On Trudinger-Moser type inequalities involving Sobolev-Lorentz spaces

Bernhard Ruf · **Cristina Tarsi**

Abstract Generalizations of the Trudinger-Moser inequality to Sobolev-Lorentz spaces with weights are considered. The weights in these spaces allow for the addition of certain lower order terms in the exponential integral. We prove an explicit relation between the weights and the lower order terms; furthermore, we show that the resulting inequalities are sharp, and that there are related phenomena of concentration-compactness.

Keywords Trudinger-Moser inequality · Lorentz spaces

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1 Introduction

1.1 The Trudinger-Moser inequality

Let $\Omega \subset \mathbb{R}^N$ be a domain of finite measure. The classical Sobolev space embeddings say that $W_0^{1,p}(\Omega) \subset L^q(\Omega)$ for $1 \le q \le \frac{Np}{N-p}$. In the limiting case $p = N$ we formally get $q = +\infty$, but easy examples show that $W_0^{1,N} \not\subset L^\infty(\Omega)$. Replacing the target L^q-space by an Orlicz space L_φ, it was shown by Yudovich [43], Pohozaev [37] and Trudinger [42] that $W_0^1(\Omega) \subset L_\varphi(\Omega)$, with the N-function $\varphi(s) = e^{s^2} - 1$. This result was improved and made sharp by J. Moser [36], obtaining what is now called the *Trudinger-Moser inequality*:

$$\sup_{\{u \in W_0^{1,N} : \|\nabla u\|_N \le 1\}} \int_\Omega e^{\alpha |u|^{\frac{N}{N-1}}} \, dx \qquad \begin{cases} < +\infty \,, & \text{if } \alpha \le \alpha_N \\ = +\infty \,, & \text{if } \alpha > \alpha_N \end{cases} \; ; \qquad (1)$$

here $\| \cdot \|_N$ denotes the norm in $L^N(\Omega)$, and $\alpha_N = N \omega_{N-1}^{1/(N-1)}$, where ω_{N-1} denotes the $(N-1)$-dimensional surface of the unit ball in $\mathbb{R}^{\mathbb{N}}$.

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Numerous generalizations, extensions and applications of the Trudinger-Moser (TM) inequality have been given in recent years:

TM-type inequalities involving higher order derivatives were given by D.R. Adams [1]. The existence of extremals in the TM-inequality was obtained by L. Carleson and A. Chang [10] for $\Omega = B_1(0) \subset \mathbb{R}^2$, by M. Flucher [19] for arbitrary bounded domains in $\mathbb{R}^2$, and by K.C. Lin [33] for bounded domains in $\mathbb{R}^N$; D.G. de Figueiredo - J.M. do Ó - B. Ruf [21] gave an alternative proof and some generalization, using an optimal normalized concentrating sequence. For extensions of the TM-inequality to manifolds, see P. Cherrier [12], L. Fontana [25], Y. Li [31,32], Y. Yang [44]. Related elliptic equations with "critical" TM growth were considered by Adimurthi [3] and de Figueiredo - O. Miyagaki - B. Ruf [20], giving sufficient conditions on the lower order terms for the existence of solutions; in D.G. de Figueiredo - B. Ruf [23] the non-existence of radial solutions was proved for equations with critical TM-growth whose lower order term does not satisfy the above existence conditions. Related existence results for elliptic systems with subcritical and critical TM-growth can be found in D.G. de Figueiredo - J.M. do Ó - B. Ruf [22] and B. Ruf [39]. For phenomena of concentration and blow-up methods in the TM-situation, see M. Struwe [41], Adimurthi - M. Struwe [5], O. Druet [17]. The usual TM-inequalities are for bounded domains; extensions to unbounded domains have been considered by D.M. Cao [9], S. Adachi - K. Tanaka [2], B. Ruf [38], Y. Li - B. Ruf [30]. For recent results on TM-inequalities with remainder terms, we refer to Adimurthi - O. Druet [4]. TM-inequalities with other boundary data and trace inequalities have been recently obtained by A. Cianchi [15], [16].

Finally, and this is closely related to the subject of the present paper, we mention TM-type inequalities in other function spaces, in particular in Orlicz spaces, Zygmund spaces, Lorentz spaces, Besov spaces etc., see e.g. A. Cianchi [14], N. Fusco - P.L. Lions - C. Sbordone [24], A. Alvino - V. Ferone - G. Trombetti [6], D.E. Edmunds - P. Gurka - B. Opic [18], S. Hencl [28], H. Brezis - S. Wainger [8].

In particular, we recall here some recent results for embeddings of Lorentz-Sobolev spaces into Orlicz spaces and the related TM-inequalities:

1.2 Sobolev-Lorentz spaces

Lorentz spaces $L^{p,q}$ are scales of interpolation spaces between the Lebesgues spaces L^p, and are obtained via *spherically decreasing rearrangement*; we refer to Section 2 for the precise definitions. We recall here only that, for $\Omega \subset \mathbb{R}^N$ of finite measure,

$$L^{p,p} = L^p \ , \ L^{p,q_1} \subset L^{p,q_2} \ , \quad \text{if } q_1 < q_2 \ ,$$

$$L^r \subset L^{p,q} \subset L^s \ , \qquad\qquad \text{if } 1 < s < p < r \ , \text{ for all } 1 \leq q \leq \infty$$

We denote the norm in $L^{p,q}$ by $\|u\|_{p,q}$.

First, we recall that the standard Sobolev embeddings can be sharpened by the use of Lorentz spaces, see e.g. [7]; denoting by $W^1 L^{p,q}(\Omega)$ the space of functions whose weak derivatives belong to $L^{p,q}$, one has

$$W^1 L^{p,q} \subset L^{p^*,q}$$

and hence in particular, since $p < p^*$

$$W^{1,p} = W^1 L^{p,p} \subset L^{p^*,p} \underset{\neq}{\subsetneq} L^{p^*,p^*} = L^{p^*} \ .$$

For the limiting case $p = N$, the following generalization of the Trudinger-Moser inequality was obtained by H. Brezis and S. Wainger [8] and A. Alvino, V. Ferone and G. Trombetti [6]: there exist numbers $\beta_q > 0$ such that

$$\sup_{\{\|\nabla u\|_{N,q} \leq 1\}} \int_\Omega e^{\beta |u(x)|^{\frac{q}{q-1}}} dx \quad \begin{cases} \leq C(N,q)|\Omega| \,, & \text{for } \beta \leq \beta_q \\ = +\infty \,, & \text{for } \beta > \beta_q \end{cases} \tag{2}$$

The Trudinger-Moser inequality corresponds to the case $W_0^{1,N}(\Omega) = W_0^1 L^{N,N}(\Omega)$. It is remarkable that in (2) the exponent depends only on the second index q of the Lorentz space.

Note that the inequalities (1) and (2) are sharp not only with respect to the coefficients α resp. β in the exponents. In fact, considering for simplicity the inequality (1) in the case $N = 2$, one notes that if $\alpha = \alpha_2 = 4\pi$, then any unbounded lower order perturbation $f(s)$ in the exponent (i.e. $f(s)$ with $\lim_{|s| \to \infty} f(s) = +\infty$ and $\lim_{s \to \infty} \frac{f(s)}{s^2} = 0$) will yield

$$\sup_{\|\nabla u\|_2 \leq 1} \int_\Omega e^{4\pi |u(x)|^2 + f(u(x))} dx = +\infty \,.$$

In this paper we aim at extending the TM-inequality (1) and the more general Brezis-Wainger inequality (2) with regard to such lower order perturbations. More precisely, concerning inequality (1) (with $N = 2$) we ask: in the limiting case $\alpha = \alpha_2 = 4\pi$, and given an unbounded lower order perturbation function $f(s)$, can we characterize a largest space $\Lambda(g)$ of Lorentz type such that

$$\sup_{\|\nabla u\|_{\Lambda(f)} \leq 1} \int_\Omega e^{4\pi |u(x)|^2 + f(u(x))} dx < +\infty \,. \tag{3}$$

This is a subtle question: note that if we replace in (1) the condition $\|\nabla u\|_2 \leq 1$ by $\|\nabla u\|_2 \leq 1 - \delta$, for an arbitrary $\delta > 0$, then $\sup_{\{\|\nabla u\|_2 \leq 1-\delta\}} \int_\Omega e^{4\pi (\frac{1}{1-\delta}|u(x)|)^2} dx \leq c$, and hence for any subquadratic perturbation $f(u)$ we get $\sup_{\{\|\nabla u\|_2 \leq 1-\delta\}} \int_\Omega e^{4\pi |u(x)|^2 + f(u(x))} dx \leq c$.

We will see that the adequate class of Lorentz spaces for this problem are *weighted Lorentz spaces*, which were proposed by G.G. Lorentz [35] already in his original paper "On the Theory of Spaces". Weighted Lorentz spaces are defined as follows (for details, see Section 2 below): Let $\phi : \Omega \to \mathbb{R}^+$ be a measurable function, and let $\phi^*(s)$ denote its decreasing rearrangement. Furthermore, let $w(t) : \mathbb{R} \to \mathbb{R}^+$ a nonnegative integrable function, such that $\int_0^t w(s) ds < +\infty$ for all $t > 0$. The weighted Lorentz space $\Lambda_p(w)$ is defined as follows: $\phi \in \Lambda^p(w)$, $1 \leq p < +\infty$, if

$$\|\phi\|_{\Lambda_p(w)} = \left(\int_0^{+\infty} (\phi^*(t))^p w(t) dt \right)^{1/p} < +\infty. \tag{4}$$

Quite surprisingly, we are able to establish a precise relation between a weight $w(s)$ and the corresponding lower order perturbation function $f(u)$ to obtain sharp TM-type inequalities.

1.3 The main results

Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function (the "weight function") such that

(H_1) $\displaystyle\lim_{t \to +\infty} \varphi(t) = 0$

(H_2) $\displaystyle\int_0^{+\infty} \varphi(t) = +\infty$

(H_3) $\varphi(t)$ is non increasing as $t \to +\infty$

We prove the following *optimal* Moser type inequality:

Theorem 1 *Let Ω be an open subset of $\mathbb{R}^N$, of finite measure, and let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying (H_1) and (H_2). Let $f(t) \in \mathscr{C}^1(\mathbb{R}^+)$ be defined by*

$$f(t) = \int_0^{\alpha_N t^{\frac{N}{N-1}}} \frac{\varphi(s)}{1 + \varphi(s)} ds \tag{5}$$

where $\alpha_N = N \omega_{N-1}^{1/(N-1)}$ and ω_{N-1} denotes the $(N-1)$-dimensional surface of the unit ball in $\mathbb{R}^N$, $N \geq 2$. Then

$$\sup_{\{u \in C_0^1(\Omega), \|\nabla u\|_{\Lambda_{N,\varphi}} \leq 1\}} \int_\Omega e^{\alpha_N |u|^{\frac{N}{N-1}} + f(u)} \leq C |\Omega| , \tag{6}$$

where

$$\|v\|_{\Lambda_{N,\varphi}}^N = \int_0^{+\infty} \left(v^*(s)\right)^N \left\{1 + \varphi\left(\left|\log\left(\frac{s}{|\Omega|}\right)\right|\right)\right\}^{N-1} ds.$$

and $C = C(\|\varphi\|_\infty)$ is a positive constant that depends only on $\|\varphi\|_\infty$.

