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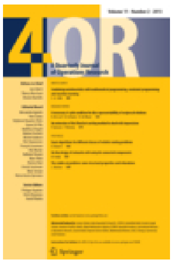

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Risk Parity for Mixed Tempered Stable distributed sources of risk

Lorenzo Mercuri · Edit Rroji

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Abstract In this paper we discuss a detailed methodology for dealing with Risk Parity in a parametric context. In particular, we use the Independent Component Analysis for a linear decomposition of portfolio risk factors. Each Independent Component is modeled with the Mixed Tempered Stable distribution. Risk Parity optimal portfolio weights are calculated for three risk measures: Volatility, modified Value At Risk and modified Expected Shortfall.
Empirical analysis is discussed in terms of out-of-sample performance and portfolio diversification.

Keywords Risk parity · Mixed Tempered Stable · Optimization

1 Introduction

Risk Parity is an approach in portfolio management which focuses on allocation of risk rather than on capital (see Denis et al., 2011, for further details). An optimization algorithm based on the risk parity approach requires the formulation of portfolio total risk in terms of marginal contributions. In this paper we exploit Euler's theorem for homogenous functions and express portfolio risk as a weighted sum of the marginal risk contributions following the approach described in Tasche (1999). In particular we focus on three standard homogeneous risk measures: Volatility, Value at Risk (VaR) and Expected Shortfall (ES). For the last two measures, we consider the modified versions proposed respectively in Zangari (1996) and in Boudt et al. (2007). Euler's principle is useful not only for portfolio

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optimization but also for internal capital allocation as suggested for example in Mizgier and Pasia (2015).
In this paper we present a general setup for obtaining risk parity portfolios by modeling directly the underlying independent factors extracted through the Independent Component Analysis (ICA) introduced in Comon (1994) [1]. We need only to model each individual component (IC) because the dependence structure of factors is captured from the mixing matrix obtained through the algorithm.
Non parametric methods for modeling time series take into account only past realizations of the variables of interest and create a dependence of the results on the length of the time interval considered. Stability issues for estimates require large sample sizes (see for example Martellini and Ziemann (2010), Hitaj and Mercuri (2013) in the context of sample moments applied to the portfolio selection problem) but on the other hand realizations observed in the farther past can be less realistic. The use of a parametric distribution is a valid alternative.
Recently a new distribution, named Mixed Tempered Stable distribution (MixedTS hereafter), has been introduced in Rroji and Mercuri (2015) as a generalization of the Normal Variance Mean Mixtures (NVMM henceforth as in Barndorff-Nielsen et al., 1982) substituting the normality assumption with the Tempered Stable distribution (see Cont and Tankov, 2003). The MixedTS is more flexible in capturing the higher moments since in the NVMM the sign of skewness depends on the drift parameter. In the MixedTS, skewness depends also on the tempering parameters of the Tempered Stable distribution. As shown in Rroji and Mercuri (2015), similar arguments hold also for kurtosis since for particular choice of the tempering parameters, the tail behavior of the MixedTS varies from semi-heavy to heavy, while the tail behavior for the NVMM depends only on the tail behavior of the mixing random variable. The Value at Risk, computed as a quantile, is less influenced from extreme values than the ES which depends on the entire left tail though in the latter a better fit of the MixedTS distribution suggests more reliable estimates of the risk measure. In addition, through likelihood ratio tests, selection of nested models is possible in our setup like for example between MixedTS and Variance Gamma distributions. Another advantage of using the MixedTS is that we do not need to know a priori if we have to consider a heavy or semi-heavy distribution for the mixing random variable differently from the NVMM. We show that portfolio moments, needed in the modified versions of risk measures, are easily derived based on the hypothesis of MixedTS distributed ICs.
The outline of the paper is as follows. In Section 2 we briefly recall the risk parity approach and its connection with other portfolio optimization methods. The main results concerning the MixedTS distribution are reviewed in Section 3 while in Section 4 we analyze the risk parity approach for portfolio optimization using modified VaR and modified ES. Empirical results are given in Section 5 and Section 6 concludes the paper.

2 Portfolio construction using the Risk Parity approach

As observed in Maillard et al. (2010), a standard approach like mean-variance optimization has two drawbacks in practice. First, optimal portfolios seem to be

[1] Details and algorithms are given in Hyvarinen et al. (2001).

concentrated in a few assets. Second, small changes in the estimated parameters give rise to relevant modifications in the optimal portfolio composition as remarked in Merton (1980). To avoid this lack of stability, researchers proposed several regularization techniques. The most used are resampling of the objective function proposed by Michaud (1989) and shrinkage estimators of the covariance matrix developed in Ledoit and Wolf (2003). Other heuristic approaches like Equally Weighted (EW), Equal Risk Contributions (ERC) or Minimum Variance (MV) portfolios put constraints directly on portfolio weights and do not face advanced programming issues.
Let us consider a linear factor model where the $1 \times T$ vector of portfolio return r is expressed as a linear combination of the $N \times T$ matrix of factors F with the portfolio exposures β, i.e.:

$$r = \beta' F. \tag{1}$$

The marginal contribution to risk (MRC) of the i-th factor, given a risk measure $R(r)$, is defined as:

$$MRC_i = \frac{\partial R(r)}{\partial \beta_i} \tag{2}$$

representing the increment in the portfolio risk for each additional unit of exposure to the i-th factor for $i = 1, \dots, N$. The product of the exposure with the marginal contribution to risk is known as total risk contribution (TRC):

$$TRC_i = \beta_i \frac{\partial R(r)}{\partial \beta_i}. \tag{3}$$

For homogeneous risk measures portfolio total risk is simply the sum of the TRCs computed for all factors. Risk parity, as other portfolio optimization rules, identifies portfolio weights (or exposures) that satisfy a certain criteria. Maillard et al. (2010) propose to perform the following minimization:

$$\begin{gathered} \min_{\beta} \sum_{i=1}^{N} \sum_{j=1}^{N} (TRC_i - TRC_j)^2 \\ \text{sub} \\ \sum_{j=1}^{N} \beta_j = 1 \\ \beta_i \geq 0;\ i = 1, \dots, N \end{gathered} \tag{4}$$

where the inequality constraints refer to the no-short selling conditions. It is worth noting that the objective function in the optimization problem (4) introduces a penalty when TRCs are different from each other. In this way, the TRCs values for all factors in the portfolio are quite similar.

