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Functional Analysis — *A density property for fractional weighted Sobolev spaces*, by Serena Dipierro and Enrico Valdinoci, communicated on 11 June 2015.[1]

Dedicated to the memory of Professor Enrico Magenes.

Abstract. — In this paper we show a density property for fractional weighted Sobolev spaces. That is, we prove that any function in a fractional weighted Sobolev space can be approximated by a smooth function with compact support.

The additional difficulty in this nonlocal setting is caused by the fact that the weights are not necessarily translation invariant.

Key words: Weighted fractional Sobolev spaces, density properties

Mathematics Subject Classification: 46E35, 35A15

Contents

1. Introduction

Goal of this paper is to provide an approximation result by smooth and compactly supported functions for a fractional Sobolev space with weights that are not necessarily translation invariant.

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The functional framework is the following. Given $s \in (0,1)$, $p \in (1,+\infty)$, such that $sp < n$, and

$$a \in \left[0, \frac{n-sp}{2}\right) \tag{1.1}$$

we introduce the semi-norm

$$[u]_{\tilde{W}_a^{s,p}(\mathbb{R}^n)} := \left(\iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} \frac{dx}{|x|^a} \frac{dy}{|y|^a} \right)^{1/p}. \tag{1.2}$$

We define the space

$$\tilde{W}_a^{s,p}(\mathbb{R}^n) := \{u : \mathbb{R}^n \to \mathbb{R} \text{ measurable s.t. } [u]_{\tilde{W}_a^{s,p}(\mathbb{R}^n)} < +\infty\}.$$

Also, we define the weighted norm

$$\|u\|_{L_a^{p_s^*}(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} \frac{|u(x)|^{p_s^*}}{|x|^{\frac{2ap_s^*}{p}}} dx \right)^{1/p_s^*}, \tag{1.3}$$

where p_s^* is the fractional critical Sobolev exponent associated to p, namely

$$p_s^* := \frac{np}{n-sp}.$$

Moreover, we set

$$L_a^{p_s^*}(\mathbb{R}^n) := \{u : \mathbb{R}^n \to \mathbb{R} \text{ measurable s.t. } \|u\|_{L_a^{p_s^*}(\mathbb{R}^n)} < +\infty\}.$$

The importance of the weighted norm in (1.3) lies in the fact that, when a lies in the range prescribed by (1.1), a weighted fractional Sobolev inequality holds true, as proved in [1]: more precisely, there exists a constant $C_{n,s,p,a} > 0$ such that

$$\|u\|_{L_a^{p_s^*}(\mathbb{R}^n)} \le C_{n,s,p,a}[u]_{\tilde{W}_a^{s,p}(\mathbb{R}^n)},$$

for any $u \in C_0^\infty(\mathbb{R}^n)$. So we define $\dot{W}_a^{s,p}(\mathbb{R}^n) := \tilde{W}_a^{s,p}(\mathbb{R}^n) \cap L_a^{p_s^*}(\mathbb{R}^n)$, which is naturally endowed with the norm

$$\|u\|_{\dot{W}_a^{s,p}(\mathbb{R}^n)} := [u]_{\tilde{W}_a^{s,p}(\mathbb{R}^n)} + \|u\|_{L_a^{p_s^*}(\mathbb{R}^n)}. \tag{1.4}$$

The space $\dot{W}_a^{s,p}(\mathbb{R}^n)$ has recently appeared in the literature in several circumstances, such as in a clever change of variable (see [11]), and in a critical and fractional Hardy equation (see [7]). Even the case with $a = 0$ presents some applications, see e.g. [6].

A natural question is whether functions with finite norm in $\dot{W}_a^{s,p}(\mathbb{R}^n)$ can be approximated by smooth functions with compact support. This is indeed the case, as stated by our main result:

THEOREM 1.1. *For any* $u \in \dot{W}_a^{s,p}(\mathbb{R}^n)$ *there exists a sequence of functions* $u_\varepsilon \in C_0^\infty(\mathbb{R}^n)$ *such that* $\|u - u_\varepsilon\|_{\dot{W}_a^{s,p}(\mathbb{R}^n)} \to 0$ *as* $\varepsilon \to 0$. *Namely,* $C_0^\infty(\mathbb{R}^n)$ *is dense in* $\dot{W}_a^{s,p}(\mathbb{R}^n)$.

We observe that Theorem 1.1 comprises also the "unweighted" case $a = 0$ (though, in this setting, the proof can be radically simplified, thanks to the translation invariance of the kernel, see e.g. [9, 13]). The result obtained in Theorem 1.1 here plays also a crucial role in [7] to obtain sharp decay estimates of the solution of a weighted equation near the singularities and at infinity.

For related results in weighted Sobolev spaces with integer exponents see for instance [3, 12, 2, 15] and the references therein.

The paper is essentially self-contained. We tried to avoid as much as possible any unnecessary complication arising from the presence of the weights and to explain all the technical details of the arguments presented.

The paper is organized as follows. In Section 2 we show a basic lemma that states that the space under consideration is not trivial. In Section 3 we show that we can perform an approximation with compactly supported functions.

The approximation with smooth functions is, in general, more difficult to obtain, due to the presence of weights that are not translation invariant. More precisely, the standard approximation techniques that rely on convolution need to be carefully reviewed, since the arguments based on the continuity under translations in the classical Lebesgue spaces fail in this case. To overcome this type of difficulties, in Section 4 we estimate the "averaged" error produced by translations of the weights and we use this estimate to control the norm of a mollification in terms of the norm of the original function.

Then, in Section 5, we perform an approximation with continuous functions, by carefully exploiting the Lusin's Theorem. The approximation with smooth functions is proved in Section 6, by using all the ingredients that were previously introduced. Finally, Section 7 is devoted to the proof of Theorem 1.1.

2. A BASIC LEMMA

In this section we consider a more general semi-norm and we show that it is bounded for functions in $C_0^\infty(\mathbb{R}^n)$. This remark shows that there is an interesting range of parameters for which the space considered here is not trivial.

We take $\alpha, \beta \in \mathbb{R}$ such that

$$-sp < \alpha, \beta < n \quad \text{and} \quad \alpha + \beta < n, \tag{2.1}$$

and we define

$$[u]_{\tilde{W}^{s,p}_{\alpha,\beta}(\mathbb{R}^n)} := \iint_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \frac{dx}{|x|^\alpha} \frac{dy}{|y|^\beta}. \tag{2.2}$$

LEMMA 2.1. *Let* $\varphi \in C_0^\infty(\mathbb{R}^n)$. *Then there exists a positive constant* C *such that*

$$[\varphi]_{\tilde{W}^{s,p}_{\alpha,\beta}(\mathbb{R}^n)} \le C.$$

PROOF. We take $\varphi \in C_0^\infty(\mathbb{R}^n)$ and we suppose that the support of φ is contained in the ball B_R (for some $R > 1$). Therefore, if $x, y \in \mathbb{R}^n \setminus B_R$ then $\varphi(x) = \varphi(y) = 0$, and so we can assume in the integral in (2.2) for φ that $x \in B_R$, up to a factor 2, i.e. we have to estimate

$$I = \iint_{B_R \times \mathbb{R}^n} \frac{|\varphi(x) - \varphi(y)|^p}{|x - y|^{n+sp}} \frac{dx}{|x|^\alpha} \frac{dy}{|y|^\beta} = I_1 + I_2, \tag{2.3}$$

where

$$I_1 := \iint_{B_R \times B_{2R}} \frac{|\varphi(x) - \varphi(y)|^p}{|x - y|^{n+sp}} \frac{dx}{|x|^\alpha} \frac{dy}{|y|^\beta} \quad \text{and}$$

$$I_2 := \iint_{B_R \times (\mathbb{R}^n \setminus B_{2R})} \frac{|\varphi(x) - \varphi(y)|^p}{|x - y|^{n+sp}} \frac{dx}{|x|^\alpha} \frac{dy}{|y|^\beta}.$$

We first estimate I_1: we have

$$I_1 \le C \iint_{B_{2R} \times B_{2R}} \frac{|x - y|^p}{|x - y|^{n+sp}} \frac{dx}{|x|^\alpha} \frac{dy}{|y|^\beta}, \tag{2.4}$$

for some constant $C > 0$ depending on the C^1-norm of u. Now, if $\alpha, \beta < 0$ then $|x|^{-\alpha} \le (2R)^{-\alpha}$ and $|y|^{-\beta} \le (2R)^{-\beta}$. Therefore, by the change of variable $z = x - y$, we get

$$I_1 \le C(2R)^{-\alpha}(2R)^{-\beta} \int_{B_{2R}} dx \int_{B_{2R}} dz |x - y|^{p-n-sp} \le C,$$

up to renaming C, that possibly depends on R.

Now we suppose that $\alpha, \beta \ge 0$. We claim that

$$I_1 \le C \iint_{B_{2R} \times B_{2R}} |x - y|^{p-n-sp} \frac{dx\, dy}{|x|^{\alpha+\beta}}. \tag{2.5}$$

Indeed, if $|x| \le |y|$, then formula (2.5) trivially follows from (2.4). On the other hand, if $|x| \ge |y|$, then

$$I_1 \le C \iint_{B_{2R} \times B_{2R}} |x - y|^{p-n-sp} \frac{dx\, dy}{|y|^{\alpha+\beta}},$$

and so by symmetry we get (2.5).

