OSCULATION FOR CONIC FIBRATIONS

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Abstract. Smooth projective surfaces fibered in conics over a smooth curve are investigated with respect to their k-th osculatory behavior. Due to the bound for the dimension of their osculating spaces they do not differ at all from a general surface for $k = 2$, while their structure plays a significant role for $k \geq 3$. The dimension of the osculating space at any point is studied taking into account the possible existence of curves of low degree transverse to the fibers, and several examples are discussed to illustrate concretely the various situations arising in this analysis. As an application, a complete description of the osculatory behavior of Castelnuovo surfaces is given. The case $k = 3$ for del Pezzo surfaces is also discussed, completing the analysis done for $k = 2$ in a previous paper of the authors (2001). Moreover, for conic fibrations $X \subset \mathbb{P}^N$, whose k-th inflectional locus has the expected codimension a precise description of this locus is provided in terms of Chern classes. In particular, for $N = 8$, it turns out that either X is hypo-osculating for $k = 3$, or its third inflectional locus is 1-dimensional.

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1. Introduction

Let $X \subset \mathbb{P}^N$ be any smooth surface. The osculatory behavior of X is determined by the rank of the k-th jet map at every point of X. Clearly, it cannot exceed $\min\{\binom{k+2}{2}, N+1\}$, the former being the rank of the k-th principal part bundle of X. If X has some special structure, the maximum rank for $x \in X$, s_k, can be even smaller (as it happens e.g., for scrolls). In particular, if X is a conic fibration, it follows from [12, Corollary 14] that $s_k \leq \min\{3k, N+1\}$. For $k = 2$ the two bounds above are the same: $s_2 \leq 6$ if $N \geq 5$. This means that if we confine to study osculation for $k = 2$, conic fibrations over curves do not play any special role among surfaces. This is not the case however for $k \geq 3$. Actually, for $N \geq 9$ we have that $s_3 \leq 10$ for a general surface, while $s_3 \leq 9$ for any conic fibration. This is the basic remark which stimulated our interest for the subject of this paper. In particular, though this paper can be regarded as a continuation of [11] and [12], we stress that this is the first contribution to the study of osculation for surfaces, except for scrolls, for $k > 2$.

Let us insist on case $k = 3$. As is well know, every surface can be embedded in $\mathbb{P}^5$. On the other hand, every conic fibration in $\mathbb{P}^N$ ($N \geq 8$) can be projected isomorphically to $\mathbb{P}^8$ without affecting the value of s_3. This suggests a parallel between the study of osculation for any surface for $k = 2$ [15] and that of conic fibrations in $\mathbb{P}^8$ for $k = 3$. An unexpected result we obtain in this context is that any conic fibration in $\mathbb{P}^8$ is either hypo-osculating, or it has a 1-dimensional third inflectional locus $\Phi_3(X)$ (Corollary 15). We face this situation in Section 6 addressing the more general framework of conic fibrations whose k-th inflectional locus $\Phi_k(X)$ has the expected codimension (k being the largest integer such that $3k \leq N + 1$). In this setting we generalize some result of [12], providing explicit expressions for the cohomology classes of $\Phi_k(X)$, by means of the Porteous formula (Theorem 11). In particular we compute the precise number of flexes for the two rational conic fibrations in $\mathbb{P}^4$, namely the quartic del Pezzo surface and the quintic Castelnuovo surface, and we describe them explicitly (Example b in Section 6, and Theorem 18(3)).

To determine the inflectional locus of X we need to compute the rank of the k-th jet map at every point $x \in X$, and this in turn translates into the computation of the codimension in $|V|$ (the linear system of hyperplane sections of $X \subset \mathbb{P}^N$) of the linear subsystem $|V - (k + 1)x|$ of hyperplane sections of X having a singular point of multiplicity $\geq k + 1$ at x. The key point is that, if $k \geq 2$, for any conic fibration X this linear system has some fixed components and it is useful to detect all of them in order to compute its dimension. For instance, the smooth fiber through x (or the component passing through x if x lies on a reducible fiber) is a fixed component of $|V - 3x|$. This fact was already taken into account in studying conic fibrations (e.g., see [4], [12]). However, in addition to these obvious curves there could be further, sometimes unexpected, fixed components of $|V - (k + 1)x|$. This leads, already for $k = 2$, to a number of possible cases, according to whether X contains or not smooth curves of low degree, passing through x, not contained in the fibers.

