PRIMES IN EXPLICIT SHORT INTERVALS ON RH

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ABSTRACT. In this paper, on the assumption of the Riemann hypothesis, we give explicit upper bounds on the difference between consecutive prime numbers.

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1. General setting and results

The computation of the maximal prime gaps given by Oliveira e Silva, Herzog and Pardi [9, Sec. 2.2] verifies that $p_{k+1}-p_k < \log^2 p_k$ for all primes $11 \le p_k \le 4\cdot 10^{18}$. This proves that for all $x \in [5, 4\cdot 10^{18}]$, there is a prime in $[x-0.5\log^2 x, x+0.5\log^2 x]$. It is the purpose of this article to furnish new explicit upper bounds on the difference between consecutive prime numbers with the assumption of the Riemann hypothesis. Specifically, we prove the following theorem.

Theorem 1.1. *Assume RH. Let $x \ge 2$ and $c := \frac{1}{2}+\frac{2}{\log x}$. Then there is a prime in $(x-c\sqrt{x}\log x, x+c\sqrt{x}\log x)$ and at least $\sqrt{x}$ primes in $(x-(c+1)\sqrt{x}\log x, x+(c+1)\sqrt{x}\log x)$.*

The mentioned conclusion coming from the computations of Oliveira e Silva et al. is stronger than the first part of our result for all $x \le 4\cdot 10^{18}$. This allows one to use $c = 0.55$ for all $x \ge 5$ when only one prime is needed.
In a recent paper [3], the first author proved Theorem 1.1 with $c = \frac{2}{\pi} = 0.6366\ldots$ and an asymptotic result in the weaker form $c = 0.5+\epsilon$ when $x \ge x(\epsilon)$, without any information on the size of $x(\epsilon)$.

In Appendix A we prove the same result with $c = 0.6102$. This value improves on the one in [3] and is stronger than Theorem 1.1 up to $2\cdot 10^8$. Despite its weakness, we believe that the method of proof is worthy of interest. Indeed, the conclusion is reached proving that the Riemann hypothesis implies a (very weak) cancellation in the exponential sum $S_\alpha(T) := \sum_{|\gamma|\le T} e^{i\alpha\gamma}$, where γ runs on the set of imaginary parts of the nontrivial zeros for the Riemann zeta function. Hypotheses assuming the existence of stronger cancellations would produce stronger conclusions. In fact, assuming the pair correlation conjecture [6], one can show that for all x there is a prime p such that $|p-x| = o(\sqrt{x\log x})$. Moreover, assuming the stronger Gonek conjecture [4], one could guarantee a prime p with $|p-x| \ll_\epsilon x^\epsilon$. This shows that the method in the appendix has a good track record and could be a promising path for this kind of result. The third author admits his surprise when we did not manage to prove the best constant $c = 1/2+\epsilon$ using this method.

We first consider the setting in which we seek to establish Theorem 1.1. Throughout, we define the von Mangoldt function as

$$\Lambda(n) := \begin{cases} \log p, & n = p^m,\ p \text{ is prime}, m \in \mathbb{N}, m \ge 1 \\ 0, & \text{otherwise}, \end{cases}$$

$\vartheta(x) := \sum_{p\le x}\log p$, where the sum is restricted to primes, and $\psi(x) := \sum_{n\le x}\Lambda(n)$. It is often convenient to work with a smoothed version of ψ, and so we define

$$\psi^{(1)}(x) := \int_0^x \psi(u)\,\mathrm{d}u = \sum_{n\le x}\Lambda(n)(x-n)$$

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through partial summation. One can recall the integral representation

$$\psi^{(1)}(x) = -\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{\zeta'}{\zeta}(s)\frac{x^{s+1}}{s(s+1)}\,\mathrm{d}s \qquad \forall x \geq 1,$$

which follows directly from an application of Perron's formula (see, for example, Ingham's classic text [7, Ch IV, Sec 4]). We let $h \in \mathbb{R}$ such that $0 < h < x$. Then

$$\psi^{(1)}(x+h) - 2\psi^{(1)}(x) + \psi^{(1)}(x-h) = \sum_n \Lambda(n)K(x-n;h),$$

where $K(u;h) := \max\{h-|u|,0\}$; one can verify this by expanding the left-hand side of the above identity. Note also that $K(u;h)$ is supported on $|u| \leq h$, positive in the open set, and has a unique maximum at $u = 0$ with $K(0;h) = h$.

From the integral representation one gets the explicit formula

$$\psi^{(1)}(x) = \frac{x^2}{2} - \sum_\rho \frac{x^{\rho+1}}{\rho(\rho+1)} - xr + r' + R^{(1)}(x)$$

where ρ runs on the set of nontrivial zeros of the Riemann zeta-function, r and r' are constants, and $|R^{(1)}(x)| \leq 0.6/x$ (one can see [5, Lemma 3.3], though this is classical). Noting that

$$(x+h)^j - 2x^j + (x-h)^j = \begin{cases} 0, & j = 0, 1 \\ 2h^2, & j = 2, \end{cases}$$

we thus have that, assuming $h \leq x/\sqrt{3}$,

$$\sum_n \Lambda(n)K(x-n;h) = h^2 - \sum_\rho \frac{(x+h)^{\rho+1} - 2x^{\rho+1} + (x-h)^{\rho+1}}{\rho(\rho+1)} + \frac{3\theta}{x}$$

for some $\theta = \theta(x,h) \in [-1,1]$, for then

$$0.6((x+h)^{-1} + 2x^{-1} + (x-h)^{-1}) \leq 3x^{-1}.$$

We split the sum over the zeros as

$$\sum_\rho \frac{(x+h)^{\rho+1} - 2x^{\rho+1} + (x-h)^{\rho+1}}{\rho(\rho+1)} =: \Sigma_1 + \Sigma_2,$$

with Σ_1 and Σ_2 representing the sums on zeros with $|\mathrm{Im}(\rho)| \leq T$ and $|\mathrm{Im}(\rho)| > T$, respectively. It is not a difficult task to bound Σ_2 (here we repeat the argument in [3]). In fact, assuming RH,

$$|\Sigma_2| \leq 4(x+h)^{3/2} \sum_{|\mathrm{Im}(\rho)|>T} \frac{1}{|\rho(\rho+1)|},$$

and since $\sum_{|\mathrm{Im}(\rho)|>T} \frac{1}{|\rho|^2} \leq \frac{\log T}{\pi T}$ (see [14, Lemma 1 (ii)]), one has

$$|\Sigma_2| \leq 4(x+h)^{3/2}\frac{\log T}{\pi T}.$$

Thus

$$\sum_n \Lambda(n)K(x-n;h) \geq h^2 - |\Sigma_1| - 4(x+h)^{3/2}\frac{\log T}{\pi T} - \frac{3}{x}.$$

