AN EXTENSION PROBLEM FOR THE FRACTIONAL DERIVATIVE DEFINED BY MARCHAUD

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Abstract

We prove that the (nonlocal) Marchaud fractional derivative in $\mathbb{R}$ can be obtained from a parabolic extension problem with an extra (positive) variable as the operator that maps the heat conduction equation to the Neumann condition. Some properties of the fractional derivative are deduced from those of the local operator. In particular we prove a Harnack inequality for Marchaud-stationary functions.

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1. Introduction

In the literature there are several definitions of fractional derivatives (see, for instance, the monographs [24], [23] or [4] for an historical introduction). In particular, we are interested in the notion given by Marchaud, see [18], who introduced two types of fractional derivatives. For a fixed $s \in (0,1)$, the left and the right Marchaud fractional derivative of order s (see [24], formulas 5.57 and 5.58) are respectively defined as follows:

$$\mathbf{D}^s_{\pm} f(t) = \frac{s}{\Gamma(1-s)} \int_0^{\infty} \frac{f(t) - f(t \mp \tau)}{\tau^{1+s}} d\tau. \tag{1.1}$$

These fractional derivatives are well defined when f is a bounded, locally Hölder continuous function in $\mathbb{R}$. In particular, we may assume that $f \in C^{\bar{\gamma}}(\mathbb{R})$, for $s < \bar{\gamma} \leq 1$ and $f \in L^{\infty}(\mathbb{R})$ (see the Appendix for further details), even though these hypotheses can be weakened. In addition, we just recall

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here that the Marchaud derivative can be defined for $s \in (0, n)$ and $n \in \mathbb{N}$, as

$$\mathbf{D}^s_{\pm} f(t) = \frac{\{s\}}{\Gamma(1-\{s\})} \int_0^\infty \frac{f^{[s]}(t) - f^{[s]}(t \mp \tau)}{\tau^{1+\{s\}}}\, d\tau,$$

where $[s]$ and $\{s\}$ denote, respectively, the integer and the fractional part of s. Our work focuses on the case $n = 1$ and, in the first part of the paper, on the left fractional derivative, that we can write using a change of variable, neglecting the constant and omitting for simplicity the subscript symbol $+$, as:

$$\mathbf{D}^s f(t) := \int_0^\infty \frac{f(t) - f(t-\tau)}{\tau^{s+1}}\, d\tau = \int_{-\infty}^t \frac{f(t) - f(\tau)}{(t-\tau)^{s+1}}\, d\tau. \tag{1.2}$$

A short remark on the right counterpart of the Marchaud fractional derivative is given in Section 5. Moreover, we consider (1.2) as the definition of our fractional derivative without taking care of what happens when $s \to 0^+$ or $s \to 1^-$. Nevertheless, in the Appendix, we briefly discuss this behavior using the definition given in (1.1), since in those cases, the constant plays a fundamental role.

The purpose of the present work is to introduce an extension operator for the fractional derivative (1.2) and to prove a Harnack inequality for stationary functions (in the sense of Marchaud). The operator $\mathbf{D}^s$ naturally arises when dealing with a family of singular/degenerate parabolic problems (which, for $s = 1/2$, reduces to the heat conduction problem) on the positive half-plane, with a positive space variable and for all times, namely for $(x,t) \in [0,\infty) \times \mathbb{R}$.

In order to construct this extension operator, we exploit the idea recently revisited in [5]. In that paper, the fractional Laplacian was characterized via an extension procedure, by means of a degenerate second order elliptic local operator.

Considering the function φ of one variable, formally representing the time variable, our approach relies on constructing a parabolic local operator by adding an extra variable, say the space variable, on the positive half-line, and working on the extended plane $[0,\infty) \times \mathbb{R}$.

The heuristic argument can be described in the simplest case $s = 1/2$ as follows. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a "good" function and U be a solution of the problem

$$\begin{cases} \dfrac{\partial U}{\partial t} = \dfrac{\partial^2 U}{\partial x^2}, & (x,t) \in (0,\infty) \times \mathbb{R} \\ U(0,t) = \varphi(t), & t \in \mathbb{R}. \end{cases} \tag{1.3}$$

We point out that this is not the usual Cauchy problem for the heat operator, but a heat conduction problem.

It is known that, without extra assumptions, we can not expect to have a unique solution of the problem (1.3), see [26], Chapter 3.3. Nevertheless, if we denote by $T_{1/2}$ the operator that associates to φ the partial derivative $\partial U/\partial x$, whenever U is sufficiently regular, we have that

$$T_{1/2}T_{1/2}\varphi = \frac{d\varphi}{dt}.$$

That is $T_{1/2}$ acts like an half derivative, indeed

$$\frac{\partial}{\partial x}\frac{\partial U}{\partial x}(x,t) = \frac{\partial U}{\partial t}(x,t) \underset{x\to 0^+}{\longrightarrow} \frac{d\varphi(t)}{dt}.$$

The solution of the problem (1.3) under the reasonable assumptions that φ is bounded and Hölder continuous, is explicitly known (check [26], Chapter 3.3) to be

$$\begin{aligned} U(x,t) &= cx\int_{-\infty}^{t} e^{-\frac{x^2}{4(t-\tau)}}(t-\tau)^{-\frac{3}{2}}\varphi(\tau)\,d\tau \\ &= cx\int_0^{\infty} e^{-\frac{x^2}{4\tau}}\tau^{-\frac{3}{2}}\varphi(t-\tau)\,d\tau, \end{aligned}$$

where the last line is obtained with a change of variables. Using $t = x^2/(4\tau)$ and the integral definition of the Gamma function (see formula 6.1.1 in [1]) we have that

$$\int_0^{\infty} xe^{-\frac{x^2}{4\tau}}\tau^{-\frac{3}{2}}\,d\tau = 2\int_0^{\infty} e^{-t}t^{-\frac{1}{2}}\,dt = 2\Gamma\left(\frac{1}{2}\right).$$

Hence,

$$\frac{U(x,t)-U(0,t)}{x} = c\int_0^{\infty} e^{-\frac{x^2}{4\tau}}\tau^{-\frac{3}{2}}\left(\varphi(t-\tau)-\varphi(t)\right)d\tau,$$

choosing c that takes into account the right normalization. This yields, by passing to the limit, that

$$-\lim_{x\to 0^+}\frac{U(x,t)-U(0,t)}{x} = c\int_0^{\infty}\frac{\varphi(t)-\varphi(t-\tau)}{\tau^{\frac{3}{2}}}d\tau.$$

Hence, with the right choice of the constant, we get exactly $\mathbf{D}^{1/2}\varphi$ (see (1.2)), i.e. the Marchaud derivative of order 1/2 of φ.