Remark 1 Formula (5) yields $f(t)$ if the weight $\varphi(s)$ is given. In principle, formula (5) can be easily inverted, giving an "inverse formula" which allows to determine $\varphi(s)$ for given $f(t)$. However, in order to obtain a well defined $\varphi(s)$ on $\mathbb{R}^+$, some suitable initial value has to be chosen, and the resulting function φ will depend on this initial value. In Theorem 10 we state a related theorem which shows that inequality (6) does not depend on this initial value (except maybe through the constant C).

Examples

1) Let $\varphi_1(s) = \dfrac{1}{2\sqrt{4\pi(s+1)} - 1}$, then $f(s) = s$, i.e.

$$\sup_{\|\nabla u\|_{\Lambda_{2,\varphi_1}} \leq 1} \int_\Omega e^{4\pi u^2 + u} \leq C |\Omega|$$

2) Let $\varphi_2(s) = \dfrac{\sqrt{\pi}\, p}{s + 4\pi\sqrt{s+p}}$, then $f(s) = p\log(1 + |u|)$, i.e.

$$\sup_{\|\nabla u\|_{\Lambda_{2,\varphi_2}} \leq 1} \int_\Omega (1 + |u|)^p\, e^{4\pi u^2} \leq C |\Omega|$$

Inequality (6) is sharp in the following sense:

Theorem 2 *Suppose that φ satisfies (H_1) and (H_2). Then*

(i) for any $\alpha > 1$

$$\sup_{\|\nabla u\|_{\Lambda_{N,\varphi}} \leq 1} \int_\Omega e^{\alpha_N u^{N/(N-1)} + \alpha f(u)} dx = +\infty \tag{7}$$

(ii) if φ satisfies also (H_3), for any $g : \mathbb{R} \to \mathbb{R}$ continuous such that

(g_1) $\qquad \lim_{t \to +\infty} \frac{g(t)}{f(t)} = 0$

(g_2) $\qquad \lim_{t \to +\infty} \dfrac{g\left(\left(\frac{t}{\alpha_N}\right)^{(N-1)/N}\right)}{\int_0^t \varphi^2(s)ds} = +\infty$

one has

$$\sup_{\|\nabla u\|_{\Lambda_{N,\varphi}} \leq 1} \int_\Omega e^{\alpha_N |u|^{N/(N-1)} + f(u) + g(u)} dx = +\infty \tag{8}$$

Finally, we show that inequality (7) has all properties of a true maximal growth: for a given weight function φ, we say that we are at *critical growth* in $W_0^1 \Lambda_{N,\varphi}$ if in (7) $\alpha = 1$ and at *subcritical growth* if $\alpha < 1$. Then we have

Theorem 3

1) For critical growth in $W_0^1 \Lambda_{N,\varphi}$ one has non-compactness: there exist sequences $(u_n) \subset W_0^{1,N}(\Omega)$ with $\|\nabla u_n\|_{\Lambda_{N,\varphi}} = 1$ converging weakly to zero in $W_0^{1,N}(\Omega)$ for which

$$\int_\Omega \left(e^{\alpha_N |u_n|^{N/(N-1)} + f(u_n)} - 1\right) dx \not\to 0$$

2) For subcritical growth in $W_0^1 \Lambda_{N,\varphi}$, there is compactness: for any sequence $(u_n) \subset W_0^{1,N}(\Omega)$ with $\|\nabla u_n\|_{\Lambda_{N,\varphi}} \leq 1$ and such that $u_n \rightharpoonup u$ in $W_0^{1,N}(\Omega)$ we have

$$\int_\Omega e^{\alpha_N |u_n|^{N/(N-1)} + \alpha f(u_n)} dx \to \int_\Omega e^{\alpha_N |u|^{N/(N-1)} + \alpha f(u)} dx \ , \ for \ \alpha < 1 \ .$$

In Section 2 we will give the precise definition and some preliminary results on weighted Lorentz spaces. In Section 3 we give the TM-inequalities for these weighted Sobolev-Lorentz spaces, i.e. we prove Theorem 1. In Section 4 we prove the sharpness of these inequalities, i.e. Theorem 2. In Section 5 we give a compactness and non compactness result for subcritical and critical growth, respectively, that is we prove Theorem 3. Finally, in Section 6 we give the generalizations of Theorem 1 to the Brezis-Wainger case of Lorentz spaces of type $L^{N,p}$ with weights, and Section 7 contains an inverse formula to determine the weight $\varphi(s)$ from $f(t)$.

2 The framework

2.1 Weighted Lorentz spaces

Let $\phi : \Omega \to \mathbb{R}^+$ be a measurable function; we denote by

$$\mu_\phi(t) = |\{x \in \Omega : \phi(x) > t\}|, \qquad t \geq 0$$

its *distribution function*. The *decreasing rearrangement* $\phi^*(s)$ of ϕ is defined by

$$\phi^*(s) = \sup\{t > 0 : \mu_\phi(t) > s\}, \qquad s \in [0, |\Omega|],$$

and the *spherically decreasing rearrangement* $\phi^\#(x)$ of ϕ is defined by

$$\phi^\#(x) = \phi^*(\omega_{N-1}|x|^N/N), \qquad x \in \Omega^\#,$$

where $\Omega^\#$ is the sphere in $\mathbb{R}^N$ such that $|\Omega^\#| = |\Omega|$.

Definition 1 Let $w(t) : \mathbb{R}^+ \to \mathbb{R}^+$ be a nonnegative integrable function, such that $\int_0^t w(s)ds < +\infty$ for all $t > 0$. The weighted Lorentz space $\Lambda_p(w)$ is given as follows: $\phi \in \Lambda_p(w)$, $1 \le p < +\infty$, if

$$\|\phi\|_{\Lambda_p(w)} = \left(\int_0^{+\infty} (\phi^*(t))^p w(t)dt \right)^{1/p} < +\infty. \tag{9}$$

The spaces $\Lambda_p(w)$ were introduced by Lorentz in [35] for $X = (0,l) \subset \mathbb{R}$, and they generalize the Lebesgue spaces L^p and the classical Lorentz spaces $L^{p,q}$. We recall the following properties of weighted Lorentz spaces (see [29] or [11] for a survey on the argument):

1) $\Lambda_p(w)$ is a Banach space and $\|\cdot\|_{\Lambda_p(w)}$ is a norm if and only if w is non-increasing; $\|\cdot\|_{\Lambda_p(w)}$ is merely equivalent to a Banach norm (see [40]) if for some $C > 0$

$$t^p \int_t^{+\infty} s^{-p}w(s)ds \le C \int_0^t w(s)ds, \text{ for all } t > 0 \, ; p > 1 \, .$$

2) $\Lambda_p(w)$ is a Banach space and $\|\cdot\|_{\Lambda_p(w)}$ is a quasi-norm if the function $W(t) = \int_0^t w(s)ds$ satisfies the Δ_2-condition, i.e.,

$$W(2t) \le CW(t) \qquad \text{for some } C > 1 \text{ and all } t \in (0, +\infty).$$

We will consider, in particular, the following weighted Lorentz norms (or quasi-norms):

$$\|v\|_{\Lambda_{N,\varphi}}^N = \int_0^{+\infty} |v^*(s)|^N \left\{ 1 + \varphi\left(\left| \log\left(\frac{s}{|\Omega|} \right) \right| \right) \right\}^{N-1} ds, \tag{10}$$

and the associated Sobolev-Lorentz spaces $W_0^1 \Lambda_{N,\varphi}$, defined as the closure of $\mathscr{C}_0^1(\Omega)$ with respect to the corresponding norm. Thanks to the continuity of φ and hypothesis $(H1)$, $\Lambda_{N,\varphi}$ is a Banach space and $\|\cdot\|_{\Lambda_{N,\varphi}}$ is a quasi norm. Note that for any $u \in W_0^{1,N}(\Omega)$,

$$\|\nabla u\|_N^N \le \|\nabla u\|_{\Lambda_{N,\varphi}}^N \le (1 + \|\varphi\|_\infty)^{N-1} \|\nabla u\|_N^N.$$

Therefore, the setting of these function spaces is nothing but $W_0^{1,N}(\Omega)$, equipped with the norms (or quasi norms) defined above, which are all equivalent to the Dirichlet norm.

2.2 Functions built from level sets

Let us introduce the following relation between nonnegative functions in $L^1(\Omega)$: we say that ϕ is dominated by ψ, and we write $\phi \prec \psi$, if

$$\int_0^s \phi^*(t)dt \leq \int_0^s \psi^*(t)dt \qquad \text{for all } s \in [0, |\Omega|)$$
$$\int_0^{|\Omega|} \phi^*(t)dt = \int_0^{|\Omega|} \psi^*(t)dt$$

This relation was first introduced by Hardy, Littlewood and Pólya in [27] for n-vectors in $\mathbb{R}^n$ and later for Lebesgue integrable functions on a finite interval. We refer to [7] and to [13] for a survey on properties and characterizations of this relation. We recall only the following theorem (see [7]):

Theorem 4 (Alvino, Lions, Trombetti) *Let ϕ, ψ two nonnegative functions in $L^1(\Omega)$. Then, the following assertions are equivalent:*

(i) $\phi \prec \psi$
(ii) for all nonnegative $\eta \in L^\infty(\Omega)$

$$\int_\Omega \phi(x)\eta(x)dx \leq \int_0^{|\Omega|} \psi^*(t)\eta^*(t)dt; \qquad \int_\Omega \phi(x)dx = \int_\Omega \psi(x)dx$$

(iii) for all nonnegative $\eta \in L^\infty(\Omega)$

$$\int_0^{|\Omega|} \phi^*(t)\eta^*(t)dt \leq \int_0^{|\Omega|} \psi^*(t)\eta^*(t)dt; \qquad \int_\Omega \phi(x)dx = \int_\Omega \psi(x)dx.$$

Following [7], we now describe a method to construct a function Φ dominated by a function ϕ.

