FAVARD THEORY FOR THE ADJOINT EQUATION AND FREDHOLM ALTERNATIVE

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Abstract. Fredholm Alternative is a classical tool of periodic linear equations, allowing to describe the existence of periodic solutions of an inhomogeneous equation in terms of the adjoint equation. A few partial extensions have been proposed in the literature for recurrent equations: our aim is to point out that they have a common root and discuss whether such a root gives rise to a general Fredholm–type Alternative. Sacker–Sell spectral theory and Favard theory are main ingredients in this discussion: a considerable effort is devoted to understand how Favard theory is affected by adjunction, at least for planar equations.

1. Introduction

Consider the inhomogeneous linear differential equation in $\mathbb{R}^N$:

$$\dot{x} = \mathfrak{A}(t)x + \mathfrak{f}(t) \tag{1.1}$$

where the matrix $\mathfrak{A}$ and the vector $\mathfrak{f}$ are bounded and uniformly continuous functions, typically enjoying some recurrence property. Our concern is solving a related boundary value problem, that is, proving the existence of a solution x which has 'the same recurrence properties' as the coefficients $\mathfrak{A}$ and $\mathfrak{f}$. In the most classical recurrent case, namely when $\mathfrak{A}$ and $\mathfrak{f}$ are both T–periodic, we are interested in solutions which are also T–periodic, and a similar condition can be formulated in the almost periodic case: see Remark 2.3 for some more details. As we will see in Section 2, the quickest way to extend this notion to more general recurrent frameworks is to consider the joint *hull* of $\mathfrak{A}$ and $\mathfrak{f}$, namely:

$$H(\mathfrak{A}, \mathfrak{f}) \;=\; \mathrm{cls}\big\{(\mathfrak{A}\tau, \mathfrak{f}\tau) : \tau \in \mathbb{R}\big\}$$

where $\mathfrak{A}\tau, \mathfrak{f}\tau$ stand for translating by τ and the closure is taken in the compact–open topology. The hull is a compact metrizable space and translations define a continuous flow on it, whose recurrence properties reflect those of $\mathfrak{A}$ and $\mathfrak{f}$. We are actually looking for a solution x which is *representable* on $H(\mathfrak{A}, \mathfrak{f})$, in the sense that a flow homomorphism exists:

$$H(\mathfrak{A}, \mathfrak{f}) \to H(x) \qquad\qquad (\mathfrak{A}, \mathfrak{f}) \mapsto x\ . \tag{1.2}$$

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Coming back to the case of T–periodic coefficients $\mathfrak{A}$ and $\mathfrak{f}$, the joint hull $H(\mathfrak{A},\mathfrak{f})$ turns out to be isomorphic to $\mathbb{R}/T\mathbb{Z}$: hence, as expected, x is representable when it is T–periodic. In this case, classical Fredholm Alternative decides which are the inhomogeneous terms $\mathfrak{f}$ for which (1.1) has representable solutions: see for instance Hale's book [12]. They are precisely those $\mathfrak{f}$ which satisfy the orthogonality condition:

$$\int_0^T \big\langle \mathfrak{f}(t), y(t)\big\rangle \, dt = 0 \tag{1.3}$$

for every T–periodic solution y of the adjoint equation:

$$\dot{y} = -\mathfrak{A}(t)^T y \ . \tag{1.4}$$

The main question we face here is whether or not a similar tool is still available when the exact periodicity is relaxed. Next to periodicity stands almost periodicity. A more general notion, which is of basic importance in this paper, is that of *Birkhoff recurrence*, or simply recurrence. A bounded and uniformly continuous function is called (Birkhoff) recurrent when its hull is *minimal* with respect to the translation flow (which means that every orbit is dense in the hull). Birkhoff Recurrence Theorem guarantees that recurrent solutions exist as soon as bounded solutions do but representability on $H(\mathfrak{A},\mathfrak{f})$ does not come for free, even if such hull is minimal, expressing the fact that $\mathfrak{A}$ and $\mathfrak{f}$ are jointly recurrent.
As usual in the aperiodic world, the answer to our question depends on the specific properties of $\mathfrak{A}$, among which there are the *spectral* properties. By σ we mean the Sacker–Sell spectrum of $\mathfrak{A}$, introduced in [25] as the set of real λ's for which the homogeneous equation:

$$\dot{x} = [\mathfrak{A}(t) - \lambda\mathfrak{I}]x$$

does not admit an exponential dichotomy on the whole $\mathbb{R}$. This is always a nonempty compact set, made up by at most N disjoint closed intervals:

$$\sigma \ = \ [a_1, b_1] \cup \cdots \cup [a_n, b_n] \qquad n \leq N$$

each spectral interval corresponding, roughly speaking, to the vector space of the solutions to:

$$\dot{x} = \mathfrak{A}(t)x \tag{1.5}$$

having Lyapunov exponents in that interval. This vector space contains all the bounded solutions and will play a relevant role in our theory; we denote its dimension by:

$$0 \ \leq \ d_S \ \leq \ N$$

and call it the *Sacker–Sell dimension* of $\mathfrak{A}$. Using a $*$ to say that we are concerned with the adjoint equation (1.4) instead of (1.5), it turns out that $\sigma^* = -\sigma$ and $d_S^* = d_S$. The easiest and most frequently used spectral assumption on $\mathfrak{A}$ is:

$$0 \notin \sigma$$

in which case we agree that $d_S = 0$. There is no need for a Fredholm Alternative to decide the solvability of the boundary value problem (1.1) in this case: whatever bounded $\mathfrak{f}$ we take, equation (1.1) admits a unique bounded solution, which is automatically recurrent and representable on $H(\mathfrak{A},\mathfrak{f})$. This follows directly from the integral representation of the unique bounded solution: see for instance Coppel's book [5]. By the way, notice that $0 \notin \sigma^*$ is also true. In particular, there are no bounded solutions to the adjoint equation (1.4) but the trivial one: any reasonable

adapted version of the orthogonality condition (1.3) for the recurrent framework should be then automatically verified for every $\mathfrak{f}$.
The opposite spectral situation, namely:

$$0 \in \sigma$$

is considerably more delicate and only few papers investigate Fredholm Alternative in this case: as far as we know, they reduce to [20], [2], [17] and [32]. Actually [20] does not apply to recurrent equations: we will discuss it in the final part of the Introduction. A reasonable candidate for a recurrent Fredhom–type Alternative is suggested in [32]. The starting point is the same integration by parts which gives (1.3) in the periodic case, that is:

$$\int_0^t \big\langle \mathfrak{f}(s), y(s) \big\rangle \, ds \;=\; \Big[\big\langle x(s), y(s) \big\rangle \Big]_0^t$$

where x, y are arbitrary solutions to (1.1) and (1.4) respectively. Because of that, to have a *bounded* x it is *necessary* that:

$$\big\langle \mathfrak{f}, y \big\rangle \in BP(\mathbb{R};\mathbb{R}) \tag{1.6}$$

for every *bounded* y, where BP stands for having bounded primitive. Notice that such y are many more than those which are representable on $H(\mathfrak{A})$ or $H(\mathfrak{A}, \mathfrak{f})$. As a consequence, when for instance $\mathfrak{A}$ and $\mathfrak{f}$ are both T–periodic, condition (1.6) looks stronger than the classical condition (1.3): in fact, the two conditions are equivalent, since the latter implies the existence of T–periodic solutions for (1.1) and hence the validity of the former. The leading idea in [32] is solving the general recurrent boundary value problem in two moves: first proving that sometimes condition (1.6) is also *sufficient* for (1.1) to admit bounded solutions, and then invoking *Favard theory* to solve the boundary value problem determined by (1.2).
Favard theory dates back to 1927 with [7] but is still the most general device to solve the boundary value problem (1.1)–(1.2). The crucial assumption is a quite involved restriction on $\mathfrak{A}$, namely that every nontrivial bounded solution to every homogeneous equation in the class:

$$\dot{x} = \mathfrak{B}(t)x \qquad\qquad \mathfrak{B} \in H(\mathfrak{A}) \tag{1.7}$$

must be separated from zero, in the sense that:

$$\inf_t |x(t)| \;>\; 0 \,.$$

This is usually called Favard separation condition and we denote it by (F). In [7] Favard proved that: *under the assumption that* (F) *holds and* $H(\mathfrak{A}, \mathfrak{f})$ *is minimal, if equation (1.1) has bounded solutions then one of them satisfies (1.2).* The actual need for (F) is an open question but optimality is well known. That is, if we omit (F), then the conclusion of the Favard result may be false: see [34], [10] and [18] for almost periodic examples where Favard condition fails, which admit bounded solutions but no almost periodic solutions. In Section 2 we show that the minimality of $H(\mathfrak{A}, \mathfrak{f})$ is also optimal for the Favard result. A handier definition of (F) has been obtained in [1] by looking at the number $d(B)$ of independent bounded solutions to equation (1.7) and to its minimum value over the hull:

$$0 \;\leq\; d_F \;=\; \min_{B \in H(\mathfrak{A})} d(B) \;\leq\; N$$

which we call *Favard dimension* of $\mathfrak{A}$.

Theorem 1.1 ([1]). *Assume $H(\mathfrak{A})$ is minimal. Then d_F is attained at a residual subset of $H(\mathfrak{A})$, whose elements are exactly the B's for which the nontrivial bounded solutions to the corresponding equation (1.7) are separated from zero.*

As a consequence (see also [33]) condition (F) holds if and only if:

$$d(B) \;=\; d_F \qquad \forall B \in H(\mathfrak{A}) \,.$$

The already commented spectral situation $0 \notin \sigma$ can be revisited in term of Favard theory. Indeed $0 \notin \sigma$ implies that (F) holds with the smallest Favard dimension $d_F = 0$, the same being automatically true for the adjoint equation. The converse implication is also true, as soon as $H(\mathfrak{A})$ is minimal: see [28] and [23]. Coming back to [32], the focus is on the opposite extremal situation: $d_F = N$, that is, the Favard dimension is as large as possible. In this case one has $\sigma = \{0\}$ and (F) holds, if $H(\mathfrak{A})$ is minimal: the same is also true for the adjoint equation since it turns out that $d^*_F = N$. The last fact is finally used to show that condition (1.6) is sufficient to get bounded solutions to (1.1) and hence to solve the associated boundary value problem (1.2).

The papers [2] and [17] support the unexpressed conjecture of [32] that the same conclusions hold for Favard dimensions which are intermediate between 0 and N. The first one deals with almost periodic $\mathfrak{A}$ and $\mathfrak{f}$ only, where:

$$\mathfrak{A}^T(t) = \mathfrak{A}(t) \le 0 \qquad \forall t \in \mathbb{R}$$

though the nonsymmetric case is also partially covered. The second one applies to a recurrent damped Hill's equation:

$$\ddot{x} + c\dot{x} + a(t)x = g(t) \qquad c \neq 0$$

whose homogeneous part is disconjugate in a strong sense. In both cases, it is possible to show that the direct and the adjoint Favard conditions hold with the same Favard dimensions and moreover (1.6) is again sufficient to solve the boundary value problem (1.1)–(1.2). Actually, the two papers use some specialized conditions whose equivalence with (1.6) is not so manifest: this and other related facts will be the subject of a forthcoming paper.

Summing up, the current literature seems to suggest that the following conclusions are generally true for recurrent equations:

1) if (F) holds then also (F^*) does;
2) if (F) and (F^*) hold then $d_F = d^*_F$;
3) if (F) and (F^*) hold and $d_F = d^*_F$ then condition (1.6) is sufficient to solve the boundary value problem (1.1)–(1.2).

We will see that, on the contrary, all these claims may be false. Planar counter–examples to 1) and 3) are provided by Propositions 8.7 and 8.4 in Section 8, while for 2) we need one more dimension: see the matrix defined by (9.14) at the end of Section 9. Some general results nevertheless survive to counter–examples and, we believe, define the scope of a recurrent Fredholm–type Alternative. The first result we prove is a kind of common root of all the positive results in the literature.

Theorem 1.2. *Assume $H(\mathfrak{A}, \mathfrak{f})$ is minimal and:*

$$d_F \;=\; d_S \,. \tag{1.8}$$

Then (F) and (F^*) hold with $d_F = d^*_F$ and condition (1.6) is sufficient for (1.1)–(1.2) to admit a solution.

We also show that the minimality of $H(\mathfrak{A},\mathfrak{f})$ is optimal for the result. More precisely, though we can prove the existence of bounded uniformly continuous solutions to (1.1) even when $\mathfrak{A}$ and $\mathfrak{f}$ are just bounded and uniformly continuous, in general none of these solutions satisfy condition (1.2): the ultimate reason is the failure of Favard theory, which is commented in Example 3.2 on the basis of the arguments of Section 2. Notice moreover that:

$$d_F \ \le\ d(B)\ \le\ d_S \qquad \forall B\in H(\mathfrak{A})\ .$$

While the first inequality cannot be sharp in the whole $H(\mathfrak{A})$, by the very definition of Favard dimension, the second is sharp when for instance the homogeneous equation (1.5) admits solutions with polynomial growth in time.
Condition (1.8) is not new in the literature: with a different formulation, it appears indeed in the Sacker and Sell paper [24], to prove a decomposition of the solution space of (1.5) into the direct sum of the stable, unstable and center manifolds, which is indeed a special instance of their future Spectral Theorem in [25]. A warning is however due about condition (1.8). On the one hand, it seems unnecessarily restrictive: when $\mathfrak{A}$ and $\mathfrak{f}$ are both T–periodic, all the conclusions of Theorem 1.2 are actually true even if (1.8) is not satisfied. On the other hand, it must be pointed out that we are dealing here with a different and harder problem: even admitting that $\mathfrak{A}$ is periodic, we are indeed trying to have the better of *every recurrent* term $\mathfrak{f}$. Next result confirms that such difference is crucial: it shows that, at least for equations with a low dimensional bounded dynamics, condition (1.8) is not only sufficient but even necessary for solving our problem.

Theorem 1.3. *Assume that $H(\mathfrak{A})$ is minimal with $d_S\le 2$, and that moreover (F) and (F^*) hold. If condition (1.6) is sufficient to solve the boundary value problem (1.1)–(1.2) for every $\mathfrak{f}$ such that $H(\mathfrak{A},\mathfrak{f})$ is minimal, then $d_F=d_S$.*

The equality $d_F=d_F^*$ is not mentioned, because it is automatic under the assumptions of the theorem. Moreover, it is not difficult to guess that the critical situation is when the Favard dimensions are 1 and the Sacker–Sell dimension is 2: in this case, the inhomogeneous term $\mathfrak{f}$ breaking down the Fredholm Alternative is such that $H(\mathfrak{A},\mathfrak{f})$ is minimal aperiodic.
A key device for proving Theorems 1.2 and 1.3 is changing variables, which permits to deal with simpler recurrent equations. For instance, it is always possible to transform (1.5) into a triangular equation or a block–diagonal one, although the price to pay is having a weaker form of recurrence than $\mathfrak{A}(t)$ and $\mathfrak{f}(t)$: see [11], [4], [19] and [6]. The general notion of change of variables is provided in Section 6, where Theorem 1.2 is also proved by distilling the arguments of [32]: see Theorem 6.9 and Remark 6.11. Theorem 1.3 is proved in Section 9, see Theorem 9.1 and Remark 9.2. The proof is based on the analysis of triangular planar recurrent equations, conducted in Section 7 and Section 8: roughly speaking, we find a restricted number of *normal forms* which account for all the relevant properties of these equations and are also essential to find the counter–examples to the aforementioned claims 1), 2) and 3).
Sections from 2 to 5 are essentially devoted to prerequisites, but all of them contain something new or at least quite overlooked by the current literature. Section 2 introduces some properties of minimal sets but also enters into the details of the notion of representability: they are common knowledge in the almost periodic framework only, while we need to understand better the general minimal case. This

better understanding is used in Section 3 to show why and how Favard theory fails for nonrecurrent equations, after having introduced such theory and spectral theory too. In Section 4 the key condition (1.6) is introduced and its properties investigated: they allow to obtain Theorem 1.2 from our general theory. Finally, Section 5 is devoted to scalar recurrent equations and to a couple of overlooked results which are also crucial for the construction of counter–examples. Both of them refer to functions that have zero in their spectrum but unbounded primitive: the first one is a general existence theorem for such functions, which is widely known in the almost periodic case but apparently not in the recurrent one, while the second one is due to Kozlov [13] and concerns some fine properties of the primitives in the quasi–periodic case.
We conclude the Introduction by commenting a result in [20], which seems to suggest that (1.6) may be not the appropriate condition to give a Fredholm–type Alternative. In a part of that paper, Palmer studies equation (1.1) when $\mathfrak{A}(t)$ and $\mathfrak{f}(t)$ are just bounded and continuous, under the assumption that (1.5) has an exponential dichotomy on both $\mathbb{R}^+$ and $\mathbb{R}^-$. The conclusion is that (1.1) admits bounded solutions if and only if:

$$\int_{-\infty}^{+\infty} \langle \mathfrak{f}(t), y(t) \rangle \, dt = 0 \tag{1.9}$$

is satisfied for every bounded solution $y(t)$ of the adjoint equation (1.4). This fact prevents (1.6) from being sufficient for the same conclusion: under Palmer assumption indeed, equations (1.5) and (1.4) may have nontrivial bounded solutions, but all of them must decay exponentially as $|t| \to \infty$ and hence condition (1.6) is empty. The point is that this cannot occur when $\mathfrak{A}(t)$ is recurrent, since having an exponential dichotomy on a half–line is equivalent to $0 \in \sigma$, implying that condition (1.9) is also empty: see Coppel's book [5] for a proof.

Notations.
The symbols $|x|$ and $\langle x, y \rangle$ stand respectively for the Euclidean norm and the inner product in $\mathbb{R}^N$, while $\mathcal{L}(N)$ and $\mathcal{GL}(N)$ denote the $N \times N$ matrices and invertible matrices with real entries respectively. Given a function $\mathfrak{f} \in C(\mathbb{R}; \mathbb{R}^N)$ we set:

$$\widetilde{\mathfrak{f}}(t) = \int_0^t \mathfrak{f}(s) \, ds$$

and we say $\mathfrak{f} \in BP(\mathbb{R}; \mathbb{R}^N)$ when the primitive $\widetilde{\mathfrak{f}}$ is bounded.
The Greek capital letters Θ, Ω, Σ denote compact metrizable spaces endowed with continuous real flows $\theta t, \omega t, \sigma t$, which we simply call compact flows. Every function $f \in C(\Theta; \mathbb{R}^N)$ gives rise to a class of functions $f_\theta \in C(\mathbb{R}; \mathbb{R}^N)$ defined by:

$$f_\theta(t) = f(\theta t) \qquad \forall t \in \mathbb{R}$$

parameterized by $\theta \in \Theta$. Moreover, by D we mean the derivative along the flow on any given compact flow. That is:

$$Df(\theta) \;=\; \lim_{t \to 0} \frac{f(\theta t) - f(\theta)}{t}$$

when the limit exists. We say that $f \in BP(\Theta; \mathbb{R}^N)$ when $f \in C(\Theta; \mathbb{R}^N)$ and there exists a function $\widehat{f} \in C(\Theta; \mathbb{R}^N)$ such that $D\widehat{f} = f$ on the whole Θ. Of course, if $f \in C(\Theta; \mathbb{R}^N)$ then $f_\theta \in BP(\mathbb{R}; \mathbb{R}^N)$ for every θ; due to Favard theory, the contrary

is true when Θ is minimal.
Most of the equations we consider are implicitly parameterized over a compact flow, say for instance Θ: if $(*)$ is one of these equations, we denote by $(*)_\theta$ the equation corresponding to the value θ of the parameter.