Now we are in position to state our main result.

THEOREM 1.1. *Let $s \in (0,1)$ and $\bar\gamma \in (s,1]$ be fixed. Let $\varphi \in C^{\bar\gamma}(\mathbb{R})$ be a bounded function and let $U\colon [0,\infty)\times\mathbb{R}\to\mathbb{R}$ be a solution of the* *problem*

$$\begin{cases} \dfrac{\partial U}{\partial t}(x,t) = \dfrac{1-2s}{x}\dfrac{\partial U}{\partial x}(x,t) + \dfrac{\partial^2 U}{\partial x^2}(x,t), & (x,t) \in (0,\infty)\times\mathbb{R} \\ U(0,t) = \varphi(t), & t \in \mathbb{R} \\ \lim\limits_{x\to+\infty} U(x,t) = 0. \end{cases} \tag{1.4}$$

Then U defines the extension operator for φ, such that

$$\mathbf{D}^s\varphi(t) = -\lim_{x\to 0^+} c_s x^{-2s}(U(x,t) - \varphi(t)), \qquad \textit{where} \qquad c_s = 4^s\Gamma(s).$$

We notice that one can write

$$\mathbf{D}^s\varphi(t) = -\lim_{x\to 0^+} c_s x^{1-2s}\frac{\partial U}{\partial x}(x,t), \tag{1.5}$$

in analogy with formula (3.1) in [5].

Remark 1.1. The extension operator satisfies, as one would expect, up to constants that

$$\mathbf{D}^{1-s}\mathbf{D}^s\varphi(t) = \varphi'(t).$$

Indeed, using (1.5) and thanks to (1.4) we have that

$$\begin{aligned} \mathbf{D}^{1-s}\mathbf{D}^s\varphi(t) &= \lim_{x\to 0^+} x^{2s-1}\frac{\partial}{\partial x}\left(x^{1-2s}\frac{\partial U}{\partial x}(x,t)\right) \\ &= \lim_{x\to 0^+} \frac{\partial^2 U}{\partial x^2}(x,t) + \frac{1-2s}{x}\frac{\partial U}{\partial x}(x,t) \\ &= \lim_{x\to 0^+} \frac{\partial U}{\partial t}(x,t) = \frac{\partial U}{\partial t}(0,t) = \varphi'(t). \end{aligned}$$

An interesting application that follows from this extension procedure is a Harnack inequality for Marchaud-stationary functions in an interval $J \subseteq \mathbb{R}$, namely for functions that satisfy $\mathbf{D}^s\varphi = 0$ in J. This fact is not obvious, indeed the set of functions determined by fractional-stationary functions (on an interval) is nontrivial, see e.g. [3].

Theorem 1.2. *Let $s \in (0,1)$. There exists a positive constant γ such that, if $\mathbf{D}^s\varphi = 0$ in an interval $J \subseteq \mathbb{R}$ and $\varphi \geq 0$ in $\mathbb{R}$, then*

$$\sup_{[t_0-\frac{3}{4}\delta, t_0-\frac{1}{4}\delta]} \varphi \leq \gamma \inf_{[t_0+\frac{3}{4}\delta, t_0+\delta]} \varphi$$

for every $t_0 \in \mathbb{R}$ and for every $\delta > 0$ such that $[t_0-\delta, t_0+\delta] \subset J$.

The previous result can be deduced from the Harnack inequality proved in [6] for some degenerate parabolic operators (see also [9] for the elliptic setting). In particular, the constant γ used in Theorem 1.2 is the same that appears in the parabolic case in [6].

In addition, we remark that Theorem 1.2 does not give the usual Harnack inequality for elliptic operators, where the comparison between the supremum and the infimum is done on the same set, e.g. the same metric ball. This Harnack inequality for the Marchaud-stationary functions inherits the behavior of its parabolic extension.

We point out at this point the very interesting paper [2]. Indeed, after we have submitted our paper, we learnt from professor José L. Torrea about the results contained in his joint paper where an extension procedure for a class of operators has been studied.

2. **The extension parabolic problem**

In this section we find a solution of the system (1.4). At first, we introduce a particular kernel, that acts as the Poisson kernel. We then look for a particular solution of the system by means of the Laplace transform, and in this way we show how the solution arises. Finally, by a straightforward check, it yields that indeed the indicated solution satisfies the problem (1.4).

2.1. **Properties of the kernel Ψ_s.**

In this section we introduce and study the properties of a kernel, that acts as the Poisson kernel for the problem (1.4). The readers can see Section 3 in [11], where this kernel is studied in a more general framework.

We define for every $x \in \mathbb{R}$,

$$\Psi_s(x,t) := \begin{cases} \dfrac{1}{4^s\Gamma(s)} x^{2s} e^{-\frac{x^2}{4t}} t^{-s-1}, & \text{if } t > 0, \\ 0, & \text{if } t \leq 0. \end{cases}$$

Also, let

$$\psi_s(t) := \begin{cases} \dfrac{1}{4^s\Gamma(s)} e^{-\frac{1}{4t}} t^{-s-1}, & \text{if } t > 0, \\ 0, & \text{if } t \leq 0 \end{cases}$$

and notice that

$$\int_{\mathbb{R}} \Psi_s(x,t)\,dt = \int_{\mathbb{R}} \psi_s(t)\,dt. \tag{2.1}$$

Indeed, we have by changing the coordinate $\tau = t/x^2$ that

$$\begin{aligned}\int_{\mathbb{R}} \Psi_s(x,t)\,dt =& \frac{1}{4^s\Gamma(s)} \int_0^\infty x^{2s} e^{-\frac{x^2}{4t}} t^{-s-1}\,dt \\ =& \frac{1}{4^s\Gamma(s)} \int_0^\infty e^{-\frac{1}{4\tau}} \tau^{-s-1}\,d\tau \\ =& \int_{\mathbb{R}} \psi_s(t)\,dt.\end{aligned}$$

The kernel Ψ_s satisfies also the following property:

$$\int_{\mathbb{R}} \Psi_s(x,t)\,dt = 1. \tag{2.2}$$

Indeed, by changing the variable $t = 1/(4\tau)$ we get that

$$\int_{\mathbb{R}} \psi_s(\tau)\,d\tau = \frac{1}{4^s\Gamma(s)} \int_0^\infty e^{-\frac{1}{4\tau}} \tau^{-s-1}\,d\tau = \frac{1}{\Gamma(s)} \int_0^\infty e^{-t} t^{s-1}\,dt = 1, \tag{2.3}$$

thanks to the integral definition of the Gamma function (see formula 6.1.1 in [1]). It follows from (2.1) that

$$\int_{\mathbb{R}} \Psi_s(x,t)\,dt = 1.$$

Taking the Laplace transform of the kernel Ψ_s (see e.g. [8] for details on this integral transform), we have the following result involving the modified Bessel function of the second kind $\mathbf{K}_s$, see [17] and [1], §9.6. We use here the notation $\Re\omega > 0$ to denote the real part of a complex number ω.