3 Mixed Tempered Stable distribution

A random variable (r.v.) $\tilde{Y}$ is a Normal Variance Mean Mixture (as in Barndorff-Nielsen et al., 1982) if its distribution has the form:

$$\tilde{Y} \stackrel{d}{=} \mu_0 + \mu V + \sigma \sqrt{V} Z \tag{5}$$

where $\mu_0, \mu \in \Re$ are constant parameters, $Z \sim N(0,1)$ and V is continuously distributed on the positive half-axis. The main idea behind the MixedTS is to substitute the normality assumption for the r.v. Z in formula (5) with the Tempered Stable distribution obtained multiplying the density of an α-Stable with a decreasing tempering function as explained in Cont and Tankov (2003). Tail behavior changes from heavy, for the α-Stable, to semi-heavy, for the Tempered Stable, characterized by an exponential instead of a power decay that ensures the existence of the conventional moments. The Tempered Stable distribution and the corresponding process have been widely applied in finance for modeling asset returns for example in Mercuri (2008); Rachev et al. (2011) and in Küchler and Tappe (2014).

Definition 1 We say that a continuous random variable Y follows a Mixed Tempered Stable distribution if:

$$Y \overset{d}{=} \mu_0 + \mu V + \sqrt{V} X \tag{6}$$

where $X\,|V \sim stdCTS(\alpha, \lambda_+\sqrt{V}, \lambda_-\sqrt{V})$ is Standardized Classical Tempered Stable distributed. The $stdCTS$ is the Classical Tempered Stable with zero mean and unit variance as reported in Kim et al. (2010) (see Küchler and Tappe, 2013, for a survey on the properties of a Classical Tempered Stable distribution and the associated Lévy process). V is an infinitely divisible distribution defined on the positive axis.

Defined the logarithm of the moment generating function (m.g.f.) of the r.v. V as:

$$\Phi_V(u) = \ln\left[E\left[\exp\left(uV\right)\right]\right] \tag{7}$$

and the characteristic exponent of a $stdCTS$:

$$L_{stdCTS}\left(u;\ \alpha,\ \lambda_+,\ \lambda_-\right) = \frac{(\lambda_+ - iu)^\alpha - \lambda_+^\alpha + (\lambda_- + iu)^\alpha - \lambda_-^\alpha}{\alpha\left(\alpha-1\right)\left(\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2}\right)} + \frac{iu\left(\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1}\right)}{\left(\alpha-1\right)\left(\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2}\right)},$$

the characteristic function of a MixedTS is computed applying the law of iterated expectation:

$$\begin{aligned} E\left[e^{iuY}\right] &= E\left[E\left[e^{iu\left(\mu_0 + \mu V + \sqrt{V}X\right)}\middle| V\right]\right] \\ &= e^{iu\mu_0} E\left[e^{\left[iu\mu + L_{stdCTS}\left(u;\ \alpha,\ \lambda_+,\ \lambda_-\right)\right]V}\right] \\ &= e^{iu\mu_0 + \Phi_V\left(iu\mu + L_{stdCTS}\left(u;\ \alpha,\ \lambda_+,\ \lambda_-\right)\right)}. \end{aligned} \tag{8}$$

The characteristic function in (8) identifies the distribution at time one of a time changed Lévy process and the distribution is infinitely divisible. The structure of the MixedTS allows a dependence of the standard higher moments not only on the mixing r.v. but also on the Standardized Classical Tempered Stable distribution parameters.

Proposition 1 *The first four central moments of the MixedTS are:*

$$\begin{cases} E\left[Y\right] = \mu_0 + \mu E\left[V\right] \\ Var\left[Y\right] = \mu^2 Var(V) + E\left[V\right] \\ m_3\left(Y\right) = \mu^3 m_3\left(V\right) + 3\mu Var(V) + (2-\alpha)\frac{\left(\lambda_+^{\alpha-3}-\lambda_-^{\alpha-3}\right)}{\left(\lambda_+^{\alpha-2}+\lambda_-^{\alpha-2}\right)}E\left[V\right] \\ m_4\left(Y\right) = \mu^4 m_4(V) + 6\mu^2 E\left[\left(V-E(V)\right)^2 V\right] + 4\mu\left(2-\alpha\right)\frac{\lambda_+^{\alpha-3}-\lambda_-^{\alpha-3}}{\lambda_+^{\alpha-2}+\lambda_-^{\alpha-2}}Var(V) \\ +(3-\alpha)(2-\alpha)\frac{\left(\lambda_+^{\alpha-4}+\lambda_-^{\alpha-4}\right)}{\left(\lambda_+^{\alpha-2}+\lambda_-^{\alpha-2}\right)}E\left[V\right]. \end{cases} \tag{9}$$

See Appendix (A) for details on the derivation of the moments. As observed in Rroji (2013), for $V \sim \Gamma(a, \sigma^2)$, the MixedTS has as special cases some well-known distributions in modeling financial returns. In particular, for $\alpha = 2$ we get the Variance Gamma distribution (see the definition in Madan and Seneta (1990) and estimation in Loregian et al. (2012)). The Standardized Classical Tempered Stable is obtained fixing $\sigma = \frac{1}{\sqrt{a}}$ and taking the limit as a goes to infinity. If the mixing r.v. V is Gamma distributed, we get explicit formulas for the right hand side quantities in (9) since:

$$\begin{aligned} E\left[V\right] &= a\sigma^2 \\ Var\left[V\right] &= a\sigma^4 \\ E\left[\left(V-E(V)\right)^2 V\right] &= E\left[\left(V-E(V)\right)^3\right] + E(V)Var(V) = \frac{2}{\sqrt{a}}a^{3/2}\sigma^6 + a^2\sigma^6 \\ E\left[\left(V-E(V)\right)^3\right] &= \frac{2}{\sqrt{a}}a^{3/2}\sigma^6 \\ E\left[\left(V-E(V)\right)^4\right] &= \left(3+\frac{6}{a}\right)a^2\sigma^8. \end{aligned}$$

Remark 1 If $V \sim \Gamma(a, \sigma^2)$ and $\tilde{V} \sim \Gamma(a, 1)$, the scale property of the Gamma r.v. ensures that:

$$V \stackrel{d}{=} \sigma^2 \tilde{V},$$

from where we notice that the definition in (6) is equivalent to:

$$Y \stackrel{d}{=} \mu_0 + \tilde{\mu}\tilde{V} + \sigma\sqrt{\tilde{V}}\tilde{X} \tag{10}$$

where $\tilde{\mu} = \mu\sigma^2$ and $\tilde{X} \sim stdCTS(\alpha, \lambda_+\sigma\sqrt{V}, \lambda_-\sigma\sqrt{V})$. The definition of NVMM in (5) with the MixedTS definition in (10) suggest a similar structure for the two distributions.