Now we remove the contribution from prime powers. Recalling that

$$0.9986\sqrt{x} \leq \psi(x) - \vartheta(x) \leq (1+10^{-6})\sqrt{x} + 3\sqrt[3]{x}$$

for every $x \geq 121$ (see [13, Th. 6] and [11, Cor. 2]), we get

$$\begin{aligned}\sum_n \Lambda(n)K(x-n;h) &\leq h \sum_{|n-x|<h} \Lambda(n) \\ &\leq h\Big(\sum_{|p-x|<h} \log p + (\psi(x+h) - \vartheta(x+h)) - (\psi(x-h) - \vartheta(x-h))\Big)\end{aligned}$$

$$\leq h\Big(\sum_{|p-x|<h}\log p + (1+10^{-6})\sqrt{x+h} + 3\sqrt[3]{x+h} - 0.9986\sqrt{x-h}\Big)$$
$$\leq h\Big(\sum_{|p-x|<h}\log p + 0.002\sqrt{x} + 3\sqrt[3]{x} + \frac{2h}{\sqrt{x}}\Big),$$

where for the last inequality we have also used that $h \leq x/\sqrt{3}$. Thus, when $x \geq 121$ we have

$$\sum_{|p-x|<h}\log p \geq h - \frac{1}{h}|\Sigma_1| - 4(x+h)^{3/2}\frac{\log T}{\pi h T} - 0.002\sqrt{x} - 3\sqrt[3]{x} - \frac{2h}{\sqrt{x}} - \frac{3}{xh}. \tag{1.1}$$

It is clear that the positivity of the right-hand side guarantees the existence of at least one prime in the interval $(x-h, x+h)$.

From before, we have that

$$\Sigma_1 = \sum_{|\mathrm{Im}(\rho)|\leq T}\frac{(x+h)^{\rho+1} - 2x^{\rho+1} + (x-h)^{\rho+1}}{\rho(\rho+1)}.$$

There are essentially two ways to bound Σ_1, both of them appearing already in [3].

The first one is based on the Taylor identity

$$(1+\epsilon)^{\frac{3}{2}+i\gamma\epsilon} - 2 + (1-\epsilon)^{\frac{3}{2}-i\gamma\epsilon} = -4\sin^2(\gamma\epsilon) + O(\gamma\epsilon^2),$$

while the second one is based on the identity

$$\frac{(x+h)^{\rho+1} - 2x^{\rho+1} + (x-h)^{\rho+1}}{\rho(\rho+1)} = \int_{x-h}^{x+h} K(x-u;h)u^{\rho-1}\,\mathrm{d}u.$$

Thus, denoting γ the imaginary part of a nontrivial zero, on the assumption of RH one gets

$$|\Sigma_1| \leq 4\sum_{|\gamma|\leq T}\frac{\sin^2(\gamma\epsilon) + O(\gamma\epsilon^2)}{\gamma^2} \tag{1.2}$$

from the first one, and

$$|\Sigma_1| \leq \int_{x-h}^{x+h} K(x-u;h)\Big|\sum_{|\gamma|\leq T}u^{i\gamma}\Big|\frac{\mathrm{d}u}{\sqrt{u}} \tag{1.3}$$

from the second one. As a consequence, the first approach takes advantage of the cancellation due to the sum of the three functions $(1+\omega\epsilon)^{\frac{3}{2}\omega i\gamma\epsilon}$ with $\omega \in \{0, \pm 1\}$ for the same zero, while the second approach takes advantage of the cancellation coming from the sum of values of the same function computed at different zeros.
The first approach is discussed in Section 2, while the second is discussed in Appendix A.

2. First bound for Σ_1

Let $N(T)$ denote the number of nontrivial zeros of $\zeta(s)$ with imaginary part in $[0,T]$, where multiplicity is included. We state the estimate of $N(T)$ done by Trudgian in [15]: let $W(T) := \frac{T}{2\pi}\log\big(\frac{T}{2\pi e}\big)$ denote what is essentially the main term of $N(T)$ and let $U(T) := N(T) - W(T)$, then the result says that

$$|U(T)| \leq 0.112\log T + 0.278\log\log T + 2.51 + \frac{0.2}{T} + \frac{7}{8} =: R(T), \qquad T \geq e.$$

Note that $U(2\pi) = 1$ because the imaginary part of the first zero is $14.13\ldots$, and that $\mathrm{d}W(T) = \log(\frac{T}{2\pi})\frac{\mathrm{d}T}{2\pi}$.

We introduce the notations

$$T = \frac{\beta}{c}\frac{\sqrt{x}}{\log x}, \qquad h = c\sqrt{x}\log x$$

for suitable β and c.

Lemma 2.1. *Let $0 \le h < x$. Then for every $\gamma \in \mathbb{R}$ there exists $\theta \in \mathbb{C}$ with $|\theta| \le 1$ such that*

$$\Big(1+\frac{h}{x}\Big)^{\frac{3}{2}+i\gamma}-2+\Big(1-\frac{h}{x}\Big)^{\frac{3}{2}+i\gamma} = -4\sin^2\Big(\frac{\gamma h}{2x}\Big)+\theta(2|\gamma|+1)\frac{h^2}{x^2}.$$

Proof. The proof is straightforward and follows from the Taylor expansion of $\log(1+u)$ and some elementary inequalities. □

Thus we get an explicit version of (1.2):

$$|\Sigma_1| \le 8x^{3/2}\sum_{0<\gamma\le T}\frac{\sin^2\big(\frac{\gamma h}{2x}\big)}{\gamma^2}+2\frac{h^2}{\sqrt{x}}\sum_{0<\gamma\le T}\frac{2\gamma+1}{\gamma^2}.$$

From [14, Lemma 1 (i,iii)], we have that $\sum_{0<\gamma<T}\frac{1}{\gamma} \le \frac{\log^2 T}{4\pi}$ and $\sum_{\gamma>0}\frac{1}{\gamma^2} \le \frac{1}{40}$ and thus

$$|\Sigma_1| \le 8x^{3/2}\sum_{0<\gamma\le T}\frac{\sin^2\big(\frac{\gamma h}{2x}\big)}{\gamma^2}+\Big(\frac{\log^2 T}{\pi}+\frac{1}{20}\Big)\frac{h^2}{\sqrt{x}}. \tag{2.1}$$