This is in fact the core of our analysis, which takes Sections 3 and 4. The general result is expressed by Theorem 5, in which the jumping loci of certain linear systems enter into the description of the second inflectional locus $\Phi_2(X)$. In Section 4 it is specialized to the case of conic fibrations containing a line or a conic transverse to the fibers (Theorem 7). Moreover, to illustrate the range of applicability of the results we characterize these surfaces (Propositions 8 and 9).

In Section 5 we present a library of examples illustrating various phenomena occurring for $k = 2, 3$, and sometimes 4. In particular, many examples we discuss (Examples 1, 2, 6, 9, 10) are conic fibrations in $\mathbb{P}^8$, in accordance with the above mentioned relevance of this space for $k = 3$. Moreover, as in our previous paper [11] osculation to del Pezzo surfaces $X \subset \mathbb{P}^N$ was studied for $k = 2$, here we take the opportunity to complete the picture for these surfaces also for $k = 3$ (Examples 1–5), at least when N is large enough to make the discussion reasonably meaningful. Other conic fibrations, rational and irrational, are also discussed (Examples 6, 7, 9, 10).

Section 7 is dedicated to Castelnuovo surfaces, namely rational surfaces whose plane model is given by a linear system of nodal quartics. They are rational conic fibrations of sectional genus 2, and the geometry of their linearly normal models is even richer than that of del Pezzo's. By applying our results, we provide a complete description of the osculatory behavior of these surfaces for $k = 2$ (Theorem 18), $k = 3$ (Theorem 22) and $k = 4$ (final comment in Section 7). We point out that for the Castelnuovo surface $X \subset \mathbb{P}^4$ of degree 5, which is isomorphic to the plane blown-up at 8 points, the notion of points in general position we use is wider than that implying the ampleness of the anticanonical bundle. In particular, the result we obtain for this surface for $k = 2$ extends the description of $\Phi_2(X)$ given at the end of [12].

2. Background material

Varieties considered in this paper are defined over the field $\mathbb{C}$ of complex numbers. We use the standard notation and terminology from algebraic geometry. Let $X \subset \mathbb{P}^N = \mathbb{P}(V)$ be a non-degenerate smooth projective variety of dimension n, let $\mathcal{L} := \mathcal{O}_{\mathbb{P}^N}(1)|_X$ be the hyperplane bundle and identify V with the subspace of $H^0(X, \mathcal{L})$ providing the embedding. Let $\mathcal{P}^k_X(\mathcal{L})$ be the kth principal parts bundle of $\mathcal{L}$ and let $j^X_k : V_X = V \otimes \mathcal{O}_X \to \mathcal{P}^k_X(\mathcal{L})$ be the sheaf homomorphism associating to every section $\sigma \in V$ its kth jet evaluated at x, for every $x \in X$. We simply write j_k instead of j^X_k when there is no ambiguity for the variety X we are dealing with.

We recall that the k-th osculating space to X at x is defined as $\mathrm{Osc}^k_x(X) := \mathbb{P}(\mathrm{Im} j_{k,x})$. Then the k-th osculating hyperplanes to X at x can be regarded as the elements of the linear system $|V - (k+1)x|$ (hyperplane sections of X having a singular point of multiplicity $\geq k+1$ at x). So we have the obvious equality

$$N = \dim(|V - (k+1)x|) + \dim(\mathrm{Osc}^k_x(X)) + 1. \tag{1}$$

First of all we stress the equivalence of the following facts for a point $x \in X$:

a) $|V - (k+1)x| = \emptyset$;
b) $\mathrm{Osc}^k_x(X) = \mathbb{P}^N$;
c) $\mathrm{rk}(j_{k,x}) = N + 1$.

