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Review Article

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On Hardy spaces on worm domains

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Abstract: In this review article we present the problem of studying Hardy spaces and the related Szegő projection on worm domains. We review the importance of the Diederich–Fornæss worm domain as a smooth bounded pseudoconvex domain whose Bergman projection does not preserve Sobolev spaces of sufficiently high order and we highlight which difficulties arise in studying the same problem for the Szegő projection. Finally, we announce and discuss the results we have obtained so far in the setting of non-smooth worm domains.

Keywords: Hardy spaces, Szegő projection, Worm domains

MSC: 32A25, 32A35, 32A40

1 Introduction

The smooth bounded pseudoconvex domain introduced by Diederich and Fornæss in [20] has a central role in complex analysis in several variables. This domain, now known in the literature as the worm domain, provided counterexamples to many important conjectures for the last 40 years. The goal of this primarily expository paper is to show that the worm domain is, once again, a good starting point to study some problems; namely, the worm domain is a good candidate to be a smooth bounded pseudoconvex domain whose Szegő projection is unbounded with respect to L^p and $W^{k,p}$ norms for some values of p and k. Here L^p denotes the classical Lebesgue space of p-integrable functions, whereas $W^{k,p}$ denotes the Sobolev space of order k with underlying L^p norm. Before presenting our problem, we briefly recall the main features of the worm domain. We do not provide the details, but we refer the reader to [17, 31] and the references therein.

Diederich and Fornæss mainly introduced the worm domain as a counterexample to a long-standing conjecture about the geometry of pseudoconvex domains. We recall that for domains whose boundary is at least $\mathcal{C}^2$, pseudoconvexity is Levi pseudoconvexity, that is, a domain is pseudoconvex if the associated Levi form is positive semi-definite. We say that a domain is strictly pseudoconvex if the associated Levi form is positive definite. For domains which are not sufficiently regular a different notion of pseudoconvexity that coincides with the Levi pseudoconvexity in the $\mathcal{C}^2$ case is formulated. We refer the reader to [29] for the details.

A pseudoconvex domain D is said to have a Stein neighborhood basis if it exists a family of smooth bounded pseudoconvex domains $\{B_j\}$ such that $B_1 \supset\supset B_2 \supset\supset \cdots \supset\supset \overline{D}$ and $\bigcap_j B_j = \overline{D}$. If such a family of domains does not exist, D is said to have nontrivial Nebenhülle.

An old and well-known example of a domain with nontrivial Nebenhülle is the Hartogs triangle [25], namely the domain

$$\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : 0 < |z_1| < |z_2| < 1\}.$$

The proof the Ω has nontrivial Nebenülle is not difficult and follows from standard arguments. The non-smooth boundary of Ω was thought to be the culprit of the lack of a Stein neighborhood basis, whereas it has been conjectured

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for decades that smooth bounded pseudoconvex domains always have a neighborhood basis of pseudoconvex domains. Diederich and Fornæss finally formulated the following unforeseen counterexample in [20].

Theorem (Diederich–Fornæss). *For any $\beta > \frac{\pi}{2}$, let $\mathcal{W}_\beta$ be the domain*

$$\mathcal{W}_\beta = \Big\{(z_1, z_2) \in \mathbb{C}^2 : \big|z_1 - e^{i \log|z_2|^2}\big|^2 < 1 - \eta\big(\log|z_2|^2\big)\Big\} \tag{1}$$

where

- η *is a non-negative even convex function;*
- $\eta^{-1}(0) = [-\beta + \frac{\pi}{2}, \beta - \frac{\pi}{2}]$*;*
- *there exists a number $a > 0$ such that $\eta(x) > 1$ if $|x| > a$;*
- $\eta'(x) \neq 0$ *if* $\eta(x) = 1$.

Then, $\mathcal{W}_\beta$ is a smooth bounded pseudoconvex domain. Moreover, if $\beta \geq \frac{3\pi}{2}$, the worm domain $\mathcal{W}_\beta$ has nontrivial Nebenhülle.

In addition to this result, Diederich and Fornæss proved some other important features of $\mathcal{W}_\beta$ which disproved some other conjectures. For instance, they proved that the worm is an example of a smooth bounded pseudoconvex domain lacking a global plurisubharmonic defining function.

After the paper by Diederich and Fornæss, the worm did not play any role for some years, but it came back powerfully in the early 90's. Inspired by a work of Kiselman [26], Barrett [1] proved that the Bergman projection associated to $\mathcal{W}_\beta$,i.e., the Hilbert space orthogonal projection from the space of square integrable functions on $\mathcal{W}_\beta$ onto the closed subspace consisting of holomorphic functions, does not preserve Sobolev spaces of sufficiently high order.

We say that the Bergman projection P_D attached to a domain D is *exactly regular* if P_D acts continuously from the Sobolev space $W^{k,2}$ to itself, i.e., $P_D : W^{k,2} \to W^{k,2}$ is bounded. We say either that P_D is *globally regular* or satisfies Bell's *Condition R* if P_D preserves the space $\mathcal{C}^\infty(\overline{D})$. Notice that if P_D is exactly regular, then it is globally regular, whereas the inverse implication does not hold.

Barrett's results can be stated as follows.

Theorem (Barrett). *The Bergman projection operator $P_{\mathcal{W}_\beta}$ is not exactly regular for $k \geq \frac{\pi}{2\beta-\pi}$.*

We stress that the geometry of $\mathcal{W}_\beta$ is reflected in Barrett's results in the sense that when $\beta \to \infty$, the critical index $k = \frac{\pi}{2\beta-\pi}$ tends to 0. The critical case $k = 0$ has been recently studied [33]. Finally, after few years, Christ [18] improved Barrett's result proving the following.

Theorem (Christ). *The Bergman projection operator $P_{\mathcal{W}_\beta}$ fails to preserve $\mathcal{C}^\infty(\overline{\mathcal{W}_\beta})$, i.e., $P_{\mathcal{W}_\beta}$ is not globally regular.*

At this point, in order to frame the importance of Barrett and Christ's results, we need a small digression. It is well-known that the Riemann mapping theorem is not valid in several complex variables, therefore it is an important problem to classify domains up to biholomorphic equivalence. In a seminal paper [21], C. Fefferman proved that any biholomorphic mapping between two smooth bounded strictly pseudoconvex domains extends to a $\mathcal{C}^\infty$ diffeomorphism of the closures. C. Fefferman's proof is complicated and requires a lot of differential geometry techniques. Some years later, Bell [5] and Bell–Ligocka [7] improved and simplified Fefferman's theorem . Namely, they proved that given two smooth bounded pseudoconvex domains D_1 and D_2 such that one of them satisfies Condition R, then a biholomorphism $\varphi : D_1 \to D_2$ extends to a smooth diffeomorphism $\varphi : \overline{D_1} \to \overline{D_2}$. We recall that the Bergman projection is deeply related to the $\bar\partial$-Neumann operator, i.e., the inverse of the complex laplacian $\overline{\partial}^*\overline{\partial} + \overline{\partial}\overline{\partial}^*$, and these two operators share the same mapping properties [12, 22]. From Kohn's work on the $\bar\partial$-Neumann operator [27, 28], we know that any strictly pseudoconvex domain satisfies Condition R, therefore we recover Fefferman's theorem from Bell and Bell–Ligocka's results.

It is now evident the centrality and importance of the regularity of the Bergman projection. To the best of the author's knowledge, the worm domain domain is the only known example of a smooth bounded pseudoconvex domain whose Bergman projection is not globally regular. Recently, different authors studied the behavior of the Bergman projection attached to different versions and generalizations of $\mathcal{W}_\beta$. We refer the reader to [2, 4, 30, 32, 33] among others.