Let $\gamma_1 = 14.13\ldots$ be the imaginary part of the first non-trivial zero of $\zeta(s)$. By partial summation we get

$$\begin{aligned}\sum_{0<\gamma\le T}\frac{\sin^2\big(\frac{\gamma h}{2x}\big)}{\gamma^2} &= \int_{\gamma_1^-}^{T^+}\frac{\sin^2\big(\frac{\gamma h}{2x}\big)}{\gamma^2}\,\mathrm{d}N(\gamma)\\ &= \Big[\frac{\sin^2\big(\frac{\gamma h}{2x}\big)}{\gamma^2}N(\gamma)\Big]\Big|_{\gamma_1^-}^{T^+}-\int_{\gamma_1}^{T}\Big[\frac{\sin^2\big(\frac{\gamma h}{2x}\big)}{\gamma^2}\Big]'N(\gamma)\,\mathrm{d}\gamma\\ &= \frac{\sin^2\big(\frac{hT}{2x}\big)}{T^2}N(T)-\int_{\gamma_1}^{T}\Big[\frac{\sin^2\big(\frac{\gamma h}{2x}\big)}{\gamma^2}\Big]'N(\gamma)\,\mathrm{d}\gamma.\end{aligned}$$

It then follows that

$$\begin{aligned}\sum_{0<\gamma\le T}\frac{\sin^2\big(\frac{\gamma h}{2x}\big)}{\gamma^2} &= \frac{\sin^2\big(\frac{hT}{2x}\big)}{T^2}N(T)-\int_{\gamma_1}^{T}\Big[\frac{\sin^2\big(\frac{\gamma h}{2x}\big)}{\gamma^2}\Big]'\frac{\gamma}{2\pi}\log\Big(\frac{\gamma}{2\pi e}\Big)\,\mathrm{d}\gamma-\int_{\gamma_1}^{T}\Big[\frac{\sin^2\big(\frac{\gamma h}{2x}\big)}{\gamma^2}\Big]'U(\gamma)\,\mathrm{d}\gamma\\ &= \frac{\sin^2\big(\frac{hT}{2x}\big)}{T^2}U(T)+\frac{\sin^2\big(\frac{\gamma_1 h}{2x}\big)}{2\pi\gamma_1}\log\Big(\frac{\gamma_1}{2\pi e}\Big)\\ &\quad+\int_{\gamma_1}^{T}\frac{\sin^2\big(\frac{\gamma h}{2x}\big)}{\gamma^2}\log\Big(\frac{\gamma}{2\pi}\Big)\frac{\mathrm{d}\gamma}{2\pi}-\int_{\gamma_1}^{T}\Big[\frac{\sin^2\big(\frac{\gamma h}{2x}\big)}{\gamma^2}\Big]'U(\gamma)\,\mathrm{d}\gamma.\end{aligned}$$

Recalling the upper bound $|U(T)| \le R(T)$ and noticing that $\gamma_1 < 2\pi e$, we get

$$\begin{aligned}\sum_{0<\gamma\le T}\frac{\sin^2\big(\frac{\gamma h}{2x}\big)}{\gamma^2} &\le \frac{R(T)}{T^2}+\int_{\gamma_1}^{T}\frac{\sin^2\big(\frac{\gamma h}{2x}\big)}{\gamma^2}\log\Big(\frac{\gamma}{2\pi}\Big)\frac{\mathrm{d}\gamma}{2\pi}-\int_{\gamma_1}^{T}\Big[\frac{\sin^2\big(\frac{\gamma h}{2x}\big)}{\gamma^2}\Big]'U(\gamma)\,\mathrm{d}\gamma\\ &\le \frac{R(T)}{T^2}+\int_{\gamma_1}^{T}\frac{\sin^2\big(\frac{\gamma h}{2x}\big)}{\gamma^2}\log\Big(\frac{\gamma}{2\pi}\Big)\frac{\mathrm{d}\gamma}{2\pi}+\int_{\gamma_1}^{T}\Big|\frac{h}{2x}\frac{\sin\big(\frac{\gamma h}{x}\big)}{\gamma^2}-2\frac{\sin^2\big(\frac{\gamma h}{2x}\big)}{\gamma^3}\Big|R(\gamma)\,\mathrm{d}\gamma.\end{aligned}$$

Using the inequality $|\sin^2 v| \le \frac{3}{4}|v|$, we simplify to get

$$\sum_{0<\gamma\le T}\frac{\sin^2\big(\frac{\gamma h}{2x}\big)}{\gamma^2} \le \frac{R(T)}{T^2}+\int_{\gamma_1}^{T}\frac{\sin^2\big(\frac{\gamma h}{2x}\big)}{\gamma^2}\log\Big(\frac{\gamma}{2\pi}\Big)\frac{\mathrm{d}\gamma}{2\pi}+\frac{5h}{4x}\int_{\gamma_1}^{T}\frac{R(\gamma)}{\gamma^2}\,\mathrm{d}\gamma.$$

Since $\int_{14}^{+\infty}\frac{R(\gamma)}{\gamma^2}\,\mathrm{d}\gamma \le 0.297$, we get

$$\begin{aligned}\sum_{0<\gamma\le T}\frac{\sin^2\big(\frac{\gamma h}{2x}\big)}{\gamma^2} &\le \frac{1}{2\pi}\int_{\gamma_1}^{T}\frac{\sin^2\big(\frac{\gamma h}{2x}\big)}{\gamma^2}\,\mathrm{d}\gamma\log\Big(\frac{T}{2\pi}\Big)+\frac{R(T)}{T^2}+\frac{2.97h}{8x}\\ &\le \frac{h}{4\pi x}\int_{0}^{\frac{hT}{2x}}\frac{\sin^2 t}{t^2}\,\mathrm{d}t\log\Big(\frac{T}{2\pi}\Big)+\frac{R(T)}{T^2}+\frac{2.97h}{8x}.\end{aligned}$$