From (2.5) we obtain that

$$I_1 \le C \int_{B_{2R}} \frac{dx}{|x|^{\alpha+\beta}} \int_{B_{4R}} \frac{dz}{|z|^{n+sp-p}} \le C,$$

thanks to (2.1), up to renaming C.

Finally, we deal with the case $\alpha \geq 0$ and $\beta \leq 0$ (the other situation is symmetric). Then, $|y|^{-\beta} \leq (2R)^{-\beta}$, and so

$$\begin{aligned} I_1 &\leq C(2R)^{-\beta} \iint_{B_{2R}\times B_{2R}} |x-y|^{p-n-sp} \frac{dx\,dy}{|x|^{\alpha}} \\ &\leq C(2R)^{-\beta} \int_{B_{2R}} \frac{dx}{|x|^{\alpha}} \int_{B_{4R}} |z|^{p-n-sp}\,dz \\ &\leq C, \end{aligned}$$

thanks to (2.1), up to relabelling C (that depends also on R).

Therefore, we have shown that for any α, β that satisfy (2.1) we have that

$$I_1 \leq C, \tag{2.6}$$

up to renaming the constant C.

Now we estimate I_2. For this, we observe that if $x \in B_R$ and $y \in \mathbb{R}^n \setminus B_{2R}$ then

$$|x-y| \geq |y| - |x| = \frac{|y|}{2} + \frac{|y|}{2} - |x| \geq \frac{|y|}{2}.$$

Thus

$$\begin{aligned} I_2 &\leq 2^{n+sp}(2\|u\|_{L^\infty(\mathbb{R}^n)})^p \iint_{B_R\times(\mathbb{R}^n\setminus B_{2R})} \frac{1}{|y|^{n+sp}} \frac{dx}{|x|^{\alpha}} \frac{dy}{|y|^{\beta}} \\ &\leq 2^{n+sp}(2\|u\|_{L^\infty(\mathbb{R}^n)})^p \int_{B_R} \frac{dx}{|x|^{\alpha}} \int_{\mathbb{R}^n\setminus B_{2R}} \frac{dy}{|y|^{n+sp+\beta}} \\ &\leq C, \end{aligned}$$

thanks to (2.1). Using this and (2.6) into (2.3) we obtain that I is bounded. □

As an obvious consequence of Lemma 2.1, we have that $C_0^\infty(\mathbb{R}^n) \subseteq \tilde{W}_a^{s,p}(\mathbb{R}^n)$, and so, by (1.1), we see that $C_0^\infty(\mathbb{R}^n) \subseteq \dot{W}_a^{s,p}(\mathbb{R}^n)$. This says that the approximation seeked by Theorem 1.1 is meaningful.

3. Approximation by compactly supported functions

In this section we will prove that we can approximate a function in $\dot{W}_a^{s,p}(\mathbb{R}^n)$ with another function with compact support, by keeping the error small.

Lemma 3.1. *Let $u \in \dot{W}_a^{s,p}(\mathbb{R}^n)$. Let $\tau \in C_0^\infty(B_2, [0,1])$ with $\tau = 1$ in B_1, and $\tau_j(x) := \tau(x/j)$. Then*

$$\lim_{j\to+\infty} \|u - \tau_j u\|_{\dot{W}_a^{s,p}(\mathbb{R}^n)} = 0.$$

PROOF. We set $\eta_j := 1 - \tau_j$. Then $u - \tau_j u = \eta_j u$, and $\eta_j(x) - \eta_j(y) = \tau_j(y) - \tau_j(x)$. Accordingly $|u(x) - \tau_j u(x)| \le 2|u(x)|$ and so

$$\lim_{j\to+\infty} \|u - \tau_j u\|_{L_a^{p_s^*}(\mathbb{R}^n)} = 0, \tag{3.1}$$

by the Dominated Convergence Theorem. Moreover,

$$|\eta_j u(x) - \eta_j u(y)| \le |\tau_j(x) - \tau_j(y)|\,|u(y)| + |u(x) - u(y)|\eta_j(x).$$

Also, we observe that if both x and y lie in B_j, then $\tau_j(x) = \tau_j(y) = 1$. Therefore

$$\begin{aligned} &\iint_{\mathbb{R}^{2n}} \frac{|(u-\tau_j u)(x) - (u-\tau_j u)(y)|^p}{|x-y|^{n+sp}} \frac{dx}{|x|^a}\frac{dy}{|y|^a} \\ &\quad\le 2\iint_{\mathbb{R}^n\times(\mathbb{R}^n\setminus B_j)} \frac{|(u-\tau_j u)(x) - (u-\tau_j u)(y)|^p}{|x-y|^{n+sp}} \frac{dx}{|x|^a}\frac{dy}{|y|^a} \\ &\quad\le C(I_j + J_j), \end{aligned} \tag{3.2}$$

where

$$I_j := \iint_{\mathbb{R}^n\times(\mathbb{R}^n\setminus B_j)} \frac{|\tau_j(x)-\tau_j(y)|^p|u(y)|^p}{|x-y|^{n+sp}} \frac{dx}{|x|^a}\frac{dy}{|y|^a}$$

$$\text{and}\quad J_j := \iint_{\mathbb{R}^n\times(\mathbb{R}^n\setminus B_j)} \frac{|u(x)-u(y)|^p\eta_j^p(x)}{|x-y|^{n+sp}} \frac{dx}{|x|^a}\frac{dy}{|y|^a}.$$

We estimate these two terms separately. First of all, we estimate I_j. For this, we define

$$\begin{aligned} &D_{j,0} := \{(x,y)\in\mathbb{R}^n\times(\mathbb{R}^n\setminus B_j) \text{ s.t. } |x|\le|y|/2\}, \\ &D_{j,1} := \{(x,y)\in\mathbb{R}^n\times(\mathbb{R}^n\setminus B_j) \text{ s.t. } |x|>|y|/2 \text{ and } |x-y|\ge j\} \\ \text{and}\quad &D_{j,2} := \{(x,y)\in\mathbb{R}^n\times(\mathbb{R}^n\setminus B_j) \text{ s.t. } |x|>|y|/2 \text{ and } |x-y|< j\}, \end{aligned}$$

and we write, for $k\in\{0,1,2\}$,

$$I_{j,k} := \iint_{D_{j,k}} \frac{|\tau_j(x)-\tau_j(y)|^p|u(y)|^p}{|x-y|^{n+sp}} \frac{dx}{|x|^a}\frac{dy}{|y|^a}.$$

Notice that

$$I_j = I_{j,0} + I_{j,1} + I_{j,2}. \tag{3.3}$$

So we define $\sigma_0 := s$, and we fix $\sigma_1 \in (0,s)$ and $\sigma_2 \in (s,1)$. We write

$$\frac{|\tau_j(x)-\tau_j(y)|^p|u(y)|^p}{|x-y|^{n+sp}|x|^a|y|^a} = \frac{|\tau_j(x)-\tau_j(y)|^p}{|x-y|^{(s+\sigma_k)p}} \cdot \frac{|u(y)|^p}{|x-y|^{n-\sigma_k p}|x|^a|y|^a}.$$

Thus we apply the Hölder inequality with exponents n/sp and p_s^*/p (which is in turn equal to $n/(n-sp)$) and, for any $k \in \{0,1,2\}$, we obtain that

$$
\begin{aligned}
(3.4)\qquad I_{j,k} \le {} & \left[\iint_{D_{j,k}} \frac{|\tau_j(x)-\tau_j(y)|^{\frac{n}{s}}}{|x-y|^{\frac{(s+\sigma_k)n}{s}}}\,dx\,dy\right]^{\frac{sp}{n}} \\
& \cdot \left[\iint_{D_{j,k}} \frac{|u(y)|^{p_s^*}}{|x-y|^{\frac{(n-\sigma_k p)n}{n-sp}}|x|^{\frac{ap_s^*}{p}}|y|^{\frac{ap_s^*}{p}}}\,dx\,dy\right]^{\frac{n-sp}{n}}.
\end{aligned}
$$

Now we change variable $X := x/j$ and we see that

$$
\begin{aligned}
\iint_{D_{j,k}} \frac{|\tau_j(x)-\tau_j(y)|^{\frac{n}{s}}}{|x-y|^{\frac{(s+\sigma_k)n}{s}}}\,dx\,dy &= \iint_{D_{j,k}} \frac{|\tau(x/j)-\tau(y/j)|^{\frac{n}{s}}}{|x-y|^{\frac{(s+\sigma_k)n}{s}}}\,dx\,dy \\
&= j^{2n-\frac{(s+\sigma_k)n}{s}} \iint_{D_{1,k}} \frac{|\tau(X)-\tau(Y)|^{\frac{n}{s}}}{|X-Y|^{\frac{(s+\sigma_k)n}{s}}}\,dX\,dY \\
&= j^{\frac{(s-\sigma_k)n}{s}} \iint_{D_{1,k}} \frac{|\tau(x)-\tau(y)|^{\frac{n}{s}}}{|x-y|^{n+\sigma_k\frac{n}{s}}}\,dx\,dy.
\end{aligned}
$$

That is, if we set $P := n/s$, we get that

$$
(3.5)\qquad \iint_{D_{j,k}} \frac{|\tau_j(x)-\tau_j(y)|^{\frac{n}{s}}}{|x-y|^{\frac{(s+\sigma_k)n}{s}}}\,dx\,dy \le j^{\frac{(s-\sigma_k)n}{s}}\|\tau\|_{\dot W^{\sigma_k,P}(\mathbb{R}^n)} \le Cj^{\frac{(s-\sigma_k)n}{s}},
$$

where $\dot W^{\sigma,P}(\mathbb{R}^n)$ is the usual Gagliardo semi-norm (which coincides with $\tilde W_a^{\sigma,P}(\mathbb{R}^n)$ with $a = 0$, see e.g. [5]).

