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One-Dimensional Super Calabi-Yau Manifolds and their Mirrors

S. Noja,[a] **S. L. Cacciatori,**[b,c] **F. Dalla Piazza,**[b] **A. Marrani,**[d,e] **R. Re**[f]

[a] *Dipartimento di Matematica, Università degli Studi di Milano, Via Saldini 50, I-20133 Milano, Italy*

[b] *Dipartimento di Scienza e Alta Tecnologia, Università dell'Insubria, Via Valleggio 11, I-22100 Como, Italy.*

[c] *INFN, Sezione di Milano, Via Celoria 16, I-20133 Milano, Italy.*

[d] *Centro Studi e Ricerche 'Enrico Fermi', Via Panisperna 89A, I-00184 Roma, Italy.*

[e] *Dipartimento di Fisica e Astronomia 'Galileo Galilei', Università di Padova, and INFN, Sezione di Padova, Via Marzolo 8, I-35131 Padova, Italy*

[f] *Dipartimento di Matematica e Informatica, Università degli Studi di Catania, Viale Andrea Doria 6, 95125 Catania, Italy*

E-mail: simone.noja@unimi.it, sergio.cacciatori@uninsubria.it, f.dallapiazza@gmail.com, Alessio.Marrani@pd.infn.it, riccardo@dmi.unict.it

Abstract: We apply a definition of generalised super Calabi-Yau variety (SCY) to supermanifolds of complex dimension one. We get that the class of all SCY's of bosonic dimension one and reduced manifold equal to $\mathbb{P}^1$ is given by $\mathbb{P}^{1|2}$ and the weighted projective super space $\mathbb{WP}^{1|1}_{(2)}$. Then we compute the corresponding sheaf cohomology of superforms, showing that the cohomology with picture number one is infinite dimensional, while the de Rham cohomology remains finite dimensional. Moreover, we provide the complete real and holomorphic de Rham cohomology for generic projective super spaces $\mathbb{P}^{n|m}$. We also determine the automorphism groups, which for $\mathbb{P}^{1|2}$ results to be larger than the projective supergroup. Finally, we show that $\mathbb{P}^{1|2}$ is self mirror, whereas $\mathbb{WP}^{1|1}_{(2)}$ has a zero dimensional mirror. The mirror map for $\mathbb{P}^{1|2}$ endows it with a structure of $N = 2$ super Riemann surface.

Keywords: Supergeometry, String Theory, Mirror Symmetry, Calabi-Yau.

Contents

1 Introduction

"Super-mathematics" has a quite long history, starting from the pioneering papers by Martin, [1, 2] and Berezin, [3, 4], before the discovery of supersymmetry in physics[1]. After its appearance in physics in the 70s, however, supergeometry has catched more attention in the mathematical community, and corresponding developments appeared not only in numerous articles but also in devoted books, see *e.g.* [6]-[15]. In most of the

[1] Even though anticommutation was proposed yet previously by Schwinger and other physicists, see [5] for a more detailed account.

concrete applications of supersymmetry, like in quantum field theory or in supergravity, algebraic properties play a key role, whereas geometry has almost always a marginal role (apart from the geometric formulation of superspace techniques; see further below). This is perhaps the reason why some subtle questions in supergeometry (see for example [12]) have not attracted too much the attention of physicists and, as a consequence, the necessity of further developments has not been stimulated.

String theory makes exception. Perturbative super string theory is expected to be described in terms of the moduli space of super Riemann surfaces, which results to be itself a supermanifold. However, some ambiguities in defining super string amplitudes at genus higher than one suggested, already in the 80s, that the geometry of such super moduli space may not be trivially obtained from the geometry of the bosonic underlying space [16]. More than twenty years of efforts have been necessary in order to be able to unambiguously compute genus two amplitudes; *cfr. e.g.* the papers by D'Hoker and Phong [17]- [27], which also include attempts in defining genus three amplitudes, without success but renewing the interest of the physical community in looking for a solution to the problem of constructing higher genus amplitudes. Through the years, various proposals have been put forward, see *e.g.* [28]-[48].

However, most of such constructions were based on the assumption that the supermoduli space is projected (see below for an explanation), but a careful analysis of perturbative string theory and of the corresponding role of supergeometry [49]-[53] suggested that this could not be the case. Indeed, it has been proved in [54] (see also [55]) that the supermoduli space is not split and not projected at least for genus $g \geq 5$. Obviously, such result gave rise to new interest in understanding the peculiarities of supergeometry with respect to the usual geometry, in particular from the viewpoint of algebraic geometry.

A second framework in which supergeometry plays a prominent role is the geometric approach to the superspace formalism, centred on *integral forms* (discussed *e.g.* in [56–58]; see below), whose application in physics can be traced back to [49, 59]. Superspace techniques are well understood and used in quantum field theory, supergravity as well as in string theory (see *e.g.* [60, 61]). They provide a very powerful method to deal with supersymmetric multiplets and to determine supersymmetric quantities, such as actions, currents, operators, vertex operators, correlators, and so on. However, even when the superspace formulation exists, it is often difficult to extract the component action. This occurs often in supergravity, in which the superdeterminant of the *supervielbein* is needed for the construction of the action, making the computation pretty cumbersome in a number of cases. On the other hand, the so-called "Ectoplasmic Integration Theorem" (EIT) [62]-[65] can be used in order the extract the component action from the superspace formulation.

Generally, supermanifolds are endowed with a tangent bundle (generated by commuting and anticommuting vector fields) and with an exterior bundle; thus, one would naïvely expect the geometric theory of integration on manifolds to be exported *tout court* in supersymmetric context. Unfortunately, such an extension is not straightforward at all, because top superforms do not exist, due to the fact that the wedge products of the differentials $d\theta$ (θ being the anticommuting coordinates) are commuting, and therefore there is no upper bound on the length of the usual exterior d-complex. In order to solve this problem, *distribution-like* quantities $\delta(d\theta)$ are introduced, for which a complete Cartan calculus can be developed. Such distributions $\delta(d\theta)$ then enter the very definition of the *integral forms* [66]-[71], which are a new type of differential forms requiring the enlargement of the conventional space spanned by the fundamental 1-forms, admitting distribution-like expressions (essentially, Dirac delta functions and Heaviside step functions). Within such an extension of the d differential, a complex with an upper bound arises, and this latter can be used to define a meaningful geometric integration theory for forms on supermanifolds. In recent years, this led to the development of a complete formalism (integral-, pseudo- and super- forms, their complexes and related integration theory) in a number of papers by Castellani, Catenacci and Grassi [58, 73, 74].

In [73], the exploitation of integral forms naturally yielded the definition of the Hodge dual operator $\bigstar$ for supermanifolds, by means of the Grassmannian Fourier transform of superforms, which in turn gave rise to new supersymmetric actions with higher derivative terms (these latter being required by the invertibility of the Hodge operator itself). Such a definition of $\bigstar$ was then converted into a Fourier-Berezin integral representation in [75], exploiting the Berezin convolution. It should also be recalled that integral forms were instrumental in the recent derivation of the superspace action of $D = 3$, $N = 1$ supergravity as an integral on a supermanifold [76].

Furthermore, in [74], the cohomology of superforms and integral forms was discussed, within a new perspective based on the Hodge dual operator introduced in [73]. Therein, it was also shown how the superspace constraints (*i.e.*, the rheonomic parametrisation) are translated from the space of superforms $\Omega^{(p|0)}$ to the space of integral forms $\Omega^{(p|m)}$ where $0 \leqslant p \leqslant n$, with n and m respectively denoting the bosonic and fermionic dimensions of the supermanifold; this naturally let to the introduction of the so-called Lowering and Picture Raising Operators (namely, the Picture Changing Operators, acting on the space of superforms and on the space of integral forms), and to their relation with the cohomology.

In light of these achievements, integral forms are crucial in a consistent geometric (superspace) approach to supergravity actions. It is here worth remarking that in

[58] the use of integral forms, in the framework of the group manifold geometrical approach [77, 78] (intermediate between the superfield and the component approaches) to supergravity, led to the proof of the aforementioned EIT, showing that the origin of that formula can be understood by interpreting the superfield action itself as an integral form. Subsequent further developments dealt with the construction of the super Hodge dual, the integral representation of Picture Changing Operators of string theories, as well as the construction of the super-Liouville form of a symplectic supermanifold [79].

A third context in which super geometry may be relevant is mirror symmetry. In [80], Sethi proposed that the extension of the concept of mirror symmetry to super Calabi-Yau manifolds (SCY's) could improve the definition of the mirror map itself, since supermanifolds may provide the correct mirrors of rigid manifolds. Such a conjecture has been strengthened by the works of Schwarz [84, 85] in the early days, but it seems to have been almost ignored afterwards, at least until the paper of Aganagic and Vafa [86] in 2004, in which a general super mirror map has been introduced and, in particular, it has been shown that the mirror of the super Calabi-Yau space $\mathbb{P}^{3|4}$ is, in a suitable limit, a quadric in $\mathbb{P}^{3|3} \times \mathbb{P}^{3|3}$. This is a quite interesting case, since these SCY's are related to amplitude computations in (super) quantum field theories, see *e.g.* [87]. Since then, a number of studies on mirror symmetry for SCY's has been carried on, see for example [88]-[91]. However, a precise definition of SCY is currently missing, and, consequently, the definition of mirror symmetry and its consequences is merely based on physical intuition.

The aim of the present paper is to provide a starting point for a systematic study of SCY's, by addressing the lowest dimensional case: SCY's whose bosonic reduction has complex dimension one.

In section 2 we collect some definitions in supergeometry and introduce the projective super spaces, which will play a major role in what follows. We will not dwell into a detailed exposition, and we address the interested reader *e.g.* to [11] and [12] for a mathematically thorough treatment of supergeometry. We also recall that an operative exposition of supergeometry, aimed at stressing its main connections with physics, is given in [49].

In section 3 we will be concerned with the geometry of the projective super space $\mathbb{P}^{1|2}$ and of the weighted projective super space $\mathbb{WP}^{1|1}_{(2)}$. Čech and de Rham cohomology of super differential forms are computed for these super varieties: here some interesting phenomena occur. Indeed we will find that on the one hand one there might be some infinite-dimensional Čech cohomology groups as soon as one deals with more than one odd coordinate (as in the case of $\mathbb{P}^{1|2}$); on the other hand this pathology gets cured at the level of de Rham cohomology, where no infinite dimensional groups occur. Our interest in these two particular supermanifolds originates from the fact that, together with the class of the so-called *N=2 super Riemann surfaces* ($N=2$ SRS's) which will be shortly addressed in what follows, $\mathbb{P}^{1|2}$ and $\mathbb{WP}^{1|1}_{(2)}$ are indeed the *unique* (non-singular) SCY's[2] having reduced manifold given by $\mathbb{P}^1$. These are therefore the simplest candidates to be considered, as one is interested into extending the mirror symmetry construction in dimension 1 to a super geometric context, pursuing a task initially suggested in [80]. Moreover, despite we keep our attention to the case $n|m = 1|2$ we also provide the de Rham cohomology of projective super spaces having generic dimension.

In section 4 we will then construct the mirrors of the projective super spaces $\mathbb{P}^{1|2}$ and $\mathbb{WP}^{1|1}_{(2)}$, following a recipe introduced in [86]. Moreover, we will show that, surprisingly, by means of the mirror construction, $\mathbb{P}^{1|2}$ actually gives a concrete example of $N=2$ SRS.

Finally, the main results and perspectives for further developments are discussed in section 5, whereas an appendix is devoted to illustrating the coherence of the adopted rule of signs.

2 Supermanifolds and Projective Super Spaces

2.1 Definitions and Notions in Supergeometry

In general, the mathematical basic notion that lies on the very basis of any physical supersymmetric theory is the one of $\mathbb{Z}_2$-grading: algebraic constructions such as rings, vector spaces and algebras and so on have their $\mathbb{Z}_2$-graded analogs, that in the context of physics are usually called *super rings, super vector spaces* and *super algebras* respectively.

A ring $(A, +, \cdot)$, for example, is called a super ring if $(A, +)$ has two subgroups A_0 and A_1, such that $A = A_0 \oplus A_1$ and

$$A_i \cdot A_j \subset A_{(i+j) mod\, 2} \qquad \forall\, i, j \in \mathbb{Z}_2. \tag{2.1}$$

The generalisation of vector spaces to super vector spaces and of algebras to super algebras follows the same lines.

Given an homogeneous element with respect to the $\mathbb{Z}_2$-grading of a super ring we can define an application, called *parity* of the element, as follows:

$$a \longmapsto |a| := \begin{cases} 0 & a \in A_0 \\ 1 & a \in A_1 \end{cases}. \tag{2.2}$$

[2] In the sense specified further below.

Elements such that $|a| = 0$ ($a \in A_0$) are called *even* or *bosonic*, and elements such that $|a| = 1$ ($a \in A_1$) are called *odd* or *fermionic*.
Notice that, up to now, there is no supersymmetric structure linked to any super commutativity of elements, which is provided by the *super commutator*, a bilinear map acting as follows on two generic homogeneous elements a, b in a super ring A,

$$(a, b) \longmapsto a \cdot b - (-1)^{|a|\cdot|b|}\, b \cdot a. \tag{2.3}$$

By additivity, this extend to a map $[\cdot,\cdot] : A \times A \to A$.
We say that a super ring is *super commutative* if *all* the super commutators among elements vanish (or in other words, the center of the super ring is the super ring itself), that is, on the homogeneous elements, one has $a \cdot b = (-1)^{|a|\cdot|b|}\, b \cdot a$, for all $a \in A_i, b \in A_i$ with $i \in \mathbb{Z}_2$. Supergeometry only deals with this class of super rings, allowing for anti-commutativity of odd elements. This has the following obvious fundamental consequence: *all odd elements are nilpotent.*
A basic but fundamental example of super commutative ring (actually algebra) is provided by the polynomial superalgebra over a certain field k which will be denoted as $k[x_1, \ldots, x_p, \theta_1, \ldots, \theta_q]$, where $x_1, \ldots, x_p$ are even generators, and $\theta_1, \ldots, \theta_q$ are odd generators. The presence of the odd anti commuting part implies the following customary picture for this super algebra:

$$k[x_1, \ldots, x_p, \theta_1, \ldots, \theta_q] \cong k[x_1, \ldots, x_p] \otimes_k \bigwedge[\theta_1, \ldots, \theta_q] \tag{2.4}$$

which makes apparent that the theta's are generators of a Grassmann algebra. Even and odd superpolynomials might be expanded into the odd (and therefore nilpotent) generators as follows

$$P_{even}(x, \theta) = f_0(x) + \sum_{i<j=1}^{q} f_{ij}(x)\theta_i\theta_j + \sum_{i<j<k<l=1}^{q} f_{ijkl}(x)\theta_i\theta_j\theta_k\theta_l + \ldots \tag{2.5}$$

$$P_{odd}(x, \theta) = \sum_{i=1}^{q} f_i(x)\theta_i + \sum_{i<j<k=1}^{q} f_{ijk}(x)\theta_i\theta_j\theta_k + \ldots \tag{2.6}$$

where the f's are usual polynomials in $k[x_1, \ldots, x_p]$ and we have written $\theta_i\theta_j$ instead of $\theta_i \wedge \theta_j$ for the sake of notation.

As one wishes to jump from pure algebra to geometry, it is customary in physics to look at a supermanifold $\mathcal{M}$ of dimension $p|q$ (that is, of even dimension p and odd dimension q) as described locally by p even coordinates and q odd coordinates, as a generalisation of the standard description of manifolds from differential geometry.

Even if this is feasible [49], the presence of nilpotent elements makes it preferable in the context of supergeometry to adopt an algebraic geometric oriented point of view and look at a supermanifold as a certain *locally ringed space* [11] [12] [14] [54].

Taking this global point of view, we define a *super space* $\mathcal{M}$ to be a $\mathbb{Z}_2$-graded locally ringed space, that is a pair $(|\mathcal{M}|, \mathcal{O}_{\mathcal{M}})$, consisting of a topological space $|\mathcal{M}|$ and a sheaf of super algebras $\mathcal{O}_{\mathcal{M}}$ over $|\mathcal{M}|$, such that the stalks $\mathcal{O}_{\mathcal{M},x}$ at every point $x \in |\mathcal{M}|$ are local rings. Notice that this makes sense as a requirement, for the odd elements are nilpotent and this reduces to ask that the even component of the stalk is a usual local commutative ring.

Morphisms between super spaces become morphisms of locally ringed spaces, that is they are given by a pair

$$(\phi, \phi^\sharp) : (|\mathcal{M}|, \mathcal{O}_{\mathcal{M}}) \longrightarrow (|\mathcal{N}|, \mathcal{O}_{\mathcal{N}}) \tag{2.7}$$

where $\phi : |\mathcal{M}| \to |\mathcal{N}|$ is a continuous function and $\phi^\sharp : \mathcal{O}_{\mathcal{N}} \to \phi_* \mathcal{O}_{\mathcal{M}}$ is morphism of sheaves (of super rings or super algebras) and it needs to preserve the $\mathbb{Z}_2$-grading.

Clearly, $\mathcal{O}_{\mathcal{M}}$ contains the subsheaf of ideals of all nilpotents, call it $\mathcal{J}_{\mathcal{M}}$, which is generated by all odd elements of the sheaf: this allows us to recover a *purely even* super space, $(|\mathcal{M}|, \mathcal{O}_{\mathcal{M}} / \mathcal{J}_{\mathcal{M}})$, which is called *reduced space* underlying $\mathcal{M}$ and denoted by $\mathcal{M}_{red}$. There always exists a close embedding $\mathcal{M}_{red} \hookrightarrow \mathcal{M}$, given by the morphism $(id_{|\mathcal{M}|}, i^\sharp) : (|\mathcal{M}|, \mathcal{O}_{\mathcal{M}} / \mathcal{J}_{\mathcal{M}}) \to (|\mathcal{M}|, \mathcal{O}_{\mathcal{M}})$, where $i^\sharp : \mathcal{O}_{\mathcal{M}} \to id_{|\mathcal{M}|*} \mathcal{O}_{\mathcal{M}} / \mathcal{J}_{\mathcal{M}} = \mathcal{O}_{\mathcal{M}} / \mathcal{J}_{\mathcal{M}}$.