Moreover, if a) holds for some point, then it holds for the general point $x \in X$. On the other hand, if $|V - (k+1)x| \neq \emptyset$ at the general point, then the same fact holds at every point $x \in X$. An immediate consequence of (1) is that

$$\mathrm{rk}(j_{k,x}) = \mathrm{codim}_{|V|}(|V - (k+1)x|), \tag{2}$$

at every point $x \in X$. Let

$$s_k := \max_{x \in X}\{\mathrm{rk}(j_{k,x})\}$$

be the maximum rank of $j_{k,x}$ on X. Clearly, $s_k \leq N+1$, and by what we said equality occurs if and only if $|V-(k+1)x| = \emptyset$ at the general point of X. According to the definition of s_k, we can now define the kth inflectional locus of X as follows:

$$\Phi_k(X) = \{x \in X \mid \mathrm{rk}(j_{k,x}) < s_k\}.$$

We say that $X \subset \mathbb{P}^N$, or $(X, \mathcal{L})$, is a *quadric fibration* (a *conic fibration* if $n=2$) over a smooth curve C if there exists a surjective morphism $\pi : X \to C$, such that any general fiber F of π is a smooth quadric hypersurface $\mathbb{Q}^{n-1} \subset \mathbb{P}^n$ and $\mathcal{L}|_F = \mathcal{O}_{\mathbb{Q}^{n-1}}(1)$. We know that every fiber of π is reduced, and also irreducible if $n \geq 3$; moreover, singular fibers, if any, are quadric cones with an isolated singular point and $\mathcal{L}$ induces the hyperplane bundle on each of them [8, Lemma 0.6]. In particular, for $n=2$ this is equivalent to saying that X is a birationally ruled surface over C, whose smooth fibers F are conics with respect to $\mathcal{L}$ (i.e. $F \cong \mathbb{P}^1$ and $F \cdot \mathcal{L} = 2$); in this case every singular fiber has the form $e_1 + e_2$, where e_1, e_2 are two distinct (-1)-curves in X with $e_i \cdot \mathcal{L} = e_1 \cdot e_2 = 1$. Note that this makes sense also when $\mathcal{L}$ is simply an ample line bundle, so, sometimes, we use the expression conic fibration also in the more general context of polarized surfaces. By *conic bundle* we mean any conic fibration with no singular fiber.

Let $X \subset \mathbb{P}^N$ be a quadric fibration over a smooth curve. In [12, Corollary 14] it is proved that

$$s_k \leq \min\{k(n+1), N+1\} \tag{3}$$

and if X is general we have in fact the equality $s_k = k(n+1)$. X is said to be k-hypo-osculating if (3) is a strict inequality. For examples of hypo-osculating conic fibrations see Section 5, Examples 10 for $k=3$, and 7 for $k=4$. According to (3), for a conic fibration $X \subset \mathbb{P}^N$ it is significant to investigate k-osculation for

$$3k - 1 \leq N.$$

In studying case $k=3$, however, we will also deviate from this bound in some instances, in order to include some special interesting surfaces in our discussion.

Let $X \subset \mathbb{P}^N$ be a conic fibration. We will denote by Σ the union of all singular fibers of X and by S the finite set consisting of their singular points. As shown in [12] (see also Theorem 5), $\Sigma \subseteq \Phi_2(X)$ if $N \geq 5$; moreover, $\mathrm{rk}(j_{2,x})$ is even smaller than 5 (namely 4 or even 3, see Example 5 in Section 5) at every point $x \in S$. As a consequence, $\Sigma \subseteq \Phi_k(X)$ for every $k \geq 3$, provided that $N \geq 3k-1$. However, if N is small it might be that $\Phi_2(X) \nsubseteq \Phi_3(X)$ if, e.g., $X \subset \mathbb{P}^N$ is hypo-osculating for $k=3$ (see Examples 3, 10 in Section 5).