We are interested in studying the Szegő projection attached to the worm $\mathcal{W}_\beta$. Let $H^2(\mathcal{W}_\beta)$ be the Hardy space on the worm domain (see, for instance, [49]). Then, the Szegő projection $S_{\mathcal{W}_\beta}$ is the Hilbert space projection operator from the space $L^2(b\mathcal{W}_\beta)$ onto the closed subspace $H^2(b\mathcal{W}_\beta)$, where $L^2(b\mathcal{W}_\beta)$ denotes the space of square integrable functions on the topological boundary $b\mathcal{W}_\beta$ of the worm and $H^2(b\mathcal{W}_\beta)$ consists of functions which are boundary values for functions in $H^2(\mathcal{W}_\beta)$. The Szegő projection $S_{\mathcal{W}_\beta}$ can be considered a boundary analogue of the Bergman projection $P_{\mathcal{W}_\beta}$ and it is reasonable to expect that the pathological geometry of $\mathcal{W}_\beta$ affects the regularity of $S_{\mathcal{W}_\beta}$ as well. Our goal is to prove an analogous of Barrett's theorem, i.e., to show that the Szegő projection of the worm does not preserve Sobolev spaces of high order. Thus, the worm domain $\mathcal{W}_\beta$ would be the first example of a smooth bounded pseudoconvex domain whose Szegő projection does not preserve the regularity of functions.

The Szegő projection has been studied in many different settings, but the worm domain does not belong to any of the known situation. We refer the reader to [3, 8–11, 16, 19, 34, 36, 37, 39, 43, 44, 50] among others.

Barrett's proof of the irregularity of the Bergman projection cannot be trivially adapted to the Hardy spaces setting and new difficulties arise. In this paper we highlight these new difficulties and we discuss the results we have obtained so far for some non-smooth versions of the Diederich-Fornæss worm.

We remark that, in analogy with the Bergman case, the regularity of the Szegő projection, at least in a certain setting, is equivalent to the regularity of the complex Green operator [24], the boundary analog of the $\bar\partial$-Neumann operator. Therefore, we hope that the study of $S_{\mathcal{W}_\beta}$ will lead to new ideas related to the complex Green operator associated to $\mathcal{W}_\beta$.

The paper is organized as follows. In Section 2 we review Barrett's proof [1] pointing out the arguments that fail in the Szegő setting. In Section 3 we discuss the problem of the regularity of the Szegő projection and in Section 4 we state our results.

2 Behavior of the Bergman projection on the smooth worm

The proof of Barrett's result has two main ingredients: the use of non-smooth unbounded model domains of $\mathcal{W}_\beta$ which allow to perform some explicit computations and a clever exhaustion argument which allows to transfer the results obtained on the model domains to the original worm $\mathcal{W}_\beta$. We briefly review in this section Barrett's argument and we refer the reader either to [1], [17] or [31] for the details.

2.1 Non-smooth worm domains

For $\beta > \frac{\pi}{2}$, we define

$$D_\beta = \left\{(z_1, z_2) \in \mathbb{C}^2 : \operatorname{Re}\left(z_1 e^{-i\log|z_2|^2}\right) > 0, \left|\log|z_2|^2\right| < \beta - \frac{\pi}{2}\right\}. \tag{2}$$

This domain is clearly unbounded and its boundary is non-smooth, but only Lipschitz. Moreover, it is not hard to see that D_β is Levi flat, i.e., its Levi form is locally constantly zero. Notice that D_β can be sliced in half-planes, that is, for each fixed z_2 we have an half-plane in the z_1 variable. The geometry of D_β is rather different from the one of $\mathcal{W}_\beta$, nonetheless the non-smooth worm D_β is a model for the original worm. Indeed, let us consider the chopped worm

$$\mathcal{W}'_\beta = \left\{(z_1, z_2) \in \mathbb{C}^2 : \left|z_1 - e^{i\log|z_2|^2}\right|^2 < 1, \left|\log|z_2|^2\right| < \beta - \frac{\pi}{2}\right\}. \tag{3}$$

We call the domain $\mathcal{W}'_\beta$ the chopped worm since it is obtained from the original smooth worm $\mathcal{W}_\beta$ by chopping the two smooth caps of $\mathcal{W}_\beta$, i.e., we replace the function η in (1) with the characteristic function of the interval $[-\beta + \frac{\pi}{2}, \beta - \frac{\pi}{2}]$. Now, for a fixed real number $\mu > 0$, we consider the following dilation of $\mathcal{W}'_\beta$

$$\mathcal{W}'_{\beta,\mu} = \left\{(z_1, z_2) \in \mathbb{C}^2 : \left(\frac{z_1}{\mu}, z_2\right) \in \mathcal{W}'_\beta\right\}. \tag{4}$$

As it is easy to check, it holds that $\mathcal{W}'_{\beta,\mu} \subseteq D_\beta$ for every μ and $\mathcal{W}'_{\beta,\mu} \nearrow D_\beta$ as $\mu \to +\infty$. The notation $\mathcal{W}'_{\beta,\mu} \nearrow D_\beta$ means that the family $\{\mathcal{W}'_{\beta,\mu}\}_\mu$ is increasing in μ, that is, $\mathcal{W}'_{\beta,\mu_1} \subseteq \mathcal{W}'_{\beta,\mu_2}$ if $\mu_1 < \mu_2$, and that $D_\beta = \cup_{\mu>0}\mathcal{W}'_{\beta,\mu}$.

Therefore, the non-smooth worm D_β is obtained from $\mathcal{W}_\beta$ by removing the smooth caps and dilating the chopped worm $\mathcal{W}'_\beta$. We use the notation with a subscript μ to also denote the domains obtained by dilation of $\mathcal{W}_\beta$, that is,

$$\mathcal{W}_{\beta,\mu} = \left\{(z_1, z_2) :\in \mathbb{C}^2 : \left(\frac{z_1}{\mu}, z_2\right) \in \mathcal{W}_\beta\right\}. \tag{5}$$

It is not hard to prove that $\mathcal{W}_{\beta,\mu} = \mathcal{W}'_{\beta,\mu} \cup B_\mu$ where the family of sets $\{B_\mu\}_\mu$ is decreasing in μ and has the property that $\cap_{\mu>0}B_\mu = \varnothing$. Moreover, if ρ_μ denotes the defining function of $\mathcal{W}_{\beta,\mu}$, it is easily seen that ρ_μ converges pointwise to the defining function ρ_{D_β} of D_β. We use the notation $\mathcal{W}_{\beta,\mu} \to D_\beta$ as $\mu \to +\infty$ to denote this property.