A special super space can be constructed as follows: given a topological space $|\mathcal{M}|$ and a locally free sheaf of $\mathcal{O}_{|\mathcal{M}|}$-modules $\mathcal{E}$, we can take $\mathcal{O}_{\mathcal{M}}$ to be the sheaf $\bigwedge^\bullet \mathcal{E}^\vee$: this makes $\mathcal{O}_{\mathcal{M}}$ out of a super commutative sheaf whose stalks are local rings. Similarly to [54], we denote super spaces constructed this way $\mathfrak{S}(|\mathcal{M}|, \mathcal{E})$.

Examples of this construction are *affine super spaces* $\mathbb{A}^{p|q} := \mathfrak{S}(\mathbb{A}^p, \mathcal{O}_{\mathbb{A}^p}^{\oplus q})$: here $\mathbb{A}^p$ is the ordinary p-dimensional affine space over $\mathbb{A}$ and $\mathcal{O}_{\mathbb{A}^p}$ is the trivial bundle over it. Super spaces like these are common in supersymmetric field theories, where one usually works with $\mathbb{R}^{p|q}$ or $\mathbb{C}^{p|q}$.

A *supermanifold* is defined as a super space which is *locally* isomorphic[3] to $\mathfrak{S}(|\mathcal{M}|, \mathcal{E})$ for some topological space $|\mathcal{M}|$ and some locally free sheaf of $\mathcal{O}_{|\mathcal{M}|}$-module $\mathcal{E}$.

Following this line of thought, then, one recovers (out of a globally defined object!) the original differential geometric induced view that physics employs, where a *real* supermanifold of dimension $p|q$ is a one that locally resembles to $\mathbb{R}^{p|q}$ and, likewise, a *complex* supermanifold of dimension $p|q$ is a one that locally resembles to $\mathbb{C}^{p|q}$, defined above: the gluing data are encoded in the cocycle condition that the structure sheaf must satisfy.

[3] In the $\mathbb{Z}_2$-graded sense: here indeed isomorphisms are isomorphisms of super algebras.

Given a supermanifold $\mathcal{M}$, we will call $\mathcal{M}_{red}$ the pair $(|M|, \mathcal{O}_{\mathcal{M}} / \mathcal{J}_{\mathcal{M}})$, which is an ordinary manifold presented as a locally ringed space of a certain type: as above, we will always have a closed embedding $\mathcal{M}_{red} \hookrightarrow \mathcal{M}$.
It is worth noticing that, on the contrary, the definition of supermanifold does not implies the existence of a projection $\mathcal{M} \to \mathcal{M}_{red}$: this would correspond to a morphism of $(id_{|\mathcal{M}|}, \pi^\sharp) : (|\mathcal{M}|, \mathcal{O}_{\mathcal{M}}) \to (|\mathcal{M}|, \mathcal{O}_{\mathcal{M}} / \mathcal{J}_{\mathcal{M}})$, where $\pi^\sharp$ is a sheaf morphism that embeds $\mathcal{O}_{\mathcal{M}} / \mathcal{J}_{\mathcal{M}}$ into $\mathcal{O}_{\mathcal{M}}$, and also to endow the sheaf $\mathcal{O}_{\mathcal{M}}$ with the structure of sheaf of $\mathcal{O}_{\mathcal{M}_{red}}$-modules. In the case such a projection does exist, the supermanifold is said to be *projected*. Thinking of the supermanifold in terms of the gluing data between open sets covering the underlying topological space, projectedness of the supermanifold implies that the even transition functions can be written as functions of the ordinary local coordinates on the reduced manifold only: there are no nilpotents (e.g. bosonisation of odd elements) at all. Obstruction to the existence of such projection for the case of the supermoduli space of super Riemann surfaces has been studied in [54] and it is an issue that has striking consequences in superstring perturbation theory, as mentioned early on in the introduction.
A stronger condition is realised when the supermanifold is *globally* isomorphic to some local model $\mathfrak{S}(|\mathcal{M}|, \mathcal{E})$. Such supermanifolds are said to be *split*. If this is the case, not only the even transition functions have no nilpotents, but the odd transition functions can be chosen in such a way that they are linear in the odd coordinates. This bears a nice geometric view of split supermanifolds: they can be looked at globally as a vector bundle $\tilde{\mathcal{E}} \to \mathcal{M}_{red}$ on the reduced manifold having purely odd fibers, as the definition of a supermanifold we have provided above suggests by itself.

2.2 Projective Super Spaces and Weighted Projective Super Spaces

The supermanifolds known as (complex) projective super spaces, call it $\mathbb{P}^{n|m}$, have been discussed extensively in the literature and introduced from several different point of view, both in mathematics and in physics, being of fundamental importance in twistor string theory.

Complex projective spaces are mostly looked at formally as a quotient of the super spaces $\mathbb{C}^{n|m}$ by the even multiplicative group $\mathbb{C}^\times$, so realising a super analog of the set of homogeneous coordinates $[X_1 : \ldots : X_n : \Theta_1 : \ldots : \Theta_m]$ obeying $[X_1 : \ldots : X_n : \Theta_1 : \ldots : \Theta_m] = [\lambda X_1 : \ldots : \lambda X_n : \lambda \Theta_1 : \ldots : \lambda \Theta_m]$, where $\lambda \in \mathbb{C}^\times$ (see for example [54], [49]). In contrast with this global construction by a quotient, a popular local construction realises $\mathbb{P}^{n|m}$ mimicking the analogous constructions of $\mathbb{P}^n$ as a complex manifold, that is by specifying it as $n+1$ copies of $\mathbb{C}^{n|m}$ glued together by the usual relations. This construction relies on the possibility to pair the usual bosonic local coordinates with q fermionic anticommuting local coordinates: such an intuitive approach can be made rigorous using the functor of points formalism [15]. A more rigorous treatment connecting the requested invariance under the action on $\mathbb{C}^\times$ with the structure of the sheaf of super commutative algebra characterising the projective super space can be found in [56].

An elegant construction of the projective super space that goes along well with the notions introduced above is given in [12]. $\mathbb{P}^{n|m}$ can actually be presented as a (split) complex supermanifold, as follows. We consider a super $\mathbb{C}$-vector space $V = V_0 \oplus V_1$ of rank $n+1|m$. As one can imagine, the topological space underlying the super projective space coincides with the usual one and it is given by the projectivization of the even part of V, we call it simply $\mathbb{P}^n$. This tells that $\mathbb{P}^n$ can be covered by $n+1$ open sets $\{U_i\}_{i=0,\ldots,n}$, characterised by

$$U_i \coloneqq \{[X_0 : \ldots : X_n] \in \mathbb{P}^n : X_i \neq 0\} \tag{2.8}$$

so one can form a system of local affine coordinates on U_i given by $z_j^{(i)} \coloneqq X_j \,/ X_i$ for $j \neq i$. Intuitively, as above, we would like to have something similar for the odd part of the geometry: this is achieved by realising a sheaf of super algebras on $\mathbb{P}^n$, as follows:

$$U_i \longmapsto \left(\bigoplus_{\ell=0}^{m} \bigwedge^{\ell} V_1^\vee \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \right) (U_i). \tag{2.9}$$

This is mapped isomorphically to the structure sheaf $\mathcal{O}_{\mathbb{P}^{n|m}}$ of the projective super space, by a map induced by

$$V_1^\vee \otimes \mathcal{O}_{\mathbb{P}^n}(-1)(U_i) \ni \Theta_\alpha \otimes X_i^{-1} \longmapsto \theta_\alpha^{(i)} \coloneqq \frac{\Theta_\alpha}{X_i} \in \mathcal{O}_{\mathbb{P}^{n|m}}(U_i), \tag{2.10}$$

where we stress that Θ_α is a generator for $V_1^\vee$, X_i^{-1} is a section of $\mathcal{O}_{\mathbb{P}^n}(-1)$ over U_i and the $\theta_\alpha^{(i)}$, where $\alpha = 1, \ldots, m$ are promoted as local odd coordinates over U_i for the projective super space $\mathbb{P}^{n|m}$.

This construction makes apparent that, in the notation introduced in the previous section, $\mathbb{P}^{n|m} = \mathfrak{S}(\mathbb{P}^n, V_1 \otimes \mathcal{O}_{\mathbb{P}^n}(1))$.

One can also read out the transition rules on $U_i \cap U_j$, even and odd, that are usually written as:

$$z_k^{(i)} = \frac{z_k^{(j)}}{z_i^{(j)}}, \qquad\qquad \theta_\alpha^{(i)} = \frac{\theta_\alpha^{(j)}}{z_i^{(j)}}. \tag{2.11}$$

In the language of morphisms of ringed spaces, we would have an isomorphism

$$(\phi_{U_i \cap U_j}, \phi^\sharp_{U_i \cap U_j}) : (U_i \cap U_j, \mathcal{O}_{\mathbb{P}^{n|m}}(U_i)\lfloor_{U_j}) \longrightarrow (U_i \cap U_j, \mathcal{O}_{\mathbb{P}^{n|m}}(U_j)\lfloor_{U_i}) \tag{2.12}$$

with $\phi_{U_i \cap U_j} : U_i \cap U_j \to U_i \cap U_j$ being the usual change of coordinate on projective space and $\phi^{\sharp}_{U_i \cap U_j} : \mathcal{O}_{\mathbb{P}^{n|m}}(U_j \lfloor_{U_i}) \to \left((\phi_{U_i \cap U_j})_* \mathcal{O}_{\mathbb{P}^{n|m}}\right)(U_i \lfloor_{U_j})$, so that

$$\left(\phi_{U_i \cap U_j}\right)_* (z_k^{(i)}) = \frac{z_k^{(j)}}{z_i^{(j)}}, \qquad \left(\phi_{U_i \cap U_j}\right)_* (\theta_\alpha^{(i)}) = \frac{\theta_\alpha^{(j)}}{z_i^{(j)}}. \tag{2.13}$$

We note, incidentally, that the cocycle relation is indeed satisfied.

Before we go on, we generalise a little the construction above, to allow us for treat in a somehow unified way also the weighted projective super spaces: we will be actually interested in the case the odd part of the geometry carries different weights compared to the even part, which is made by an ordinary projective space.
Since above we have taken $V = V_0 \oplus V_1$ to be a super $\mathbb{C}$-vector space, then it has a well defined notion of dimension, namely $n + 1|m$, and we can actually take a basis for it. Focusing on the odd part, we take $\{\Theta_\alpha\}_{\alpha=1,\ldots,m}$ as a system of generators for V_1. Then, we might realise a more general sheaf of super algebras by

$$U_i \longmapsto \bigwedge^{\bullet} \left(\bigoplus_{\alpha=1}^{m} \Theta_\alpha^\vee \otimes \mathcal{O}_{\mathbb{P}^n}(-w_\alpha) \right) (U_i). \tag{2.14}$$

In other words, each odd variable has been assigned a weight w_α, which reverberates in the transition functions: the ordinary case of $\mathbb{P}^{n|m}$ is recovered assigning $w_\alpha = 1$ for each $\alpha = 1, \ldots, m$.
We will call this space weighted projective super space and we will denote it by $\mathbb{WP}^{n|m}_{(w_1,\ldots,w_m)}$, where the string $(w_1, \ldots, w_m)$ gives the fermionic weights. In this paper we will be particularly concerned with low dimensional examples of projective and weighted projective super space, namely $\mathbb{P}^{1|2}$ and $\mathbb{WP}^{1|1}_{(2)}$, whose geometry will be studied in some details in the following section.

2.3 Vector Bundles over $\mathbb{P}^1$, Grothendieck's Theorem and cohomology of $\mathcal{O}_{\mathbb{P}^n}(k)$-bundles

In this section we recall and comment a classification result due to Grothendieck which will be heavily exploited to study projective super spaces having reduced space given by $\mathbb{P}^1$. Moreover, for future use, the cohomology of $\mathcal{O}_{\mathbb{P}^n}(k)$-bundles is given.

The main result about vector bundles on $\mathbb{P}^1$ is that any holomorphic vector bundle of rank n is isomorphic to the direct sum of n line bundles and the decomposition is unique up to permutations of the line bundles, that is

$$\mathcal{E} \cong \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^1}(k_i), \tag{2.15}$$

where the ordered sequence $k_1 \geq k_2 \geq \ldots \geq k_n$ is uniquely determined (see [92] for a complete proof). We will refer at it as Grothendieck's Theorem. Basically, it guarantees that the only interesting vector bundles on $\mathbb{P}^1$ are the line bundles on it, which in turn are all of the form $\mathcal{O}_{\mathbb{P}^1}(k)$ for some $k \in \mathbb{Z}$ (recall that $\mathrm{Pic}(\mathbb{P}^1) \cong \mathbb{Z}$).
Concretely, since every (algebraic) vector bundle over $\mathbb{C}$ is trivial, the restriction of a vector bundle $\mathcal{E}$ over $\mathbb{P}^1$ of rank n to the standard open sets $U_0 \coloneqq \{[X_0 : X_1] : X_0 \neq 0\} \cong \mathbb{C}$ and $U_1 \coloneqq \{[X_0 : X_1] : X_1 \neq 0\} \cong \mathbb{C}$ is trivial. Choosing the coordinates $z \coloneqq \frac{X_1}{X_0}$ on U_0 and $w \coloneqq \frac{X_0}{X_1} = \frac{1}{z}$ on U_1 we have that an isomorphism $\mathcal{E}\lfloor_{U_0} \to \mathcal{O}_{\mathbb{P}^1}^{\oplus n}\lfloor_{U_0}$ is equivalent to an isomorphism of $\mathcal{O}_{\mathbb{P}^1}(U_0) = \mathbb{C}[z]$-modules as follows

$$\phi_0 : \mathcal{E}(U_0) \xrightarrow{\cong} \mathcal{O}_{\mathbb{P}^1}(U_0)^{\oplus n} = \mathbb{C}[z]^{\oplus n}. \tag{2.16}$$

Likewise, we have an isomorphism of $\mathcal{O}_{\mathbb{P}^1}(U_1) = \mathbb{C}\left[z^{-1}\right]$-modules

$$\phi_1 : \mathcal{E}(U_1) \xrightarrow{\cong} \mathcal{O}_{\mathbb{P}^1}(U_1)^{\oplus n} = \mathbb{C}\left[z^{-1}\right]^{\oplus n}. \tag{2.17}$$

Clearly, two such isomorphisms ϕ_i and ϕ_i' gives an automorphism $\phi_i \circ \phi_i' : \mathbb{C}[t]^{\oplus n} \to \mathbb{C}[t]^{\oplus n}$ where $t = z$ for $i = 0$ and $t = z^{-1}$ for $i = 1$, so $\phi_i \circ \phi_i'$ determines an invertible $n \times n$ matrix, having coefficients in $\mathbb{C}[t]$.
The composition $\phi_{01} \coloneqq \phi_0 \circ \phi_1^{-1}$ gives the glueing relation between the two trivial bundles over U_0 and U_1: it is again given by an invertible $n \times n$ matrix having coefficient in $\mathbb{C}[z, z^{-1}]$: its determinant is equal to z^k for some $k \in \mathbb{Z}$ up to a non-zero constant. Thus, classifying rank n vector bundles over $\mathbb{P}^1$ corresponds to classifying invertible matrices $M \in GL(n, \mathbb{C}[z, z^{-1}])$ up to the following equivalence:

$$M(z, z^{-1}) \sim A(z)M(z, z^{-1})B(z^{-1}) \qquad A(z) \in GL(n, \mathbb{C}[z]), B(z^{-1}) \in GL(n, \mathbb{C}[z^{-1}]).$$

By a theorem due to Birkhoff, $M(z, z)$ belongs to the same class of a *diagonal* matrix $M_d = \mathrm{diag}(z^{k_1}, \ldots, z^{k_n})$ where $k_i \in \mathbb{Z}$. Therefore any bundle over $\mathbb{P}^1$ is isomorphic to a direct sum of line bundles $\mathcal{O}_{\mathbb{P}^1}(k_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(k_n)$.

By looking at vector bundles over $\mathbb{P}^1$ as sheaves of locally free $\mathcal{O}_{\mathbb{P}^1}$-modules, the theorem reduces the problem of computing sheaf cohomology over $\mathbb{P}^1$ to computing the sheaf cohomology of $\mathcal{O}_{\mathbb{P}^1}(k)$, which is well-known. In general, for $k \geq 0$ one has

$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) = \mathbb{C}[x_0, \ldots, x_n]^{(k)}$, the *degree-k linear subspace of the polynomial ring*, therefore

$$h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) = \binom{k+n}{k} = \frac{(n+k)!}{k! \cdot n!} \qquad k \geq 0, \tag{2.18}$$

and if $k < 0$ one has $H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) = \left\langle x_0^{i_0} \cdot \ldots \cdot x_n^{i_n} : i_j < 0, \quad \sum_{i=0}^n i_j = k \right\rangle_{\mathbb{C}}$. It is an exercise in combinatorics to see that

$$h^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) = \binom{|k|-1}{|k|-n-1} \qquad k < 0,\ |k| \geq n+1. \tag{2.19}$$

These results will be used to compute the cohomology in the following section.