Once we have the characteristic function ϕ_Y of a r.v. Y, the distribution function of the r.v. Y denoted with $F_Y(y)$ is computed using the Inverse Fourier Transform, i.e.:

$$F_Y\left(y\right) = \frac{1}{2} - \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\left[e^{-ity}\phi_Y\left(t\right)\right]}{it}\mathrm{d}t.$$

Let us now suppose Y to be the vector of returns of an asset or of a portfolio. The Value at Risk at the confidence level α is obtained by inverting the distribution function:

$$VaR_\alpha(Y) = -F_Y^{-1}(\alpha).$$

Under the assumption of existence for the $E(Y)$, the Expected Shortfall is computed using the formula:

$$ES_{\alpha}(Y) = E\left[Y \mid Y \leq -VaR_{\alpha}(Y)\right] = -VaR_{\alpha}(Y) - \frac{1}{\alpha}\int_{-\infty}^{-VaR_{\alpha}(Y)} F_Y(u)\,\mathrm{d}u.$$

For an application of the univariate MixedTS in risk measurement using see Mercuri and Rroji (2014a).

4 Parametric risk decomposition

Let $R(r)$ be a positive homogeneous risk measure. Applying Euler's theorem as in Tasche (1999) we get:

$$R(r) = \sum_{i=1}^{N} \beta_i \frac{\partial R(r)}{\partial \beta_i} = \sum_{i=1}^{N} TRC_i \tag{11}$$

where the Total Risk Contribution of the i-th risk factor is defined in equation (3). In particular the TRC_i for the risk measures considered in this paper are listed below based in the linear model in (1).

- For the Volatility, defined as $R(r) = \sqrt{\beta' \Sigma \beta}$, we get:

$$TRC_i = \frac{\left(\Sigma\beta\right)_i}{\sqrt{\beta'\Sigma\beta}}\beta_i \tag{12}$$

  where Σ is the variance-covariance matrix of the factors.
- For the Value-at-Risk as in Gouriéroux and Laurent (2000):

$$TRC_i = -E\left[F_i \left| r = VaR_{\alpha}(r)\right.\right]\beta_i. \tag{13}$$

- For the Expected Shortfall as in Tasche (2002):

$$TRC_i = -E\left[F_i \left| r \leq -VaR_{\alpha}(r)\right.\right]\beta_i. \tag{14}$$

The computation of the total risk contribution for a given factor is easy in the historical approach[2] but this method has been criticized in Boudt et al. (2007) for the Value at Risk and the Expected Shortfall. Indeed the high estimation error in the historical approach, especially for small sample size, is reflected into a larger variation in the simulated future values compared to those generated from a correctly specified parametric distribution.

In a non-Gaussian parametric framework, the modified VaR proposed in Zangari (1996) and the modified ES developed in Boudt et al. (2007) both preserve the homogeneity property and they can be easily computed once the multivariate moments of the factors are available. In our approach, starting from (1), portfolio

[2] For the Value at Risk and the Expected Shortfall, the historical approach involves the following steps. Start with a data matrix containing in the first row the vector r while the others are the N vectors with the realized returns for each factor. Then, sort all rows of the data matrix in ascending order of the first row.
Looking at the first row estimate the portfolio $VaR_{\alpha}(r)$ at the confidence level α. Marginal contributions are then computed on the sorted rows using the sample estimators in (13) and (14).

return is a weighted average of factor returns. The mean vector for the factors is μ while Σ is their variance-covariance matrix of dimension $N \times N$. The co-skewness matrix for the factors:

$$M_3 = E[(F-\mu)(F-\mu)' \otimes (F-\mu)'] \tag{15}$$

is of dimension $N \times N^2$ while their co-kurtosis matrix is of dimension $N \times N^3$:

$$M_4 = E[(F-\mu)(F-\mu)' \otimes (F-\mu)' \otimes (F-\mu)'] \tag{16}$$

where $\otimes$ denotes the Kronecker product. The second, third and fourth order centered moments of the returns in vector r are computed respectively as:

$$\begin{cases} m_2 = \beta' \Sigma \beta \\ m_3 = \beta' M_3 (\beta \otimes \beta) \\ m_4 = \beta' M_4 (\beta \otimes \beta \otimes \beta). \end{cases} \tag{17}$$

The skewness (skew) and kurtosis (kurt) are defined based on the centered moments, respectively:

$$skew = \frac{m_3}{m_2^{\frac{2}{3}}} \tag{18}$$

and

$$kurt = \frac{m_4}{m_2^2} - 3. \tag{19}$$

In order to compute Σ, M_3 and M_4 and consequently the centered moments, we need the multivariate distribution for the factor returns F or their dependence structure by means of a copula function. Here we face the problem from a different point of view, that is, we look for the underlying independent factors that generate the observed returns. In practice, the ICA analysis Hyvarinen (1999) applied to the factors simplifies the computation of Σ, M_3 and M_4 since it yields:

$$F = AS \tag{20}$$

where in $S = [s_1 \ldots\ldots s_N]'$ we have the ICs and A is the mixing matrix to be estimated. The procedure is based on the maximization of a non Gaussian measure computed for each component as for example the negentropy. Each signal is then modeled using the MixedTS, i.e.:

$$s_i \sim \mu_0^i + \mu^i V^i + \sqrt{V^i} \tilde{X}^i. \tag{21}$$

As shown in Appendix B, the computation of the elements in the matrices Σ, M_3 and M_4 is quite easy and fast due to the independence of the ICs. In particular, we get:

$$\begin{cases} \Sigma^{ik} = \sum_{j=1}^N a_{ij} a_{kj} \sigma^2(s_j) \\ \\ M_3^{ikl} = \sum_{j=1}^N a_{ij} a_{kj} a_{lj} skew(s_j) \\ \\ M_4^{iklm} = \sum_{j=1}^N a_{ij} a_{kj} a_{lj} a_{mj} kurt(s_j). \end{cases} \tag{22}$$

where a_{ij} is the ij-th element of the mixing matrix A. The quantities $\sigma^2(s_j)$, $skew(s_j)$ and $kurt(s_j)$ denote respectively variance, skewness and kurtosis of the

j-th IC. Computed the moments and co-moments, we use them in the modified VaR definition introduced in Zangari (1996):