Let $u(x)$ be a measurable function in Ω; then (see [7]) there exists a family $\{D(s)\}$, $s \in [0, |\Omega|]$ of subsets of Ω satisfying the following properties:

(i) $|D(s)| = s$
(ii) $s_1 < s_2 \Rightarrow D(s_1) \subset D(s_2)$
(iii) $D(s) = \{x \in \Omega : |u(x)| > t\}$, if $s = \mu_u(t)$

For a fixed nonnegative function $\phi \in L^1(\Omega)$, let $\Phi(t)$ be the function defined by

$$\int_{D(s)} \phi(x)dx = \int_0^s \Phi(t)dt, \quad s \in [0, |\Omega|]. \tag{11}$$

We will say that Φ *is built from ϕ on the level sets of $|u|$*. One shows that

$$\Phi \prec \phi \tag{12}$$

3 Proof of Theorem 1

3.1 Some known results

The aim of this section is to prove Theorem 1. To begin with, we construct a function $v(x)$ such that $u^*(s) \leq v^*(s)$, and such that $|\nabla v|$ is dominated by $|\nabla u|$.

Let $u \in \mathscr{C}_0^1(\Omega)$, and let $U(x)$ be the function built from $|\nabla u|$ on the level sets of u, that is as in (11)

$$\int_{|u|>t} |\nabla u| dx = \int_0^{|\{|u|>t\}|} U(s) ds. \tag{13}$$

Then we have

Theorem 5

$$u^*(s) \leq \frac{1}{N^{(N-1)/N} \omega_{N-1}^{1/N}} \int_s^{|\Omega|} \frac{U(t)}{t^{1-1/N}} dt =: v^*(s) \tag{14}$$

Proof The proof of this theorem can be found in [26]. We briefly sketch it.

Using (13) we obtain

$$-\frac{d}{dt} \int_{|u|>t} |\nabla u| dx = -\mu_u'(t) \cdot U(\mu_u(t)).$$

Applying the Fleming-Rishel formula and the isoperimetric inequality yields

$$NC_N^{1/N} \mu_u(t)^{1-1/N} \leq \int_{\partial\{|u|>t\}} dH_1(t) = -\frac{d}{dt} \int_{|u|>t} |\nabla u| dx = -\mu_u'(t) \cdot U(\mu_u(t)),$$

where C_N denotes the measure of the unit ball in $\mathbb{R}^N$, so that

$$-(u^*)'(s) \leq \frac{1}{NC_N^{1/N}} \frac{U(s)}{s^{1-1/N}}.$$

(14) follows immediately, recalling that $NC_N = \omega_{N-1}$.

By Theorem 5, in order to estimate $u(x)$ we can estimate the radial decreasing function

$$v(x) = \frac{1}{N^{\frac{N-1}{N}} \omega_{N-1}^{1/N}} \int_{\frac{\omega_{N-1}}{N}|x|^N}^{|\Omega|} \frac{U(t)}{t^{1-1/N}} dt. \tag{15}$$

Note that, in general, $|\nabla v|^* \neq |\nabla u|^*$, but $|\nabla v|$ is dominated by $|\nabla u|$. This fact, thanks to the following lemma, allows us to estimate $u(x)$ with a function involving $|\nabla u|^*$.

Lemma 1 *Let*

$$v^{**}(s) = \frac{1}{s} \int_0^s v^*(t) dt.$$

Then

$$v^{**}(s) \leq \frac{1}{N^{\frac{N-1}{N}} \omega_{N-1}^{1/N}} \left\{ \int_s^{|\Omega|} |\nabla u|^*(t) \frac{dt}{t^{(N-1)/N}} + \frac{1}{s^{(N-1)/N}} \int_0^s |\nabla u|^*(t) dt \right\}. \tag{16}$$

Proof The proof of this lemma can be found in [6]. We briefly sketch it.

By the definition of $v(x)$,

$$v^{**}(s) = \frac{1}{N^{\frac{N-1}{N}} \omega_{N-1}^{1/N}} \left\{ \int_s^{|\Omega|} U(t) \frac{dt}{t^{(N-1)/N}} + \frac{1}{s} \int_0^s U(t) t^{1/N} dt \right\}$$

$$\leq \frac{1}{N^{\frac{N-1}{N}} \omega_{N-1}^{1/N}} \int_0^{|\Omega|} U(t) g(t,s) dt$$

where

$$g(t,s) = \begin{cases} s^{-\frac{N-1}{N}} & 0 \leq t \leq s \\ t^{-\frac{N-1}{N}} & s < t \leq |\Omega| \end{cases}$$

Since $U \prec |\nabla u|$, Theorem 4 implies (16) directly.

3.2 A Lemma by D.R. Adams

We recall that J. Moser used symmetrization for proving his result (1), thereby reducing the problem to the following one-dimensional calculus inequality: for any measurable function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\int_0^\infty (\phi(t))^N dt \leq 1$$

holds

$$\int_0^\infty e^{-F(t)} dt \leq c_0 , \quad \text{where} \quad F(t) = t - \left(\int_0^t \phi(s) ds \right)^{N/(N-1)}$$

For the extension to higher order derivatives, the method of symmetrization is not available. But working with Riesz potentials, D.R. Adams [1] was again able to reduce the problem to a one-dimensional calculus inequality, namely: let $a : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^+$ be a measurable function such that

$$a(s,t) \leq 1, \text{ if } 0 < s < t, \text{ and } \sup_{t>0} \left(\int_{-\infty}^0 + \int_t^\infty a(s,t)^{p'} ds \right)^{1/p'} = b < \infty$$

Then there exists a constant $c_0(p,b)$ such that for $\phi : \mathbb{R} \to \mathbb{R}^+$ satisfying

$$\int_{-\infty}^\infty \phi(s)^p ds \leq 1$$

holds

$$\int_0^\infty e^{-F(t)} dt \leq c_0 , \quad \text{where} \quad F(t) = t - \left(\int_{-\infty}^\infty a(s,t) \phi(s) ds \right)^{p'}$$

Notice that the above one-dimensional inequality of J. Moser corresponds to the case $a(s,t) = 1$, if $0 < s < t$, and zero otherwise in Adams' inequality.

The proof of our Theorem 1 relies on a generalization of Adams' inequality.

Lemma 2 *Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying hypotheses $(H1), (H2)$, and let $f(t)$ be defined by* (5).
Let $a(s,t)$ be a non-negative measurable function on $\mathbb{R} \times [0, +\infty)$ such that

$$a(s,t) \leq 1, \quad for \ a.e. \quad 0 < s < t \tag{17}$$

$$\sup_{t>0} \left(\int_{-\infty}^0 + \int_t^{+\infty} \frac{a^{\frac{N}{N-1}}(s,t)}{1 + \varphi(s)} ds \right)^{\frac{N-1}{N}} = \gamma < \infty \tag{18}$$

Then there exists a constant $c_0 = c_0(\|\varphi\|_\infty, \gamma)$ such that for $\phi \geq 0$ with

$$\int_{-\infty}^{+\infty} \phi^N(s) \left(1 + \varphi(s) \right)^{N-1} ds \leq 1 \tag{19}$$

one has

$$\int_0^{+\infty} e^{-\Psi(t)} dt \leq c_0 , \tag{20}$$

where

$$\Psi(t) = t - \left\{ \left(\int_{-\infty}^{+\infty} a(s,t)\phi(s)ds \right)^{\frac{N}{N-1}} + f\left(\frac{1}{\alpha_N^{\frac{N-1}{N}}} \int_{-\infty}^{+\infty} a(s,t)\phi(s)ds \right) \right\} \tag{21}$$

Note that for $\varphi(s) \equiv 0$ we have $f(t) \equiv 0$, and hence $\Psi(t) = F(t)$ in Adams' inequality.

Proof The integral in (20) can be written as

$$\int_{-\infty}^{+\infty} |E_\lambda| e^{-\lambda} d\lambda , \tag{22}$$

where $E_\lambda = \{ t \geq 0 : \Psi(t) \leq \lambda \}$. The proof is divided into three steps:

(i) there is a constant $c = c(\gamma, \|\varphi\|_\infty)$ such that $\Psi(t) \geq -c$ for all $t > 0$
(ii) if $t \in E_\lambda$ then

$$\int_t^{+\infty} \phi^N(s) \left(1 + \varphi(s) \right)^{N-1} ds \leq \frac{C_1 + C_2 |\lambda|}{\gamma^{\frac{N}{N-1}} + t - \int_0^t \frac{\varphi}{1+\varphi} ds}$$

where C_1, C_2 are positive constants depending only on $\|\varphi\|_\infty$ and γ.
(iii) $|E_\lambda| \leq A + B|\lambda| + C|\lambda|^{\frac{2N-1}{N}}$ where A, B, C are constants depending only on $\|\varphi\|_\infty, \gamma$ and N.

Proof of (i):
By (17), (18), (19) and Hölder's inequality,

$$\int_{-\infty}^{+\infty} a(s,t)\phi(s)ds \leq \left\{ \int_{-\infty}^0 + \int_t^{+\infty} + \int_0^t \frac{a^{N/(N-1)}(s,t)}{1 + \varphi(s)} \right\}^{\frac{N-1}{N}} \cdot$$
$$\cdot \left\{ \int_{-\infty}^{+\infty} \phi^N(s) \left(1 + \varphi(s) \right)^{N-1} ds \right\}^{1/N}$$
$$\leq \left\{ \gamma^{\frac{N}{N-1}} + t - \int_0^t \frac{\varphi(s)}{1 + \varphi(s)} ds \right\}^{\frac{N-1}{N}},$$

so that

$$\left\{ \int_{-\infty}^{+\infty} a(s,t)\phi(s)ds \right\}^{\frac{N}{N-1}} \leq \gamma^{\frac{N}{N-1}} + t - \int_0^t \frac{\varphi(s)}{1+\varphi(s)}ds \qquad (23)$$

Since $x + f((x/\alpha_N)^{\frac{N-1}{N}})$ is an increasing function on $[0,+\infty)$, (23) implies

$$\Psi(t) \geq -\gamma^{\frac{N}{N-1}} + \int_0^t \frac{\varphi(s)}{1+\varphi(s)}ds - f\left(\left(\frac{t + \gamma^{\frac{N}{N-1}} - \int_0^t \frac{\varphi}{1+\varphi}ds}{\alpha_N}\right)^{\frac{N-1}{N}}\right). \qquad (24)$$

By $(H1)$, $t + \gamma^{\frac{N}{N-1}} - \int_0^t \frac{\varphi}{1+\varphi}ds < t$ if $t > t_\gamma$ large enough, and since $f(t)$ is increasing for $t > t_\gamma$ we get

$$\Psi(t) \geq -\gamma^{\frac{N}{N-1}} + \int_0^t \frac{\varphi(s)}{1+\varphi(s)}ds - f\left((t/\alpha_N)^{\frac{N-1}{N}}\right) = -\gamma^{\frac{N}{N-1}} \; ;$$

on the other hand, if $t \leq t_\gamma$, (24) implies directly that Ψ is bounded from below, and so (i) follows.