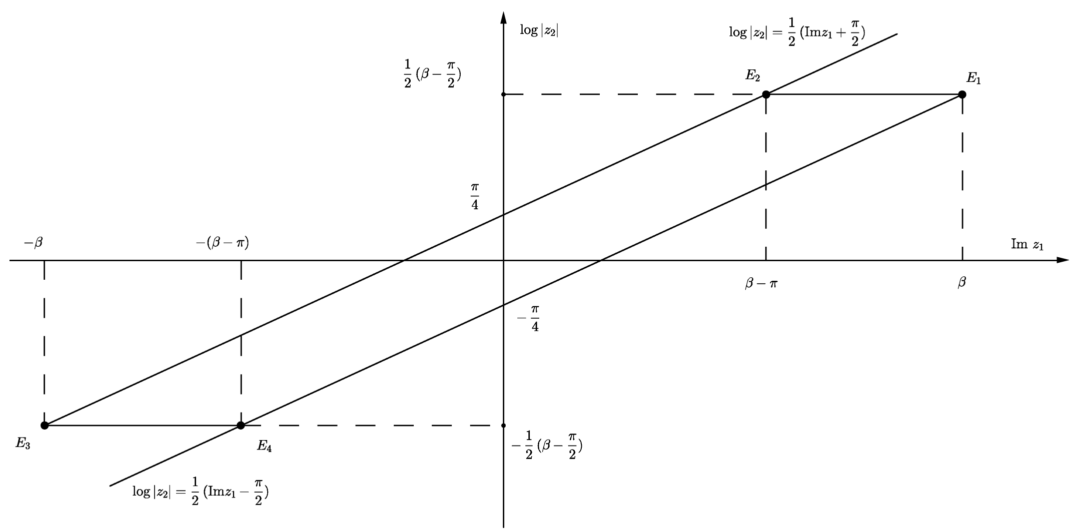
We point out that if f is a smooth compactly supported function on D_β, then there exists a positive number $\mu = \mu(f)$ such that the support of f is contained in $\mathcal{W}'_{\beta,\mu_f}$. Thus $f \in \mathcal{C}^\infty_0(\mathcal{W}'_{\beta,\mu_f}) \subseteq \mathcal{C}^\infty_0(\mathcal{W}_{\beta,\mu_f})$. This is a trivial, but important remark.

Besides the domain D_β, we also consider the domain

$$D'_\beta = \left\{(z_1, z_2) \in \mathbb{C}^2 : \left|\operatorname{Im} z_1 - \log|z_2|^2\right| < \frac{\pi}{2}, \left|\log|z_2|^2\right|^2 < \beta - \frac{\pi}{2}\right\}. \tag{6}$$

We notice that this domain can be sliced in strips, that is, for every fixed z_2 we have a strip in the variable z_1.

Fig. 1. A representation of the domain D'_β in the $(\operatorname{Im} z_1, \log|z_2|)$-plane.



The domains D_β and D'_β are biholomorphically equivalent via the map

$$\begin{aligned} \varphi : D'_\beta &\to D_\beta \\ (z_1, z_2) &\mapsto (e^{z_1}, z_2). \end{aligned} \tag{7}$$

A well defined inverse of φ is the function

$$\varphi^{-1}(z_1, z_2) = \left(\operatorname{Log}[z_1 e^{-i\log|z_2|^2}] + i\log|z_2|^2, z_2\right) \tag{8}$$

where $\operatorname{Log}(z)$ is the principal branch of the complex logarithm.

Despite being biholomorphically equivalent, the domains D'_β and D_β differ in a really important aspect for our purposes. For each fixed z_1, the fiber in the second component of D'_β, i.e., the set $\{z_2 \in \mathbb{C} : (z_1, z_2) \in D'_\beta\}$, is connected, whereas the similar property for D_β does not hold.

2.2 Estimates for the Bergman kernel of non-smooth worm domains

Let A^2 be the Bergman space either of the domain D'_β or D_β, that is, the space of square integrable functions with respect to the Lebesgue volume measure dV. Using the rotational invariance of D_β and D'_β, we obtain that each function $F \in A^2$ admits a decomposition

$$F(z_1, z_2) = \sum_{j \in \mathbb{Z}} F_j(z_1, z_2)$$

where each F_j satisfies $F_j(z_1, e^{i\theta} z_2) = e^{ij\theta} F(z_1, z_2)$. Furthermore, each of the function F_j must have the form

$$F_j(z_1, z_2) = f_j(z_1, |z_2|) z_2^j, \tag{9}$$

where f_j is holomorphic in z_1 and locally constant in z_2. Therefore, the Bergman space A^2 admits a decomposition

$$A^2 = \bigoplus_{j \in \mathbb{Z}} A_j^2 \tag{10}$$

where

$$A_j^2 = \left\{ G \in L^2 : G \text{ is holomorphic and } G(z_1, z_2) = G(z_1, |z_2|) z_2^j \right\}.$$

Everything we said so far holds either for the Bergman space on D'_β or on D_β. The gimmick is to restrict our focus to the domain D'_β; since the fiber in the second component of D'_β are connected for each fixed z_1, the functions f_j are actually independent of z_2, that is, $f_j(z_1, |z_2|) = f_j(z_1)$. From the decomposition (10) we obtain that if K' is the Bergman kernel of $A^2(D'_\beta)$, i.e., the kernel of the integral representation of $P_{D'_\beta}$, then $K' = \sum_{j \in \mathbb{Z}} K'_j$ where each K'_j is the Bergman kernel of the corresponding space $A_j^2(D'_\beta)$ and

$$K'_j[(z_1, z_2), (w_1, w_2)] = k'_j(z_1, w_1) z_2^j \overline{w}_2^j.$$

Hence, the task is now to compute each k_j. The following proposition holds.

Proposition 2.1. *Let $\beta > \frac{\pi}{2}$. Then,*

$$k'_j(z_1, w_1) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{\xi(\xi - \frac{j+1}{2}) e^{i(z_1 - \overline{w_1})\xi}}{\mathrm{Sh}[\pi\xi] \sinh[(2\beta - \pi)(\xi - \frac{j+1}{2})]} \, d\xi. \tag{11}$$

Kiselman [26] and Barrett [1] studied the easier case $j = -1$ and this was enough for their purposes. In particular, using the method of contour integral, it is proved in [1] that

$$K'_{-1}[(z_1, z_2), (w_1, w_2)] = \left[e^{-\nu_\beta |z_1 - \overline{w}_1|} + \mathcal{O}(e^{-\nu |\operatorname{Re} z_1 - \operatorname{Re} w_1|}) \right] (z_2 \overline{w}_2)^{-1}$$

as $|\operatorname{Re} z_1 - \operatorname{Re} w_1| \to +\infty$ and this estimate is uniform for z_1, w_1 varying in a closed strip and $\nu_\beta = \frac{\pi}{2\beta - \pi} < \nu$. In [32], Krantz and Peloso improved this analysis performing the computation for every j.

Thanks to the transformation rule for the Bergman kernel under biholomorphic mappings (see, e.g., [29]), we obtain an asymptotic expansion for the kernel K_{-1} associated to the space $A_{-1}^2(D_\beta)$. It holds,

$$K_{-1}[(z_1, z_2), (\omega_1, \omega_2)] = (|z_1||\omega_1|)^{-1} \Big[\frac{|\omega_1|^{\nu_\beta}}{|z_1|^{\nu_\beta}} + \mathcal{O}\Big(\frac{|\omega_1|^{\nu_\beta}}{|z_1|^{\nu_\beta}} \Big)^{-\nu} \Big] (z_2 \overline{\omega}_2)^{-1}$$

as $|z_1| - |\omega_1| \to 0^+$. Using this expansion, we finally observe that for any positive integer m it holds

$$|\operatorname{Re}(z_1 e^{-i \log |z_2|^2})|^s \left(\frac{\partial}{\partial z_1} \right)^m K_{-1}[(z_1, z_2), (w_1, w_2)] \notin L^2(D_\beta), \text{ for } s \leq m - \nu_\beta. \tag{12}$$

As a consequence,

$$|\operatorname{Re}(z_1 e^{-i \log |z_2|^2})|^s \left(\frac{\partial}{\partial z_1} \right)^m K[(z_1, z_2), (w_1, w_2)] \notin L^2(D_\beta), \text{ for } s \leq m - \nu_\beta. \tag{13}$$

where K is the Bergman kernel of $A^2(D_\beta)$. The next step is to use this information on the integrability of K to obtain information on the Bergman projection $P_{\mathcal{W}_\beta}$ of the smooth worm.