In addition, if $(x,y) \in D_{j,0}$, we have that $|x-y| \ge |y|-|x| \ge |y|/2$ and so

$$
\begin{aligned}
(3.6)\qquad & \iint_{D_{j,0}} \frac{|u(y)|^{p_s^*}}{|x-y|^{\frac{(n-\sigma_0 p)n}{n-sp}}|x|^{\frac{ap_s^*}{p}}|y|^{\frac{ap_s^*}{p}}}\,dx\,dy \\
&\le C\iint_{D_{j,0}} \frac{|u(y)|^{p_s^*}}{|x|^{\frac{ap_s^*}{p}}|y|^{\frac{(n-\sigma_0 p)n}{n-sp}+\frac{ap_s^*}{p}}}\,dx\,dy \\
&\le C\int_{\mathbb{R}^n\setminus B_j}\left[\int_0^{|y|/2} \rho^{n-1-\frac{ap_s^*}{p}} \frac{|u(y)|^{p_s^*}}{|y|^{\frac{n(n-sp+a)}{n-sp}}}\,d\rho\right]dy \\
&= C\int_{\mathbb{R}^n\setminus B_j} \frac{|y|^{\frac{n(n-sp-a)}{n-sp}}|u(y)|^{p_s^*}}{|y|^{\frac{n(n-sp+a)}{n-sp}}}\,dy \\
&= C\int_{\mathbb{R}^n\setminus B_j} \frac{|u(y)|^{p_s^*}}{|y|^{\frac{2ap_s^*}{p}}}\,dy.
\end{aligned}
$$

Moreover, if $k \in \{1, 2\}$, using the change of variable $z := x - y$ (and integrating in $y \in \mathbb{R}^n \setminus B_j$ separately), we see that

$$\begin{aligned}
&\iint_{D_{j,k}} \frac{|u(y)|^{p_s^*}}{|x-y|^{\frac{(n-\sigma_k p)n}{n-sp}} |x|^{\frac{ap_s^*}{p}} |y|^{\frac{ap_s^*}{p}}} \, dx\, dy \\
&\quad \le C \iint_{D_{j,k}} \frac{|u(y)|^{p_s^*}}{|x-y|^{\frac{(n-\sigma_k p)n}{n-sp}} |y|^{\frac{2ap_s^*}{p}}} \, dx\, dy \\
&\quad = C \iint_{D_{j,k}} \frac{|u(y)|^{p_s^*}}{|x-y|^{n+\frac{(s-\sigma_k)pn}{n-sp}} |y|^{\frac{2ap_s^*}{p}}} \, dx\, dy \\
&\quad \le \begin{cases} C\|u\|_{L_a^{p_s^*}(\mathbb{R}^n \setminus B_j)} \displaystyle\int_{\mathbb{R}^n \setminus B_j} \frac{dz}{|z|^{n+\frac{(s-\sigma_1)pn}{n-sp}}} & \text{if } k = 1, \\ C\|u\|_{L_a^{p_s^*}(\mathbb{R}^n \setminus B_j)} \displaystyle\int_{B_j} \frac{dz}{|z|^{n+\frac{(s-\sigma_2)pn}{n-sp}}} & \text{if } k = 2. \end{cases}
\end{aligned}$$

Thus, recalling that $\sigma_1 < s < \sigma_2$, we conclude that, for any $k \in \{1, 2\}$,

$$\iint_{D_{j,k}} \frac{|u(y)|^{p_s^*}}{|x-y|^{\frac{(n-\sigma_k p)n}{n-sp}} |x|^{\frac{ap_s^*}{p}} |y|^{\frac{ap_s^*}{p}}} \, dx\, dy \le C\|u\|_{L_a^{p_s^*}(\mathbb{R}^n \setminus B_j)} \, j^{\frac{(\sigma_k - s)pn}{n-sp}}. \tag{3.7}$$

As a matter of fact, in virtue of (3.6), and recalling that $\sigma_0 = s$, we have that the above equation is valid also for $k = 0$.

So, for $k \in \{0, 1, 2\}$, we insert formulas (3.7) and (3.5) into (3.4) and we conclude that

$$I_{j,k} \le C\left(j^{\frac{(s-\sigma_k)n}{s}}\right)^{\frac{sp}{n}} \cdot \left(\|u\|_{L_a^{p_s^*}(\mathbb{R}^n \setminus B_j)} \, j^{\frac{(\sigma_k - s)pn}{n-sp}}\right)^{\frac{n-sp}{n}} \le C\|u\|^{\frac{n-sp}{n}}_{L_a^{p_s^*}(\mathbb{R}^n \setminus B_j)}.$$

Thus, by (3.3), we obtain

$$I_j \le C\|u\|^{\frac{n-sp}{n}}_{L_a^{p_s^*}(\mathbb{R}^n \setminus B_j)} \to 0 \quad \text{as } j \to +\infty. \tag{3.8}$$

Now we consider J_j. For this, we define

$$\psi_j(x, y) := \chi_{\mathbb{R}^n \times (\mathbb{R}^n \setminus B_j)}(x, y) \, \frac{|u(x) - u(y)|^p \, \eta_j^p(x)}{|x-y|^{n+sp} |x|^a |y|^a}.$$

Notice that

$$|\psi_j(x, y)| \le \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp} |x|^a |y|^a} \in L^1(\mathbb{R}^{2n}),$$

thus, by the Dominated Convergence Theorem,

$$J_j = \iint_{\mathbb{R}^{2n}} \psi_j(x, y)\, dx\, dy \to 0 \quad \text{as } j \to +\infty.$$

This, (3.2) and (3.8) give that

$$\iint_{\mathbb{R}^{2n}} \frac{|(u - \tau_j u)(x) - (u - \tau_j u)(y)|^p}{|x - y|^{n+sp}|x|^a|y|^a}\, dx\, dy \to 0 \quad \text{as } j \to +\infty. \tag{3.9}$$

The latter formula and (3.1) give the desired result. □

4. Estimates in average and control of the convolution

Here we perform some detailed estimate on the "averaged" effect of the weights under consideration. Roughly speaking, the weights themselves are not translation invariant, but we will be able to estimate the averaged effect of the translations in a somehow uniform way.

From this, we will be able to control the norm of the mollification by the norm of the original function, and this fact will in turn play a crucial role in the approximation with smooth functions performed in Section 6 (namely, one will approximate first a given function in the space with a continuous and compactly supported function, so one will have to bound the convolution of this difference in terms of the difference of the original functions).

Due to the presence of two types of weights, the arguments of this part are quite technical. We start with an averaged weighted estimate:

PROPOSITION 4.1. *There exists $C > 0$ such that*

$$\sup_{r>0} \frac{1}{r^n} \int_{B_r} \frac{dz}{|x + z|^a |y + z|^a} \le \frac{C}{|x|^a |y|^a},$$

for every $x, y \in \mathbb{R}^n \setminus \{0\}$.

PROOF. Fixed $r > 0$, consider the following four domains:

$$D_0 := \left\{ z \in B_r \text{ s.t. } |x + z| \ge \frac{|x|}{2} \text{ and } |y + z| \ge \frac{|y|}{2} \right\},$$

$$D_1 := \left\{ z \in B_r \text{ s.t. } |x + z| \le \frac{|x|}{2} \text{ and } |y + z| \ge \frac{|y|}{2} \right\},$$

$$D_2 := \left\{ z \in B_r \text{ s.t. } |x + z| \ge \frac{|x|}{2} \text{ and } |y + z| \le \frac{|y|}{2} \right\}$$

$$\text{and} \quad D_3 := \left\{ z \in B_r \text{ s.t. } |x + z| \le \frac{|x|}{2} \text{ and } |y + z| \le \frac{|y|}{2} \right\}.$$

Then

$$\int_{D_0} \frac{dz}{|x+z|^a |y+z|^a} \le \int_{B_r} \frac{dz}{(|x|/2)^a (|y|/2)^a} \le \frac{4^a |B_r|}{|x|^a |y|^a}. \tag{4.1}$$

Now we observe that

$$\begin{aligned} &\text{if there exists } z \in B_r \text{ such that } |x+z| \le \frac{|x|}{2}, \\ &\text{then } r \ge |z| \ge |x| - |x+z| \ge \frac{|x|}{2}. \end{aligned} \tag{4.2}$$

From this, we observe that if $D_1 \neq \emptyset$ it follows that $r \ge |x|/2$ and so, using the substitution $\zeta := x + z$,

$$\begin{aligned} \int_{D_1} \frac{dz}{|x+z|^a |y+z|^a} &\le \frac{2^a}{|y|^a} \int_{B_r} \frac{dz}{|x+z|^a} \le \frac{2^a}{|y|^a} \int_{B_{r+|x|}} \frac{d\zeta}{|\zeta|^a} \\ &\le \frac{C_1 (r+|x|)^{n-a}}{|y|^a} \le \frac{C_2 (3r)^n}{(r+|x|)^a |y|^a} \\ &\le \frac{C_2 (3r)^n}{|x|^a |y|^a} = \frac{C_3 r^n}{|x|^a |y|^a}, \end{aligned} \tag{4.3}$$

for some constants $C_1, C_2, C_3 > 0$. Similarly, by exchanging the roles of x and y, we see that

$$\int_{D_2} \frac{dz}{|x+z|^a |y+z|^a} \le \frac{C_4 r^n}{|x|^a |y|^a}. \tag{4.4}$$