We can bound the integral in the above equation with ease, for

$$\begin{aligned}\int_0^y \frac{\sin^2 t}{t^2}\,\mathrm{d}t &= \frac{\pi}{2}-\int_y^\infty \frac{\sin^2 t}{t^2}\,\mathrm{d}t = \frac{\pi}{2}-\int_y^\infty \frac{1-\cos(2t)}{2t^2}\,\mathrm{d}t\\ &= \frac{\pi}{2}-\frac{1}{2y}+\int_y^\infty \frac{\cos(2t)}{2t^2}\,\mathrm{d}t = \frac{\pi}{2}-\frac{1}{2y}-\frac{\sin(2y)}{4y^2}+\int_y^\infty \frac{\sin(2t)}{2t^3}\,\mathrm{d}t\\ &= \frac{\pi}{2}-\frac{1}{2y}-\frac{\sin(2y)}{4y^2}+\frac{\theta}{4y^2} = \frac{\pi}{2}-\frac{1}{2y}+\frac{\theta}{2y^2}\end{aligned}$$

for some $\theta\in[-1,1]$. As such, we can now use $R(T)\le 1.5\log T$ for every $T\ge\gamma_1$ to get

$$\sum_{0<\gamma\le T}\frac{\sin^2\left(\frac{\gamma h}{2x}\right)}{\gamma^2}\le \frac{h}{8x}\Big(1-\frac{2}{\pi}\frac{x}{hT}+\frac{4}{\pi}\frac{x^2}{h^2T^2}\Big)\log\left(\frac{T}{2\pi}\right)+1.5\frac{\log T}{T^2}+\frac{2.97h}{8x}$$

so that (2.1) becomes

$$\begin{aligned}|\Sigma_1| &\le 8x^{3/2}\Big(\frac{h}{8x}\Big(1-\frac{2}{\pi}\frac{x}{hT}+\frac{4}{\pi}\frac{x^2}{h^2T^2}\Big)\log\left(\frac{T}{2\pi}\right)+1.5\frac{\log T}{T^2}+\frac{2.97h}{8x}\Big)+\Big(\frac{\log^2 T}{\pi}+\frac{1}{20}\Big)\frac{h^2}{\sqrt{x}}\\ &= h\sqrt{x}\Big(1-\frac{2}{\pi}\frac{x}{hT}+\frac{4}{\pi}\frac{x^2}{h^2T^2}\Big)\log\left(\frac{T}{2\pi}\right)+12x^{3/2}\frac{\log T}{T^2}+2.97h\sqrt{x}+\Big(\frac{\log^2 T}{\pi}+\frac{1}{20}\Big)\frac{h^2}{\sqrt{x}}\\ &= h\sqrt{x}\log\left(\frac{T}{2\pi}\right)-\frac{2}{\pi}h\sqrt{x}\frac{x}{hT}\log\left(\frac{T}{2\pi}\right)+\frac{4}{\pi}h\sqrt{x}\frac{x^2}{h^2T^2}\log\left(\frac{T}{2\pi}\right)\\ &\quad +2.97h\sqrt{x}+\frac{\log^2 T}{\pi}\frac{h^2}{\sqrt{x}}+12x^{\frac{3}{2}}\frac{\log T}{T^2}+\frac{h^2}{20\sqrt{x}}.\end{aligned}\tag{2.2}$$

3. Proof of Theorem 1.1

Substituting (2.2) into (1.1) we get

$$\begin{aligned}\sum_{|p-x|<h}\log p \ge h-\Big[&\sqrt{x}\log\left(\frac{T}{2\pi}\right)-\frac{2}{\pi}\sqrt{x}\frac{x}{hT}\log\left(\frac{T}{2\pi}\right)+\frac{4}{\pi}\sqrt{x}\frac{x^2}{h^2T^2}\log\left(\frac{T}{2\pi}\right)\\ &+2.97\sqrt{x}+\frac{\log^2 T}{\pi}\frac{h}{\sqrt{x}}+12x^{3/2}\frac{\log T}{hT^2}+\frac{h}{20\sqrt{x}}\Big]-4(x+h)^{3/2}\frac{\log T}{\pi hT}-0.002\sqrt{x}-3\sqrt[3]{x}-\frac{2h}{\sqrt{x}}-\frac{3}{xh}.\end{aligned}$$

Recalling that we have set $h=c\sqrt{x}\log x$, $T=\frac{\beta}{c}\frac{\sqrt{x}}{\log x}$ (so that $hT=\beta x$), and estimating $(x+h)^{3/2}\le x^{3/2}(1+2\frac{h}{x})$ (which holds whenever $h/x\le 1.6$), we have that

$$\begin{aligned}\sum_{|p-x|<h}\log p \ge h-\sqrt{x}\log\left(\frac{T}{2\pi}\right)+\frac{2}{\pi}\frac{\sqrt{x}}{\beta}\log\left(\frac{T}{2\pi}\right)-\frac{4}{\pi}\Big(1+2\frac{h}{x}\Big)\sqrt{x}\frac{\log T}{\beta}-3\sqrt{x}\\ -\frac{4}{\pi}\frac{\sqrt{x}}{\beta^2}\log\left(\frac{T}{2\pi}\right)-\frac{\log^2 T}{\pi}\frac{h}{\sqrt{x}}-12\frac{h}{\beta^2\sqrt{x}}\log T-3\sqrt[3]{x}-2.05\frac{h}{\sqrt{x}}-\frac{3}{xh},\end{aligned}$$

or, upon gathering like terms, that

$$\begin{aligned}\sum_{|p-x|<h}\log p &\ge h-\sqrt{x}\log T-\frac{2}{\pi}\frac{\sqrt{x}}{\beta}\log T-(3-\log(2\pi))\sqrt{x}\\ &-\frac{2}{\pi}\log(2\pi)\frac{\sqrt{x}}{\beta}-\frac{4}{\pi}\frac{\sqrt{x}}{\beta^2}\log\left(\frac{T}{2\pi}\right)-3\sqrt[3]{x}-\frac{8}{\pi}\frac{\log T}{\beta}\frac{h}{\sqrt{x}}-\frac{\log^2 T}{\pi}\frac{h}{\sqrt{x}}-\Big(2.05+12\frac{\log T}{\beta^2}\Big)\frac{h}{\sqrt{x}}-\frac{3}{xh}.\end{aligned}\tag{3.1}$$

For this computation it is convenient to take $\beta=\beta(x)$ and diverging as x goes to infinity. To ensure the best result we have to set β so that the sum $\log T+\frac{2}{\pi}\frac{\log T}{\beta}$ is minimised. This sum is, up to terms of lower order in β,

$$\log\beta+\frac{\log x}{\pi\beta}.$$

This last sum is minimum when

$$\beta = \frac{1}{\pi}\log x.$$

Thus we have $T = \frac{1}{\pi c}\sqrt{x}$. With this, the lower bound (3.1) becomes

$$\sum_{|p-x|<h} \log p \geq h - \frac{1}{2}\sqrt{x}\log x - (4-\log(2c\pi^2))\sqrt{x} + o(\sqrt{x}).$$

Using the fact that $c \geq 1/2$, which is the best we can do in this setting, in order to have a positive lower bound it is sufficient to take

$$h \geq \Big(\frac{1}{2} + \frac{d}{\log x}\Big)\sqrt{x}\log x$$

for any $d > 4-2\log\pi = 1.7105\ldots$, when x is large enough. Actually, the choice $d = 1.72$ holds only for $x \geq \exp(590) \approx 2{\cdot}10^{256}$. On the contrary, the choice $d = 2$ holds for $x \geq 7.5{\cdot}10^{8}$. Thus the claim asserting the existence of a prime when $c = 1/2+2/\log x$ is proved for $x \geq 7.5{\cdot}10^{8}$. Moreover, the upper bound $\log(x+h)\sum_{|p-x|<h} 1 \geq \sum_{|p-x|<h}\log p$ and (3.1) prove the existence of $\sqrt{x}$ primes in $(x-(c+1)\sqrt{x}\log x, x+(c+1)\sqrt{x}\log x)$ for $x \geq 1.4{\cdot}10^{5}$. Lastly, for $x \in [2, 1.4{\cdot}10^{5}]$ it is sufficient to check that $p_{k+1}-p_k \leq 2c\sqrt{p_k}\log p_k$ (which gives the claim for $x \in [p_k, p_{k+1}]$) when $k \leq 13010$.

