

2 Uplifting solutions to six dimensions

Let us consider the family of $\mathcal{N} = 2, d = 5$ theories coupled to $n_{V5} = n_V + 2$ vector multiplets and symmetric tensor C_{IJK}, $I = 0, \cdots, n_V + 2$ given by $C_{0\,r+1\,s+1} = \frac{1}{3!}\eta_{r+1\,s+1}$, $r, s, \ldots = 0, \cdots, n_V + 1$. The scalar functions h^I can be parametrized in terms of the physical scalars by

$$h^0 = 2e^{-2\phi^1/\sqrt{6}} \qquad h^{r+1} = L^r e^{\phi^1/\sqrt{6}}\,, \tag{2.1}$$

where the functions L^r only depend on the scalars $\phi^2, \cdots, \phi^{n_V+2}$, and satisfy

$$L^r L^s \eta_{rs} = 1\,. \tag{2.2}$$

The action can be written in terms of these functions and the scalar ϕ^1 and takes the form

$$\begin{aligned} S = \int d^5x\sqrt{|g|}\bigg\{ & R + \frac{1}{2}(\partial\phi^1)^2 - \partial_\mu L^r\partial^\mu L_r - \frac{1}{48}e^{4\phi^1/\sqrt{6}}F^0F^0 - \frac{1}{6}e^{-2\phi^1/\sqrt{6}}\mathcal{M}_{rs}F^{r+1}F^{s+1} \\ & + \frac{\epsilon}{24\sqrt{3}\sqrt{|g|}}\eta_{rs}F^{r+1}F^{s+1}A^0\bigg\}\,, \end{aligned} \tag{2.3}$$

where

$$L_r = \eta_{rs}L^s\,, \quad \text{and} \quad \mathcal{M}_{rs} = 2L_rL_s - \eta_{rs}\,. \tag{2.4}$$

For our purposes, though, it is convenient to express everything in terms of the h^I:

$$L^r = h^{r+1}\sqrt{h^0/2}\,, \qquad L_r = h_{r+1}/\sqrt{h^0/2}\,, \qquad \mathcal{M}_{rs} = 4\frac{h_{r+1}h_{s+1}}{h_0} - \eta_{rs}. \tag{2.5}$$

According to our previous discussion, this theory can be uplifted to two different 6-dimensional theories.

2.1 Uplift to $\mathcal{N} = 2B, d = 6$ supergravity

$\mathcal{N} = 2B, d = 6$ supergravity is the name that we have given to the theories of $\mathcal{N} = (2,0), d = 6$ supergravity coupled to $n_T = n_V + 1$ tensor multiplets only. The equations of motion of this theory can be obtained form the pseudoaction

$$\hat{S} = \int d^6\hat{x}\sqrt{|\hat{g}|}\left\{\hat{R} - \partial_{\hat{a}}\hat{L}^r\partial^{\hat{a}}\hat{L}_r + \frac{1}{3}\hat{\mathcal{M}}_{rs}\hat{H}^r{}_{\hat{a}\hat{b}\hat{c}}\hat{H}^{s\ \hat{a}\hat{b}\hat{c}}\right\}\,, \tag{2.6}$$

supplemented by the (anti-) selfduality conditions

$$\hat{\mathcal{M}}_{rs}\hat{H}^r = -\eta_{rs} \star \hat{H}^s \,. \tag{2.7}$$

Then, according to the results in section 1, the 6-dimensional fields of this theory can be expressed in terms of those of the 5-dimensional theory eq. (2.3) as follows:

Scalars. The physical scalars $\hat{\varphi}^{\underline{\alpha}}$, and the functions $\hat{L}^r$, with $\underline{\alpha} = 1, \cdots, n_V + 1$ and $r = 0, \cdots, n_V + 1$, are given by

$$\begin{aligned} \hat{\varphi}^{\underline{\alpha}} &= \phi^{\underline{\alpha}+1} \,, \\ \hat{L}^r(\varphi^{\underline{\alpha}}) &= h^{r+1}(h^0/2)^{1/2} \,. \end{aligned} \tag{2.8}$$

Metric. The 6-dimensional metric components are the following

$$\begin{aligned} \hat{g}_{\underline{zz}} &= -\left(h^0/2\right)^{-3/2} \,, \\ \hat{g}_{\mu\underline{z}} &= -\frac{1}{\sqrt{12}}\left(h^0/2\right)^{-3/2} A^0{}_\mu \,, \\ \hat{g}_{\mu\nu} &= (h^0/2)^{1/2} g_{\mu\nu} - \frac{1}{12}\left(h^0/2\right)^{-3/2} A^0{}_\mu A^0{}_\nu \,, \end{aligned} \tag{2.9}$$

or, equivalently

$$d\hat{s}^2 = -(h^0/2)^{-3/2}\left[dz + \frac{1}{\sqrt{12}}A^0\right]^2 + (h^0/2)^{1/2} ds^2 \,. \tag{2.10}$$

2-forms. We only need to know the component $\hat{B}^r{}_{\mu\underline{z}}$ of the 2-forms, because the rest of components are determined through the duality relations eqs. (2.7). We have

$$\hat{B}^r{}_{\mu\underline{z}} = \frac{1}{\sqrt{12}} A^{r+1}{}_\mu \,. \tag{2.11}$$

It can also be useful to have the expression of the 3-form field strengths in the Vielbein basis:

$$\begin{aligned} \hat{H}^r{}_{abz} &= \frac{1}{\sqrt{12}}(h^0/2)^2 F^{r+1}{}_{ab} \,, \\ \hat{H}^r{}_{abc} &= -\frac{1}{2\sqrt{12}}(h^0/2)\mathcal{M}^r{}_s \epsilon_{abcde} F^{s+1\;de} \,, \end{aligned} \tag{2.12}$$

where one has to take into account that $F^{s+1\;de}$ and ϵ_{abcde} are five-dimensional quantities.

2.2 Uplift to $\mathcal{N} = 2A, d = 6$ supergravity

$\mathcal{N} = 2A, d = 6$ supergravity is the name that we have given to the theories of $\mathcal{N} = (2,0), d = 6$ supergravity coupled to $n_T = 1$ tensor multiplets and n_V vector multiplets with $c_{0\;ij} = c_{1\;ij} = \delta_{ij}$ with $i = 1, \cdots, n_V$. Since in this case the two 2-forms have opposite chirality, they can be combined into a single, unrestricted, 2-form that we are going to denote by $\tilde{B}$ (no indices) and there is a covariant action from which one can derive directly the equations of motion. It takes the form

$$\tilde{S} = \int d^6\tilde{x}\sqrt{|\tilde{g}|}\left\{\tilde{R} + \frac{1}{2}(\partial\tilde{\varphi})^2 + \frac{1}{3}e^{\sqrt{2}\tilde{\varphi}}\tilde{H}^2 - e^{\tilde{\varphi}/\sqrt{2}}\tilde{F}^i\tilde{F}^i\right\}, \tag{2.13}$$

where now we are using tildes instead of hats in order to distinguish these fields from the previous ones and from the 5-dimensional ones. In this action, $i = 1, \cdots, n_V$ and the 3-form field strength is defined as

$$\tilde{H} = d\tilde{B} + \tilde{F}^i \wedge \tilde{A}^i\,. \tag{2.14}$$

This theory is obtained when we parametrize the functions $\tilde{L}^r$, $r = 0, 1$ as

$$\tilde{L}^0 = \cosh\left(\tilde{\varphi}/\sqrt{2}\right), \qquad \tilde{L}^1 = \sinh\left(\tilde{\varphi}/\sqrt{2}\right), \tag{2.15}$$

and $\tilde{H}$ and $\tilde{B}$ are related to the fields $\tilde{H}^r$ and $\tilde{B}^r$ (which appear in (1.13)) by

$$\tilde{B} = \tilde{B}^0 - \tilde{B}^1\,, \qquad \tilde{H} = \tilde{H}^0 - \tilde{H}^1\,. \tag{2.16}$$