Lemma 2.1. *The Laplace transform of the function $\psi_s \in L^1(\mathbb{R})$ is*

$$\mathcal{L}(\psi_s)(\omega) = \frac{1}{2^{s-1}\Gamma(s)} \omega^{\frac{s}{2}} \mathbf{K}_s(\sqrt{\omega}) \text{ for } \Re\omega > 0. \tag{2.4}$$

Moreover, the Laplace transform with respect to the variable t of the kernel $\Psi_s \in L^1(\mathbb{R}, dt)$ is

$$\mathcal{L}(\Psi_s)(x,\omega) = \frac{1}{2^{s-1}\Gamma(s)} x^s \omega^{\frac{s}{2}} \mathbf{K}_s(x\sqrt{\omega}) \text{ for } \Re\omega > 0. \tag{2.5}$$

P r o o f. If one proves claim (2.4), the identity (2.5) follows by changing the variable $\tau = t/x^2$. For $\Re a > 0$ and $\omega \in \mathbb{C}$ with $\Re\omega > 0$, as stated in formula 5.34 in [20], we have that

$$\mathcal{L}\left(t^{\gamma-1} e^{-\frac{a}{t}}\right) = 2\left(\frac{a}{\omega}\right)^{\frac{\gamma}{2}} \mathbf{K}_\gamma\left(2(a\omega)^{\frac{1}{2}}\right).$$

Taking $\gamma = -s$ and $a = 1/4$, recalling that $\mathbf{K}_s = \mathbf{K}_{-s}$, we obtain that

$$\mathcal{L}(\psi_s)(\omega) = \frac{1}{4^s\Gamma(s)}\mathcal{L}\Big(e^{-\frac{1}{4\tau}}\tau^{-s-1}\Big) = \frac{1}{2^{s-1}\Gamma(s)}\omega^{\frac{s}{2}}\mathbf{K}_s(\sqrt{\omega})$$

and thus (2.4). This concludes the proof of the Lemma. □

2.2. **Existence of the solution.** We prove in this section the existence of a solution of the system (1.4).

We recall at first a useful result (see [10], Proposition 4.1) involving the modified Bessel function of the second kind.

Proposition 2.1. *If $-\infty < \alpha < 1$, the boundary value problem*

$$\begin{cases} x^\alpha y''(x) = y(x), & \text{in } (0,\infty) \\ y(0) = 1, \\ \lim\limits_{x\to\infty} y(x) = 0. \end{cases} \tag{2.6}$$

has a solution $y \in C^{2-\alpha}([0,\infty))$ of the form

$$y(x) = c_k x^{\frac{1}{2}}\mathbf{K}_{\frac{1}{2k}}\left(\frac{t^k}{k}\right),$$

where c_k is the positive constant

$$c_k = \frac{2^{1-\frac{1}{2k}}k^{-\frac{1}{2k}}}{\Gamma\left(\frac{1}{2k}\right)} \quad \text{and} \quad k := \frac{2-\alpha}{2}.$$

We show in the next rows how the solution of the problem (1.4) arises, using the Laplace transform. So, we look for a possible candidate of a solution in the simplified situation in which U has a sub-exponential growth in t, and in which the function φ is zero on the negative semi-axis $(-\infty, 0]$. Under this additional hypothesis, we take the Laplace transform in t of the system (1.4). Since the Laplace transform of the derivative of a function gives

$$\mathcal{L}(f')(\omega) = \omega\mathcal{L}f(\omega),$$

we get that

$$\begin{cases} \omega\mathcal{L}U(x,\omega) = \dfrac{1-2s}{x}\dfrac{\partial\mathcal{L}U}{\partial x}(x,\omega) + \dfrac{\partial^2\mathcal{L}U}{\partial x^2}(x,\omega), & \text{in} \quad (0,\infty)\times\mathbb{C} \\ \mathcal{L}U(0,\omega) = \mathcal{L}\varphi(\omega), & \text{in} \quad \mathbb{C} \\ \lim\limits_{x\to+\infty}\mathcal{L}U(x,\omega) = 0, & \text{in} \quad \mathbb{C}. \end{cases}$$

We define for any fixed $\omega \in \mathbb{C}$

$$f(x) := \mathcal{L}U(x,\omega), \tag{2.7}$$

then f must be a solution of the system

$$\begin{cases} \omega f(x) = \dfrac{1-2s}{x} f'(x) + f''(x), & \text{in } (0,\infty) \\ f(0) = \mathcal{L}\varphi(\omega) \\ \lim\limits_{x\to+\infty} f(x) = 0. \end{cases} \tag{2.8}$$

We assume here that for any $\omega \in \mathbb{C}$, $\mathcal{L}\varphi(\omega) \neq 0$.

We take in Proposition 2.1, $\alpha = (2s-1)/s$ (notice for $s \in (0,1)$ that $\alpha \in (-\infty, 1)$) and $y(x)$ to be the solution there introduced. We claim that taking

$$f(x) = \mathcal{L}\varphi(\omega) y \left(\omega^s \left(\frac{x}{2s} \right)^{2s} \right),$$

$f(x)$ is a solution of the system (2.8). Indeed, $f(0) = \mathcal{L}\varphi(\omega)$ and

$$\begin{aligned} y'' \left(\omega^s \left(\frac{x}{2s} \right)^{2s} \right) &= f''(x) \frac{1}{\mathcal{L}\varphi(\omega)} \omega^{-2s} (2s)^{4s-2} x^{2-4s} \\ &\quad + f'(x) \frac{1-2s}{\mathcal{L}\varphi(\omega)} (2s)^{4s-2} \omega^{-2s} x^{1-4s}. \end{aligned}$$

Since $y(x)$ satisfies the system (2.6) we have that

$$\left(\omega^s \left(\frac{x}{2s} \right)^{2s} \right)^{\frac{2s-1}{s}} y'' \left(\omega^s \left(\frac{x}{2s} \right)^{2s} \right) = y \left(\omega^s \left(\frac{x}{2s} \right)^{2s} \right).$$

This implies that

$$\omega f(x) = f''(x) + (1-2s) x^{-1} f'(x),$$

which yields that f is a solution of (2.8).