We conclude the section with an important remark. For each $w \in D_\beta$, let $f_w \in \mathcal{C}_0^\infty(D_\beta)$ be a smooth real-valued function radially symmetric with respect to the center w and such that $\int_{D_\beta} f_w(z)dV(z) = 1$. Then, using the mean value property of $K(z,\cdot)$, polar coordinates and the radial symmetry of f_w, we obtain

$$K(z,w) = \int_{D_\beta} K(z,\eta) f_w(\eta)\, dV(\eta) = P_{D_\beta} f_w(z). \tag{14}$$

It is thanks to this property of the kernel K that we can study the regularity of the Bergman projection by studying the integrability of the kernel itself. Hence, the analysis performed to obtain (13) is justified.

2.3 Irregularity of the Bergman projection on the smooth worm

In this section we briefly review the main steps of the exhaustion argument used by Barrett to obtain the irregularity of $P_{\mathcal{W}_\beta}$. A fundamental tool is a characterization of Sobolev norms due to Ligocka. Since we always deal with L^2 norm, from now on, W^k denotes the Sobolev space $W^{k,2}$.

In [38] it is proved that given a smooth bounded domain D with a defining function ρ, a non-negative integer m and a real number $s \geq 0$, then the norm

$$\sum_{|\alpha|\leq m} \| |\rho|^s \partial_z^\alpha f \|_{L^2(D)} \tag{15}$$

is equivalent to the $W^m(D)$ norm of the holomorphic function f.

Assume now that $P_{\mathcal{W}_\beta}$ is continuous on $W^k(\mathcal{W}_\beta)$ for $k \geq \frac{\pi}{2\beta-\pi}$, i.e.,

$$\|P_{\mathcal{W}_\beta} f\|_{W^k(\mathcal{W}_\beta)} \leq C_k \|f\|_{W^k(\mathcal{W}_\beta)} \tag{16}$$

for every function $f \in \mathcal{W}^k(\mathcal{W}_\beta)$. Then, if τ_μ denotes the operator $T_\mu f(z_1,z_2) = f(\mu z_1, z_2)$, it is possible to prove the relationship

$$P_\mu = T_\mu^{-1} P T_\mu, \tag{17}$$

where P_μ is the Bergman projection attached to the dilated smooth worm $\mathcal{W}_{\beta,\mu}$ defined in (5). If ρ_μ denotes the defining function of $\mathcal{W}_{\beta,\mu}$, then, using (15), (16) and (17), it is possible to obtain for any $f \in \mathcal{C}_0^\infty(\mathcal{W}'_{\beta,\mu}) \subseteq \mathcal{C}_0^\infty(\mathcal{W}_{\beta,\mu})$ the estimate

$$\| |\rho_\mu|^s \Big(\frac{\partial}{\partial z_1}\Big)^m P_\mu f\|_{L^2(\mathcal{W}_{\beta,\mu})} \leq C_\mu \|f\|_{W^k(\mathcal{W}_{\beta,\mu})} \tag{18}$$

with $k = m - s$ where m is an integer, $s \geq 0$ and the constant C is independent of μ. The final step of the proof is to prove that

$$\| |\rho_\mu|^s \Big(\frac{\partial}{\partial z_1}\Big)^m P_\mu f\|_{L^2(\mathcal{W}_{\beta,\mu})} \to \| |\operatorname{Re}(z_1 e^{-i\log|z_2|^2})|^s \Big(\frac{\partial}{\partial z_1}\Big)^m P_{D_\beta} f\|_{L^2(D_\beta)}. \tag{19}$$

as $\mu \to \infty$. If we assume (19), then we obtain a contradiction from (13),(14) and (18). Hence, the irregularity of $P_{\mathcal{W}_\beta}$ follows.

The limit (19) is obtained by showing that $P_\mu f \rightharpoonup P_{D_\beta}$ weakly in $L^2(\mathbb{C}^2)$ where P_{D_β} is set to be zero outside D_β and $P_\mu f = 0$ outside $\mathcal{W}_{\beta,\mu}$. In order to prove this weak convergence, the trivial remark after (5) is also used.

3 The problem of the regularity of the Szegő projection

In this section we illustrate the difficulties we face when dealing with the study of the Szegő projection $S_{\mathcal{W}_\beta}$. Given a smooth bounded domain $D \subset \mathbb{C}^n$ with a defining function ρ,i.e., $D = \{z \in \mathbb{C}^{\cdot} \rho(z) < 0\}$ where ρ is smooth and $\nabla\rho \neq 0$ on the boundary of D, a standard way to define the Hardy space $H^2(D)$ is to consider a family of approximating domains $D_\varepsilon = \{z \in \mathbb{C}^n : \rho(z) < -\varepsilon\}$, $\varepsilon > 0$ together with the quantity

$$\sup_{\varepsilon>0} \int_{bD_\varepsilon} |F(\zeta)|^2 \, d_\varepsilon\sigma \tag{20}$$

where bD_ε denotes the topological boundary of D_ε and $d_\varepsilon\sigma$ is the induced Euclidean measure on bD_ε. Then, $H^2(D)$ is the function space

$$H^2(D) = \left\{ F \text{ holomorphic in } D : \|F\|^2_{H^2(D)} = \sup_{\varepsilon>0} \int_{bD_\varepsilon} |F(\zeta)|^2 \, d_\varepsilon\sigma < \infty \right\}.$$

For every function $F \in H^2(D)$ the pointwise limit $\lim_{z\to\zeta_0} f(\zeta)$ exists for almost every ζ_0 in bD when $z \to \zeta_0$ non-tangentially. Thus, if we denote by $\widetilde{F}$ the boundary value function of F, then the function space

$$H^2(bD) = \{G \in L^2(bD) : G = \widetilde{F} \text{ for some } G \in H^2(D)\}$$

is a closed subspace of $L^2(bD)$ and the Szegő projection attached to D is the Hilbert space orthogonal projection $S_D : L^2(bD) \to H^2(bD)$. We refer the reader to [49] for the details. Since $\mathcal{W}_\beta$ is smooth and bounded, the projection $S_{\mathcal{W}_\beta}$ we are interested in is the one obtained with this standard construction. As in the Bergman setting, it seems reasonable to obtain information on $S_{\mathcal{W}_\beta}$ by studying the model domains D_β and D'_β, but, unlike the Bergman case, it is not clear what is the Szegő projection attached to D_β or D'_β since they are unbounded and non-smooth domains.

Let us focus for a moment on the domain D'_β; a similar discussion is also valid for the domain D_β. We want to define a space $H^2(D'_\beta)$ by means of a condition similar to (20). The novelty here is that we can consider different restrictions of a function F holomorphic in D'_β. One option, of course, is to use the very same condition (20), that is, to integrate the restriction of a function F on copies of the topological boundary

$$\begin{aligned} bD'_\beta = &\left\{(z_1,z_2) : \left|\operatorname{Im} z_1 - \log|z_2|^2\right| \le \frac{\pi}{2}, \left|\log|z_2|^2\right|^2 = \beta - \frac{\pi}{2}\right\} \\ &\cup \left\{(z_1,z_2) : \left|\operatorname{Im} z_1 - \log|z_2|^2\right| = \frac{\pi}{2}, \left|\log|z_2|^2\right|^2 < \beta - \frac{\pi}{2}\right\}. \end{aligned} \tag{21}$$

A different option is to integrate the restriction of F on copies of the distinguished boundary

$$\partial D'_\beta = \left\{(z_1,z_2) \in \mathbb{C}^2 : \left|\operatorname{Im} z_1 - \log|z_2|^2\right| = \frac{\pi}{2}, \left|\log|z_2|^2\right|^2 = \beta - \frac{\pi}{2}\right\}. \tag{22}$$

We point out that $\partial D'_\beta$ corresponds to the union of the four vertices $E_\ell, \ell = 1,\ldots,4$ of the quadrilateral in Figure 1. When dealing with a product domain, it is a standard choice to define Hardy spaces using the distinguished boundary of the domain; for instance, we refer the reader to [48] for the case of the polydisc. Another option is to consider a mixed condition and to restrict and integrate a function f on copies of a set B intermediate between $\partial D'_\beta$ and bD_β, that is, $\partial D'_\beta \subseteq B \subseteq bD'_\beta$. Of course, every different definition of $H^2(D'_\beta)$ gives rise to a different Szegő projection $S_{D'_\beta}$. Whichever strategy we use to build an H^2 space on D'_β and D_β, we have to deal with the lack of a transformation rule for the Szegő projection under biholomorphic mappings. Hence, the information we eventually obtain on the Szegő projection $S_{D'_\beta}$ cannot be immediately transferred to S_{D_β} as in the Bergman setting. We do have a transformation rule for the Szegő projection attached to biholomorphic domains in [6, Theorem 12.3] , but it holds only for smooth bounded domains in the plane.