3 The Geometry and Cohomology of $\mathbb{P}^{1|2}$ and $\mathbb{WP}^{1|1}_{(2)}$

3.1 Super Calabi-Yau Varieties

The physical approach to Calabi-Yau's geometries in a supersymmetric context has employed the differential geometric point of view, defining a super Calabi-Yau manifold (SCY) as a Ricci-flat supermanifold. Indeed there exists a generalisation of tensor calculus to a supersymmetric context, making sense out of the notion of super Riemannian manifold, super curvature tensor and super Ricci tensor.
The crucial observation regarding projective super spaces is that there exists a generalisation of the Fubini-Study metric to the supersymmetric context. Let us consider $\mathbb{P}^{n|m}$, then it can be given a super Kähler potential

$$\mathcal{K}^s = \log\left(\sum_{i=0}^n X_i \bar{X}_i + \sum_{j=1}^m \Theta_j \bar{\Theta}_j\right) \tag{3.1}$$

defined everywhere on $\mathbb{P}^{n|m}$ that reduces to the ordinary Fubini-Study potential as one restrict it to the underlying reduced manifolds. Locally, on a patch of the projective super space, it takes the form

$$K^s = \log\left(\sum_{i=1}^n z_i \bar{z}_i + \sum_{j=1}^m \theta_j \bar{\theta}_j\right). \tag{3.2}$$

Notice that in this complex differential geometric context the variables are paired with their anti-holomorphic analogs, as customary in theoretical physics.
The super Kähler form is defined in the local patch to be

$$\Omega^s := \partial\bar{\partial}K^s \quad \text{or analogously} \quad \Omega^s := \partial_A \partial_{\bar{B}} K^s dX^A dX^{\bar{B}} \tag{3.3}$$

where ∂ and $\bar{\partial}$ are the holomorphic and anti-holomorphic super derivatives: we refer to the Appendix A for details. Then the super metric tensor is simply given by

$$H^s_{A\bar{B}} \coloneqq \partial_A \partial_{\bar{B}} K^s, \tag{3.4}$$

and the Ricci tensor reads

$$\mathrm{Ric}_{A\bar{B}} \coloneqq \partial_A \partial_{\bar{B}} \log \left(\mathrm{Ber} H^s \right). \tag{3.5}$$

Notice that the only modification compared to the ordinary complex geometric case is that the determinant of the metric has been substituted by the Berezinian [11] [49] of the super metric.
Remarkably, as one chooses the projective super spaces of the form $\mathbb{P}^{n|n+1}$ for $n \geq 1$ (note that $\mathbb{P}^{0|1} \cong \mathbb{C}^{0|1}$), endowed with the super Fubini-Study metric defined as above, then one gets a vanishing super Ricci tensor!
We stress that, as it is common in the context of super geometry, even an easy calculation might present some difficulties, due to the anti-commutativity of some variables. One needs to establish and keep coherent conventions throughout the calculations. As an example, a detailed computation of the vanishing of the super Ricci tensor in the case of $\mathbb{P}^{1|2}$ is reported in Appendix A. The same calculation can be easily generalised to any $\mathbb{P}^{n|n+1}$ for $n \geq 2$.

Actually, defining a SCY manifold by requiring that its super Ricci tensor vanishes appears as a very strong request. Moreover, it is not that useful, for it is often hard to write down super metrics for interesting classes of supermanifolds: for example there is not straightforward generalisation of the super Fubini-Study metric to the case of weighted projective super spaces. Moreover, by the result in [88] a Ricci-flat Kälher supermetric on $\mathbb{WP}^{1|1}_{(2)}$ does not exist. Still, it is possible to give a weaker but certainly more useful definition:

Definition 1 *We say that an orientable super projective variety X is super Calabi-Yau if its Berezinian sheaf is trivial, that is $\mathcal{B}er_X \cong \mathcal{O}_X$.*

Notice that this definition is again the super analog of the usual algebraic geometric definition of an ordinary CY variety, that calls for a trivial canonical sheaf. Indeed, the Berezinian sheaf is, in some sense (see [11] [12] [49]), the super analog of the canonical sheaf, since the sections of the Berezinian transform as densities and it turns out that they are the right objects to define a meaningful notion of super integration, the Berezin integral. In other words, then, a SCY variety is one whose Berezinian sheaf has an everywhere non-zero global section.

We now start making use of the Grothendieck splitting theorem to prove the triviality of the Berezian sheaf of the supermanifolds $\mathbb{P}^{1|2}$ and $\mathbb{WP}^{1|1}_{(2)}$ and hence confirm that they are both SCY varieties. The strategy we will use throughout to compute the cohomologies will be to look at the varieties as split supermanifolds, therefore, in this case, as the total space of vector bundles over $\mathbb{P}^1$, the reduced manifold, having odd fibers. Then, we will achieve the splitting into line bundles over $\mathbb{P}^1$ and we will compute their cohomology.
We start considering $\mathbb{P}^{1|2}$. Following what established above, we have two patches, we now call U_z and U_w: passing from one patch to the other, yields the following transformations

$$\begin{cases} w \longmapsto z = \frac{1}{w} \\ \phi_0 \longmapsto \theta_0 = \frac{\phi_1}{w} \\ \phi_1 \longmapsto \theta_1 = \frac{\phi_2}{w} \end{cases} . \tag{3.6}$$

The structure sheaf, $\mathcal{O}_{\mathbb{P}^{1|2}}$, is therefore locally generated by

$$\mathcal{O}_{\mathbb{P}^{1|2}}(U_z) = \langle 1 \rangle_{\mathcal{O}_{\mathbb{P}^{1|2}}(U_z)} = \langle 1, \theta_0, \theta_1, \theta_0\theta_1 \rangle_{\mathcal{O}_{\mathbb{P}^1}(U_z)}, \tag{3.7}$$

where in the last equality we are looking at it as a locally free $\mathcal{O}_{\mathbb{P}^1}$-module. Considering the transformation rules in the intersection, we find the following factorisation as $\mathcal{O}_{\mathbb{P}^1}$-modules:

$$\mathcal{O}_{\mathbb{P}^{1|2}} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(-2). \tag{3.8}$$

Notice that the cohomology can be readily computed as $h^0(\mathbb{P}^{1|2}, \mathcal{O}_{\mathbb{P}^{1|2}}) = 1$ and
$h^1(\mathbb{P}^{1|2}, \mathcal{O}_{\mathbb{P}^{1|2}}) = 1$.
Using a notation due to Witten [49], the Berezinian sheaf over $\mathbb{P}^{1|2}$ is locally generated by an element of the form

$$\mathcal{B}er_{\mathbb{P}^{1|2}}(U_z) = \langle [dz|d\theta_0, d\theta_2] \rangle_{\mathcal{O}_{\mathbb{P}^{1|2}}(U_z)}. \tag{3.9}$$

Under a coordinate transformation, call it Φ, taking local coordinates $w|\phi_0, \phi_1$ to $z|\theta_0, \theta_1$ as above, the Berezinian transforms as follows:

$$[dw|d\phi_0, d\phi_1] \longmapsto [dz|d\theta_0, d\theta_1] = \mathrm{Ber}(\Phi)[dw|d\phi_0, d\phi_1]. \tag{3.10}$$

Therefore one gets:

$$\mathrm{Ber}\left[\begin{pmatrix} -1/w^2 & 0 & 0 \\ -\theta_0/w^2 & 1/w & 0 \\ -\theta_1/w^2 & 0 & 1/w \end{pmatrix}\right] = \frac{-1/w^2}{1/w \cdot 1/w} = -1. \tag{3.11}$$

This trivial transformation implies the triviality. More precisely, viewing $\mathcal{B}er_{\mathbb{P}^{1|2}}$ as a $\mathcal{O}_{\mathbb{P}^1}$-module, one finds the following factorisation:

$$\mathcal{B}er_{\mathbb{P}^{1|2}} \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \cong \mathcal{O}_{\mathbb{P}^{1|2}}, \tag{3.12}$$

and under the correspondence $1 \mapsto [dz|d\theta_0, d\theta_1]$, one has

$$\mathcal{B}er_{\mathbb{P}^{1|2}} \cong \mathcal{O}_{\mathbb{P}^{1|2}}, \tag{3.13}$$

as expected. The cohomology is obviously the same as the one of the structure sheaf. Things go in same way as one consider the structure sheaf and the Berezinian sheaf of $\mathbb{WP}^{1|1}_{(2)}$, remembering that one has transformations of the following form

$$\begin{cases} w \longrightarrow z = \frac{1}{w} \\ \phi \longrightarrow \theta = \frac{\phi}{w^2}. \end{cases} \tag{3.14}$$

Again, one finds that the Berezinian has a trivial transformation on the intersection and we have a correspondence $1 \mapsto [dz|d\theta]$ and an isomorphism

$$\mathcal{B}er_{\mathbb{WP}^{1|1}_{(2)}} \cong \mathcal{O}_{\mathbb{WP}^{1|1}_{(2)}}. \tag{3.15}$$

This confirm that also the weighted projective space $\mathbb{WP}^{1|1}_{(2)}$ is a SCY variety, in the weak sense.

3.2 The Sheaf Cohomology of Differential and Integral Forms

To a large extent, generalisation of the ordinary commuting geometry to the richer context of supergeometry is pretty straightforward and it boils down to an application of the "rule of sign" [11] [12]. One issue stands out for its peculiarity: the theory of differential form and integration. The problematics concerned with this topic have been recently investigated by Catenacci et al. in a series of papers (here we will particularly refer to [56]) and reviewed by Witten in [49]. We briefly sketch the main points, by leaving the details of the constructions to the literature.
As one tries to generalise the complex of forms $(\Omega^\bullet, d^\bullet)$ to supergeometry using the 1-superforms $\{d\theta^i\}_{i\in I}$ constructed out of the θ^i, then it comes natural to define wedge products such as $d\theta^1 \wedge \ldots \wedge d\theta^n$ to be *commutative* in the $d\theta$'s, since the θ's are odd elements. This bears a very interesting consequence: the complex of superforms $(\Omega^\bullet_s, d^\bullet_s)$ is bounded from below but not from above! For example, superforms such as $(d\theta^i)^n := d\theta^i \wedge \ldots \wedge d\theta^i$ do make sense and they are not zero, such as their bosonic counterparts. The problematic aspect resides in that there is no notion of a top-form, therefore a coherent notion of "super integration" is obtained only at the cost of

enlarging the complex of superforms and supplementing it with the so-called *integral forms*. Using the notation of [56], the basic integral form are given by $\{\delta(d\theta^i)\}_{i\in I}$ and its higher derivatives $\{\delta^{(n)}(d\theta^i)\}_{i\in I}$, for $n > 0$. Here the use of the symbol δ should remind the Dirac delta distribution - and indeed an integral form satisfies similar properties [56] -: it sets to zero terms in $d\theta^i$ and therefore, in some sense, it lowers the degree of a superform. For this reason an integral form is assigned a *non-positive degree*: indeed in the context of supergeometry one can also have negative degree forms. This is better understood by mean of an example. Let us take the super space $\mathbb{C}^{2|2}$, then we can consider the following superform

$$\omega_s = dz_1 dz_2 (d\theta^2)^4 \delta^{(2)}(d\theta^1), \tag{3.16}$$

where the wedge products are understood. Then $dz_1dz_2(d\theta^2)^4$ carries a degree of 6, while the integral form $\delta^{(2)}(d\theta^1)$ lower the degree by 2, so as a whole, we say that ω_s has degree 4 and we signal the presence of an integral form (of any degree) by saying that it has *picture number* equal to 1. Therefore, this enlarged complex of superforms is characterised by two numbers, the degree of the form n and their picture number s and we have that $\omega_s \in \Omega^{n=4;s=1}_{\mathbb{C}^{2|2}}$. Notice, incidentally, that the picture number cannot exceed the odd dimension of the supermanifold and operators linking complexes having different picture numbers - called *picture changing operators* - can be defined.

In [56] the sheaf cohomology of superforms and integral forms of $\mathbb{P}^{1|1}$ has been studied, proving that just by adding an anti-commuting dimension, the cohomology becomes far richer. There is, though, a substantial hole in the literature: no sheaf cohomology of superforms and integral forms has ever been computed for supermanifolds having extended supersymmetries, that is more than one odd dimensions. In this scenario the computation of the cohomology for the case of $\mathbb{P}^{1|2}$ acquires value, besides being an example of cohomology of a SCY variety.
We will see indeed that as soon as one has more than a single odd dimension, when the picture number is *middle-dimensional* (that is, it is non-zero and not equal to the odd dimension of the manifold), then one finds that the space of superforms is infinitely generated and its cohomology may be infinite-dimensional!
This result calls, from a mathematical side, for a better understanding of the (algebraic) geometry of the complex of superforms and integral forms. Moreover, from the physical side, the possible usage and purposes of forms having middle-dimensional picture number should be investigated and clarified.

We now compute the sheaf cohomology of superforms of $\mathbb{P}^{1|2}$. We will carry out the computation in some details for the first case, namely the space of superforms having

null picture number, $\Omega^{n;0}_{\mathbb{P}^{1|2}}$, to elucidate our method and we leave to the reader all the other cases that follow the same pattern.
As a $\mathcal{O}_{\mathbb{P}^{1|2}}$-module, $\Omega^{n;0}_{\mathbb{P}^{1|2}}$ is locally generated by:

$$\Omega^{n;0}_{\mathbb{P}^{1|2}}(U_z) = \left\langle \{d\theta_0^i d\theta_1^{n-i}\}_{i=0,\dots,n}, \{dz d\theta_0^j d\theta_1^{n-1-j}\}_{j=0,\dots,n-1} \right\rangle_{\mathcal{O}_{\mathbb{P}^{1|2}(U_z)}}. \tag{3.17}$$

By looking at it as a (locally free) $\mathcal{O}_{\mathbb{P}^1}$-module, we might find the transformations of its generators. The first block of generators transform as (up to unimportant constants and signs)

$$\begin{aligned}
d\theta_0^i d\theta_1^{n-i} &= \frac{1}{w^n} d\phi_0^i d\phi_i^{n-i} + \frac{1}{w^{n+1}} \phi_0 dw d\phi_0^{i-1} d\phi_1^{n-i} + \frac{1}{w^{n+1}} \phi_1 dw d\phi_0^n d\phi_1^{n-1-i}, \\
\theta_0 d\theta_0^i d\theta_1^{n-i} &= \frac{1}{w^{n+1}} \phi_0 d\phi_0^i d\phi_1^{n-i} + \frac{1}{w^{n+2}} \phi_0 \phi_1 dw d\phi_0^i d\phi_1^{n-1-i}, \\
\theta_1 d\theta_0^i d\theta_1^{n-i} &= \frac{1}{w^{n+1}} \phi_1 d\phi_0^i d\phi_1^{n-i} + \frac{1}{w^{n+2}} \phi_0 \phi_1 dw d\phi_0^{i-1} d\phi_1^{n-i}, \\
\theta_0 \theta_1 d\theta_0^i d\theta_1^{n-i} &= \frac{1}{w^{n+2}} \phi_0 \phi_1 d\phi_0^i d\phi_1^{n-i}.
\end{aligned}$$

The second block, instead, has only diagonal terms:

$$\begin{aligned}
dz d\theta_0^j d\theta_1^{n-1-j} &= \frac{1}{w^{n+1}} dw d\phi_0^i d\phi_1^{n-1-j}, \\
\theta_0 dz d\theta_0^j d\theta_1^{n-1-j} &= \frac{1}{w^{n+2}} \phi_0 dw d\phi_0^i d\phi_1^{n-1-j}, \\
\theta_1 dz d\theta_0^j d\theta_1^{n-1-j} &= \frac{1}{w^{n+2}} \phi_1 dw d\phi_0^i d\phi_1^{n-1-j}, \\
\theta_0 \theta_1 dz d\theta_0^j d\theta_1^{n-1-j} &= \frac{1}{w^{n+2}} \phi_0 \phi_1 dw d\phi_0^i d\phi_1^{n-1-j},
\end{aligned}$$

where we recall that $i = 0, \dots, n$ and $j = 0, \dots, n-1$. Before going on we observe that, for n fixed, looking at $\Omega^{n;0}_{\mathbb{P}^{1|2}}$ as a $\mathcal{O}_{\mathbb{P}^1}$-module, we will have

$$\dim_{\mathcal{O}_{\mathbb{P}^1}} \Omega^{n;0}_{\mathbb{P}^{1|2}} = 4(n+1) + 4n = 8n + 4 \tag{3.18}$$

terms in the factorisation. This is its dimension as a vector bundle/locally free sheaf of $\mathcal{O}_{\mathbb{P}^1}$-modules.
The strategy that we will follow will be to group together pieces having similar form, evaluating their transformations and afterwards factorising them into a direct sum of line bundles over $\mathbb{P}^1$ by means of Grothendieck splitting theorem, by treating the off-diagonal terms in the transition functions matrix: we recall that, in the notation above, we will be free to perform $\mathbb{C}[w]$-linear operations in the columns and $\mathbb{C}[1/w]$-linear operations in the rows.