Now, from Proposition 2.1 we have $k = 1/(2s)$ and

$$y(x) = \frac{2^{1-s} (2s)^s}{\Gamma(s)} x^{\frac{1}{2}} \mathbf{K}_s \left(2s x^{\frac{1}{2s}} \right).$$

And so we get that

$$f(x) = \mathcal{L}\varphi(\omega) \frac{2^{1-s}}{\Gamma(s)} \omega^{\frac{s}{2}} x^s \mathbf{K}_s(x\sqrt{\omega}).$$

We use (2.7), take the inverse Laplace transform, and recall that the pointwise product is taken into the convolution product to obtain that

$$U(x,t) = \frac{2^{1-s}}{\Gamma(s)} \varphi * \mathcal{L}^{-1} \left(\omega^{\frac{s}{2}} x^s \mathbf{K}_s(x\sqrt{\omega}) \right)(t).$$

And so, using (2.5), we get the following representation formula for the system (1.4):

$$U(x,t) = \varphi * \Psi_s(x,t) = \int_0^t \Psi(x,\tau)\varphi(t-\tau)\,d\tau.$$

We recall that we obtained the above formula by taking the function φ to be vanishing in $(-\infty,0)$. However, it is reasonable to suppose that this formula holds true also for a function that is not a signal. Hence, we take φ that does not vanish in $(-\infty,0)$ and claim that $\varphi * \Psi_s$ still defines a solution of the problem (1.4). Indeed, we show the following existence theorem:

Theorem 2.1. *There exists a continuous solution of the problem* (1.4) *given by*

$$U(x,t) = \Psi_s(x,\cdot) * \varphi(t) := \int_{\mathbb{R}} \Psi_s(x,\tau)\varphi(t-\tau)\,d\tau.$$

More precisely (inserting the definition (2.1)*) we have that*

$$U(x,t) = \frac{1}{4^s\Gamma(s)} x^{2s} \int_0^\infty e^{-\frac{x^2}{4\tau}} \tau^{-s-1} \varphi(t-\tau)\,d\tau. \tag{2.9}$$

P r o o f. We define

$$A_{x,\tau} := \begin{cases} e^{-\frac{x^2}{4\tau}}\tau^{-s-1}, & \text{if } \tau > 0 \\ 0, & \text{if } \tau \le 0 \end{cases}$$

and notice that

$$\frac{\partial A_{x,\tau}}{\partial x} = \begin{cases} -\dfrac{x}{2\tau} A_{x,\tau}, & \text{if } \tau > 0 \\ 0, & \text{if } \tau \le 0. \end{cases}$$

Let

$$V(x,t) := 4^s\Gamma(s)U(x,t) = x^{2s}\int_{\mathbb{R}} A_{x,\tau}\varphi(t-\tau)\,d\tau,$$

where we have introduced the notation $A_{x,\tau}$ into (2.9). Taking the derivative with respect to x of $V(x,t)$ we have that

$$\frac{\partial V}{\partial x}(x,t) = 2sx^{2s-1}\int_{\mathbb{R}} A_{x,\tau}\varphi(t-\tau)\,d\tau - \frac{x^{2s+1}}{2}\int_{\mathbb{R}} \frac{A_{x,\tau}}{\tau}\varphi(t-\tau)\,d\tau,$$

and that

$$\begin{aligned}\frac{\partial^2 V}{\partial x^2}(x,t) =& 2s(2s-1)x^{2s-2}\int_{\mathbb{R}} A_{x,\tau}\varphi(t-\tau)\,d\tau \\ &- \frac{(4s+1)x^{2s}}{2}\int_{\mathbb{R}} \frac{A_{x,\tau}}{\tau}\varphi(t-\tau)\,d\tau + \frac{x^{2s+2}}{4}\int_{\mathbb{R}} \frac{A_{x,\tau}}{\tau^2}\varphi(t-\tau)\,d\tau.\end{aligned}$$

Then, by changing variables, we write

$$V(x,t) = x^{2s} \int_{\mathbb{R}} A_{x,t-\tau} \varphi(\tau)\, d\tau,$$

and taking the derivative with respect to t, we get that

$$\frac{\partial V}{\partial t}(x,t) = x^{2s} \int_{\mathbb{R}} \left[x^2 \frac{A_{x,t-\tau}}{4(t-\tau)^2} \varphi(\tau) - (s+1) \frac{A_{x,t-\tau}}{(t-\tau)} \varphi(\tau) \right] d\tau.$$

We change back variables to obtain

$$\frac{\partial V}{\partial t}(x,t) = x^{2s+2} \int_{\mathbb{R}} \frac{A_{x,\tau}}{4\tau^2} \varphi(t-\tau)\, d\tau - (s+1) x^{2s} \int_{\mathbb{R}} \frac{A_{x,\tau}}{\tau} \varphi(t-\tau)\, d\tau.$$

By substituting these computations, we obtain that indeed V, hence U by the definition of V, satisfies the equation

$$\frac{\partial U}{\partial t}(x,t) = \frac{1-2s}{x} \frac{\partial U}{\partial x}(x,t) + \frac{\partial^2 U}{\partial x^2}(x,t).$$

Moreover, using for x large enough the bound

$$x^{2s} e^{-\frac{x^2}{4\tau}} \leq M e^{-\frac{1}{4\tau}},$$

thanks to the Dominated Convergence Theorem and the limit

$$\lim_{x \to +\infty} x^{2s} e^{-\frac{x^2}{4\tau}} = 0,$$

it yields that

$$\lim_{x \to +\infty} U(x,t) = 0.$$

Furthermore, in (2.9) by changing the variable $\tilde{\tau} = \tau / x^2$ (but still writing τ as the variable of integration), we have that

$$U(x,t) = \frac{1}{4^s \Gamma(s)} \int_0^\infty e^{-\frac{1}{4\tau}} \tau^{-s-1} \varphi(t - \tau x^2)\, d\tau.$$

Since φ is bounded, by the Dominated Convergence Theorem, we have that

$$\lim_{x \to 0^+} U(x,t) = \frac{\varphi(t)}{4^s \Gamma(s)} \int_0^\infty e^{-\frac{1}{4\tau}} \tau^{-s-1}\, d\tau = \varphi(t),$$

according to (2.3). This proves the continuity up to the boundary of the solution U, concluding the proof of the Theorem. $\square$

3. Relation with the Marchaud fractional derivative

We prove here the relation between the parabolic equation studied in Subsection 2.2 and the Marchaud fractional derivative. Namely, the Marchaud derivative is obtained as the trace operator of the extension given by the solution of (1.4).