Moreover, if $D_3 \neq \emptyset$, we deduce from (4.2) (and the similar formula for y) that

$$r \ge \max\left\{ \frac{|x|}{2}, \frac{|y|}{2} \right\}$$

and therefore

$$\begin{aligned} \int_{D_3} \frac{dz}{|x+z|^a |y+z|^a} &\le \sqrt{\int_{B_r} \frac{dz}{|x+z|^{2a}}} \sqrt{\int_{B_r} \frac{dz}{|y+z|^{2a}}} \\ &\le \sqrt{\int_{B_{r+|x|}} \frac{dz}{|\zeta|^{2a}}} \sqrt{\int_{B_{r+|y|}} \frac{dz}{|\zeta|^{2a}}} \\ &\le C_5 \sqrt{(r+|x|)^{n-2a}} \sqrt{(r+|y|)^{n-2a}} \\ &= \frac{C_5 (r+|x|)^{n/2} (r+|y|)^{n/2}}{(r+|x|)^a (r+|y|)^a} \\ &\le \frac{C_5 (3r)^{n/2} (3r)^{n/2}}{|x|^a |y|^a} = \frac{C_6 r^n}{|x|^a |y|^a}. \end{aligned} \tag{4.5}$$

Notice that we have used all over in the integrals that $a \le 2a < n$, thanks to (1.1).

The desired result now follows by combining (4.1), (4.3), (4.4), (4.5) and the fact that $B_r = D_0 \cup D_1 \cup D_2 \cup D_3$. $\square$

A simpler (but still useful for our purposes) version of Proposition 4.1 is the following:

Proposition 4.2. *Let* $b := \frac{2ap_s^*}{p} = \frac{2an}{n-sp}$. *There exists* $C > 0$ *such that*

$$\sup_{r>0} \frac{1}{r^n} \int_{B_r} \frac{dz}{|x+z|^b} \le \frac{C}{|x|^b},$$

for every $x \in \mathbb{R}^n \setminus \{0\}$.

Proof. The proof is similar to the one of Proposition 4.1, just dropping the dependence in y. We give the details for the facility of the reader. Fixed $r > 0$, consider the following two domains:

$$D_0 := \left\{ z \in B_r \text{ s.t. } |x+z| \ge \frac{|x|}{2} \right\},$$

$$\text{and} \quad D_1 := \left\{ z \in B_r \text{ s.t. } |x+z| \le \frac{|x|}{2} \right\}.$$

Then

$$\int_{D_0} \frac{dz}{|x+z|^b} \le \int_{B_r} \frac{dz}{(|x|/2)^b} \le \frac{2^b |B_r|}{|x|^b}. \tag{4.6}$$

Now we observe that if there exists $z \in B_r$ such that $|x+z| \le |x|/2$, then $r \ge |z| \ge |x| - |x+z| \ge |x|/2$. From this, we observe that if $D_1 \ne \emptyset$ it follows that $r \ge |x|/2$ and so

$$\int_{D_1} \frac{dz}{|x+z|^b} \le \int_{B_{r+|x|}} \frac{d\zeta}{|\zeta|^b} \le C_1 (r+|x|)^{n-b} = \frac{C_1 (r+|x|)^n}{(r+|x|)^b} \le \frac{C_1 (3r)^n}{|x|^b} \tag{4.7}$$

for some constant $C_1 > 0$. We observe that we have used here above that $b < n$, thanks to (1.1). Then, formulas (4.7) and (4.6) imply the desired result. $\square$

Now, we observe that, in this paper, two types of "different" weighted norms appear all over, namely (1.2) and (1.3). In order to deal with both of them at the same time, we introduce now an "abstract" notation, by working in $\mathbb{R}^N$ (then, in our application, we will choose either $N = n$ or $N = 2n$). Also, we will consider two functions $\varpi : \mathbb{R}^n \to \mathbb{R}^N$ and $\Theta : \mathbb{R}^N \to [0, +\infty]$. The main assumption that we will take is that

$$\sup_{r>0} \frac{1}{r^n} \int_{B_r} \frac{dz}{\Theta(X + \varpi(z))} \le \frac{C}{\Theta(X)}, \tag{4.8}$$

for a suitable $C > 0$, for a.e. $X \in \mathbb{R}^N$. We point out that the integral in (4.8) is always performed on an n-dimensional ball B_r (i.e., in that notation, $z \in B_r \subset \mathbb{R}^n$), but the point X lies in $\mathbb{R}^N$ (and n and N may be different).

Concretely, in the light of Propositions 4.1 and 4.2, we have that condition (4.8) holds true when

$$N = 2n, \quad \varpi(z) = (z, z), \quad \Theta(X) = |x|^a |y|^a, \quad X = (x, y) \in \mathbb{R}^n \times \mathbb{R}^n,$$

and when

$$N = n, \quad \varpi(z) = z, \quad \Theta(x) = |x|^b, \quad b = \frac{2ap_s^*}{p}. \tag{4.9}$$

From (4.8), we obtain a useful bound on (a suitable variant of) the maximal function in $\mathbb{R}^n \times \mathbb{R}^n$:

Lemma 4.3. *Assume that condition* (4.8) *holds true. Let* $q > 1$. *Let* V *be a measurable function from* $\mathbb{R}^N$ *to* $\mathbb{R}$. *Then, for any* $r > 0$,

$$\int_{\mathbb{R}^N} \left[\frac{1}{r^n} \int_{B_r} |V(X - \varpi(z))| \, dz \right]^q \frac{dX}{\Theta(X)} \leq C \int_{\mathbb{R}^N} \frac{|V(X)|^q}{\Theta(X)} \, dX, \tag{4.10}$$

for a suitable $C > 0$.

Proof. We may suppose that the right hand side of (4.10) is finite, otherwise we are done. We use the Hölder inequality with exponents q and $q/(q-1)$, to see that

$$\begin{aligned} \frac{1}{r^n} \int_{B_r} |V(X - \varpi(z))| \, dz &\leq \frac{1}{r^n} \left[\int_{B_r} |V(X - \varpi(z))|^q \, dz \right]^{1/q} \left[\int_{B_r} 1 \, dz \right]^{(q-1)/q} \\ &= \frac{C_1}{r^{n/q}} \left[\int_{B_r} |V(X - \varpi(z))|^q \, dz \right]^{1/q}, \end{aligned}$$

for some $C_1 > 0$, and so, by (4.8), and using the change of variable $\tilde{X} := X - \varpi(z)$ over $\mathbb{R}^N$, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^N} \left[\frac{1}{r^n} \int_{B_r} |V(X - \varpi(z))| \, dz \right]^q \frac{dX}{\Theta(X)} \\ &\quad \leq \frac{C_1^q}{r^n} \int_{\mathbb{R}^N} \left[\int_{B_r} |V(X - \varpi(z))|^q \, dz \right] \frac{dX}{\Theta(X)} \\ &\quad = \frac{C_1^q}{r^n} \int_{B_r} \left[\int_{\mathbb{R}^N} |V(X - \varpi(z))|^q \frac{dX}{\Theta(X)} \right] dz \end{aligned}$$

$$
\begin{aligned}
&= \frac{C_1^q}{r^n} \int_{B_r} \left[\int_{\mathbb{R}^N} |V(\tilde{X})|^q \frac{d\tilde{X}}{\Theta(\varpi(z) + \tilde{X})} \right] dz \\
&= \frac{C_1^q}{r^n} \int_{\mathbb{R}^N} |V(\tilde{X})|^q \left[\int_{B_r} \frac{dz}{\Theta(\varpi(z) + \tilde{X})} \right] d\tilde{X} \\
&\le \int_{\mathbb{R}^N} |V(\tilde{X})|^q \frac{C_2}{\Theta(\tilde{X})} d\tilde{X},
\end{aligned}
$$

as desired. □

With the estimate in Lemma 4.3, we are in the position of bounding a (suitable variant of) the standard mollification. For this, we take a radially symmetric, radially decreasing function $\eta_o \in C^\infty(\mathbb{R}^n)$, with $\eta \ge 0$, $\operatorname{supp} \eta_o \subseteq B_1$ and

$$
\int_{\mathbb{R}^n} \eta_o(x)\, dx = 1 \tag{4.11}
$$

With a slight abuse of notation, we write $\eta_o(r) = \eta_o(x)$ whenever $|x| = r$. Given a measurable function $v = v(x, y)$ from $\mathbb{R}^{2n}$ to $\mathbb{R}$, we also define

$$
v \star \eta_o(x, y) := \int_{\mathbb{R}^n} v(x - z, y - z)\, \eta_o(z)\, dz. \tag{4.12}
$$

Then we have:

Proposition 4.4. *For every measurable function $v = v(x, y)$ from $\mathbb{R}^{2n}$ to $\mathbb{R}$, we have that*

$$
\iint_{\mathbb{R}^{2n}} \frac{|v \star \eta_o(x, y)|^p}{|x|^a |y|^a} dx\, dy \le C \iint_{\mathbb{R}^{2n}} \frac{|v(x, y)|^p}{|x|^a |y|^a} dx\, dy,
$$

for a suitable $C > 0$.