2. Minimal hulls and representability

The aim of the section is twofold: to recall some basic properties of minimal flows and to introduce and comment the appropriate notion of representability of functions on these flows.

Standing assumption. In all the paper Θ, Ω ... stand for metrizable topological spaces which are at least compact, endowed with real flows $\theta t, \omega t$ ... which are continuous. We call them *compact flows*.

A compact flow Θ is *pointed* when $\overline{\theta_0\mathbb{R}} = \Theta$ for some $\theta_0 \in \Theta$; in this case, we also say that Θ is pointed at θ_0. In concrete applications pointed flows appear as hulls of some suitable functions. Consider indeed the class $C(\mathbb{R}; X)$ of the continuous functions with values in a finite dimensional Banach space X, endowed with the compact–open topology. This is a metrizable topology. Setting:

$$(\mathfrak{u}\tau)(t) = \mathfrak{u}(t+\tau) \qquad\qquad \forall \mathfrak{u} \in C(\mathbb{R}; X) \quad \forall \tau, t \in \mathbb{R}$$

defines a continuous flow on $C(\mathbb{R}; X)$, which is usually called Bebutov flow. Given $\mathfrak{u} \in C(\mathbb{R}; X)$ we define the *hull* of $\mathfrak{u}$ as the closed subset:

$$H(\mathfrak{u}) \;=\; \overline{\mathfrak{u}\mathbb{R}}\,.$$

The space $H(\mathfrak{u})$ is connected and naturally pointed at $\mathfrak{u}$. If moreover it is compact, then we can give it the name of pointed flow: this happens if and only if $\mathfrak{u}$ is bounded and uniformly continuous, see [29] for a proof.
Here we are mostly interested in functions $\mathfrak{u}$ which are *recurrent*, in the sense that when their hull $H(\mathfrak{u})$ is not only compact but also minimal for the Bebutov flow. By *minimal* subset of a compact flow Θ we mean a nonempty closed invariant $M \subset \Theta$ which does not admit any proper subset of the same type: such M's always exist due to the Birkhoff Recurrence Theorem. When the only possible M is Θ itself, we say that Θ is minimal. This is equivalent to saying that $\overline{\theta\mathbb{R}} = \Theta$ for every θ: in other words, Θ is pointed with respect to every point θ.
Periodicity and almost periodic are the most important cases of recurrence. In the literature, almost periodicity has different and often nonequivalent meanings. Here we choose the stronger one: according to [31], we say that Θ is *almost periodic* when there exists a $\theta \in \Theta$ such that the flow line $t \mapsto \theta t$ is an almost periodic function in the classical sense of Bohr and moreover its orbit is dense in Θ. All the flow lines can be easily proved to share the same properties, so that Θ is actually minimal. By further specializing almost periodicity we finally get a *periodic* Θ: now all the flow lines are obtained by translating a single periodic one, and then all of them have the same period and the same orbit, that is the whole Θ.
Coming back to the general case, it is well known that Θ is minimal if and only if all its points are recurrent. Given a compatible metric d, the recurrence of the

point θ means that for every $\varepsilon > 0$ the set of τ for which:

$$d(\theta\tau, \theta) < \varepsilon$$

is relatively dense in $\mathbb{R}$, that is, there is a (inclusion) length $L > 0$ such that the set intersects every interval of length L. See [30] for a proof. Next lemma states a minor variation of the recurrence property, which we state without proof: we will need this technical fact in Section 7 only.

Lemma 2.1. *Let Θ be minimal and $\theta_0, \theta_1 \in \Theta$ two arbitrary points. Then for every $\delta > 0$ there exist a relatively dense $\mathcal{T} \subset \mathbb{R}$ and $\rho > 0$ such that:*

$$d\big(\theta_0(\tau + s), \theta_1\big) < \delta$$

for every $\tau \in \mathcal{T}$ and $|s| < \rho$.

Compact flows have a natural order, which will be crucial to define the recurrent Fredholm Alternative in Section 6. To introduce such order, let us first define a homomorphism $\varphi : \Omega \to \Theta$ between compact flows as a continuous map preserving the flows:

$$\varphi(\omega t) = \varphi(\omega)t \qquad \forall\omega \in \Omega \quad \forall t \in \mathbb{R}\ .$$

Notice that, by combining the compactness of Ω and the continuity of φ one gets the following useful technical fact:

$$\varphi\left(\overline{\omega\mathbb{R}}\right) = \overline{\varphi(\omega)\mathbb{R}} \qquad \forall\omega \in \Omega\ . \tag{2.1}$$

This implies that homomorphisms send minimal sets into minimal sets, and hence are epimorphisms when the target space is minimal. Similarly, if Θ is pointed with respect to θ_0 and $\varphi(\omega_0) = \theta_0$, then φ is again an epimorphism:

$$\varphi(\Omega) \supset \varphi\left(\overline{\omega_0\mathbb{R}}\right) = \overline{\theta_0\mathbb{R}} = \Theta\ .$$

When there exists an *epimorphism* $\Omega \to \Theta$ we say that Ω *extends* Θ and we write:

$$\Omega \succ \Theta\ .$$

This order structure is *directed* in the category of compact flows. Given indeed any two compact flows Θ and $\widehat{\Theta}$, we can construct the product flow:

$$\Omega = \Theta \times \widehat{\Theta} \qquad \big(\theta, \widehat{\theta}\big)t = \big(\theta t, \widehat{\theta} t\big) \tag{2.2}$$

and observe that $\Omega \succ \Theta$ and $\Omega \succ \widehat{\Theta}$ with projections in the role of epimorphisms. By slightly modifying the arguments, we can also direct the order in the smaller category of minimal flows: for it is enough to replace Ω in (2.2) with any minimal subset of itself: the restricted projections are again surjective, since the target spaces Θ and $\widehat{\Theta}$ are minimal.

After these premises, we start now with the main concern of the section: the representability of functions on compact flows and its properties. We say that $\mathfrak{u} \in C(\mathbb{R}; X)$ is *representable on* Θ *at a given* $\theta_0 \in \Theta$ when there exists a function $u \in C\big(\overline{\theta_0\mathbb{R}}\, ; X\big)$ such that:

$$\mathfrak{u}(t) \ = \ u(\theta_0 t) \qquad \forall t \in \mathbb{R}\ . \tag{2.3}$$

As it will be clear after Proposition 2.2, this definition is not in contrast with that given in the Introduction. The represented function $\mathfrak{u}$ is automatically uniformly continuous. Since the representing function u is clearly unique, with a little abuse we say that $C\big(\overline{\theta_0\mathbb{R}}\, ; X\big)$ *is* the class of representable functions at θ_0. Notice that Tietze's Theorem would allow to extend u to the whole Θ but uniqueness is lost

and characterization becomes more problematic.
Two other features of representability deserve some attention. The first one is that representability is preserved under extensions. Assume indeed that (2.3) holds and denote by φ the epimorphism responsible for $\Omega \succ \Theta$. After choosing any $\omega_0 \in \Omega$ such that $\varphi(\omega_0) = \theta_0$ we have:

$$u(\varphi(\omega_0 t)) = u(\theta_0 t) = \mathfrak{u}(t) \qquad \forall t \in \mathbb{R}$$

and hence $u \circ \varphi$ represents $\mathfrak{u}$ on Ω at ω_0 and is well defined on $\overline{\omega_0 \mathbb{R}}$ because of (2.1). The second feature is that $H(\mathfrak{u})$ is the most obvious compact flow where to represent a bounded and uniformly continuous function $\mathfrak{u}$. Indeed $u(\mathfrak{v}) = \mathfrak{v}(0)$ is a continuous function on the whole $H(\mathfrak{u})$ satisfying:

$$u(\mathfrak{u}t) = (\mathfrak{u}t)(0) = \mathfrak{u}(t) \qquad \forall t \in \mathbb{R}$$

and hence it is the unique function representing $\mathfrak{u}$ on $H(\mathfrak{u})$ at the point $\mathfrak{u}$ itself. This is a kind of minimal representation, as the next proposition suggests. Both the statement and the proof are variations of some results by Ščerbakov in [26].

Proposition 2.2. *A bounded and uniformly continuous* $\mathfrak{u}$ *is representable on* Θ *at* θ_0 *if and only if one of the following equivalent conditions is satisfied:*

(1) *for every choice of the involved time sequences, if* $d(\theta_0\tau_n, \theta_0 s_n) \to 0$ *then also* $|\mathfrak{u}\tau_n - \mathfrak{u}s_n| \to 0$*;*
(2) *a homomorphism* $\varphi : \overline{\theta_0\mathbb{R}} \to H(\mathfrak{u})$ *exists with* $\varphi(\theta_0) = \mathfrak{u}$*.*

The homomorphism in (2) is automatically surjective and hence $\overline{\theta_0\mathbb{R}} \succ H(\mathfrak{u})$. In particular, $H(\mathfrak{u})$ is the smallest compact flow where $\mathfrak{u}$ can be represented and is minimal when $\overline{\theta_0\mathbb{R}}$. Finally, the property (2) is the ultimate reason for which it is not convenient thinking of a representing function as defined on the whole Θ: indeed, extending a flow homomorphism from $\overline{\theta_0\mathbb{R}}$ to the whole Θ may be rather problematic and the equivalence with representability seems no longer guaranteed.

Proof. That representability implies (1) follows from the uniform continuity of the representing function, while (2) implies representability since the latter is preserved by extensions. To close the circle, it is enough to prove that (1) implies (2). For that, notice that the map:

$$\theta_0\tau \in \theta_0\mathbb{R} \;\mapsto\; \mathfrak{u}\tau \in H(\mathfrak{u})$$

is well defined and uniformly continuous because of (1), and hence extends to a unique continuous map $\varphi : \overline{\theta_0\mathbb{R}} \to H(\mathfrak{u})$. By definition $\varphi(\theta_0) = \mathfrak{u}$ and, to prove that it is a homomorphism, suppose $\theta_0\tau_n \to \theta$ and observe that $\theta_0(\tau + \tau_n) \to \theta\tau$ for every τ. Thus:

$$\varphi(\theta\tau) \;\leftarrow\; (\mathfrak{u}\tau_n)\tau \;\to\; \varphi(\theta)\tau$$

follows from the continuity of the extension. □

Remark 2.3. Let us consider a compact flow Θ pointed at θ_0. Since $\overline{\theta_0\mathbb{R}} = \Theta$, property (2) in Proposition 2.2 coincides with the notion of representability we used in the Introduction. The equivalence with (1) accounts for the inheritance of the recurrence properties which is typical of boundary value problems. Notice indeed that, in particular, the returning sequences are preserved in the sense that:

$$\theta_0\tau_n \to \theta_0 \qquad \text{implies} \qquad \mathfrak{u}\tau_n \to \mathfrak{u}\ .$$

In the last limit, the topology is that of the uniform convergence on compact sets. However, when Θ and hence $\mathfrak{u}$ are almost periodic, the uniform convergence on the whole real line can be equivalently used: this follows from the classical Bochner characterization of almost periodicity. Given $\varepsilon > 0$, denote now by $P_\varepsilon(\mathfrak{u})$ the class of the so–called ε–periods of $\mathfrak{u}$, that is the τ such that:

$$\sup_{t\in\mathbb{R}} |\mathfrak{u}(t+\tau) - \mathfrak{u}(t)| \ \leq \ \varepsilon$$

and define similarly $P_\varepsilon(\theta_0)$ for the flow line $t \to \theta_0 t$. The aforementioned inheritance of the returning sequences can be re–written as the inclusion:

$$\forall \varepsilon > 0 \ \exists \delta > 0 : \quad P_\delta(\theta_0) \subset P_\varepsilon(\mathfrak{u}) \ .$$

This inclusion is commonly accepted as a boundary–type data in the almost periodic framework, though the equivalent module containment of the Fourier coefficients is more frequently invoked. In the periodic framework, this is actually equivalent to the more classical inclusion:

$$P(\theta_0) \subset P(\mathfrak{u})$$

between the exact periods.

Next we tune representability on the application to differential equations. Given a compact flow Θ and two maps $A \in C(\Theta; \mathcal{L}(N))$ and $f \in C(\Theta; \mathbb{R}^N)$, we consider the family of differential equations:

$$\dot{x} = A(\theta t)x + f(\theta t) \tag{2.4}$$

where the parameter θ ranges in the whole Θ. Imagine that we know a solution $x_0(t)$ of the equation $(2.4)_{\theta_0}$ which is representable on Θ at the point θ_0, and denote by $x \in C\big(\overline{\theta_0\mathbb{R}}\,; \mathbb{R}^N\big)$ the representing function. For every $\theta \in \overline{\theta_0\mathbb{R}}$ the slice:

$$x_\theta(t) = x(\theta t)$$

is again a representable function on Θ, but now at the point θ. Moreover, standard arguments apply to show that it is a solution to the corresponding equation $(2.4)_\theta$. An equivalent but more intrinsic way to express this fact, is by introducing the *derivative along the flow*:

$$Dx(\theta) \ = \ \lim_{t\to 0} \frac{x(\theta t) - x(\theta)}{t}$$

and asking that x is a *continuous* solution of the abstract differential equation:

$$Dx = A(\theta)x + f(\theta) \tag{2.5}$$

though not in the whole Θ but in the compact subset $\overline{\theta_0\mathbb{R}}$ only. The desirable notion of *representable solution* is that, on the contrary, this happens *for every* θ: this is automatic when Θ is minimal or at least pointed at θ_0, which are the cases we are really interested in, but not in all the other cases. In Section 3 we will present a concrete example where the gap cannot be filled, using it to show that minimality of Θ is optimal for Favard theory: see Example 3.2 there. These comments justify the following definition, whose role is just expressing a very classical notion in a different and more compact guise.

Definition 2.4. *By a representable solution of (2.4) we mean a function $x \in C(\Theta; \mathbb{R}^N)$ such that Dx exists on the whole Θ and satisfies equation (2.5).*

Notice that Dx is also continuous from the equation: we denote this fact by saying that $x \in C^1(\Theta;\mathbb{R}^N)$. Similarly to representable functions at a given point of a compact flow, also representable solutions are preserved by extensions. To be more precise, assume that:

$$\Omega \succ \Theta$$

and use the involved epimorphism φ to *extend* equation (2.4) into:

$$\dot{z} = (A \circ \varphi)(\omega t)z + (f \circ \varphi)(\omega t) \qquad \omega \in \Omega \ . \tag{2.6}$$

Equation $(2.6)_\omega$ coincides with equation $(2.4)_{\varphi(\omega)}$ for every $\omega \in \Omega$, so that solutions are exactly the same. However, Ω has in general weaker recurrence properties than Θ and this may affect the representability of solutions. Of course all continuous solutions x of the equation (2.5) give rise to continuous solutions $z = x \circ \varphi$ of equation:

$$Dz = A(\varphi(\omega))z + f(\varphi(\omega))$$

but the latter may have other continuous solutions.
Before ending the section, we come back to the framework of the Introduction: next remark recall the classical construction, allowing to merge the single equation (1.1) into a parameterized family of the type (2.4).

Remark 2.5. Let $\mathfrak{A}$ and $\mathfrak{f}$ be bounded and uniformly continuous functions and consider the equation:

$$\dot{x} = \mathfrak{A}(t)x + \mathfrak{f}(t) \ . \tag{2.7}$$

The smallest compact flow where we can represent $\mathfrak{A}$ and $\mathfrak{f}$ is their joint hull $H(\mathfrak{A},\mathfrak{f})$. Notice also that $H(\mathfrak{A},\mathfrak{f})$ may be not minimal, also when $H(\mathfrak{A})$ and $H(\mathfrak{f})$ separately are. The representing functions at the point $(\mathfrak{A},\mathfrak{f})$ are respectively:

$$A(\mathfrak{B},\mathfrak{g}) = \mathfrak{B}(0) \qquad\qquad f(\mathfrak{B},\mathfrak{g}) = \mathfrak{g}(0)$$

and give the way to merge (2.7) into the continuous family of equation:

$$\begin{aligned} \dot{x} &= A\big((\mathfrak{B},\mathfrak{g})t\big)x + f\big((\mathfrak{B},\mathfrak{g})t\big) \\ &= \mathfrak{B}(t)x + \mathfrak{g}(t) \end{aligned} \qquad (\mathfrak{B},\mathfrak{g}) \in H(\mathfrak{A},\mathfrak{f}) \ . \tag{2.8}$$

Because of Proposition 2.2, the representable solutions we considered in the Introduction are the solutions of (2.7) which are representable functions on $H(\mathfrak{A},\mathfrak{f})$ at the point $(\mathfrak{A},\mathfrak{f})$: since $H(\mathfrak{A},\mathfrak{f})$ is pointed at $(\mathfrak{A},\mathfrak{f})$, they coincide with the representable solutions of which in Definition 2.4. Finally notice that the homogeneous equation:

$$\dot{x} = \mathfrak{A}(t)x$$

can be represented either in $H(\mathfrak{A},\mathfrak{f})$ or in $H(\mathfrak{A})$. Since $H(\mathfrak{A},\mathfrak{f}) \succ H(\mathfrak{A})$ by means of the obvious projection, to some extent the choice is immaterial.