Proof of (ii):
If $t \in E_\lambda$, then

$$t - \lambda \leq \left(\int_{-\infty}^{+\infty} a(s,t)\phi(s)ds\right)^{\frac{N}{N-1}} + f\left(\frac{\int_{-\infty}^{+\infty} a(s,t)\phi(s)ds}{\alpha_N^{(N-1)/N}}\right). \qquad (25)$$

Let us define

$$L(t) = \int_t^{+\infty} \phi^N(s)\left(1+\varphi(s)\right)^{N-1}ds \; ; \qquad (26)$$

note that $L(t) \leq 1$, by (19).
If $t \in E_\lambda$, by (17), (18), (19) and Hölder's inequality,

$$\int_{-\infty}^{+\infty} a(s,t)\phi(s)ds = \int_{-\infty}^t + \int_t^{+\infty} a(s,t)\phi(s)ds$$

$$\leq \left\{ \int_{-\infty}^0 + \int_0^t \frac{a^{\frac{N}{N-1}}(s,t)}{1+\varphi(s)} \right\}^{\frac{N-1}{N}} \left\{ \int_{-\infty}^t \phi^N(s)\left(1+\varphi(s)\right)^{N-1}ds \right\}^{1/N}$$

$$+ \left\{ \int_t^{+\infty} \frac{a^{\frac{N}{N-1}}(s,t)}{1+\varphi(s)} \right\}^{\frac{N-1}{N}} \left\{ \int_t^{+\infty} \phi^N(s)\left(1+\varphi(s)\right)^{N-1}ds \right\}^{1/N}$$

$$\leq \left\{ \gamma^{\frac{N}{N-1}} + t - \int_0^t \frac{\varphi(s)}{1+\varphi(s)}ds \right\}^{\frac{N-1}{N}} \left\{ 1 - L(t) \right\}^{1/N} + \gamma L(t)^{1/N}.$$

Let us now observe that for all $1 < \beta \leq 2$ there exists $c_\beta > 0$ such that

$$(a+b)^\beta \leq a^\beta + b^\beta + c_\beta a^{\beta-1}b \quad \forall a,b \geq 0. \qquad (27)$$

Indeed, by scaling it suffices to show that

$$(1+t)^\beta \leq 1 + t^\beta + c_{1,\beta}t \quad \forall 0 < t \leq 1 \quad \text{and}$$
$$(1+t)^\beta \leq 1 + t^\beta + c_{2,\beta}t \quad \forall t \geq 1;$$

this follows from the fact that

$$\lim_{t \to 0} \frac{(1+t)^\beta - 1 - t^\beta}{t} = \beta > 0 \quad \text{and}$$

$$\lim_{t \to +\infty} \frac{(1+t)^\beta - 1 - t^\beta}{t} = \begin{cases} 2, & \beta = 2 \\ 0, & 1 < \beta < 2 \end{cases}$$

Thus

$$\left\{ \int_{-\infty}^{+\infty} a(s,t)\phi(s)ds \right\}^{\frac{N}{N-1}} \leq \alpha(t)\left(1 - L(t)\right)^{\frac{1}{N-1}} + c_N \gamma L(t)^{1/N}\alpha(t)^{1/N} + \gamma^{\frac{N}{N-1}} \qquad (28)$$

where

$$\alpha(t) = \gamma^{\frac{N}{N-1}} + t - \int_0^t \frac{\varphi(s)}{1 + \varphi(s)}ds. \qquad (29)$$

Note that $\alpha(t) = \gamma^{\frac{N}{N-1}} + \int_0^t \frac{1}{1+\varphi}ds$, so that

$$\alpha(t) \geq \gamma^{\frac{N}{N-1}} \qquad \text{for all } t \in (0, +\infty). \qquad (30)$$

Inserting (28) into (25), and recalling that $x + f((x/\alpha_N)^{\frac{N-1}{N}})$ is increasing, we have

$$t - \lambda \leq \alpha(t)\left(1 - L(t)\right)^{\frac{1}{N-1}} + c_N \gamma L(t)^{1/N}\alpha(t)^{1/N} + \gamma^{\frac{N}{N-1}}$$
$$+ f\left(\frac{\alpha(t)^{\frac{N-1}{N}}(1 - L(t))^{1/N} + \gamma L(t)^{1/N}}{\alpha_N^{(N-1)N}} \right).$$

It is easy to verify that for $0 < p \leq 1$,

$$(1 - \varepsilon)^p \leq 1 - \frac{p}{2}\varepsilon \quad 0 \leq \varepsilon \leq 1, \qquad (31)$$

so that

$$\left(1 - L(t)\right)^{\frac{1}{N-1}} < 1 - \frac{L(t)}{2(N-1)}.$$

Hence, recalling the definition of $\alpha(t)$, (29), we have

$$L(t)\alpha(t) - 2(N-1)c_N\gamma\left(L(t)\alpha(t)\right)^{\frac{1}{N}} \leq 4(N-1)\gamma^{\frac{N}{N-1}} - 2(N-1)\int_0^t \frac{\varphi(s)}{1 + \varphi(s)}ds$$
$$+ 2(N-1)\lambda + 2(N-1)f\left(\frac{\alpha(t)^{\frac{N-1}{N}}(1-L(t))^{1/N} + \gamma L(t)^{1/N}}{\alpha_N^{(N-1)/N}} \right). \qquad (32)$$

Observe now that by definition and (27),

$$f\left(\frac{\alpha(t)^{\frac{N-1}{N}}(1-L(t))^{1/N} + \gamma L(t)^{1/N}}{\alpha_N^{(N-1)/N}} \right) = \int_0^{\left(\alpha(t)^{\frac{N-1}{N}}(1-L(t))^{1/N} + \gamma L(t)^{1/N} \right)^{N/(N-1)}} \frac{\varphi(s)}{1 + \varphi(s)}ds$$

$$\leq \int_0^{\alpha(t) + c_N\gamma L(t)^{1/N}\alpha(t)^{1/N} + \gamma^{\frac{N}{N-1}}} \frac{\varphi(s)}{1 + \varphi(s)}ds$$

$$= \int_0^{\alpha(t)} + \int_{\alpha(t)}^{\alpha(t) + c_N\gamma L(t)^{1/N}\alpha(t)^{1/N} + \gamma^{\frac{N}{N-1}}} \frac{\varphi}{1 + \varphi}ds.$$

On one hand,

$$\int_0^{\alpha(t)} \frac{\varphi(s)}{1+\varphi(s)}ds = \int_0^{\gamma^{\frac{N}{N-1}}+t-\int_0^t \frac{\varphi}{1+\varphi}} \frac{\varphi(s)}{1+\varphi(s)}ds$$

$$\leq \int_0^t \frac{\varphi(s)}{1+\varphi(s)}ds$$

when $t > t_\gamma$ large enough; on the other,

$$\int_{\alpha(t)}^{\alpha(t)+c_N\gamma L(t)^{1/N}\alpha(t)^{1/N}+\gamma^{\frac{N}{N-1}}} \frac{\varphi(s)}{1+\varphi(s)}ds \leq c_N\gamma L(t)^{1/N}\alpha(t)^{1/N}+\gamma^{\frac{N}{N-1}}.$$

Combining these inequalities with (32) we obtain

$$L(t)\alpha(t) - 4(N-1)c_N\gamma\Big(L(t)\alpha(t)\Big)^{\frac{1}{N}} \leq 6(N-1)\gamma^{\frac{N}{N-1}} + c(\gamma, \|\varphi\|_\infty) + 2(N-1)\lambda$$

Let us now observe that there exists $c(N,\gamma)$ such that $\frac{1}{2}x + c(N,\gamma) \geq 4(N-1)c_N\gamma\sqrt[N]{x}$ for any $x \in [0, +\infty)$, that is, $x - 4(N-1)c_N\gamma\sqrt[N]{x} \geq \frac{1}{2}x - c(N,\gamma)$. Thus

$$\tfrac{1}{2}\alpha(t)L(t) \leq c(N,\gamma,\|\varphi\|_\infty) + 2(N-1)|\lambda|,$$

which yields (ii) directly.

Proof of (iii):

It suffices to prove that there exist $C_3, C_4, C_5, C_6 > 0$, depending only on γ and $\|\varphi\|_\infty$, such that

$$\begin{cases} t_1, t_2 \in E_\lambda, \\ t_2 > t_1 > C_3|\lambda| \end{cases} \Rightarrow t_2 - t_1 \leq C_4 + C_5|\lambda| + C_6|\lambda|^{\frac{2N-1}{N}}. \tag{33}$$

Since $t_2 \in E_\lambda$, by (17), (18), (19) and Hölder's inequality,

$$\int_{-\infty}^{+\infty} a(s,t_2)\phi(s)ds = \int_{-\infty}^{t_1} + \int_{t_1}^{t_2} + \int_{t_2}^{+\infty} a(s,t_2)\phi(s)ds$$

$$\leq \Big\{\int_{-\infty}^0 + \int_0^{t_1} \frac{a^{\frac{N}{N-1}}(s,t_2)}{1+\varphi(s)}\Big\}^{\frac{N-1}{N}} + \Big\{\int_{t_1}^{t_2} \frac{a^{\frac{N}{N-1}}(s,t_2)}{1+\varphi(s)}\Big\}^{\frac{N-1}{N}} L(t_1)^{\frac{1}{N}} + \gamma L(t_1)^{\frac{1}{N}}$$

$$\leq \alpha(t_1)^{\frac{N-1}{N}} + \Big(t_2 - t_1 - \int_{t_1}^{t_2} \frac{\varphi(s)}{1+\varphi(s)}ds\Big)^{\frac{N-1}{N}} L(t_1)^{\frac{1}{N}} + \gamma L(t_1)^{\frac{1}{N}}$$

$$\leq \alpha(t_1)^{\frac{N-1}{N}} + \Big((t_2-t_1)^{\frac{N-1}{N}} + \gamma\Big)L(t_1)^{\frac{1}{N}}.$$

Therefore, by (27) and (ii)

$$\Big\{\int_{-\infty}^{+\infty} a(s,t_2)\phi(s)ds\Big\}^{\frac{N}{N-1}} \leq \alpha(t_1) + \Big((t_2-t_1)^{\frac{N-1}{N}} + \gamma\Big)^{\frac{N}{N-1}} L(t_1)^{\frac{1}{N-1}}$$

$$+ c_N\Big((t_2-t_1)^{\frac{N-1}{N}} + \gamma\Big)L(t_1)^{\frac{1}{N}}\alpha(t_1)^{\frac{1}{N}}$$

$$\leq \alpha(t_1) + \Big((t_2-t_1)^{\frac{N-1}{N}} + \gamma\Big)^{\frac{N}{N-1}}\Big(\frac{C_1+C_2|\lambda|}{\alpha(t_1)}\Big)^{\frac{1}{N-1}} + c_N\Big((t_2-t_1)^{\frac{N-1}{N}} + \gamma\Big)(C_1+C_2|\lambda|)^{\frac{1}{N}}.$$