This theory can be obtained from the compactification of $\mathcal{N} = 1, d = 10$ supergravity coupled to vector multiplets (the effective field theory of the Heterotic String) on T^4 followed by a truncation. In particular, the scalar $\tilde{\varphi}$ is related to the dilaton field of the Heterotic String by

$$\tilde{\varphi} = -\sqrt{2}\,\phi_{\text{Het}}\,. \tag{2.17}$$

Now, as we have seen, this theory, also gives (2.3) when reduced to five dimensions. In order to find the relations among the fields, we have to use the linear transformation (1.63). This gives us directly the transformation of vector fields. Also, on taking into account the parametrizations (2.1) and (1.61) we get the relation between the different scalar fields. This leads to the following expressions for the 6-dimensional fields in terms of the 5-dimensional ones:

Scalar. The dilaton is related to the five-dimensional scalars by

$$e^{\tilde{\varphi}/\sqrt{2}} = 2^{-1/2}h^0\left(h^1 + h^2\right)^{1/2}. \tag{2.18}$$

Metric. The KK scalar ϕ and the KK vector A_μ are given by

$$e^{-2\phi/\sqrt{6}} = \frac{1}{2}\left(h^1 + h^2\right), \qquad A_\mu = \frac{1}{\sqrt{12}}\left(A^1_\mu + A^2_\mu\right), \tag{2.19}$$

and, therefore, the metric is given by

$$\begin{aligned}
\tilde{g}_{\underline{zz}} &= -2^{3/2}\left(h^1 + h^2\right)^{-3/2}, \\
\tilde{g}_{\mu\underline{z}} &= -\sqrt{2/3}\left(h^1 + h^2\right)^{-3/2}\left({A^1}_\mu + {A^2}_\mu\right), \\
\tilde{g}_{\mu\nu} &= \frac{1}{\sqrt{2}}\left(h^1 + h^2\right)^{1/2} g_{\mu\nu} - \frac{1}{3\sqrt{2}}\left(h^1 + h^2\right)^{-3/2}\left(A^1 + A^2\right)_\mu\left(A^1 + A^2\right)_\nu
\end{aligned} \tag{2.20}$$

or, equivalently, by

$$d\tilde{s}^2 = -2^{3/2}\left(h^1 + h^2\right)^{-3/2}\left[dz + \frac{1}{\sqrt{12}}\left(A^1 + A^2\right)\right]^2 + 2^{-1/2}\left(h^1 + h^2\right)^{1/2} ds^2\,. \tag{2.21}$$

Vectors. The 1-form potentials are given by

$$\begin{aligned}\tilde{A}^{i}{}_{\underline{z}} &= -\frac{h^{i+2}}{h^1+h^2}\,,\\ \tilde{A}^{i}{}_{\mu} &= \frac{1}{\sqrt{12}}\left[A^{i+2}{}_{\mu}+\tilde{A}^{i}{}_{\underline{z}}\left(A^1{}_{\mu}+A^2{}_{\mu}\right)\right]\,,\end{aligned}\tag{2.22}$$

or, equivalently, by

$$\tilde{A}^{i} = \frac{1}{\sqrt{12}}A^{i+2} - \frac{h^{i+2}}{h^1+h^2}\left[dz+\frac{1}{\sqrt{12}}\left(A^1+A^2\right)\right]\,.\tag{2.23}$$

2-form. The components $\tilde{B}_{\mu\underline{z}}$ can be easily found to be

$$\tilde{B}_{\mu\underline{z}} = \frac{1}{\sqrt{12}}\left(A^1{}_{\mu}-A^2{}_{\mu}\right)\,.\tag{2.24}$$

Now the components $\tilde{B}_{\mu\nu}$ are independent and have to be explicitly given. They do not have a simple expression, though, and we must content ourselves with the field strength components instead:

$$\begin{aligned}\tilde{H}_{\mu\nu\underline{z}} &= \frac{1}{\sqrt{3}}\left(h^1+h^2\right)^{-1}\left\{\left[h^1-\left[h^0\left(h^1+h^2\right)\right]^{-1}\right]F^1{}_{\mu\nu}\right.\\ &\quad\left.-\left[h^2+\left[h^0\left(h^1+h^2\right)\right]^{-1}\right]F^2{}_{\mu\nu}+h^iF^i{}_{\mu\nu}\right\}, \qquad i\geq 3\,,\\ \tilde{H}_{\mu\nu\rho} &= -\frac{1}{4\sqrt{3}}(h^0)^{-2}\frac{\epsilon_{\mu\nu\rho\alpha\beta}}{\sqrt{|g|}}F^{0\,\alpha\beta}+\frac{\sqrt{3}}{2}(A^1{}_{[\rho}+A^2{}_{[\rho})\tilde{H}_{\mu\nu]\underline{z}}\,.\end{aligned}\tag{2.25}$$

2.3 Uplift to $\mathcal{N}=2A^*, d=6$ supergravity

The theory that we have called $\mathcal{N}=2A, d=6$ supergravity is not uniquely defined. One can obtain another theory that we are going to call $\mathcal{N}=2A^*, d=6$ supergravity by dualizing the field strength $\tilde{H}$ into another field strength $\breve{H}$ given by[12]

$$\breve{H} = -e^{\sqrt{2}\tilde{\varphi}}\star\tilde{H}\,.\tag{2.26}$$

It turns out that this new field strength is an exact 3-form:

$$\breve{H} = d\breve{B}\,,\tag{2.27}$$

and $\breve{H}$ and $\breve{B}$ are related to $\hat{H}^r$ and $\hat{B}^r$ in the theory of eq. (1.13) with $n_T=1$, arbitrary n_V and $c_{0\,ij}=c_{1\,ij}=\delta_{ij}$ by[13]

$$\breve{H} = \hat{H}^0+\hat{H}^1\,,\qquad \breve{B} = \hat{B}^0+\hat{B}^1\,.\tag{2.28}$$

The action for this theory is

$$\breve{S} = \int d^6\breve{x}\sqrt{|\breve{g}|}\left\{\breve{R}+\frac{1}{2}(\partial\breve{\varphi})^2+\frac{1}{3}e^{-\sqrt{2}\breve{\varphi}}\breve{H}^2-e^{\breve{\varphi}/\sqrt{2}}\breve{F}^i\breve{F}^i-\frac{\epsilon}{3\sqrt{|g|}}\breve{H}\breve{F}^i\breve{A}^i\right\}.\tag{2.29}$$

[12] In the Einstein frame this is the only field which is modified in this transformation. We will denote all the field of this theory with $\breve{}$ accents anyway.