Proof of Theorem 1.1. By inserting the expression of $U(x,t)$ from (2.9), we compute

$$\begin{aligned}&\lim_{x\to 0^+} x^{-2s}\left(U(x,t)-\varphi(t)\right)\\ &= \lim_{x\to 0^+} x^{-2s}\left(\frac{1}{4^s\Gamma(s)}\int_0^\infty x^{2s}e^{-\frac{x^2}{4\tau}}\tau^{-s-1}\varphi(t-\tau)\,d\tau-\varphi(t)\right).\end{aligned}$$

Recalling property (2.1) of the kernel, we have that

$$\begin{aligned}&\lim_{x\to 0^+} x^{-2s}\left(U(x,t)-\varphi(t)\right)\\ &\qquad= \lim_{x\to 0^+}\frac{x^{-2s}}{4^s\Gamma(s)}\int_0^\infty x^{2s}e^{-\frac{x^2}{4\tau}}\tau^{-s-1}\left(\varphi(t-\tau)-\varphi(t)\right)\,d\tau\\ &\qquad= \lim_{x\to 0^+}\frac{1}{4^s\Gamma(s)}\int_0^\infty e^{-\frac{x^2}{4\tau}}\frac{\varphi(t-\tau)-\varphi(t)}{\tau^{s+1}}\,d\tau.\end{aligned}$$

Now

$$e^{-\frac{x^2}{4\tau}}\le 1$$

and since φ is bounded, we have that

$$\frac{|\varphi(t-\tau)-\varphi(t)|}{\tau^{s+1}}\le 2M\tau^{-s-1}\in L^1\left((1,\infty)\right).$$

On the other hand, recalling that φ is $C^{\bar\gamma}(\mathbb{R})$ we have that

$$|\varphi(t)-\varphi(t-\tau)|\le c\tau^{\bar\gamma}.$$

Hence, since $\bar\gamma>s$,

$$\frac{|\varphi(t-\tau)-\varphi(t)|}{\tau^{s+1}}\le c\tau^{\bar\gamma-s-1}\in L^1\left((0,1)\right).$$

Using the Dominated Converge Theorem, we obtain

$$\begin{aligned}\lim_{x\to 0^+} x^{-2s}\left(U(x,t)-\varphi(t)\right) &= \frac{1}{4^s\Gamma(s)}\int_0^\infty \lim_{x\to 0^+} e^{-\frac{x^2}{4\tau}}\frac{\varphi(t-\tau)-\varphi(t)}{\tau^{s+1}}\,d\tau\\ &= \frac{1}{4^s\Gamma(s)}\int_0^\infty \frac{\varphi(t-\tau)-\varphi(t)}{\tau^{s+1}}\,d\tau.\end{aligned} \tag{3.1}$$

And so for $c_s = 4^s\Gamma(s)$,

$$-c_s \lim_{x\to 0^+} x^{-2s}\left(U(x,t)-\varphi(t)\right) = \int_0^\infty \frac{\varphi(t)-\varphi(t-\tau)}{\tau^{s+1}}\,d\tau = \mathbf{D}^s\varphi(t)$$

by definition (1.2). This concludes the proof of Theorem 1.1. □

4. Applications: a Harnack inequality for Marchaud-stationary functions

In this part of the paper we prove a Harnack inequality for functions that have a vanishing Marchaud derivative in a bounded interval J, namely we prove here Theorem 1.2. At this purpose, we use a known Harnack inequality for degenerate parabolic operators, that can be found in [6], see Theorem 2.1. There, the result is given in its generality, in $\mathbb{R}^n$. For the reader's convenience we recall in Proposition 4.1 this result in the case $n = 1$.

4.1. **Preliminary notions.** We would like to point out that the result given in [6] was proved for $n \geq 3$. Nevertheless the same proof works also for $n = 1$ with some adjustments. We recall here the hypotheses we need, adapted in our case $n = 1$. It is worth to say that this problem has been studied in a more general fashion in [13] and [14].

The degenerate parabolic

$$w(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(w(x)\frac{\partial u}{\partial x}\right), \tag{4.1}$$

is given in $Q = (-R,R)\times(0,T)$, for $R>0$. The weight w has to satisfy an integrability condition (also known as a Muckehoupt, or A_2 weight condition), given by

$$\sup_J \left(\frac{1}{|J|}\int_J w(x)\,dx\right)\left(\frac{1}{|J|}\int_J \frac{1}{w(x)}\,dx\right) = c_0 < \infty, \tag{4.2}$$

for any interval $J \subseteq (-R,R)$. The constant c_0 is indicated as the A_2 constant of w.

In this particular case we give here in (4.1), the conductivity coefficient (i.e. the coefficient in front of the x derivative) and the specific heat (the coefficient of the t derivative) coincide. A more general form of the equation in $\mathbb{R}$ can be given in these terms:

$$w(x)\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\left(a(x)\frac{\partial u}{\partial x}\right), \tag{4.3}$$

i.e. when the conductivity and the specific heat are not equal. In that case, one has to require, besides condition (4.2), that

$$\lambda^{-1}w(x) \leq a(x) \leq \lambda w(x).$$

In addition we consider the functional space

$$W := \left\{ u \in L^2(0,T; H^1_0(J,w)) \text{ s.t. } \quad \frac{\partial u}{\partial t} \in L^2(0,T; L^2(J,w)) \right\}.$$

We denote here by $L^2(J,w)$, the Banach space of measurable functions u with finite weighted norm

$$\|u\|_{2,w;J} = \left(\int_J |u|^2 w \, dx \right)^{1/2} < \infty,$$

by $H^1(J,w)$ the completion of $C^\infty(\overline{J})$ under the norm

$$\|u\|_{1,w;J} = \left(\int_J (u^2 + |\partial_x u|^2) w \, dx \right)^{1/2}$$

and by $H^1_0(J,w)$ the completion of $C^\infty_0(J)$ under the norm

$$\|u\|_{1,w;J} = \left(\int_J |\partial_x u|^2 w \, dx \right)^{1/2}.$$

The time dependent Sobolev space $L^2\left(0,T; H^1_0(J,w)\right)$ is defined as the set of all measurable functions u such that

$$\|u\|_{L^2\left(0,T;H^1_0(J,w)\right)} := \left(\iint_{J\times(0,T)} |u(x,t)|^2 w(x) \, dx \, dt \right)^{\frac{1}{2}} < \infty.$$

In this setting, we introduce the notion of weak solution of the problem (4.1).

Definition 4.1. We say that $u \in L^2(0,T; H^1(J,w))$ is a weak solution of (4.1) in $J \times (0,T)$ if, for every $\eta \in W$, such that $\eta(x,0) = \eta(x,t)$ for any $x \in J$, we have that

$$\iint_{J\times(0,T)} w(x) \left(\frac{\partial u}{\partial x}\frac{\partial \eta}{\partial x} - u\frac{\partial \eta}{\partial t} \right) dx \, dt = 0.$$

We have the next proposition (see for the proof Theorem 2.1 in [6]).