4. An application

On the Riemann hypothesis, Cramér [2] was the first to prove the bound $p_{n+1}-p_n \ll \sqrt{p_n}\log p_n$, and he noted the implication that there exists some constant $\alpha > 0$ such that there will be a prime in the interval

$$(n^2, (n+\alpha\log n)^2)$$

for all sufficiently large n. This was intended for comparison to Legendre's conjecture that there is a prime in the interval $(n^2, (n+1)^2)$ for all n. The following corollary of Theorem 1.1 states that one can take $\alpha = 1+o(1)$.

Corollary 4.1. *Assume RH. Then for every integer $n \geq 2$ there is a prime in the interval*

$$(n^2, (n+\alpha\log n)^2),$$

where

$$\alpha := 1 + \frac{2}{\log n} + \frac{1}{\log^2 n}.$$

Proof. Let

$$x := \frac{n^2 + (n+\alpha\log n)^2}{2}$$

be the mid-point of the interval. We will prove that $(x-c\sqrt{x}\log x, x+c\sqrt{x}\log x) \subseteq (n^2, (n+\alpha\log n)^2)$ with $c = \frac{1}{2}+\frac{2}{\log x}$ so that the corollary will be a consequence of the theorem. Let $\beta := \alpha\frac{\log n}{n}$ and observe that $x = n^2(1+\beta+\frac{\beta^2}{2})$. We just need to prove that $n^2 \leq x - c\sqrt{x}\log x$. It holds if and only if

$$\sqrt{x}(\log x + 4) \leq n^2(2\beta+\beta^2)$$

which is equivalent to

$$\sqrt{1+\beta+\frac{\beta^2}{2}}\Big(2\log n + 4 + \log\Big(1+\beta+\frac{\beta^2}{2}\Big)\Big) \leq 2\log n + 4 + \frac{2}{\log n} + n\beta^2.$$

The last inequality is elementary and is true for any $n \geq 2$. □

Now, upon setting $n = p_k$ in the above corollary, it follows that there is a prime in the interval

$$(p_k^2, (p_k+\alpha\log p_k)^2)$$

for all $k \geq 1$. It should be noted, as $\alpha = 1+o(1)$ and the average gap between p_k and p_{k+1} is $\log p_k$, that something can be said here about the existence of primes in the interval (p_k^2, p_{k+1}^2);

this is related to the so-called Brocard conjecture predicting the existence of four primes at least in this interval (see for instance Ribenboim [12, p. 248]).

It was first proven by Cramér [2], on RH, that the number of $n < x$ such that there is no prime in the interval $(n^2, (n+1)^2)$ is $O(x^{2/3+\epsilon})$ (improved to $O(x^{1/2+\epsilon})$ unconditionally and to $O((\log x)^{2+\epsilon})$ on RH in [1]), and from this it follows that there is a prime in *almost all* intervals of the form (p_k^2, p_{k+1}^2). However, there may still be infinitely many exceptions, though the following corollary assures us that the exceptions must occur when the prime gap is essentially less than the average gap.

Corollary 4.2. *Assume RH. Suppose that p_k and p_{k+1} are consecutive primes satisfying*

$$p_{k+1}-p_k \geq \alpha \log p_k + \frac{\alpha^2 \log^2 p_k}{2p_k},$$

where $\alpha := \big(1+\frac{1}{\log p_k}\big)^2$. Then there is a prime in the interval (p_k^2, p_{k+1}^2).

Proof. First, it follows that

$$p_{k+1}-p_k > \frac{2\alpha p_k \log p_k + \alpha^2 \log^2 p_k}{p_k+p_{k+1}}.$$

It is straightforward to rearrange this so that

$$p_{k+1}^2 > (p_k+\alpha \log p_k)^2$$

and, with reference to Corollary 4.1, this completes the proof. □

Appendix A. Second bound for Σ_1

Since $N(T) \leq \frac{T}{2\pi}\log T$ (see [15, Corollary 1]), from (1.3) one has

$$|\Sigma_1| \leq \frac{2N(T)}{\sqrt{x-h}} \int_{x-h}^{x+h} K(x-u;h)\,\mathrm{d}u = \frac{2h^2}{\sqrt{x-h}} N(T) \leq \frac{h^2 T}{\pi\sqrt{x-h}} \log T,$$

which is the way this sum is estimated in [3]. We improve the result by proving the existence of a cancellation for the sum $\sum_{|\gamma|\leq T} u^{i\gamma}$. The structure of the counting function $N(T)$ alone, that is the fact that $N(T) = \frac{T}{2\pi}\log T + O(\log T)$, is not sufficient to ensure a cancellation in $\sum_{|\gamma|\leq T} u^{i\gamma}$ *for every u*. To see this, one can consider a set of points generated in this way: in the neighborhood of every integer n there is a cloud of $\lfloor \frac{1}{\pi}\log n \rfloor$ points which are placed very close to n. Their counting function satisfies the same formula as $N(T)$, size of the remainder included. For this set, however, one has $\sum_{|\gamma|\leq T} u^{i\gamma} \gg T\log T$ when $u = e^{2\pi}$, and similarly for every $u = e^{2k\pi}$ when $k \in \mathbb{N}$ is small with respect to T.

Thus, we can furnish a cancellation essentially in two ways: either we assume some hypothesis about the distribution of the imaginary parts of the zeroes of $\zeta(s)$ (for example the Pair Correlation Conjecture, as done in [6] and in [8], or the stronger Gonek conjecture [4]), or we try to prove a cancellation in some mean sense. The second possibility appears promising since in our computation the estimated object appears naturally in an integral and produces a result not depending on a further unproved hypothesis.
In this way we can prove Theorem 1.1 with $c = 0.6102$.