To this end, we now keep our attention on the diagonal terms that do not need any further investigation: we will get $n+1$ terms and n standing-alone terms out of $\theta_0\theta_1 d\theta_0^i d\theta_1^{n-i}$ and $dzd\theta_0^j d\theta_1^{n-1-j}$, so these contribute to the factorisation with terms of the form

$$\mathcal{O}_{\mathbb{P}^1}(-n-2)^{\oplus n+1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n-1)^{\oplus n}. \tag{3.19}$$

So we are left with $8n+4-(n+1)-n = 6n+3$ terms to give account to.
The other terms need some carefulness. We start dealing with the terms coming from the transformation of $d\theta_0^i d\theta_1^{n-i}$: these couples with the ones coming from $\theta_0 dzd\theta_0^j d\theta_1^{n-1-j}$ and $\theta_1 dzd\theta_0^j d\theta_1^{n-1-j}$ whenever $i=j$ in the pairing with $\theta_0 dzd\theta_0^j d\theta_1^{n-1-j}$ and $i=j+1$ in the pairing with $\theta_1 dzd\theta_0^j d\theta_1^{n-1-j}$. We therefore need to consider $n-1$ (since this holds true in the case $i=1,\ldots,n-1$) identical 3×3 matrices of the following form:

$$\begin{pmatrix} 1/w^n & 1/w^{n+1} & 1/w^{n+1} \\ 0 & 1/w^{n+2} & 0 \\ 0 & 0 & 1/w^{n+2} \end{pmatrix} \xrightarrow{C_1 - wC_2} \begin{pmatrix} 0 & 1/w^{n+1} & 1/w^{n+1} \\ -1/w^{n+1} & 1/w^{n+2} & 0 \\ 0 & 0 & 1/w^{n+2} \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1/w^{n+1} & 1/w^{n+1} \\ -1/w^{n+1} & 1/w^{n+2} & 0 \\ 0 & 0 & 1/w^{n+2} \end{pmatrix} \xrightarrow{C_1 \leftrightarrow C_2} \begin{pmatrix} 1/w^{n+1} & 0 & 1/w^{n+1} \\ 1/w^{n+2} & -1/w^{n+1} & 0 \\ 0 & 0 & 1/w^{n+2} \end{pmatrix},$$

$$\begin{pmatrix} 1/w^{n+1} & 0 & 1/w^{n+1} \\ 1/w^{n+2} & -1/w^{n+1} & 0 \\ 0 & 0 & 1/w^{n+2} \end{pmatrix} \xrightarrow{R_2 - 1/wR_1} \begin{pmatrix} 1/w^{n+1} & 0 & 1/w^{n+1} \\ 0 & -1/w^{n+1} & -1/w^{n+2} \\ 0 & 0 & 1/w^{n+2} \end{pmatrix},$$

$$\begin{pmatrix} 1/w^{n+1} & 0 & 1/w^{n+1} \\ 0 & -1/w^{n+1} & -1/w^{n+2} \\ 0 & 0 & 1/w^{n+2} \end{pmatrix} \xrightarrow{C_3 - C_1} \begin{pmatrix} 1/w^{n+1} & 0 & 0 \\ 0 & -1/w^{n+1} & -1/w^{n+2} \\ 0 & 0 & 1/w^{n+2} \end{pmatrix},$$

$$\begin{pmatrix} 1/w^{n+1} & 0 & 0 \\ 0 & -1/w^{n+1} & -1/w^{n+2} \\ 0 & 0 & 1/w^{n+2} \end{pmatrix} \xrightarrow{R_2 + R_3} \begin{pmatrix} 1/w^{n+1} & 0 & 0 \\ 0 & -1/w^{n+1} & 0 \\ 0 & 0 & 1/w^{n+2} \end{pmatrix}.$$

So this bit contributes with terms of the following forms:

$$\mathcal{O}_{\mathbb{P}^1}(-n-1)^{\oplus 2n-2} \oplus \mathcal{O}_{\mathbb{P}^1}(-n-2)^{\oplus n-1} \tag{3.20}$$

to be added to the previous ones. This boils the number of the remaining pieces down to $6n+3-(3n-3)=3n+6$.

Now it is important to notice that we have not given account for some terms in the counting above yet: we need indeed to consider separately 4 terms that group into two identical 2×2 matrices. Indeed the term $i=0$ of $d\theta_0^i d\theta_1^{n-i}$, that is $d\theta_1^n$, couples to the term $j=0$ (which was left out of the counting above) into the term $\theta_1 dz d\theta_0^j d\theta_1^{n-1-j}$, that is $\theta_1 dz d\theta_1^{n-1}$. This gives a 2×2 matrix of the form:

$$\begin{pmatrix} 1/w^n & 1/w^{n+1} \\ 0 & 1/w^{n+2} \end{pmatrix} \xrightarrow{C_1-wC_2} \begin{pmatrix} 0 & 1/w^{n+1} \\ -1/w^{n+1} & 1/w^{n+2} \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1/w^{n+1} \\ -1/w^{n+1} & 1/w^{n+2} \end{pmatrix} \xrightarrow{C_1\leftrightarrow C_2} \begin{pmatrix} 1/w^{n+1} & 0 \\ 1/w^{n+2} & -1/w^{n+1} \end{pmatrix},$$

$$\begin{pmatrix} 1/w^{n+1} & 0 \\ 1/w^{n+2} & -1/w^{n+1} \end{pmatrix} \xrightarrow{R_2-1/wR_1} \begin{pmatrix} 1/w^{n+1} & 0 \\ 0 & -1/w^{n+1} \end{pmatrix}.$$

The very same holds true in the case $i=n$ for $d\theta_0^i d\theta_1^{n-i}$, that is $d\theta_0^n$, and $j=n$ for $\theta_0 dz d\theta_0^j d\theta_1^{n-1-j}$, that is $\theta_0 dz \theta_0^{n-1}$. So we have a pair of identical contributions that sums up to the ones already accounted:

$$\mathcal{O}_{\mathbb{P}^1}(-n-1)^{\oplus 4}. \tag{3.21}$$

So this adds up 4 terms to the counting above, leaving us with $3n+2$ terms to be accounted for.

The terms $\theta_0 d\theta_0^i d\theta_1^{n-i}$ and $\theta_1 d\theta_0^i d\theta_1^{n-i}$, couple with the last term, $\theta_0\theta_1 dz d\theta_0^j d\theta_1^{n-1-j}$, in the cases $i=0,\dots n-1$ for $\theta_0 d\theta_0^i d\theta_1^{n-i}$ and $i=1,\dots,n$ for $\theta_1 d\theta_0^i d\theta_1^{n-i}$ and for all j. Therefore we have $3n$ identical 3×3 matrices of the form:

$$\begin{pmatrix} 1/w^{n+1} & 0 & 1/w^{n+2} \\ 0 & 1/w^{n+1} & 1/w^{n+2} \\ 0 & 0 & 1/w^{n+3} \end{pmatrix} \xrightarrow{R_2-R_1} \begin{pmatrix} 1/w^{n+1} & 0 & 1/w^{n+2} \\ -1/w^{n+1} & 1/w^{n+1} & 0 \\ 0 & 0 & 1/w^{n+3} \end{pmatrix},$$

$$\begin{pmatrix} 1/w^{n+1} & 0 & 1/w^{n+2} \\ -1/w^{n+1} & 1/w^{n+1} & 0 \\ 0 & 0 & 1/w^{n+3} \end{pmatrix} \xrightarrow{C_1-C_2} \begin{pmatrix} 1/w^{n+1} & 0 & 1/w^{n+2} \\ 0 & 1/w^{n+1} & 0 \\ 0 & 0 & 1/w^{n+3} \end{pmatrix},$$

$$\begin{pmatrix} 1/w^{n+1} & 0 & 1/w^{n+2} \\ 0 & 1/w^{n+1} & 0 \\ 0 & 0 & 1/w^{n+3} \end{pmatrix} \xrightarrow{C_1-wC_3} \begin{pmatrix} 0 & 0 & 1/w^{n+2} \\ 0 & 1/w^{n+1} & 0 \\ 1/w^{n+2} & 0 & 1/w^{n+3} \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 1/w^{n+2} \\ 0 & 1/w^{n+1} & 0 \\ 1/w^{n+2} & 0 & 1/w^{n+3} \end{pmatrix} \xrightarrow{C_1 \leftrightarrow C_3} \begin{pmatrix} 1/w^{n+2} & 0 & 0 \\ 0 & 1/w^{n+1} & 0 \\ 1/w^{n+3} & 0 & 1/w^{n+2} \end{pmatrix},$$

$$\begin{pmatrix} 1/w^{n+2} & 0 & 0 \\ 0 & 1/w^{n+1} & 0 \\ 1/w^{n+3} & 0 & 1/w^{n+2} \end{pmatrix} \xrightarrow{R_1 - 1/wR_3} \begin{pmatrix} 1/w^{n+2} & 0 & 0 \\ 0 & 1/w^{n+1} & 0 \\ 0 & 0 & 1/w^{n+2} \end{pmatrix}.$$

So we have the following contribution to the factorisation:

$$\mathcal{O}_{\mathbb{P}^1}(-n-1)^{\oplus n} \oplus \mathcal{O}_{\mathbb{P}^1}(-n-2)^{\oplus 2n}. \tag{3.22}$$

Notice that we have to take into account separately the terms corresponding to $i = n$ for $\theta_0 d\theta_0^i d\theta_1^{n-i}$ and $i = 0$ for $\theta_1 d\theta_0^i d\theta_1^{n-i}$, yielding an identical (diagonal) contribution of the form: $\mathcal{O}_{\mathbb{P}^1}(-n-1)^{\oplus 2}$.
These last $2n+2$ terms complete the enumeration. We are therefore ready to write down the whole factorisation for $n > 0$:

$$\Omega^{n;0}_{\mathbb{P}^{1|2}} \cong \mathcal{O}_{\mathbb{P}^1}(-n-2)^{\oplus 4n} \oplus \mathcal{O}_{\mathbb{P}^1}(-n-1)^{\oplus 4n+4}.$$

We are finally in the position to count the dimensions of the cohomology groups:

$$h^0(\Omega^{n;0}_{\mathbb{P}^{1|2}}) = 0, \qquad h^1(\Omega^{n;0}_{\mathbb{P}^{1|2}}) = 8n^2 + 8n. \tag{3.23}$$

This terminates the discussion of the form with null picture number. All the other cases, having non-null picture number are treated in an analogous way, by remembering the transformation of the integral forms of type $\delta^{(n)}(d\theta^i)$ [56]. We now list their factorisation as $\mathcal{O}_{\mathbb{P}^1}$-modules.
The space of superforms having maximal picture number and degree is locally generated by

$$\Omega^{1;2}_{\mathbb{P}^{1|2}}(U_z) = \left\langle dz\delta(d\theta_0)\delta(d\theta_1)\right\rangle_{\mathcal{O}_{\mathbb{P}^{1|2}}(U_z)}. \tag{3.24}$$

Considering the transformation among the two charts yields the factorisation

$$\Omega^{1;2}_{\mathbb{P}^{1|2}} \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \cong \mathcal{O}_{\mathbb{P}^{1|2}}. \tag{3.25}$$

So we can easily compute the dimensions of the cohomology groups, which are exactly the same as the ones of the structural sheaf:

$$h^0(\Omega^{1;2}_{\mathbb{P}^{1|2}}) = 1, \qquad h^1(\Omega^{1;2}_{\mathbb{P}^{1|2}}) = 1. \tag{3.26}$$

It is not surprising that this is the same as the Berezinian line bundle over $\mathbb{P}^{1|2}$: indeed elements of this sheaf are in some sense the supersymmetric analog of the ordinary top-form for a manifold, and we (Berezin-)integrate them, as one can integrate sections of the Berezinian sheaf. These two peculiar supersymmetric sheaves are fundamental in theory of integration on supermanifold.

We are left with the group $\Omega^{-n;2}_{\mathbb{P}^{1|2}}$, for $n \geq 0$ (which deserve some attention and book-keeping such as $\Omega^{n;0}_{\mathbb{P}^{1|2}}$). It is locally generated by

$$\Omega^{-n;2}_{\mathbb{P}^{1|2}}(U_z) = \big\langle \{\delta^{(i)}(d\theta_0)\delta^{(n-i)}(d\theta_1)\}_{i=0,\dots,n}, dz\{\delta^{(j)}(d\theta_0)\delta^{(n+1-j)}(d\theta_1)\}_{j=0,\dots,n+1}\big\rangle_{\mathcal{O}_{\mathbb{P}^{1|2}(U_z)}},$$

which give the following factorisation

$$\Omega^{-n;2}_{\mathbb{P}^{1|2}} \cong \mathcal{O}_{\mathbb{P}^1}(n+1)^{\oplus 4n+5} \oplus \mathcal{O}_{\mathbb{P}^1}(n)^{\oplus 4n+6} \oplus \mathcal{O}_{\mathbb{P}^1}(n-1).$$

The dimensions of the cohomology groups then read

$$h^0(\Omega^{-n;2}_{\mathbb{P}^{1|2}}) = 8(n+2)(n+1), \qquad h^1(\Omega^{-n;2}_{\mathbb{P}^{1|2}}) = 0. \tag{3.27}$$

By pulling together all the cohomologies, we have the following result:

$$h^0(\Omega^{n;m}_{\mathbb{P}^{1|2}}) = \begin{cases} 1 & n=0,\ m=0 \\ 0 & n>0, m=0 \\ 8(n+2)(n+1) & n \leq 0,\ m=2 \\ 1 & n=1,\ m=2 \end{cases}, \tag{3.28}$$

$$h^1(\Omega^{n;m}_{\mathbb{P}^{1|2}}) = \begin{cases} 1 & n=0,\ m=0 \\ 8n(n+1) & n>0, m=0 \\ 0 & n \leq 0, m=2 \\ 1 & n=1,\ m=2 \end{cases}. \tag{3.29}$$

Notice that so far we have not carried out the computation of superforms having picture number equal to 1: as anticipated, these are infinitely generated as a locally free sheaf and they give infinite dimensional cohomology.

The generators read

$$\begin{aligned}\Omega^{n\geq 0;1}_{\mathbb{P}^{1|2}}(U_z) =& \big\langle \{\delta^{(i)}(d\theta_0)d\theta_1^{n+i}\}_{i\in\mathbb{N}},\ dz\{\delta^{(i+1)}(d\theta_0)d\theta_1^{n+i}\}_{i\in\mathbb{N}}, \\ & \{\delta^{(i)}(d\theta_1)d\theta_0^{n+i}\}_{i\in\mathbb{N}},\ dz\{\delta^{(i+1)}(d\theta_1)d\theta_0^{n+i}\}_{i\in\mathbb{N}}\big\rangle_{\mathcal{O}_{\mathbb{P}^{1|2}(U_z)}}.\end{aligned} \tag{3.30}$$

This is factorised as

$$\bigoplus_{i\in\mathbb{N}} \big(\mathcal{O}^{\oplus 8}_{\mathbb{P}^1}(-n-1) \oplus \mathcal{O}^{\oplus 8}_{\mathbb{P}^1}(-n)\big) \tag{3.31}$$

this gives $h^0(\Omega^{n>0;1}_{\mathbb{P}^{1|2}}) = 0$, while, remarkably, $h^0(\Omega^{0;1}_{\mathbb{P}^{1|2}}) = h^1(\Omega^{n\geq 0;1}_{\mathbb{P}^{1|2}}) = \infty$!
Similarly, one finds

$$\begin{aligned}\Omega^{n<0;1}_{\mathbb{P}^{1|2}}(U_z) =& \Big\langle \{\delta^{(|n|+i)}(d\theta_0)d\theta_1^i\}_{i\in\mathbb{N}},\ dz\{\delta^{(|n|+i+1)}(d\theta_0)d\theta_1^i\}_{i\in\mathbb{N}},\\ & \{\delta^{(|n|+i)}(d\theta_1)d\theta_0^i\}_{i\in\mathbb{N}},\ dz\{\delta^{(|n|+i+1)}(d\theta_1)d\theta_0^i\}_{i\in\mathbb{N}}\Big\rangle_{\mathcal{O}_{\mathbb{P}^{1|2}(U_z)}}\end{aligned} \tag{3.32}$$

having factorisation

$$\bigoplus_{i\in\mathbb{N}} \left(\mathcal{O}^{\oplus 8}_{\mathbb{P}^1}(|n|-1) \oplus \mathcal{O}^{\oplus 8}_{\mathbb{P}^1}(|n|)\right) \tag{3.33}$$

which again gives infinite dimensional cohomology.

The computation of the cohomology of $\mathbb{WP}^{1|1}_{(2)}$ is much easier and it can be performed following the same lines as above. Also, having no middle picture number, there are no infinitely generated modules and infinite cohomologies.
By means of Grothendieck theorem, the complete sheaf cohomology is thus given by

$$h^0(\Omega^{n;m}_{\mathbb{WP}^{1|1}_{(2)}}) = \begin{cases} 1 & n=0,\ m=0 \\ 0 & n>0, m=0 \\ 4n+6 & n\leq 0,\ m=1 \\ 1 & n=1,\ m=1 \end{cases}, \tag{3.34}$$

$$h^1(\Omega^{n;m}_{\mathbb{WP}^{1|1}_{(2)}}) = \begin{cases} 1 & n=0,\ m=0 \\ 8n & n>0, m=0 \\ 0 & n\leq 0,\ m=1 \\ 0 & n=1,\ m=1 \end{cases}. \tag{3.35}$$

We signal, by the way, a pathology which looks like it can apply to any weighted projective space.
While the Berezinian sheaf is isomorphic to the structural sheaf of $\mathbb{WP}^{1|1}_{(2)}$ - and indeed it has analogous factorisation and cohomology - one finds instead that the sheaf of the "top-superform" $\Omega^{1;1}_{\mathbb{WP}^{1|1}_{(2)}}$ is not! To see that, it is enough to check the different factorisation and therefore the different cohomologies: one finds indeed that $h^1(\Omega^{0;0}_{\mathbb{WP}^{1|1}_{(2)}}) = 1$ while $h^1(\Omega^{1;1}_{\mathbb{WP}^{1|1}_{(2)}}) = 0$.

3.3 de Rham Cohomology of $\mathbb{WP}^{1|1}_{(2)}$ and $\mathbb{P}^{1|2}$

Having at disposal the sheaf cohomology of superforms on the super varieties $\mathbb{WP}^{1|1}_{(2)}$ and $\mathbb{P}^{1|2}$, we now aim to compute their *holomorphic* de Rham cohomology.