Here are some Chern class computations we need in the sequel.

Lemma 1. *Let $X \subset \mathbb{P}^N$ be any smooth projective surface, let $\mathcal{L}$ be the hyperplane bundle and set $L := c_1(\mathcal{L})$.*

(i) $c_1(\mathcal{P}^2_X(\mathcal{L})) = 4K_X + 6L$ *and* $c_1(\mathcal{P}^3_X(\mathcal{L})) = 10K_X + 10L$;
(ii) $c_2(\mathcal{P}^2_X(\mathcal{L})) = 5c_2(X) + 5K_X^2 + 20K_XL + 15L^2$ *and* $c_2(\mathcal{P}^3_X(\mathcal{L})) = 15c_2(X) + 40K_X^2 + 90K_XL + 45L^2$.

Proof. The assertion follows from standard computations, using recursively the following exact sequence (e. g., see [9, p. 70])

$$0 \to S^m(\Omega_X \otimes \mathcal{L}) \to \mathcal{P}^m_X(\mathcal{L}) \to \mathcal{P}^{m-1}_X(\mathcal{L}) \to 0. \tag{4}$$

□

Lemma 2. *Let $X \subset \mathbb{P}^N = \mathbb{P}(V)$ be a smooth surface, with hyperplane bundle $\mathcal{L}$, let $x \in X$ and suppose that $|V - (k+1)x| \neq \emptyset$ for some integer $k \geq 2$. Let $\gamma \subset X$ be an irreducible curve passing through x and smooth at x.*

(1) *If $\gamma \cdot \mathcal{L} \leq k$, then γ is a fixed component of $|V - (k+1)x|$;*
(2) *if $k = 3$ and either*
  a) *$\gamma \cdot \mathcal{L} = 2$ and $\gamma^2 = 0$, or*
  b) *$\gamma \cdot \mathcal{L} = 1$ and $\gamma^2 = -1$,*
then 2γ is in the fixed part of $|V - 4x|$.

Proof. Consider a general element $D \in |V-(k+1)x|$. If γ were not contained in D, then we would get

$$k \geq \mathcal{L} \cdot \gamma = D \cdot \gamma \geq \mathrm{mult}_x(D) \geq k+1,$$

a contradiction. This proves (1). Now let $k = 3$; by (1) γ is a fortiori a fixed component of $|V - 4x|$ and $|V - 4x| = \gamma + |V - \gamma - 3x|$, since γ is smooth at x. Clearly, $|V - \gamma - 3x| \neq \emptyset$. Let $D' \in |V - \gamma - 3x|$ be a general element. If γ were not contained in D', then in both cases a) and b) we would get

$$2 = \mathcal{L} \cdot \gamma - \gamma^2 = (\mathcal{L} - \gamma) \cdot \gamma = D' \cdot \gamma \geq \mathrm{mult}_x(D') \geq 3,$$

a contradiction. Therefore γ is a fixed component of $|V - \gamma - 3x|$ and this proves (2). □

For any nonnegative integer e, we denote by $\mathbb{F}_e$ the Segre–Hirzebruch surface of invariant e, i.e., $\mathbb{F}_e = \mathbb{P}(\mathcal{E})$, where $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)$. By s and f we will denote the (a, if $e = 0$) tautological section of minimal self-intersection $s^2 = -e$, and a fiber, respectively. We recall that the classes of s and f generate the Picard group of $\mathbb{F}_e$, hence, for any line bundle $\mathcal{L} \in \mathrm{Pic}(\mathbb{F}_e)$ we can write $\mathcal{L} = [\alpha s + \beta f]$ for some integers α, β. According to [7, Corollary 2.18, p. 380], $\mathcal{L}$ is ample, if and only if it is very ample, if and only if $\alpha > 0$ and $\beta > \alpha e$. We will use these conditions over and over in the paper, without any further reference. Clearly, if $\alpha = 2$ (and $\beta > 2e$), then $(\mathbb{F}_e, \mathcal{L})$ is a conic bundle. By the projection formula we have $h^0(\mathcal{L}) = h^0(\pi_*\mathcal{L}) = h^0(S^2\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^1}(\beta))$, where $\pi : \mathbb{F}_e \to \mathbb{P}^1$ is the bundle projection and S^2 stands for the second symmetric power. Note that