$$mVaR_{\alpha}(r) = -\beta' \mu - \sqrt{m_2}\Phi^{-1}(\alpha) + \sqrt{m_2} C(z_{\alpha}, skew, kurt) \tag{23}$$

where the quantity:

$$C(z_{\alpha}, skew, kurt) = \left[-\frac{1}{6}(z_{\alpha}^2 - 1) skew - \frac{1}{24}(z_{\alpha}^3 - 3z_{\alpha}) kurt + \frac{1}{36}(2z_{\alpha}^3 - 5z_{\alpha}) skew^2 \right] \tag{24}$$

corrects the Gaussian VaR by considering skewness ($skew$) and kurtosis ($kurt$) of the return vector r and $z_{\alpha} = \Phi^{-1}(\alpha)$. Observe that with Φ we denote the distribution of a standard normal while its inverse is the quantile. Modified ES, introduced in Boudt et al. (2007), is a linear transformation of the expected value of the returns below the α Cornish-Fisher quantile where the second order Edgeworth expansion of the true distribution G_2 is considered. It is computed as:

$$mES_{\alpha}(r) = -\beta' \mu - \sqrt{m_2} E_{G_2}[z \,|z \leq g_{\alpha}] \tag{25}$$

with $g_{\alpha} = G_2^{-1}(\alpha)$. The extended formula is:

$$\begin{aligned} E_{G_2}[z \,|z \leq g_{\alpha}] = -\frac{1}{\alpha} \Big\{ & \phi(g_{\alpha}) + \frac{1}{24}\left[I^4 - 6I^2 + 3\phi(g_{\alpha}) \right] kurt + \frac{1}{6}\left[I^3 - 3I \right] skew \\ & + \frac{1}{72}\left[I^6 - 15I^4 + 45I^2 - 15\phi(g_{\alpha}) \right] skew^2 \Big\} \end{aligned} \tag{26}$$

where

$$I^q = \begin{cases} \prod_{j=1}^{q/2} (\frac{\prod_{j=1}^{q/2} 2j}{\prod_{j=1}^{i} 2j}) g_{\alpha}^{2i} \phi(g_{\alpha}) + (\prod_{j=1}^{q/2} 2j)\phi(g_{\alpha}) & for \;\; q \;\; even \\ \prod_{j=0}^{q^*} (\frac{\prod_{j=0}^{q^*}(2j+1)}{\prod_{j=0}^{i}(2j+1)}) g_{\alpha}^{2i+1} \phi(g_{\alpha}) - (\prod_{j=0}^{q^*}(2j+1))\phi(g_{\alpha}) & for \;\; q \;\; odd. \end{cases} \tag{27}$$

and $q^* = \frac{q-1}{2}$. The partial derivatives formulas for the centered moments are:

$$\begin{cases} \frac{\partial m_2}{\partial \beta_i} = 2\left(\Sigma \beta\right)_i \\ \frac{\partial m_3}{\partial \beta_i} = 3\left[M_3(\beta \otimes \beta)\right]_i \\ \frac{\partial m_4}{\partial \beta_i} = 4\left[M_4(\beta \otimes \beta \otimes \beta)\right]_i \end{cases} \tag{28}$$

Partial derivatives in (28) allow us to obtain the total risk contribution for modified VaR using the following formula:

$$\begin{aligned} \frac{\partial mVaR_{\alpha}(r)}{\partial \beta_i} = & -\mu_i - \frac{\partial m_2}{\partial \beta_i}\frac{1}{2\sqrt{m_2}}\Phi^{-1}(\alpha) \\ & + \frac{\partial m_2}{\partial \beta_i}\frac{1}{\sqrt{m_2}}\left[-\frac{1}{12}(z_{\alpha}^2 - 1) skew - \frac{1}{48}(z_{\alpha}^3 - 3z_{\alpha}) kurt + \frac{1}{72}(2z_{\alpha}^3 - 5z_{\alpha}) skew^2 \right] \\ & + \sqrt{m_2}\left[-\frac{1}{6}(z_{\alpha}^2 - 1)\frac{\partial skew}{\partial \beta_i} - \frac{1}{24}(z_{\alpha}^3 - 3z_{\alpha})\frac{\partial kurt}{\partial \beta_i} + \frac{1}{18}(2z_{\alpha}^3 - 5z_{\alpha}) skew \frac{\partial skew}{\partial \beta_i} \right] \end{aligned} \tag{29}$$

Total risk contribution for modified ES is expressed through a similar formula given in Boudt et al. (2007). The derivative of (25) requires straightforward but long calculations that can be implemented directly using standard algebra in any programming language.
In Figure 1 we give a detailed sketch of the entire procedure described in this paper.

Insert here Figure 1.

Through the ICA algorithm, we get the linear decomposition in (20) which rules out the possibility of having errors due to incorrect specification of the multivariate distribution for factors. There remain three sources of error in the procedure: the fitting of each IC distribution, the approximation quality of the modified risk measure and the minimization problem in (4). Empirical results in Rroji and Mercuri (2015) suggest a better fit of the MixedTS compared to the VG distribution. A detailed discussion on how well modified VaR and modified ES approximate respectively VaR and ES is given in Boudt et al. In particular they find that the approximation error increases for extreme skewness and excess kurtosis. Modified ES is more sensitive to extreme deviations from normality than modied VaR, and therefore should only be used in the case of moderate deviation from normality.
An alternative portfolio optimization problem based on the computation of partial derivatives of homogeneous risk measures is defined in Kim et al. (2012). Instead of a linear decomposition of factors, they consider a multivariate Normal Tempered Stable distribution for modeling asset returns. Variation of portfolio risk measure is approximated through marginal risk contributions using first order Taylor expansion while in our approach an approximation is used directly in the definition of modified VaR and modified ES. The idea of these models is not simply to extract optimal portfolio weights based on advanced mathematical tools, as for example in Babaeia et al. (2015) that look at the optimization problem as a multi-objective mixed integer programming, but also to manage risk allocation.

5 Empirical analysis

In the first part of this section we empirically investigate the ability of the MixedTS in capturing tail behavior through a comparison with the historical approach in the computation of risk measures for univariate time series. Then we apply the proposed methodology based on the hypothesis that the extracted ICs are MixedTS distributed.
We consider a dataset composed by daily log returns, for the period going from 24/06/2010 to 10/07/2013, of the Vanguard fund (VFIAX) and ten sector indices: Utility, Telecommunications, Materials, Information Technology, Industrial, Health, Financial, Energy, Consumption Staple and Consumption Discretionary. The VFIAX fund replicates the performance of the S&P500 index and the weights reflect market capitalization of the constituent sector indices. In Table 1 we report the main statistics of the dataset.