Since $\alpha(t) = t + o(t)$ as $t \to +\infty$, there exists for any $\varepsilon > 0$ a constant $C_\varepsilon > 0$ such that for any $t > C_\varepsilon |\lambda|$,

$$\frac{C_1 + C_2 |\lambda|}{\alpha(t)} < \varepsilon \; ; \tag{34}$$

then we have

$$\left\{ \int_{-\infty}^{+\infty} a(s, t_2) \phi(s) ds \right\}^{\frac{N}{N-1}} \leq \alpha(t_1) + \varepsilon \left((t_2 - t_1)^{\frac{N-1}{N}} + \gamma \right)^{\frac{N}{N-1}}$$
$$+ c_N \left((t_2 - t_1)^{\frac{N-1}{N}} + \gamma \right) (C_1 + C_2 |\lambda|)^{\frac{1}{N}}. \tag{35}$$

Since $t_2 \in E_\lambda$, $-\lambda \leq -\Psi(t_2)$; applying (35), we have for any $t_1 > 0$

$$t_2 - t_1 \leq \gamma^{\frac{N}{N-1}} - \int_0^{t_1} \frac{\varphi(s)}{1 + \varphi(s)} ds + \varepsilon((t_2 - t_1)^{\frac{N-1}{N}} + \gamma)^{\frac{N}{N-1}}$$
$$+ c_N((t_2 - t_1)^{\frac{N-1}{N}} + \gamma)(C_1 + C_2 |\lambda|)^{\frac{1}{N}} + \lambda$$
$$+ f\left(\frac{\{ \alpha(t_1) + \varepsilon((t_2 - t_1)^{\frac{N-1}{N}} + \gamma)^{\frac{N}{N-1}} + c_N((t_2 - t_1)^{\frac{N-1}{N}} + \gamma)(C_1 + C_2 |\lambda|)^{\frac{1}{N}} \}^{(N-1)/N}}{\alpha_N^{(N-1)/N}} \right)$$

As in the proof of (ii), the last term can be estimated as follows

$$f\left(\frac{\{ \alpha(t) + ... \}^{(N-1)/N}}{\alpha_N^{(N-1)/N}} \right) = \int_0^{\alpha(t_1) + \varepsilon((t_2 - t_1)^{\frac{N-1}{N}} + \gamma)^{\frac{N}{N-1}} + c_N((t_2 - t_1)^{\frac{N-1}{N}} + \gamma)(C_1 + C_2 |\lambda|)^{\frac{1}{N}}} \frac{\varphi}{1 + \varphi} ds$$

$$\leq \int_0^{\alpha(t_1)} \frac{\varphi(s)}{1 + \varphi(s)} ds$$
$$+ \varepsilon((t_2 - t_1)^{\frac{N-1}{N}} + \gamma)^{\frac{N}{N-1}} + c_N((t_2 - t_1)^{\frac{N-1}{N}} + \gamma)(C_1 + C_2 |\lambda|)^{\frac{1}{N}}$$

$$\leq \int_0^{t_1} \frac{\varphi(s)}{1 + \varphi(s)} ds + c(N, \gamma, \|\varphi\|_\infty)$$
$$+ \varepsilon((t_2 - t_1)^{\frac{N-1}{N}} + \gamma)^{\frac{N}{N-1}} + c_N((t_2 - t_1)^{\frac{N-1}{N}} + \gamma)(C_1 + C_2 |\lambda|)^{\frac{1}{N}}$$

so that

$$t_2 - t_1 \leq c(N, \gamma, \|\varphi\|_\infty) + \gamma^{\frac{N}{N-1}} + 2\varepsilon((t_2 - t_1)^{\frac{N-1}{N}} + \gamma)^{\frac{N}{N-1}}$$
$$+ 2c_N((t_2 - t_1)^{\frac{N-1}{N}} + \gamma)(C_1 + C_2 |\lambda|)^{\frac{1}{N}} + \lambda.$$

Observe now that, by the Young inequality, for any $\eta > 0$

$$2c_N((t_2 - t_1)^{\frac{N-1}{N}} + \gamma)(C_1 + C_2 |\lambda|)^{\frac{1}{N}} \leq \eta^{\frac{2N-1}{2N-2}} \frac{2N-2}{2N-1}((t_2 - t_1)^{\frac{N-1}{N}} + \gamma)^{\frac{2N-1}{2N-2}}$$
$$+ \left(\frac{2c_N}{\eta} \right)^{2N-1} \frac{(C_1 + C_2 |\lambda|)^{\frac{2N-1}{N}}}{2N-1}$$

$$\leq (2\eta)^{\frac{2N-1}{2N-2}} \frac{2N-2}{2N-1}((t_2 - t_1)^{\frac{2N-1}{2N}} + \gamma^{\frac{2N-1}{2N-2}}) + \left(\frac{2c_N}{\eta} \right)^{\frac{2N-1}{N}} \frac{(C_1 + C_2 |\lambda|)^{\frac{2N-1}{N}}}{2N-1}$$

$$\leq (2\eta)^{\frac{2N-1}{2N-2}} \frac{2N-2}{2N-1}((t_2 - t_1) + 1 + \gamma^{\frac{2N-1}{2N-2}}) + \left(\frac{2c_N}{\eta} \right)^{\frac{2N-1}{N}} \frac{(C_1 + C_2 |\lambda|)^{\frac{2N-1}{N}}}{2N-1}$$

since $(a+b)^p \leq 2^p(a^p + b^p)$ for any $a, b > 0$ and $p > 0$. On the other hand,

$$2\varepsilon((t_2 - t_1)^{\frac{N-1}{N}} + \gamma)^{\frac{N}{N-1}} \leq 2^{\frac{2N-1}{N-1}} \varepsilon((t_2 - t_1) + \gamma^{\frac{N}{N-1}})$$

Therefore,

$$\left(1-(2\eta)^{\frac{2N-1}{2N-2}}\frac{2N-2}{2N-1}-2^{\frac{2N-1}{N-1}}\varepsilon\right)(t_2-t_1)\leq c(N,\gamma,\|\varphi\|_\infty,\eta,\varepsilon)$$

$$+\left(\frac{2c_N}{\eta}\right)^{\frac{2N-1}{N}}\frac{(C_1+C_2|\lambda|)^{\frac{2N-1}{N}}}{2N-1}+|\lambda|$$

(33) follows directly, choosing ε,η such that $1-(2\eta)^{\frac{2N-1}{2N-2}}\frac{2N-2}{2N-1}-2^{\frac{2N-1}{N-1}}\varepsilon>0$.

Combining (22) with (i) and (iii) we have

$$\int_0^{+\infty}e^{-\Psi(t)}dt=\int_{-\infty}^{+\infty}|E_\lambda|e^{-\lambda}d\lambda$$

$$=\int_{-c}^{+\infty}|E_\lambda|e^{-\lambda}d\lambda\leq c_0,$$

that is our thesis.

3.3 Proof of Theorem 1

Without loss of generality, we may suppose that $u\geq 0$. By (14), and recalling the definition of v^{**} given in Lemma 1,

$$u^*(s)\leq v^*(s)\leq v^{**}(s)\,.$$

Therefore (recall that $\alpha_N x^{\frac{N}{N-1}}+f(x)$ is increasing in $[0,+\infty)$)

$$\int_\Omega e^{\alpha_N|u|^{\frac{N}{N-1}}+f(u)}dx=\int_0^{|\Omega|}e^{\alpha_N(u^*)^{\frac{N}{N-1}}+f(u^*)}ds\leq\int_0^{|\Omega|}e^{\alpha_N(v^{**})^{\frac{N}{N-1}}+f(v^{**})}ds$$

$$=|\Omega|\int_0^{+\infty}e^{\alpha_N(v^{**}(|\Omega|e^{-t}))^{\frac{N}{N-1}}+f(v^{**}(|\Omega|e^{-t}))-t}dt$$

$$=|\Omega|\int_0^{+\infty}e^{w^{\frac{N}{N-1}}+f(w/N^{\frac{N-1}{N}}\omega_N^{1/N})-t}dt\,,$$

where

$$w(t)=\alpha_N^{\frac{N-1}{N}}v^{**}(|\Omega|e^{-t})=N^{\frac{N-1}{N}}\omega_{N-1}^{1/N}v^{**}(|\Omega|e^{-t})\,.$$

Lemma 1 implies

$$w(t)\leq|\Omega|^{1/N}\left\{\int_0^t|\nabla u|^*(|\Omega|e^{-s})e^{-s/N}ds+e^{t(1-1/N)}\int_t^{+\infty}|\nabla u|^*(|\Omega|e^{-s})e^{-s}ds\right\}$$

$$=\int_{-\infty}^{+\infty}\Phi(s)a(s,t)ds\,,$$

where

$$a(s,t)=\begin{cases}0 & \text{if }s\leq 0\\e^{(t-s)\frac{N-1}{N}} & \text{if }t<s<+\infty\\1 & \text{if }0<s<t\end{cases}$$

and

$$\Phi(s)=\begin{cases}|\Omega|^{1/N}|\nabla u|^*(|\Omega|e^{-s})e^{-s/N} & \text{if }s\geq 0\\0 & \text{if }s<0\end{cases}$$

Hence, the proof of (6) will be concluded if we check that the hypotheses of Lemma 2 are satisfied.

Hypothesis (17) is clearly verified. As regards (18),

$$
\int_{-\infty}^{0} + \int_{t}^{+\infty} \frac{a^{\frac{N}{N-1}}(s,t)}{1+\varphi(s)} ds = \int_{t}^{+\infty} \frac{e^{t-s}}{1+\varphi(s)} ds
$$
$$
\leq \int_{t}^{+\infty} e^{t-s} ds = 1.
$$

Finally,

$$
\int_{-\infty}^{+\infty} \Phi^{N}(s) \left(1+\varphi(s)\right)^{N-1} ds = |\Omega| \int_{0}^{+\infty} \left(|\nabla u|^{*}(|\Omega|e^{-s})\right)^{N} \left(1+\varphi(s)\right)^{N-1} e^{-s} ds
$$
$$
= \int_{0}^{|\Omega|} \left(|\nabla u|^{*}(t)\right)^{N} \left(1+\varphi\left(\left|\log\left(\frac{t}{|\Omega|}\right)\right|\right)\right)^{N-1} dt
$$
$$
= \|\nabla u\|_{\Lambda_{N,\varphi}}^{N} \leq 1.
$$

Therefore, (17), (18) and (19) are satisfied; Lemma 2 yields (6). □

4 Sharpness: proof of Theorem 2

Let us suppose now that φ satisfies (H_1) and (H_2) ; we will prove that inequality (6) is sharp, that is there exists a sequence of functions $(u_n) \subset W_0^{1,N}(\Omega)$ such that $\|\nabla u_n\|_{\Lambda_{N,\varphi}} \leq 1$ and

(i) for any $\alpha > 1$

$$
\lim_{n\to+\infty} \int_{\Omega} e^{\alpha_N u_n^{N/(N-1)} + \alpha f(u_n)} dx = +\infty \tag{36}
$$