[13] Observe that the absence of Chern-Simons term in $\breve{H}$ is due to the cancellation of those in $\hat{H}^0$ and $\hat{H}^1$ and not to the vanishing of the constants $c^r{}_{ij}$.

This theory can be obtained from the effective field theory of the type IIA superstrings compactified on K3 [39, 62–65] followed by a truncation. In particular, the scalar $\breve{\varphi}$ (which is equal to $\tilde{\varphi}$), is related to the dilaton of that superstring theory by

$$\breve{\varphi} = \sqrt{2}\,\phi_{IIA}\,. \tag{2.30}$$

The different coupling of the dilaton field to the vector fields (comparing with the $\mathcal{N} = 2A$ case) is mainly due to the fact that they are RR fields in this case instead of NSNS fields.

All the fields have the same relation with the five-dimensional ones as the tilded ones, except for the 2-form $\breve{B}$, whose components $\mu\underline{z}$ now are given by

$$\breve{B}_{\mu\underline{z}} = \frac{1}{\sqrt{12}} A^{0}{}_{\mu}\,. \tag{2.31}$$

The 3-form field strength is given by

$$\begin{aligned}
\breve{H}_{\mu\nu\underline{z}} =& \frac{1}{\sqrt{12}} F^{0}{}_{\mu\nu}\,,\\
\breve{H}_{\mu\nu\rho} =& -\frac{1}{8\sqrt{3}}(h^0)^2(h^1+h^2)\frac{\epsilon_{\mu\nu\rho\alpha\beta}}{\sqrt{|g|}}\Big\{\left[h^1-\left[h^0\left(h^1+h^2\right)\right]^{-1}\right]F^{1\,\alpha\beta}\\
&-\left[h^2+\left[h^0\left(h^1+h^2\right)\right]^{-1}\right]F^{2\,\alpha\beta}+h^iF^{i\,\alpha\beta}\Big\}+\frac{\sqrt{3}}{2}\left(A^1{}_{[\rho}+A^2{}_{[\rho}\right)\breve{H}_{\mu\nu]\underline{z}}\,,\qquad i\geq 3\,.
\end{aligned} \tag{2.32}$$

3 Maps between six-dimensional theories

Putting together all our results we can write the following generalization of the Buscher rules between the $\mathcal{N} = 2A, 2A^*$ and $2B$ theories:

From $\mathcal{N} = 2B$ to $\mathcal{N} = 2A$.

$$\begin{aligned}
e^{\sqrt{2}\tilde{\varphi}} =& -2\left(\hat{L}^0+\hat{L}^1\right)/\hat{g}_{\underline{zz}}\,,\\
\tilde{g}_{\underline{zz}} =& -2^{3/2}\left(\hat{L}^0+\hat{L}^1\right)^{-3/2}|\hat{g}_{\underline{zz}}|^{-1/2}\,,\\
\tilde{g}_{\mu\underline{z}} =& -2^{3/2}\left(\hat{L}^0+\hat{L}^1\right)^{-3/2}|\hat{g}_{\underline{zz}}|^{-1/2}\left(\hat{B}^0+\hat{B}^1\right)_{\mu\underline{z}}\,,\\
\tilde{g}_{\mu\nu} =& 2^{-1/2}\left(\hat{L}^0+\hat{L}^1\right)^{1/2}\left[|\hat{g}_{\underline{zz}}|^{1/2}\hat{g}_{\mu\nu}+|\hat{g}_{\underline{zz}}|^{-1/2}\hat{g}_{\mu\underline{z}}\hat{g}_{\nu\underline{z}}\right]\\
&-2^{3/2}\left(\hat{L}^0+\hat{L}^1\right)^{-3/2}|\hat{g}_{\underline{zz}}|^{-1/2}\left(\hat{B}^0+\hat{B}^1\right)_{\mu\underline{z}}\left(\hat{B}^0+\hat{B}^1\right)_{\nu\underline{z}}\,,\\
\tilde{A}^i{}_{\underline{z}} =& -\hat{L}^{i+1}/\left(\hat{L}^0+\hat{L}^1\right)\,,\\
\tilde{A}^i{}_{\mu} =& \hat{B}^{i+1}{}_{\mu\underline{z}}-\hat{L}^{i+1}\left(\hat{B}^0+\hat{B}^1\right)_{\mu\underline{z}}/\left(\hat{L}^0+\hat{L}^1\right)\,,\\
\tilde{B}_{\mu\underline{z}} =& \left(\hat{B}^0-\hat{B}^1\right)_{\mu\underline{z}}\,.
\end{aligned} \tag{3.1}$$

From $\mathcal{N}=2A$ to $\mathcal{N}=2B$.

$$
\begin{aligned}
\left|\hat{g}_{\underline{zz}}\right| &= 2^{\frac{3}{2}} e^{-\frac{3}{2\sqrt{2}}\tilde{\varphi}} \left|\tilde{g}_{\underline{zz}}\right|^{-\frac{1}{2}} \,, \\
\hat{g}_{\mu\underline{z}} &= -2^{\frac{3}{2}} e^{-\frac{3}{2\sqrt{2}}\tilde{\varphi}} \left|\tilde{g}_{\underline{zz}}\right|^{-\frac{1}{2}} \left(\tilde{B}^0+\tilde{B}^1\right)_{\mu\underline{z}} \,, \\
\hat{g}_{\mu\nu} &= 2^{-\frac{1}{2}} \left|\tilde{g}_{\underline{zz}}\right|^{\frac{1}{2}} e^{\frac{\tilde{\varphi}}{2\sqrt{2}}} \left(\tilde{g}_{\mu\nu} - \tilde{g}_{\mu\underline{z}}\tilde{g}_{\nu\underline{z}}/\tilde{g}_{\underline{zz}}\right) + 2^{\frac{3}{2}} e^{-\frac{3}{2\sqrt{2}}\tilde{\varphi}} \left|\tilde{g}_{\underline{zz}}\right|^{-\frac{1}{2}} \left(\tilde{B}^0+\tilde{B}^1\right)_{\mu\underline{z}} \left(\tilde{B}^0+\tilde{B}^1\right)_{\nu\underline{z}} \,, \\
\hat{L}^0 &= 2^{-\frac{3}{2}} e^{-\frac{\tilde{\varphi}}{2\sqrt{2}}} \left|\tilde{g}_{\underline{zz}}\right|^{\frac{1}{2}} + 2^{-\frac{1}{2}} e^{\frac{\tilde{\varphi}}{2\sqrt{2}}} \left|\tilde{g}_{\underline{zz}}\right|^{-\frac{1}{2}} \left(1+\tilde{A}^r{}_{\underline{z}}\tilde{A}^r{}_{\underline{z}}\right) \,, \qquad r>1\,, \\
\hat{L}^1 &= -2^{-\frac{3}{2}} e^{-\frac{\tilde{\varphi}}{2\sqrt{2}}} \left|\tilde{g}_{\underline{zz}}\right|^{\frac{1}{2}} + 2^{-\frac{1}{2}} e^{\frac{\tilde{\varphi}}{2\sqrt{2}}} \left|\tilde{g}_{\underline{zz}}\right|^{-\frac{1}{2}} \left(1-\tilde{A}^r{}_{\underline{z}}\tilde{A}^r{}_{\underline{z}}\right) \,, \qquad r>1\,, \\
\hat{L}^r &= -\sqrt{2}\left|\tilde{g}_{\underline{zz}}\right|^{-\frac{1}{2}} e^{\frac{\tilde{\varphi}}{2\sqrt{2}}} \tilde{A}^{r-1}{}_{\underline{z}} \,, \qquad r\geq 2\,, \\
\hat{B}^0{}_{\mu\underline{z}} &= \frac{1}{2}\left(\tilde{B}_{\mu\underline{z}} + \tilde{g}_{\mu\underline{z}}/\tilde{g}_{\underline{zz}}\right) \,, \\
\hat{B}^1{}_{\mu\underline{z}} &= \frac{1}{2}\left(-\tilde{B}_{\mu\underline{z}} + \tilde{g}_{\mu\underline{z}}/\tilde{g}_{\underline{zz}}\right) \,, \\
\hat{B}^r{}_{\mu\underline{z}} &= \tilde{A}^{r-1}{}_{\mu} - \tilde{A}^{r-1}{}_{\underline{z}}\,\tilde{g}_{\mu\underline{z}}/\tilde{g}_{\underline{zz}} \,, \qquad r\geq 2\,.
\end{aligned}
\tag{3.2}
$$