Insert here Table 1.

We fit the MixedTS distribution to the log returns, computed using the entire dataset, of the VFIAX fund and compare the historical VaR and ES with modified VaR and modified ES using respectively the formulas in (23) and in (25). The confidence level parameter α ranges in the interval $(0.01; 0.1)$. From Figure 2 we observe that for $\alpha < 0.08$ the historical VaR is lower than the modified VaR computed using the MixedTS. Notice that from its definition ES is highly influenced by extreme values. We consider the comparison with the empirical (or robust) ES introduced in Cont et al. (2010)) that is a trimmed mean since for $0 < \alpha_1 < \alpha < 1$ it reads:

$$ES_{\alpha}^{Empi}(Y) = \frac{1}{(\alpha - \alpha_1)} \int_{\alpha_1}^{\alpha} VaR_u(Y) du. \tag{30}$$

Insert Figure 2.

We move now to the portfolio optimization problem. We consider the returns of the VFIAX fund as a linear combination of the returns of ten sector indices based on model (1) and perform an ICA analysis on the factor returns for the period from 24/06/2010 to 24/06/2011. The output of this algorithm is the mixing matrix in Table 2.

Insert here Table 2.

We fit the MixedTS to each independent component as in (21). The fitted parameters obtained through maximum likelihood estimation are reported in Table 3. Particular attention deserves the parameter α since for $\alpha = 2$ we get the Variance Gamma distribution. We notice that only the fourth and the fifth components can be modeled with the Variance Gamma while the others behave differently. The first four moments of each component are computed once we have the parameters.

Insert here Table 3.

Insert here Figure 3.

We compare out-of-sample performance of the VFIAX fund with the three risk parity portfolios based respectively on Volatility, modified VaR and modified ES. We consider 250 closing prices as in sample data and the following 50 closing prices as out of sample data starting from 24/06/2011 till 10/07/2013 in rolling windows. In Table 4 we find mean of the out-of-sample log returns respectively of the S&P500 index, the VFIAX fund index, and of the three risk parity portfolios. First we give the results for each out-of-sample window and then the mean and standard deviation of all out-of-sample results. In Figure 3 we depict the composition of the four portfolios at on June 24th 2011.

Insert here Table 4.

In Figure 4 we display the cumulative out-of-sample performance of two portfolios: the VFIAX fund and the risk parity portfolio when the risk measure considered is the modified ES.

Insert here Figure 3.

Insert here Figure 4.

The Gini index G defined as:

$$G = \frac{1}{N-1}\left(N+1-2\left(\frac{\sum_{i=1}^{N}(N+1-i)y_i}{\sum_{i=1}^{N} y_i}\right)\right) \tag{31}$$

where the observations are ordered, i.e $y_i \leq y_{i+1}$, is used for measuring diversification. The Gini index for equally weighted portfolios equals 0 and 1 when all the weight is given to one asset, i.e for perfectly concentrated portfolios. In Table 5 we report the Gini index G for each window in the rolling analysis.

Insert here Table 5.

We notice that risk parity portfolios based on the two risk measures, Volatility and modified ES, are less concentrated almost in all windows. The VFIAX fund weights follow the market capitalization of the sectors in the S&P500 reflected in the Gini index values.

6 Conclusion

In this paper we describe the steps required in a parametric risk decomposition framework. The idea of applying the ICA analysis on the factors and modeling each source signal with the MixedTS distribution gives rise to the possibility of having analytical formulas for the portfolio return moments and flexibility in capturing tail behavior. This approach can be applied to any setup that considers a homogeneous risk measure. In order to make an investment decision we have to consider both performance and portfolio concentration. Based on our results, we have that risk parity portfolios are less concentrated and show better out-of sample performances than the passive strategy of investing the entire wealth on the VFIAX fund. We also observed that the decision of which risk measure to consider is not so relevant in terms of portfolio composition. Risk parity portfolios based on modified VaR resulted to be more concentrated than the alternative portfolios here considered. Finally, some remarks on possible future research starting from this paper are listed below. One can generalize the proposed procedure into a dynamic context through the use of heteroscedastic models, for instance a GARCH with MixedTS noise for modeling each independent component. Another possible extension is the introduction of a multivariate MixedTS distribution based on similar structures of multivariate NVMM already existing in literature.

Appendix A Proof of Proposition 1

We recall that the formula for the cumulant of order n of the Standardized Tempered Stable r.v. X with parameters $(\alpha, \lambda_+, \lambda_-)$ derived in Kim et al. (2010) is:

$$c_n(X) = \Gamma(n-\alpha)\left(\lambda_+^{\alpha-n} + (-1)^n \lambda_-^{\alpha-n}\right) C, \quad n \geq 2$$

where the constant C is fixed in order to ensure the standardization condition, i.e:

$$C = \frac{1}{\Gamma\left(2-\alpha\right)\left(\lambda_{+}^{\alpha-2}+\lambda_{-}^{\alpha-2}\right)}.$$

In the following we show how to compute the first four moments of the MixedTS defined in (6). The mean is computed directly as:

$$E\left[Y\right] = \mu_0 + \mu E\left[V\right],$$

while the variance is:

$$\begin{aligned} Var\left[Y\right] &= E\left\{\left[\mu\left(V-E(V)\right)+\sqrt{V}X\right]^2\right\} \\ &= E\left\{\mu^2\left(V-E(V)\right)^2 + VX^2 + 2\mu\left(V-E(V)\right)\sqrt{V}X\right\}. \end{aligned}$$

Exploiting the linearity and the iteration property of the expected value, we get the variance of the MixedTS as a function of the first two moments of the mixing r.v. V computed as:

$$\begin{aligned} Var\left[Y\right] &= \mu^2 Var(V) + E\left[VE\left(X^2\left|V\right.\right)\right] \\ &= \mu^2 Var(V) + E\left[V\right]. \end{aligned}$$

The third central moment is:

$$\begin{aligned} m_3\left(Y\right) &= E\left\{\left[\mu\left(V-E(V)\right)+\sqrt{V}X\right]^3\right\} \\ &= E\left\{\left[\mu^3\left(V-E(V)\right)^3 + 3\mu^2\left(V-E(V)\right)^2\sqrt{V}X + 3\mu\left(V-E(V)\right)VX^2 + V^{3/2}X^3\right]\right\}. \end{aligned}$$