Before we start, a nod to the adopted notation is due: given a supermanifold $\mathcal{M}$, we will denote its de Rham cohomology groups as $H^{n;m}_{dR}(\mathcal{M})$ where n refers to the usual degree of the forms and m refers to their picture number.
We also stress that the boundary operator of the complex, acts as $d : \mathcal{A}^{n;m}_{\mathcal{M}} \to \mathcal{A}^{n+1;m}_{\mathcal{M}}$, where $\mathcal{A}^{n;m}_{\mathcal{M}}$ is the freely generated module of the n-forms having *fixed* picture number m that are defined *everywhere*, that is $\mathcal{A}^{n;m}_{\mathcal{M}} \cong H^0(\Omega^{n;m}_{\mathcal{M}})$. In other words, the boundary operator d does *not* change the picture number of the form, and it just raises the degree of the form, so - as in ordinary purely bosonic geometry - we are just moving *horizontally* on the complex and we cannot jump from on complex to the other, by picture changing procedure.

For the sake of clarity, we start from where we left, and we take on the computation of the de Rham cohomology of the weighted projective super space $\mathbb{WP}^{1|1}_{(2)}$, where the sheaf cohomology of superforms is always finite. We will adopt a cumbersome but effective method, that has the advantages to display explicitly a basis of generators for the various de Rham groups. This is remarkable for it possibly sets a more concrete ground for the observations in [82] and especially in the interesting [83], where it is observed that the *BRST cohomology* of a (super) A-model is isomorphic to the cohomology of the superforms on the target space, that is a supermanifold $\mathcal{M}$.
The starting point to compute the de Rham cohomology of the weighted projective super space $\mathbb{WP}^{1|1}_{(2)}$ is to look at its zeroth Čech cohomology, computed above. This actually gives us two results for free: the first one is that $H^{0;0}_{dR}(\mathbb{WP}^{1|1}_{(2)}) = \mathbb{C}$, and it is generated by the constant function 1. The second result that can be easily red is that we have $H^{n;0}_{dR}(\mathbb{WP}^{1|1}_{(2)}) = 0$ for $n > 0$, indeed Čech cohomology guarantees that there are no everywhere defined forms of degree $n > 0$.
Let us now consider $H^{1;1}_{dR}(\mathbb{WP}^{1|1}_{(2)})$, Čech cohomology tells that there is one everywhere defined form, which is locally given by $dz\delta^{(0)}(d\theta)$, that generates the group: this is also trivially *closed* for in particular $d(\delta^{(n)}(d\theta)) = 0$ (see [56]), so the question is whether $dz\delta^{(0)}(d\theta)$ is exact or not. To answer the question one needs to look at $H^0(\Omega^{0;1}_{\mathbb{WP}^{1|1}_{(2)}}) = \mathbb{C}^6$.
In this case it is easy to see that

$$dz\delta^{(0)}(d\theta) = d(z\delta^{(0)}(d\theta)) \tag{3.36}$$

where $z\delta^{(0)}(d\theta) \in H^0(\Omega^{0;1}_{\mathbb{WP}^{1|1}_{(2)}})$, so the form is exact and one gets $H^{1;1}_{dR}(\mathbb{WP}^{1|1}_{(2)}) = 0$.
We now look with some more attention to the de Rham cohomology connected to $H^{n;1}(\mathbb{WP}^{1|1}_{(2)}) = \mathbb{C}^{4n+6}$, where $n \leq 0$. Let us consider a generic form expanded as it

belongs to a freely-generated module over $\mathcal{O}_{\mathbb{P}^1}$. We have that:

$$\begin{aligned}\omega =& \left(F_0(z) + \theta F_1(z)\right) \delta^{(n)}(d\theta) + \left(G_0(z) + \theta G_1(z)\right) dz \delta^{(n+1)}(d\theta) \\ =& \left(w^{n+2} F_0(1/w) + \phi w^n F_1(1/z)\right) \delta^{(n)}(d\phi) + \\ &+ \left(-w^{n+1} G_0(1/w) + \phi \left(-w^{n+1} F_0(1/w) - w^{n-1} G_1(1/w)\right)\right) dw \delta^{(n+1)}(d\phi) \end{aligned} \tag{3.37}$$

where F_0, F_1, G_0, G_1 are polynomials. By changing the coordinates to U_w we find that the form remains everywhere defined if and only if $\deg F_0 = n+2, \deg F_1 = n, \deg G_0 = n+1, \deg G_1 = n$, where G_1 has the constraint that the coefficient of its highest degree monomial is equal to the coefficient of the highest degree of F_0, which indeed yields a total of $4n+6$ free parameters, as already computed above. So far, this is nothing but another method to find the zeroth-dimensional Čech cohomology, without using Grothendieck's theorem, as done in [56] for the case of $\mathbb{P}^{1|1}$. If on the one hand it is certainly not efficient - especially as one needs to deal with more than one fermionic dimension -, it is true that on the other hand, in the context of the de Rham cohomology, it has the advantage to make explicit in terms of the coefficients the basis of the zeroth cohomology group of everywhere defined forms.
However, we stress that a careful analysis of the various pieces involved in the computation carried out by mean of Grothendieck's theorem would have led to the same result in term of the basis of the space of everywhere defined forms. Here we opted for this more rough method as long as the computations are easy-to-follow.
The most interesting group is the zeroth: we find that $\deg F_0 = 2, \deg F_1 = 0, \deg G_0 = 1, \deg G_1 = 1$, and explicitly, we have

$$F_0(z) = az^2 + bz + c, \tag{3.38}$$

$$F_1(z) = d, \tag{3.39}$$

$$G_0(z) = ez + f, \tag{3.40}$$

$$G_1(z) = -a. \tag{3.41}$$

Gathering together the terms having the same coefficients we find the following basis:

$$a \to \delta^{(0)}(d\theta) - \theta dz \delta^{(1)}(d\theta), \tag{3.42}$$

$$b \to z\delta^{(0)}(d\theta), \tag{3.43}$$

$$c \to z^2 \delta^{(0)}(d\theta), \tag{3.44}$$

$$d \to \theta \delta^{(0)}(d\theta), \tag{3.45}$$

$$e \to dz \delta^{(1)}(d\theta), \tag{3.46}$$

$$f \to z dz \delta^{(1)}(d\theta). \tag{3.47}$$

Then one can verify that the module of the closed forms is generated by

$$Z^{0;1}_{dR}(\mathbb{WP}^{1|1}_{(2)}) = \left\langle \theta\delta^{(0)}(d\theta), dz\delta^{(1)}(d\theta), zdz\delta^{(1)}(d\theta)\right\rangle_{\mathcal{O}_{\mathbb{P}^1}}. \tag{3.48}$$

Actually, the forms $dz\delta^{(1)}(d\theta), zdz\delta^{(1)}(d\theta)$ are easily seen to be exact, indeed

$$dz\delta^{(1)}(d\theta) = d\left(z\delta^{(1)}(d\theta)\right), \tag{3.49}$$

$$zdz\delta^{(1)}(d\theta) = d\left(\frac{1}{2}z^2\delta^{(1)}(d\theta)\right), \tag{3.50}$$

and both the forms on the right-hand side are everywhere defined, that is they are in $H^0(\Omega^{-1;1}_{\mathbb{WP}^{1|1}_{(2)}})$. So we conclude that $H^{0;1}_{dR}(\mathbb{WP}^{1|1}_{(2)}) = \mathbb{C}$ and the group is generated by the closed form $\theta\delta^{(0)}(\theta)$.
Writing explicitly the forms, we can see that *all* the other groups $H^{n;1}_{dR}(\mathbb{WP}^{1|1}_{(2)})$ for $n > 0$ are trivial: one finds that $Z^{n;1}_{dR}(\mathbb{WP}^{1|1}_{(2)})$ is actually non-zero - there are closed forms -, but $Z^{n;1}_{dR}(\mathbb{WP}^{1|1}_{(2)}) \cong B^{n;1}_{dR}(\mathbb{WP}^{1|1}_{(2)})$ - all the closed forms are exact and do not contribute to the de Rham cohomology. Summing up, we have:

$$h^{n;m}_{dR}(\mathbb{WP}^{1|1}_{(2)}) = \begin{cases} 1 & n = 0,\ m = 0 \\ 0 & n > 0, m = 0 \\ 1 & n = 0,\ m = 1 \\ 0 & n \neq 0,\ m = 1. \end{cases} \tag{3.51}$$

We now move to the holomorphic de Rham cohomology of $\mathbb{P}^{1|2}$: again, the starting point will be to look at the forms defined everywhere. By mean of our previous computations in sheaf cohomology of superforms, we see that $H^{0;0}_{dR}(\mathbb{P}^{1|2}) = \mathbb{C}$ and it is generated by the constant function 1 and $H^{n;0}_{dR}(\mathbb{P}^{1|2}) = 0$, indeed there are no globally defined forms.
Let us now consider the case $n = 1, m = 2$ - corresponding, as observed, to a sort of top-form -: the relative group is locally generated by the superform $dz\delta^{(0)}(d\theta_0)\delta^{(0)}(d\theta_1)$, which extends globally: this is certainly closed and moreover, one can easily see it is exact, for $d(z\delta^{(0)}(d\theta_0)\delta^{(0)}(d\theta_1)) = dz\delta^{(0)}(d\theta_0)\delta^{(0)}(d\theta_1)$ and $z\delta^{(0)}(d\theta_0)\delta^{(0)}(d\theta_1) \in H^0(\Omega^{0;2}_{\mathbb{P}^{1|2}})$. This tells that $H^{1;2}_{dR}(\mathbb{P}^{1|2}) = 0$.
We now take on the groups $H^{n;2}_{dR}(\mathbb{P}^{1|2})$ for $n \leq 0$. The most interesting case is given by $H^{0;2}_{dR}(\mathbb{P}^{1|2})$: the relative Čech cohomology group has dimension 16 and we will study it carefully. We should be considering forms of the kind

$$\begin{aligned}\omega =&(F_0(z) + F_1(z)\theta_0 + F_2(z)\theta_1 + F_3(z)\theta_0\theta_1)\delta^{(0)}(d\theta_0)\delta^{(0)}(d\theta_1) + \\ &(G_0(z) + G_1(z)\theta_0 + G_2(z)\theta_1 + G_3(z)\theta_0\theta_1)dz\delta^{(0)}(d\theta_1)\delta^{(1)}(d\theta_1) + \\ &(H_0(z) + H_1(z)\theta_0 + H_2(z)\theta_1 + H_3(z)\theta_0\theta_1)dz\delta^{(1)}(d\theta_1)\delta^{(0)}(d\theta_1),\end{aligned}$$

where the F's, G's and H's are all polynomials, whose degree is identified as above, by studying whenever the form remains defined everywhere under a change of local chart, from U_z to U_w.
There are 10 closed forms:

$$\begin{aligned} Z_{dR}^{0;2}(\mathbb{P}^{1|2}) =& \big\langle \delta^{(0)}(d\theta_0)\delta^{(0)}(d\theta_1), \theta_0 dz \delta^{(0)}(d\theta_0)\delta^{(1)}(d\theta_1), \theta_1 dz \delta^{(1)}(d\theta_0)\delta^{(0)}(d\theta_2), \\ & \theta_0 \delta^{(0)}(d\theta_0)\delta^{(0)}(d\theta_1), \theta_0\delta^{(0)}(d\theta_0)\delta^{(0)}(d\theta_1), z dz \delta^{(0)}(d\theta_0)\delta^{(1)}(d\theta_1), \\ & dz \delta^{(0)}(d\theta_0)\delta^{(1)}(d\theta_1), z dz \delta^{(1)}(d\theta_0)\delta^{(0)}(d\theta_1), dz\delta^{(1)}(d\theta_0)\delta^{(0)}(d\theta_1), \\ & \theta_1\theta_2 \delta^{(0)}(d\theta_0)\delta^{(0)}(d\theta_1)\big\rangle_{\mathcal{O}_{\mathbb{P}^1}}. \end{aligned}$$

The only closed form that it is not exact is given by $\theta_0\theta_1\delta^{(0)}(d\theta_0)\delta^{(0)}(d\theta_1)$, which is therefore a generator for the group $H_{dR}^{0;2}(\mathbb{P}^{1|2}) = \mathbb{C}$.
Indeed, let us consider for example the closed form $dz\delta^{(1)}(d\theta_0)\delta^{(0)}(d\theta_1)$, one has:

$$dz\delta^{(1)}(d\theta_0)\delta^{(0)}(d\theta_1) = d(-\theta_1 dz \delta^{(1)}(d\theta_0)\delta^{(1)}(d\theta_1)), \tag{3.52}$$

remembering that $d\theta_1\delta^{(1)}(d\theta_1) = -\delta^{(0)}(d\theta_1)$.
As in the case of the weighted projective super space, going on in the negative degree, one finds that the closed forms are *all* exact, and we have $H_{dR}^{n;2}(\mathbb{P}^{1|2}) = 0$ for $n \leq -1$. This is to be connected, at the end of the day, to the dimension of the space $H^0(\Omega^{n;2}_{\mathbb{P}^{1|2}})$ for $n \leq -1$, and again in turns to the transformation properties of the integral forms, which allow for a huge space of globally defined forms.

We now consider the space of everywhere defined forms having picture number equal to 1, which is somehow the most sensitive one, for as we have seen above, it yielded an infinite dimensional sheaf cohomology. Before going on, we recall that $\Omega^{n\geq 0;1}_{\mathbb{P}^{1|2}}$ is infinitely generated as locally free sheaf, and its generators read

$$\begin{aligned} \Omega^{n\geq 0;1}_{\mathbb{P}^{1|2}}(U_z) =& \big\langle \{\delta^{(i)}(d\theta_0) d\theta_1^{n+i}\}_{i\in\mathbb{N}},\ dz\{\delta^{(i+1)}(d\theta_0)d\theta_1^{n+i}\}_{i\in\mathbb{N}}, \\ & \{\delta^{(i)}(d\theta_1)d\theta_0^{n+i}\}_{i\in\mathbb{N}},\ dz\{\delta^{(i+1)}(d\theta_1)d\theta_0^{n+i}\}_{i\in\mathbb{N}}\big\rangle_{\mathcal{O}_{\mathbb{P}^{1|2}(U_z)}}. \end{aligned} \tag{3.53}$$

The factorisation is

$$\bigoplus_{i\in\mathbb{N}} \left(\mathcal{O}^{\oplus 8}_{\mathbb{P}^1}(-n-1) \oplus \mathcal{O}^{\oplus 8}_{\mathbb{P}^1}(-n)\right). \tag{3.54}$$

First thing to observe is therefore that there are *no* globally defined forms for $n > 0$! It follows that the de Rham cohomology is $H_{dR}^{n>0;1}(\mathbb{P}^{1|2}) = 0$.
All the other modules, for $n \leq 0$ gives an infinite dimensional zeroth (Čech) cohomology

group.
We start analysing, as usual, the $n = 0$ module. Since the generators read

$$\begin{aligned}\Omega^{0;1}_{\mathbb{P}^{1|2}}(U_z) =& \Big\langle \{\delta^{(i)}(d\theta_0)d\theta_1^i\}_{i\in\mathbb{N}},\ dz\{\delta^{(i+1)}(d\theta_0)d\theta_1^i\}_{i\in\mathbb{N}}, \\ & \{\delta^{(i)}(d\theta_1)d\theta_0^i\}_{i\in\mathbb{N}},\ dz\{\delta^{(i+1)}(d\theta_1)d\theta_0^i\}_{i\in\mathbb{N}}\Big\rangle_{\mathcal{O}_{\mathbb{P}^{1|2}(U_z)}},\end{aligned} \tag{3.55}$$

we can just deal with the first two block, and the other ones are symmetric up to the exchange $\theta_0 \leftrightarrow \theta_1$.
For the sake of convenience, let us consider separately the case $i = 0$ and $i > 0$, for some attention is requested as one deals with $i = 0$ in the transformations.
In this case, $i = 0$, one has:

$$\omega =(F_0(z) + \theta_0 F_1(z) + \theta_1 F_2(z) + \theta_0\theta_1 F_3(z))\delta^{(0)}(d\theta_0)+ \tag{3.56}$$

$$(G_0(z) + \theta_0 G_1(z) + \theta_1 G_2(z) + \theta_0\theta_1 G_3(z))dz\delta^{(1)}(d\theta_0). \tag{3.57}$$

From Čech cohomology computation we expect 4 free parameters that yield:

$$H^0(\Omega^{0;1}_{\mathbb{P}^{1|2}})\lfloor_{i=0} = \Big\langle z\delta^{(0)}(d\theta_0) - dz\delta^{(1)}(d\theta_0), \delta^{(0)}(d\theta_0), \theta_0\delta^{(0)}(d\theta_0), dz\delta^{(1)}(d\theta_0)\Big\rangle. \tag{3.58}$$

The last three forms are closed, but only $\theta_0\delta^{(0)}(d\theta_0)$ is *not* exact, indeed

$$\delta^{(0)}(d\theta_0) = d(-\theta_0\delta^{(1)}(d\theta_0)) \qquad dz\delta^{(1)}(d\theta_0) = d(z\delta^{(1)}(d\theta_0)) \tag{3.59}$$

and $-\theta_0\delta^{(1)}(d\theta_0), z\delta^{(1)}(d\theta_0)$ are globally defined. Analogously, we have that $\theta_1\delta^{(0)}(d\theta_1)$ is closed and not exact, therefore it is non-zero in the quotient.
In the case $i \neq 0$ one is led to consider the transformation of

$$\omega = (F_0(z) + F_1(z)\theta_0 + F_2(z)\theta_1 + F_3(z)\theta_0\theta_1)\,\delta^{(i)}(d\theta_0)d\theta_1^i+ \tag{3.60}$$

$$(G_0(z) + G_1(z)\theta_0 + G_2(z)\theta_1 + G_3(z)\theta_0\theta_1)\,dz\delta^{(i+1)}(d\theta_0)d\theta_1^i. \tag{3.61}$$