(ii) if φ satisfies (H_3), for any continuous function $g : \mathbb{R} \to \mathbb{R}$ satisfying $(g_1), (g_2)$

$$
\lim_{n\to+\infty} \int_{\Omega} e^{\alpha_N u_n^{N/(N-1)} + f(u_n) + g(u_n)} dx = +\infty \tag{37}
$$

The sequence of functions we exhibit is obtained normalizing in $W_0^1 \Lambda_{N,\varphi}$ the sequence used by J. Moser in [36]. Consider for example the ball B centered at the origin and such that $|B| = 1$. Let us define

$$
v_n(x) = \begin{cases} \dfrac{(1-\delta_n)^{\frac{N-1}{N}}}{N^{\frac{N-1}{N}} \omega_{N-1}^{1/N}} n^{\frac{N-1}{N}} & , \ \frac{\omega_{N-1}}{N}|x|^N < e^{-n} \\[3ex] \dfrac{(1-\delta_n)^{\frac{N-1}{N}}}{N^{\frac{N-1}{N}} \omega_{N-1}^{1/N} n^{1/N}} \log\left(\dfrac{N}{\omega_{N-1}|x|^N}\right) & , \ e^{-n} \leq \frac{\omega_{N-1}}{N}|x|^N \leq 1 , \end{cases} \tag{38}
$$

where $\delta_n \in (0,1)$ will be fixed later. We have

$$
v_n^*(s) = \begin{cases} \dfrac{(1-\delta_n)^{\frac{N-1}{N}}}{N^{\frac{N-1}{N}} \omega_{N-1}^{1/N}} n^{\frac{N-1}{N}} & , \ 0 < s < e^{-n} \\[3ex] \dfrac{(1-\delta_n)^{\frac{N-1}{N}}}{N^{\frac{N-1}{N}} \omega_{N-1}^{1/N} n^{1/N}} \log\left(\dfrac{1}{s}\right) & , \ e^{-n} \leq s \leq 1 ; \end{cases}
$$

furthermore

$$|\nabla v_n|(x) = \begin{cases} 0 & , \ \frac{\omega_{N-1}}{N}|x|^N < e^{-n} \\[3mm] \dfrac{N^{1/N}(1-\delta_n)^{\frac{N-1}{N}}}{\omega_{N-1}^{1/N}n^{1/N}|x|} & , \ e^{-n} < \frac{\omega_{N-1}}{N}|x|^N < 1 \end{cases}$$

and

$$|\nabla v_n|^*(s) = \begin{cases} 0 & , \ 1-e^{-n} \leq s \leq 1 \\[3mm] \dfrac{(1-\delta_n)^{\frac{N-1}{N}}}{(s+e^{-n})^{1/N}n^{1/N}} & , \ 0 \leq s < 1-e^{-n} \ . \end{cases}$$

Therefore

$$\begin{aligned} \|v_n\|_{\Lambda_{N,\varphi}}^N &= \int_0^1 \big(|\nabla v_n|^*\big)^N \big(1+\varphi(|\log s|)\big)^{N-1} ds \\ &= (1-\delta_n)^{N-1} + \\ &\quad + \frac{(1-\delta_n)^{N-1}}{n}\int_0^{1-e^{-n}} \frac{1}{(s+e^{-n})}\sum_{k=1}^{N-1}\binom{N-1}{k}\varphi^k(-\log s)ds. \end{aligned} \tag{39}$$

We claim that

$$\begin{aligned} I_N(n) &:= \int_0^{1-e^{-n}} \frac{1}{(s+e^{-n})}\sum_{k=1}^{N-1}\binom{N-1}{k}\varphi^k(-\log s)ds \\ &\sim (N-1)\int_0^n \varphi(s)ds \ , \ \text{ as } n \to +\infty \ . \end{aligned} \tag{40}$$

Indeed, on one hand we have

$$\begin{aligned} I_N(n) &\geq (N-1)\int_0^{1-e^{-n}} \frac{\varphi(-\log s)}{s+e^{-n}}ds = (N-1)\int_{-\log(1-e^{-n})}^{+\infty} \frac{\varphi(t)}{1+e^{t-n}}dt \\ &\geq (N-1)\int_{-\log(1-e^{-n})}^{n} \frac{\varphi(t)}{1+e^{t-n}}dt \\ &= (N-1)\int_{-\log(1-e^{-n})}^{n} \varphi(t)dt - (N-1)\int_{-\log(1-e^{-n})}^{n} \varphi(t)\frac{e^{t-n}}{1+e^{t-n}}dt \\ &= (N-1)\Big(\int_0^n \varphi(t)dt - \int_0^{-\log(1-e^{-n})} \varphi(t)dt - \|\varphi\|_\infty \int_{-\log(1-e^{-n})}^{n} \frac{e^{t-n}}{1+e^{t-n}}dt\Big) \\ &= (N-1)\Big(\int_0^n \varphi(t)dt - \log 2\|\varphi\|_\infty + \mathrm{o}(1)\Big) \\ &\sim (N-1)\int_0^n \varphi(t)dt \quad \text{ as } n \to +\infty \end{aligned}$$

On the other hand,

$$
\begin{aligned}
I_N(n) = {} & (N-1) \int_{-\log(1-e^{-n})}^{+\infty} \frac{\varphi(t)}{1+e^{t-n}} dt \\
& + \int_{-\log(1-e^{-n})}^{+\infty} \sum_{k=2}^{N-1} \binom{N-1}{k} \frac{\varphi^k(t)}{1+e^{t-n}} dt \\
\leq {} & (N-1)\left(\int_0^n \varphi(t)dt + \|\varphi\|_\infty \int_n^{+\infty} \frac{dt}{1+e^{t-n}} \right) \\
& + \sum_{k=2}^{N-1} \binom{N-1}{k} \int_0^n \varphi^k(t)dt + \sum_{k=2}^{N-1} \binom{N-1}{k} \|\varphi\|_\infty^k \int_n^{+\infty} \frac{1}{1+e^{t-n}} dt \\
= {} & (N-1) \int_0^n \varphi(t)dt + \mathrm{o}\left(\int_0^n \varphi \right) + \mathrm{O}(1) \\
\sim {} & (N-1) \int_0^n \varphi(t)dt \quad \text{as } n \to +\infty
\end{aligned}
\tag{41}
$$

by (H_2), so that (40) is proved.

Let us now choose

$$
\delta_n = \frac{1}{n(N-1)} I_N(n) \, ;
\tag{42}
$$

then by (40)

$$
\delta_n \sim \frac{1}{n} \int_0^n \varphi(s)ds \to 0 \quad \text{as} \quad n \to +\infty \, ;
\tag{43}
$$

furthermore, recalling (39)

$$
\|v_n\|_{\Lambda_{N,\varphi}}^N = 1 - \frac{N(N-1)}{2} \delta_n^2 + \mathrm{o}\left(\delta_n^2\right) < 1 \, .
\tag{44}
$$

Consider now

$$
u_n(x) = \frac{v_n(x)}{\|v_n\|_{\Lambda_{N,\varphi}}} \, ;
$$

then clearly

$$
\|u_n\|_{\Lambda_{N,\varphi}} = 1
$$

and, by (43), for any $\alpha > 1$

$$
\begin{aligned}
\int_B e^{\alpha_N u_n^{\frac{N}{N-1}} + \alpha f(u_n)} dx &= \int_0^1 e^{\alpha_N (u_n^*)^{\frac{N}{N-1}} + \alpha f(u_n^*)} ds \\
&\geq \exp\left(\frac{1-\delta_n}{\|v_n\|_{\Lambda_{N,\varphi}}^{N/(N-1)}} n + \alpha f\left(\frac{(1-\delta_n)^{\frac{N-1}{N}} n^{\frac{N-1}{N}}}{\|v_n\|_{\Lambda_{N,\varphi}} (\alpha_N)^{\frac{N-1}{N}}} \right) - n \right).
\end{aligned}
\tag{45}
$$

By definition (42), (43) and (44),

$$
f\left(\frac{(1-\delta_n)^{\frac{N-1}{N}} n^{\frac{N-1}{N}}}{\|v_n\|_{\Lambda_{N,\varphi}} (\alpha_N)^{\frac{N-1}{N}}}\right) = \int_0^{\frac{1-\delta_n}{\|v_n\|_{\Lambda_{N,\varphi}}^{N/(N-1)}} n} \frac{\varphi(s)}{1+\varphi(s)} ds
$$

$$
= \int_0^{\frac{1-\delta_n}{\|v_n\|_{\Lambda_{N,\varphi}}^{N/(N-1)}} n} \left\{\varphi(s) - \frac{\varphi^2(s)}{1+\varphi(s)}\right\} ds
$$

$$
\geq \int_0^{\frac{1-\delta_n}{\|v_n\|_{\Lambda_{N,\varphi}}^{N/(N-1)}} n} \left\{\varphi(s) - \varphi^2(s)\right\} ds
$$

$$
= \int_0^n \left\{\varphi(s) - \varphi^2(s)\right\} ds - \int_{\frac{1-\delta_n}{\|v_n\|_{\Lambda_{N,\varphi}}^{N/(N-1)}} n}^n \left\{\varphi(s) - \varphi^2(s)\right\} ds
$$

$$
= \int_0^n \left\{\varphi(s) - \varphi^2(s)\right\} ds - \frac{n\delta_n}{\|v_n\|_{\Lambda_{N,\varphi}}^{N/(N-1)}} \left\{\varphi(t_n) - \varphi^2(t_n)\right\} \quad \text{where } t_n \in \left[\frac{1-\delta_n}{\|v_n\|_{\Lambda_{N,\varphi}}^{N/(N-1)}} n, n\right]
$$

$$
= \int_0^n \varphi(s) ds + \mathrm{o}\left(\int_0^n \varphi\right). \tag{46}
$$

Combining (45) with (46) and (42) yields

$$
\int_B e^{\alpha_N u_n^{\frac{N}{N-1}} + \alpha f(u_n)} dx \geq \exp\left(\frac{1-\delta_n}{1+\mathrm{o}(\delta_n)} n + \alpha\left(n\delta_n + \mathrm{o}(n\delta_n)\right) - n\right)
$$

$$
= \exp\left((\alpha - 1)n\delta_n + \mathrm{o}(n\delta_n)\right) \longrightarrow +\infty
$$

for any $\alpha > 1$.