Proof. The argument is a careful modification of the one on pages 63–65 of [14]. First of all, we use an integration by parts to notice that

$$
\begin{aligned}
\int_0^1 r^n |\eta_o'(r)|\, dr &= -\int_0^1 r^n \eta_o'(r)\, dr = n \int_0^1 r^{n-1} \eta_o(r)\, dr \\
&= C_0 \int_{B_1} \eta_o(x)\, dx = C_0,
\end{aligned} \tag{4.13}
$$

for some $C_0 > 0$, due to (4.11). We define

$$\lambda(r,x,y) := r^{n-1} \int_{S^{n-1}} |v(x-r\omega, y-r\omega)| \, d\mathscr{H}^{n-1}(\omega)$$

$$\text{and} \quad \Lambda(r,x,y) := \int_{B_r} |v(x-z, y-z)| \, dz.$$

Now we use Lemma 4.3 with $N := 2n$, $\varpi(z) := (z,z)$, $X := (x,y)$, $\Theta(X) := |x|^a|y|^a$, $q := p$ and $V(X) := v(x,y)$, see (4.9). In this way we obtain that

$$\iint_{\mathbb{R}^{2n}} \left[\frac{\Lambda(r,x,y)}{r^n}\right]^p \frac{dx\,dy}{|x|^a|y|^a} \le C_1 \iint_{\mathbb{R}^{2n}} \frac{|v(x,y)|^p}{|x|^a|y|^a} dx\,dy, \tag{4.14}$$

for some $C_1 > 0$. Moreover, by polar coordinates,

$$\begin{aligned}\Lambda(r,x,y) &= C_2 \int_0^r \left[\rho^{n-1} \int_{S^{n-1}} |v(x-\rho\omega, y-\rho\omega)| \, d\mathscr{H}^{n-1}(\omega)\right] d\rho \\ &= C_2 \int_0^r \lambda(\rho,x,y) \, d\rho,\end{aligned}$$

and therefore

$$\frac{\partial}{\partial r}\Lambda(r,x,y) = C_2 \lambda(r,x,y).$$

Notice also that $\Lambda(0,x,y) = 0 = \eta_o(1)$. Consequently, using again polar coordinates and an integration by parts, we obtain

$$\begin{aligned}|v \star \eta_o(x,y)| &\le \int_{B_1} |v(x-z, y-z)|\eta_o(z) \, dz \\ &= C_3 \int_0^1 \left[\int_{S^{n-1}} r^{n-1}|v(x-r\omega, y-r\omega)|\eta_o(r) \, d\mathscr{H}^{n-1}(\omega)\right] dr \\ &= C_3 \int_0^1 \lambda(r,x,y)\eta_o(r) \, dr \\ &= C_4 \int_0^1 \frac{\partial \Lambda}{\partial r}(r,x,y)\eta_o(r) \, dr \\ &= -C_4 \int_0^1 \Lambda(r,x,y)\eta_o'(r) \, dr.\end{aligned} \tag{4.15}$$

We recall that $\eta_o' \le 0$, so the latter term is indeed non-negative. Now we use the Minkowski integral inequality (see e.g. Appendix A.1 in [14]): this gives that, for a given $F = F(r,x,y)$, and $d\mu(x,y) := \frac{dx\,dy}{|x|^a|y|^a}$, we have

$$\left[\iint_{\mathbb{R}^{2n}}\left[\int_0^1 |F(r,x,y)|\,dr\right]^p d\mu(x,y)\right]^{1/p}$$
$$\le \int_0^1\left[\iint_{\mathbb{R}^{2n}}|F(r,x,y)|^p\,d\mu(x,y)\right]^{1/p} dr.$$

Using this with $F(r,x,y) := \Lambda(r,x,y)\eta_o'(r)$ and recalling (4.15), we conclude that

$$\left[\iint_{\mathbb{R}^{2n}}\frac{|v\star\eta_o(x,y)|^p}{|x|^a|y|^a}\,dx\,dy\right]^{1/p}$$
$$\le C_5\left[\iint_{\mathbb{R}^{2n}}\left[\int_0^1\Lambda(r,x,y)|\eta_o'(r)|\,dr\right]^p\frac{dx\,dy}{|x|^a|y|^a}\right]^{1/p}$$
$$\le C_5\int_0^1\left[\iint_{\mathbb{R}^{2n}}|\Lambda(r,x,y)|^p|\eta_o'(r)|^p\,\frac{dx\,dy}{|x|^a|y|^a}\right]^{1/p} dr$$
$$= C_5\int_0^1\left[\iint_{\mathbb{R}^{2n}}\left[\frac{\Lambda(r,x,y)}{r^n}\right]^p\frac{dx\,dy}{|x|^a|y|^a}\right]^{1/p} r^n|\eta_o'(r)|\,dr.$$

Therefore, recalling (4.14),

$$\left[\iint_{\mathbb{R}^{2n}}\frac{|v\star\eta_o(x,y)|^p}{|x|^a|y|^a}\,dx\,dy\right]^{1/p}$$
$$\le C_6\int_0^1\left[\iint_{\mathbb{R}^{2n}}\frac{|v(x,y)|^p}{|x|^a|y|^a}\,dx\,dy\right]^{1/p} r^n|\eta_o'(r)|\,dr.$$

This and (4.13) give the desired result. □

A simpler, but still useful, version of Proposition 4.4 holds for the standard convolution of a function $u:\mathbb{R}^n\to\mathbb{R}$, i.e.

$$u*\eta_o(x) := \int_{\mathbb{R}^n} u(x-z)\eta_o(z)\,dz.$$

The reader may compare the latter formula with (4.12). In this more standard setting, we have:

PROPOSITION 4.5. *Let $b := \frac{2ap_s^*}{p}$. For every measurable function u from $\mathbb{R}^n$ to $\mathbb{R}$, we have that*

$$\int_{\mathbb{R}^n}\frac{|u*\eta_o(x)|^{p_s^*}}{|x|^b}\,dx \le C\int_{\mathbb{R}^n}\frac{|u(x)|^{p_s^*}}{|x|^b}\,dx,$$

for a suitable $C>0$.

PROOF. The argument is a simplification of the one given for Proposition 4.4. For the convenience of the reader, we provide all the details. We define

$$\lambda(r,x) := r^{n-1} \int_{S^{n-1}} |u(x - r\omega)| \, d\mathscr{H}^{n-1}(\omega)$$
$$\text{and} \quad \Lambda(r,x) := \int_{B_r} |u(x-z)| \, dz.$$

Here we use Lemma 4.3 with $N := n$, $\varpi(z) := z$, $X := x$, $\Theta(X) := |x|^b$, $q := p_s^*$ and $V(X) := u(x)$, see (4.9). In this way we obtain that

$$\int_{\mathbb{R}^n} \left[\frac{\Lambda(r,x)}{r^n}\right]^{p_s^*} \frac{dx}{|x|^b} \le C_1 \int_{\mathbb{R}^n} \frac{|u(x)|^{p_s^*}}{|x|^b} \, dx, \tag{4.16}$$

for some $C_1 > 0$. Moreover, by polar coordinates,

$$\Lambda(r,x) = C_2 \int_0^r \left[\rho^{n-1} \int_{S^{n-1}} |u(x-\rho\omega)| \, d\mathscr{H}^{n-1}(\omega)\right] d\rho = C_2 \int_0^r \lambda(\rho,x) \, d\rho,$$

and therefore

$$\frac{\partial}{\partial r}\Lambda(r,x) = C_2 \lambda(r,x).$$

Notice also that $\Lambda(0,x) = 0 = \eta_o(1)$. Consequently, using again polar coordinates and an integration by parts, we obtain

$$\begin{aligned}
|u * \eta_o(x)| &\le \int_{B_1} |u(x-z)| \eta_o(z) \, dz \\
&= C_3 \int_0^1 \left[\int_{S^{n-1}} r^{n-1} |u(x - r\omega)| \eta_o(r) \, d\mathscr{H}^{n-1}(\omega)\right] dr \\
&= C_3 \int_0^1 \lambda(r,x) \eta_o(r) \, dr \\
&= C_4 \int_0^1 \frac{\partial \Lambda}{\partial r}(r,x) \eta_o(r) \, dr \\
&= -C_4 \int_0^1 \Lambda(r,x) \eta_o'(r) \, dr.
\end{aligned}$$

Now we use the Minkowski integral inequality (see e.g. Appendix A.1 in [14]) and we conclude that

$$\begin{aligned}\left[\int_{\mathbb{R}^n}\frac{|u*\eta_o(x)|^{p_s^*}}{|x|^b}\,dx\right]^{1/p_s^*} &\le C_5\left[\int_{\mathbb{R}^n}\left[\int_0^1\Lambda(r,x)|\eta_o'(r)|\,dr\right]^{p_s^*}\frac{dx}{|x|^b}\right]^{1/p_s^*}\\ &\le C_5\int_0^1\left[\int_{\mathbb{R}^n}|\Lambda(r,x)|^{p_s^*}|\eta_o'(r)|^{p_s^*}\frac{dx}{|x|^b}\right]^{1/p_s^*}dr\\ &= C_5\int_0^1\left[\int_{\mathbb{R}^n}\left[\frac{\Lambda(r,x)}{r^n}\right]^{p_s^*}\frac{dx}{|x|^b}\right]^{1/p_s^*} r^n|\eta_o'(r)|\,dr.\end{aligned}$$

So, recalling (4.16),

$$\left[\int_{\mathbb{R}^n}\frac{|u*\eta_o(x)|^{p_s^*}}{|x|^b}\,dx\right]^{1/p_s^*} \le C_6\int_0^1\left[\int_{\mathbb{R}^n}\frac{|u(x)|^{p_s^*}}{|x|^b}\,dx\right]^{1/p_s^*} r^n|\eta_o'(r)|\,dr.$$

From this and (4.13) we obtain the desired result. □

5. Approximation in weighted Lebesgue spaces by continuous functions

In order to deal with the semi-norm in (1.2), it is often convenient to introduce a weighted norm over $\mathbb{R}^{2n}$, by proceeding as follows. Given a measurable function $v = v(x, y)$ from $\mathbb{R}^{2n}$ to $\mathbb{R}$, we define

$$\|v\|_{L^p_{a,a}(\mathbb{R}^{2n})} := \left(\iint_{\mathbb{R}^{2n}} |v(x,y)|^p\,\frac{dx}{|x|^a}\,\frac{dy}{|y|^a}\right)^{1/p}. \tag{5.1}$$

When $\|v\|_{L^p_{a,a}(\mathbb{R}^{2n})}$ is finite, we say that v belongs to $L^p_{a,a}(\mathbb{R}^{2n})$. Notice that

(5.2) if $v^{(u)}(x,y) := \dfrac{u(x)-u(y)}{|x-y|^{\frac{n}{p}+s}}$, then formula (5.1) reduces to (1.2),

namely $\|v^{(u)}\|_{L^p_{a,a}(\mathbb{R}^{2n})} = [u]_{\tilde{W}^{s,p}_a(\mathbb{R}^n)}$.