It is easy to show that:

$$E\left[\sigma\mu^2\left(V-E(V)\right)^2\sqrt{V}X\right] = 0,$$

from where we get:

$$m_3\left(Y\right) = \mu^3 m_3\left(V\right) + 3\mu E\left[\left(V-E(V)\right)VX^2\right] + E\left[V^{3/2}X^3\right].$$

Since:

$$E\left[\left(V-E(V)\right)VX^2\right] = Var(V)$$

and

$$\begin{aligned} E\left[V^{3/2}X^3\right] &= E\left[V^{3/2}E\left(X^3\left|V\right.\right)\right] \\ &= E\left[V^{3/2}\frac{\Gamma\left(3-\alpha\right)\left(\lambda_{+}^{\alpha-3}+(-1)^3\lambda_{-}^{\alpha-3}\right)}{\Gamma\left(2-\alpha\right)\left(\lambda_{+}^{\alpha-2}+\lambda_{-}^{\alpha-2}\right)}\frac{V^{\alpha/2-3/2}}{V^{\alpha/2-2/2}}\right] \\ &= \left(2-\alpha\right)\frac{\left(\lambda_{+}^{\alpha-3}-\lambda_{-}^{\alpha-3}\right)}{\left(\lambda_{+}^{\alpha-2}+\lambda_{-}^{\alpha-2}\right)}E\left[V\right], \end{aligned}$$

the third central moment depends on the first three moments of the mixing random variable V:

$$m_3(Y) = \mu^3 m_3(V) + 3\mu Var(V) + (2-\alpha)\frac{\left(\lambda_+^{\alpha-3} - \lambda_-^{\alpha-3}\right)}{\left(\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2}\right)} E[V].$$

For the fourth central moment we start from its definition:

$$\begin{aligned} m_4(Y) &= E\left\{\left[\mu(V - E(V)) + \sqrt{V}X\right]^4\right\} \\ &= \mu^4 k(V) + 4E\left\{\left[\mu^3 (V - E(V))^3 \sqrt{V}X\right]\right\} + 6E\left\{\left[\mu^2 (V - E(V))^2 VX^2\right]\right\} \\ &+ 4\mu E\left\{\left[(V - E(V)) V^{3/2} X^3\right]\right\} + E\left[V^2 X^4\right]. \end{aligned}$$

and observe that since:

$$\begin{aligned} E\left[(V - E(V)) V^{3/2} X^3\right] &= (2-\alpha)\frac{\lambda_+^{\alpha-3} - \lambda_-^{\alpha-3}}{\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2}} Var(V) \\ E\left[V^2 X^4\right] &= E\left[V^2 E\left[X^4 \,|V\right]\right] \\ &= E\left[V^2 \frac{\Gamma(4-\alpha)}{\Gamma(2-\alpha)} \frac{\left(\lambda_+^{\alpha-4} + \lambda_-^{\alpha-4}\right)}{\left(\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2}\right)} \frac{V^{\alpha/2-2}}{V^{\alpha/2-1}}\right] \\ &= (3-\alpha)(2-\alpha)\frac{\left(\lambda_+^{\alpha-4} + \lambda_-^{\alpha-4}\right)}{\left(\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2}\right)} E[V], \end{aligned}$$

the fourth central moment of the MixedTS depends on the first four moments of the r.v. V. Indeed:

$$\begin{aligned} m_4(Y) &= \mu^4 m_4(V) + 6\mu^2 E\left[(V - E(V))^2 V\right] + 4\mu(2-\alpha)\frac{\lambda_+^{\alpha-3} - \lambda_-^{\alpha-3}}{\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2}} Var(V) \\ &+ (3-\alpha)(2-\alpha)\frac{\left(\lambda_+^{\alpha-4} + \lambda_-^{\alpha-4}\right)}{\left(\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2}\right)} E[V]. \end{aligned}$$

Appendix B Moments using ICA

From the independence of the ICs, the formula of the ik-th element of the variance-covariance matrix Σ of factors reads:

$$\begin{aligned} \Sigma^{ik} &= E\left[\{F_i - E[F_i]\}\{F_k - E[F_k]\}\right] \\ &= E\left[\left\{\sum_{j=1}^{N} a_{ij}(s_j - E[s_j])\right\}\left\{\sum_{j=1}^{N} a_{kj}(s_j - E[s_j])\right\}\right] \\ &= \sum_{j=1}^{N} a_{ij} a_{kj} \sigma^2(s_j). \end{aligned}$$

The element M_3^{ikl} is:

$$\begin{aligned} M_3^{ikl} &= E\left[\{F_i - E[F_i]\}\{F_k - E[F_k]\}\{F_l - E[F_l]\}\right] \\ &= E\left[\left\{\sum_{j=1}^{N} a_{ij}\left(s_j - E[s_j]\right)\right\}\left\{\sum_{j=1}^{N} a_{kj}\left(s_j - E[s_j]\right)\right\}\left\{\sum_{j=1}^{N} a_{lj}\left(s_j - E[s_j]\right)\right\}\right] \\ &= \sum_{j=1}^{N} a_{ij}a_{kj}a_{lj}skew(s_j). \end{aligned}$$

The element M_4^{iklm} is computed as:

$$\begin{aligned} M_4^{iklm} &= E\left[\{F_i - E[F_i]\}\{F_k - E[F_k]\}\{F_l - E[F_l]\}\{F_m - E[F_m]\}\right] \\ &= \sum_{j=1}^{N} a_{ij}a_{kj}a_{lj}a_{mj}kurt(s_j). \end{aligned}$$

	Mean	Std	Skewness	Kurtosis	Max	Min
VFIAX	5.22E-04	0.0111	-0.4990	7.4284	0.0463	-0.0690
COND	7.93E-04	0.0119	-0.5873	6.4336	0.0472	-0.0690
CONS	5.67E-04	0.0076	-0.4175	6.0214	0.0332	-0.0390
ENRS	4.77E-04	0.0145	-0.4215	6.8501	0.0687	-0.0864
FINL	4.00E-04	0.0159	-0.3977	7.9692	0.0789	-0.1052
HLTH	6.69E-04	0.0096	-0.4605	6.7295	0.0456	-0.0540
INDU	5.08E-04	0.0129	-0.4854	6.3092	0.0495	-0.0711
INFT	4.31E-04	0.0121	-0.2512	5.2089	0.0445	-0.0596
MATR	3.73E-04	0.0147	-0.3828	5.9989	0.0593	-0.0756
TELS	5.25E-04	0.0096	-0.2754	5.5523	0.0426	-0.0550
UTIL	3.07E-04	0.0086	-0.1836	7.2391	0.0414	-0.0563

Table 1 Main statistics of the VFIAX fund and of the Sector Indices for the period 24/06/2010-10/07/2013.