3. Spectral theory and Favard theory

In this section we fix a compact flow Θ and a continuous map $A : \Theta \to \mathcal{L}(N)$ into the space of the $N \times N$ real matrices, with the aim of studying the class of linear homogeneous equations:

$$\dot{x} = A(\theta t)x \tag{3.1}$$

for θ varying in Θ. We say that Θ is the *hull* of equation (3.1) and, to refer to a single equation in the class, we use the symbol $(3.1)_\theta$.
We denote by $\phi_A(t,\theta)$ the Cauchy operator associated to equation (3.1), that is, the unique matrix solution satisfying:

$$\phi_A(0,\theta) = I \qquad \forall \theta \in \Theta\ .$$

With it, one can introduce a so–called linear skew–product flow:

$$(\theta,\xi)t \ = \ (\theta t, \phi_A(t,\theta)\xi)$$

namely an autonomous flow on $\Theta \times \mathbb{R}^N$, which retains all the features of the non–autonomous equation (3.1) and has been extensively studied in the literature. Before summarizing some results we need later on, let us introduce the following notation:

$$\mathcal{B}_\theta(A) \ = \ \big\{\xi \in \mathbb{R}^N : \sup_t |\phi_A(t,\theta)\xi| < +\infty\big\}$$

to denote the initial data producing bounded solutions to $(3.1)_\theta$ for a given $\theta \in \Theta$. As we will see, the consistency of these vector spaces determines most of the features we are interested in.
This is for instance the case of the classical notion of exponential dichotomy. The equation (3.1) or the matrix A are said to have this property over $\mathbb{R}$ when constants $K, \alpha > 0$ exist such that the following exponential estimates hold:

$$\begin{aligned} \big\|\phi_A(t,\theta)P_\theta\, \phi_A(s,\theta)^{-1}\big\| &\le Ke^{-\alpha(t-s)} \qquad \forall t \ge s \\ \big\|\phi_A(t,\theta)(I-P_\theta)\phi_A(s,\theta)^{-1}\big\| &\le Ke^{-\alpha(s-t)} \quad\ \ \forall s \ge t \end{aligned} \tag{3.2}$$

for every $\theta \in \Theta$ and for a suitable choice of the projections $P_\theta \in \mathcal{L}(N)$. Each P_θ is easily seen to be uniquely defined and to depend continuously on θ: see [5] and [23] for more details on the subject. When A has an exponential dichotomy, it is clear that:

$$\mathcal{B}_\theta(A) \ = \ \{0\} \qquad \forall \theta \in \Theta$$

while the converse is true for minimal Θ's, giving rise to a very convenient characterization of exponential dichotomy. The proof of this fact has been independently obtained in [23] and [28].
The Sacker–Sell spectrum $\sigma(A)$ has been introduced in [25] as the set of real λ's such that the equation:

$$\dot{x} = \big[A(\theta t) - \lambda I\big]x$$

does not admit an exponential dichotomy. In view of the previous characterization, when Θ is minimal:

$$\sigma(A) \ = \ \big\{\lambda \in \mathbb{R} : \mathcal{B}_\theta(A-\lambda I) \ne \{0\} \text{ for some } \theta \in \Theta\big\}\ . \tag{3.3}$$

For a general but *connected* compact flow Θ, the spectrum is a nonempty compact subset of $\mathbb{R}$, made by at most N closed intervals:

$$\sigma(A) \ = \ [a_1,b_1] \cup \cdots \cup [a_n,b_n] \qquad n \le N$$

which are possibly degenerate. To each interval $[a_k,b_k]$ there correspond an invariant vector subbundle of $\Theta \times \mathbb{R}^N$ which, roughly speaking, consists of the initial data of solutions having Lyapunov exponents in this interval. The Spectral Theorem in [25] asserts that these vector subbundles are independent and decompose the whole

$\Theta \times \mathbb{R}^N$. For our purposes, the most relevant of these subbundles is that associated to $0 \in \sigma(A)$. We denote it by $\mathcal{V}(A)$ and recall that its fibers are given by:

$$\mathcal{V}_\theta(A) \;=\; \left\{\xi \in \mathbb{R}^N : \lim_{t\to-\infty} e^{-\mu t}\phi_A(t,\theta)\xi = 0 = \lim_{t\to+\infty} e^{-\lambda t}\phi_A(t,\theta)\xi\right\} \tag{3.4}$$

where $\mu < a_k \leq b_k < \lambda$ defines any open neighborhood of the spectral interval $[a_k, b_k] \ni 0$ which avoids any other spectral interval. Its dimension is independent of θ since Θ is connected. We call it *Sacker–Sell* dimension of A and denote it by:

$$d_S(A) \;=\; \dim\left(\mathcal{V}_\theta(\mathrm{A})\right) .$$

Of course we agree that $\mathcal{V}_\theta(A) = \{0\}$ and $d_S(A) = 0$ when $0 \notin \sigma(A)$. With this agreement, the following inclusion becomes always true:

$$\mathcal{B}_\theta(A) \;\subset\; \mathcal{V}_\theta(A) \qquad\quad \forall\theta\in\Theta .$$

The second ingredient we need is Favard theory, which is also concerned with the behavior of the bounded solutions to (3.1). The so–called *Favard condition* reads as:

$$\inf\nolimits_t |\phi_A(t,\theta)\xi| > 0 \qquad\quad \forall\theta\in\Theta,\quad \forall\xi\in B_\theta(A)\setminus\{0\} \tag{F_A}$$

and then it is automatically satisfied for those θ where $B_\theta(A) = \{0\}$. Given a term $f \in C(\Theta; \mathbb{R}^N)$, Favard condition is crucial to guarantee the existence of representable solutions of the inhomogeneous equation:

$$\dot{x} = A(\theta t)x + f(\theta t) \tag{3.5}$$

in the sense that we already introduced in Section 2. Next result has been stated and proved in [7] for almost periodic flows, but the same proof actually works for minimal flows: see [21].

Theorem 3.1. *Assume Θ is minimal and (3.5) admits bounded solutions. If moreover (F_A) is satisfied then:*

$$Dx = A(\theta)x + f(\theta)$$

admits a solution $x \in C(\Theta; \mathbb{R}^N)$.

Notice that, due to minimality of Θ, either all the equations $(3.5)_\theta$ admit bounded solutions or no one does: this follows from standard compactness arguments and clarifies the previous statement. Next example shows that Favard Theorem may fail when Θ is not minimal, even if each equation $(3.5)_\theta$ admits bounded solutions and some of them are representable on Θ at some suitable point. This is rather in contrast with spectral theory that, to a large extent, applies to every connected compact flow.

Example 3.2. Let $\mathfrak{f} > 0$ be an even bounded and uniformly continuous function, which vanishes and is integrable at infinity, and set and assume:

$$\widetilde{\mathfrak{f}}(t) = \int_0^t \mathfrak{f}(s)\,ds \qquad\quad c = \widetilde{\mathfrak{f}}(+\infty) = -\widetilde{\mathfrak{f}}(-\infty) > 0 .$$

As explained in Remark 2.5 (see equation (2.8)) the inhomogeneous equation:

$$\dot{x} = \mathfrak{g}(t) \qquad\quad \mathfrak{g}\in H(\mathfrak{f}) \tag{3.6}$$

is of the type (3.5). We claim that (3.6) satisfies all the assumptions of Theorem 3.1 but for the non minimality of the pointed flow:

$$\Theta = H(\mathfrak{f}) = \{\mathfrak{f}\tau : \tau \in \mathbb{R}\} \cup \{0\}$$

while conclusion fails. Favard separation condition clearly holds, since $\dot{z} = 0$ is the unique homogeneous equation involved. Moreover, the general solution of (3.6) is:

$$x(t) \;=\; x_0 + \begin{cases} \widetilde{\mathfrak{f}}(t+\tau) - \widetilde{\mathfrak{f}}(\tau) & \mathfrak{g} = \mathfrak{f}\tau \\ 0 & \mathfrak{g} = 0 \end{cases} \tag{3.7}$$

where x_0 stands for the initial value. All these solutions are bounded. Those corresponding to $\mathfrak{g} = 0$ are constant and then representable on $H(\mathfrak{f})$ at any point $\mathfrak{g} \in H(\mathfrak{f})$ we like and in particular at $\mathfrak{g} = 0$, which is the only point we are interested in. However, since 0 is a fixed point in $H(\mathfrak{f})$, they possibly do not extend to representable solutions of (3.6), in the sense of Definition 2.4. After an integration, representable solutions are easily seen to be the $u \in C(H(\mathfrak{f}); \mathbb{R})$ satisfying:

$$u(\mathfrak{g}t) - u(\mathfrak{g}) \;=\; \begin{cases} \widetilde{\mathfrak{f}}(t+\tau) - \widetilde{\mathfrak{f}}(\tau) & \mathfrak{g} = \mathfrak{f}\tau \\ 0 & \mathfrak{g} = 0 \end{cases} .$$

Next we show that actually such solutions cannot exist. Suppose by contradiction that we have one and consider the slice $x(t) = u(\mathfrak{f}t)$. Since $\mathfrak{f}t \to 0$ as $t \to \pm\infty$ by continuity we should have:

$$x(+\infty) = u(0) = x(-\infty) .$$

On the other hand $\dot{x} = \mathfrak{f}(t)$ so that from (3.7) we have:

$$x(+\infty) - x(-\infty) = \{x_0 + \widetilde{\mathfrak{f}}(+\infty)\} - \{x_0 + \widetilde{\mathfrak{f}}(-\infty)\} = 2c > 0$$

which contradicts the previous conclusion, proving the claim.

Although necessity is an open problem, it is well known since longtime that (F_A) is optimal for the validity of Theorem 3.1, even if we restrict ourselves to the Θ's which are almost periodic: see [34], [10] and [18]. However, testing (F_A) in concrete situations is not always an easy task. A quite helpful tool for that has been provided in [33] and [1]. To introduce it, let us give the name:

$$d_\theta(A) \;=\; \dim\big(\mathcal{B}_\theta(A)\big)$$

to the number of independent bounded solutions to $(3.1)_\theta$ and consider their minimal value, together with the subset of Θ where it is attained:

$$d_F(A) \;= \min_{\theta\in\Theta} d_\theta(A) \qquad\qquad \Theta_F(A) = \big\{\theta \in \Theta :\, d_\theta(A) = d_F(A)\big\} .$$

The number $d_F(A)$ will be called the *Favard dimension* of A or of equation (3.1). The corresponding set $\Theta_F(A)$ is clearly an invariant subset of Θ. In [1] the following result is proved, that coincides with Theorem 1.1 in the concrete situation of the Introduction (see also Remark 2.5).

Theorem 3.3. *Assume Θ is minimal. Then the set $\Theta_F(A)$ is residual in Θ and $\theta \in \Theta_F(A)$ if and only if:*

$$\inf_t |\phi_A(t,\theta)\xi| > 0 \qquad\qquad \forall \xi \in \mathcal{B}_\theta(A) \setminus \{0\} .$$

Here *residual* must be understood as large in the classical Baire sense, that is a countable intersection of open and dense subsets of Θ. Because of this theorem, in minimal Θ's Favard condition (F_A) reads equivalently as $\Theta_F(A) = \Theta$ or, more explicitly, as the following purely dimensional fact:

$$d_\theta(A) = d_F(A) \qquad \forall \theta \in \Theta \; . \tag{3.8}$$

This is also equivalent to saying that the *bounded* fiber space:

$$\mathcal{B}(A) \; = \; \bigcup_\theta \{\theta\} \times \mathcal{B}_\theta(A)$$

is actually a subbundle of $\Theta \times \mathbb{R}^N$. Characterization (3.8) was already obtained in [33] and allows to quickly reobtain all the known cases were (F_A) is satisfied, like for instance a periodic Θ or a minimal Θ but $d_F(A) = N$: it will be crucial in many parts of the present paper, like for instance the proof of Theorem 7.5 and its consequences. It is worth stressing that the validity of this characterization and Theorem 3.3 depends on the minimality of Θ: see [1]. We conclude by pointing out a frequent case in the applications.

Lemma 3.4. *Assume Θ is minimal. If $0 \in \sigma(A)$ and $d_F(A) = 0$ then (F_A) fails.*

Proof. The spectral characterization (3.3) says that $d_{\theta_0}(A) \geq 1$ for some suitable $\theta_0 \in \Omega$. On the other hand, we know that $d_\theta(A) = 0$ for a residual set of θ's. Then the conclusion follows from Theorem 3.3. $\square$

4. Adjoint equation and extremal Favard dimensions

The adjoint equation to (3.1) is the equation:

$$\dot{y} = -A(\theta t)^T y \tag{4.1}$$

whose Cauchy operator is:

$$\phi^*_A(t,\theta) \; := \; \phi_{-A^T}(t,\theta) \; = \; \left\{\phi_A(t,\theta)^T\right\}^{-1} . \tag{4.2}$$

We agree to label with $*$ all the quantities when they are referred to the adjoint equation, instead of the direct one. For instance, we write:

$$\sigma^*(A) = \sigma(-A^T) \qquad d^*_{F/S}(A) = d_{F/S}(-A^T) \qquad (F^*_A) = (F_{-A^T})$$

to denote the spectrum, the Sacker–Sell/Favard dimensions and the Favard separation condition for the adjoint equation (4.1), respectively. A well known point is that spectral theory behaves quite smoothly with respect to adjunction, since for instance:

$$\sigma^*(A) = -\sigma(A) \qquad d^*_S(A) = d_S(A)$$

where connectedness of Θ is required for the second statement.
Whether similar conclusions hold for Favard theory, is one of the main questions we face in the present paper. For instance, this is certainly the case when Θ is periodic. We know indeed that (F^*_A) and (F_A) hold simultaneously and, using Floquet theory, it is not difficult to check that:

$$d^*_F(A) = d_F(A) \; .$$

The equality is true and (F_A) and (F_A^*) are equivalent also for scalar equations on a minimal Θ (see Section 5) while things are much more complicated in higher dimensions: see the discussion at the end of the section. On the contrary, it is not difficult to check that things may go wrong even in the scalar case, if Θ is not minimal: see for instance the subsequent Example 4.2.
Let us now consider the inhomogeneous equation:

$$\dot{x} = A(\theta t)x + f(\theta t) \ . \tag{4.3}$$

Our aim is to use Favard theory to construct representable solutions on Θ, but for that we first need to guarantee that bounded solutions for (4.3) do exist. By means of a straightforward integration by parts one sees that, if for some θ the equation $(4.3)_\theta$ admits a bounded solution, then the following condition:

$$\Big\langle f_\theta \, , \, \phi_A^*(\cdot,\theta)\zeta \Big\rangle \in BP(\mathbb{R};\mathbb{R}) \qquad \forall \zeta \in \mathcal{B}_\theta^*(A) \tag{4.4}$$

must be satisfied for the same θ. A large part of our investigation is deciding whether this condition is also *sufficient* to get a bounded solution for $(4.3)_\theta$.
However, before facing the question an obstruction must be removed. Notice indeed that, at least when Θ is minimal, having bounded solutions to (4.3) is independent of θ. If we pretend that $(4.3)_\theta$ has bounded solutions *if and only if* condition $(4.4)_\theta$ is satisfied, then also the latter must be independent of θ. Next lemma says that this is actually the case when the appropriate Favard condition is satisfied.

Lemma 4.1. *Assume Θ is minimal and (F_A^*) holds. If condition (4.4) is satisfied for some $\theta_0 \in \Theta$ then it is satisfied for all $\theta \in \Theta$.*

Proof. Use the minimality of Θ to find a sequence τ_n such that $\theta_0\tau_n \to \theta$. Since $\mathcal{B}_{\theta_0}^*(A)$ is finite dimensional, possibly passing to a subsequence the following limit:

$$L\zeta_0 \ := \ \lim_{n\to+\infty} \phi_A^*(\tau_n,\theta_0)\zeta_0$$

exists for every $\zeta_0 \in \mathcal{B}_{\theta_0}^*(A)$. Standard a priori estimates show that L is a linear map $\mathcal{B}_{\theta_0}^*(A) \to \mathcal{B}_\theta^*(A)$. In fact, we can say more: due to the validity of (F_A^*), the map L is an isomorphism. Consider indeed an arbitrary $\zeta_0 \neq 0$ and notice that:

$$\delta_0 := \inf_t |\phi_A^*(t,\theta_0)\zeta_0| > 0$$

due to (F_A^*). By the very definition of L we deduce $|L\zeta_0| \geq \delta_0 > 0$ and hence that $L\zeta_0 \neq 0$. Summing up, L is injective and to conclude it's enough to remember that $\mathcal{B}_{\theta_0}^*(A)$ and $\mathcal{B}_\theta^*(A)$ have the same dimension, again due to (F_A^*): see Theorem 3.3 and the comments thereafter.
Consider now an arbitrary $\zeta \in B_\theta^*(A)$ and define $\zeta_0 = L^{-1}(\zeta) \in B_{\theta_0}^*(A)$. By hypothesis we know that there exists $M \geq 0$ such that:

$$\left| \int_0^t \big\langle f(\theta_0 s), \phi_A^*(s,\theta_0)\zeta_0 \big\rangle \, ds \right| \leq M \qquad \forall t \in \mathbb{R} \ .$$

Thus, as a consequence of the classical cocycle identity we get:

$$\left| \int_0^t \big\langle f((\theta_0\tau_n)s), \phi_A^*(s,\theta_0\tau_n)\phi_A^*(\tau_n,\theta_0)\zeta_0 \big\rangle \, ds \right| = \left| \int_0^t \big\langle f(\theta_0(s+\tau_n)), \phi_A^*(s+\tau_n,\theta_0)\zeta_0 \big\rangle \, ds \right|$$

$$= \left| \int_0^{t+\tau_n} \big\langle f(\theta_0 s), \phi_A^*(s,\theta_0)\zeta_0 \big\rangle \, ds - \int_0^t \big\langle f(\theta_0 s), \phi_A^*(s,\theta_0)\zeta_0 \big\rangle \, ds \right| \leq 2M$$

independently of n. Passing to the limit as $n \to +\infty$ we finally obtain:

$$\left| \int_0^t \langle f(\theta s), \phi_A^*(s,\theta)\zeta \rangle \, ds \right| \leq 2M$$

for every $t \in \mathbb{R}$, showing that $(4.4)_\theta$ is satisfied. $\square$

The assumptions of Lemma 4.1 are optimal, even for scalar equations. The counter–example showing the optimality of (F_A^*) needs Koszlov functions, so that we postpone it till the end of Section 5. Next we show that things may go wrong, when minimality of Θ is weakened into being pointed at θ_0.

Example 4.2. As usual, the counter–example is done in the concrete framework described in Remark 2.5. Let $\mathfrak{a}$ be a bounded and uniformly continuous function such that:

$$\mathfrak{a}(-\infty) \;=\; 0 \;=\; \mathfrak{a}(+\infty) \qquad\qquad \widetilde{\mathfrak{a}}(\pm\infty) = -\infty \, .$$

where $\widetilde{\mathfrak{a}}(t) = \int_0^t \mathfrak{a}(s)\, ds$. On the nonminimal pointed flow:

$$H(\mathfrak{a}) \;=\; \{\mathfrak{a}\tau : \tau \in \mathbb{R}\} \cup \{0\}$$

we consider the inhomogeneous equation:

$$\dot{x} = \mathfrak{b}(t)x + 1 \qquad\qquad \mathfrak{b} \in H(\mathfrak{a}) \, .$$

The involved adjoint equation is $\dot{y} = -\mathfrak{b}(t)y$ and the general solution is:

$$y(t) \;=\; y_0 \begin{cases} e^{\widetilde{\mathfrak{a}}(\tau) - \widetilde{\mathfrak{a}}(t+\tau)} & \mathfrak{b} = \mathfrak{a}\tau \\ 1 & \mathfrak{b} = 0 \end{cases}$$

where y_0 stands for the initial value. All these equations satisfy the Favard separation condition, those corresponding to $\mathfrak{b} = \mathfrak{a}\tau$ since they have no bounded solutions but the trivial one. Consider now $\mathfrak{f} \equiv 1$: then condition (4.4) is satisfied at the point $\mathfrak{a} \in H(\mathfrak{a})$ while clearly is not at $0 \in H(\mathfrak{a})$.