Cancellation in mean. We let

$$S_\alpha(T) := \sum_{|\gamma|\leq T} e^{i\alpha\gamma},$$

keep the notations

$$T = \frac{\beta}{c}\frac{\sqrt{x}}{\log x}, \qquad h = c\sqrt{x}\log x,$$

and introduce

$$a := \log(x-h), \qquad b := \log(x+h),$$

$$A := \frac{b-a}{2} = \frac{1}{2}\log\Big(\frac{x+h}{x-h}\Big), \qquad B := \frac{a+b}{2} = \frac{1}{2}\log(x^2-h^2).$$

Notice that $A \sim h/x \approx T^{-1}$ and $B \sim \log x$ as x diverges to infinity.

Proposition A.1. *Assume RH. Suppose $\beta \geq 1$, $c \leq 1$ and $x \geq 2$. Then*

$$\int_a^b |S_\alpha(T)|^2 \,\mathrm{d}\alpha \leq \frac{1}{\pi^2} F(AT) T \log^2\Big(\frac{T}{2\pi}\Big) + H(A,T)$$

with

$$F(y) := \frac{1}{y}\int_0^y \int_{-y}^y |\operatorname{sinc}(u-u')| \,\mathrm{d}u\,\mathrm{d}u'$$

where $\operatorname{sinc}(x) := \frac{\sin x}{x}$, *and*

$$H(A,T) := \frac{4}{\pi}(2+AT)AT(R(T)+1)\log\Big(\frac{T}{2\pi}\Big) + 8A\Big(1+AT+\frac{1}{3}(AT)^2\Big)(R(T)+1)^2.$$

Remark. Once the orders of h and T as functions of x are considered, the trivial bound and the new bound are respectively

$$\int_a^b |S_\alpha(T)|^2 \,\mathrm{d}\alpha \leq \frac{2\beta}{\pi^2} T\log^2 T \quad v.s. \quad \int_a^b |S_\alpha(T)|^2 \,\mathrm{d}\alpha \leq \frac{1+o(1)}{\pi^2} F(\beta) T \log^2 T,$$

and it is easy to see that the second one improves on the first one for every $\beta > 0$ as $T \to \infty$.

Proof. First, we have the series of equations:

$$\begin{aligned}\int_a^b |S_\alpha(T)|^2 \,\mathrm{d}\alpha &= \operatorname{Re}\int_a^b |S_\alpha(T)|^2 \,\mathrm{d}\alpha = \operatorname{Re}\sum_{|\gamma|,|\gamma'|\leq T}\int_a^b e^{i\alpha(\gamma-\gamma')}\,\mathrm{d}\alpha \\ &= \operatorname{Re}\sum_{|\gamma|,|\gamma'|\leq T}\frac{e^{ib(\gamma-\gamma')}-e^{ia(\gamma-\gamma')}}{i(\gamma-\gamma')} = 2\operatorname{Re}\sum_{|\gamma|,|\gamma'|\leq T} e^{iB(\gamma-\gamma')}\frac{\sin\big(A(\gamma-\gamma')\big)}{\gamma-\gamma'} \\ &= 4\sum_{\substack{0<\gamma\leq T\\ -T\leq\gamma'\leq T}}\frac{\cos\big(B(\gamma-\gamma')\big)\sin\big(A(\gamma-\gamma')\big)}{\gamma-\gamma'} \leq 4A\sum_{\substack{0<\gamma\leq T\\ -T\leq\gamma'\leq T}}|\operatorname{sinc}\big(A(\gamma-\gamma')\big)|.\end{aligned}$$

We will use below the following bounds for $\operatorname{sinc}(x)$:

$$\|\operatorname{sinc}\|_\infty \leq 1, \qquad \|\operatorname{sinc}'\|_\infty \leq 1/2, \qquad \|\operatorname{sinc}''\|_\infty \leq 1/3.$$

These are an immediate consequence of the representation $2\operatorname{sinc}(x) = \int_{-1}^1 e^{ixy} dy$. We thus write the double sum on zeros as a Stieltjes integral. Recalling that the imaginary part of the first zero exceeds 2π we get

$$\begin{aligned}\int_a^b |S_\alpha(T)|^2 \,\mathrm{d}\alpha \leq& 4A\int_{\substack{0<\gamma\leq T\\ -T\leq\gamma'\leq T}} |\operatorname{sinc}\big(A(\gamma-\gamma')\big)|\,\mathrm{d}N(\gamma)\,\mathrm{d}N(\gamma') \\ =& 4A\int_{\substack{2\pi<\gamma\leq T\\ 2\pi<\gamma'\leq T}}\Big(|\operatorname{sinc}\big(A(\gamma-\gamma')\big)|+|\operatorname{sinc}\big(A(\gamma+\gamma')\big)|\Big)\,\mathrm{d}N(\gamma)\,\mathrm{d}N(\gamma').\end{aligned}$$

To ease matters, we employ the notation

$$f(t_1,t_2) := |\operatorname{sinc}(A(t_1-t_2))| + |\operatorname{sinc}(A(t_1+t_2))|$$

which allows us to write that

$$\begin{aligned}\int_a^b |S_\alpha(T)|^2 \,\mathrm{d}\alpha =& 4A\int_{\substack{2\pi<\gamma\leq T\\ 2\pi<\gamma'\leq T}} f(\gamma,\gamma')\,\mathrm{d}W(\gamma)\,\mathrm{d}W(\gamma') + 4A\int_{\substack{2\pi<\gamma\leq T\\ 2\pi<\gamma'\leq T}} f(\gamma,\gamma')\,\mathrm{d}W(\gamma)\,\mathrm{d}U(\gamma') \\ &+ 4A\int_{\substack{2\pi<\gamma\leq T\\ 2\pi<\gamma'\leq T}} f(\gamma,\gamma')\,\mathrm{d}U(\gamma)\,\mathrm{d}W(\gamma') + 4A\int_{\substack{2\pi<\gamma\leq T\\ 2\pi<\gamma'\leq T}} f(\gamma,\gamma')\,\mathrm{d}U(\gamma)\,\mathrm{d}U(\gamma').\end{aligned}$$

We write the sum of the above four integrals as $I{+}II{+}III{+}IV$ where the order is kept. It thus remains to estimate separately the contribution of each integral. The first one produces the main term, since

$$\begin{aligned} I &\leq \frac{4A}{(2\pi)^2}\Big[\int_{\substack{2\pi<\gamma\leq T\\ 2\pi<\gamma'\leq T}} f(\gamma,\gamma')\,\mathrm{d}\gamma\,\mathrm{d}\gamma'\Big]\log^2\Big(\frac{T}{2\pi}\Big)\\ &\leq \frac{1}{\pi^2 A}\Big[\int_{\substack{0<u\leq AT\\ -AT\leq u'\leq AT}} |\operatorname{sinc}(u{-}u')|\,\mathrm{d}u\,\mathrm{d}u'\Big]\log^2\Big(\frac{T}{2\pi}\Big) = \frac{1}{\pi^2}F(AT)T\log^2\Big(\frac{T}{2\pi}\Big). \end{aligned}$$