One has:

$$\begin{aligned}H^0(\Omega^{0;1}_{\mathbb{P}^{1|2}})\lfloor_{i\neq 0} = & \Big\langle z\delta^{(i)}(d\theta_0)d\theta_1^i + \theta_1 dz\delta^{(i+1)}(d\theta_0)d\theta_1^i + \theta_2 dz\delta^{(i)}(d\theta_0)d\theta_1^{i-1}, \delta^{(i)}(d\theta_0)d\theta_1^i, \\ & \theta_0\delta^{(0)}(d\theta_0)d\theta_1^i + \theta_1\delta^{(i-1)}(d\theta_0)d\theta_1^{i-1}, dz\delta^{(i+1)}(d\theta_0)d\theta_1^i\Big\rangle.\end{aligned} \tag{3.62}$$

It can be seen that $\delta^{(i)}(d\theta_0)d\theta_1^i$ and $dz\delta^{(i+1)}(d\theta_0)d\theta_1^i$ are closed forms for every $i \geq 1$, but they are also exact, for

$$\delta^{(i)}(d\theta_0)d\theta_1^i = d(-\theta_0\delta^{(i+1)}(d\theta_0)d\theta_1^i), \qquad dz\delta^{(i+1)}(d\theta_0)d\theta_1^i = d(z\delta^{(i+1)}(d\theta_0)d\theta_1^i),$$

so there is no contribution to the cohomology.
This applies at each $n < 0$, so there are no closed and not exact forms, and the complete holomorphic de Rham cohomology of $\mathbb{P}^{1|2}$ reads

$$h^{n;m}_{dR}(\mathbb{P}^{1|2}) = \begin{cases} 1 & n = 0,\ m = 0, \\ 0 & n > 0, m = 0, \\ 2 & n = 0,\ m = 1, \\ 0 & n \neq 0,\ m = 1, \\ 1 & n = 0,\ m = 2, \\ 0 & n \neq 0,\ m = 2. \end{cases} \tag{3.63}$$

The generators of the non-trivial groups are

$$H^{0;0}_{dR}(\mathbb{P}^{1|2}) = \big\langle 1 \big\rangle_{\mathcal{O}_{\mathbb{P}^1}}, \tag{3.64}$$

$$H^{0;1}_{dR}(\mathbb{P}^{1|2}) = \big\langle \theta_1 \delta^{(0)}(d\theta_1),\ \theta_2 \delta^{(0)}(d\theta_2) \big\rangle_{\mathcal{O}_{\mathbb{P}^1}}, \tag{3.65}$$

$$H^{0;2}_{dR}(\mathbb{P}^{1|2}) = \big\langle \theta_1 \theta_2 \delta^{(0)}(d\theta_1) \delta^{(0)}(d\theta_2) \big\rangle_{\mathcal{O}_{\mathbb{P}^1}}. \tag{3.66}$$

As anticipated above, this is an interesting result, for it shows that the oddity connected to an infinite-dimensional Čech cohomology, is cured at the level of the de Rham cohomology, which is what really matters from the physical point of view, since it is connected to the physical observables and it enters the evaluation of correlation functions [83]. We expect this kind of behaviour to be characteristic for supermanifolds having more than one fermionic dimension.

3.4 The complete de Rham cohomology of $\mathbb{P}^{n|m}$

For the sake of completeness and for future reference we write down the whole holomorphic and real de Rham cohomology for general projective superspaces $\mathbb{P}^{n|m}$. This can be computed by using the same tedious direct method as above (see also [56]).
In the holomorphic case one gets (notice that for $j = 0$, i cannot be negative)

$$H^{i;j}_{dR}(\mathbb{P}^{n|m}) = \begin{cases} \mathbb{C}^{\binom{m}{j}} & i = 0,\ j = 0, \dots, m, \\ 0 & i \neq 0,\ j = 0, \dots, m. \end{cases} \tag{3.67}$$

In the real case one gets instead

$$H^{i;j}_{dR}(\mathbb{P}^{n|m}) = \begin{cases} \mathbb{R}^{\binom{m}{j}} & i = 2k,\ k = 0, \dots, n,\ j = 0, \dots, m, \\ 0 & i = 2k+1,\ k = 0, \dots, n-1,\ j = 0, \dots, m. \end{cases} \tag{3.68}$$

The generators in the holomorphic case are given by a straightforward generalisation of the case $\mathbb{P}^{1|2}$ displayed above. In the real case they are

$$\omega_{k,I_j} := \wedge^k \omega_{FS} \otimes \bigwedge_{\ell \in I_j} \theta_\ell \delta(d\theta_\ell) \tag{3.69}$$

where $I_j \subseteq \{0, 1, \ldots, m\}$ has cardinality j, and ω_{FS} is the ordinary Fubini-Study form.

3.5 Automorphisms and Deformations of $\mathbb{P}^{1|m}$

The method we developed to compute the cohomology of projective super spaces over $\mathbb{P}^1$ easily allows also to evaluate the cohomology of the super tangent space.
Computing the super Jacobian of the change of coordinates, we get

$$\partial_z = -w^2\partial_w - w\sum_{i=1}^{n} \phi_i \partial_{\phi_i} \tag{3.70}$$

$$\partial_{\theta_i} = w\partial_{\phi_i} \tag{3.71}$$

for $i = 1, \ldots m$. The super tangent sheaf is locally generated by the following elements:

$$\mathcal{T}_{\mathbb{P}^{1|n}} U_z = \left\langle \partial_z, \{\theta_J \partial_z\}_{J=(j_1,\ldots,j_m)}, \{\partial_{\theta_i}\}^{i=1,\ldots,m}, \{\theta_J \partial_{\theta_i}\}^{i=1,\ldots,m}_{J=(j_1,\ldots,j_m)} \right\rangle_{\mathcal{O}_{\mathbb{P}^1}(U_z)} \tag{3.72}$$

where $J = (j_1, \ldots, j_m)$ is a multi-index such that $|J| = 1, \ldots, m$ and $j_i = \{0, 1\}$. For example, we can have elements like this: $\theta_1\theta_3\partial_z = \theta_{J=(1,0,1,0\ldots,0)}\partial_z$. Notice there are a total of $(m + 1) \cdot 2^m$ generators.
These have the following transformation rules:

$$\begin{aligned}
\partial_z &= -w^2\partial_w - w\sum_{i=1}^{m} \phi_i \partial_{\phi_i} \\
\theta_J \partial_z &= \left(\frac{1}{w}\right)^{|J|-1} \phi_J \left(-w\partial_w - \sum_{i=1}^{m} \phi_i \partial_{\phi_i}\right) \\
\partial_{\theta_i} &= w\partial_{\phi_i} \\
\theta_J \partial_{\theta_i} &= \left(\frac{1}{w}\right)^{|J|-1} \phi_J \partial_{\phi_i},
\end{aligned} \tag{3.73}$$

where we stress that, depending on J, many terms might be zero in the transformation of $\theta_J\partial_z$, namely all the terms in the sum over i such that $i \in J$.
Using Grothendieck's theorem as above, one can compute the zeroth cohomology group of the tangent sheaf, whose dimension is:

$$h^0(\mathcal{T}_{\mathbb{P}^{1|m}}) = (m + 2)^2 - 1 + \delta_{m,2}. \tag{3.74}$$

Notice that $(m+2)^2 - 1$ is just the number of generators of the Lie algebra associated to the super group $PGL(2|m)$, which is the supersymmetric generalisation of the ordinary Möbius group $PGL(2, \mathbb{C})$, the automorphisms group of the projective line $\mathbb{P}^1$.

It is worth to notice the presence of the "correction" $\delta_{n,2}$, which, incidentally, makes its appearance in the case of the super CY variety $\mathbb{P}^{1|2}$. This correspond to the presence of a further global vector field, (locally) given by $\theta_1\theta_2\partial_z \in H^0(\mathcal{T}_{\mathbb{P}^{1|2}})$, which clearly does not belong to $\mathfrak{sl}(2|2)$, the Lie algebra of $PGL(2|2)$, as already noticed in [13] and more recently in [72].
Integrating this global vector field, we get the "finite" version of the automorphism $\psi : \mathbb{P}^{1|2} \to \mathbb{P}^{1|2}$, which is what is called a "bosonisation" in physics; locally it is given by:

$$\psi\lfloor_{U_z}: (z, \theta_1, \theta_2) \longmapsto (z + \theta_1\theta_2, \theta_1, \theta_2), \tag{3.75}$$

$$\psi\lfloor_{U_w}: (w, \phi_1, \phi_2) \longmapsto (w - \phi_1\phi_2, \phi_1, \phi_2). \tag{3.76}$$

Before we go on, it is important to stress that among *all* the projective super spaces $\mathbb{P}^{n|m}$ - not only among $\mathbb{P}^{1|m}$! -, the case of $\mathbb{P}^{1|2}$ represents, remarkably, the *unique exception*: indeed, it is the only case in which the automorphism group is larger than $PGL(n+1|m, \mathbb{C})$,[4] unlike to what stated in [72]. For reduced dimension 1 this exception has been first observed in [13], page 41.
This and other issues will be the subject of a forthcoming paper, where different methods to compute the cohomology of projective super spaces in a more general setting will be introduced and discussed.

As for the deformations, given by $h^1(\mathcal{T}_{\mathbb{P}^{1|m}})$, one finds

$$h^1(\mathcal{T}_{\mathbb{P}^{1|m}}) = (m+2)\left[(m+2) + (m-4)2^{m-1}\right] - (m-2)2^{m-1} - 1. \tag{3.77}$$

We can see therefore that $\mathbb{P}^{1|1}$, together with $\mathbb{P}^{1|3}$ and the super CY variety $\mathbb{P}^{1|2}$ are *rigid* as they have no deformations, while in the case $m \geq 4$, we start finding a non-zero $h^1(\mathcal{T}_{\mathbb{P}^{1|m}})$. For instance, for $m = 4$ we find $h^1(\mathcal{T}_{\mathbb{P}^{1|4}}) = 19$. We leave to future works a careful investigation of the structure of these deformations.

4 A Super Mirror Map for SCY in Reduced Dimension 1

In [80] has been suggested that the puzzle of mirror of rigid (ordinary) CY manifolds could be solved by enlarging the category of interest to mirror symmetry as to include also super manifolds, in particular SCY manifolds. Later on, triggered by the previous [93] and [87], Aganagic and Vafa proposed a path integral argument to obtain the mirror of Calabi-Yau supermanifolds as super Landau-Ginzburg theories [86]: the construction is exploited to compute the mirror of SCY manifolds in toric varieties and

[4] the bosonic reduction of $PGL(n+1|m)$

in particular to compute the mirror of the "twistorial" (actually super) Calabi-Yau $\mathbb{P}^{3|4}$ [87]. Remarkably, after a suitable limit of the Kälher parameter t, the mirror has a geometric interpretation: it is a quadric in the product space $\mathbb{P}^{3|3} \times \mathbb{P}^{3|3}$ and it is again a SCY manifold.

Being us interested into enlarging the mirror symmetry map for elliptic curves to a supersymmetric context, here we will apply the construction of [86] in the case of bosonic dimension equal to 1 and reduced manifold given by $\mathbb{P}^1$, that is to the two SCY's $\mathbb{P}^{1|2}$ and $\mathbb{WP}^{1|1}_{(2)}$. In doing that, in contrast with [86], we will not need to take any limit of the Kähler parameter: a further geometric investigation carried out by some suitable change of coordinates, shows that $\mathbb{P}^{1|2}$ is actually self-mirror and it is mapped to itself. The mirror of the weighted projective super space $\mathbb{WP}^{1|1}_{(2)}$ instead is not a geometry.

Before we go into the actual computation, we underline that a further, mathematically oriented, analysis needs to be carried out. Despite the effort in [86], many issues remain indeed not that clear, such as for example the role of the Kähler parameter t. It is indeed a matter of question how to define mathematically and in full generality a super analog of the ordinary Kähler condition in the supersymmetric context and therefore how to identify a super Kähler variety.

4.1 Mirror Construction for $\mathbb{P}^{1|2}$

Following [86], we construct the dual of the *LG model* associated to $\mathbb{P}^{1|2}$: it turns out this is given by a σ*-model* on a super Calabi-Yau variety in $\mathbb{P}^{1|1} \times \mathbb{P}^{1|1}$, which is again a SCY variety given by $\mathbb{P}^{1|2}$. In other words, $\mathbb{P}^{1|2}$ gets mapped to itself!

We will focus on the holomorphic part of the potential, where X_I, Y_I for $I = 0, 1$ are *bosonic/even* super fields and η_I, χ_I for $I = 0, 1$ are *fermionic/odd* super fields (that is, the lowest component of their expansion is a bosonic field and a fermionic field respectively), while t is the Kähler parameter, mentioned above. This is given by

$$\mathcal{W}_{\mathbb{P}^{1|2}}(X, Y, \eta, \xi) = \int \prod_{I=0}^{1} \mathcal{D}Y_I \mathcal{D}X_I \mathcal{D}\eta_I \mathcal{D}\chi_I \delta \left(\sum_{I=0}^{1} (Y_I - X_I) - t \right) \cdot \exp \left\{ \sum_{I=0}^{1} e^{-Y_I} + e^{-X_I} + e^{-X_I} \eta_I \chi_I \right\}.$$

By a field redefinition,

$$X_1 = \hat{X}_1 + Y_0, \qquad Y_1 = \hat{Y}_1 + Y_0, \tag{4.1}$$

the path-integral above takes the following form

$$\int \mathcal{D}Y_0 \mathcal{D}X_0 \mathcal{D}\hat{Y}_1 \mathcal{D}\hat{X}_1 \prod_{I=0}^{1} \mathcal{D}\eta_I \mathcal{D}\chi_I \delta\left(Y_0 - X_0 + Y_1 - X_1 - t\right)$$
$$\cdot \exp\left\{e^{-Y_0} + e^{-X_0} + e^{-\hat{Y}_1 - Y_0} + e^{-\hat{X}_1 - Y_0} + e^{-X_0}\eta_0\chi_0 + \eta_1\chi_1 e^{-\hat{X}_1 - Y_0}\right\}.$$

Integrating in X_0, the delta imposes the following constraint on the bosonic fields:

$$X_0 = Y_0 + (Y_1 - X_1) - t. \tag{4.2}$$

Plugging this inside the previous path integral one gets

$$\int \mathcal{D}Y_0 \mathcal{D}\hat{Y}_1 \mathcal{D}\hat{X}_1 \prod_{I=0}^{1} \mathcal{D}\eta_I \mathcal{D}\chi_I \exp\left\{e^{-Y_0} + e^{-Y_0-(Y_1-X_1)+t} + e^{-\hat{Y}_1 - Y_0} + e^{-\hat{X}_1 - Y_0}\right\}$$
$$\cdot \exp\left\{e^{-Y_0-(Y_1-X_1)+t}\eta_0\chi_0 + \eta_1\chi_1 e^{-\hat{X}_1 - Y_0}\right\}.$$

We now perform the fermionic $\mathcal{D}\eta_0\mathcal{D}\chi_0$ integration. We have that

$$\int \mathcal{D}\eta_0 \mathcal{D}\chi_0 \exp\left\{e^{-Y_0-(Y_1-X_1)+t}\eta_0\chi_0\right\} =$$
$$= \int \mathcal{D}\eta_0 \mathcal{D}\chi_0 e^{-Y_0-(Y_1-X_1)+t}\left(1 + \eta_0\chi_0\right) = -e^{-Y_0-(Y_1-X_1)+t}. \tag{4.3}$$

Therefore one gets,

$$-\int \mathcal{D}Y_0 \mathcal{D}\hat{Y}_1 \mathcal{D}\hat{X}_1 \mathcal{D}\eta_1 \mathcal{D}\chi_1 e^{-Y_0-(Y_1-X_1)+t}$$
$$\cdot \exp\left\{e^{-Y_0}\left(1 + e^{-(Y_1-X_1)+t} + e^{-\hat{Y}_1} + e^{-\hat{X}_1} + \eta_1\chi_1 e^{-\hat{X}_1}\right)\right\}.$$

Now, e^{-Y_0} might be interpreted as a multiplier, and we change coordinates to

$$e^{-Y_0} = \Lambda, \qquad \mathcal{D}Y_0 = -\Lambda^{-1}\mathcal{D}\Lambda, \tag{4.4}$$

therefore the integral reads

$$\int \Lambda^{-1} \mathcal{D}\Lambda \mathcal{D}\hat{Y}_1 \mathcal{D}\hat{X}_1 \mathcal{D}\eta_1 \mathcal{D}\chi_1 \Lambda e^{-(Y_1-X_1)+t}$$
$$\cdot \exp\left\{\Lambda\left(1 + e^{-(Y_1-X_1)+t} + e^{-\hat{Y}_1} + e^{-\hat{X}_1} + \eta_1\chi_1 e^{-\hat{X}_1}\right)\right\}.$$

Finally, we change coordinates by redefining the fields as

$$e^{-\hat{X}_1} = x_1, \qquad \mathcal{D}\hat{X}_1 = -\frac{\mathcal{D}x_1}{x_1}, \tag{4.5}$$

$$e^{-\hat{Y}_1} = x_1 y_1, \qquad \mathcal{D}\hat{Y}_1 = -\frac{\mathcal{D}y_1}{y_1}, \tag{4.6}$$

$$\eta_1 = \frac{\tilde{\eta}_1}{x_1}, \qquad \mathcal{D}\eta = x_1 \mathcal{D}\tilde{\eta}. \tag{4.7}$$