Summing up, when Θ is minimal and (F_A^*) is satisfied, we can hope that the necessary condition (4.4) is also sufficient for (4.3) having bounded solutions. If this is the case and (F_A) is moreover satisfied, then Theorem 3.1 applies to guarantee that:

$$Dx = A(\theta)x + f(\theta) \tag{4.5}$$

admits continuous solutions. Whether or not such approach really succeeds, it turns out to depend on the matrix $A \in C(\Theta; \mathcal{L}(N))$: next we present a couple of known cases where this happens. The first one is:

$$0 \notin \sigma(A) \, .$$

In this case, everything works fine for a general compact flow Θ, possibly not minimal. Indeed condition $(4.4)_\theta$ is empty for every $\theta \in \Theta$ and equation $(4.3)_\theta$ is well known to admit a unique bounded solution x_θ for every choice of $f \in C(\Theta; \mathbb{R}^N)$. From the integral representation of such solutions one gets that (4.5) admits a unique continuous solution:

$$x(\theta) \;=\; \int_{-\infty}^0 P_\theta \, \phi_A(s,\theta)^{-1} f(\theta s)\, ds \;-\; \int_0^{+\infty} (I - P_\theta)\phi_A(s,\theta)^{-1} f(\theta s)\, ds$$

where the P_θ's are the projectors involved in the definition of exponential dichotomy, see formula (3.2). In all that, Favard theory plays no role; in any case, conditions

(F_A) and (F_A^*) both hold wiht Favard dimension zero.
The second case where things work fine is:

$$d_F(A) = N\,.$$

In [32] the following result is proved, for a general compact flow Θ: we sketch however the proof, since the formulation there is slightly different and to make explicit the role of the key condition (4.4) in our approach.

Proposition 4.3. *Assume $d_F(A) = N$ and (F_A) holds. Then $\sigma(A) = \{0\}$ and (F_A^*) also holds with $d_F^*(A) = N$. If moreover $(4.4)_\theta$ is satisfied for a given θ, then all the solutions of $(4.3)_\theta$ are bounded.*

Proof. Fix an arbitrary $\theta \in \Theta$. There exist constants $0 < m \leq M < +\infty$ such that:

$$m|\xi| \;\leq\; |\phi_A(t,\theta)\xi| \;\leq\; M|\xi|$$

for every $\xi \in \mathbb{R}^N$ and every $t \in \mathbb{R}$. The right inequality follows from $d_F(A) = N$ and the finite dimension of $\mathbb{R}^N$. The left inequality is obtained by contradiction, using the right one and the validity of (F_A). In particular, from the right inequality:

$$\lim_{t\to-\infty} e^{-\mu t}\,\phi_A(t\theta) \;=\; 0 \;=\; \lim_{t\to+\infty} e^{-\lambda t}\,\phi_A(t\theta)$$

for every $\mu < 0 < \lambda$, which in turn implies $\sigma(A) = \{0\}$. Using now (4.2) we deduce:

$$\frac{1}{M}|\xi| \;\leq\; |\phi_A^*(t,\theta)\xi| \;\leq\; \frac{1}{m}|\xi|$$

for every $\xi \in \mathbb{R}^N$ and every $t \in \mathbb{R}$. In particular (F_A^*) holds with $d_F^*(A) = N$. To conclude, assume now that $(4.4)_\theta$ is satisfied and observe that:

$$\Big\langle \int_0^t \phi_A^{-1}(s,\theta) f(\theta s)\,ds,\, \zeta \Big\rangle \;=\; \int_0^t \Big\langle f(\theta s),\, \phi_A^*(s,\theta)\zeta \Big\rangle\,ds$$

for every $\zeta \in \mathbb{R}^N$, showing that the integral on the left hand side is uniformly bounded in t. □

The optimality of (F_A) for the proposition is also proved in [32]. Due to Theorem 3.3, when Θ is minimal the validity of (F_A) actually follows from $d_F(A) = N$. In this case, Theorem 3.1 applies to get the desired conclusion.

Corollary 4.4. *Assume Θ is minimal and $d_F(A) = N$. Then (F_A) and (F_A^*) hold with $d_F^*(A) = N$ and, if moreover (4.4) is satisfied, then (4.5) admits continuous solutions.*

In the statement, no more reference to any particular θ is done: this is possible due to Lemma 4.1. As for Favard Theorem, next example shows that the minimality of Θ is optimal for the validity of Corollary 4.4.

Example 4.5. It is sufficient to look a bit closer at Example 3.2. The homogeneous equation $\dot z = 0$ is self–adjoint and fulfills the Favard separation condition with maximal Favard dimension $N = 1$. Condition (4.4) reads as:

$$\mathfrak{g} \in BP(\mathbb{R};\mathbb{R})$$

and then is satisfied for every $\mathfrak{g} \in H(\mathfrak{f})$, in view of the integrability of $\mathfrak{f}$. All the solutions to (3.6) are bounded, according to Proposition 4.3, but we know from Example 3.2 that there are no representable solutions.

The pending question is whether or not similar conclusions hold when Θ is minimal and:

$$0 \in \sigma(A) \qquad\qquad 0 < d_F(A) < N\ .$$

On the one hand, we will show that anything that can go wrong, will go wrong sometimes. For instance, there are cases where (F_A) and (F_A^*) are not equivalent. Moreover, when they are both satisfied, it may happen that $d_F(A) \neq d_F^*(A)$. The lowest dimensions are however exceptional, from this point of view.

Lemma 4.6. *Assume that Θ is minimal and $N \leq 2$. If (F_A) and (F_A^*) are both satisfied, then $d_F(A) = d_F^*(A)$.*

Proof. The thesis is trivially true when $0 \notin \sigma(A)$. The case $N = 1$ then follows from Corollary 4.4. The same corollary covers also the case $N = 2$ with $d_F(A) = 2$. It remains only the case $N = 2$ with $d_F(A) = 1$. We cannot have $d_F^*(A) = 0$ since otherwise $0 \notin \sigma(A)$, and we cannot have $d_F^*(A) = 2$ since otherwise Corollary 4.4 applies to the adjoint equation: thus necessarily $d_F^*(A) = 1$. □

The construction of the counter–examples rests on a classification of planar recurrent equations, which allows to localize all the possible troubles into a restricted number of normal forms: this is essentially done in Sections 7 and 8. As a consequence of this classification, it will be clear that there are no troubles for most planar equations.
Before producing all the mentioned counter–examples, we will distill Corollary 4.4 into the main positive result of the paper: this is done in Section 6 by means of some suitable change of variables. By using the above mentioned classification, in Section 9 we will also show that our positive result is the best one for planar equations. Next section is devoted to scalar equations, where clearly the change of variables plays no role.

5. Scalar equations

In all the section Θ stands for a given *minimal* flow and $a \in C(\Theta;\mathbb{R})$. We summarize some well known results about the scalar equation:

$$\dot{x} = a(\theta t)x \tag{5.1}$$

for $\theta \in \Theta$, and present a couple of less known or new results. It is well known [25] that $\sigma(a)$ is a single closed interval, which can be also described as the set of the mean values:

$$\int_\Theta a\, d\mu$$

where μ ranges over the invariant probability measures on Θ. The spectrum can degenerate to a single point, when for instance Θ is uniquely ergodic: in this case, the unique mean value is denoted by $\overline{a}$.
Coming back to the general minimal case, from the point of view of Favard theory we have to distinguish three different situations. The first one is:

$$0 \notin \sigma(a)\ . \tag{5.2}$$

The scalar equation (5.1) admits an exponential dichotomy, so that the Favard condition (F_a) holds in the minimal sense $d_F(a) = 0$.
A more interesting situation is obtained when we assume $0 \in \sigma(a)$ but for some θ the primitive:

$$\widetilde{a_\theta}(t) = \int_0^t a(\theta s)\, ds$$

is bounded in t. In this case, due to the minimality of Θ, the same is true for every $\theta \in \Theta$. In particular, Favard condition (F_a) is again satisfied but now in the maximal sense $d_F(a) = 1$. Moreover, a straightforward application of Favard Theorem 3.1 shows that a function $\widehat{a} \in C(\Theta; \mathbb{R})$ exists such that:

$$D\widehat{a} = a$$

in the whole Θ, see also [8] or [9]. We will use many times such function in the next sections and, to denote the situation where it does exist, we simply write:

$$a \in BP(\Theta; \mathbb{R}) \tag{5.3}$$

with no reference to the initial assumption $0 \in \sigma(a)$. The reason is that, as explained at the end of the previous section, the maximality of the Favard dimension automatically yields that $BP(\Theta; \mathbb{R})$ is a subset of:

$$C_0(\Theta; \mathbb{R}) \;=\; \Big\{ a \in C(\Theta; \mathbb{R}) : \; \sigma(a) = \{0\} \Big\} \; .$$

The next result guarantees that the inclusion is always strict when Θ is aperidoic: this conclusion will be used in the proof of Proposition 8.4 and is actually of common knowledge when Θ is almost periodic, but seems to have been overlooked in the general minimal case.

Lemma 5.1. *Let Θ be minimal aperiodic. Then $BP(\Theta; \mathbb{R})$ is a dense subset of $C_0(\Theta; \mathbb{R})$ of first Baire category in $C_0(\Theta; \mathbb{R})$.*

Since $C_0(\Theta; \mathbb{R})$ is a closed meager subset of $C(\Theta; \mathbb{R})$, the category information transfers from the former to the latter. Moreover, clearly $BP(\Theta; \mathbb{R}) = C_0(\Theta; \mathbb{R})$ when Θ is periodic.

Proof. In [27] Schwartzman proved that the closure of $BP(\Theta; \mathbb{R})$ is the intersection of all the spaces:

$$C_0^\mu(\Theta; \mathbb{R}) \;=\; \left\{ a \in C(\Theta; \mathbb{R}) : \int_\Theta a\, d\mu = 0 \right\}$$

where μ ranges over the invariant probability measure on Θ, independently of the minimality of the latter. But, in the minimal case, this intersection is $C_0(\Theta; \mathbb{R})$ due to the spectral characterization given at the beginning of the section: the proof of the density claim is then complete.
To prove the second part of the theorem, we exploit the minimality of Θ to measure the boundedness of primitives at a given $\theta_0 \in \Theta$ only. For every $M \geq 0$ let S_M be the subset of $C(\Theta; \mathbb{R})$ defined by:

$$\sup_t \left| \int_0^t f(\theta_0 s)\, ds \right| \;\leq\; M \; .$$

The S_M are closed in $C(\Theta; \mathbb{R})$ and their union along any diverging sequence of M's is exactly $BP(\Theta; \mathbb{R})$. Due to classical Baire's Theorem, to conclude it is enough to show that each S_M has empty interior in $C_0(\Theta;\mathbb{R})$. We claim that, for every $\tau > 0$ there exists $\varphi \in C_0(\Theta;\mathbb{R})$ such that:

$$\|\varphi\|_\infty = 1 \qquad \int_0^\tau \varphi(\theta_0 t)\,dt = \tau\ . \tag{5.4}$$

To construct this function, use the aperiodicity of Θ to guarantee that $\theta_0[0,\tau]$ does not intersect $\theta_0[2\tau,3\tau]$. Then choose an open neighborhood U of $\theta_0[0,\tau]$ such that $U \cap U(2\tau) = \emptyset$. Finally consider an Urysohn function satisfying:

$$\psi(\theta) \ = \ \begin{cases} 1 & \theta \in \theta_0[0,\tau] \\ 0 & \theta \notin U \end{cases}$$

and use it to define:

$$\varphi(\theta) \ = \ \psi(\theta) - \psi(\theta(2\tau))\ .$$

Conditions (5.4) are trivial to check. Moreover:

$$\int_\Theta \varphi\,d\mu \ = \ 0$$

for every invariant probability measure μ on Θ. Hence $\sigma(\varphi) = \{0\}$ follows from the characterization of the spectrum given at the beginning of the section, so proving the claim.

To conclude the proof, assume by contradiction that some S_M has nonempty interior in $C(\Theta;\mathbb{R})$. Choose $f \in S_M$ and $\varepsilon > 0$ such that $\|f - g\|_\infty \le \varepsilon$ implies $g \in S_M$. Then take any $\tau > 2M/\varepsilon$ and consider the corresponding function φ. Finally set $g = f + \varepsilon\varphi$ and notice that:

$$M \ \ge \ \int_0^\tau g(\theta_0 s)\,ds \ \ge \ -M + \varepsilon\tau \ > \ M$$

giving the desired contradiction. □

The two situations considered until now cover all the cases where (F_a) is satisfied, since no other Favard dimensions than 0 and 1 are available in the scalar case. Thus, in the third and last situation to be considered:

$$\begin{cases} 0 \in \sigma(a) \\ a \notin BP(\Theta;\mathbb{R}) \end{cases} \tag{5.5}$$

the Favard separation condition (F_a) must fail. This situation cannot occur when Θ is periodic. When on the contrary Θ is aperiodic, Lemma 5.1 says that this is the most common situation. In this case, the dimension $d_\theta(a)$ must vary in Θ, taking both the values 0 and 1. In particular the Favard dimension is $d_F(a) = 0$ and is attained at the residual invariant subset of Θ which we already denoted by $\Theta_A(a)$ in Section 3. Its complement, which is again invariant and dense though of first Baire category, is where the dimension is 1. In other words, $\theta_0 \notin \Theta_F(a)$ means:

$$\sup_t \ \widetilde{a_{\theta_0}}(t) < +\infty\ . \tag{5.6}$$

Since $a \notin BP(\Theta;\mathbb{R})$, the primitive $\widetilde{a_{\theta_0}}(t)$ must be unbounded from below; in fact, it is not difficult to prove that this must happens bilaterally, in the sense that:

$$\liminf_{t\to\pm\infty} \ \widetilde{a_{\theta_0}}(t) = -\infty\ .$$

A special but relevant case is given by:

$$\lim_{|t|\to+\infty} \widetilde{a_{\theta_0}}(t) = -\infty\ . \tag{5.7}$$

From a dynamical point of view, this means that equation $(5.1)_{\theta_0}$ has a nontrivial homoclinic solution to zero. Many concrete examples of such strong failure of the Favard separation condition can be found in the literature, typically aimed to show that the almost periodic world behaves very differently from the periodic one: see for instance [3], [34], [10] and [18].

There is another way for (F_a) to fail that is relevant to the present paper and, in some sense, is transversal to condition (5.7). Under the assumption (5.5), Johnson proved in [9] that:

$$\liminf_{t\to\pm\infty} \widetilde{a_\theta}(t) = -\infty \qquad \limsup_{t\to\pm\infty} \widetilde{a_\theta}(t) = +\infty \tag{5.8}$$

holds for all the θ in a residual subset of Θ. As a consequence of this fact, condition (5.7) can be only rarely satisfied in Θ: hereafter, we are interested in those cases where it is *never* satisfied. More precisely, to construct some crucial counter–examples in Section 7 and Section 9, we have to consider functions $a \in C(\Theta;\mathbb{R})$ satisfying (5.5) and such that moreover:

$$\forall\theta\in\Theta \qquad \lim_{|t|\to+\infty} \widetilde{a_\theta}(t) \quad \text{does not exist} . \tag{5.9}$$

For reasons which will be clear in a while, we call them *Kozlov functions*. To be more clear, saying that the limit in (5.9) exists means that both the limits as $t \to \pm\infty$ exist and, in addition, they are equal: in view of (5.5), such common value must be infinite. In other words, for a function to be Kozlov, either one of the two aforementioned limits does not exist, or both do exist but their values are different: moreover, this must happen for every $\theta \in \Theta$.

Starting from the seminal paper [13] by Kozlov, the Russian literature has provided a very interesting class of functions which satisfy simultaneously condition (5.5) and condition (5.9). The context there is the quasi–periodic one:

$$\Theta = \mathbb{T}^N = \mathbb{R}^N/\mathbb{Z}^N \qquad \theta t = \theta + \nu t$$

where $\nu \in \mathbb{R}^N$ is a nonresonant vector, namely its components are independent over $\mathbb{Z}$. Suppose now that $a \in C(\mathbb{T}^N)$ and $\overline{a} = 0$, so that the first part of condition (5.5) is satisfied. The question to be discussed is whether the primitive:

$$\widetilde{a}(t) = \int_0^t a(\nu s)\, ds$$

is *Poisson stable*, in the sense that it returns near to the initial position $\widetilde{a}(0) = 0$ for arbitrarily large times. More explicitly, the question is the existence of sequences of times such that:

$$t_n^\pm \to \pm\infty \qquad \widetilde{a}(t_n^\pm) \to 0 .$$

This is certainly true if $a \in BP(\mathbb{T}^N;\mathbb{R})$, since in this case $\widetilde{a}(t)$ is a quasi–periodic function with zero mean value. The problem becomes challenging when on the contrary $a \notin BP(\mathbb{T}^N;\mathbb{R})$: the general answer is well known to be negative (see for instance [34]) but Kozlov proved that it becomes positive, as soon as sufficiently smooth functions are considered.

Theorem 5.2. *Assume that $a \in C^d(\mathbb{T}^N;\mathbb{R})$ with d large enough (depending on N only) and that moreover $\overline{a} = 0$. Then the primitive $\widetilde{a}(t)$ is Poisson stable.*

This theorem has been proved by Kozlov in [13] for the case $N = d = 2$, with an elementary and very attractive approach. The general form is due to Moshchevitin in [16], where the history of the intermediate steps can be also traced. Two crucial facts have to be stressed, about the statement. The first one is that, since smoothness is not affected by translations in $\mathbb{T}^N$, the same conclusion clearly holds *for every* primitive:

$$\widetilde{a_\theta}(t) = \int_0^t a(\theta + \nu s)\, ds$$

whichever $\theta \in \mathbb{T}^N$ we take. The second fact is that no assumptions are made on the frequency vector ν but its nonresonance. For instance, highly nonresonant vectors ν are covered by the Kozlov result, in which case the smoothness of $a(\theta)$ implies the boundedness of the primitive: this is a classical result in the so–called K.A.M. theory. However, when ν is less nonresonant, it is possible to find analytic functions $a(\theta)$ with zero mean value and unbounded primitive: see [8] for a concrete example. Also these functions are in the scope of Theorem 5.2 and they are exactly those we are looking for: indeed, they satisfy condition (5.5) and condition (5.9) at the same time.

Passing to the adjoint equation, in view of the scalar character of (5.1) it simply reads:

$$\dot{y} = -a(\theta t) y \; . \tag{5.10}$$

It is manifest that none of the conditions (5.2), (5.3) and (5.5) is affected by the change in the sign of a. The Favard conditions (F_a) and (F_a^*) are then equivalent and, independently of their validity, the equality $d_F(a) = d_F^*(a)$ holds true. This equivalence restore the expected symmetry in the class of primitives of $a \in C(\Theta; \mathbb{R})$. Just to make a trivial example notice that, when Favard condition fails, there exists not only a θ_0 satisfying (5.6), but also a θ_1 such that:

$$\inf_t \, \widetilde{a_{\theta_1}}(t) > -\infty \; .$$

The oscillation properties (5.8) are also unaffected by changing the sign of a, and the same is true also for the notion of Kozlov function and Theorem 5.2.