We decided to define the space $H^2(D'_\beta)$ using the distinguished boundary of D'_β. Due to the geometry of the domain, this seems to be a natural choice: the distinguished boundary $\partial D'_\beta$ has four different components –the vertices of the quadrilateral in Figure 1–and each of these components has a nice product structure since it can be identified with the cartesian product $\mathbb{R} \times \mathbb{T}$. Here $\mathbb{T}$ denotes the 1–dimensional torus. This product structure of the components of $\partial D'_\beta$ allows to perform precise computations and to obtain an explicit formula for the Szegő projection operator $S_{D'_\beta}$. Moreover, the space $H^2(D'_\beta)$ results to be isometric to the analogously defined space $H^2(D_\beta)$ and this allows us to prove a transformation rule for the Szegő projections $S_{D'_\beta}$ and S_{D_β}.

We define the Hardy space $H^p(D'_\beta)$, $p \in (1,\infty)$ as the function space

$$H^p(D'_\beta) = \left\{ F \text{ holomorphic in } D'_\beta : \|F\|^p_{H^p(D'_\beta)} = \sup_{(t,s)\in[0,\frac{\pi}{2})\times[0,\beta-\frac{\pi}{2})} \mathcal{L}'_p F(t,s) < \infty \right\} \tag{23}$$

where

$$\mathcal{L}'_p F(t,s) = \int_{\mathbb{R}}\int_0^1 \left| F\left(x+i(s+t), e^{\frac{s}{2}} e^{2\pi i\theta}\right)\right|^p d\theta dx + \int_{\mathbb{R}}\int_0^1 \left| F\left(x-i(s+t), e^{-\frac{s}{2}} e^{2\pi i\theta}\right)\right|^p d\theta dx$$
$$+ \int_{\mathbb{R}}\int_0^1 \left| F\left(x+i(s-t), e^{\frac{s}{2}} e^{2\pi i\theta}\right)\right|^p d\theta dx + \int_{\mathbb{R}}\int_0^1 \left| F\left(x-i(s-t), e^{-\frac{s}{2}} e^{2\pi i\theta}\right)\right|^p d\theta dx.$$

The distinguished boundary of D_β is the set

$$\partial D_\beta = \left\{ (z_1,z_2)\in\mathbb{C}^2 : \operatorname{Re}(z_1 e^{-i\log|z_2|^2}) = 0, \left|\log|z_2|^2\right| = \beta - \frac{\pi}{2} \right\}. \tag{24}$$

Therefore, for every $p \in (1,\infty)$, we define the Hardy space $H^p(D_\beta)$ as the function space

$$H^p(D_\beta) := \left\{ F \text{ holomorphic in } D_\beta : \|F\|^p_{H^p(D_\beta)} = \sup_{(t,s)\in(0,\frac{\pi}{2})\times[0,\beta-\frac{\pi}{2})} \mathcal{L}_p F(t,s) < \infty \right\} \tag{25}$$

where

$$\mathcal{L}_p F(t,s) = \int_0^\infty\int_0^1 |F[re^{i(s+t)}, e^{\frac{s}{2}} e^{2\pi i\theta}]|^2 \, d\theta dr + \int_0^\infty\int_0^1 F[re^{i(s-t)}, e^{\frac{s}{2}} e^{2\pi i\theta}] \, d\theta dr$$
$$+ \int_0^\infty\int_0^1 |F[re^{-i(s+t)}, e^{-\frac{s}{2}} e^{2\pi i\theta}]|^2 \, d\theta dr + \int_0^\infty\int_0^1 F[re^{-i(s-t)}, e^{-\frac{s}{2}} e^{2\pi i\theta}] \, d\theta dr.$$

It can be proved that any function $F \in H^2(D'_\beta)$ admits a boundary value function $\widetilde{F}$. If we denote by $H^2(\partial D'_\beta)$ the subspace of $L^2(\partial D'_\beta)$ consisting of these boundary value functions, then the Szegő projection $S_{D'_\beta}$ is the Hilbert space orthogonal projection operator $S_{D'_\beta} : L^2(\partial D'_\beta) \to H^2(\partial D'_\beta)$. The projection $S_{D'_\beta}$ is orthogonal with respect to the natural inner product in $L^2(\partial D'_\beta)$.

We point out that the norm $\|\cdot\|_{H^2(D'_\beta)}$ in (23) can be proved to be equal to the norm induced by the following inner product on $H^2(D'_\beta)$: let F, G be two functions in $H^2(D'_\beta)$ and denotes by $\widetilde{F}$ and $\widetilde{G}$ their boundary value functions. Then, we set

$$\langle F, G\rangle_{H^2(D'_\beta)} := \int_{\partial D'_\beta} \widetilde{F}(\zeta)\overline{\widetilde{G}(\zeta)} \, d\zeta.$$

The Szegő projection $S_{D'_\beta}$ has an integral representation by means of the Szegő kernel $K_{D'_\beta}$; namely, for every $F \in L^2(\partial D'_\beta)$ and $\zeta \in \partial D'_\beta$,

$$S_{D'_\beta} F(\zeta) = \int_{\partial D'_\beta} K_{D'_\beta}(\zeta,\omega) F(\omega) \, d\zeta. \tag{26}$$

We remark that, by definition, the function $S_{D'_\beta} F$ is a function defined on the distinguished boundary $\partial D'_\beta$.

Similarly, S_{D_β} denotes the Hilbert space orthogonal projection operator $S_{D_\beta} : L^2(\partial D_\beta) \to H^2(\partial D_\beta)$ and S_{D_β} has an integral representation by means of the Szegő kernel K_{D_β}.

In the next section we announce and discuss some results regarding the boundedness of $S_{D'_\beta}$ and S_{D_β} in L^p and Sobolev scale, whereas we conclude this section highlighting which difficulties arise when applying Barrett's exhaustion argument in this setting.

In the Bergman situation we used that fact that the Bergman kernel $K(\cdot, w)$ of $A^2(D_\beta)$ belongs to $P_{D_\beta}(\mathcal{C}_0^\infty(D_\beta))$. The argument that it is used to prove (14) cannot work in this context. In particular, to prove (14) it is used the mean value property of the holomorphic function $K(\cdot, w)$; the Szegő projection of a function $F \in \mathcal{C}_0^\infty(\partial D_\beta)$ is a new function $S_{D_\beta} F$ defined on ∂D_β given by the integration against the Szegő kernel on the distinguished boundary ∂D_β. The fact that we are integrating on the boundary D_β prevents us to exploit the holomorphicity of the function $K_{D_\beta}(\cdot, w)$ in D_β, thus to use the mean value property and obtain a conclusion similar to (14).