Mixing Matrix									
-0.0113	0.0024	-0.0040	0.0029	-0.0016	-0.0009	-0.0078	-0.0033	0.0012	-0.0022
-0.0066	0.0010	-0.0007	0.0038	-0.0039	-0.0004	-0.0029	-0.0005	0.0015	-0.0002
-0.0140	0.0007	-0.0069	0.0008	-0.0062	0.0013	-0.0072	-0.0023	0.0051	0.0023
-0.0173	0.0010	-0.0091	0.0050	-0.0030	0.0037	-0.0034	-0.0037	0.0039	-0.0040
-0.0095	0.0011	-0.0039	0.0032	-0.0032	-0.0004	-0.0048	0.0019	0.0004	-0.0010
-0.0117	-0.0004	-0.0053	0.0037	-0.0038	0.0019	-0.0085	-0.0014	0.0040	-0.0031
-0.0103	0.0032	-0.0053	0.0024	-0.0005	-0.0024	-0.0069	-0.0009	0.0061	-0.0017
-0.0128	0.0022	-0.0074	0.0000	-0.0068	0.0004	-0.0078	-0.0021	0.0046	-0.0043
-0.0091	-0.0036	-0.0017	0.0022	-0.0026	-0.0026	-0.0018	-0.0012	0.0016	-0.0011
-0.0095	0.0000	0.0017	0.0009	-0.0025	0.0011	-0.0019	0.0009	0.0014	-0.0006

Table 2 Mixing Matrix obtained applying the ICA algorithm on the matrix whose vectors are the historical time series of log returns from 24/06/2010 to 24/06/2011 of the ten sector indices of the S&P500.

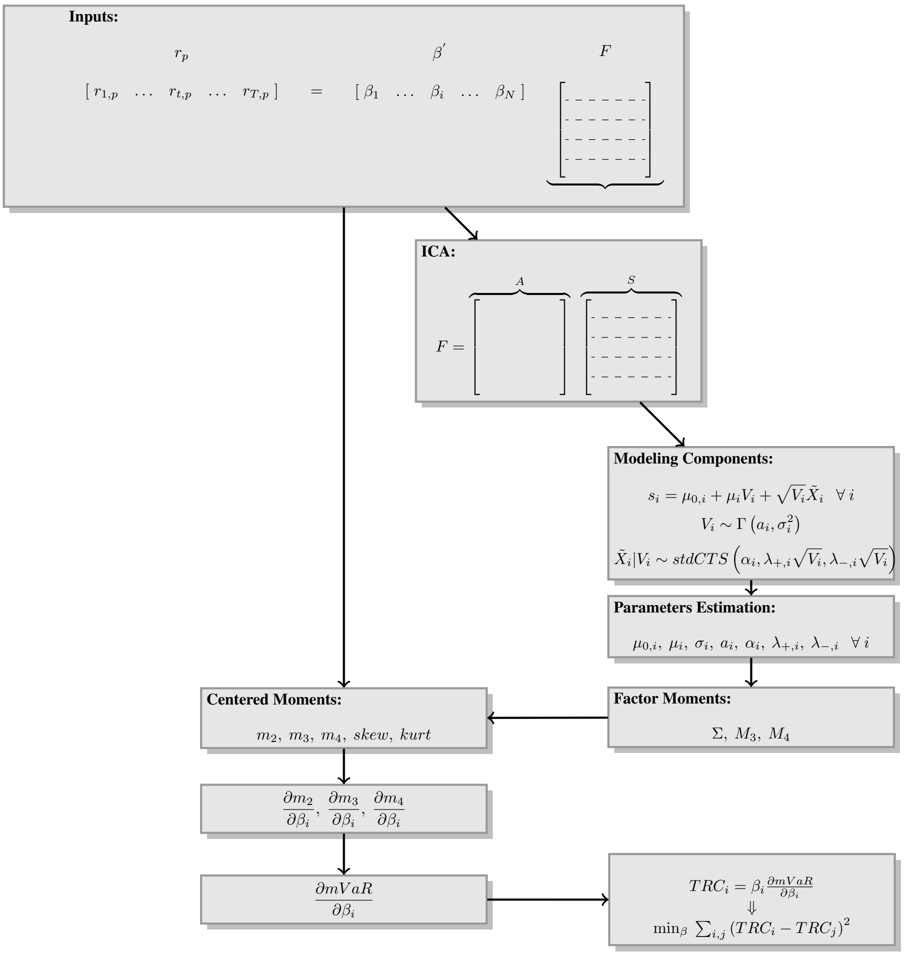

Fig. 1 Main steps required in parametric risk parity portfolio construction. Start with a linear factor model for portfolio returns as in (1). Based on (20), use the ICA algorithm. Each ICs s_i for $i = 1, .., N$ is then modeled using the MixedTS distribution as described in (21). The fitted parameters on the time series of each s_i are used for the computation of the moments in (22). Marginal risk contribution formula in (29) (for the modified VaR) requires the partial derivatives of the centered moments in (28). The last step for the portfolio construction is the optimization problem in (4).

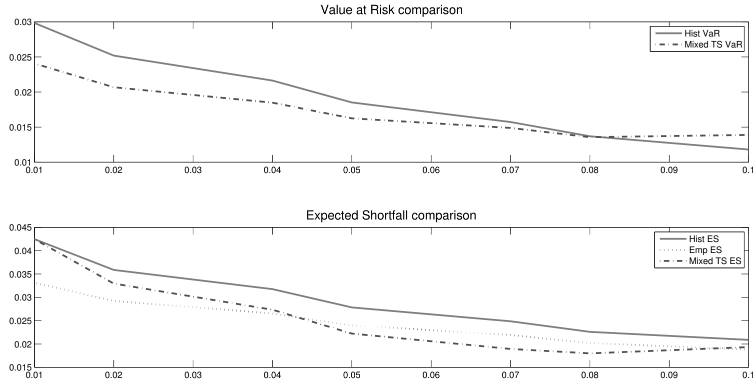

Fig. 2 In the upper plot Value at Risk of the VFIAX fund index is computed for the period 24/06/2010 - 10/07/2013 for $\alpha \in (0.01 : 0.1)$ using both the historical approach and the formula in (23) for MixedTS distributed log returns. In the lower plot both historical and MixedTS based ES in (25) for $\alpha \in (0.01 : 0.1)$ together with the empirical (robust) ES for $\alpha_1 = 0.005$ are reported.