If $\alpha = 1$, let φ satisfies (H_3) and let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying $(g_1), (g_2)$. By (42) and observing that

$$
I_N(n) \leq (N-1)\int_0^n \varphi(s) ds + \frac{(N-1)(N-2)}{2}\int_0^n \varphi^2(s) ds + \mathrm{o}\left(\int_0^n \varphi^2\right)
$$

we have

$$
\int_0^n \varphi(s) ds \geq n\delta_n - \frac{N-2}{2}\int_0^n \varphi^2(s) ds + \mathrm{o}\left(\int_0^n \varphi^2\right). \tag{47}
$$

On the other hand, as in (46) we have

$$
f\left(\frac{(1-\delta_n)^{\frac{N-1}{N}} n^{\frac{N-1}{N}}}{\|v_n\|_{\Lambda_{N,\varphi}} (\alpha_N)^{\frac{N-1}{N}}}\right) \geq \int_0^{\frac{1-\delta_n}{\|v_n\|_{\Lambda_{N,\varphi}}^{N/(N-1)}} n} \varphi(s) ds - \int_0^{\frac{1-\delta_n}{\|v_n\|_{\Lambda_{N,\varphi}}^{N/(N-1)}} n} \varphi^2(s) ds;
$$

by (H_3), for any $\lambda \in (0,1)$

$$
\int_0^{\lambda n} \varphi(s) ds \geq \int_{\lambda t_0}^{\lambda n} \varphi(s) ds = \lambda \int_{t_0}^n \varphi(\lambda r) dr
$$

$$
\geq \lambda \int_{t_0}^n \varphi(r) dr
$$

$$
= \lambda \int_0^n \varphi(r) dr - \lambda \int_0^{t_0} \varphi(r) dr
$$

$$
\geq \lambda \int_0^n \varphi(r) dr - \lambda t_0 \|\varphi\|_\infty \tag{48}
$$

so that, by (44) and (47),

$$
f\left(\frac{(1-\delta_n)^{\frac{N-1}{N}} n^{\frac{N-1}{N}}}{\|v_n\|_{\Lambda_{N,\varphi}}(\alpha_N)^{\frac{N-1}{N}}}\right) \geq \frac{1-\delta_n}{\|v_n\|_{\Lambda_{N,\varphi}}^{N/(N-1)}} \int_0^n \varphi(s)ds - \int_0^{\frac{1-\delta_n}{\|v_n\|_{\Lambda_{N,\varphi}}^{N/(N-1)}} n} \varphi^2(s)ds - C
$$

$$
\geq \frac{1-\delta_n}{\|v_n\|_{\Lambda_{N,\varphi}}^{N/(N-1)}} \int_0^n \varphi(s)ds - \int_0^n \varphi^2(s)ds - C
$$

$$
\geq \left(1-\delta_n + \mathrm{O}(\delta_n^2)\right)\left(n\delta_n - \frac{N-2}{2}\int_0^n \varphi^2(s)ds + \mathrm{o}(\textstyle\int_0^n \varphi^2)\right)
$$

$$
- \int_0^n \varphi^2(s)ds - C \tag{49}
$$

Hence, by (49),

$$
\int_B e^{\alpha_N u_n^{\frac{N}{N-1}} + f(u_n) + g(u_n)} dx \geq \exp\left(\frac{1-\delta_n}{1+\mathrm{O}(\delta_n^2)}n + f\left(\frac{(n(1-\delta_n))^{(N-1)/N}}{\|v_n\|_{\Lambda_{N,\varphi}}\alpha_N^{(N-1)/N}}\right)\right.
$$

$$
+ g\left(\frac{(n(1-\delta_n))^{(N-1)/N}}{(1+\mathrm{O}(\delta_n^2))\alpha_N^{(N-1)/N}}\right) - n\Bigg)
$$

$$
\geq \exp\left(-n\delta_n^2 - \frac{N}{2}\int_0^n \varphi^2(s)ds + \mathrm{O}(n\delta_n^2) - C\right.
$$

$$
+ \mathrm{o}\left(\int_0^n \varphi^2\right) + g\left(\frac{(n(1-\delta_n))^{(N-1)/N}}{(1+\mathrm{O}(\delta_n^2))\alpha_N^{(N-1)/N}}\right)\Bigg)
$$

$$
\geq \exp\left(-c\int_0^n \varphi^2 + g\left(\frac{(n(1-\delta_n))^{(N-1)/N}}{(1+\mathrm{O}(\delta_n^2))\alpha_N^{(N-1)/N}}\right)\right),
$$

since

$$
n\delta_n^2 \sim \frac{1}{n}\left(\int_0^n \varphi(s)ds\right)^2 \leq \int_0^n \varphi^2(s)ds .
$$

Finally, using (g_2) we get

$$
\lim_{n\to+\infty}\int_B e^{\alpha_N u_n^{\frac{N}{N-1}} + f(u_n) + g(u_n)} dx = +\infty.
$$

$\square$

5 Proof of Theorem 3

5.1 Concentration-compactness

Let us first recall the following concentration-compactness result due to P.L. Lions [34]:

Theorem 6 *(P.L.Lions) Let Ω be a bounded domain in $\mathbb{R}^N$, and let $\{u_n\}$ be a sequence in $W_0^{1,N}(\Omega)$ such that $\|u_n\|_N \leq 1$ for all n. We may suppose that $u_n \rightharpoonup u$ weakly in $W_0^{1,N}(\Omega)$, $|\nabla u_n|^N \to \mu$ weakly in measure. Then either*
(i) $\mu = \delta_{x_0}$, the Dirac measure of mass 1 concentrated at some $x_0 \in \overline{\Omega}$, and $u \equiv 0$, or
(ii) there exists $\beta > \alpha_N$ such that the family $v_n = e^{|u_n|^{N/(N-1)}}$ is uniformly bounded in $L^\beta(\Omega)$, and thus $\int_\Omega e^{\alpha_N |u_n|^{N/(N-1)}} \to \int_\Omega e^{\alpha_N |u|^{N/(N-1)}}$ as $n \to +\infty$. In particular, this is the case if u is different from 0.

5.2 Compactness

A consequence of the concentration-compactness principle is the following compactness result.

Theorem 7 *Let Ω be a bounded domain in $\mathbb{R}^N$, and let (u_n) be a sequence in $W_0^{1,N}(\Omega)$ such that $\|\nabla u_n\|_{\Lambda_{N,\varphi}} \leq 1$ for all n. We may suppose that $u_n \rightharpoonup u$ weakly in $W_0^{1,N}(\Omega)$, $|\nabla u_n|^N \to \mu$ weakly in measure. Then for any $0 < \alpha < 1$*

$$\int_\Omega e^{\alpha_N |u_n|^{N/(N-1)} + \alpha f(u_n)} dx \longrightarrow \int_\Omega e^{\alpha_N |u|^{N/(N-1)} + \alpha f(u)} dx \qquad (50)$$

as $n \to +\infty$.

Proof As observed in Section 2

$$\|\nabla u_n\|_N^N \leq \|u_n\|_{\Lambda_{N,\varphi}}^N \ ,$$

so that we can apply the concentration-compactness principle. If $e^{|u_n|^{N/(N-1)}}$ is uniformly bounded in L^β, with $\beta > \alpha_N$, the claim is an obvious consequence of Theorem 6,(ii). Otherwise, by (i), $|\nabla u_n| \to \delta_{x_0}$ weakly in measure and $u_n \to 0$ in L^r, for any $r \geq 1$, and a.e. (up to a subsequence).

Since $\alpha < 1$ and $f(t) \to +\infty$ as $t \to +\infty$ by (H_2), for any $\varepsilon > 0$ there is a constant $K = K(\varepsilon)$ such that

$$e^{\alpha_N t^{N/(N-1)} + \alpha f(t)} < \varepsilon \cdot e^{\alpha_N t^{N/(N-1)} + f(t)} \quad , \text{ if } t > K \ .$$

Therefore

$$\begin{aligned}
\int_\Omega e^{\alpha_N |u_n|^{N/(N-1)} + \alpha f(u_n)} dx &= \int_{|u_n| \leq K} + \int_{|u_n| > K} e^{\alpha_N |u_n|^{N/(N-1)} + \alpha f(u_n)} dx \\
&\leq \int_{|u_n| \leq K} e^{\alpha_N |u_n|^{N/(N-1)} + \alpha f(u_n)} dx + \varepsilon \int_\Omega e^{\alpha_N |u_n|^{N/(N-1)} + f(u_n)} dx \\
&\leq \int_{|u_n| \leq K} e^{\alpha_N |u_n|^{N/(N-1)} + \alpha f(u_n)} dx + \varepsilon C.
\end{aligned}$$

But

$$\int_{|u_n| \leq K} \left(e^{\alpha_N |u_n|^{N/(N-1)} + \alpha f(u_n)} - 1 \right) dx = \int_\Omega \left(e^{\alpha_N |v_n|^{N/(N-1)} + \alpha f(v_n)} - 1 \right) dx$$

where

$$v_n = \begin{cases} u_n \ , & \text{if } |u_n| \leq K \\ 0 \ , & \text{if } |u_n| > K \ , \end{cases}$$

so that, by Lebesgue's dominated convergence theorem

$$\int_{|u_n| \leq K} \left(e^{\alpha_N |u_n|^{N/(N-1)} + \alpha f(u_n)} - 1 \right) dx \to 0 \quad \text{ as } n \to +\infty \ .$$

5.3 Concentration and non-compactness

We now show that in the critical case, i.e. $\alpha = 1$ in (7), there is a loss of compactness:

Theorem 8 *Assume that φ satisfies (H_1), (H_2), (H_3) and $\varphi \in L^2(\mathbb{R}^+)$. Then there exists a sequence $(u_n) \subset W_0^{1,N}(\Omega)$ with $\|u_n\|_{\Lambda_{N,\varphi}} \leq 1$, $u_n \rightharpoonup u$ weakly in $W_0^{1,N}$, and such that*

$$\int_\Omega e^{\alpha_N |u_n|^{N/(N-1)} + f(u_n)} \not\to \int_\Omega e^{\alpha_N |u|^{N/(N-1)} + f(u)} .$$