$$S^2\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^1}(\beta) = \mathcal{O}_{\mathbb{P}^1}(\beta) \oplus \mathcal{O}_{\mathbb{P}^1}(\beta - e) \oplus \mathcal{O}_{\mathbb{P}^1}(\beta - 2e).$$

Moreover, all summands have positive degree, since $\beta > 2e$. Therefore,

$$h^0(\mathcal{L}) = 3(\beta - e + 1). \tag{5}$$

Hence $|\mathcal{L}|$ embeds X into $\mathbb{P}^N$, with $N = 3(\beta - e) + 2$. In particular, consider $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$. In this case we set $\mathcal{O}(\alpha, \beta) := [\alpha s + \beta f]$. If $\alpha = \beta = 2$, then $(\mathbb{F}_0, \mathcal{L})$ has two distinct conic bundle structures.

By a *del Pezzo surface* we mean a smooth projective surface X, whose anticanonical bundle $-K_X$ is ample. The degree of X is the degree of the polarized surface $(X, -K_X)$, namely K_X^2. Let X be a del Pezzo surface. According to the classification [5], either $X = \mathbb{P}^2$ (degree 9), $X = \mathbb{P}^1 \times \mathbb{P}^1$ (degree 8), or X is obtained by blowing-up $\mathbb{P}^2$ at no more than 8 distinct points in general position. In accordance with the literature, $r+1 \leq 8$ points of $\mathbb{P}^2$, say $p_0, p_1, \dots, p_r$, are said to be *in general position* to mean that