3.1 From $\mathcal{N}=2B$ to $\mathcal{N}=2A^*$

$$
\begin{aligned}
e^{\sqrt{2}\breve{\varphi}} =&\; -2\left(\hat{L}^0+\hat{L}^1\right)/\hat{g}_{\underline{zz}}\,, \\
\breve{g}_{\underline{zz}} =&\; -2^{3/2}\left(\hat{L}^0+\hat{L}^1\right)^{-3/2}|\hat{g}_{\underline{zz}}|^{-1/2}\,, \\
\breve{g}_{\mu\underline{z}} =&\; -2^{3/2}\left(\hat{L}^0+\hat{L}^1\right)^{-3/2}|\hat{g}_{\underline{zz}}|^{-1/2}\left(\hat{B}^0+\hat{B}^1\right)_{\mu\underline{z}}\,, \\
\breve{g}_{\mu\nu} =&\; 2^{-1/2}\left(\hat{L}^0+\hat{L}^1\right)^{1/2}\left[|\hat{g}_{\underline{zz}}|^{1/2}\hat{g}_{\mu\nu} + |\hat{g}_{\underline{zz}}|^{-1/2}\hat{g}_{\mu\underline{z}}\hat{g}_{\nu\underline{z}}\right] \\
&\; -2^{3/2}\left(\hat{L}^0+\hat{L}^1\right)^{-3/2}|\hat{g}_{\underline{zz}}|^{-1/2}\left(\hat{B}^0+\hat{B}^1\right)_{\mu\underline{z}}\left(\hat{B}^0+\hat{B}^1\right)_{\nu\underline{z}}\,, \\
\breve{A}^i{}_{\underline{z}} =&\; -\hat{L}^{i+1}/\left(\hat{L}^0+\hat{L}^1\right)\,, \\
\breve{A}^i{}_{\mu} =&\; \hat{B}^{i+1}{}_{\mu\underline{z}} - \hat{L}^{i+1}\left(\hat{B}^0+\hat{B}^1\right)_{\mu\underline{z}}/\left(\hat{L}^0+\hat{L}^1\right)\,, \\
\breve{B}_{\mu\underline{z}} =&\; \hat{g}_{\mu\underline{z}}/\hat{g}_{\underline{zz}}\,.
\end{aligned}
\tag{3.3}
$$

From $\mathcal{N}=2A^*$ to $\mathcal{N}=2B$.

$$\begin{aligned}
\left|\hat{g}_{\underline{zz}}\right| &= 2^{\frac{3}{2}} e^{-\frac{3}{2\sqrt{2}}\breve{\varphi}} \left|\breve{g}_{\underline{zz}}\right|^{-\frac{1}{2}} , \\
\hat{g}_{\mu\underline{z}} &= -2^{\frac{3}{2}} e^{-\frac{3}{2\sqrt{2}}\breve{\varphi}} \left|\breve{g}_{\underline{zz}}\right|^{-\frac{1}{2}} \breve{B}_{\mu\underline{z}} , \\
\hat{g}_{\mu\nu} &= 2^{-\frac{1}{2}} \left|\breve{g}_{\underline{zz}}\right|^{\frac{1}{2}} e^{\frac{\breve{\varphi}}{2\sqrt{2}}} \left(\breve{g}_{\mu\nu} - \breve{g}_{\mu\underline{z}}\breve{g}_{\nu\underline{z}}/\breve{g}_{\underline{zz}}\right) + 2^{\frac{3}{2}} e^{-\frac{3}{2\sqrt{2}}\breve{\varphi}} \left|\breve{g}_{\underline{zz}}\right|^{-\frac{1}{2}} \breve{B}_{\mu\underline{z}}\breve{B}_{\nu\underline{z}} , \\
\hat{L}^0 &= 2^{-\frac{3}{2}} e^{-\frac{\breve{\varphi}}{2\sqrt{2}}} \left|\breve{g}_{\underline{zz}}\right|^{\frac{1}{2}} + 2^{-\frac{1}{2}} e^{\frac{\breve{\varphi}}{2\sqrt{2}}} \left|\breve{g}_{\underline{zz}}\right|^{-\frac{1}{2}} \left(1 + \breve{A}^r{}_{\underline{z}}\breve{A}^r{}_{\underline{z}}\right) , \qquad r>1 , \\
\hat{L}^1 &= -2^{-\frac{3}{2}} e^{-\frac{\breve{\varphi}}{2\sqrt{2}}} \left|\breve{g}_{\underline{zz}}\right|^{\frac{1}{2}} + 2^{-\frac{1}{2}} e^{\frac{\breve{\varphi}}{2\sqrt{2}}} \left|\breve{g}_{\underline{zz}}\right|^{-\frac{1}{2}} \left(1 - \breve{A}^r{}_{\underline{z}}\breve{A}^r{}_{\underline{z}}\right) , \qquad r>1 , \\
\hat{L}^r &= -\sqrt{2}\left|\breve{g}_{\underline{zz}}\right|^{-\frac{1}{2}} e^{\frac{\breve{\varphi}}{2\sqrt{2}}} \breve{A}^{r-1}_{\underline{z}} , \qquad r\geq 2 , \\
\hat{B}^0{}_{\mu\underline{z}} &= \frac{1}{2}\left(\tilde{B}_{\mu\underline{z}} + \breve{g}_{\mu\underline{z}}/\breve{g}_{\underline{zz}}\right) , \\
\hat{B}^1{}_{\mu\underline{z}} &= \frac{1}{2}\left(-\tilde{B}_{\mu\underline{z}} + \breve{g}_{\mu\underline{z}}/\breve{g}_{\underline{zz}}\right) , \\
\hat{B}^r{}_{\mu\underline{z}} &= \breve{A}^{r-1}{}_{\mu} - \breve{A}^{r-1}{}_{\underline{z}}\breve{g}_{\mu\underline{z}}/\breve{g}_{\underline{zz}} , \qquad r\geq 2 .
\end{aligned} \tag{3.4}$$

4 Applications

We are now ready to exploit the relations between 5- and 6-dimensional theories that we have uncovered. There is one more twist that we can add to them, though: observe that if we had dimensionally reduced the *gauged* $\mathcal{N}=2A, d=6$ theory we would have obtained a gauged $\mathcal{N}=2, d=5$ supergravity theory and the relation between the physical fields of these two gauged theories would be exactly the same we have obtained in the ungauged case. This is true as long as the gauge group does not change in the process of dimensional reduction (as in the case of generalized dimensional reduction [59]). Then, we can use the formulae we have obtained to uplift solutions of the 5-dimensional gauged theories to solutions of the 6-dimensional gauged theories and vice-versa.