In estimating the integral II, an application of integration by parts gives (note that $\partial_{\gamma'}|\operatorname{sinc}(A(\gamma\pm\gamma'))|$ has only jump singularities, so the formula still holds)

$$\begin{aligned} II &= \frac{4A}{2\pi}\int_{2\pi}^{T}\Big[\int_{2\pi}^{T} f(\gamma,\gamma')\log\Big(\frac{\gamma}{2\pi}\Big)\,\mathrm{d}\gamma\Big]\,\mathrm{d}U(\gamma')\\ &= \frac{2A}{\pi}\Big[\int_{2\pi}^{T} f(\gamma,\gamma')\log\Big(\frac{\gamma}{2\pi}\Big)\,\mathrm{d}\gamma\Big]U(\gamma')\Big|_{2\pi}^{T} - \frac{2A}{\pi}\int_{2\pi}^{T}\Big[\int_{2\pi}^{T}\partial_{\gamma'}f(\gamma,\gamma')\log\Big(\frac{\gamma}{2\pi}\Big)\,\mathrm{d}\gamma\Big]U(\gamma')\,\mathrm{d}\gamma'. \end{aligned}$$

We can estimate it as (recall that $U(2\pi)=1$)

$$\begin{aligned} &\leq \frac{2A}{\pi}\int_{2\pi}^{T} 2\|\operatorname{sinc}\|_\infty \log\Big(\frac{\gamma}{2\pi}\Big)\,\mathrm{d}\gamma(R(T){+}1) + \frac{2A^2}{\pi}\int_{2\pi}^{T}\int_{2\pi}^{T} 2\|\operatorname{sinc}'\|_\infty \log\Big(\frac{\gamma}{2\pi}\Big)|U(\gamma')|\,\mathrm{d}\gamma\,\mathrm{d}\gamma'\\ &\leq \frac{4}{\pi}AT(R(T){+}1)\log\Big(\frac{T}{2\pi}\Big) + \frac{2}{\pi}(AT)^2R(T)\log\Big(\frac{T}{2\pi}\Big)\\ &\leq \frac{2}{\pi}(2{+}AT)AT(R(T){+}1)\log\Big(\frac{T}{2\pi}\Big). \end{aligned}$$

The contribution of III equals that of II, for we note the symmetry of the integral under the transposition $\gamma\leftrightarrow\gamma'$. And so, lastly we have

$$\begin{aligned} IV =&4A\int_{2\pi}^{T}\Big[\int_{2\pi}^{T} f(\gamma,\gamma')\,\mathrm{d}U(\gamma)\Big]\,\mathrm{d}U(\gamma')\\ =&4A\int_{2\pi}^{T}\Big[f(\gamma,\gamma')U(\gamma)\Big|_{2\pi}^{T}\Big]\,\mathrm{d}U(\gamma') - 4A\int_{2\pi}^{T}\Big[\int_{2\pi}^{T}\partial_\gamma\Big(f(\gamma,\gamma')\Big)U(\gamma)\,\mathrm{d}\gamma\Big]\,\mathrm{d}U(\gamma')\\ =&4A\int_{2\pi}^{T} f(T,\gamma')U(T)\,\mathrm{d}U(\gamma') - 4A\int_{2\pi}^{T} f(2\pi,\gamma')\,\mathrm{d}U(\gamma')\\ &-4A\int_{2\pi}^{T}\Big[\int_{2\pi}^{T}\partial_\gamma\Big(f(\gamma,\gamma')\Big)U(\gamma)\,\mathrm{d}\gamma\Big]\,\mathrm{d}U(\gamma'), \end{aligned}$$

where a second integration by parts gives

$$\begin{aligned} IV =&4Af(T,\gamma')U(T)U(\gamma')\Big|_{2\pi}^{T} - 4A\int_{2\pi}^{T}\partial_{\gamma'}(f(T,\gamma'))U(T)U(\gamma')\,\mathrm{d}\gamma'\\ &-4Af(2\pi,\gamma')U(\gamma')\Big|_{2\pi}^{T} + 4A\int_{2\pi}^{T}\partial_{\gamma'}(f(2\pi,\gamma'))U(\gamma')\,\mathrm{d}\gamma'\\ &-4A\Big[\int_{2\pi}^{T}\partial_\gamma\Big(f(\gamma,\gamma')\Big)U(\gamma)\,\mathrm{d}\gamma\Big]U(\gamma')\Big|_{2\pi}^{T} + 4A\int_{2\pi}^{T}\Big[\int_{2\pi}^{T}\partial_{\gamma'}\partial_\gamma\Big(f(\gamma,\gamma')\Big)U(\gamma)\,\mathrm{d}\gamma\Big]U(\gamma')\,\mathrm{d}\gamma'. \end{aligned}$$

Thus, one may establish the bound

$$\begin{aligned} IV \leq&\big[8AR^2(T){+}8AR(T)\big] + 4A^2\int_{2\pi}^{T} 2\|\operatorname{sinc}'\|_\infty R(T)|U(\gamma')|\,\mathrm{d}\gamma'\\ &+\big[8AR(T){+}8A\big] + 4A^2\int_{2\pi}^{T} 2\|\operatorname{sinc}'\|_\infty|U(\gamma')|\,\mathrm{d}\gamma'\\ &+4A^2\Big[\int_{2\pi}^{T} 2\|\operatorname{sinc}'\|_\infty|U(\gamma)|\,\mathrm{d}\gamma\Big](R(T){+}1) + 4A^3\int_{2\pi}^{T}\int_{2\pi}^{T} 2\|\operatorname{sinc}''\|_\infty|U(\gamma)U(\gamma')|\,\mathrm{d}\gamma\,\mathrm{d}\gamma'. \end{aligned}$$

Estimating the integrals, one has that

$$\begin{aligned}IV \le &8AR^2(T)+8AR(T)+4A^2TR^2(T)+8AR(T)+8A+4A^2TR(T)\\ &+4A^2TR(T)(R(T)+1)+\frac{8}{3}A^3T^2R^2(T)\\ =&8A\Big((R(T)+1)^2+AT(R(T)+1)R(T)+\frac{1}{3}(AT)^2R^2(T)\Big)\\ \le&8A\Big(1+AT+\frac{1}{3}(AT)^2\Big)(R(T)+1)^2.\end{aligned}$$