(6) no three of them are collinear and no six lie on a conic

if $r \leq 6$, and they satisfy the further condition that

(7) not all lie on a cubic having a double point at one of them,

if $r = 7$. Conditions (6) and (7) insure that $-K_X$ is ample. Let $\sigma : X \to \mathbb{P}^2$ be the morphism expressing X as the plane blown-up at $p_0, p_1, \dots p_r$, and let e_i be the exceptional curve corresponding to p_i; then $-K_X = \sigma^*\mathcal{O}_{\mathbb{P}^2}(3) - \sum_{j=0}^r e_j$. In particular, the degree of X is $K_X^2 = 8 - r \leq 8$, so, in degree 8 there are two distinct del Pezzo surfaces, namely $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{F}_1$, i.e. the plane blown-up at a point p_0, in which case $-K_X$ can be rewritten as $[2s + 3f]$ identifying the (-1)-section s with the exceptional curve e_0 and f with the proper transform via σ of a line through p_0. For any del Pezzo surface X, we have $\dim(|-K_X|) = K_X^2$. Clearly, if $X = \mathbb{P}^2$, then $-K_X = \mathcal{O}_{\mathbb{P}^2}(3)$ is 3-very ample and embeds X in $\mathbb{P}^9$. If $X = \mathbb{P}^1 \times \mathbb{P}^1$, then $-K_X = \mathcal{O}(2,2)$ is 2-very ample and embeds X in $\mathbb{P}^8$; moreover, each of the projections onto the two factors makes X a conic bundle over $\mathbb{P}^1$. Finally, if X is $\mathbb{P}^2$ blown-up at $r+1$ points as above, we recall the following facts: $-K_X$ is very ample for $r \leq 5$, ample and spanned for $r = 6$, with $|-K_X|$ defining a double cover of $\mathbb{P}^2$ branched along a smooth plane quartic, while for $r = 7$, $-K_X$ is just ample, $|-K_X|$ consists of a pencil with a single base point, say x_0, all elements of $|-K_X|$ are smooth at x_0, and, in general, $|-K_X|$ contains exactly twelve singular curves, each having one singular point. Note that X is a conic fibration with respect to $-K_X$ for any $r \geq 0$, regardless whether $-K_X$ is very ample or not. Actually the projection of $\mathbb{P}^2 \setminus \{p_0\}$ from p_0 onto a general line induces a morphism $\pi : X \to \mathbb{P}^1$. Its general fiber F is the proper transform via σ of a general line of the pencil through p_0, hence F is a smooth rational curve with $-K_X \cdot F = 2$. Moreover, the exceptional curve e_0 is a section of π. Note that π has exactly $\mu = r$ singular fibers, namely $F_i = e_i + \widetilde{\ell}_i$, where $\widetilde{\ell}_i$ is the proper transform via σ of the line $\ell_i := \langle p_0, p_i \rangle \subset \mathbb{P}^2$ joining p_0 and p_i $(i = 1, \dots, r)$. In particular, $\Sigma = \cup_{i=1}^r F_i$. Sometimes, if $r > 1$ it is useful also to consider the lines $\ell_{j,h} := \langle p_j, p_h \rangle$ joining p_j and p_h $(1 \leq j < h \leq r)$, and their proper transforms $\widetilde{\ell_{j,h}}$ via σ. Note that $\widetilde{\ell_{j,h}}$ is a line of $(X, -K_X)$: actually, as $\sigma^*\ell_{j,h} = \widetilde{\ell_{j,h}} + e_j + e_h$ we get $-K_X \cdot \widetilde{\ell_{j,h}} = 1$. Moreover, each of them is a section of π. Finally, we point out that if $r > 0$ then X admits r further distinct conic fibration structures $\pi_i : X \to \mathbb{P}^1$ $(i = 1, \ldots, r)$, each being induced by the projection of $\mathbb{P}^2 \setminus \{p_i\}$ from p_i onto a general line: what we said for π applies also to π_i; in particular, there are $r + 1$ distinct smooth conics passing though the general point $x \in X$.

For our need in Section 7, in case $r = 7$ we have also to consider the surface X obtained by blowing-up $\mathbb{P}^2$ at $p_0, \ldots, p_7$, when our points satisfy (6) but not (7). In this case there exists an irreducible plane cubic $\Gamma \in |\mathcal{O}_{\mathbb{P}^2}(3) - 2p_{i_0} - \sum_{j=0;j\neq i_0}^{7} p_j|$ passing through our 8 points and having a double point at one of them, say p_{i_0}, $(0 \leq i_0 \leq r)$. Its proper transform, say G, is a (-2)-curve on X. In this case, $-K_X$ is nef and big, but not ample, since $-K_X \cdot G = 0$. The anticanonical system $|-K_X|$ is a pencil with a single base point x_0, where all its elements meet transversally; moreover, $|-K_X|$ contains a finite number of singular elements, as before. However, as $G = \sigma^*\mathcal{O}_{\mathbb{P}^2}(3) - 2e_{i_0} - \sum_{j=0;j\neq i_0}^{7} e_j$, we see that $|-K_X|$ contains the divisor $G + e_{i_0}$. Note that $G \cdot e_{i_0} = 2$, and $x_0 \in e_{i_0} \setminus G$. In fact $G \cap e_{i_0}$ consists either of two distinct points or of a single point, according to whether Γ has a node or a cusp at p_{i_0}. Accordingly, $G + e_{i_0}$ has two or one singular points. Moreover, in connection with the conic fibration structure of $(X, -K_X)$ defined by π, it turns out that such points lie outside the singular fibers of π if and only if $i_0 = 0$.