There are some points to be discussed and clarified before carrying out this program.

First of all we must discuss the possible gaugings of these theories. The $\mathcal{N}=2A, d=6$ theories can be gauged in essentially two ways:

1. We could just gauge a subgroup of the $\mathrm{SO}(n_V)$ group that rotates the vector fields among themselves. The only fermion fields this global symmetry acts on are the gaugini, which carry the same indices as the vector fields and an $\mathrm{Sp}(1)\sim\mathrm{SU}(2)$ R-symmetry index which remains inert under these transformations. Observe that the only scalar of the theory, the dilaton, is also inert.

2. We can gauge the whole R-symmetry group, SO(3) or a SO(2) subgroup of it using Fayet-Iliopoulos terms.[14] Observe that one needs vectors transforming in the same fashion. Thus, in this case one would be gauging SO(3) or a SO(2) subgroup of $\mathrm{SO}(n_V)$ which, on top of acting on some the $\mathrm{SO}(n_V)$ indices of the vectors and gaugini, would also act on the R-symmetry indices of all the fermions of the theory, which would now be charged.

[14] This is the theory considered in ref. [33], for instance.

The dimensional reduction of these gauged 6-dimensional theories would be the models of $\mathcal{N}=1, d=5$ supergravity that we have found, characterized by the C_{IJK} tensor with non-vanishing indices $C_{0rs} = \frac{1}{3!}\eta_{rs}$, with exactly the same kind of gaugings (with our without Fayet-Iliopoulos terms). The main difference with the 6-dimensional theories is that, in the non-Abelian case, the gauge group acts on the scalars that originate in the 6th component of the 6-dimensional vector fields and these transformations are isometries of the σ-model metric. The relations between 5- and 6-dimensional fields can be used directly in the gauged case but we must take into account that in order to get the C_{IJK} tensor in the form $C_{0rs} = \frac{1}{3!}\eta_{rs}$ we had to make linear combinations of several different vector fields. This can only be done if they have the same transformation properties under the group to be gauged, which is not the case. Thus, we only must gauge vector fields not involved in these redefinitions.

The $\mathcal{N}=2B, d=6$ theories cannot be gauged, at least in a conventional way. However, it is believed that there are 6-dimensional gauge theories based on chiral 2-forms associated to coincident M5-branes. The main reason is that, when compactified on a circle, M5-branes behave as D4-branes and the Born-Infeld fields of coincident D4-branes are non-Abelian. This means that, at least, the non-Abelian theory of 2-forms exists when one of the 6 dimensions is compactified on a circle and, in those conditions, the massless modes are essentially non-Abelian 1-forms. Actually, there have been several proposals of non-Abelian theories of 2-forms in 6 dimensions [43–45] and, in general, they consider that one of the 6 dimensions is compactified.

The situation we are facing here is similar and, probably, directly related to the worldvolume theories of the M5-branes. It is clear that, when these theories are compactified on a circle, at least the massless part of the spectrum (1-forms in $d=5$) can be gauged. We do not know how to formulate the gauging using chiral 2-forms directly in 6 uncompactified dimensions but we do know that, at lowest order, the relation between the 6- and 5-dimensional non-Abelian fields is the same as between the Abelian ones. We can, therefore, use the uplifting formulae to construct non-Abelian solutions of a "SO(3)-gauged" $\mathcal{N}=2B, d=6$ theory whose exact 6-dimensional formulation we do not know. Actually, we can use this relation as a lowest-order formulation of that theory which probably only exists anyway when one of the 6 dimensions is compactified on a circle.

4.1 Solutions of the SO(3)-gauged $\mathcal{N}=2A^*, d=6$ theory

The supersymmetric solutions of the gauged $\mathcal{N}=2A, d=6$ theory with Fayet-Iliopoulos (FI) terms were classified in ref. [33], where some interesting examples were also constructed. We can dimensionally reduce them to 5 dimensions using our results but we prefer to construct supersymmetric solutions of the SO(3)-gauged $\mathcal{N}=2A, d=6$ theory without FI terms by uplifting some of the supersymmetric solutions of the similarly gauged (no FI terms) $\mathcal{N}=2, d=5$ supergravity with no hypermultiplets[15] recently constructed in

[15]These theories are the simplest supersymmetrization of the Einstein-Yang-Mills (EYM) theory and have been called $\mathcal{N}=2, d=5$ Super-Einstein-Yang-Mills (SEYM) theories in ref. [46]. They are related by dimensional reduction to the $\mathcal{N}=2, d=4$ SEYM theories [14, 48–52]. The same relation applies to the 4- and 5-dimensional solutions.

ref. [46]. In particular, we are going to uplift an extremal black hole sourced by a BPST instanton [66].

Thus, let us consider the $\mathcal{N}=2, d=5$ SEYM theory with $n_{V5}=5$ vectors labeled by $x=1,\cdots,5$ or $x=1,2,A$ where $A,B,\ldots$ label the three directions gauged with the group SO(3) and with non-vanishing components of C_{IJK} given by $C_{0xy}=\frac{1}{3!}\eta_{xy}$, $\eta=\mathrm{diag}(+-----)$. The solution that we are going to uplift was obtained in a model with one vector multiplet less but, here, for the reasons explained above, we cannot gauge the first vector multiplets and so we add one more ($x=2$) whose fields will vanish identically.

The metric is static and spherically symmetric

$$ds^2 = f^2 dt^2 - f^{-1}\left(d\rho^2 + \rho^2 d\Omega^2_{(3)}\right)\,, \tag{4.1}$$

where the metric function f is given by

$$f^{-1} = 3\cdot 2^{-1/3}\left\{L_1^2\left[L_0 - \frac{9}{2g^2}\left(\rho+\frac{\lambda^2}{4}\rho^3\right)^{-2}\right]\right\}^{1/3}\,, \tag{4.2}$$

where L_0 and L_1 are two spherically symmetric harmonic functions[16] on $\mathbb{R}^4$

$$L_{0,1} = a_{0,1} + q_{0,1}/\rho^2\,, \tag{4.3}$$

$a_{0,1}$ being integration constants and $q_{0,1}$ being electric charges. The integration constants are constrained by the normalization of the metric at infinity, but we are are not going to impose this condition in 5 dimensions.

There is only one non-trivial scalar that we can write as h^1/h^0, for instance. In terms of the scalar functions h^I we have

$$h^0 = 2^{-1/3}\left[\frac{L_1}{L_0 - \frac{9}{2g^2}\left(\rho+\frac{\lambda^2}{4}\rho^3\right)^{-2}}\right]^{2/3}\,, \tag{4.4}$$

$$h^1 = 2^{2/3}\left[\frac{L_1}{L_0 - \frac{9}{2g^2}\left(\rho+\frac{\lambda^2}{4}\rho^3\right)^{-2}}\right]^{-1/3}\,, \tag{4.5}$$

$$h^2 = h^A = 0\,, \tag{4.6}$$

and

$$\phi^1 = 2\frac{L_0 - \frac{9}{2g^2}\left(\rho+\frac{\lambda^2}{4}\rho^3\right)^{-2}}{L_1}\,. \tag{4.7}$$

[16] Not to be confused with the 6-dimensional scalar functions $\hat{L}_r$.