Now we give two approximation results (namely Lemmata 5.1 and 5.2) with respect to the norm in (5.1).

Lemma 5.1. *Let $v \in L^p_{a,a}(\mathbb{R}^{2n})$. Then there exists a sequence of functions $v_M \in L^p_{a,a}(\mathbb{R}^{2n}) \cap L^\infty(\mathbb{R}^{2n})$ such that $\|v - v_M\|_{L^p_{a,a}(\mathbb{R}^{2n})} \to 0$ as $M \to +\infty$.*

Proof. We set

$$v_M(x,y) := \begin{cases} M & \text{if } v(x,y) \ge M,\\ v(x,y) & \text{if } v(x,y) \in (-M, M),\\ -M & \text{if } v(x,y) \le -M.\end{cases}$$

We have that $v_M \to v$ a.e. in $\mathbb{R}^{2n}$ and

$$\frac{|v_M(x,y)|^p}{|x|^a|y|^a} \leq \frac{|v(x,y)|^p}{|x|^a|y|^a} \in L^1(\mathbb{R}^{2n}),$$

thus the claim follows from the Dominated Convergence Theorem. □

LEMMA 5.2. *Let $v \in L^p_{a,a}(\mathbb{R}^{2n})$. Then there exists a sequence of continuous and compactly supported functions $v_\delta : \mathbb{R}^{2n} \to \mathbb{R}$ such that $\|v - v_\delta\|_{L^p_{a,a}(\mathbb{R}^{2n})} \to 0$ as $\delta \to 0$.*

PROOF. In the light of Lemma 5.1, we can also assume that

$$v \in L^\infty(\mathbb{R}^{2n}). \tag{5.3}$$

Let $\tau_j \in C^\infty(\mathbb{R}^{2n}, [0,1])$, with $\tau_j(P) = 1$ if $|P| \leq j$ and $\tau(P) = 0$ if $|P| \geq j+1$. Let $v_j := \tau_j u$. Then $v_j \to v$ pointwise in $\mathbb{R}^{2n}$ as $j \to +\infty$, and

$$\frac{|v(x,y) - v_j(x,y)|^p}{|x|^a|y|^a} \leq \frac{2^p|v(x,y)|^p}{|x|^a|y|^a} \in L^1(\mathbb{R}^{2n}).$$

As a consequence, by the Dominated Convergence Theorem,

$$\lim_{j\to+\infty} \|v - v_j\|_{L^p_{a,a}(\mathbb{R}^{2n})} = 0.$$

So, fixed $\delta > 0$, we find $j_\delta \in \mathbb{N}$ such that

$$\|v - v_{j_\delta}\|_{L^p_{a,a}(\mathbb{R}^{2n})} \leq \delta. \tag{5.4}$$

Notice that v_{j_δ} is supported in $\{P \in \mathbb{R}^{2n} \text{ s.t. } |P| \leq j_\delta + 1\}$.

Also, given a set $A \subseteq \mathbb{R}^{2n}$, we set

$$\mu_{a,a}(A) := \iint_A \frac{dx\,dy}{|x|^a|y|^a}.$$

By (1.1), we see that $\mu_{a,a}$ is finite over compact sets. So, we can use Lusin's Theorem (see e.g. Theorem 7.10 in [10], and page 121 there for the definition of the uniform norm). We obtain that there exist a closed set $E_\delta \subset \mathbb{R}^{2n}$ and a continuous and compactly supported function $v_\delta : \mathbb{R}^{2n} \to \mathbb{R}$ such that $v_\delta = v_{j_\delta}$ in $\mathbb{R}^{2n} \backslash E_\delta$, $\mu_{a,a}(E_\delta) \leq \delta^p$ and $\|v_\delta\|_{L^\infty(\mathbb{R}^{2n})} \leq \|v_{j_\delta}\|_{L^\infty(\mathbb{R}^{2n})}$.

In particular, since $\tau_{j_\delta} \in [0,1]$, we have that $\|v_\delta\|_{L^\infty(\mathbb{R}^{2n})} \leq \|v\|_{L^\infty(\mathbb{R}^{2n})}$, and this quantity is finite, due to (5.3). Therefore

$$\begin{aligned}
\|v_{j_\delta} - v_\delta\|^p_{L^p_{a,a}(\mathbb{R}^{2n})} &= \iint_{E_\delta} |v_{j_\delta}(x,y) - v_\delta(x,y)|^p \frac{dx}{|x|^a}\frac{dy}{|y|^a} \\
&\leq 2^p(\|v_{j_\delta}\|^p_{L^\infty(\mathbb{R}^{2n})} + \|v_\delta\|^p_{L^\infty(\mathbb{R}^{2n})})\mu_{a,a}(E_\delta) \leq 2^{p+1}\|v\|^p_{L^\infty(\mathbb{R}^{2n})}\delta^p.
\end{aligned}$$

From this and (5.4), we obtain that $\|v - v_\delta\|_{L^p_{a,a}(\mathbb{R}^{2n})} \le (1 + 4\|v\|_{L^\infty(\mathbb{R}^{2n})})\delta$, which concludes the proof. $\square$

We remark that a simpler version of Lemma 5.2 also holds true in $L_a^{p_s^*}(\mathbb{R}^n)$. We state the result explicitly as follows:

LEMMA 5.3. *Let $u \in L_a^{p_s^*}(\mathbb{R}^n)$. Then there exists a sequence of continuous and compactly supported functions $u_\delta : \mathbb{R}^n \to \mathbb{R}$ such that $\|u - u_\delta\|_{L_a^{p_s^*}(\mathbb{R}^n)} \to 0$ as $\delta \to 0$.*

PROOF. The argument is a simplified version of the one given for Lemma 5.2. Full details are provided for the reader's convenience. First of all, by the Dominated Convergence Theorem, we can approximate u in $L_a^{p_s^*}(\mathbb{R}^n)$ with a sequence of bounded functions

$$u_M(x) := \begin{cases} M & \text{if } u(x) \ge M, \\ u(x) & \text{if } u(x) \in (-M, M), \\ -M & \text{if } u(x) \le -M. \end{cases}$$

Consequently, we can also assume that

$$u \in L^\infty(\mathbb{R}^n). \tag{5.5}$$

Let $\tau_j \in C^\infty(\mathbb{R}^n, [0,1])$, with $\tau_j(P) = 1$ if $|P| \le j$ and $\tau(P) = 0$ if $|P| \ge j+1$. Let $u_j := \tau_j u$. Then $u_j \to u$ pointwise in $\mathbb{R}^n$ as $j \to +\infty$, and

$$\frac{|u(x) - u_j(x)|^{p_s^*}}{|x|^{\frac{2ap_s^*}{p}}} \le \frac{2^{p_s^*}|u(x)|^{p_s^*}}{|x|^{\frac{2ap_s^*}{p}}} \in L^1(\mathbb{R}^{2n}).$$

As a consequence, by the Dominated Convergence Theorem,

$$\lim_{j \to +\infty} \|u - u_j\|_{L_a^{p_s^*}(\mathbb{R}^n)} = 0.$$

So, fixed $\delta > 0$, we find $j_\delta \in \mathbb{N}$ such that

$$\|u - u_{j_\delta}\|_{L_a^{p_s^*}(\mathbb{R}^n)} \le \delta. \tag{5.6}$$

Notice that u_{j_δ} is supported in $\overline{B_{j_\delta+1}}$. Also, given a set $A \subseteq \mathbb{R}^n$, we set

$$\mu_a(A) := \int_A \frac{dx}{|x|^{\frac{2ap_s^*}{p}}}.$$

By (1.1), we see that μ_a is finite over compact sets. So, we can use Lusin's Theorem (see e.g. Theorem 7.10 in [10], and page 121 there for the definition of the uniform norm). We obtain that there exist a closed set $E_\delta \subset \mathbb{R}^n$ and a continuous and compactly supported function $u_\delta : \mathbb{R}^n \to \mathbb{R}$ such that $u_\delta = u_{j_\delta}$ in $\mathbb{R}^n \setminus E_\delta$, $\mu_a(E_\delta) \le \delta^{p_s^*}$ and $\|u_\delta\|_{L^\infty(\mathbb{R}^n)} \le \|u_{j_\delta}\|_{L^\infty(\mathbb{R}^n)}$.