A class of conic fibrations relevant for this paper is that of *Castelnuovo surfaces*, namely, rational surfaces whose hyperplane sections correspond to a linear system of plane quartics with a double point. In Section 7 we provide a complete account of their inflectional loci. To give a precise description, consider a finite subset $\mathcal{P} := \{p_0, p_1, \ldots, p_r\}$ of $\mathbb{P}^2$ consisting of $r + 1 \leq 8$ points satisfying (6). We stress that in case of 8 points we do not require the further condition (7). Actually, let $\theta : X \to \mathbb{P}^2$ be the blowing-up of $\mathbb{P}^2$ at $\mathcal{P}$, let e_i be the exceptional curve corresponding to p_i $(i = 0, 1, \ldots, r)$, and let $\mathcal{L} = \theta^*\mathcal{O}_{\mathbb{P}^2}(4) - 2e_0 - e_1 - \cdots - e_r$. The line bundle $\mathcal{L}$ is very ample, since the points satisfy (6) [2, Section 3 and Remark 3.4.1, fifth case in Table I], and we have $\mathcal{L}^2 = 12 - r$ and $h^0(\mathcal{L}) = 11 - r$. Embedding X by $|\mathcal{L}|$ provides a linearly normal surface $X \subset \mathbb{P}^N$, which we call Castelnuovo surface. Clearly, $N = d - 1$ and X has degree $d = 12 - r$ $(5 \leq d \leq 12)$ and sectional genus 2. For the classification of rational surfaces of sectional genus 2, we refer to [8, Proposition 3.1, i)]: Castelnuovo surfaces correspond to case $e = 1$ in that statement. Note that X is a conic fibration (in a unique way) via the morphism $\pi : X \to \mathbb{P}^1$ induced by the pencil of lines through p_0. Here there are $\mu = r = 12 - d$ singular fibers, all of them having the form $\widetilde{\ell_i} + e_i$ $(i = 1, \ldots, r)$, where $\widetilde{\ell_i}$ is the proper transform via θ of the line $\ell_i := \langle p_0, p_i \rangle \subset \mathbb{P}^2$, $i = 1, \ldots, r$. Clearly, e_0 is a section of π, and it is a smooth conic since $e_0 \cdot \mathcal{L} = 2$. This makes the role of p_0 different from that of the remaining points $p_i \in \mathcal{P}$; hence, sometimes it is useful to consider also the subset $P := \mathcal{P} \setminus \{p_0\}$. Due to condition (6), from the abstract point of view, for $r \leq 6$ our X is a del Pezzo surface, while for $r = 7$, we can only claim that $-K_X$ is nef and big. In particular $|-K_X|$ is a pencil with a single base point, say x_0, where all elements meet transversally, and containing a finite number of singular elements (see the previous paragraph). The (-2)-curve G preventing $-K_X$ from being ample is either a line or a conic of $(X, \mathcal{L})$, according to whether $i_0 = 0$ or $i_0 \geq 1$, respectively. These facts will be useful in Section 7. As noted discussing del Pezzo surfaces, for $r > 1$, sometimes we will need to consider also the lines $\ell_{j,h} := \langle p_j, p_h \rangle$ for $1 \leq j < h \leq r$, and their proper transforms $\widetilde{\ell_{j,h}}$ on X via θ. Note that all these curves are (-1)-conics on $(X, \mathcal{L})$.

Finally, let us point out the following fact. We can factor θ as $\theta = \eta \circ \sigma$ where $\sigma : \mathbb{F}_1 \to \mathbb{P}^2$ is the blowing-up at p_0 and $\eta : X \to \mathbb{F}_1$ is the blowing-up of $\mathbb{F}_1$ at r points, each corresponding via σ to a point p_i $(i = 1, \dots, r)$. These r points do not lie on the (-1)-section s of $\mathbb{F}_1$. So, the conic e_0 is simply the proper transform of s via η, and at the same time, $e_0 = \eta^*(s)$. Since $\sigma^*\mathcal{O}_{\mathbb{P}^2}(1) = [s + f]$ we thus get $\mathcal{L} = \eta^*(\sigma^*\mathcal{O}_{\mathbb{P}^2}(4) - 2s) - \sum_{i=1}^r e_i = \eta^*\mathcal{L}_0 - \sum_{i=1}^r e_i$, where $\mathcal{L}_0$ is the line bundle on $\mathbb{F}_1$ given by $\sigma^*\mathcal{O}_{\mathbb{P}^2}(4) - 2s = [4(s + f) - 2s] = [2s + 4f]$.