Given an $f \in C(\Theta; \mathbb{R})$, consider finally the inhomogeneous equation:

$$\dot{x} = a(\theta t) x + f(\theta t) \; . \tag{5.11}$$

When Favard condition is satisfied, either $0 \notin \sigma(a)$ or Corollary 4.4 applies: thus condition (4.4) is sufficient for the existence of representable solutions. The same conclusion is also manifest from the explicit expression of the solutions to (5.11), that is:

$$x(t) = e^{\widetilde{a_\theta}(t)} \left\{ x(0) + \int_0^t e^{-\widetilde{a_\theta}(s)} f(\theta s)\, ds \right\} \; . \tag{5.12}$$

We end the section by proving that Favard separation condition is optimal for the validity of Lemma 4.1, as already anticipated in Section 4. Start noticing that, if the Favard condition fails, then the key condition (4.4) reads differently, according to θ. Precisely, it is trivially satisfied for every $\theta \in \Theta_F^*(a)$, while reads:

$$e^{-\widetilde{a_\theta}} f_\theta \, \in BP(\mathbb{R}; \mathbb{R}) \tag{5.13}$$

for every $\theta \notin \Theta_F^*(a)$. Suppose now a is a Kozlov function and $f \equiv 1$: clearly (F_a^*) fails and we claim that condition (5.13) also fails outside $\Theta_F^*(a)$, showing that the validity of (4.4) actually depends on θ. To prove the claim observe that for every $\theta \notin \Theta^*_F(a)$ we have:

$$\sup_t \{-\widetilde{a_\theta}(t)\} < +\infty \qquad\qquad \liminf_{t\to\pm\infty} \{-\widetilde{a_\theta}(t)\} = -\infty$$

and hence, by definition of Kozlov function, at least one of the two following conditions must be satisfied:

$$\limsup_{t\to-\infty} \{-\widetilde{a_\theta}(t)\} > -\infty \qquad\qquad \limsup_{t\to+\infty} \{-\widetilde{a_\theta}(t)\} > -\infty\ .$$

Since $\widetilde{a_\theta}$ is Lipschitz by construction, it is not difficult to see that the primitive of $e^{-\widetilde{a_\theta}}$ must explode in both cases.
We suspect that the Favard condition is also optimal for the validity of Corollary 4.4: finding an f such that (5.13) is satisfied but none of the (5.12) is representable on Θ, seems however a difficult task.

6. Change of variables and Fredholm Alternative

Given a matrix $A \in C(\Theta; \mathcal{L}(N))$ with $N > 1$, the standard way to study the dynamical properties of:

$$\dot{x} = A(\theta t)x \tag{6.1}$$

for $\theta \in \Theta$ is by making use of changes of variables. We will see that actually they are also relevant to define an appropriate notion of Fredholm Alternative. Here by change of variable, we mean the result of two consecutive steps. The first step is taking an epimorphism $\varphi : \Omega \to \Theta$ and considering the new equation:

$$\dot{z} = (A \circ \varphi)(\omega t)z \tag{6.2}$$

where now $\omega \in \Omega$. We say that $\mathcal{A} = A \circ \varphi$ *extends* A, using the same terminology for the corresponding equations, and we write:

$$\mathcal{A} \succ A\ .$$

By *minimal extension* we mean of course that Ω is minimal, which is only possible when Θ is minimal too. Notice that, since φ respects the flow, the extended equation $(6.2)_\omega$ is nothing else than the old equation $(6.1)_{\varphi(\omega)}$ for every $\omega \in \Omega$. Their solutions are then exactly the same, giving to extensions the look of a rather useless operation: the point is that, the larger Ω is the easier is finding a Lypapunov–Perron transformation giving equation (6.2) some convenient form. As usual, by *Lypapunov–Perron transformation* on Ω we mean a map $Q \in C(\Omega; \mathcal{GL}(N))$ such that DQ exists and is also continuous. The time–dependent change of variable $z = Q(\omega t)\, u$ then transforms the extended equation (6.2) into:

$$\dot{u} = B(\omega t)\, u \tag{6.3}$$

where the continuous matrix B is given by:

$$B(\omega) = Q(\omega)^{-1}\big\{A(\varphi(\omega))Q(\omega) - DQ(\omega)\big\}\ . \tag{6.4}$$

Such B is called *kinematic extension* of A and we write:

$$B > A$$

adding the adjective minimal to denote that the underling extension is minimal. Clearly extensions are just particular cases of kinematic extensions, but two different symbols turn out to be convenient. It is trivial to check that the above relations are reflexive and transitive. They become symmetric when φ is an isomorphism: in this case, we talk about *similarity* and *kinematic similarity*, writing $B \sim A$ for the latter. With this language, we can decompose $B > A$ into the chain:

$$B \;\sim\; \mathcal{A} = A \circ \varphi \;\succ\; A$$

where the first is a kinematic similarity of special type, the involved epimorphism being the identity on Ω.

A crucial point is that kinematic extensions do not affect neither spectral theory, nor Favard theory. Start indeed noticing that the Cauchy operator associated to equation (6.3) reads:

$$\phi_B(\omega, t) \;=\; Q(\omega t)^{-1}\, \phi_A\big(t, \varphi(\omega)\big)\, Q(\omega)\ .$$

Since Q and Q^{-1} are bounded on Ω, it is for instance clear that:

$$\sigma(B) = \sigma(A) \qquad\qquad d_S(B) = d_S(A)$$

where d_S stands for the Sacker–Sell dimension we introduced in Section 3. The same argument shows that (F_B) and (F_A) are equivalent and:

$$d_F(B) = d_F(A)$$

while to preserve the full force of Favard theory, namely Theorem 3.1 and Theorem 3.3, we need that Ω is a minimal extension of Θ. Observe moreover that similar conclusions hold for starred quantities, since:

$$Q^*(\omega) = \Big(Q(\omega)^{-1}\Big)^T \tag{6.5}$$

is again a Lyapunov–Perron transformation, taking now the first adjoint equation into the second one:

$$\dot y = -A(\theta t)^T y \qquad\qquad \dot v = -B(\omega t)^T\, v\ .$$

We are now ready to introduce a Fredholm–type Alternative for the recurrent setting. As already pointed out in the Introduction, admitting or not such alternative is a property of the matrix A: based on the arguments of Section 4, the following definition is expected to identify such good A's.

Definition 6.1. *We say that* $A \in C(\Theta; \mathcal{L}(N))$ *has the property* (C_A) *when, whatever* $f \in C(\Theta; \mathbb{R}^N)$ *we take, if condition:*

$$\Big\langle f_\theta\,,\, \phi_A^*(\cdot, \theta)\zeta \Big\rangle \in BP(\mathbb{R}; \mathbb{R}) \qquad\qquad \forall \zeta \in \mathcal{B}_\theta^*(A) \tag{6.6}$$

is satisfied for every $\theta \in \Theta$*, then equation:*

$$\dot x = A(\theta t)x + f(\theta t) \tag{6.7}$$

admits bounded solutions for every $\theta \in \Theta$*.*

Indeed, if moreover Θ is minimal and (F_A) holds, then Favard Theorem 3.1 guarantees that the abstract equation:

$$Dx = A(\theta)x + f(\theta)$$

admits continuous solutions, which is actually the final goal of all our efforts. In spite of that, condition (C_A) alone is unfit to solve the problem considered in the

Introduction: as it will be clear after reading Remark 6.3, we have to take care of the whole class of minimal extensions of A. This leads to the following definition.

Definition 6.2. *Let Θ be minimal and $A \in C(\Theta; \mathcal{L}(N))$. We say that A has the recurrent Fredholm Alternative property when:*

(a) *conditions (F_A) and (F_A^*) are satisfied;*
(b) *every minimal extension $\mathcal{A} \succ A$ has the property $(C_{\mathcal{A}})$.*

Though the equality $d_F(A) = d_F^*(A)$ turns out to be satisfied in all the cases where things work fine, we are not sure about its role in the theory we are trying to develop: because of that, we decided not to insert the equality into the definition, discussing it in all the results we prove.

Remark 6.3. As in the Introduction, consider the equation:

$$\dot{x} = \mathfrak{A}(t)x + \mathfrak{f}(t) \tag{6.8}$$

where the matrix $\mathfrak{A}$ and the vector $\mathfrak{f}$ are recurrent, and suppose now that the integral condition:

$$\langle \mathfrak{f}, y \rangle \in BP(\mathbb{R}; \mathbb{R}) \tag{6.9}$$

is satisfied for every bounded solution y of the adjoint equation $\dot{y} = -\mathfrak{A}(t)^T y$. As it is clear from Remark 2.5, the most suitable Θ for Definition 6.1 is $H(\mathfrak{A})$, the representing function being:

$$A(\mathfrak{B}) = \mathfrak{B}(0) \qquad \forall \mathfrak{B} \in H(\mathfrak{A}) \ . \tag{6.10}$$

On the other hand, to represent equation (6.8) we need at least $H(\mathfrak{A}, \mathfrak{f})$ and the problem stated in the Introduction is indeed deciding which matrices $\mathfrak{A}$'s have the following property: for every $\mathfrak{f}$ satisfying (6.9) and such that $H(\mathfrak{A}, \mathfrak{f})$ is minimal, equation (6.8) has a representable solution on $H(\mathfrak{A}, \mathfrak{f})$.
The glitch when trying to use condition (C_A) is that $H(\mathfrak{A}, \mathfrak{f})$ may become larger and larger by varying $\mathfrak{f}$, so that possibly it exceeds $H(\mathfrak{A})$. More generally, there is no hope to find a compact flows Θ satisfying $\Theta \succ H(\mathfrak{A}, \mathfrak{f})$ for every $\mathfrak{f}$.
On the contrary, $H(\mathfrak{A}, \mathfrak{f}) \succ H(\mathfrak{A})$ whatever $\mathfrak{f}$ is. The representing function of $\mathfrak{A}$ on $H(\mathfrak{A}, \mathfrak{f})$ is:

$$\mathcal{A}(\mathfrak{B}, \mathfrak{g}) = \mathfrak{B}(0) \qquad (\mathfrak{B}, \mathfrak{g}) \in H(\mathfrak{A}, \mathfrak{f}) \tag{6.11}$$

and $(C_{\mathcal{A}})$ is then the relevant condition to take care of the specific $\mathfrak{f}$. This suggests the effectiveness of the recurrent Fredholm Alternative property to find solutions of (6.8) that are representable on $H(\mathfrak{A}, \mathfrak{f})$: the proof will be done in Remark 6.11, where the role of the direct and the adjoint Favard separation conditions will be also explained.

A class of A's satisfying the recurrent Fredholm Alternative property has already been considered in Section 4. To be precise, there it was proved that, when A verifies one of the following two conditions:

$$0 \notin \sigma(A) \qquad d_F(A) = N$$

then A has the property (C_A). But these conditions are preserved under extensions, so that $(C_{\mathcal{A}})$ is also satisfied for every $\mathcal{A} \succ A$. To go further, we have to understand what happens to property (C_A) under extensions. This is done in the next two lemmas.

Lemma 6.4. *Assume that $\mathcal{A} \succ A$. If $(C_{\mathcal{A}})$ holds, then (C_A) holds too.*

Proof. Write $\mathcal{A} = A \circ \varphi$ where $\varphi : \Omega \to \Theta$ is an epimorphism. Take an arbitrary $f \in C(\Theta; \mathbb{R}^N)$ and suppose that $(6.6)_\theta$ is satisfied. Since $f \circ \varphi \in C(\Omega; \mathbb{R}^N)$ we are in the scope of condition $(C_{\mathcal{A}})$. To conclude, set $\varphi(\omega) = \theta$ and observe that:

$$(f \circ \varphi)_\omega = f_\theta \qquad \phi^*_{A\circ\varphi}(t, \omega) = \phi^*_A(t, \theta)$$

while equation:

$$\dot{z} = (A \circ \varphi)(\omega t)z + (f \circ \varphi)(\omega t)$$

is just another way to write equation $(6.6)_\theta$. □

Remark 6.5. It is worth stressing that the same lemma could have been false if property (C_A) were referred to representable solutions of (6.7), instead of bounded solutions. The reason is that, in principle, a bounded solution to $(6.6)_\theta$ may be representable on Ω but not on Θ. We need that Θ is minimal and (F_A) holds to deduce, via Favard theory, the existence of another bounded solution which is representable on Θ.

Lemma 6.6. *Assume that $B \sim A$. Then (C_B) is equivalent to (C_A).*

Proof. Let $\varphi : \Omega \cong \Theta$ be the underlying isomorphism and $Q : \Omega \to \mathcal{GL}(N)$ the Lyapunov–Perron transformation. Condition (C_B) refers to the existence of bounded solutions of the equation:

$$\dot{u} = B(\omega t)u + g(\omega t) \tag{6.12}$$

where B is given by (6.4) and g is an arbitrary element of $C(\Omega; \mathbb{R}^N)$. Since φ is invertible, it is easy to check that $f \mapsto g$ where:

$$g(\omega) = Q(\omega)^{-1} f(\varphi(\omega))$$

defines a bijection $C(\Theta; \mathbb{R}^N) \cong C(\Omega; \mathbb{R}^N)$. Assume from now on that g has this form and that θ is chosen according to $\varphi(\omega) = \theta$. The change of variables $x = Q(\omega t)u$ transforms equation $(6.6)_\theta$ into $(6.12)_\omega$ and defines a bijection between their bounded solutions. Moreover, the change of variables $y = Q^*(\omega t)v$ with Q^* as in (6.5) does exactly the same job for the adjoint equations. To conclude it is then enough to observe that:

$$\big\langle g(\omega t), v(t)\big\rangle = \big\langle Q(\omega t)^{-1} f(\varphi(\omega t)), v(t)\big\rangle = \big\langle f(\theta t), Q^*(\omega t)v(t)\big\rangle = \big\langle f(\theta t), y(t)\big\rangle$$

for every t. □

On the basis of the previous lemmas, the arbitrariness of the extensions in Definition 6.2 can be relaxed to some extent: this is done in next proposition, which is the technical key for the subsequent characterization of recurrent Fredholm Alternative property. We state the result for minimal flows but it also holds in other categories, like that of compact or almost periodic flows.

Proposition 6.7. *Let $B > A$ be a given minimal kinematic extension. The two following facts are equivalent:*

(1) *every minimal extension $\mathcal{A} \succ A$ has the property $(C_{\mathcal{A}})$;*
(2) *every minimal extension $\mathcal{B} \succ B$ has the property $(C_{\mathcal{B}})$.*

As a consequence, the matrix A has the recurrent Fredholm Alternative property if and only if B has it.

Proof. Let $\varphi : \Omega \to \Theta$ be the epimorphism underlying $B > A$, and $Q : \Omega \to \mathcal{GL}(N)$ the Lypapunov–Perron transformation allowing to write B as in (6.4).
Assume first (1) is satisfied and consider an arbitrary minimal extension $\mathcal{B} \succ B$. Write $\mathcal{B} = B \circ \psi$ where $\psi : \Sigma \to \Omega$ is the involved epimorphism. From $B \sim A \circ \varphi$ we deduce:

$$\mathcal{B} = B \circ \psi \sim (A \circ \varphi) \circ \psi = A \circ (\varphi \circ \psi) =: \mathcal{A} \succ A\ .$$

Since $(C_{\mathcal{A}})$ holds by assumption, the same is true for $(C_{\mathcal{B}})$ because of Lemma 6.6. Assume now (2) is satisfied and consider an arbitrary minimal extension $\mathcal{A} \succ A$. Write $\mathcal{A} = A \circ \psi$ where $\psi : \Sigma \to \Theta$ is the involved epimorphism. To prove that property $(C_{\mathcal{A}})$ is satisfied, we construct a special common extension of Ω and Σ. Consider the product flow in $\Omega \times \Sigma$ and denote by p and q the projections on Ω and Σ respectively, which are clearly epimorphisms. The subset:

$$\{(\omega, \sigma) \in \Omega \times \Sigma :\ \phi(\omega) = \psi(\sigma)\}$$

is closed invariant and nonempty, so that it admits a minimal subset M. Redefine p and q to be their restrictions to M: they are again surjective, since the target spaces Ω and Σ are minimal. Moreover, by construction the following identity:

$$\varphi \circ p \equiv \psi \circ q \tag{6.13}$$

holds in M. From $B \sim A \circ \psi$ we then deduce:

$$\mathcal{B} := B \circ p \sim (A \circ \varphi) \circ p = (A \circ \psi) \circ q = \mathcal{A} \circ q \succ \mathcal{A}$$

where the central equality depends on (6.13). Property $(C_{\mathcal{B}})$ holds by assumption. Lemma 6.6 and Lemma 6.4 then apply to show that $(C_{\mathcal{A} \circ q})$ and $(C_{\mathcal{A}})$ hold too. □

Remark 6.8. Let us come back to the setting of Remark 6.3. Because of Proposition 6.7, having the recurrent Fredholm Alternative property is an intrinsic feature of the matrix $\mathfrak{A}$, totally independent of the particular representation we choose for it.

We are now ready for the main result of the section.

Theorem 6.9. *Assume Θ is minimal and $A \in C(\Theta; \mathcal{L}(N))$. If:*

$$d_F(A) = d_S(A) \tag{6.14}$$

then A has the recurrent Fredholm Alternative property and $d_F(A) = d_F^*(A)$.

The minimality of Θ is optimal for the result: notice indeed that Corollary 4.4 is a particular case of Theorem 6.9, and hence Example 4.5 can be used to conclude. Finally, a partial inverse of Theorem 6.9 will be given in Section 9, based on the results of Sections 7 and 8.