Finally, the vector fields of the solution are given by

$$
\begin{aligned}
A^0 &= -\frac{1}{\sqrt{3}}\left[L_0 - \frac{9}{2g^2}\left(\rho + \frac{\lambda^2}{4}\rho^3\right)^{-2}\right]^{-1} dt\,, \\
A^1 &= -\frac{2}{\sqrt{3}}L_1^{-1}\, dt\,, \\
A^2 &= 0\,, \\
A^A &= -\frac{1}{g}\left(1 + \frac{\lambda^2}{4}\rho^2\right)^{-1} v^A{}_L\,,
\end{aligned}
\tag{4.8}
$$

where the $v^A{}_L$ are the left-invariant Maurer-Cartan 1-forms of the Lie group SU(2), given in our conventions in the appendix of ref. [52]. A^A is the potential of the BPST instanton and g is the 5-dimensional gauge coupling constant.

It is now straightforward to uplift this solution to a solution of the $\mathcal{N} = 2A, d = 6$ theory with $n_T = 1$ (by definition) and $n_V = n_{V5} - 2 = 3$ (one of the six 5-dimensional vectors is the KK vector and the other two come from the non-chiral 2-form) and the 3 vectors are the gauge field of the SO(3) gauge group[17]

Using eqs. (2.18), (2.21), (2.23) and (2.24), we find the following 6-dimensional fields:[18]

$$
\begin{aligned}
d\breve{s}^2 &= 2\breve{f}du\left[dv' - \frac{3}{2}(L_1 - a_1)du\right] - \breve{f}^{-1}\left(d\rho^2 + \rho^2 d\Omega^2_{(3)}\right)\,, \\
\breve{f} &= \frac{\sqrt{2}}{3}\left\{L_1\left[L_0 - \frac{2}{9g^2}\left(\rho + \frac{\lambda^2}{4}\rho^3\right)^{-2}\right]\right\}^{-1/2}\,, \\
e^{\sqrt{2}\breve{\varphi}} &= \frac{1}{2}L_1\left[L_0 - \frac{2}{9g^2}\left(\rho + \frac{\lambda^2}{4}\rho^3\right)^{-2}\right]^{-1}\,, \\
\breve{A}^A &= -\frac{1}{\sqrt{12}g}\left(1 + \frac{\lambda^2}{4}\rho^2\right)^{-1} v^A{}_L\,, \\
\breve{H} &= -\frac{1}{6}dv' \wedge du \wedge d\left[L_0 - \frac{2}{9g^2}\left(\rho + \frac{\lambda^2}{4}\rho^3\right)^{-2}\right]^{-1} + \frac{3}{2}q_1\omega_3\,,
\end{aligned}
\tag{4.10}
$$

and where ω_3 is the volume form of the round 3-sphere of unit radius whose metric is $d\Omega^2_{(3)}$. If, for instance, we use the Euler coordinates (θ, ϕ, ψ) such that

$$
d\Omega^2_{(3)} = \frac{1}{4}\left[(d\psi + \cos\theta d\phi)^2 + d\theta^2 + \sin^2\theta d\phi^2\right]\,, \tag{4.11}
$$

[17] Globally, the instanton solution requires the group to be SU(2).

[18] We have renamed the coordinates z and t as u and v, respectively, since they are conjugate null coordinates in 6 dimensions. Then, we have shifted one of them $v = v' + \frac{3}{2}a_1 u$. The null coordinates u and v' can be expressed in terms of time (τ) and space (y) coordinate as

$$
u = \frac{1}{\sqrt{2}}(\tau + y)\,, \qquad v = \frac{1}{\sqrt{2}}(\tau - y)\,. \tag{4.9}
$$

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then $\omega_3 = \frac{1}{8}\sin\theta d\theta \wedge d\phi \wedge d\psi$, and the 2-form $\breve{B}$ can be written in this coordinate patch, up to gauge transformations, as

$$\breve{B} = -\frac{1}{6}\left[L_0 - \frac{2}{9g^2}\left(\rho + \frac{\lambda^2}{4}\rho^3\right)^{-2}\right]^{-1} dv' \wedge du + \frac{3}{16}q_1 \cos\theta d\psi \wedge d\phi\,. \tag{4.12}$$

Observe that now $\breve{A}^A$ carries a factor of $1/\sqrt{12}$ with respect to the potential of the BPST instanton. The reason behind this apparent inconsistency is that the rescaling of the potentials is harmless in the Abelian case but brings the non-Abelian 2-form field strength to an unconventional form. To bring it back to the standard form we just have to rescale the coupling constant. Thus, the 6-dimensional coupling constant is given in terms of the 5-dimensional one by

$$\breve{g} = \sqrt{12}g\,. \tag{4.13}$$

The metric $d\breve{s}^2$ is typical that of a superposition of a string lying in the z direction and a wave with momentum $\sim q_1$ in the same direction. The 3-form field strength $\breve{H}$ indicates that the string is dyonic, with electric and magnetic charges $\sim q_0, q_1$. This kind of solutions are very well known as they are particular cases of 3-charge configurations dual to the D1D6W one.[19] The additional ingredient here is the BPST instanton that modifies the metric function $\breve{f}$. The string part of this solution is also clearly related to the "gauge dyonic string" solution of the Heterotic string effective action compactified to 6 dimensions constructed in ref. [67] by adding Yang-Mills instantons in the transverse directions to the dyonic string found in ref. [68] (see also ref. [69]).

We have left intentionally undetermined the integration constants a_0, a_1 because different choices can leave, as we are going to see, to physically inequivalent solutions, depending on whether we demand asymptotic flatness or not.

Asymptotic limit. Let us first consider the $\rho \to \infty$ limit. There are two possibilities:

1. If we choose the two integration constants in the harmonic functions $L_{0,1}$ to be non-vanishing, $a_0 a_1 > 0$

$$\breve{f} \sim \frac{\sqrt{2}/3}{\sqrt{a_0 a_1}}\,, \qquad e^{\sqrt{2}\breve{\varphi}_\infty} = \frac{a_1}{2a_0}\,, \qquad \text{and} \qquad \breve{H}_{\rho v' u} \sim \frac{q_0}{3a_0^2}\frac{1}{\rho^3}\,. \tag{4.14}$$

First of all, we see that the metric is asymptotically flat. The normalization $\breve{f} = 1$ fixes the integration constants in terms of just $\breve{\varphi}_\infty$:

$$a_0 = \frac{1}{3}e^{-\breve{\varphi}_\infty\sqrt{2}}\,, \qquad a_1 = \frac{2}{3}e^{+\breve{\varphi}_\infty\sqrt{2}}\,. \tag{4.15}$$

This solution describes the superposition of the dyonic string and *pp*-wave mentioned above. The charges of the string can be easily computed and are given by

$$Q \equiv \frac{1}{2\pi^2}\int_{S^3_\infty} e^{-\sqrt{2}\breve{\varphi}} \star \breve{H} = -3q_0\,, \qquad P \equiv \frac{1}{2\pi^2}\int_{S^3_\infty} \breve{H} = \frac{3}{2}q_1\,. \tag{4.16}$$

[19] Only two out of the three different charges are independent in this solution. This is necessary to have a consistent truncation to minimal supergravity.