When we defined the Hardy spaces on D_β and D'_β we pointed out that the use of the distinguished boundary allows to perform some explicit computations thanks to its product structure. However, it is not clear how to relate S_{D_β} and S_μ, the Szegő projection attached to the dilated smooth worm $\mathcal{W}_{\beta,\mu}$. Since $\mathcal{W}_{\beta,\mu}$ is a smooth bounded domain, it is clear from the standard theory how to define the operator S_μ. In particular, the operator S_μ acts on functions defined on the topological boundary $b\mathcal{W}_{\beta,\mu}$ of $\mathcal{W}_{\beta,\mu}$. The main drawback of our definition of S_{D_β} is the following. For every fixed $\mu > 0$, if f is a smooth function compactly supported in the chopped and dilated worm $\mathcal{W}'_{\beta,\mu}$ (see (4)), then f belongs to both $\mathcal{C}_0^\infty(\mathcal{W}_{\beta,\mu})$ and $\mathcal{C}_0^\infty(D_\beta)$. Therefore, the Bergman projections $P_\mu f$ and $P_{D_\beta} f$ attached to $\mathcal{W}_{\beta,\mu}$ and D_β respectively are both well-defined. Finally, we exploit the fact that $\mathcal{W}_{\beta,\mu} \to D_\beta$ as $\mu \to +\infty$ to relate P_μ and P_{D_β}.

When dealing with functions defined on the boundaries of the domains the above argument immediately fails. A function F supported, say, on $b\mathcal{W}'_{\beta,\mu} \cap b\mathcal{W}_{\beta,\mu}$, the intersection of the topological boundaries of $\mathcal{W}'_{\beta,\mu}$ and $\mathcal{W}_{\beta,\mu}$, is *a priori* not even defined on the distinguished boundary ∂D_β of D_β since $b\mathcal{W}'_{\beta,\mu} \cap b\mathcal{W}_{\beta,\mu} \not\subseteq \partial D_\beta$. Therefore, the Szegő projections $S_\mu F$ and $S_{D_\beta} F$ are not well-defined for the same function F.

Moreover, even if $\mathcal{W}_{\beta,\mu} \to D_\beta$ as $\mu \to +\infty$, it is clear and easy to check that $b\mathcal{W}_{\beta,\mu} \nrightarrow \partial D_\beta$ as $\mu \to +\infty$. Indeed, as expected, it is only a submanifold of $b\mathcal{W}_{\beta,\mu}$ that coincides with ∂D_β when $\mu \to +\infty$. In detail, let us consider the distinguished boundary of the chopped worm

$$\partial \mathcal{W}'_\beta = \left\{ (z_1, z_2) \in \mathbb{C}^2 : \left| z_1 - e^{i \log |z_2|^2} \right|^2 = 1, \left| \log |z_2|^2 \right| = \beta - \frac{\pi}{2} \right\}. \tag{27}$$

If $\partial \mathcal{W}'_{\beta,\mu}$ denotes the distinguished boundary of the dilated chopped worm, it holds $\partial \mathcal{W}'_{\beta,\mu} \subseteq b\mathcal{W}_{\beta,\mu}$ and $\partial \mathcal{W}'_{\beta,\mu} \to \partial D_\beta$ when $\mu \to +\infty$ as we wished. Nonetheless, we remark that $\partial \mathcal{W}'_{\beta,\mu} \not\subseteq \partial D_\beta$ for every $\mu > 0$.

It should be now clear that the argument used by Barrett cannot be applied in this new setting and a new strategy is necessary. This is currently work in progress.

4 Results on non-smooth worms

In this section we state some results on the regularity of the Szegő projections attached to D'_β and D_β. The results regarding D'_β are proved in [41] together with a detailed analysis of the space $H^2(D'_\beta)$, whereas the results on D_β are new and greater details will appear in a forthcoming paper [42].

4.1 The Szegő projection of D'_β

As in the Bergman setting, it can be proved that the Hardy space $H^2(D'_\beta)$ admits an orthogonal decomposition

$$H^2(D'_\beta) = \bigoplus_{j \in \mathbb{Z}} \mathcal{H}_j^2 \tag{28}$$

where each $\mathcal{H}_j^2$ is a subspace of $H^2(D'_\beta)$ such that

$$\mathcal{H}_j^2 = \left\{ F \in H^2(D'_\beta) : F(z_1, e^{i\theta} z_2) = e^{ij\theta} F(z_1, z_2) \right\}. \tag{29}$$

Using again the connectedness of the fibers in the second component of D'_β, it turns out that if F_j denotes a function in the subspace $\mathcal{H}_j^2$, then F_j is of the form $F_j(z_1, z_2) = f_j(z_1) z_2^j$ where the function f_j belongs to the Hardy space H^2 on the strip $S_\beta = \{x + iy \in \mathbb{C} : |y| < \beta\}$. Thus, it can be proved that the Szegő kernel $K_{D'_\beta}$ is given by

$$K_{D'_\beta}[(w_1, w_2), (z_1, z_2)] = \sum_{j\in\mathbb{Z}} w_2^j \overline{z_2}^j k_j(w_1, z_1) \tag{30}$$

where

$$k_j(z_1, z_2) = \frac{1}{8\pi} \int_{\mathbb{R}} \frac{e^{i(w_1-\overline{z_1})\xi}}{\operatorname{Ch}[\pi\xi]\operatorname{Ch}[(2\beta-\pi)(\xi-\frac{j}{2})]}\, d\xi.$$

Before writing an explicit formula for the Szegő projection $S_{D'_\beta}$, we need a remark. Because of the way we defined the space $H^2(D'_\beta)$, the Szegő projection $S_{D'_\beta}F$ of a function defined on the distinguished boundary $\partial D'_\beta$ is a new function still defined, of course, on $\partial D'_\beta$. We mentioned that $\partial D'_\beta$ has 4 different components, the 4 vertices $E_\ell, \ell = 1, \ldots, 4$, of the quadrilateral in Figure 1, thus a function F defined on $\partial D'_\beta$ can be identified with a vector, namely $F = (F_1, F_2, F_3, F_4)$ where each $F_\ell, \ell = 1, 2, 3, 4$ is thought as defined on the component E_ℓ of $\partial D'_\beta$. Hence, the operator $S_{D'_\beta}$ associates to a vector of functions $F = (F_1, F_2, F_3, F_4)$ another vector of functions $S_{D'_\beta}F = [(S_{D'_\beta}F)_1, (S_{D'_\beta}F)_2, (S_{D'_\beta}F)_3, (S_{D'_\beta}F)_4]$.

Now, we write an explicit formula for $(S_{D'_\beta}F)_1$ assuming that the starting function F is of the form $F = (F_1, 0, 0, 0)$, that is, F is constantly zero on the components E_2, E_3 and E_4 of $\partial D'_\beta$. We refer the reader to [41] for the details on how to obtain this formula and for its most general version.