Proof. Since we already know that conclusions are otherwise true, we suppose from now on that $0 \in \sigma(A)$ and $0 < d_S(A) < N$ and we set:

$$n = d_S(A) \qquad m = N - n$$

Let $\mathcal{V}(A)$ be the spectral subbundle corresponding to the spectral interval containing zero: see Section 3 and in particular formula (3.4). Assumption (6.14) says that:

$$\mathcal{B}_\theta(A) = \mathcal{V}_\theta(A) \qquad \forall \theta \in \Theta\ .$$

As a consequence, the involved spectral interval reduces to $\{0\}$ and (F_A) holds with $d_F(A) = n$. Consider now the spectral decomposition:

$$\Theta \times \mathbb{R}^N = \mathcal{V}(A) \oplus \mathcal{W}(A) \tag{6.15}$$

where $\mathcal{W}(A)$ is the direct sum of the invariant subbundles corresponding to the spectral intervals in $\sigma(A) \setminus \{0\}$. Palmer proved in [19] that $\mathcal{V}(A)$ and $\mathcal{W}(A)$ can be untwisted by means of a kinematic extension $B > A$ on a minimal $\Omega \succ \Theta$, see also [6]. That is, the resulting B is block–diagonal:

$$B = \begin{pmatrix} B_{\mathcal{V}} & 0 \\ 0 & B_{\mathcal{W}} \end{pmatrix}$$

with blocks $B_{\mathcal{V}}$ and $B_{\mathcal{W}}$ having dimensions n and m respectively, and the solutions of the two uncoupled equations:

$$\begin{cases} \dot{v} = B_{\mathcal{V}}(\omega t)\, v \\ \dot{w} = B_{\mathcal{W}}(\omega t)\, w \end{cases} \tag{6.16}$$

are, modulo the change of variables underlying B, the solutions of (6.1) that lie in $\mathcal{V}(A)$ and $\mathcal{W}(A)$ respectively. Because of that, we have:

$$d_F(B_{\mathcal{V}}) = n \qquad\qquad 0 \notin \sigma(B_{\mathcal{W}})\ . \tag{6.17}$$

Proposition 6.7 says that A has the recurrent Fredholm Alternative property if and only if B has it. To prove that the property is possessed by B, we have to consider an arbitrary epimorphism $\psi : \Sigma \to \Omega$ and the associated minimal extension $\mathcal{B} = B \circ \psi \succ B$. The decomposition (6.15) transfers to $\Sigma \times \mathbb{R}^N$ and the corresponding $\mathcal{B}_{\mathcal{V}}$ and $\mathcal{B}_{\mathcal{W}}$ satisfy again condition (6.17): since this is actually all we need to conclude, we can forget $\mathcal{B}$ and work directly with B.
The adjoint equation to (6.16) is of course:

$$\begin{cases} \dot{y} = -B_{\mathcal{V}}(\omega t)^T\, y \\ \dot{z} = -B_{\mathcal{W}}(\omega t)^T\, z \end{cases}$$

and a solution is bounded if and only if y is bounded and $z \equiv 0$. Corollary 4.4 applies to the first equation in (6.16) and says that $(F_{B_{\mathcal{V}}})$ and $(F^*_{B_{\mathcal{V}}})$ are satisfied with the same Favard dimensions:

$$d_F(B_{\mathcal{V}}) = n = d^*_F(B_{\mathcal{V}})\ .$$

Thus (F^*_B) holds with dimension $d^*_F(B) = d^*_F(B_{\mathcal{V}}) = n$. Since these conclusions are invariant by kinematic extensions, also (F_A) holds with $d^*_F(A) = n$.
Consider moreover an arbitrary $f \in C(\Omega; \mathbb{R}^N)$ and decompose it as $f = (g, h)$ according to $\mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^m$. The integral condition involved in property (C_B) reads as:

$$\left\langle g_\omega\, ,\, \phi^*_{B_{\mathcal{V}}}(\cdot, \omega)\zeta \right\rangle \in BP(\mathbb{R}; \mathbb{R}) \qquad\qquad \forall \zeta \in \mathcal{B}^*_\omega(B_{\mathcal{V}})$$

and, because of Proposition 4.3, is equivalent to the existence of bounded solutions for the equation $\dot{v} = B_{\mathcal{V}}(\omega t)v + g(\omega t)$. Let now w be the unique bounded solution of $\dot{w} = B_{\mathcal{W}}(\omega t)w + h(\omega t)$, which exists due to the second part of (6.17). Then $x = (v, w)$ is a bounded solution of:

$$\dot{x} = B(\omega t)x + f(\omega t)$$

concluding the proof that B has the property (C_B). □

Remark 6.10. A notion of almost periodic Fredholm Alternative can be introduced by specializing Definition 6.2, by taking Θ almost periodic and restricting to almost periodic extensions of Θ. Proposition 6.7 also holds in the category of almost periodic flows and then Theorem 6.9 states a sufficient condition for the validity of the almost periodic Fredholm Alternative. It is worth stressing that, however, the proof is anyway outside the almost periodic framework. The reason is that the extension $\Omega \succ \Theta$ where A diagonalizes by blocks may fail to be almost periodic, even when Θ is: see the already mentioned Palmer paper [19]. A similar comment applies to the triangularization procedure we will use in Section 9.

Remark 6.11. Theorem 1.2 in the Introduction is a corollary of Theorem 6.9. To show why, let us use the notations of Remark 6.3, setting $\Theta = H(\mathfrak{A})$ and denoting by A the matrix defined by (6.10). The hypothesis (1.8) of Theorem 1.2 writes now $d_F(A) = d_S(A)$ and hence from Theorem 6.9 we get that A has the recurrent Fredholm Alternative property.
Consider now the minimal compact flow $H(\mathfrak{A}, \mathfrak{f}) \succ H(\mathfrak{A})$ and, on it, the matrix $\mathcal{A} \succ A$ defined by (6.11) and the equation:

$$\dot{x} = \mathfrak{B}(t)x + \mathfrak{g}(t) \qquad (\mathfrak{B}, \mathfrak{g}) \in H(\mathfrak{A}, \mathfrak{f}) \ . \tag{6.18}$$

We claim that, due to the recurrent Fredholm Alternative property, each of the above equations possesses a bounded solution. To this aim, we have just to activate condition $(C_{\mathcal{A}})$, showing that the analogous of condition (6.6) is satisfied. That is, we have to prove that, whatever $(\mathfrak{B}, \mathfrak{g}) \in H(\mathfrak{A}, \mathfrak{f})$ we take, the integral condition:

$$\big\langle \mathfrak{g}, z \big\rangle \in BP(\mathbb{R}; \mathbb{R})$$

is satisfied for every bounded z solving the adjoint equation $\dot{z} = -\mathfrak{B}(t)^T z$. In general, this is a more restrictive condition than the assumption (1.6) of Theorem 1.2. The two conditions are actually equivalent in our concrete case, as a consequence of Lemma 4.1: we know indeed that the adjoint Favard separation condition (F_A^*) is satisfied and hence, as we explained at the beginning of the section, the same happens to $(F_{\mathcal{A}}^*)$.
The conclusion now follows from Favard theory. Every equation (6.18) has indeed a bounded solution and the direct Favard separation condition (F_A) is satisfied, so $(F_{\mathcal{A}})$ is satisfied too: Theorem 3.1 then applies to prove the existence of solutions of (6.18) which are representable on $H(\mathfrak{A}, \mathfrak{f})$.

As we already anticipated in the Introduction, an equivalent condition to (6.14) has already been used in [24]. Two assumptions are actually made by Sacker and Sell in this paper. The first one is that (F_A) holds, saying that the invariant fiber space $\mathcal{B}(A)$ is indeed a subbundle of $\Theta \times \mathbb{R}^N$. In general $\mathcal{B}(A)^\perp$ is not invariant but an induced flow can be defined on it, by projecting the Cauchy operator ϕ_A. The second assumption in [24] is that this induced flow has no bounded solutions but the trivial one. Sacker and Sell prove that these assumptions are equivalent to the existence of a trichotomy. That is, the stable and unstable fiber spaces $\mathcal{U}(A)$ and $\mathcal{S}(A)$ defined in [24] are also subbundles and moreover:

$$\Theta \times \mathbb{R}^N \ = \ \mathcal{U}(A) \oplus \mathcal{B}(A) \oplus \mathcal{S}(A) \ .$$

Since by construction $\mathcal{U}(A)$ and $\mathcal{S}(A)$ have spectra strictly to the left and to the right of 0 respectively, the decomposition implies $d_F(A) = d_S(A)$. Finally, the Spectral Theorem in [25] allows to reverse the conclusion.
In a couple of forthcoming papers, we will show that those parts of [2] and [17], which are concerned with Fredholm Alternative, are covered by Theorem 6.9: actually, we will also provide some extensions of these results. We end the present section by noticing that (6.14) implies:

$$d_F^*(A) = d_S^*(A) \ .$$

Thus the conclusion of Theorem 6.9 can be strengthened by saying that also the adjoint equation satisfies the recurrent Fredholm Alternative: this is another good reason, besides that considered in Remark 6.3, to require that both the direct and the adjoint Favard conditions enter in the definition of recurrent Fredholm Alternative.

7. Favard condition for planar triangular equations

In this and the next section we consider a *minimal flow* Ω and an upper triangular matrix:

$$B = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \tag{7.1}$$

where $a, b, c \in C(\Omega; \mathbb{R})$. The aim hereafter is to investigate the validity of the Favard separation condition for the corresponding planar equation:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} a(\omega t) & b(\omega t) \\ 0 & c(\omega t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{7.2}$$

The two scalar diagonal equations:

$$\dot{x}_1 = a(\omega t) x_1 \qquad\qquad \dot{x}_2 = c(\omega t) x_2 \tag{7.3}$$

are expected to drive, to some extent, the behavior of the whole equation (7.2). For instance, if x_1 solves the first equation in (7.3) then $x = (x_1, 0)$ solves (7.2), while if $x = (x_1, x_2)$ solves (7.2) then x_2 solves the second equation. The first fact proves the following lemma.

Lemma 7.1. *The inequality $d_F(B) \geq d_F(a)$ holds and moreover condition (F_B) implies condition (F_a).*

The inequality in the above statement may be strict. A partial converse of the lemma can be also easily proved.

Lemma 7.2. *If conditions (F_a) and (F_c) hold simultaneously, then condition (F_B) holds too.*

Concerning Favard dimensions, the inequality $d_F(B) \leq d_F(a) + d_F(c)$ is expected to hold: the claim will be proved for higher dimensional equations in Section 9, where an example of strict inequality will also be provided.

Proof. Let $x = (x_1, x_2)$ be a nontrivial bounded solution to (7.2). Then x_2 is a bounded solution to the second equation in (7.3) and, since (F_c) holds, either x_2 is nontrivial and separated from zero or $x_2 \equiv 0$. In the first case x is separated from zero too. In the second case $x = (x_1, 0)$ and the conclusion follows from the validity of (F_a). □

A crucial benefit of the triangular form is that (7.2) admits an exponential dichotomy if and only if the two scalar equations in (7.3) do. This is well known in the literature (see for instance [11]) and generalizes to the following property:

$$\sigma(B) = \sigma(a) \cup \sigma(c) .$$

The next two lemmas make use of exponential dichotomy to strengthen the conclusions of Lemma 7.2.

Lemma 7.3. *Assume that* $0 \notin \sigma(c)$*. Then* $d_F(B) = d_F(a)$ *and moreover* (F_B) *is equivalent to* (F_a)*.*

Proof. Let $x = (x_1, x_2)$ be a bounded solution of $(7.2)_\omega$. Since x_2 is a bounded solution to the second equation in $(7.3)_\omega$ and $0 \notin \sigma(c)$, we must have $x_2 \equiv 0$. That is $x = (x_1, 0)$ where x_1 is a bounded solution to the first equation in $(7.3)_\omega$. □

Lemma 7.4. *Assume that* $0 \notin \sigma(a)$*. Then* $d_F(B) = d_F(c)$ *and moreover* (F_B) *is equivalent to* (F_c)*.*

Proof. Let $x = (x_1, x_2)$ be a bounded solution of equation $(7.2)_\omega$. Since $0 \notin \sigma(a)$ and:

$$\dot{x}_1 = a(\omega t)x_1 + b(\omega t)x_2$$

the component x_1 is uniquely determined by components x_2. As a consequence, $x_2 \mapsto x$ is an isomorphism between the bounded solutions of the second equation in $(7.3)_\omega$ and the bounded solutions of $(7.2)_\omega$: this implies that $d_\omega(B) = d_\omega(c)$. Since ω is arbitrary, the thesis follows. □

It remains a pending question: *does condition* (F_B) *implies* (F_c)*?*
The answer is negative, but constructing an explicit counter–example is not a trivial task. To start with, notice that Lemma 7.1 and Lemma 7.4 give a couple of prescriptions: that is, condition (F_a) must be satisfied and moreover $0 \in \sigma(a)$. In other words, according to Section 5 we must choose:

$$a \in BP(\Omega; \mathbb{R}) \; . \tag{7.4}$$

According again to Section 5, the failure of (F_c) means:

$$0 \in \sigma(c) \qquad c \notin BP(\Omega; \mathbb{R}) \; . \tag{7.5}$$

To construct the desired counter–example, we add the further restriction that c is a Kozlov function, that is:

$$\forall \omega \in \Omega \qquad \lim_{|t| \to +\infty} \int_0^t c(\omega s)\, ds \quad \text{does not exist} \; . \tag{7.6}$$

See the final part of Section 2 for comments about this type of functions. Finally, we will choose a nontrivial b with sign:

$$b \geq 0 \qquad b \not\equiv 0 \; . \tag{7.7}$$

Theorem 7.5. *Under the assumptions (7.4)–(7.5) we have* $d_F(B) = 1$*. If moreover we assume (7.6)–(7.7), then condition* (F_B) *is satisfied.*

Proof. Use assumption (7.4) to construct $\widehat{a} \in C(\Omega; \mathbb{R})$ with $D\widehat{a} = a$, and then change the variables as follows:

$$x_1 = e^{\widehat{a}(\omega t)} u_1 \qquad\qquad x_2 = u_2$$

obtaining the new equation:

$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix} = \begin{pmatrix} 0 & b_*(\omega t) \\ 0 & c(\omega t) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad \text{where} \qquad b_* = b e^{-\widehat{a}} \; . \tag{7.8}$$

Denote by C the coefficients' matrix of this equation. Since $C \sim B$ and b_* satisfies again the sign condition (7.7), it is enough to prove the proposition for the equation (7.8). The general solution is:

$$u_1 \;=\; u_{10} + u_{20} \int_0^t e^{\widetilde{c_\omega}(s)} b_*(\omega s)\, ds \qquad\qquad u_2 \;=\; u_{20}\, e^{\widetilde{c_\omega}(t)}$$

where u_{10}, u_{20} are the initial data and we set:

$$\widetilde{c_\omega}(t) \;=\; \int_0^t c(\omega s)\, ds \;.$$

By taking $u_{20} = 0$, we get the constant solution $u = (u_{10}, 0)$. Thus:

$$\mathbb{R} \times \{0\} \;\subset\; \mathcal{B}_\omega(C)$$

for every $\omega \in \Omega$ and in particular $d_F(C) \geq 1$. We know from Section 5 that $\widetilde{c_\omega}$ is unbounded from above for a residual set of ω's. The same is then true for u_2, as long as $u_{20} \neq 0$. This implies that $d_\omega(C) < 2$ for the same ω's so that we may conclude that $d_F(C) = 1$. Because of Theorem 3.3, condition (F_C) is then satisfied if and only if:

$$d_\omega(C) = 1 \qquad\quad \forall \omega \in \Omega \;.$$

Assume now by contradiction that $d_{\omega_0}(C) = 2$ for some ω_0. This is equivalent to requiring that conditions:

$$\sup_t \widetilde{c_{\omega_0}}(t) \;<\; \infty \tag{7.9}$$

and:

$$\sup_t \left| \int_0^t e^{\widetilde{c_{\omega_0}}(s)} b_*(\omega_0 s)\, ds \right| \;<\; +\infty \tag{7.10}$$

are simultaneously satisfied. We claim that, due to the sign condition (7.7), condition (7.10) implies a much more restrictive condition than (7.9) that is:

$$\widetilde{c_{\omega_0}}\big(\pm\infty\big) = -\infty \;. \tag{7.11}$$

To prove the claim, start observing that (7.7) guarantees the existence of $\omega_1 \in \Omega$ where $b_*(\omega_1) > 0$. By continuity, we can always chose $\varepsilon > 0$ and $\delta > 0$ such that:

$$d(\omega, \omega_1) < \delta \qquad \text{implies} \qquad b_*(\omega) \geq \varepsilon \;.$$

We can now use Lemma 2.1 to find a relatively dense $\mathcal{T} \subset \mathbb{R}$ and $\rho > 0$ such that:

$$b_*\big(\omega_0(\tau + s)\big) \geq \varepsilon \tag{7.12}$$

for every $\tau \in \mathcal{T}$ and every $|s| < \rho$. Denote by $L > 0$ an inclusion length for $\mathcal{T}$ and assume that ρ is chosen in such a way that $\rho < L$.
Assume now by contradiction that $\widetilde{c_{\omega_0}}\big(+\infty\big) = -\infty$ is false, the other case being similar. Then we can find a constant M and a sequence of times $0 < t_n \to +\infty$ such that:

$$\widetilde{c_{\omega_0}}\big(t_n\big) \geq M \qquad\qquad t_{n+1} \geq t_n + 2L \;.$$

Use the relative density of $\mathcal{T}$ to find for every n a time:

$$\tau_n \;\in\; \mathcal{T} \cap \left[t_n - \frac{L}{2}\,,\, t_n + \frac{L}{2}\right]$$

and then use $\rho < L$ to see that all the intervals:

$$(\tau_n - \rho, \tau_n + \rho) \;\subset\; [t_n - L\,,\, t_n + L]$$

are pairwise disjoint. Moreover notice that, by Lipschitz–type estimates:

$$\widetilde{c_{\omega_0}}(\tau_n + s) \;\geq\; M - L\|c\|_\infty$$

for every $|s| < \rho$ and hence:

$$e^{\widetilde{c_{\omega_0}}(\tau_n+s)} b_*(\omega_0(\tau_n + s)) \;\geq\; \varepsilon e^{M-L\|c\|_\infty}$$

due to (7.12). Thus:

$$\int_0^{\tau_n+L} e^{\widetilde{c_{\omega_0}}(t)} b_*(\omega_0 t)\, ds \;\geq\; n\varepsilon e^{M-L\|c\|_\infty}$$

which explodes as $n \to +\infty$. This contradicts assumption (7.10), so proving that $\widetilde{c_{\omega_0}}(+\infty) = -\infty$ is true.
Summing up, we have that (7.11) is true: however, this fact contradicts the initial assumption (7.6) and hence concludes the proof. □

8. Fredholm Alternative and normal forms for planar triangular equations

As in the previous section, Ω is a minimal flow and B is given by (7.1). Hereafter we continue the analysis of the triangular planar equation (7.2) by studying how its properties are related to those of the adjoint equation:

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} -a(\omega t) & 0 \\ -b(\omega t) & -c(\omega t) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} . \tag{8.1}$$

The aim is to answer, in this particular case, all the questions raised in the Introduction for general recurrent equations. The strategy is finding a restricted set of normal forms, which captures the core of the problem: a number of changes of variables is necessary for that, but we stress that *no extensions of the hull Ω are required.*
The coefficient matrix in (8.1) is again triangular, though of lower type: the results of the previous section must be translated accordingly to this difference. To this aim, start remembering from Section 5 that Favard condition and dimension of scalar equations are preserved under adjunction. Thus by adapting Lemma 7.1 one finds that:

$$d_F^*(B) \geq d_F^*(c) = d_F(c)$$

and moreover (F_B^*) implies (F_c^*), which in turn is equivalent to (F_c). On the other hand, the same assumptions of Lemma 7.2 imply that (F_B^*) is also satisfied. Together with Lemma 4.6, this proves the following result.

Lemma 8.1. *Conditions (F_B) and (F_B^*) hold simultaneously if and only if (F_a) and (F_c) do the same. In this case moreover $d_F^*(B) = d_F(B)$.*

It remains to introduce the adjoint version of Lemma 7.3 and Lemma 7.4: it is not difficult to check that the same statements hold but for replacing (F_B) and $d_F(B)$ with the corresponding (F_B^*) and $d_F^*(B)$, while the proofs are swapped. Using these facts, we are now ready to introduce the first couple of normal forms.

Theorem 8.2. *Assume that $0 \in \sigma(B)$. Then conditions (F_B) and (F_B^*) are jointly satisfied if and only if B is kinetically similar on Ω to either:*

$$A_* = \begin{pmatrix} a_* & 0 \\ 0 & 0 \end{pmatrix} \qquad \textit{with} \quad 0 \notin \sigma(a_*) \tag{8.2}$$

or to:

$$B_* = \begin{pmatrix} 0 & b_* \\ 0 & 0 \end{pmatrix} \tag{8.3}$$

where $a_*, b_* \in C(\Omega;\mathbb{R})$.