The instanton field falls too fast at infinity to give any contributions to charges, masses or momenta.

2. If both integration constants vanish $a_0 = a_1 = 0$,[20] as long as $q_1\left(q_0 - \frac{8}{3\breve{g}^2}\right)\breve{f}$ remains always finite and strictly real and positive for all finite values of ρ and the whole metric is regular. In the $\rho \to \infty$ limit the fields behave as

$$\breve{f} \sim \frac{\rho^2}{R_\infty^2}\,, \qquad e^{\sqrt{2}\breve{\varphi}_\infty} = \frac{q_1}{2q_0}\,, \quad \text{and} \quad \breve{H}_{\rho v' u} \sim -\frac{1}{3q_0}\rho\,, \tag{4.17}$$

where we have defined the constant

$$R_\infty^2 \equiv \sqrt{\frac{9q_0q_1}{2}}\,, \tag{4.18}$$

which depends on the charges but not on the modulus $\breve{\varphi}_\infty$, and the metric takes a direct product form

$$d\breve{s}_\infty^2 = R_\infty^2\left(2du'dv''\rho^2 - 3q_1 du'^2 - \frac{d\rho^2}{\rho^2}\right) - R_\infty^2 d\Omega_{(3)}^2\,, \tag{4.19}$$

where $u = R_\infty^2 u'$ and $v' = R_\infty^2 v''$.

The transverse part of the metric is that of a round 3-sphere of radius R_∞. The rest turns out to be the metric of an AdS$_3$ space of radius R_∞ as well: computing its Riemann tensor we find

$$R^{(3)}_{\mu\nu\rho\sigma} = -\frac{2}{R_\infty^2}\, g^{(3)}_{\mu[\rho}g^{(3)}_{\sigma]\nu}\,. \tag{4.20}$$

Thus, the second choice of integration constants gives a solution which is asymptotically AdS$_3 \times S^3$ with radii equal to R_∞. Observe that, in the Abelian case (which we can always recover by eliminating the instanton field) the solution would be globally, and not just asymptotically, AdS$_3 \times S^3$. In the $\rho \to \infty$ limit we recover essentially this Abelian solution because the instanton field vanishes and, in particular, the 3-form field strength $\breve{H}$ takes the form

$$\breve{H} = \frac{3}{2}q_1\left[-\pi_3 + \omega_3\right]\,, \tag{4.21}$$

where π_3 and ω_3 are the volume forms of unit-radii AdS$_3$ and S^3, respectively. In the coordinates we are using, the first is given by

$$\pi_3 = \rho d\rho \wedge dv'' \wedge du'\,. \tag{4.22}$$

Now we are interested in studying the near-horizon ($\rho \to 0$) limits of these two solutions.

[20] If only one of them vanished, the dilaton would not be well defined.

Near-horizon limit. For any values of the integration constants a_0, a_1 (that is: for the two different solutions identified above), in the limit $\rho \to 0$, the Ricci scalar and the Kretschmann invariant of the full metric remain finite. Thus, we expect to have a well-defined $\rho \to 0$ metric which in the asymptotically-flat case will be interpreted as a near-horizon metric. In both cases we have the the following asymptotic expansions:

$$L_{0,1} \sim \frac{q_{0,1}}{\rho^2} + \mathcal{O}(1), \quad \breve{f} = \rho^2/R_{\rm h}^2 + \mathcal{O}(\rho^4)\,, \tag{4.23}$$

where[21]

$$R_{\rm h}^2 \equiv \sqrt{\frac{9q_1(q_0 - 8/(3\breve{g}^2))}{2}}\,, \tag{4.24}$$

which is well defined as long as $q_1(q_0 - 8/(3\breve{g}^2)) > 0$ (in particular, $q_1 \neq 0$). We will assume that this condition holds. Then, rescaling the null coordinates as $u = R_{\rm h}^2 u'$, $v' = R_{\rm h}^2 v''$ the metric takes the same form we found above

$$d\breve{s}_{\rm h}^2 = R_{\rm h}^2 \left(2\rho^2 du' dv'' - 3q_1 du'^2 - \frac{d\rho^2}{\rho^2} \right) - R_{\rm h}^2 d\Omega_{(3)}^2\,, \tag{4.25}$$

which is that of $\mathrm{AdS}_3 \times S^3$ with radii equal to $R_{\rm h}$. The fact that this near-horizon limit is the same as in the case of the pure dyonic string solutions (with no *pp*-wave) [70] is somewhat surprising.

In this limit the dilaton takes a constant and finite value,

$$e^{\sqrt{2}\breve{\varphi}} = \frac{q_1}{2\left(q_0 - \frac{8}{3\breve{g}^2}\right)}\,, \tag{4.26}$$

while the vectors are simply proportional to the left-invariant Maurer-Cartan 1-forms $\breve{A}^A = \frac{1}{\breve{g}} v^A{}_L$. Recalling the definition of the left-invariant Maurer-Cartan forms $V = v^A T_A = -u^{-1}du$ for the SU(2) group representative u and the $\mathfrak{su}(2)$ generators T_A, we conclude that the gauge fields are proportional to a pure gauge configuration, i.e. they describe a meron field, analogous to the one found in ref. [33]. Finally, in the $\rho \to 0$ limit the 3-form field strength $\breve{H}$ takes exactly the same form as in the $\rho \to \infty$ limit eq. (4.21), but we should notice that the coordinates we are using in the AdS_3 are different.

Summarizing, we have found two solutions:

1. The first solution, which is asymptotically flat and has a regular horizon. Asymptotically it cannot be distinguished from the well-known dyonic string solution (plus *pp*-wave) that one can obtain by eliminating the instanton field. This behaviour is similar to that of the colored black holes constructed in refs. [46, 49, 50]. In the near-horizon limit it has an $\mathrm{AdS}_3 \times S^3$ metric with radius $R_{\rm h}$ whose value, given in eq. (4.24) does have a contribution from the instanton field.

2. The second solution is a globally regular metric that interpolates between two $\mathrm{AdS}_3 \times S^3$ solutions with radii R_∞ and R_h given, respectively, in eq. (4.18) and eq. (4.24).

We will discuss these solutions further in the Conclusions section.

[21] Compare this expression with eq. (4.18).