Recall that each component $E_\ell, \ell = 1, \ldots 4$, can be identified with $\mathbb{R}\times\mathbb{T}$. If ψ is a smooth compactly supported function on $\mathbb{R}\times\mathbb{T}$, we denote by $\mathcal{F}_{\mathbb{R}}\psi(\xi, \widehat{j})$ the Fourier transform of ψ in the first variable and the j-th Fourier coefficient in the second, i.e.,

$$\mathcal{F}_{\mathbb{R}}\psi(\xi, \widehat{j}) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_0^1 \psi(x, \gamma) e^{-ix\xi} e^{-2\pi ij\gamma}\, d\gamma dx.$$

Finally, let $F = (F_1, 0, 0, 0)$ be a function in $\mathcal{C}_0^\infty(\partial D'_\beta)$. Then,

$$(S_{D'_\beta}F)_1(x, \gamma) = \sum_{j\in\mathbb{Z}} e^{2\pi ij\gamma} \mathcal{F}_{\mathbb{R}}^{-1} \left[\frac{e^{-(2\beta-\pi)(\cdot-\frac{j}{2})} e^{-\pi(\cdot)} \mathcal{F}_{\mathbb{R}} F_1(\cdot, \widehat{j})}{4\operatorname{Ch}[\pi\cdot]\operatorname{Ch}[(2\beta-\pi)(\cdot-\frac{j}{2})]} \right](x). \tag{31}$$

Even if (31) refers to a function of a specific form, it already contains the main features of the operator $S_{D'_\beta}$. In [41] it can be seen that $(S_{D'_\beta})_\ell, \ell = 1, \ldots, 4$, is given by a sum similar to the one in(31), but with more addends. Although, each of these addends is a minor modification of the ones already appearing in (31).

A few word on the features of the formula (31). If g is a smooth compactly supported function on the real line, $g \in \mathcal{C}_0^\infty(\mathbb{R})$, we say that M is a Fourier multiplier operator on the real line if M is of the form

$$Mg(x) := \mathcal{F}_{\mathbb{R}}^{-1}\left[m(\cdot)\mathcal{F}_{\mathbb{R}}g(\cdot)\right](x)$$

where m belongs to $L^\infty(\mathbb{R})$ and is called the multiplier associated to M. Similarly, if h is a smooth function on the one dimensional torus $\mathbb{T}$, we say that N is a Fourier multiplier operator on the torus, if N is of the form

$$Nh(\gamma) = \sum_{j\in\mathbb{Z}} n(j)\widehat{h}(j) e^{2\pi ij\gamma}$$

where $\widehat{h}(j)$ denotes the j-th Fourier coefficient of h and $\{n(j)\}_{j\in\mathbb{Z}}$ is a sequence in ℓ^∞. Therefore, we say from (31) that $S_{D'_\beta}$ is a Fourier multiplier operator on $\mathbb{R}\times\mathbb{T}$.

Using techniques from harmonic analysis and classical results for Fourier multiplier operators (see, e.g., [23]), we study the boundedness of $S_{D'_\beta}$ on $L^p(\partial D'_\beta)$ and $W^{k,p}(\partial D'_\beta)$. Given $p \in (1, \infty)$ and a real number $k > 0$, the Sobolev space $W^{k,p}(\partial D'_\beta)$ is the function space

$$W^{k,p}(\partial D'_\beta) = \left\{ F = (F_1, F_2, F_3, F_4) : \|F\|^p_{W^{k,p}(\partial D'_\beta)} = \sum_{\ell=1}^4 \|F_\ell\|^p_{W^{k,p}(\mathbb{R}\times\mathbb{T})} < \infty \right\} \tag{32}$$

where

$$\|F_\ell\|^p_{W^{k,p}(\mathbb{R}\times\mathbb{T})} = \int_{\mathbb{R}\times\mathbb{T}} \left| \sum_{j\in\mathbb{Z}} e^{2\pi i j\gamma} \mathcal{F}^{-1}_{\mathbb{R}} \Big[[1 + j^2 + (\cdot)^2]^{\frac{k}{2}} \mathcal{F}_{\mathbb{R}} F_\ell(\cdot, \widehat{j}) \Big](x) \right|^p dx d\gamma.$$

In [41] the following theorem is proved.

Theorem 4.1. *The Szegő projection extends to a linear bounded operator $S_{D'_\beta} : X \to X$ where X denotes either the Lebesgue space $L^p(\partial D'_\beta)$, $p \in (1,\infty)$, or the Sobolev space $W^{k,p}(\partial D'_\beta)$, $p \in (1,\infty)$ and k positive real number.*

We point out that Krantz and Peloso studied in [32] the L^p mapping properties of the Bergman projection $P_{D'_\beta}$ by computing and analyzing a precise asymptotic expansion of the Bergman kernel. Differently, in the proof of Theorem 4.1, we do not rely on an asymptotic expansion of the Szegő kernel $K_{D'_\beta}$. Nonetheless, we do obtain an asymptotic expansion for $K_{D'_\beta}$ in [40] and we compare it with the one obtained by Krantz and Peloso in the Bergman setting.

4.2 The Szegő projection of D_β

In this section we announce some results on the mapping properties of the Szegő projection S_{D_β} obtained in a joint work with Marco M. Peloso. The details of the proofs will appear in a forthcoming paper [42]

First, we need to find an explicit formula for the operator S_{D_β}. We do it by showing that the biholomorphism (7) induces an isometry between the spaces $H^2(D'_\beta)$ and $H^2(D_\beta)$. As a consequence, we also obtain a transformation rule for $S_{D'_\beta}$ and S_{D_β}.

We need the following elementary lemma contained in [33].

Lemma 4.2. *The function*

$$\psi(z_1, z_2) := e^{-\frac{1}{2}\log|z_2|^2} \Big[z_1 e^{-i\log|z_2|^2} \Big]^{-\frac{1}{2}} \tag{33}$$

is a well-defined holomorphic function on D_β.

We have the following proposition.

Proposition 4.3. *Let F be a function in $H^2(D'_\beta)$, $\varphi^{-1} : D_\beta \to D'_\beta$ the biholomorphism* (8) *and $\psi : D_\beta \to \mathbb{C}$ the function* (33)*. Then, the operator T defined by*

$$TF := \psi[F \circ \varphi^{-1}] \tag{34}$$

is an isometry $T : H^2(D'_\beta) \to H^2(D_\beta)$.

Proof. The holomorphicity of TF on D_β follows immediately from the lemma and the holomorphicity of φ^{-1}. It remains to prove that $\|F\|_{H^2(D'_\beta)} = \|TF\|_{H^2(D_\beta)}$. This is immediate since, by easy computations, it can be showed that $\mathcal{L}'_2(t,s) = \mathcal{L}_2 TF(t,s)$ for every $(t,s) \in (0, \frac{\pi}{2}) \times [0, \beta - \frac{\pi}{2})$. □

From Proposition 4.3, the following transformation rule for $S_{D'_\beta}$ and S_{D_β} can be deduced.

Theorem 4.4. *Let F be a function in $L^2(\partial D_\beta)$ and let T be the isometry* (34)*. Then,*

$$S_{D'_\beta}[(T^{-1}F)] = T^{-1}[S_{D_\beta} F]. \tag{35}$$

We remark that this transformation rule is similar to the one proved by Bell [6] for smooth bounded domains of $\mathbb{C}$. From (31) and (35) we can deduce an explicit formula for S_{D_β}. Before doing it, we fix some notation. It is clear from (24) that ∂D_β has two boundary components, namely

$$E_+ = \Big\{ (\rho e^{i\beta}, e^{\frac{1}{2}(\beta - \frac{\pi}{2})} e^{2\pi i\gamma} I \in \mathbb{C}^2 : \rho \in \mathbb{R}, \gamma \in [0, 2\pi) \Big\};$$

$$E_- = \left\{(\rho e^{-i\beta}, e^{-\frac{1}{2}(\beta-\frac{\pi}{2})}e^{2\pi i\gamma}I \in \mathbb{C}^2 : \rho \in \mathbb{R}, \gamma \in [0, 2\pi)\right\}.$$

Both E_+ and E_- can be identified with $\mathbb{R} \times \mathbb{T}$. Therefore, a function $F \in L^2(\partial D_\beta)$ can be identified with a vector of function (F_+, F_-) where F_+ and F_- are in $L^2(\mathbb{R} \times \mathbb{T})$. Similarly, the Szegő projection $S_{D_\beta}F$ of F can be identified with a vector $[(S_{D_\beta}F)_+, (S_{D_\beta}F)_-]$ where $(S_{D_\beta}F)_+$ is thought as defined on E_+, whereas $(S_{D_\beta}F)_-$ is thought as defined on E_-. We are now ready to write down an explicit formula for S_{D_β}. As in the case of $S_{D'_\beta}$, in order to avoid long and complicated formulas, we do not write $S_{D_\beta}F$ for the most general function F. Indeed, we suppose that F is of the form $F = (F_+, 0)$, i.e., F is constantly 0 on the component E_- of ∂D_β, and we suppose that F_+ is supported on $(0, \infty) \times \mathbb{T} \subseteq E_+$. Moreover, we only write explicitly the component $(S_{D_\beta}F)_+$ of $S_{D_\beta}F$. The expression of $S_{D_\beta}F$ for the most general F can be deduced from (35) and the formula for $S_{D'_\beta}$ contained in [41].