Proof. That the normal forms A_* and B_* satisfy the direct and adjoint Favard conditions, follows from Lemma 8.1. Assume now that (F_B) and (F_B^*) are satisfied and use Lemma 8.1 to guarantee that (F_a) and (F_b) are satisfied too. Since by construction $0 \in \sigma(B) = \sigma(a) \cup \sigma(c)$ we can distinguish, according to Section 5, three different cases which we treat separately. In the first case:

$$0 \notin \sigma(a) \qquad\qquad c \in BP(\Omega;\mathbb{R})$$

it is possible to construct two functions $\widehat{c}$, $p \in C(\Omega;\mathbb{R})$ satisfying:

$$D\widehat{c} = c \qquad\qquad Dp = ap + be^{\widehat{c}}\ .$$

A direct computation shows that the change of variables:

$$x_1 = u_1 + p(\omega t)\, u_2 \qquad\qquad x_2 = e^{\widehat{c}(\omega t)} u_2$$

transforms the equation (7.2) into:

$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix} = \begin{pmatrix} a(\omega t) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} .$$

Thus B is kinematically similar to A_* in (8.2) with $a_* = a$.
The second case:

$$a \in BP(\Omega;\mathbb{R}) \qquad\qquad 0 \notin \sigma(c)\ .$$

is specular to the first one. The idea is to act exactly as in the first case but on the adjoint equation (8.1) instead of (7.2). The needed change of variables is now:

$$y_1 = e^{-\widehat{a}(\omega t)} v_1 \qquad\qquad y_2 = q(\omega t) v_1 + v_2$$

where $\widehat{a}$, $q \in C(\Omega;\mathbb{R})$ solve:

$$D\widehat{a} = a \qquad\qquad Dq = -cq - be^{-\widehat{a}}\ .$$

The final effect is transforming the adjoint equation (8.1) into:

$$\begin{pmatrix} \dot{v}_1 \\ \dot{v}_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -c(\omega t) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} .$$

After swapping the two components and taking the adjoint, we deduce that B is again kinematically similar to A_* in (8.2) but now with $a_* = c$.
The third and last case is:

$$a \in BP(\Omega;\mathbb{R}) \qquad\qquad c \in BP(\Omega;\mathbb{R})\ .$$

The diagonal change of variables:

$$x_1 = e^{\widehat{a}(\omega t)} u_1 \qquad\qquad x_2 = e^{\widehat{c}(\omega t)} u_2$$

transforms equation (7.2) into:

$$\begin{pmatrix} \dot u_1 \\ \dot u_2 \end{pmatrix} = \begin{pmatrix} 0 & b\, e^{\widehat{c}(\omega t)-\widehat{a}(\omega t)} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} .$$

Thus B is kinematically similar to B_* in (8.3) where $b_* = b\, e^{\widehat{c}-\widehat{a}}$. □

We are now ready to discuss which B's have the recurrent Fredholm Alternative property. We already know from Section 4 that this is true whenever:

$$0 \notin \sigma(B)$$

independently of any triangularity. In the more interesting case $0 \in \sigma(B)$, Proposition 6.7 guarantees that it's enough to discuss the normal forms determined in Theorem 8.2: this is done in the next two propositions.

Proposition 8.3. *Let Ω be minimal and A_* as in (8.2). Then:*

$$\sigma(A_*) = \{0\} \cup \sigma(a_*) \qquad d_F(A_*) = 1 = d_S(A_*)$$

and A_* has the recurrent Fredholm Alternative property.

The proof is straightforward and then omitted: it follows from the spectral characterization (3.4) and Theorem 6.9. The other normal form is where Fredholm Alternative may fail.

Proposition 8.4. *Let Ω be minimal and B_* as in (8.3). Then:*

$$\sigma(B_*) = \{0\} \qquad d_S(B_*) = 2 \qquad d_F(B_*) = \begin{cases} 1 & \text{if } \ b_* \notin BP(\Omega;\mathbb{R}) \\ 2 & \text{if } \ b_* \in BP(\Omega;\mathbb{R}) \end{cases}$$

and B_* has the recurrent Fredholm Alternative property if and only if $b_* \in BP(\Omega;\mathbb{R})$.

Proof. That $d_S(B_*) = 2$ follows from $\sigma(B_*) = \{0\}$, which in turn is trivially true. Since the general solution of $\dot x = B_*(\omega t)x$ is:

$$x_1 = x_{10} + x_{20}\int_0^t b_*(\omega s)\, ds \qquad x_2 = x_{20}$$

the computation of $d_F(B_*)$ also follows. Thus Theorem 6.9 applies to conclude when $b_* \in BP(\Omega;\mathbb{R})$.
Assume now that $b_* \notin BP(\Omega;\mathbb{R})$. We claim that (C_{B_*}) fails as soon as Ω is aperiodic: since any periodic Ω admits many aperiodic but nevertheless almost periodic extensions, it follows that B_* cannot have the recurrent Fredholm Alternative property. To prove the claim, start noticing that the general solution of the adjoint equation $\dot y = -B_*(\omega t)^T y$ is:

$$y_1 = y_{10} \qquad y_2 = y_{20} - y_{10}\int_0^t b_*(\omega s)\, ds$$

and then it is bounded if and only if $y_{10} = 0$. As a consequence, considered any two functions $f, g \in C(\Omega;\mathbb{R})$ and the corresponding equation:

$$\begin{cases} \dot x_1 = b_*(\omega t)x_2 + f(\omega t) \\ \dot x_2 = g(\omega t) \end{cases} \tag{8.4}$$

the necessary integral condition for the existence of bounded solutions writes as:

$$g \in BP(\Omega;\mathbb{R}) .$$

This is exactly condition (6.6) in Definition 6.2. Suppose now that this condition is satisfied and take $\widehat{g} \in C(\Omega;\mathbb{R})$ satisfying $D\widehat{g} = g$. The general solution of the second equation in (8.4) is $x_2 = x_{20} + \widehat{g}(\omega t)$ which is bounded for every initial data x_{20}. Inserting this information into the first equation, we get:

$$\dot{x}_1 = b_*(\omega t)\big\{x_{20} + \widehat{g}(\omega t)\big\} + f(\omega t)$$

and the existence of bounded solutions reads as:

$$b_*\big(x_{20} + \widehat{g}\big) + f \ \in \ BP(\Omega;\mathbb{R}) \tag{8.5}$$

for some suitable choice of the initial data x_{20}. When Ω is periodic, belonging to $BP(\Omega;\mathbb{R})$ is the same as having mean value zero: since the mean value of b_* is different from zero, we may always choose x_{20} such that the left hand side of (8.5) has zero mean value, and we have no problem. When on the contrary Ω is aperiodic, we can take $g \equiv 0$ and choose $f \in C(\Omega;\mathbb{R})$ such that:

$$\lambda b_* + f \notin BP(\Omega;\mathbb{R})$$

for every $\lambda \in \mathbb{R}$. The concrete choice of f depends on why $b_* \notin BP(\Omega;\mathbb{R})$. If the reason is that $0 \notin \sigma(b_*)$ then we take any f satisfying:

$$0 \in \sigma(f) \qquad\qquad f \notin BP(\Omega;\mathbb{R})$$

while the other case is obtained by swapping the conditions of b_* and f. That these choices are always possible is guaranteed by Lemma 5.1. □

Summing up, when B is such that both the Favard separation conditions (F_B) and (F_B^*) are satisfied, three different normal forms are possible: one of the type (8.2) and two of the type (8.3), with the recurrent Fredholm Alternative failing in one case only. Notice however that, in order that this statement makes full sense, associating a given B with its normal form must be an unambiguous process. This is also guaranteed by the two last propositions: they show indeed that the three normal forms differ either for spectral type features or for Favard type features, which are both invariant by kinematic extensions.

With Theorem 8.2, we exhausted all the cases where (F_B) and (F_B^*) hold jointly. Next we explore the complementary situation, with the aim of understanding if (F_B) and (F_B^*) are jointly false or may have different truth values, and moreover how the corresponding Favard dimensions are related. We already know from Lemma 8.1 that (F_a) and (F_c) cannot be jointly true: we start considering the case where they are jointly false.

Proposition 8.5. *Assume that:*

$$\begin{cases} 0 \in \sigma(a) \\ a \notin BP(\Omega;\mathbb{R}) \end{cases} \qquad\qquad \begin{cases} 0 \in \sigma(c) \\ c \notin BP(\Omega;\mathbb{R}) \end{cases} . \tag{8.6}$$

Then (F_B) and (F_B^*) are simultaneously false and $d_F(B) = 0 = d_F^*(B)$.

From (8.6) it is clear that $\sigma(B)$ is a single interval and hence $d_S(B) = 2$.

Proof. The first conclusion follows from Lemma 7.1; or even from the second conclusion, due to Lemma 3.4. Concerning the Favard dimension, fix an $\omega \in \Omega$ such that both the primitives of the slices a_ω and c_ω are both unbounded, either from below or from above. From Section 5 we know that such ω exists: indeed, it can be chosen in the intersection of two residual sets, which is itself residual. Consider now a bounded solution $x = (x_1, x_2)$ of the corresponding equation $(7.2)_\omega$. Since $\dot{x}_2 = c(\omega t)x_2$ we must have $x_2 \equiv 0$. Inserting this information into the first equation we get $\dot{x}_1 = a(\omega t)x_1$ showing that also $x_1 \equiv 0$. Hence $d_F(B) \le d_\omega(B) = 0$. The proof that $d_F^*(B) = 0$ proceeds in a similar way. □

The next step is looking at the case where the truth values of (F_a) and (F_c) are different. We start considering the case where one of them fails, while the other is satisfied but in a trivial way, that is, because of an exponential dichotomy.

Proposition 8.6. *Assume that either:*

$$0 \notin \sigma(a) \qquad \begin{cases} 0 \in \sigma(c) \\ c \notin BP(\Omega;\mathbb{R}) \end{cases} \tag{8.7}$$

or:

$$\begin{cases} 0 \in \sigma(a) \\ a \notin BP(\Omega;\mathbb{R}) \end{cases} \qquad 0 \notin \sigma(c)\ . \tag{8.8}$$

Then (F_B) and (F_B^*) are simultaneously false and $d_F(B) = 0 = d_F^*(B)$.

Nothing special can be said about $\sigma(B)$ and $d_S(B)$, since in general the two intervals $\sigma(a)$ and $\sigma(c)$ may overlap or not.

Proof. When the case (8.7) is considered, apply the direct and the adjoint versions of Lemma 7.4, by taking into account $d_F(c) = 0$, as explained in Section 5. For the case (8.8) do the same but with Lemma 7.3. □

Summing up, in all the cases considered until now we found that (F_B) is equivalent to (F_B^*) and moreover the corresponding Favard dimensions satisfy $d_F(B) = d_F^*(B)$. Notice that there are only two cases remaining, one being the case where (F_c) fails while (F_a) holds nontrivially:

$$a \in BP(\Omega;\mathbb{R}) \qquad \begin{cases} 0 \in \sigma(c) \\ c \notin BP(\Omega;\mathbb{R})) \end{cases} \tag{8.9}$$

and the other being the specular case:

$$\begin{cases} 0 \in \sigma(a) \\ a \notin BP(\Omega;\mathbb{R}) \end{cases} \qquad c \in BP(\Omega;\mathbb{R})\ . \tag{8.10}$$

In both cases $\sigma(B)$ is a single interval and hence $d_S(B) = 2$. Reversing the perspective, we know that: *if either the Favard separation conditions are not equivalent, or the Favard dimensions are different, then we are either in case (8.9) or in (8.10).* That these pathologies can really occur, is a consequence of the arguments developed in Section 7.

Proposition 8.7. *Under assumption (8.9) one has $d_F(B) = 1$ and condition (F_B^*) fails. If in addition $b \ge 0$ is nontrivial then:*

$$d_F^*(B) = 0$$

and, when moreover c is a Kozlov function, condition (F_B) holds.

By removing the assumption that c is a Kozlov function, it may happen that (F_B) fails too: see Example 8.9.

Proof. All the conclusions follow directly from the adjoint version of Lemma 7.1 and Theorem 7.5, but for $d_F^*(B) = 0$. To prove this last claim, recall from the proof of Theorem 7.5 that B is kinematically similar on Ω to the matrix:

$$C = \begin{pmatrix} 0 & be^{-\widehat{a}} \\ 0 & c \end{pmatrix} \tag{8.11}$$

so that we can equivalently show that $d_F^*(C) = 0$. The general solution of the adjoint equation $\dot{y} = -C(\omega t)^T y$ is:

$$\begin{cases} y_1(t) \;=\; y_{10} \\ y_2(t) \;=\; e^{-\widetilde{c_\omega}(t)} \left\{ y_{20} - y_{10} \displaystyle\int_0^t e^{\widetilde{c_\omega}(s)} b_*(\omega s)\, ds \right\} \end{cases} \tag{8.12}$$

where y_{10}, y_{20} are the initial data, $\widetilde{c_\omega}(t) = \int_0^t c(\omega s)\, ds$ and $b_* = b\, e^{-\widehat{a}}$ is again a nonnegative and nontrivial function. Now it is clear that $d_F^*(C) \leq 1$ since otherwise $d_F^*(B) = d_F^*(C) = 2$ and hence (F_B^*) would be satisfied, while it is not. Assume now by contradiction that $d_F^*(C) = 1$. Then for every $\omega \in \Omega$ there must exist initial data y_{10}, y_{20} which are not both zero and such that y_2 is bounded. Let ω be such that:

$$\liminf_{t \to \pm\infty} \widetilde{c_\omega}(t) = -\infty \qquad\qquad \limsup_{t \to \pm\infty} \widetilde{c_\omega}(t) = +\infty\ . \tag{8.13}$$

As explained in Section 5, there is a residual subset of such ω's due to the second part of condition (8.9). By taking a sequence $t_n \to +\infty$ where $\widetilde{c_\omega}(t_n) \to -\infty$, from the boundedness of y_2 it follows that:

$$y_{10} \int_0^{t_n} e^{\widetilde{c_\omega}(s)} b_*(\omega s)\, ds \;\to\; y_{20}\ .$$

This implies that $y_{10} \neq 0$, since otherwise also $y_{20} = 0$, so that we get:

$$\int_0^{t_n} e^{\widetilde{c_\omega}(s)} b_*(\omega s)\, ds \;\to\; y_{20}/y_{10}\ .$$

Since the integrand is nonnegative, we deduce integrability at $+\infty$. Doing the same argument at $-\infty$ we finally get:

$$\sup_t \left| \int_0^t e^{\widetilde{c_\omega}(s)} b_*(\omega s)\, ds \right| \;<\; +\infty\ .$$

This is exactly condition (7.10) in the proof of Theorem 7.5. From that proof we know that $\widetilde{c_\omega}(\pm\infty) = -\infty$ follows, contradicting (8.13) and hence proving our initial claim. □

With the previous proposition, we provided a class of examples where direct and adjoint Favard conditions are not equivalent *and* Favard dimensions are different: actually, the two facts are not independent.

Proposition 8.8. *If (F_B) holds and (F_B^*) fails, then (8.9) must be true and:*

$$d_F(B) = 1 \qquad\qquad d_F^*(B) = 0\ .$$

The example after the proof shows that the implication cannot be reversed: both the Favard conditions may fail with different Favard dimensions.

Proof. Condition (8.10) must be excluded, since otherwise (F_B) fails: thus (8.9) must be true. Due to Proposition 8.7, it only remains to show that the validity of (F_B) implies $d_F^*(B) = 0$. We prove this implication for the matrix C defined by (8.11), by using the arguments and the notations in the proofs of Proposition 8.7 and Theorem 7.5.
If we assume that $d_F^*(C) = 1$ then, for every $\omega \in \Omega$, there must exist initial data y_{10}, y_{20} which are not both zero and such that the solution y_2 given in (8.12) is bounded. Specializing this fact to any $\omega = \omega_0$ where:

$$\sup_t \widetilde{c_{\omega_0}}(t) < +\infty \tag{8.14}$$

we get that:

$$\sup_t \left| y_{20} - y_{10} \int_0^t e^{\widetilde{c_{\omega_0}}(s)} b_*(\omega_0 s)\, ds \right| < +\infty$$

and moreover $y_{10} \neq 0$. For the last conclusion notice that, if $y_{10} = 0$ then $y_2(t) = y_{20} e^{-\widetilde{c_{\omega_0}}(t)}$ which must be unbounded for every $y_{20} \neq 0$. Summing up, we must have:

$$\sup_t \left| \int_0^t e^{\widetilde{c_{\omega_0}}(s)} b_*(\omega_0 s)\, ds \right| < +\infty\ . \tag{8.15}$$

Conditions (8.14)–(8.15), which are exactly conditions (7.9) and (7.10) in the proof of Theorem 7.5, are verified: from that proof, it follows that $d_{\omega_0}(C) = 2$ and (F_C) fails. □

Example 8.9. Consider the same assumptions of Proposition 8.7 but for c being Kozlov, so that again (F_B^*) fails with $d_F^*(B) = 0$ and $d_F(B) = 1$. Suppose moreover that an $\omega_0 \in \Omega$ exists such that:

$$\limsup_{|t| \to +\infty}\ |t|^{-\alpha}\, \widetilde{c_{\omega_0}}(t)\ <\ 0$$

for some suitable $0 < \alpha < 1$. Many examples of this type are known in the literature, when Ω is almost periodic: see for instance [22] or [34]. Conditions (8.14)–(8.15) are clearly satisfied and, as in the proof of Proposition 8.7, we may conclude that (F_C) fails: thus (F_B) fails too.

Similar results to Proposition 8.7 and Proposition 8.8 hold when (8.9) is replaced by the specular assumption (8.10): now it is (F_B) that fails and (F_B^*) that holds under some suitable conditions. In this case, the correct normal form for B is:

$$\begin{pmatrix} a & b\, e^{\widehat{c}} \\ 0 & 0 \end{pmatrix} \tag{8.16}$$

while formula (8.11) gives the normal form of the matrix associated to the adjoint equation. Notice that the two normal forms (8.11) and (8.16) may be kinematically similar: for instance, this is certainly the case when $b \equiv 0$, since we obtain the normal forms one from the other by swapping the variables. However, it is clear that this cannot happen for the counter–examples we are really interested in: though their spectrum may be equal, no change of variables can invert the truth values of the Favard conditions. A similar conclusion holds for the Favard dimensions.

9. Conclusions and complementary results

The aim of this final section is twofold. The first one is to conclude the discussion about the recurrent Fredholm Alternative for the equation:

$$\dot{x} = A(\theta t)x$$

where Θ is a minimal flow and $A \in C(\Theta; \mathcal{L}(N)$. A sufficient condition for A having such property has been stated in Theorem 6.9: here, on the basis of the results of Section 8, we show that such condition is also necessary at least for low values of the Saker–Sell dimension.