4.2 Solutions of the SO(3)-gauged $\mathcal{N}=2A, d=6$ theory

Dualizing the 3-form field strength of the $\mathcal{N}=2A^*, d=6$ theory solutions we just obtained according to eq. (2.26) we can get very similar solutions of the $\mathcal{N}=2A, d=6$ theory which will have, however, very different string-frame metrics and (possibly) Kalb-Ramond field.

$$\tilde{H} = -\frac{1}{3}dv\wedge du\wedge dL_1^{-1} - \frac{3}{2}\rho^3\partial_\rho\left[L_0 - \frac{2}{9g^2}\left(\rho+\frac{\lambda^2}{4}\rho^3\right)^{-2}\right]\omega_3\,. \tag{4.27}$$

Since, in this case, the 3- and 2-form field strengths are defined as

$$\tilde{H} = d\tilde{B} + \tilde{F}^A\wedge\tilde{A}^A + \frac{1}{3!}\tilde{g}\varepsilon_{ABC}\tilde{A}^A\wedge\tilde{A}^B\wedge\tilde{A}^C\,, \tag{4.28}$$

$$\tilde{F}^A = d\tilde{A}^A - \frac{1}{2}\tilde{g}\varepsilon^A{}_{BC}\tilde{A}^B\wedge\tilde{A}^C\,, \tag{4.29}$$

and the gauge fields are those of the BPS instanton

$$\tilde{A}^A = -\frac{1}{\tilde{g}}\frac{1}{1+\frac{\lambda^2}{4}\rho^2}v^A{}_L\,, \tag{4.30}$$

we find that

$$d\tilde{B} = -\frac{1}{3}dv\wedge du\wedge dL_1^{-1} + 3q_0\omega_3\,, \tag{4.31}$$

and using the Euler coordinates as in eq. (4.12), we obtain the 2-form field

$$\tilde{B} = -\frac{1}{3}L_1^{-1}dv\wedge du + \frac{3}{8}q_0\cos\theta d\psi\wedge d\phi\,, \tag{4.32}$$

which has no non-Abelian contributions.

4.3 Solutions of the "SO(3)-gauged" $\mathcal{N}=2B, d=6$ theory

As we have already mentioned, there is no possible gauging in any conventional sense of the $\mathcal{N}=2B, d=6$ supergravity theory because it has no vector fields. However, it can be argued that, at least when the theory is compactified in a circle, a gauged $\mathcal{N}=2B, d=6$ supergravity theory exists whose massless (in the 5-dimensional sense) sector is given by a gauged $\mathcal{N}=2, d=5$ theory related to the former by dimensional reduction in the Abelian case.

We have also stressed that the relation between the fields of two gauged supergravities is the same as in the ungauged case, as long as their gauge groups are identical. Then, we can use the formulae obtained in the dimensional reduction of the standard $\mathcal{N}=2B, d=6$ to ungauged $\mathcal{N}=2, d=5$ supergravity to uplift solutions of the SO(3)-gauged 5-dimensional theory to this conjectured SO(3)- gauged $\mathcal{N}=2B, d=6$ supergravity. We are going to apply this idea to the non-Abelian black-hole solution we have uplifted to the gauged $\mathcal{N}=2A$ and $\mathcal{N}=2A^*, d=6$ theories. Eliminating the BPST instanton from the solution we obtain a solution of the standard (ungauged) $\mathcal{N}=2B, d=6$ theory.

Thus, using eqs. (2.8), (2.10), (2.11), calling u and v the coordinates z and t and shifting $v' = v + 3a_0 u$ we get the following solution

$$
\begin{aligned}
d\hat{s}^2 &= \left(\frac{2}{3L_1}\right) 2du \left\{ dv' - 3\left[(L_0 - a_0) - \frac{2}{9g^2}\left(\rho + \frac{\lambda^2}{4}\rho^3\right)^{-2}\right] du \right\} \\
&\quad - \left(\frac{2}{3L_1}\right)^{-1} \left(d\rho^2 + \rho^2 d\Omega^2_{(3)}\right) , \\
\hat{L}^r &= \delta^r{}_1 , \\
\hat{B}^1{}_{uv'} &= \frac{1}{3} L_1^{-1} , \\
\hat{B}^A{}_{\mu u} dx^\mu &= -\frac{1}{2\sqrt{6}g} v^A{}_L .
\end{aligned}
\tag{4.33}
$$

This solution has the typical form of a solution describing the superposition of a self-dual string with charge $\sim q_1$ and a *pp*-wave with momentum $\sim q_0$ but there is a non-conventional non-Abelian contribution to this wave which can be interpreted as an instanton expressed in 2-form variables. This non-Abelian contribution, as in the previous cases, falls off too fast at infinity to give a contribution to the wave's momentum and, therefore, the solution has the same asymptotic behaviour as the standard solution with no non-Abelian contribution. It also seems to be regular everywhere as long as $L_1 \neq 0$ (but we always choose a_1 and q_1 with equal signs).

In this solution the string charge and the *pp*-wave momentum are independent and can be set to zero independently.Setting both to zero gives a non-standard, purely non-Abelian *pp*-wave solution.

Asymptotic limit. There are two possible choices of the integration constant a_1 which give physically inequivalent solutions:[22]

1. $a_1 = 1$ gives an asymptotically ($\rho \to \infty$ limit) flat metric with the string-plus-wave interpretation mentioned above.

2. $a_1 = 0$ gives a metric that, with the usual rescaling of u and v', takes the form

$$
d\hat{s}^2 = R^2 \left\{ \left[2du'^2 dv'' \rho^2 - 3\left(q_0 - \frac{2}{9g^2}\left(1 + \frac{\lambda^2}{4}\rho^2\right)^{-2}\right) du' - \frac{d\rho^2}{\rho^2} \right] - d\Omega^2_{(3)} \right\} .
\tag{4.34}
$$

In the $\rho \to \infty$ limit this metric is that of $\mathrm{AdS}_3 \times \mathrm{S}^3$ with radii

$$
R^2 = 3q_1/2 ,
\tag{4.35}
$$

but, for all finite values of ρ it is different from it, except when the non-Abelian contribution is eliminated.

[22]Observe that a_0 has disappeared from the solution.

Near-horizon limit. For the two solutions $a_1 = 1, 0$ one obtains the same metric in the $\rho \to 0$ (near-horizon) limit: an $\mathrm{AdS}_3 \times \mathrm{S}^3$ with radii R given by eq. (4.35). The difference between this metric and the one obtained in the $\rho \to \infty$ limit for the second solution is that in the near-horizon limit there is a non-Abelian contribution in the g_{uu} component, although this does not affect the value of the radii of the factor spaces.

5 Conclusions

We have found a very interesting relation between two families of models of $\mathcal{N} = (2, 0), d = 6$ supergravity that can be used to transform solutions of one of them admitting one isometry into solutions of the other. The relation is based on the fact that they reduce to the same family of models of $\mathcal{N} = 2, d = 5$ supergravity, a fact that we have used to construct new 6-dimensional supersymmetric non-Abelian solutions by uplifting a known 5-dimensional solution.

It is natural to expect that the relation between 6-dimensional supergravities is related to a string duality, but more work is necessary in order to identify the string compactifications that produce the 6-dimensional theories that only have chiral 2-forms.

We have only uplifted the simplest non-Abelian 5-dimensional solution (a black hole), but one should consider more possibilities like the non-Abelian black ring or rotating black hole of ref. [47]. As in the 5- and 4-dimensional cases, the non-Abelian does not contribute to any of the quantities one can measure at infinity, like the mass, but it does modify the near-horizon geometry, with a negative contribution to the entropy. This means that, for the same asymptotic data there are several black-body configurations with different entropies and the non-Abelian one, having the least entropy, should be unstable. An intriguing possibility is that the solution that interpolates between two different $\mathrm{AdS}_3 \times \mathrm{S}_3$ geometries is somehow related to an instanton associated to that instability. Work in this direction is underway [71].

Finally, a long-standing problem that remains unsolved as yet is the microscopical interpretation of the entropy of all the black objects with non-Abelian fields found so far. We believe that the work presented here will help to find the embedding of these solutions in a string theory, providing the first step to solve it.

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