Proof The modified Moser sequence (u_n) defined in the proof of Theorem 1 furnishes a counterexample. Thus, assume $B \subset \Omega$ (B as in (38)), and let

$$u_n = \frac{v_n}{\|v_n\|_{\Lambda_{N,\varphi}}} = \frac{v_n}{\left(1 - \frac{N(N-1)}{2}\delta_n^2 + \mathrm{o}(\delta_n^2)\right)^{1/N}}$$

with v_n, δ_n given by (38) and (42). Obviously $\|u_n\|_{\Lambda_{N,\varphi}} = 1$, and

$$u_n \rightharpoonup 0 \quad \text{in} \quad W_0^{1,N}(B).$$

On the other hand, by (49)

$$\int_B e^{\alpha_N |u_n|^{N/(N-1)} + f(u_n)} dx = \int_0^1 e^{\alpha_N |u_n^*|^{N/(N-1)} + f(u_n^*)} ds$$

$$= \int_{e^{-n}}^1 \exp\left\{ \frac{1-\delta_n}{n^{1/(N-1)}(1 - \frac{N}{2}\delta_n^2 + \mathrm{o}(\delta_n^2))} |\log s|^{N/(N-1)} \right.$$

$$\left. + f\left(\frac{-\log s(1-\delta_n)^{(N-1)/N}}{\alpha_N^{(N-1)/N} n^{1/N}(1 - \frac{N-1}{2}\delta_n^2 + \mathrm{o}(\delta_n^2))} \right) \right\} ds$$

$$+ \exp\left\{ \left(\frac{(1-\delta_n)}{1 - \frac{N}{2}\delta_n^2 + \mathrm{o}(\delta_n^2)} - 1 \right)n + f\left(\frac{((1-\delta_n)n)^{(N-1)/N}}{\alpha_N^{(N-1)/N}(1 - \frac{N-1}{2}\delta_n^2 + \mathrm{o}(\delta_n^2))} \right) \right\}$$

$$\geq \int_0^n \exp\left\{ \frac{1-\delta_n}{n^{1/(N-1)}(1 - \frac{N}{2}\delta_n^2 + \mathrm{o}(\delta_n^2))} t^{N/(N-1)} \right.$$

$$\left. + f\left(\frac{t(1-\delta_n)^{(N-1)/N}}{\alpha_N^{(N-1)/N} n^{1/N}(1 - \frac{N-1}{2}\delta_n^2 + \mathrm{o}(\delta_n^2))} \right) - t \right\} ds + \exp\left\{ -n\delta_n + \frac{N}{2}n\delta_n^2 + \mathrm{o}(n\delta_n^2) \right.$$

$$\left. + \left(1 - \delta_n + \frac{N}{2}\delta_n^2 + \mathrm{o}(\delta_n^2) \right)\left(n\delta_n - \frac{N-2}{2}\int_0^n \varphi^2 + \mathrm{o}(\int_0^n \varphi^2) \right) - \int_0^n \varphi^2 + \mathrm{o}(\int_0^n \varphi^2) \right\}$$

$$\geq \int_0^n e^{-t} dt + \exp\left\{ \frac{N-2}{2}n\delta_n^2 + \mathrm{o}(n\delta_n^2) - \frac{N}{2}\int_0^n \varphi^2 + \mathrm{o}(\int_0^n \varphi^2) \right\}$$

$$\geq \int_0^n e^{-t} dt + \exp\left\{ -\frac{N}{2}\int_0^n \varphi^2 + \mathrm{o}(\int_0^n \varphi^2) \right\} \quad (\text{since } n\delta_n^2 \leq \int_0^n \varphi^2 + \mathrm{o}(\int_0^n \varphi^2))$$

$$\geq \int_0^n e^{-t} dt + e^{-C_1} \quad (\text{since } \varphi \in L^2(\mathbb{R}^+))$$

$$= 1 - e^{-n} + e^{-C_1} \to 1 + e^{-C_1} > 1 = \int_B e^{\alpha_N |u|^{N/(N-1)} + f(u)} dx .$$

6 Generalizations: the Brezis-Wainger case

Theorem 1 can be generalized to the setting of $L(N,q)$ spaces as follows:

Theorem 9 *Let $\Omega \subset \mathbb{R}^N$ be a domain of finite measure, and let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying (H_1) and (H_2). Let $q > 1$, and let $f(t) \in \mathscr{C}^1(\mathbb{R}^+)$ be defined by*

$$f(t) = \int_0^{\alpha_{N,q} t^{\frac{q}{q-1}}} \frac{\varphi(s)}{1 + \varphi(s)} ds \tag{51}$$

where $\alpha_{N,q} = (N^{\frac{N-1}{N}} \omega_{N-1}^{1/N})^{\frac{q}{q-1}}$ and ω_{N-1} denotes the $(N-1)$-dimensional surface of the unit ball in $\mathbb{R}^N$. Then

$$\sup_{\{u \in C_0^1(\Omega), \|\nabla u\|_{\Lambda_{N,q,\varphi}} \leq 1\}} \int_\Omega e^{\alpha_{N,q} |u|^{\frac{q}{q-1}} + f(u)} \leq C|\Omega| \tag{52}$$

where

$$\|v\|_{\Lambda_{N,q,\varphi}}^q = \int_0^{+\infty} (v^*(s) s^{1/N})^q \left\{ 1 + \varphi\left(\left| \log\left(\frac{s}{|\Omega|} \right) \right| \right) \right\}^{q-1} \frac{ds}{s} .$$

and $C = C(\|\varphi\|_\infty)$ is a positive constant that depends only on $\|\varphi\|_\infty$.

Furthermore, the inequality is sharp in the following sense:

(i) for any $\alpha > 1$

$$\sup_{\|\nabla u\|_{\Lambda_{N,q,\varphi}} \leq 1} \int_\Omega e^{\alpha_{N,q} |u|^{q/(q-1)} + \alpha f(u)} dx = +\infty$$

(ii) for $\alpha = 1$: assume that φ satisfies (H_3) and that $g : \mathbb{R} \to \mathbb{R}$ is continuous such that

$(g_{1,q})$ $\lim_{t \to +\infty} \frac{g(t)}{f(t)} = 0$

$(g_{2,q})$ $\lim_{t \to +\infty} \dfrac{g\left(\left(\frac{t}{\alpha_{N,q}} \right)^{(q-1)/q} \right)}{\int_0^t \varphi^2(s) ds} = +\infty$;

then

$$\sup_{\|\nabla u\|_{\Lambda_{N,q,\varphi}} \leq 1} \int_\Omega e^{\alpha_{N,q} |u|^{q/(q-1)} + f(u) + g(u)} dx = +\infty.$$

Proof (Proof of Theorem 9) The proof follows the lines of Theorem 1 (with slight modifications), replacing Lemma 2 with the following (N,q)−version:

Lemma 3 *Let $a(s,t)$ be a non-negative measurable function on $\mathbb{R} \times [0,+\infty)$ such that*

$$a(s,t) \leq 1, \quad \text{for a.e.} \quad 0 < s < t \tag{53}$$

$$\sup_{t > 0} \left(\int_{-\infty}^0 + \int_t^{+\infty} \frac{a^{\frac{q}{q-1}}(s,t)}{1 + \varphi(s)} ds \right)^{\frac{q-1}{q}} = \gamma < \infty \tag{54}$$

Then there exists a constant $c_0 = c_0(\|\varphi\|_\infty, \gamma)$ such that for $\phi \geq 0$ with

$$\int_{-\infty}^{+\infty} \phi^q(s) (1 + \varphi(s))^{q-1} ds \leq 1 \tag{55}$$

one has

$$\int_0^{+\infty} e^{-\Psi(t)} dt \le c_0 , \tag{56}$$

where

$$\Psi(t) = t - \left\{ \left(\int_{-\infty}^{+\infty} a(s,t)\phi(s)ds \right)^{\frac{q}{q-1}} + f\left(\frac{1}{\alpha_{N,q}^{\frac{q-1}{q}}} \int_{-\infty}^{+\infty} a(s,t)\phi(s)ds \right) \right\} \tag{57}$$

6.1 Sharpness

Following the proof of Theorem 2 , combined with the estimates obtained in [6], it is not hard to prove that the sequence of functions

$$u_n = \frac{v_n}{\|v_n\|_{\Lambda_{N,q,\varphi}}}$$

furnishes a counterexample for Theorem 9, where

$$v_n(x) = \begin{cases} \dfrac{(1-\delta_n)^{\frac{q-1}{q}}}{N^{\frac{N-1}{N}}\omega_{N-1}^{1/N}} n^{\frac{q-1}{q}} & , \ \frac{\omega_{N-1}}{N}|x|^N < e^{-n} \\[3mm] \dfrac{(1-\delta_n)^{\frac{q-1}{q}}}{N^{\frac{N-1}{N}}\omega_{N-1}^{1/N}n^{1/q}} \log\left(\dfrac{N}{\omega_{N-1}|x|^N} \right) & , \ e^{-n} \le \frac{\omega_{N-1}}{N}|x|^N \le 1 \end{cases} \tag{58}$$

and

$$\delta_n = \frac{1}{q-1} \frac{I_q(n)}{n} \tag{59}$$

with

$$\begin{aligned} I_q(n) &= \int_0^{1-e^{-n}} \frac{s^{q/N-1}}{(s+e^{-n})^{q/N}} \left[\left(1 + \varphi(-\log s)\right)^{q-1} - 1 \right] ds \\ &= \int_0^n \left(1 - e^{t-n}\right)^{q/N-1} \left[\left(1 + \varphi\left(t - \log(1 - e^{t-n})\right)\right)^{q-1} - 1 \right] dt. \end{aligned}$$

7 The inverse case: determining φ from f

Relation (5) in Theorem 1 gives a formula for the perturbation $f(t)$ when the weight-function $\varphi(s)$ in the Lorentz space is given. We now give a formula for the inverse situation, i.e. on how to determine $\varphi(s)$ when $f(t)$ is given (cf. Remark (1)). Indeed, with slight modifications in the proof of Theorem 1, we have the following result

Theorem 10 *Let Ω be an open subset of $\mathbb{R}^N$, of finite measure, and let $f \in \mathscr{C}^1(\mathbb{R}^+)$ such that*

(F_1) $\qquad f(t) \ge 0$

(F_2) $\qquad \displaystyle\lim_{t \to +\infty} \frac{f'(t)}{t^{1/(N-1)}} = 0 .$

Let $\varphi(t) \in \mathscr{C}^1(\mathbb{R}^+)$ be defined by

$$\varphi(t) = \frac{f'\left(\left(\frac{t+t_0}{\alpha_N}\right)^{\frac{N-1}{N}}\right)}{\frac{N}{N-1}\alpha_N^{\frac{N}{N-1}}(t+t_0)^{1/N} - f'\left(\left(\frac{t+t_0}{\alpha_N}\right)^{\frac{N-1}{N}}\right)}\ , \tag{60}$$

where t_0 is such that

$$\inf_{t \geq 0}\left\{\frac{N}{N-1}\alpha_N^{\frac{N}{N-1}}(t+t_0)^{1/N} - f'\left(\left(\frac{t+t_0}{\alpha_N}\right)^{\frac{N-1}{N}}\right)\right\} > 0\ .$$

Then

$$\sup_{\|\nabla u\|_{\Lambda_{N,\varphi}} \leq 1} \int_\Omega e^{\alpha_N|u|^{\frac{N}{N-1}}+f(u)} \leq C(f,t_0)\,|\Omega| \tag{61}$$

where

$$\|v\|_{\Lambda_{N,\varphi}}^N = \int_0^{+\infty}(v^*(s))^N\left\{1 + \varphi\left(\left|\log\left(\frac{s}{|\Omega|}\right)\right|\right)\right\}^{N-1}ds\ .$$

The inequality is sharp: for any $\alpha > 1$

$$\sup_{\|\nabla u\|_{\Lambda_{N,\varphi}} \leq 1} \int_\Omega e^{\alpha_N|u|^{\frac{N}{N-1}}+\alpha f(u)}dx = +\infty\ . \tag{62}$$

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