Theorem 9.1. *Assume that Θ is minimal and $d_S(A) \le 2$. If A has the recurrent Fredholm Alternative property then $d_F(A) = d_S(A)$.*

Notice that, due to Theorem 6.9, the equality $d_F^*(A) = d_F(A)$ is automatically satisfied.

Proof. Suppose that $0 \in \sigma(A)$ and $0 < n = d_S(A) < N$, since otherwise we already know that the thesis is true. Let now $B > A$ be the same kinematic extension on $\Omega \succ \Theta$ as the one we considered in the proof of Theorem 6.9. We recall that B is block–diagonal:

$$B \;=\; \begin{pmatrix} B_{\mathcal{V}} & 0 \\ 0 & B_{\mathcal{W}} \end{pmatrix}$$

where the block $B_{\mathcal{V}}$ has dimension $n = d_S(A)$ and:

$$0 \in \sigma(B_{\mathcal{V}}) \qquad\qquad 0 \notin \sigma(B_{\mathcal{W}}) \; .$$

Since nontrivial bounded solutions are confined to the first block, it is clear that:

$$d_F(A) = d_F(B) = d_F(B_{\mathcal{V}}) \qquad\qquad d_S(A) = d_S(B) = d_S(B_{\mathcal{V}}) \; .$$

Concerning the recurrent Fredholm Alternative property, Proposition 6.7 guarantees that it is possessed by B and we claim that the same is true for the block $B_{\mathcal{V}}$. On the one hand, the validity of (F_A) and (F_A^*) implies that of $(F_{B_{\mathcal{V}}})$ and $(F_{B_{\mathcal{V}}}^*)$. Suppose now by contradiction that, given a minimal extension $\psi : \Sigma \to \Omega$ and defined $\mathcal{B} = B \circ \psi$, the corresponding block $\mathcal{B}_{\mathcal{V}}$ does not satisfy property $(C_{\mathcal{B}_{\mathcal{V}}})$. That is, for some $g \in C(\Sigma; \mathbb{R}^N)$ condition:

$$\left\langle g_\sigma \, , \, \phi^*_{\mathcal{B}_{\mathcal{V}}}(\cdot, \sigma)\zeta \right\rangle \in BP(\mathbb{R}; \mathbb{R}) \qquad \forall \zeta \in \mathcal{B}^*_\sigma(\mathcal{B}_{\mathcal{V}}) \tag{9.1}$$

is satisfied but the equation:

$$\dot{v} = \mathcal{B}_{\mathcal{V}}(\sigma t)v + g(\sigma t)$$

does not admit any bounded solution. If we set $f = (g, 0) \in C(\Sigma; \mathbb{R}^N)$, then it is clear that also the equation:

$$\dot{x} = \mathcal{B}(\sigma t)x + f(\sigma t)$$

does not admit any bounded solution. On the other hand, for such f, the necessary condition involved in $(C_{\mathcal{B}})$ is exactly (9.1) and hence is satisfied: this contradicts the validity of $(C_{\mathcal{B}})$, proving the claim.

Let us finally use the assumption $1 \le n \le 2$ in connection with the fact that $B_{\mathcal{V}}$ has the recurrent Fredholm Alternative property. Notice that, since $0 \in \sigma(B_{\mathcal{V}})$ and $(F_{B_{\mathcal{V}}})$ is satisfied, we must have $d_F(B_{\mathcal{V}}) > 0$. When $n = 1$ this automatically gives

the desired equality $1 = d_F(B_{\mathcal{V}}) = d_F(A)$.
Assume now that $n = 2$. Since also $(F^*_{B_{\mathcal{V}}})$ is satisfied, Theorem 8.2 says that $B_{\mathcal{V}}$ is kinematically similar to either A_* or B_* as defined by (8.2) and (8.3) respectively. But A_* has be excluded, since $d_S(A_*) = 1$ from Proposition 8.3, while we know that $d_S(B_{\mathcal{V}}) = d_S(A) = n = 2$. Thus $B_{\mathcal{V}}$ must be kinematically similar to B_*. Since such B_* inherits the recurrent Fredholm Alternative property from $B_{\mathcal{V}}$, Proposition 8.4 guarantees that we are in the case $d_F(B_*) = 2$: thus the desired equality $2 = d_F(B_*) = d_F(B_{\mathcal{V}}) = d_F(A)$ is again satisfied. □

Remark 9.2. Theorem 1.3 in the Introduction follows from Theorem 9.1. Suppose indeed that the assumptions of Theorem 1.3 are satisfied, then set $\Theta = H(\mathfrak{A})$ and define:

$$A(\mathfrak{B}) = \mathfrak{B}(0) \qquad \forall \mathfrak{B} \in H(\mathfrak{A}) \ .$$

Thus Θ is minimal and Theorem 9.1 applies to conclude, as soon as we show that the hypotheses of Theorem 1.3 are enough to guarantee that A has the recurrent Fredholm Alternative property.
To prove it, take an arbitrary minimal compact flow Ω with a flow epimorphism $\varphi : \Omega \to H(\mathfrak{A})$ and define $\mathcal{A} = A \circ \varphi$. Take moreover an arbitrary $f \in C(\Omega; \mathbb{R}^N)$ such that:

$$\Big\langle f_\omega \, , \, \phi^*_{\mathcal{A}}(\cdot, \omega)\zeta \Big\rangle \in BP(\mathbb{R}; \mathbb{R}) \tag{9.2}$$

for every $\omega \in \Omega$ and every $\zeta \in \mathcal{B}^*_\omega(\mathcal{A})$. All we need to prove is that, as a consequences of the hypotheses of Theorem 1.3, the equation:

$$\dot{x} \ = \ \mathcal{A}(\omega t)x + f(\omega t) \tag{9.3}$$

has a bounded solution for every $\omega \in \Omega$. Due to the minimality of Ω, it is sufficient to prove it for a single element of Ω: it turns out to be convenient to choose such element as any $\omega_0 \in \varphi^{-1}(\mathfrak{A})$, a set which is nonempty since φ is surjective. With this choice we have indeed $\mathcal{A}(\omega_0 t) = A(\mathfrak{A}t) = \mathfrak{A}(t)$ and, if we define:

$$\mathfrak{f}(t) = f(\omega_0 t)$$

we can use it as the inhomogeneous term inside equation (1.1) in the Introduction. Since by construction the pair $(\mathfrak{A}, \mathfrak{f})$ is representable on the minimal Ω, Proposition 2.2 implies that $\Omega \succ H(\mathfrak{A}, \mathfrak{f})$ and hence shows that $H(\mathfrak{A}, \mathfrak{f})$ is minimal too. Moreover, again by construction equation (1.1) coincides with $(9.3)_{\omega_0}$, while condition (1.6) is nothing else than $(9.2)_{\omega_0}$ and hence is satisfied. The hypotheses of Theorem 1.3 then say that (1.1) alias $(9.3)_{\omega_0}$ admits bounded solutions, as we claimed.

The second aim of the present section is trying to extend to higher dimensions some results we proved for planar triangular equations, understanding to which extent this is possible. We consider:

$$\dot{x} = B(\omega t)x \tag{9.4}$$

where B is an *upper triangular* $N \times N$ matrix with entries $b_{hk} \in C(\Omega; \mathbb{R})$. Here Ω stands for a minimal flow, as usual, and we remember that:

$$\sigma(B) \ = \ \sigma(b_{11}) \cup \ \cdots \ \cup \sigma(b_{NN}) \ . \tag{9.5}$$

Next proposition provides a general estimate from above of the Favard dimension of B, which extends those given in Section 7 and is independent of the validity of (F_B).

Proposition 9.3. *The following estimate holds:*

$$d_F(B) \;\le\; \#\left\{h : \; b_{hh} \in BP(\Omega;\mathbb{R})\right\} . \tag{9.6}$$

Proof. The proof is by induction on the dimension N of the matrix B. The case $N = 1$ is trivial. Assume that the conclusion is true for $N - 1$ and let B be an $N \times N$ matrix. In particular, we know that:

$$d_F(C) \;\le\; d \;=\; \#\left\{h \le N-1 : \; b_{hh} \in BP(\Omega;\mathbb{R})\right\}$$

where C is the square matrix obtained by taking the first $N - 1$ rows and columns of B. To count the bounded solutions to (9.4), begin by noticing that their last component is a bounded solution to:

$$\dot{x}_N \;=\; b_{NN}(\omega t)\, x_N \; . \tag{9.7}$$

We may then distinguish two different situations. The first one is when:

$$0 \notin \sigma(b_{NN}) \qquad \text{or} \qquad \left\{ \begin{array}{l} 0 \in \sigma(b_{NN}) \\ b_{NN} \notin BP(\Omega;\mathbb{R}) \end{array} \right. \; .$$

In this case, there exists a set Ω_0 which is (at least) residual in Ω such that, if x_N is a bounded solution to $(9.7)_\omega$ and $\omega \in \Omega_0$, then $x_N \equiv 0$. Thus for every $\omega \in \Omega_0$ the bounded solutions to $(9.4)_\omega$ are indeed in the form $x = (u, 0)$ where u is a bounded solutions to:

$$\dot{u} = C(\omega t) u \; . \tag{9.8}$$

Consider now just the $\omega \in \Omega_0 \cap \Omega_F(C)$, which is again residual and then nonempty. We have:

$$d_F(B) \;\le\; d_\omega(B) \;=\; d_\omega(C) \;=\; d_F(C) \;\le\; d$$

and the conclusion follows from the fact that, by construction, the right hand side of (9.6) is exactly d.

The second case is when instead $b_{NN} \in BP(\Omega;\mathbb{R})$. It is not difficult to guess and prove that, in this case:

$$d_\omega(B) = d_\omega(C) + 1$$

for every $\omega \in \Omega$. Then the conclusion follows by taking the minimum over Ω. □

The inequality in (9.6) becomes an equality for diagonal matrices but it may be strict otherwise, even when (F_B) and (F_B^*) are both satisfied and we are in the planar case. The easiest example is:

$$B \;=\; \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

where the right hand side of (9.6) is 2 but $d_F(B) = 1$. A similar result with lower Favard dimensions can be obtained, when Favard condition fails. For instance we may take:

$$B \;=\; \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$$

where $a \in C(\Omega;\mathbb{R})$ satisfies $0 \in \sigma(a)$ and $a \notin BP(\Omega;\mathbb{R})$, while $b \in C(\Omega;\mathbb{R})$ is such that the equation:

$$\dot{x} = a(\omega t)x + b(\omega t)$$

does not admit bounded solutions for any $\omega \in \Omega$. Such term b always exists since otherwise $0 \notin \sigma(a)$, due to the functional characterization of exponential dichotomies given in [15]: see also [5] for a more direct approach. By taking into

account that $a \notin BP(\Omega;\mathbb{R})$, it is not difficult to check that $d_F(B) = 0$ while the right hand side of (9.6) is 1.
The same ingredients of Proposition 9.3 may be used for a test, which extends Lemma 7.2 to higher dimensions.

Proposition 9.4. *For every $1 \le h \le N$ assume that $b_{hh} \in BP(\Omega;\mathbb{R})$ whenever $0 \in \sigma(b_{hh})$. Then (F_B) and (F_B^*) hold jointly.*

An equivalent way to formulate the assumption is saying that, for each diagonal element b_{hh}, the scalar Favard condition $(F_{b_{hh}})$ is satisfied. As a consequence of (9.5), either 0 does not belong to $\sigma(B)$ or is an isolated point of it.

Proof. The proof is again by induction on the dimension N of the matrix B, and is trivial when $N = 1$. Suppose now that the conclusion is true for matrices of dimension $N-1$ and take B of dimension N. To prove that (F_B) holds, start considering the same matrix C as in the proof of Proposition 9.3: by the inductive hypothesis (F_C) is true. Assume now that x is a nontrivial bounded solution to (9.4) and distinguish two cases. If $x_N \equiv 0$ then $x = (u, 0)$ where u is a nontrivial bounded solution to (9.8): since (F_C) holds, u and hence x are separated from zero. On the other hand, if $x_N \not\equiv 0$ then $b_{NN} \in BP(\Omega;\mathbb{R})$ must occur and then $(F_{b_{NN}})$ is satisfied: thus x_N and hence x are separated from zero.
A specular approach, working for lower triangular matrices, allows to prove that (F_B^*) is also true. □

We now consider the problem of reversing Proposition 9.4, proving that the simultaneous validity of (F_B) and (F_B^*) implies that of $(F_{b_{hh}})$ for every h. Lemma 8.1 says that this is true when $N = 2$ and an extension to higher dimensions would be desirable. Unfortunately, a class of counter–examples may be obtained by taking:

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{pmatrix}$$

where the entries are continuous functions on Ω which satisfy the following conditions. First of all, we choose the central entry to be a Kozlov function, that is, satisfying:

$$0 \in \sigma(b_{22}) \qquad b_{22} \notin BP(\Omega;\mathbb{R}) \tag{9.9}$$

and moreover:

$$\forall \omega \in \Omega \qquad \lim_{|t|\to+\infty} \int_0^t b_{22}(\omega s)\, ds \quad \text{does not exist .}$$

See the final part of Section 2 for comments about this type of functions. Secondly, we assume that the remaining diagonal entries satisfy:

$$b_{11},\, b_{33} \in BP(\Omega;\mathbb{R}) \tag{9.10}$$

while the entries above the diagonal are nontrivial and have a constant sign on Ω, say for instance:

$$\begin{cases} b_{12} \ge 0 \\ b_{12} \not\equiv 0 \end{cases} \qquad \begin{cases} b_{23} \ge 0 \\ b_{23} \not\equiv 0 \end{cases} \tag{9.11}$$

though any choice of the signs is admitted. Finally, no conditions are imposed on the term b_{13}, which is then arbitrary.

Proposition 9.5. *If assumptions (9.9)–(9.11) are satisfied, then* (F_B) *and* (F_B^*) *hold simultaneously.*

Since $(F_{b_{22}})$ is false by construction, we have the announced counter–example.

Proof. Use condition (9.10) to construct functions $\widehat{b}_{11}, \widehat{b}_{33} \in C(\Omega)$ such that:

$$\widehat{b}_{11}(\omega t) - \widehat{b}_{11}(\omega) = \int_0^t b_{11}(\omega s)\, ds$$
$$\widehat{b}_{33}(\omega t) - \widehat{b}_{33}(\omega) = \int_0^t b_{33}(\omega s)\, ds$$

for every $\omega \in \Omega$ and every $t \in \mathbb{R}$. Then make the diagonal change of variables:

$$x_1 = e^{\widehat{b}_{11}(\omega t)} u_1 \qquad x_2 = u_2 \qquad x_3 = e^{\widehat{b}_{33}(\omega t)} u_3$$

to transform (9.4) into the new equation:

$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{pmatrix} = \begin{pmatrix} 0 & c_{12}(\omega t) & c_{13}(\omega t) \\ 0 & b_{22}(\omega t) & c_{23}(\omega t) \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \tag{9.12}$$

where the entries c_{12} and c_{23} satisfy again (9.11). To show that this equation satisfies the Favard condition, consider an arbitrary $\omega \in \Omega$ and a corresponding bounded solution $u = (u_1, u_2, u_3)$. Since $\dot{u}_3 = 0$ then $u_3(t) = c$ for every t. If $c \neq 0$ then u is trivially separated from zero. When $c = 0$ the solution is $u = (u_1, u_2, 0)$ where the nontrivial components satisfy:

$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix} = \begin{pmatrix} 0 & c_{12}(\omega t) \\ 0 & b_{22}(\omega t) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

This equation satisfies the Favard condition due to Theorem 7.5 and hence either $u_1 \equiv u_2 \equiv 0$ or (u_1, u_2) is separated from zero. This implies that (9.12) satisfies the Favard condition.
Let us now consider the adjoint equation of (9.12), namely:

$$\begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \\ \dot{w}_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ -c_{12}(\omega t) & -b_{22}(\omega t) & 0 \\ -c_{13}(\omega t) & -c_{23}(\omega t) & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}. \tag{9.13}$$

To show that it satisfies the Favard condition, consider again an arbitrary $\omega \in \Omega$ and a corresponding bounded solution $w = (w_1, w_2, w_3)$. This time $\dot{w}_1 = 0$ so that $w_1(t) = c$ for every t. If $c \neq 0$ then w is trivially separated from zero, so that we will assume $w = (0, w_2, w_3)$ from now on, where the nontrivial components satisfy:

$$\begin{pmatrix} \dot{w}_2 \\ \dot{w}_3 \end{pmatrix} = \begin{pmatrix} -b_{22}(\omega t) & 0 \\ -c_{23}(\omega t) & 0 \end{pmatrix} \begin{pmatrix} w_2 \\ w_3 \end{pmatrix}$$

Swapping the order of variables we get the equation:

$$\begin{pmatrix} \dot{w}_3 \\ \dot{w}_2 \end{pmatrix} = \begin{pmatrix} 0 & -c_{23}(\omega t) \\ 0 & -b_{22}(\omega t) \end{pmatrix} \begin{pmatrix} w_3 \\ w_2 \end{pmatrix}$$

which satisfies the Favard condition again due to Theorem 7.5. Then also (9.13) satisfies the Favard condition. $\square$

The final question concerns the equality of the Favard dimensions:

$$d_F(B) = d_F^*(B)$$

which is expected when both the Favard conditions (F_B) and (F_B^*) hold. By using Floquet theory, it is indeed possible to prove that equality holds when Ω is periodic. On the other hand, the arguments of Section 5 show that the same is true for minimal Ω's when $N = 1$, while Lemma 4.6 covers the case $N = 2$ independently of the possible triangularity. Increasing the value of N, Proposition 9.4 provides a sufficient condition in order (F_B) and (F_B^*) to hold jointly, but unfortunately the statement does not give any information about the corresponding Favard dimensions: next example makes clear the reason of such omission. Consider indeed equation (9.4) where:

$$B = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{9.14}$$

and $b \in C(\Omega; \mathbb{R})$ is such that:

$$0 \in \sigma(b) \qquad\qquad b \notin BP(\Omega; \mathbb{R}).$$

Because of Proposition 9.4, both the Favard conditions (F_B) and (F_B^*) hold. We claim that however:

$$d_F(B) = 2 \qquad\qquad d_F^*(B) = 1\ .$$

Start observing that the general solution of the direct equation is:

$$x_1 = x_{10} + x_{30}\,\widetilde{b_\omega}(t) \qquad\quad x_2 = x_{20} + x_{30}t \qquad\quad x_3 = x_{30}\ .$$

Since each $\widetilde{b_\omega}$ is unbounded, the only bounded solution are constants with $x_{30} = 0$ and hence (F_B) holds with $d_F(B) = 2$. On the other hand, the adjoint equation corresponds to the matrix:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -b & -1 & 0 \end{pmatrix}$$

and hence the associated flow is now:

$$y_1 = y_{10} \qquad\quad y_2 = y_{20} \qquad\quad y_3 = -y_{10}\,\widetilde{b_\omega}(t) - y_{20}t + y_{30}\ .$$

Again the only bounded solutions are constants, but now they correspond to the choice $y_{10} = y_{20} = 0$: thus (F_B^*) holds with $d_F^*(B) = 1$, proving the claim.

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