Notice that the Berezinian enters the transformation of the measure! This brings the path-integral in the following form

$$\begin{aligned} \mathcal{W}_{\mathbb{P}^{1|2}} &= \int \mathcal{D}\Lambda \frac{\mathcal{D}y_1}{y_1}\frac{\mathcal{D}x_1}{x_1}(x_1\mathcal{D}\tilde{\eta}_1)\mathcal{D}\chi_1\left(y_1 e^t\right)\exp\left\{\Lambda\left(1+e^t y_1 + x_1 + x_1 y_1 + \tilde{\eta}_1\chi_1\right)\right\} \\ &= \int \mathcal{D}\Lambda\mathcal{D}y_1\mathcal{D}x_1\mathcal{D}\tilde{\eta}_1\mathcal{D}\chi_1 e^t \exp\left\{\Lambda\left(1+e^t y_1 + x_1 + x_1 y_1 + \tilde{\eta}_1\chi_1\right)\right\}. \end{aligned} \tag{4.8}$$

We can actually throw the factor e^t, which is not integrated over, in the normalisation and perform the integration over the Lagrange multiplier Λ, that constraints the theory on the following hypersurface

$$1 + x_1 + x_1 y_1 + \tilde{\eta}\chi + e^t y_1 = 0. \tag{4.9}$$

By redefining the field $\tilde{y}_1 = 1 + y_1$ we obtain the more symmetric form

$$1 + x_1\tilde{y}_1 + \tilde{\eta}\chi + e^t(\tilde{y}_1 - 1) = 0. \tag{4.10}$$

Putting the equation in homogeneous form, we have

$$\mathbb{P}^{1|1}\times\mathbb{P}^{1|1} \supset X_0\tilde{Y}_0 + X_1\tilde{Y}_1 + \tilde{\eta}\chi + e^t(X_0\tilde{Y}_1 - X_0\tilde{Y}_0) = 0. \tag{4.11}$$

This is a quadric, call it $\mathcal{Q}$, in $\mathbb{P}^{1|1}\times\mathbb{P}^{1|1}$, with homogeneous coordinates $[X_0 : X_1 : \tilde{\eta}]$ and $[\tilde{Y}_0 : \tilde{Y}_1 : \chi]$ respectively, and it is a super Calabi-Yau manifold. In the following we will drop the tildes and we just call the homogenous coordinates of the super projective spaces $[X_0 : X_1 : \eta] \equiv [X_0 : X_1 : \tilde{\eta}]$ and $[Y_0 : Y_1 : \eta] \equiv [\tilde{Y}_0 : \tilde{Y}_1 : \chi]$. We now re-write the equation for $\mathcal{Q}$ in the following form:

$$X_0((1-e^t)Y_0 + e^t Y_1) + X_1 Y_1 + \eta\chi = 0. \tag{4.12}$$

If we set

$$\ell(Y_0, Y_1) := (1-e^t)Y_0 + e^t Y_1 \tag{4.13}$$

it is not hard to see that the reduced part $\mathcal{Q}_{red}$ in $\mathbb{P}^1 \times \mathbb{P}^1$ is obtained just by putting the odd coordinates to zero, as

$$\mathbb{P}^1 \times \mathbb{P}^1 \supset X_0\, \ell(Y_0, Y_1) + X_1 Y_1 = 0 \tag{4.14}$$

and one can see that $\mathcal{Q}_{red} \cong \mathbb{P}^1$.
We are now interested into fully identifying $\mathcal{Q}$ as a known variety: we observe that as embedded into $\mathbb{P}^{1|1} \times \mathbb{P}^{1|1}$, it is covered by the cartesian product of the usual four open sets:

$$U_0 \times V_0 = \{[X_0 : X_1 : \eta] : X_0 \neq 0\} \times \{[Y_0 : Y_1 : \chi] : Y_0 \neq 0\}, \tag{4.15}$$
$$U_0 \times V_1 = \{[X_0 : X_1 : \eta] : X_0 \neq 0\} \times \{[Y_0 : Y_1 : \chi] : Y_1 \neq 0\}, \tag{4.16}$$
$$U_1 \times V_0 = \{[X_0 : X_1 : \eta] : X_1 \neq 0\} \times \{[Y_0 : Y_1 : \chi] : Y_0 \neq 0\}, \tag{4.17}$$
$$U_1 \times V_1 = \{[X_0 : X_1 : \eta] : X_1 \neq 0\} \times \{[Y_0 : Y_1 : \chi] : Y_1 \neq 0\}. \tag{4.18}$$

Notice that so far we need all of the four open sets to cover $\mathcal{Q}$, for indeed:

$$\mathcal{Q}_{red} \cap \{X_0 = 0\} = [0 : 1] \times [1 : 0] \in U_1 \times V_0, \tag{4.19}$$
$$\mathcal{Q}_{red} \cap \{X_1 = 0\} = [1 : 0] \times [1 : 1 - e^{-t}] \in U_0 \times V_0, \tag{4.20}$$
$$\mathcal{Q}_{red} \cap \{Y_0 = 0\} = [1 : -e^t] \times [0 : 1] \in U_0 \times V_1, \tag{4.21}$$
$$\mathcal{Q}_{red} \cap \{X_0 = X_1 = 1\} = [1 : 1] \times [e^t + 1 : e^t - 1] \in U_1 \times V_1. \tag{4.22}$$

We aim to find a suitable change of coordinates that allow us to use just two open sets. Let us now change coordinates to

$$Y_0' := \ell(Y_0, Y_1), \qquad Y_1' := Y_1, \tag{4.23}$$
$$X_0' := X_0, \qquad X_1' := X_1, \tag{4.24}$$
$$\eta' := \eta, \qquad \chi' := \chi, \tag{4.25}$$

so that the equation for $\mathcal{Q}$ becomes

$$X_0' Y_0' + X_1' Y_1' + \eta' \chi' = 0. \tag{4.26}$$

Now, changing again the coordinates, by exchanging Y_0' with Y_1', and dropping the primes for convenience, we get the following equation for $\mathcal{Q}$

$$X_0 Y_1 + X_1 Y_0 + \eta \chi = 0. \tag{4.27}$$

It is clear that

$$\mathcal{Q}_{red} \cap \{X_0 = 0\} = \mathcal{Q}_{red} \cap \{Y_0 = 0\} = [0 : 1] \times [0 : 1] \in U_1 \times V_1, \tag{4.28}$$
$$\mathcal{Q}_{red} \cap \{X_1 = 0\} = \mathcal{Q}_{red} \cap \{Y_1 = 0\} = [1 : 0] \times [1 : 0] \in U_0 \times V_0. \tag{4.29}$$

Therefore, this change of coordinates allows us to cover $\mathcal{Q}$ by just two open sets:

$$U_{\mathcal{Q}} := \mathcal{Q} \cap (U_0 \times V_0), \tag{4.30}$$
$$V_{\mathcal{Q}} := \mathcal{Q} \cap (U_1 \times V_1), \tag{4.31}$$

we can therefore choose the following (affine) coordinates:

$$U_{\mathcal{Q}}: \; z := \frac{X_1}{X_0}, \quad u := \frac{Y_1}{Y_0}, \quad \theta_0 := \frac{\eta}{X_0}, \quad \theta_1 := \frac{\chi}{Y_0}, \tag{4.32}$$
$$V_{\mathcal{Q}}: \; w := \frac{X_0}{X_1}, \quad v := \frac{Y_0}{Y_1}, \quad \phi_0 := -\frac{\eta}{X_1}, \quad \phi_1 := \frac{\chi}{Y_1}. \tag{4.33}$$

Upon using these affine coordinates, we get the following two affine equation for $\mathcal{Q}$ of $U_{\mathcal{Q}}$ and $V_{\mathcal{Q}}$ respectively:

$$U_{\mathcal{Q}}: \; z + u + \theta_0\theta_1 = 0, \tag{4.34}$$
$$V_{\mathcal{Q}}: \; w + v - \phi_0\phi_1 = 0, \tag{4.35}$$

which describe lines in $\mathbb{C}^{2|2}$. We notice that this two equations are glued together using the relations

$$w = \frac{1}{z}, \qquad v = \frac{1}{u}, \tag{4.36}$$
$$\phi_0 = -w\theta_0, \qquad \phi_1 = v\theta_1. \tag{4.37}$$

We now would like to characterise the variety $\mathcal{Q}$ by its transition functions in order to identify it with a well-known one. By the previous equation, we may take as *proper* bosonic coordinates u and v, as

$$z = -u - \theta_0\theta_1, \tag{4.38}$$
$$w = -v + \phi_0\phi_1. \tag{4.39}$$

We already know that $v = \frac{1}{u}$ and $\phi_1 = \frac{\theta_1}{u}$, so we still have to deal with ϕ_0 :

$$\phi_0 = -\frac{\theta_0}{z} = \frac{\theta_0}{u + \theta_0\theta_1} = \frac{\theta_0(u - \theta_0\theta_1)}{(u + \theta_0\theta_1)(u - \theta_0\theta_1)} = \frac{\theta_0 u}{u^2} = \frac{\theta_0}{u}, \tag{4.40}$$

which tells that the variety $\mathcal{Q} \subset \mathbb{P}^1 \times \mathbb{P}^1$ is actually nothing but $\mathbb{P}^{1|2}$.
This shows that the super mirror map proposed by Vafa and Aganagic makes the supermanifold $\mathbb{P}^{1|2}$ self-mirror, actually it is mapped to itself. This goes along well with what happens in the case of elliptic curves: an elliptic curve is mirror of another elliptic curve.

4.2 $\mathbb{P}^{1|2}$ as a $N = 2$ Super Riemann Surface

We recall that a $N = 2$ super Riemann surface is, by definition, a $1|2$ complex supermanifold M such that the super tangent sheaf $\mathcal{T}_M$ has two $0|1$ subbundles $\mathcal{D}_1$ and $\mathcal{D}_2$, locally generated by vector fields D_1, D_2 that are integrable, i.e. $D_i^2 = fD_i$ for some odd function, and $\mathcal{D}_1 \otimes \mathcal{D}_2, \mathcal{D}_1, \mathcal{D}_2$ generate $\mathcal{T}_M$ at any point. The reader may look into [81] and [50] for details and the more recent articles [82] and [83] for further developments and some physical interpretations.

We now show that $\mathbb{P}^{1|2}$ is indeed a $N = 2$ super Riemann surface. In order to find the needed $0|1$ line bundles $\mathcal{D}_1$ and $\mathcal{D}_2$ we adopt the method envisaged in [50] on page 107, that is we find two maps $p_1 : \mathbb{P}^{1|2} \to X_1$ and $p_2 : \mathbb{P}^{1|2} \to X_2$, with X_1, X_2 two suitable $1|1$ supermanifolds, and will define $\mathcal{D}_i$ as the sheaf kernel of the differential $dp_i : \mathcal{T}_{\mathbb{P}^{1|2}} \to p_i^* \mathcal{T}_{X_i}$. These two maps are immediately available from the model of $\mathbb{P}^{1|2}$ contained in $\mathbb{P}^{1|1} \times \mathbb{P}^{1|1}$ found in the previous section, when we computed the mirror of $\mathbb{P}^{1|2}$. Indeed we can set $X_1 = X_2 = \mathbb{P}^{1|1}$ and the map p_i equal to the restriction of the i-th projection $\pi_i : \mathbb{P}^{1|1} \times \mathbb{P}^{1|1} \to \mathbb{P}^{1|1}$ to $\mathbb{P}^{1|2}$. To give explicit local calculations of the vector fields D_1, D_2 that generate the line bundles $\mathcal{D}_1, \mathcal{D}_2$ and to show that they have all the required properties, we can use the equations (4.34) and (4.35) of the open sets U_Q and V_Q as sub-supermanifolds of $\mathbb{A}^{2|2}$. For example from the equation

$$z + u + \theta_0\theta_1 = 0$$

in the open affine $\mathbb{A}^{2|2} \subset \mathbb{P}^{1|1} \times \mathbb{P}^{1|1}$ with coordinates z, u, θ_0, θ_1, we see that

$$p_1(z, u, \theta_0, \theta_1) = (z, \theta_0) \tag{4.41}$$

$$p_2(z, u, \theta_0, \theta_1) = (u, \theta_1). \tag{4.42}$$

Then $\mathcal{D}_1$ has sections given by those vector fields $\alpha\partial_z + \beta\partial_u + \gamma\partial_{\theta_0} + \delta\partial_{\theta_1}$ that evaluate to 0 on the elements $z,\ \theta_0,\ z + u + \theta_0\theta_1$. Then $\alpha = \gamma = 0$ and $\beta = \delta\theta_0$. They are the multiples of

$$D_1 = \partial_{\theta_1} + \theta_0\partial_u.$$

Similarly one finds that the vector field

$$D_2 = \partial_{\theta_0} - \theta_1\partial_z$$

generates all the vector fields on U_Q that vanish on $u,\ \theta_1,\ z + u + \theta_0\theta_1$. Since D_1 and D_2 vanish on $z + u + \theta_0\theta_1$, they are tangent vector fields on U_Q that by construction generate the kernels $\mathcal{D}_1$ and $\mathcal{D}_2$ of the differentials dp_1 and dp_2. The reader can easily check that $D_1^2 = D_2^2 = 0$ and that

$$\{D_1, D_2\} = D_1D_2 + D_2D_1 = \partial_u - \partial_z$$

and this latter is equal to ∂_u when evaluated on an element of $\mathcal{O}_{U_{\mathcal{Q}}}$. As u is a bosonic coordinate for $U_{\mathcal{Q}}$, we see that $\{D_1, D_2\}, D_1, D_2$ generate $\mathcal{T}_{\mathbb{P}^{1|2}}$ at any point of $U_{\mathcal{Q}}$. Similar formulas are obtained for the open $V_{\mathcal{Q}}$.

4.3 Mirror Construction for $\mathbb{WP}^{1|1}_{(2)}$

In the case of weighted projective super space, we need to evaluate the following potential in order to find the dual theory:

$$\mathcal{W}_{\mathbb{WCP}^{1|1}_{(1,1|2)}} = \int (\mathcal{D}Y_1 \mathcal{D}Y_2) \mathcal{D}X \mathcal{D}\eta \mathcal{D}\chi \delta\left(Y_1 + Y_2 - 2X - t\right) \cdot \exp\left\{e^{-Y_1} + e^{-Y_2} + e^{-X}(1 + \eta\chi)\right\}. \quad (4.43)$$

Performing the integration in the fermionic variables, yields

$$\mathcal{W}_{\mathbb{WCP}^{1|1}_{(1,1|2)}} = \int (\mathcal{D}Y_1 \mathcal{D}Y_2) \mathcal{D}X e^{-X} \delta\left(Y_1 + Y_2 - 2X - t\right) \cdot \exp\left\{e^{-Y_1} + e^{-Y_2} + e^{-X}\right\}. \quad (4.44)$$

Now we can integrate the field X. Up to factors to throw away in the normalisation, the delta gives:

$$\mathcal{W}_{\mathbb{WCP}^{1|1}_{(1,1|2)}} = \int (\mathcal{D}Y_1 \mathcal{D}Y_2) e^{-Y_1/2 - Y_2/2} \exp\left\{e^{-Y_1} + e^{-Y_2} + e^{-Y_1/2 - Y_2/2 + t/2}\right\}. \quad (4.45)$$

We then define the new variables

$$y_i = e^{-Y_i/2}, \qquad i = 1, 2. \quad (4.46)$$

The measure changes as follows $-\frac{1}{2} y_i^{-1} \mathcal{D}y_i = \mathcal{D}Y_i$, therefore, up to factors in the normalisation one gets:

$$\mathcal{W}_{\mathbb{WCP}^{1|1}_{(1,1|2)}} = \int (\mathcal{D}y_1 \mathcal{D}y_2) \exp\left\{y_1^2 + y_2^2 + e^{t/2} y_1 y_2\right\}. \quad (4.47)$$

We therefore see that in the case of $\mathbb{WP}^{1|1}_{(2)}$ we do not get directly a geometry. However, we can further introduce the new variables λ and x defined by

$$y_1 = y_2 x, \qquad y_2^2 = \lambda, \quad (4.48)$$

so that, omitting an inessential constant factor, we get

$$\mathcal{W}_{\mathbb{WCP}^{1|1}_{(1,1|2)}} = \int (\mathcal{D}x \mathcal{D}\lambda) \exp\left\{\lambda\left(x^2 + 1 + e^{t/2} x\right)\right\}. \quad (4.49)$$

Thus, λ is a multiplier and the geometric phase reduces to two points parametrized by t. This is a zero dimensional bosonic model in accordance with the results of Schwarz [84].

5 Conclusions

In the present paper we have investigated some basic questions about super Calabi-Yau varieties (SCY's). We have introduced a very general definition of a SCY, which contains a large class of varieties, including the usual Calabi-Yau manifolds and several projective super spaces. We then restricted our analysis to the SCY with complex bosonic dimension 1, proving that - beyond the usual elliptic curves - it contains the class of $N = 2$ SRS's and the projective super spaces $\mathbb{P}^{1|2}$ and $\mathbb{WP}^{1|1}_{(2)}$. As a byproduct of the mirror map construction, we realised at the very end that $\mathbb{P}^{1|2}$ is indeed a $N = 2$ SRS: this provides a concrete realisation of a $N = 2$ SRS by a map - the mirror map - into the cartesian product of two copies of $\mathbb{P}^{1|1}$. A comment is in order here. In the present paper we have referred to [50] for the definition of $N = 2$ SRS: in this case the proof of triviality of the Berezinian bundle is given in [82]. Nevertheless there exists a more general definition of $N = 2$ SRS given in [81]. To our best knowledge, it is not completely obvious that the two definitions do actually coincide: indeed the definition of $N = 2$ SRS in [81] includes the definition in [50] and, as a consequence, this should imply that all the $N = 2$ SRS's in [50][82][83] are holomorphically *split*. Still, we feel like this topic deserve some more study.