	I	II	III	IV	V	VI	VII	VIII	IX	X
μ_0	0.0989	0.1915	1.0361	-0.0555	0.4227	0.5418	0.9911	0.7190	0.3449	0.7476
μ	-0.0719	-0.0745	-0.3914	0.0579	-0.0674	-0.0991	-0.1763	-0.1094	-0.0688	-0.1386
σ	0.6847	0.5991	0.5766	0.5132	0.3285	0.4095	0.3798	0.3729	0.4490	0.4705
a	2.1983	2.5824	2.6360	3.8144	6.6537	6.0530	5.8454	6.3537	5.0876	5.0049
α	0.8740	1.7955	0.6383	2.0000	1.9904	0.0594	0.0100	1.5698	0.0100	0.1282
λ_+	1.1631	1.3175	1.2307	1.2924	1.2891	1.5148	1.9890	1.6767	1.6033	1.8090
λ_-	1.2186	1.4375	2.1308	2.9084	2.9103	2.6869	2.4690	4.0004	2.5576	2.4291
LogLik	-354.4313	-342.0764	-371.4771	-403.3216	-403.7799	-374.1327	-344.5811	-494.5449	-336.7657	-360.1470

Table 3 MixedTS fitted parameters of the independent components obtained by applying the ICA algorithm to the matrix containing the returns from 24/06/2010 to 24/06/2011 of the ten sector indices of the S&P500.

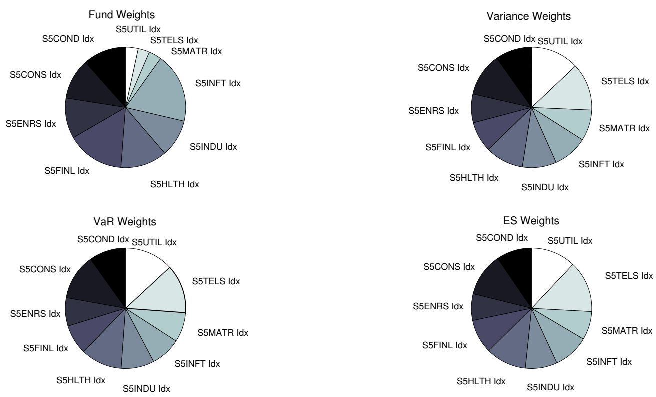

Fig. 3 Portfolio composition respectively of the VFIAX fund and of the three risk parity portfolios based on the homogeneous risk measures: Volatility, modified VaR and modified ES. The fund weights refer to the closing date 24/06/2011 and the risk parity portfolios are computed at the same date based on the previous year of daily data.

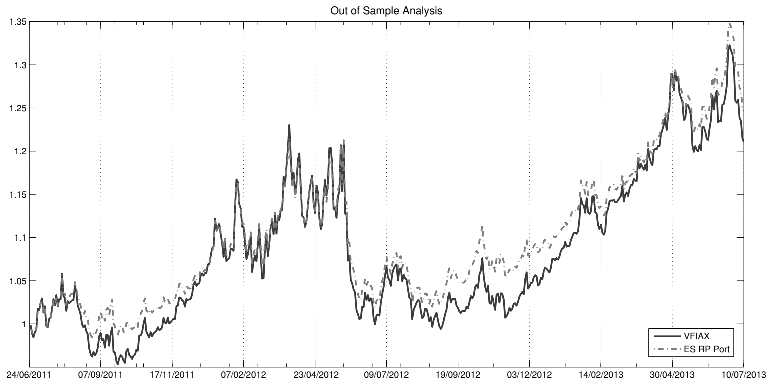

Fig. 4 Out of sample performance of two portfolios: the VFIAX fund and the risk parity portfolio when the risk measure considered is the modified ES. The analysis refers to the period 24/06/2011 till 10/07/2013 considering rolling windows of 250 closing prices as in sample data and the following 50 closing prices as out of sample data.

Out-of-sample results for each window					
w	mean SPX	mean VFIAX	mean $RP_{Volatility}$	mean RP_{VaR}	mean RP_{ES}
1	-0.0213%	-0.0209%	0.0278%	0.0312%	0.0302%
2	0.0293%	0.0311%	0.0189%	0.0208%	0.0200%
3	0.2045%	0.2058%	0.1654%	0.1631%	0.1698%
4	0.0290%	0.0289%	0.0235%	0.0229%	0.0231%
5	-0.1132%	-0.1102%	-0.0876%	-0.0895%	-0.0934%
6	-0.0920%	-0.0867%	-0.0442%	-0.0455%	-0.0491%
7	0.0481%	0.0466%	0.0503%	0.0502%	0.0509%
8	0.1327%	0.1315%	0.1015%	0.1008%	0.1034%
9	0.2913%	0.2940%	0.2467%	0.2473%	0.2564%
10	-0.1267%	-0.1275%	-0.0672%	-0.0672%	-0.0719%
Global out-of-sample results					
	SPX	VFIAX	$RP_{Volatility}$	RP_{VaR}	RP_{ES}
mean	0.0382%	0.0393%	0.0435%	0.0434%	0.0439%
s.d.	0.01242	0.01241	0.01090	0.010862	0.011040

Table 4 Mean of log returns for the S&P500, VFIAX fund and risk parity portfolios for three risk measures: Volatility, modified VaR and modified ES for the rolling windows analysis in the period 24/06/2011 till 10/07/2013 with 250 closing prices as in sample data and the following 50 closing prices as out of sample data. In the last two rows the mean and standard deviation (s.d.) of all out of sample results are given.

w	G^{VFIAX}	$G^{Vol_{RP}}$	$G^{VaR_{RP}}$	$G^{ES_{RP}}$
1	0.301	0.194	0.247	0.197
2	0.301	0.166	0.248	0.235
3	0.302	0.178	0.222	0.189
4	0.301	0.194	0.247	0.197
5	0.300	0.198	0.244	0.185
6	0.297	0.200	0.231	0.198
7	0.297	0.186	0.218	0.203
8	0.294	0.181	0.206	0.177
9	0.299	0.193	0.246	0.150
10	0.301	0.179	0.233	0.187

Table 5 Gini index computed for each rolling window, in the period 24/06/2011 till 10/07/2013 with 250 closing prices as in sample data and the following 50 closing prices as out of sample data, for the VFIAX fund and for the three risk parity portfolios based respectively on the homogeneous risk measures: Volatility, modified VaR and modified ES.

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