Therefore, the contribution of II, III and IV is bounded by

$$2\cdot\frac{2}{\pi}(2+AT)AT(R(T)+1)\log\Big(\frac{T}{2\pi}\Big)+8A\Big(1+AT+\frac{1}{3}(AT)^2\Big)(R(T)+1)^2,$$

which is $H(A,T)$. □

Estimation of Σ_1. We now use the above result on cancellation to estimate the first sum over the zeroes. The Cauchy-Schwarz inequality yields the bound

$$\begin{aligned}|\Sigma_1| &\le \int_{x-h}^{x+h} K(x-u;h)\Big|\sum_{|\gamma|\le T} e^{i\gamma\log u}\Big|\frac{\mathrm{d}u}{\sqrt{u}}\\ &= \int_{\log(x-h)}^{\log(x+h)} e^{\alpha/2}K(x-e^\alpha;h)\Big|\sum_{|\gamma|\le T} e^{i\alpha\gamma}\Big|\,\mathrm{d}\alpha\\ &\le \Big[\int_{\log(x-h)}^{\log(x+h)} e^{\alpha}K^2(x-e^\alpha;h)\,\mathrm{d}\alpha\Big]^{1/2}\Big[\int_{\log(x-h)}^{\log(x+h)} |S_\alpha(T)|^2\,\mathrm{d}\alpha\Big]^{1/2},\end{aligned}$$

and so we have that

$$\begin{aligned}|\Sigma_1| &\le \Big[\int_{-h}^{h} K^2(u;h)\,\mathrm{d}u\Big]^{1/2}\Big[\int_{\log(x-h)}^{\log(x+h)} |S_\alpha(T)|^2\,\mathrm{d}\alpha\Big]^{1/2}\\ &= \sqrt{\frac{2}{3}}\,h^{3/2}\Big[\int_{\log(x-h)}^{\log(x+h)} |S_\alpha(T)|^2\,\mathrm{d}\alpha\Big]^{1/2}.\end{aligned}$$

We can now apply Proposition A.1 to get the estimate

$$\begin{aligned}|\Sigma_1| &\le \sqrt{\frac{2}{3}}\,h^{3/2}\Big(\frac{1}{\pi^2}F(AT)T\log^2\Big(\frac{T}{2\pi}\Big)+H(A,T)\Big)^{1/2}\\ &= \frac{1}{\pi}\Big(\frac{2}{3}F(AT)+\frac{2\pi^2}{3}\frac{H(A,T)}{T\log^2(T/2\pi)}\Big)^{1/2}h\sqrt{hT}\log\Big(\frac{T}{2\pi}\Big)\\ &= \frac{1}{\pi}\Big(\frac{2\beta}{3}F(AT)+\frac{2\beta\pi^2}{3}\frac{H(A,T)}{T\log^2(T/2\pi)}\Big)^{1/2}h\sqrt{x}\log\Big(\frac{T}{2\pi}\Big).\end{aligned}\tag{A.1}$$

We now proceed to prove the analog of Theorem 1.1.

First claim. We want to prove that there is a prime in $(x-c\sqrt{x}\log x, x+c\sqrt{x}\log x)$ with $c = 0.6102$. From (1.1) and the bound (A.1) for Σ_1 we get

(A.2)

$$h^{-1}\sum_{|p-x|<h}\log p \ge 1-\Big(\frac{2\beta}{3}F(AT)+\frac{2\beta\pi^2}{3}\frac{H(A,T)}{T\log^2(T/2\pi)}\Big)^{1/2}\frac{\sqrt{x}}{\pi h}\log\Big(\frac{T}{2\pi}\Big)-4(x+h)^{3/2}\frac{\log T}{\pi h^2T}$$
$$-0.002\frac{\sqrt{x}}{h}-3\frac{\sqrt[3]{x}}{h}-\frac{2}{\sqrt{x}}-\frac{3}{xh^2}$$

when $x \geq 121$. When x diverges to infinity, Inequality (A.2) becomes

$$h^{-1} \sum_{|p-x|<h} \log p \geq 1-\frac{\alpha}{c}+\Big(\frac{2\alpha}{c}+o(1)\Big)\frac{\log\log x}{\log x}, \tag{A.3}$$

where $\alpha := \frac{1}{\pi}\big(\sqrt{\frac{\beta}{6}F(\beta)}+\frac{2}{\beta}\big)$, uniformly for β and c in any compact set of $(0,+\infty)$. The minimum of α is attained for $\beta = \beta_{\min} := 2.4934\ldots$, and is $\alpha_{\min} := 0.61019\ldots$. Thus, setting $\beta = \beta_{\min}$ and $c = \alpha_{\min}$, the right-hand side of (A.3) is positive when x is large enough. Inserting $\beta = 2.493$ and $c = 0.6102$ directly into (A.2) one obtains an inequality where the function appearing on the right-hand side is positive whenever $x \geq 16000$, so that the claim is proved in this range. Lastly, for $x \in [2, 16000]$ it is sufficient to check that $p_{k+1}-p_k \leq 2c\sqrt{p_k}\log p_k$ (which gives the claim for $x \in [p_k, p_{k+1}]$) when $k \leq 2000$.

Second claim. We want to prove that there are at least $\sqrt{x}$ primes in $(x-(c+1)\sqrt{x}\log x, x+(c+1)\sqrt{x}\log x)$ with $c = 0.6102$. Since

$$h^{-1} \sum_{|p-x|<h} \log p \leq \frac{1+O(\frac{\log x}{\sqrt{x}})}{c\sqrt{x}} \sum_{|p-x|<h} 1,$$

from (A.3) we also get that

$$\sum_{|p-x|<h} 1 \geq \Big(c-\alpha+(2\alpha+o(1))\frac{\log\log x}{\log x}\Big)\sqrt{x}.$$

In particular, setting $\alpha = \alpha_{\min}$ and $c = \alpha_{\min}+1$ this shows that

$$\sum_{|p-x|<(\alpha_{\min}+1)\sqrt{x}\log x} 1 \geq \sqrt{x},$$

when x is large enough. Once again, choosing these values directly in (A.3) one gets an explicit inequality which can be proved for $x \geq 1500$, proving the statement in this range. The claim for $x \in [2, 1600]$ may be checked directly by noticing that $p_{n+\lceil\sqrt{p_n}\rceil}-p_n \leq 2(c+1)\sqrt{p_n}\log p_n$ (giving the claim for $x \in [p_n, p_{n+\lceil\sqrt{p_n}\rceil}]$) for $n = 1, \ldots, 251$.

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