Proposition 4.1. *Let u be a positive solution in $(-R,R)\times(0,T)$ of (4.1) and assume that condition* (4.2) *holds, with constant c_0. Then there exists $\gamma = \gamma(c_0) > 0$ such that*

$$\sup_{(-\frac{\rho}{2},\frac{\rho}{2})\times\left(t_0-\frac{3\rho^2}{4},t_0-\frac{\rho^2}{4}\right)} u \le \gamma \inf_{(-\frac{\rho}{2},\frac{\rho}{2})\times\left(t_0+\frac{3\rho^2}{4},t_0+\rho^2\right)} u$$

holds for $t_0 \in (0,T)$ *and any* ρ *such that* $0<\rho<R/2$ *and* $[t_0-\rho^2, t_0+\rho^2] \subset (0,T)$.

Remark 4.1. The reader can easily imagine the general situation in any dimension as explicated in Theorem 2.1 in [6], where the coefficient $a(x)$ in (4.3) is a matrix and the domains are cylinders. We have stated the Harnack inequality in $(0,T)$. Nevertheless with a change of coordinates in space and time, we can always say that the Harnack inequality holds in any subset of $(R_1,R_2)\times(\tau_1,\tau_2)$, where $R_1,R_2,\tau_1,\tau_2\in\mathbb{R}$.

4.2. **Reflection of the solution.** We consider here that $\mathbf{D}^s\varphi(t)=0$ in an interval J. By taking the reflection of the solution of problem (1.4), we obtain a solution in a weak sense of (1.4) across $x=0$.

It is useful to introduce a weak version of the limit $\lim\limits_{x\to 0^+} x^{1-2s}\partial_x U(x,t)$. In this sense, we have:

Definition 4.2. We say that in a weak sense

$$\lim_{x\to 0^+} x^{1-2s}\frac{\partial U}{\partial x}(x,t)=0$$

if and only if, for any $\eta\in W$ such that $\eta(x,0)=\eta(x,t)$ for any $x\in J$, we have that

$$\lim_{x\to 0^+}\int_0^T x^{1-2s}\frac{\partial U}{\partial x}\,\eta\,dt=0. \tag{4.4}$$

Lemma 4.1. *Let* $U\colon \mathbb{R}\times[0,\infty)\to\mathbb{R}$ *be a solution of the problem (1.4) such that, in a weak sense,* $\lim\limits_{x\to 0^+} x^{1-2s}\partial_x U(x,t)=0$. *Then the extension*

$$\tilde U(x,t):=\begin{cases} U(x,t), & (x,t)\in[0,+\infty)\times(0,T)\\ U(-x,t), & (x,t)\in(-\infty,0)\times(0,T)\end{cases}$$

is a weak solution of

$$\frac{\partial(|x|^{1-2s}U)}{\partial t}(x,t)=\frac{\partial}{\partial x}\left(|x|^{1-2s}\frac{\partial U}{\partial x}(x,t)\right) \tag{4.5}$$

in $(-R,R)\times(0,T)$.

P r o o f. We claim that the extension $\tilde U$ is a weak solution of (4.5), hence that

$$\int_{(-R,R)\times(0,T)} |x|^{1-2s}\left(\frac{\partial\tilde U}{\partial x}\frac{\partial\eta}{\partial x}-\tilde U\frac{\partial\eta}{\partial t}\right)dx\,dt=0. \tag{4.6}$$

We compute, integrating by parts

$$
\begin{aligned}
&\int_0^T \left(\int_0^R x^{1-2s} \frac{\partial \tilde{U}}{\partial x} \frac{\partial \eta}{\partial x}\, dx \right) dt \\
&\quad = \int_0^T R^{1-2s} \frac{\partial \tilde{U}}{\partial x}(R,t)\, \eta(R,t)\, dt - \lim_{x \to 0} \int_0^T x^{1-2s} \frac{\partial U}{\partial x}\, \eta\, dt \\
&\quad - \int_0^T \left(\int_0^R \frac{\partial}{\partial x} \left(x^{1-2s} \frac{\partial \tilde{U}}{\partial x} \right) \eta\, dx \right) dt \\
&\quad = \int_0^T R^{1-2s} \frac{\partial \tilde{U}}{\partial x}(R,t) \eta(R,t)\, dt - \int_0^T \left(\int_0^R x^{1-2s} \frac{\partial \tilde{U}}{\partial t}\, \eta\, dx \right) dt,
\end{aligned}
$$

where we have used the weak limit in (4.4) and the fact that $\tilde{U}$ solves equation (4.5). In the same way, one obtains that

$$
\begin{aligned}
&\int_0^T \left(\int_{-R}^0 (-x)^{1-2s} \frac{\partial \tilde{U}}{\partial x} \frac{\partial \eta}{\partial x}\, dx \right) dt = \int_0^T R^{1-2s} \frac{\partial \tilde{U}}{\partial x}(-R,t) \eta(-R,t)\, dt \\
&\quad - \int_0^T \left(\int_{-R}^0 (-x)^{1-2s} \frac{\partial \tilde{U}}{\partial t}\, \eta\, dx \right) dt,
\end{aligned}
$$

therefore, by summing up,

$$
\begin{aligned}
&\int_{(-R,R)\times(0,T)} |x|^{1-2s} \frac{\partial \tilde{U}}{\partial x} \frac{\partial \eta}{\partial x}\, dx\, dt \\
&\qquad = \int_0^T R^{1-2s} \left(\frac{\partial \tilde{U}}{\partial x}(R,t) \eta(R,t) - \frac{\partial \tilde{U}}{\partial x}(-R,t) \eta(-R,t) \right) dt \\
&\qquad - \int_0^T \left(\int_{-R}^R |x|^{1-2s} \frac{\partial \tilde{U}}{\partial t}\, \eta\, dx \right) dt.
\end{aligned}
$$