The assumption that F_+ is supported in $(0, \infty) \times \mathbb{T}$ is easily explained. We want to deduce $(S_{D_\beta}F)_+$ from (35) and (31), but (31) is the formula for a function which is non-zero only on the component E_1 of $\partial D'_\beta$. If φ is the biholomorphism (7) is immediate to check that $\varphi(E_1) = (0, \infty) \times \mathbb{T}$. Hence, our assumption on the support of F_+ is motivated.

We need one last remark before stating our results. Given a function g defined on $(0, \infty)$, the operator $\mathcal{C}$ defined by $\mathcal{C}g(x) = e^{\frac{x}{2}}g(e^x)$ is an isometry $\mathcal{C} : L^2(0, \infty) \to L^2(\mathbb{R})$. Since we assume that F_+ is supported on $(0, \infty) \times \mathbb{T}$, with an abuse of notation, we write $\mathcal{C}F_+$ to denote the function

$$\begin{aligned} \mathcal{C}F_+ &: \mathbb{R} \times \mathbb{T} \to \mathbb{C} \\ (x, \gamma) &\mapsto e^{\frac{x}{2}}F_+(e^x, \gamma). \end{aligned} \tag{36}$$

Finally, the following theorem holds.

Theorem 4.5. *Let $F = (F_+, 0)$ be a function in $L^2(\partial D_\beta)$ such that F_+ is supported in $(0, \infty) \times \mathbb{T}$. Then,*

$$(S_{D_\beta}F)_+(x, \gamma) = \begin{cases} \sum_{j \in \mathbb{Z}} e^{2\pi i\gamma}\mathcal{C}^{-1}\mathcal{F}_{\mathbb{R}}^{-1}\left[\frac{e^{-\pi(\cdot)}e^{-(2\beta-\pi)(\cdot-\frac{j}{2})}\mathcal{F}_{\mathbb{R}}\mathcal{C}F_+(\cdot,\widehat{j})}{4\,\mathrm{Ch}[\pi\cdot]\,\mathrm{Ch}[(2\beta-\pi)(\cdot-\frac{j}{2})]}\right](x) & \text{if } x > 0 \\ e^{i\frac{\pi}{2}}\sum_{j \in \mathbb{Z}} e^{2\pi ij\gamma}\mathcal{C}^{-1}\mathcal{F}_{\mathbb{R}}^{-1}\left[\frac{e^{-(2\beta-\pi)(\cdot-\frac{j}{2})}\mathcal{F}_{\mathbb{R}}[\mathcal{C}F_+(\cdot,\widehat{j})]}{4\,\mathrm{Ch}[\pi\cdot]\,\mathrm{Ch}[(2\beta-\pi)(\cdot-\frac{j}{2})]}\right](-x) & \text{if } x < 0. \end{cases} \tag{37}$$

Thus, $(S_{D_\beta}F)_+(x, \gamma)$, $x > 0$, is given by a Fourier multiplier operator on $\mathbb{R} \times \mathbb{T}$ acting on the function $\mathcal{C}F_+$ composed with the operator $\mathcal{C}^{-1}$. Similarly, we obtain $(S_{D_\beta}F)_+(x, \gamma)$ for $x < 0$.

It is not difficult to rewrite (37) in terms of the Mellin transform instead of the composition of $\mathcal{F}_{\mathbb{R}}$ and $\mathcal{C}$ (and their inverse). Here, given a function $g \in \mathcal{C}_0^\infty(0, \infty)$, we call the Mellin transform of g the function $\mathcal{M}_{\mathbb{R}}g$ defined by

$$\mathcal{M}_{\mathbb{R}}g(s) := \int_0^\infty g(x)x^s\,\frac{dx}{x}, \qquad s = \frac{1}{2} - it, t \in \mathbb{R}. \tag{38}$$

For a more general and precise discussion on the Mellin transform we refer the reader to [14, 15]. By simple change of variables, we immediately obtain the relationship

$$\mathcal{F}_{\mathbb{R}}\mathcal{C}g(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{\frac{x}{2}}g(e^x)e^{-ix\xi}\,dx = \frac{1}{2\pi}\int_0^\infty g(t)t^{[\frac{1}{2}-i\xi]}\,\frac{dt}{t} = \frac{1}{2\pi}\mathcal{M}_{\mathbb{R}}g(\frac{1}{2} - i\xi),$$

Reinterpreting (37) using the Mellin transform, we successfully apply some results by Rooney [45–47] to prove that the Szegő projection S_{D_β} is bounded on some L^p spaces.

Theorem 4.6. *The operator S_{D_β} extends to a linear bounded operator $S_{D_\beta} : L^p(\partial D_\beta) \to L^p(\partial D_\beta)$ if*

$$-\frac{\nu_\beta}{2} < \frac{1}{2} - \frac{1}{p} < \frac{\nu_\beta}{2}$$

where $\nu_\beta = \frac{\pi}{2\beta-\pi}$.

Subsequently, we provide an explicit counterexample which allows to improve the result in L^p scale and to obtain a partial result in Sobolev scale. For $p \in (1,\infty)$ and $k > 0$, the Sobolev space $W^{k,p}(\partial D_\beta)$ is defined similarly to $W^{k,p}(\partial D'_\beta)$ (see (32)). Recall that $\nu_\beta = \frac{\pi}{2\beta-\pi}$.

Theorem 4.7. *The operator S_{D_β} extends to a linear bounded operator $S_{D_\beta} : L^p(\partial D_\beta) \to L^p(\partial D_\beta)$ if and only if*

$$-\frac{\nu_\beta}{2} < \frac{1}{2} - \frac{1}{p} < \frac{\nu_\beta}{2}.$$

Moreover, if S_{D_β} extends to a linear bounded operator

$$S_{D_\beta} : W^{k,p}(\partial D_\beta) \to W^{k,p}(\partial D_\beta)$$

where $p \geq 2$ and $k > 0$, then

$$0 < k + \frac{1}{2} - \frac{1}{p} < \frac{\nu_\beta}{2}. \tag{39}$$

If we assume $p = 2$, then we improve the result in Sobolev scale showing that condition (39) is necessary and sufficient.

Theorem 4.8. *The operator S_{D_β} extends to a linear bounded operator $S_{D_\beta} : W^k(\partial D_\beta) \to W^k(\partial D_\beta)$ if and only if*

$$0 < k < \frac{\nu_\beta}{2}. \tag{40}$$

This last theorem is the analogous for the non-smooth worm D_β of Barrett's result. We remark that Barrett only proved the irregularity of the Bergman projection $P_{\mathcal{W}_\beta}$, whereas the easier geometry of D_β allows also to prove the regularity of S_{D_β}. The proofs of Theorem 4.6,Theorem 4.7 and Theorem 4.8 will appear in [42].

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