In particular, since $\tau_{j_\delta} \in [0,1]$, we have that $\|u_\delta\|_{L^\infty(\mathbb{R}^n)} \le \|u\|_{L^\infty(\mathbb{R}^n)}$, and this quantity is finite, due to (5.5). Therefore

$$\begin{aligned}\|u_{j_\delta} - u_\delta\|^{p_s^*}_{L_a^{p_s^*}(\mathbb{R}^n)} &= \int_{E_\delta} |u_{j_\delta}(x) - u_\delta(x)|^{p_s^*} \frac{dx}{|x|^{\frac{2ap_s^*}{p}}} \\ &\le 2^{p_s^*}(\|u_{j_\delta}\|^{p_s^*}_{L^\infty(\mathbb{R}^n)} + \|u_\delta\|^{p_s^*}_{L^\infty(\mathbb{R}^n)})\mu_a(E_\delta) \le 2^{p_s^*+1}\|v\|^{p_s^*}_{L^\infty(\mathbb{R}^n)}\delta^{p_s^*}.\end{aligned}$$

From this and (5.6), we obtain that $\|u - u_\delta\|_{L_a^{p_s^*}(\mathbb{R}^n)} \le (1 + 4\|u\|_{L^\infty(\mathbb{R}^n)})\delta$, which concludes the proof. □

6. Approximation by smooth functions

In this section we show that we can approximate a function in the space $\dot{W}_a^{s,p}(\mathbb{R}^n)$ with a smooth one. We remark that, if there are no weights, smooth approximations are much more standard, since one can use directly the continuity of the translations in $L^p(\mathbb{R}^{2n})$. Since the weights are not translation invariant, and the continuity of the translations in Lebesgue spaces is, in general, not uniform, a more careful procedure is needed in our case (namely, to overcome this difficulty we exploit the techniques developed in Sections 4 and 5).

We take a radially symmetric, radially decreasing function $\eta \in C_0^\infty(\mathbb{R}^n)$ such that $\eta \ge 0$, $\operatorname{supp}\eta \subseteq B_1$ and

$$\int_{B_1} \eta(x)\,dx = 1, \tag{6.1}$$

and, for $\varepsilon > 0$, we define the mollifier η_ε as

$$\eta_\varepsilon(x) := \frac{1}{\varepsilon^n}\eta\left(\frac{x}{\varepsilon}\right), \quad \text{for any } x \in \mathbb{R}^n.$$

Then, given $u \in \dot{W}_a^{s,p}(\mathbb{R}^n)$, we consider its standard convolution with the mollifier η_ε. That is, for any $\varepsilon > 0$, we define

$$u_\varepsilon(x) := (u * \eta_\varepsilon)(x) = \int_{\mathbb{R}^n} u(x-z)\eta_\varepsilon(z)\,dz, \quad \text{for any } x \in \mathbb{R}^n. \tag{6.2}$$

By construction, $u_\varepsilon \in C^\infty(\mathbb{R}^n)$. We will show that, if ε is sufficiently small, then the error made approximating u with u_ε is "small". The rigorous result is the following:

Lemma 6.1. *Let $u \in \dot{W}_a^{s,p}(\mathbb{R}^n)$. Then*

$$\lim_{\varepsilon\to 0} \|u - u_\varepsilon\|_{\dot{W}_a^{s,p}(\mathbb{R}^n)} = 0.$$

PROOF. We first check that

$$\lim_{\varepsilon\to 0} \|u - u_\varepsilon\|_{L_a^{p_s^*}(\mathbb{R}^n)} = 0. \tag{6.3}$$

To this scope, we start by proving that

(6.4) if $\tilde{u} : \mathbb{R}^n \to \mathbb{R}$ is continuous and compactly supported, then
$$\lim_{\varepsilon\to 0} \|\tilde{u} - \tilde{u} * \eta_\varepsilon\|_{L_a^{p_s^*}(\mathbb{R}^n)} = 0.$$

For this, we fix $\varepsilon_o > 0$ and we use the fact that $\tilde{u}$ is uniformly continuous to write that

$$\sup_{z\in B_1} |\tilde{u}(x - \varepsilon z) - \tilde{u}(x)| \le \varepsilon_o,$$

provided that ε is small enough (possibly in dependence of ε_o). Also, since $\tilde{u}$ is compactly supported, say in B_R, and writing $b := \frac{2ap_s^*}{p}$, we obtain that

$$\begin{aligned}\int_{\mathbb{R}^n} |\tilde{u}(x) - \tilde{u} * \eta_\varepsilon(x)|^{p_s^*} \frac{dx}{|x|^b} &\le \int_{B_{R+1}} \left[\int_{B_1} |\tilde{u}(x) - \tilde{u}(x - \varepsilon z)|\eta(z)\, dz\right]^{p_s^*} \frac{dx}{|x|^b} \\ &\le \varepsilon_o^{p_s^*} \int_{B_{R+1}} \frac{dx}{|x|^b} = C\varepsilon_o^{p_s^*},\end{aligned}$$

with C independent of ε and ε_o. Since ε_o can be taken arbitrarily small, the proof of (6.4) is complete.

Now we prove (6.3). For this, we fix $\varepsilon_o > 0$, to be taken as small as we wish in the sequel, and we use Lemma 5.3 to find a continuous and compactly supported function $\tilde{u} : \mathbb{R}^n \to \mathbb{R}$ such that $\|u - \tilde{u}\|_{L_a^{p_s^*}(\mathbb{R}^n)} \le \varepsilon_o$.

By Proposition 4.5, we deduce that

$$\|u * \eta_\varepsilon - \tilde{u} * \eta_\varepsilon\|_{L_a^{p_s^*}(\mathbb{R}^n)} = \|(u - \tilde{u}) * \eta_\varepsilon\|_{L_a^{p_s^*}(\mathbb{R}^n)} \le C\|u - \tilde{u}\|_{L_a^{p_s^*}(\mathbb{R}^n)} \le C\varepsilon_o.$$

Furthermore, by (6.4), we know that

$$\|\tilde{u} - \tilde{u} * \eta_\varepsilon\|_{L_a^{p_s^*}(\mathbb{R}^n)} \le \varepsilon_o,$$

as long as ε is sufficiently small. By collecting these pieces of information, we conclude that

$$\begin{aligned}\|u - u_\varepsilon\|_{L_a^{p_s^*}(\mathbb{R}^n)} &\le \|u - \tilde{u}\|_{L_a^{p_s^*}(\mathbb{R}^n)} + \|\tilde{u} - \tilde{u} * \eta_\varepsilon\|_{L_a^{p_s^*}(\mathbb{R}^n)} + \|\tilde{u} * \eta_\varepsilon - u * \eta_\varepsilon\|_{L_a^{p_s^*}(\mathbb{R}^n)} \\ &\le (2 + C)\varepsilon_o.\end{aligned}$$

This completes the proof of (6.3).

Now we recall the notation in (4.12) and we prove that

$$\text{(6.5)} \qquad \text{if } v : \mathbb{R}^{2n} \to \mathbb{R} \text{ is continuous and compactly supported, then} \quad \lim_{\varepsilon \to 0} \|v - v \star \eta_\varepsilon\|_{L^p_{a,a}(\mathbb{R}^{2n})} = 0.$$

For this, we fix $\varepsilon_o > 0$ and we use the fact that v is uniformly continuous to write that

$$\sup_{z \in B_1} |v(x - \varepsilon z, y - \varepsilon z) - v(x, y)| \le \varepsilon_o,$$

provided that ε is small enough (possibly in dependence of ε_o). Also, since v is compactly supported, say in $\{|(x, y)| \le R\}$, for some $R > 0$, we have that

$$v(x, y) = 0 = v(x - \varepsilon z, y - \varepsilon z)$$

if $z \in B_1$ and $\max\{|x|, |y|\} \ge R + 1$, as long as $\varepsilon < 1$. Moreover

$$v(x, y) - v \star \eta_\varepsilon(x, y) = \int_{B_1} (v(x, y) - v(x - \varepsilon z, y - \varepsilon z))\eta(z)\, dz,$$

and, as a consequence,

$$\begin{aligned}
&\iint_{\mathbb{R}^{2n}} |v(x, y) - v \star \eta_\varepsilon(x, y)|^p \frac{dx\, dy}{|x|^a |y|^a} \\
&\le \iint_{B_{R+1} \times B_{R+1}} \left[\int_{B_1} \Big| v(x, y) - v(x - \varepsilon z, y - \varepsilon z) \Big| \eta(z)\, dz \right]^p \frac{dx\, dy}{|x|^a |y|^a} \\
&\le \varepsilon_o^p \iint_{B_{R+1} \times B_{R+1}} \frac{dx\, dy}{|x|^a |y|^a} \\
&= C\varepsilon_o^p,
\end{aligned}$$

with C depending on v, but independent of ε and ε_o. Since ε_o can be taken arbitrarily small, the proof of (6.5) is complete.