Hence

$$\begin{aligned}
\int_{(-R,R)\times(0,T)} & |x|^{1-2s} \left(\frac{\partial \tilde{U}}{\partial x} \frac{\partial \eta}{\partial x} - \tilde{U} \frac{\partial \eta}{\partial t} \right) dx\, dt \\
&= \int_0^T R^{1-2s} \left(\frac{\partial \tilde{U}}{\partial x}(R,t)\eta(R,t) - \frac{\partial \tilde{U}}{\partial x}(-R,t)\eta(-R,t) \right) dt \\
&\quad - \int_0^T \left(\int_{-R}^R |x|^{1-2s} \left(\frac{\partial \tilde{U}}{\partial t}\, \eta - \tilde{U} \frac{\partial \eta}{\partial t} \right) dx \right) dt \\
&= \int_0^T R^{1-2s} \left(\frac{\partial \tilde{U}}{\partial x}(R,t)\eta(R,t) - \frac{\partial \tilde{U}}{\partial x}(-R,t)\eta(-R,t) \right) dt \\
&\quad - \int_{-R}^R |x|^{1-2s} \left(\tilde{U}(x,T)\eta(x,T) - \tilde{U}(x,0)\eta(x,0) \right) dx \\
&= 0,
\end{aligned}$$

since $\eta(x,T) = \eta(x,0) = 0$ and $\eta(R,t) = \eta(-R,t) = 0$. This is the claim in (4.6), and we conclude the proof of the Lemma. $\square$

4.3. **The Harnack inequality for Marchaud stationary functions.** We show here that the Harnack inequality for Marchaud stationary functions can be deduced from the Harnack inequality associated with the extension operator.

The interested reader can also see [5] for the proof (using the extension operator) of the Harnack inequality for the fractional Laplacian, and [7] for the inequality for other types of nonlocal operators. In addition, we also point out [10] for the case of the fractional subelliptic operators in Carnot groups and [25] for the fractional harmonic oscillator.

Proof of Theorem 1.2. We consider U to be the extension of φ, as introduced in Theorem 1.1. Since φ is nonnegative, given the explicit solution U in Theorem 2.1, the function U is also positive. Now, we reflect U and obtain $\tilde{U}$, as we have done in Lemma 4.1.

We prove at first the theorem when $J = (0,T)$. Since $\mathbf{D}^s\varphi(t) = 0$ in $(0,T)$, we have by definition that

$$\lim_{x\to 0^+} x^{-2s} \frac{\partial U}{\partial x}(x,t) = 0,$$

and thanks to Lemma 4.1, we obtain that $\tilde{U}$ is a weak solution of (4.5) in, say, $(-R,R)\times(0,T)$ for a fixed arbitrary $R > 0$. Moreover, the function $|x|^{1-2s}$ satisfies the condition (4.2), and according to Proposition 4.1, we have that

$$\sup_{(-\frac{\rho}{2},\frac{\rho}{2})\times\left(t_0-\frac{3\rho^2}{4},t_0-\frac{\rho^2}{4}\right)} \tilde{U} \leq \gamma \inf_{(-\frac{\rho}{2},\frac{\rho}{2})\times\left(t_0+\frac{3\rho^2}{4},t_0+\rho^2\right)} \tilde{U}.$$

It suffices now to slice the domain at $x = 0$ to obtain that

$$\sup_{\left(t_0-\frac{3\rho^2}{4},t_0-\frac{\rho^2}{4}\right)} U(0,t) \leq \gamma \inf_{\left(t_0+\frac{3\rho^2}{4},t_0+\rho^2\right)} U(0,t),$$

hence

$$\sup_{\left(t_0-\frac{3\rho^2}{4},t_0-\frac{\rho^2}{4}\right)} \varphi(t) \leq \gamma \inf_{\left(t_0+\frac{3\rho^2}{4},t_0+\rho^2\right)} \varphi(t)$$

for any ρ such that $0 < \rho < R/2$ and $[t_0 - \rho^2, t_0 + \rho^2] \subset (0,T)$.

Now, in order to prove that the Harnack inequality holds on any interval $J \subset \mathbb{R}$, one considers a translation of U, namely for any $\theta \in \mathbb{R}$, the function $U_\theta(x,t) := U(x,t+\theta)$, and reflects it as in Lemma 4.1. Then $\tilde{U}_\theta$ is a weak solution of (4.5), and $\tilde{U}_\theta(0,t) = \varphi(t+\theta)$. One obtains then, as a consequence of the Harnack inequality for the solution U_θ, the following:

$$\sup_{\left(t_0-\frac{3\rho^2}{4},t_0-\frac{\rho^2}{4}\right)} \varphi(t+\theta) \leq \gamma \inf_{\left(t_0+\frac{3\rho^2}{4},t_0+\rho^2\right)} \varphi(t+\theta)$$

for any ρ such that $0 < \rho < R/2$ and $[t_0 - \rho^2, t_0 + \rho^2] \subset (0,T)$. Therefore

$$\sup_{\left(t_0-\frac{3\rho^2}{4},t_0-\frac{\rho^2}{4}\right)} \varphi(t) \leq \gamma \inf_{\left(t_0+\frac{3\rho^2}{4},t_0+\rho^2\right)} \varphi(t)$$

for any ρ such that $0 < \rho < R/2$ and $[t_0 - \rho^2, t_0 + \rho^2] \subset (\theta, T+\theta)$. As θ and R are arbitrary, one concludes that

$$\sup_{\left(t_0-\frac{3\delta}{4},t_0-\frac{\delta}{4}\right)} \varphi(t) \leq \gamma \inf_{\left(t_0+\frac{3\delta}{4},t_0+\delta\right)} \varphi(t)$$

for any $\delta > 0$ such that $[t_0 - \delta, t_0 + \delta] \subset J$. This concludes the proof of Theorem 1.2. □

Remark 4.2. We would like to point out that the Harnack type inequality obtained in Theorem 1.2 can be equivalently stated as follows. Let us define for every $\delta > 0$ and for every $\tau \in \mathbb{R}$ the sets:

$$I(\tau,\delta) = [\tau - \frac{15}{8}\delta, \tau + \frac{1}{8}\delta],$$
$$I^+(\tau,\delta) = [\tau - \frac{15}{8}\delta, \tau - \frac{7}{4}\delta],$$
$$I^-(\tau,\delta) = [\tau - \frac{1}{8}\delta, \tau + \frac{1}{8}\delta].$$

With this notation, the Harnack inequality gives that for every $I(\tau,\delta)\subset J$

$$\sup_{I^+(\tau,\delta)} \varphi \leq \gamma \inf_{I^-(\tau,\delta)} \varphi.$$

5. **Backward equation**

We now consider the case of the right Marchaud fractional derivative, denoted by $\mathbf{D}^s_-\varphi$. The following result is true:

THEOREM 5.1. *Let $s\in(0,1)$ and $\bar\gamma\in(s,1]$ be fixed. Let $\varphi\in C^{\bar\gamma}(\mathbb{R})$ be a bounded function and let $U_-\colon [0,\infty)\times\mathbb{R}\to\mathbb{R}$ be a solution of the problem*

$$\begin{cases} -\dfrac{\partial U(x,t)}{\partial t} = \dfrac{1-2s}{x}\dfrac{\partial U(x,t)}{\partial x} + \dfrac{\partial^2 U(x,t)}{\partial x^2}, & (x,t)\in(0,\infty)\times\mathbb{R} \\ U(0,t)=\varphi(t), & t\in\mathbb{R} \\ \lim\limits_{x\to+\infty} U(x,t)=0. \end{cases} \tag{5.1}$$

Then U_- defines the extension operator for φ, such that

$$\mathbf{D}^s_-\varphi(t) = -\lim_{x\to0^+} c_s x^{-2s}(U_-(x,t)-\varphi(t)), \qquad \text{where} \qquad c_s = 4^s\Gamma(s).$$

We do not repeat all the computations, that are very similar to the case of the left Marchaud-derivative $\mathbf{D}^s\equiv\mathbf{D}^s_+$.