Next, we have computed the super cohomology groups, which include integral forms, showing that for extended supersymmetric varieties a puzzle arises: when the picture number is not maximal nor vanishing, then the corresponding Čech cohomology groups are infinitely generated. Surely, this result will deserve a much deeper investigation; for instance, it would be interesting to understand if it enjoys a geometrical interpretation. Anyway, remarkably, we have shown that this sort of pathology is cured whenever one considers the de Rham cohomology of superforms, which is always finite, even when the corresponding group in Čech cohomology is infinite-dimensional. The same phenomenon occurs in arbitrary dimension $n|m$ as we have seen by explicitly computing the de Rham cohomology of $\mathbb{P}^{n|m}$. The computation of the sheaf cohomology also allowed us to determine the automorphisms of $\mathbb{P}^{1|2}$ and $\mathbb{WP}^{1|1}_{(2)}$, which, on the other hand, are rigid manifolds. It is interesting to note that for SCY with fermionic dimension larger than 1, the automorphism supergroup is larger than the superprojective group. A more systematic analysis of the automorphism group will be presented in a deserved paper. Finally, we have applied the mirror map defined in [86], showing that $\mathbb{P}^{1|2}$ is self-mirror (and, indeed, mapped to itself), whereas $\mathbb{WP}^{1|1}_{(2)}$ is mapped to a zero dimensional bosonic model.

Even though we have chosen to work with an apparently elementary example, we see that highly non trivial aspects appear and some questions remains unsatisfied. For example, we have not been able to provide a suitable definition of Kähler structure (or

Kähler moduli space) for SCY varieties. On one hand, SCY's of bosonic dimension $n = 1$ having $\mathbb{P}^1$ as reduced space, are simple enough in order to allow a complete analysis and shed some light on new interesting properties of supermanifolds; on the other hand, they are too simple for providing a rich list of examples hinting to suitable solutions to the unanswered questions. The natural prosecution would then be to include properly the whole class of $N = 2$ super Riemann surfaces, that are indeed SCY's having bosonic dimension 1, and, more interestingly, to analyse SCY's with bosonic dimension 2, that is *super K3 varieties*.

Despite the results discussed above, we still cannot take our definition of SCY manifold as a definitive one. At the moment, indeed, the triviality of the Berezinian bundle alone appears as a provisional condition, maybe allowing for too many varieties to enter the class. From this point of view, our definition might be considered as a pre-SCY condition. In this context, one might wonder whether the existence of a Ricci-flat metric is a natural condition to add, but in some meaningful example, such as $\mathbb{WP}^{1|1}_{(2)}$, it does not even exist. This may suggest that Ricci-flatness is not the natural condition to add to the triviality of the Berezinian bundle. These and other topics are currently under investigation.

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A Super Fubini-Study Metric and Ricci Flatness of $\mathbb{P}^{1|2}$

We take on the computation of the super Ricci tensor for $\mathbb{P}^{1|2}$ starting from the local form, say in U_z, of the Kähler potential, given by

$$K^s = \log(1 + z\bar{z} + \theta_1\bar{\theta}_1 + \theta_2\bar{\theta}_2). \tag{A.1}$$

This can of course be expanded in power of the anticommuting variables as in [80], but it is not strictly necessary to our end.
In the following we will adopt this convention: we use latin letters $i, j, \ldots$ for bosonic indices, Greek letters $\alpha, \beta, \ldots$ for fermionic indices and capital Latin letters $A, B, \ldots$ will gather both of them. The convention on the unbarred and barred indices goes as usual.
The *holomorphic* and *anti-holomorphic* super derivatives are defined as follows (in the local patch):

$$\partial \coloneqq \partial_z dz + \partial_{\theta_\alpha} d\theta_\alpha, \qquad \bar{\partial} \coloneqq \partial_{\bar{z}} d\bar{z} + \partial_{\bar{\theta}_{\bar{\alpha}}} d\bar{\theta}_{\bar{\alpha}}, \tag{A.2}$$

where $\alpha, \bar{\alpha} = 1, 2$: in other words we have $\partial \coloneqq \partial_A dX^A$ and $\bar{\partial} \coloneqq \partial_{\bar{A}} d\bar{X}^{\bar{A}}$ with $dX^A = (dz|d\theta_1, d\theta_2)$ and $d\bar{X}^{\bar{A}} = (d\bar{z}|d\bar{\theta}_1, d\bar{\theta}_2)$. It is important to stress that while the holomorphic derivative ∂ acts as usual from the left to the right, the anti-holomorphic derivative $\bar{\partial}$ acts from the right to the left instead (even if it is written on left of the function acted on). We also stress that ∂ and $\bar{\partial}$ behave as a standard exterior derivative d on forms. As such the derivatives "do not talk" at all with the forms and only acts on functions, while the forms in ∂ or $\bar{\partial}$ are moved to the right and in turn do not talk to the functions acted by the derivatives. This means that, for example, considering the local expression for a (holomorphic) 1-form acted on by ∂, we will find:

$$\partial(f(z|\theta_1, \theta_1)d\theta_1) = (\partial_B f(z|\theta_1, \theta_1))dX^B d\theta_1. \tag{A.3}$$

Coherently, we will *never* consider expression of the kind $dX^B f(z|\theta)d\theta_1$, so that we will never have to commute or anti-commute a form with a function to get $\pm f(z|\theta)dX^B d\theta_1$: forms and functions just don't talk to each other and the form in ∂ and $\bar{\partial}$ are moved the right.
We now define the super Kähler form as

$$\Omega^s \coloneqq \partial\bar{\partial}K^s \quad \text{or analogously} \quad \Omega^s = \partial_A \partial_{\bar{B}} K^s dX^A d\bar{X}^{\bar{B}}. \tag{A.4}$$

The super metric tensor $H^s_{A\bar{B}}$ can then be red out of it, similarly to the ordinary complex geometric case:

$$H^s_{A\bar{B}} = \partial_A \partial_{\bar{B}} K^s. \tag{A.5}$$

We now deal with the derivative of the super Kähler potential K^s. Remembering that $\partial_{\bar{B}}$ acts from the right, it is straightforward to check that:

$$\partial_{\bar{B}} K^s = \partial_{\bar{B}} \log(1 + z\bar{z} + \theta_1\bar{\theta}_1 + \theta_2\bar{\theta}_2) = \frac{z d\bar{z} + \theta_1 d\bar{\theta}_1 + \theta_2 d\bar{\theta}_2}{1 + z\bar{z} + \theta_1\bar{\theta}_1 + \theta_2\bar{\theta}_2}. \tag{A.6}$$

We now have a product of functions: since we are dealing with anti-commuting objects we need to make a careful use of the generalized Leibniz rule

$$\partial(f \cdot g) = (\partial f) \cdot g + (-1)^{|\partial||f|} f \cdot (\partial g). \tag{A.7}$$

We will put $f \coloneqq z d\bar{z} + \theta_1 d\bar{\theta}_1 + \theta_2 d\bar{\theta}_2$ and $g \coloneqq 1/(1 + z\bar{z} + \theta_1\bar{\theta}_1 + \theta_2\bar{\theta}_2)$ in the following computation.
While the first bit of the ∂ derivative is pretty straightforward and simply gives

$$(\partial f) \cdot g = \frac{1}{1 + z\bar{z} + \theta_1\bar{\theta}_1 + \theta_2\bar{\theta}_2} \left(dz d\bar{z} + d\theta_1 d\bar{\theta}_1 + d\theta_2 d\bar{\theta}_2 \right), \tag{A.8}$$

the second contribution need some extra care: to avoid errors, we may split the derivatives in ∂ by linearity, bearing in mind the non-trivial commutation relation in the generalised Leibniz rule above.
We have the following contribution from $(-1)^{|\partial||f|} f \cdot (\partial g)$:

$$\partial_z \left(\frac{z}{1 + |z|^2 + \theta^2} \right) dz d\bar{z} + \partial_z \left(\frac{\theta_1}{1 + |z|^2 + \theta^2} \right) dz d\bar{\theta}_1 + \partial_z \left(\frac{\theta_2}{1 + |z|^2 + \theta^2} \right) dz d\bar{\theta}_2$$

$$= \frac{-|z|^2 dz d\bar{z} - \theta_1 \bar{z} dz d\bar{\theta}_1 - \theta_2 \bar{z} dz d\bar{\theta}_2}{\left(1 + |z|^2 + \theta^2\right)^2} \tag{A.9}$$

where we have written $\theta^2 \coloneqq \theta_1\bar{\theta}_1 + \theta_2\bar{\theta}_2$. Notice, incidentally that the minus signs above do not come from the commutation relation, but just from the derivative: the commutation relation gives contribution when ∂_{θ_i} is involved

$$\partial_{\theta_1} \left(\frac{z}{1 + |z|^2 + \theta^2} \right) d\theta_1 d\bar{z} + \partial_{\theta_1} \left(\frac{\theta_1}{1 + |z|^2 + \theta^2} \right) d\theta_1 d\bar{\theta}_1 + \partial_{\theta_1} \left(\frac{\theta_2}{1 + |z|^2 + \theta^2} \right) d\theta_1 d\bar{\theta}_2$$

$$= \frac{-z\theta_1 d\theta_1 d\bar{z} + \theta_1\bar{\theta}_1 d\theta_1 d\bar{\theta}_1 + \theta_2\bar{\theta}_1 d\theta_1 d\bar{\theta}_2}{\left(1 + |z|^2 + \theta^2\right)^2}, \tag{A.10}$$

$$\partial_{\theta_2} \left(\frac{z}{1 + |z|^2 + \theta^2} \right) d\theta_2 d\bar{z} + \partial_{\theta_2} \left(\frac{\theta_1}{1 + |z|^2 + \theta^2} \right) d\theta_2 d\bar{\theta}_1 + \partial_{\theta_2} \left(\frac{\theta_2}{1 + |z|^2 + \theta^2} \right) d\theta_2 d\bar{\theta}_2$$

$$= \frac{-z\theta_2 d\theta_2 d\bar{z} + \theta_2\bar{\theta}_1 d\theta_2 d\bar{\theta}_1 + \theta_2\bar{\theta}_2 d\theta_2 d\bar{\theta}_2}{(1+|z|^2+\theta^2)^2}. \tag{A.11}$$

Putting together all the pieces we have:

$$\begin{aligned}\partial\bar{\partial}K^s = \frac{1}{(1+|z|^2+\theta^2)^2}\Big[&(1+\theta^2)dzd\bar{z} + (1+|z|^2+2\theta_1\bar{\theta}_1+\theta_2\bar{\theta}_2)d\theta_1 d\bar{\theta}_1 + \\ &+ (1+|z|^2+\theta_1\bar{\theta}_1+2\theta_2\bar{\theta}_2)d\theta_2 d\bar{\theta}_2 - \theta_1\bar{z}dzd\theta_1 + \\ &- \theta_2\bar{z}dzd\theta_2 - z\bar{\theta}_1 d\theta_1 d\bar{z} - z\bar{\theta}_2 d\theta_2 d\bar{z} + \theta_2\bar{\theta}_1 d\theta_1 d\bar{\theta}_2 + \theta_1\theta_2 d\theta_2 d\bar{\theta}_1\Big],\end{aligned}$$

so the supermetric reads

$$H^s_{A\bar{B}} = \left(\begin{array}{c|cc} 1+\theta^2 & -\theta_1\bar{z} & -\theta_2\bar{z} \\ \hline -z\bar{\theta}_1 & 1+|z|^2+2\theta_1\bar{\theta}_1+\theta_2\bar{\theta}_2 & \theta_2\bar{\theta}_1 \\ -z\bar{\theta}_2 & \theta_1\bar{\theta}_2 & 1+|z|^2+\theta_1\bar{\theta}_1+2\theta_2\bar{\theta}_2 \end{array}\right). \tag{A.12}$$

Using the metric one can generalise the expression for the Ricci tensor one has in ordinary complex geometry, by substituting the determinant with the Berezinian:

$$\mathrm{Ric}_{A\bar{B}} = \partial_A\partial_{\bar{B}} \log\left(\mathrm{Ber}\, H^s\right). \tag{A.13}$$

So the first thing we need to evaluate to prove the (super) Ricci flatness of $\mathbb{P}^{1|2}$ is the Berezinian of the super metric above.
We recall that in general, considering a generic square matrix X valued in a super commutative ring, we have

$$\mathrm{Ber}(X) = \det(A)\det(D - CA^{-1}B) \tag{A.14}$$

where A, B, C, D are the blocks as enlightened above. Notice that A and D are *even* while B and C are *odd*.
We underline that in our case, to make sense out of the expression above we have to look at $CA^{-1}B$ as Kronecker product, as follows:

$$CA^{-1}B \to A^{-1}\cdot C\otimes B, \tag{A.15}$$

A^{-1} consisting of a single even element.
We start from the computation of $A^{-1}\cdot C\otimes B$. We have:

$$A^{-1} = \left(\frac{1+\theta^2}{(1+|z|^2+\theta^2)^2}\right)^{-1}, \tag{A.16}$$

$$C = \frac{1}{(1+|z|^2+\theta^2)^2} \begin{pmatrix} -z\bar{\theta}_1 \\ -z\bar{\theta}_2 \end{pmatrix}, \tag{A.17}$$

$$B = \frac{1}{(1+|z|^2+\theta^2)^2} (-\theta_1 \bar{z},\ -\theta_2 \bar{z}). \tag{A.18}$$

This leads to:

$$\begin{aligned} A^{-1} \cdot C \otimes B &= \frac{1}{(1+\theta^2)(1+|z|^2+\theta^2)^2} \begin{pmatrix} -z\bar{\theta}_1 \\ -z\bar{\theta}_2 \end{pmatrix} \otimes (-\theta_1\bar{z},\ -\theta_2\bar{z}) \\ &= -\frac{|z|^2}{(1+\theta^2)(1+|z|^2+\theta^2)^2} \begin{pmatrix} \theta_1\bar{\theta}_1 & \theta_2\bar{\theta}_1 \\ \\ \theta_1\bar{\theta}_2 & \theta_2\bar{\theta}_2 \end{pmatrix} \end{aligned} \tag{A.19}$$

where the overall minus sign comes from the commutation relation of the theta's. It is actually convenient to multiply $(1+\theta^2)^{-1}$ out: first of all we observe

$$\frac{1}{(1+\theta^2)} = 1 - \theta^2 + 2\theta^4 \tag{A.20}$$

where $\theta^4 := \theta_1\bar{\theta}_1\theta_2\bar{\theta}_2$. So the product above becomes:

$$A^{-1} \cdot C \otimes B = -\frac{|z|^2}{(1+|z|^2+\theta^2)^2} \begin{pmatrix} \theta_1\bar{\theta}_1 - \theta^4 & \theta_2\bar{\theta}_1 \\ \\ \theta_1\bar{\theta}_2 & \theta_2\bar{\theta}_2 - \theta^4 \end{pmatrix}. \tag{A.21}$$

Therefore one has the following expression:

$$D - CA^{-1}B = \frac{1}{(1+|z|^2+\theta^2)^2}$$
$$\cdot \begin{pmatrix} 1+|z|^2+(2+|z|^2)\theta_1\bar{\theta}_1 + \theta_2\bar{\theta}_2 - |z|^2\theta^4 & (1+|z|^2)\theta_2\bar{\theta}_1 \\ \\ (1+|z|^2)\theta_1\bar{\theta}_2 & 1+|z|^2+\theta_1\bar{\theta}_1+(2+|z|^2)\theta_2\bar{\theta}_2 - |z|^2\theta^4 \end{pmatrix}.$$

We now need to evaluate the determinant of the square matrix above:

$$\begin{aligned} \det(D - CA^{-1}B) = \frac{1}{(1+|z|^2+\theta^2)^4} \Big[&(1+|z|^2)^2 + (1+|z|^2)\theta_1\bar{\theta}_1 \\ &+ (1+|z|^2)(2+|z|^2)\theta_2\bar{\theta}_2 + (1+|z|^2)(2+|z|^2)\theta_1\bar{\theta}_1 + (1+|z|^2)\theta_2\bar{\theta}_2 + \\ &+ \theta^4 + (2+|z|^2)^2\theta^4 - 2|z|^2(1+|z|^2)\theta^4 + (1+|z|^2)^2\theta^4 \Big] \end{aligned} \tag{A.22}$$

where we have isolated on different lines the zeroth, quadratic and quartic contribution in the theta's. We can simplify a little the expression above to get:

$$\det(D - CA^{-1}B) = \frac{(1+|z|^2)^2}{(1+|z|^2+\theta^2)^4}\Big[1 + \frac{3+|z|^2}{1+|z|^2}\theta^2 + \frac{6+4|z|^2}{(1+|z|^2)^2}\theta^4\Big]. \tag{A.23}$$

To evaluate the full Berezinian we need to invert the determinant we just got. This yields:

$$\frac{1}{\det(D - CA^{-1}B)} = (1+|z|^2+\theta^2)^4\Big[1 - \frac{3+|z|^2}{1+|z|^2}\theta^2 - 2\frac{6+4|z|^2+|z|^4}{(1+|z|^2)^2}\theta^4\Big]. \tag{A.24}$$

Putting together the pieces, we can evaluate the full Berezinian:

$$\mathrm{Ber}(H^s) = \frac{(1+|z|^2+\theta^2)^2(1+\theta^2)}{(1+|z|^2)^2}\Big[1 - \theta^2 - \frac{2}{1+|z|^2}\theta^2 - 2\frac{6+4|z|^2+|z|^4}{(1+|z|^2)^2}\theta^4\Big] = 1.$$

Remembering that $\mathrm{Ric}_{A\bar{B}} = \partial_A\partial_{\bar{B}}\log(\mathrm{Ber}(H^s))$, since we have found that $\mathrm{Ber}(H^s) = 1$, this leads us the the conclusion:

$$\mathrm{Ric}_{A\bar{B}} = 0. \tag{A.25}$$

$\mathbb{P}^{1|2}$ is Ricci-flat and therefore it is a super Calabi-Yau manifold in the strong sense.

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