Now we are in the position of completing the proof of Lemma 6.1. We remark that, by (6.3), and recalling (1.2) and (1.4), in order to prove Lemma 6.1, it only remains to show that

$$\text{(6.6)} \qquad \lim_{\varepsilon \to 0} \iint_{\mathbb{R}^{2n}} \frac{|u(x) - u_\varepsilon(x) - u(y) + u_\varepsilon(y)|^p}{|x - y|^{n+sp}} \frac{dx}{|x|^a} \frac{dy}{|y|^a} = 0.$$

To this goal, we let

$$v^{(u)}(x, y) := \frac{u(x) - u(y)}{|x - y|^{\frac{n}{p}+s}}.$$

By comparing (4.12) and (6.2), we see that

$$v^{(u)} \star \eta_\varepsilon(x, y) = \int_{\mathbb{R}^n} v^{(u)}(x - z, y - z)\eta_\varepsilon(z)\, dz \tag{6.7}$$
$$= \int_{\mathbb{R}^n} \frac{u(x - z) - u(y - z)}{|x - y|^{\frac{n}{p}+s}} \eta_\varepsilon(z)\, dz$$
$$= \frac{u * \eta_\varepsilon(x) - u * \eta_\varepsilon(y)}{|x - y|^{\frac{n}{p}+s}}$$
$$= v^{(u*\eta_\varepsilon)}(x, y).$$

We fix $\varepsilon_o > 0$, to be taken as small as we wish in the sequel, and use Lemma 5.2, to find a continuous and compactly supported function v such that

$$\|v^{(u)} - v\|_{L^p_{a,a}(\mathbb{R}^{2n})} \le \varepsilon_o. \tag{6.8}$$

Notice that, by (6.5),

$$\|v - v \star \eta_\varepsilon\|_{L^p_{a,a}(\mathbb{R}^{2n})} \le \varepsilon_o, \tag{6.9}$$

as long as ε is sufficiently small.

Moreover, by Proposition 4.4 (applied here to the function $v^{(u)} - v$) and (6.8), we have that

$$\|(v^{(u)} - v) \star \eta_\varepsilon\|_{L^p_{a,a}(\mathbb{R}^{2n})} \le C\|v^{(u)} - v\|_{L^p_{a,a}(\mathbb{R}^{2n})} \le C\varepsilon_o. \tag{6.10}$$

Also, by (5.2)

$$[u - u * \eta_\varepsilon]_{\tilde{W}^{s,p}_a(\mathbb{R}^n)} = \|v^{(u-u*\eta_\varepsilon)}\|_{L^p_{a,a}(\mathbb{R}^{2n})} = \|v^{(u)} - v^{(u*\eta_\varepsilon)}\|_{L^p_{a,a}(\mathbb{R}^{2n})}.$$

Thus, recalling (6.7),

$$[u - u * \eta_\varepsilon]_{\tilde{W}^{s,p}_a(\mathbb{R}^n)} = \|v^{(u)} - v^{(u)} \star \eta_\varepsilon\|_{L^p_{a,a}(\mathbb{R}^{2n})}.$$

Accordingly, by (6.8), (6.9) and (6.10),

$$\begin{aligned} &[u - u * \eta_\varepsilon]_{\tilde{W}^{s,p}_a(\mathbb{R}^n)} \\ &\quad\le \|v^{(u)} - v\|_{L^p_{a,a}(\mathbb{R}^{2n})} + \|v - v \star \eta_\varepsilon\|_{L^p_{a,a}(\mathbb{R}^{2n})} + \|v \star \eta_\varepsilon - v^{(u)} \star \eta_\varepsilon\|_{L^p_{a,a}(\mathbb{R}^{2n})} \\ &\quad\le (2 + C)\varepsilon_o. \end{aligned}$$

Since ε_o can be taken arbitrarily small, we have proved (6.6), and therefore the proof of Lemma 6.1 is complete. □

7. Proof of Theorem 1.1

Let $u \in \dot{W}_a^{s,p}(\mathbb{R}^n)$, and fix $\delta > 0$. If τ_j is as in Lemma 3.1, then for j large enough we have that

$$\|u - \tau_j u\|_{\dot{W}_a^{s,p}(\mathbb{R}^n)} < \frac{\delta}{2}, \tag{7.1}$$

thanks to Lemma 3.1.

Now, for any $\varepsilon > 0$, let η_ε be the mollifier defined at the beginning of Section 6. We set

$$\rho_\varepsilon := \tau_j u * \eta_\varepsilon.$$

By construction, $\rho_\varepsilon \in C^\infty(\mathbb{R}^n)$. Moreover, standard properties of the convolution imply that

$$\operatorname{supp} \rho_\varepsilon \subseteq \operatorname{supp}(\tau_j u) + \bar{B}_\varepsilon.$$

Also (see e.g. Lemma 9 in [9]) one sees that

$$\operatorname{supp}(\tau_j u) \subseteq (\operatorname{supp} \tau_j) \cap (\operatorname{supp} u) \subseteq \bar{B}_{2j} \cap (\operatorname{supp} u).$$

Hence

$$\operatorname{supp} \rho_\varepsilon \subseteq (\bar{B}_{2j} \cap (\operatorname{supp} u)) + \bar{B}_\varepsilon.$$

As a consequence, $\rho_\varepsilon \in C_0^\infty(\mathbb{R}^n)$.

Furthermore, Lemma 6.1 gives that

$$\|\rho_\varepsilon - \tau_j u\|_{\dot{W}_a^{s,p}(\mathbb{R}^n)} < \frac{\delta}{2},$$

if ε is sufficiently small. Therefore, from this and (7.1) we obtain that

$$\|u - \rho_\varepsilon\|_{\dot{W}_a^{s,p}(\mathbb{R}^n)} \leq \|u - \tau_j u\|_{\dot{W}_a^{s,p}(\mathbb{R}^n)} + \|\tau_j u - \rho_\varepsilon\|_{\dot{W}_a^{s,p}(\mathbb{R}^n)} < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Since δ can be taken arbitrarily small, this concludes the proof of Theorem 1.1.

Remark 7.1. We point out that some of the statements of this paper may be interpreted in the light of the theory of the maximal function and in terms of the Muckenhoupt weights A_1.

For instance, Proposition 4.2 could be written equivalently in terms of the maximal function (it is sufficient to make a change of variable in the integral, translating the center of the balls in x), and it is related to the fact that the weight $|x|^{-b}$ is in the Muckenhoupt class A_1 if and only if $0 \leq b < n$ (see e.g. page 141 in [8]).

Similarly, formula (4.8) states that the weight $1/\Theta$ is in A_1 and Lemma 4.3 is related to the boundedness of the maximal operator for A_q weights, since the average over the balls can be majored by the maximal operator (see also page 136 of [8]).

In this sense, Proposition 4.4 is also related to the theory of convolutions in functional spaces (see e.g. Corollaire 7.20 on page 387 of [4]).

It is an interesting problem to generalize the results given here to the class of Muckenhoupt weights in a wider functional setting.

References

[1] B. Abdellaoui - R. Bentifour: *Caffarelli–Kohn–Nirenberg type inequalities of fractional order and applications*, Preprint (2014).

[2] V. I. Bogachev: *Differentiable measures and the Malliavin calculus*, Mathematical Surveys and Monographs 164. American Mathematical Society, Providence (2010).

[3] V. Chiadò Piat - F. Serra Cassano: *Some remarks about the density of smooth functions in weighted Sobolev spaces*, J. Convex Anal. 1 (1994), no. 2, 135–142.

[4] F. Demengel - G. Demengel: *Function spaces. Usage for the solution of partial differential equations*, EDP Sciences, Paris (2007).

[5] E. Di Nezza - G. Palatucci - E. Valdinoci: *Hitchhikers guide to the fractional Sobolev spaces*, Bull. Sci. Math. 136 (2012), no. 5, 521–573.

[6] S. Dipierro - M. Medina - I. Peral - E. Valdinoci: *Bifurcation results for a fractional elliptic equation with critical exponent in $\mathbb{R}^n$*, Preprint (2014).

[7] S. Dipierro - L. Montoro - I. Peral - B. Sciunzi: *Qualitative properties of positive solutions to nonlocal critical problems involving the Hardy–Leray potential*, Preprint (2015).

[8] J. Duoandikoetxea: *Fourier analysis*, American Mathematical Society, Providence (2001).

[9] A. Fiscella - R. Servadei - E. Valdinoci: *Density properties for fractional Sobolev spaces*, Ann. Acad. Sci. Fenn. Math., 40 (2015), 235–253.

[10] G. B. Folland: *Real Analysis. Modern techniques and their applications*, J. Wiley, New York (1984).

[11] R. Frank - E. H. Lieb - R. Seiringer: *Hardy-Lieb-Thirring inequalities for fractional Schrdinger operators*, J. Am. Math. Soc. 21 (2008), no. 4, 925–950.

[12] T. Kilpeläinen: *Weighted Sobolev spaces and capacity*, Ann. Acad. Sci. Fenn. Math. 19 (1994), 95–113.

[13] H. Luiro - V. Vahakangas: *Beyond local maximal operators*, Preprint (2015).

[14] E. M. Stein: *Singular integrals and differentiability properties of functions*, Princeton University Press, Princeton (1970).

[15] J. M. Tölle: *Uniqueness of weighted Sobolev spaces with weakly differentiable weights*, J. Funct. Anal. 263 (2012), no. 10, 3195–3223.

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Serena Dipierro
Maxwell Institute for Mathematical Sciences and School of Mathematics
University of Edinburgh
James Clerk Maxwell Building
Peter Guthrie Tait Road
Edinburgh EH9 3FD, United Kingdom
serena.dipierro@ed.ac.uk

Enrico Valdinoci
Weierstrass Institute for Applied Analysis and Stochastics
Mohrenstrasse 39
10117 Berlin, Germany
and
Dipartimento di Matematica
Università di Milano
Via Cesare Saldini 50
20133 Milano, Italy
enrico@mat.uniroma3.it