We only point out that if U_- is a solution of (5.1), then $U_-(x,t)=U(x,-t)$, where U is the solution of the differential equation in (1.4). As in Theorem 2.1 we get that

$$U_-(x,t) = \frac{1}{4^s\Gamma(s)} x^{2s}\int_0^\infty e^{-\frac{x^2}{4\tau}}\tau^{-s-1}\varphi(t+\tau)\,d\tau.$$

Recalling the computations in (3.1) and the properties of the kernel Ψ_s (see formula (2.2)), we obtain that

$$\begin{aligned}
&\lim_{x\to0^+} x^{-2s}\left(U_-(x,t)-\varphi(t)\right) \\
&\qquad = \lim_{x\to0^+} \frac{x^{-2s}}{4^s\Gamma(s)}\int_0^\infty x^{2s}e^{-\frac{x^2}{4\tau}}\tau^{-s-1}\left(\varphi(t+\tau)-\varphi(t)\right)\,d\tau \\
&\qquad = \lim_{x\to0^+} \frac{1}{4^s\Gamma(s)}\int_0^\infty e^{-\frac{x^2}{4\tau}}\frac{\varphi(t+\tau)-\varphi(t)}{\tau^{s+1}}\,d\tau.
\end{aligned}$$

Then, using the same argument as in the proof of Theorem 1.1, we conclude that

$$\lim_{x\to 0^+} x^{-2s}\left(U_-(x,t)-\varphi(t)\right) = \frac{1}{4^s\Gamma(s)}\int_0^\infty \frac{\varphi(t+\tau)-\varphi(t)}{\tau^{s+1}}\,d\tau,$$

that is

$$\mathbf{D}^s_-\varphi(t) = -c_s \lim_{x\to 0^+} x^{-2s}\left(U_-(x,t)-\varphi(t)\right).$$

It is worth to say that $\mathbf{D}^{1-s}_-\mathbf{D}^s_-\varphi(t) = -\dfrac{d\varphi}{dt}$. Hence, using a different notation we can write that

$$\mathbf{D}^s_+\varphi(t) = \left(\frac{d}{dt}\right)^s\varphi, \quad \mathbf{D}^s_-\varphi(t) = \left(-\frac{d}{dt}\right)^s\varphi.$$

6. **Appendix**

In the Appendix, we provide some details on the Marchaud derivative.

First of all, as stated in the introductory Section 1, the Marchaud fractional operator $\mathbf{D}^s\varphi$ is well defined for a bounded function $\varphi\in C^{\bar\gamma}(\mathbb{R})$, with $\bar\gamma > s$. Indeed, we have that:

$$\begin{aligned}\int_0^\infty \frac{\varphi(t)-\varphi(t-\tau)}{\tau^{s+1}}\,d\tau &= \int_1^\infty \frac{\varphi(t)-\varphi(t-\tau)}{\tau^{s+1}}\,d\tau + \int_0^1 \frac{\varphi(t)-\varphi(t-\tau)}{\tau^{s+1}}\,d\tau\\ &= I_1 + I_2.\end{aligned}$$

Since φ is bounded, we have

$$I_1 \le 2\|\varphi\|_{L^\infty(\mathbb{R})}\int_1^\infty \frac{1}{\tau^{s+1}}\,d\tau = C_{s,\varphi}.$$

Moreover, φ is Hölder hence in I_2 we may write

$$|\varphi(t)-\varphi(t-\tau)| \le c\tau^{\bar\gamma}.$$

Therefore

$$I_2 \le c\int_0^1 \tau^{\bar\gamma-s-1}\,d\tau \le C_{s,\bar\gamma},$$

recalling that $\bar\gamma > s$.

There are in the literature many other definitions of fractional derivatives. The interested reader can consult, for instance, [19] or [24] for further details and possibly [21] or [22] for some recent remarks. Here, we recall only the Riemann-Liouville fractional derivative, defined as

$$\mathcal{D}^s_\pm f(t) = \frac{\pm 1}{\Gamma(1-s)}\frac{d}{dt}\int_0^\infty \frac{f(t\mp\tau)}{\tau^s}d\tau$$

for $s \in \mathbb{C}$, $0 < \Re s < 1$ (see [23], Definition 1.16) and the Caputo derivative (see formulas 2.4.17 and 2.4.18 in [15]), given by

$$D^s_\pm f(t) := \frac{\pm 1}{\Gamma(1-s)} \int_0^\infty \frac{f'(t \mp \tau)}{\tau^s}\, d\tau.$$

The definitions of Caputo and Riemann-Liouville are related to the Marchaud definition. Indeed, as one can see in formula (13.2) in the monograph [24], the Marchaud derivative is an extension of Riemann-Liouville's, with weaker conditions on the function f. For a sufficiently smooth f (say absolutely continuous, for instance), integrating by parts in the Riemann-Liouville definition, one can deduce the Marchaud derivative (see also Theorem 1.17 in [23]).

As a further remark, the Marchaud derivative coincides with the notion of fractional derivative given by Grünwald and Letnikov, see [12], [16], and Theorem 20.4 in [24] for the proof.

We like also to remark that, just adapting the constant c_s given in Theorem 1.1 by fixing $c_s = \frac{4^s\Gamma(s)s}{\Gamma(1-s)}$, in (1.5) we straightforwardly obtain the definition (1.1). The advantage of this choice is that $\mathbf{D}^s_\pm \varphi \to \varphi$ as $s \to 0^+$ and $D^s_\pm \varphi \to \varphi'$ as $s \to 1^-$. Indeed, it is well known that the Marchaud derivative is not defined for $s = 0$ and $s = 1$ because in those cases the integral term in (1.1) (as in (1.2)) does not converge. However, one is able to pass to the limit by using the constant term from definition (1.1), that in this sense plays a fundamental role.

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