At last let us recall few facts on the *jumping sets*, since they will often occur in our discussion. Let $\mathcal{M}$ be an ample line bundle on a smooth surface X and suppose that there exists a vector subspace W of sections spanning $\mathcal{M}$ at every point. In this case we say that the corresponding linear system $|W|$ is ample and spanned. Clearly, in this situation, $\mathrm{codim}_{|W|}(|W - x|) = 1$ for every $x \in X$. Let φ be the morphism defined by $|W|$. For $i = 1, 2$ the two loci

$$\mathcal{J}_i(W) = \{x \in X \mid \mathrm{rk}(d\varphi)(x) \leq 2 - i\}$$

are called the jumping sets of W [13, Section 1]. Clearly, $\mathcal{J}_2(W) \subset \mathcal{J}_1(W)$, moreover $\dim(\mathcal{J}_i(W)) \leq 2 - i$, and

$$\mathcal{J}_i(W) = \{x \in X \mid \mathrm{codim}_{|W-x|}(|W - 2x|) \leq 2 - i\}.$$

So, $\mathcal{J}_2(W) = \{x \in X \mid |W - x| = |W - 2x|\}$, while $\mathcal{J}_1(W)$ can be identified with the ramification locus of φ.

3. The osculatory behavior of conic fibrations

Specializing Lemma 2 to conic fibrations, we obtain the following two propositions, which are very useful in the recognition of the inflectional loci for $k = 2$ and $k = 3$, respectively. In particular, they will come up over and over in the next Sections.

Proposition 3. *Let $X \subset \mathbb{P}^N$ be a conic fibration over a smooth curve C and let $x \in X$. Suppose that $|V - 3x| \neq \emptyset$. Any line and any smooth conic* *of X passing through x are fixed components of $|V-3x|$. In particular, let F be the fiber containing x.*

(i) *if either F is smooth or x is the singular point of F then F is a fixed component of $|V-3x|$, and the residual part is either $|V-F-2x|$ or $|V-F-x|$ according to the two cases respectively;*

(ii) *let F be a singular fiber, i.e., $F=e_1+e_2$, where e_1 and e_2 are two lines, and let x be a smooth point of F, i.e., $x \in e_i \setminus e_j$. Then e_i is a fixed component of $|V-3x|$, the residual part being $|V-e_i-2x|$.*

Proof. The first assertion follows from Lemma 2, part (1). In particular, if $\gamma = F$ is a smooth fiber then we can write every element of $|V-3x|$ as $D=F+R$ where R must have a double point at x. This shows that the residual part of $|V-3x|$ is $|V-F-2x|$. If $x \in S$, letting $\gamma = e_i$ for $i=1$ or 2, we have that both lines constituting F are fixed components of $|V-3x|$. Then F is in the fixed part of $|V-3x|$ again, hence we can write $D=F+R$ as before. But now F has a double point at x, so the only condition R must satisfy it that of passing through x. Therefore the residual part of $|V-3x|$ is $|V-F-x|$. Finally, if x is a smooth point of a singular fiber, let γ be the component of F containing x. Then γ is a fixed component of $|V-3x|$ and we can write $D=\gamma+R$, where R must have a singular point at x, γ being smooth. We thus conclude that the residual part of γ in $|V-3x|$ is $|V-\gamma-2x|$. □

Proposition 4. *Let $X \subset \mathbb{P}^N$ be a conic fibration over a smooth curve C and let $x \in X$. Suppose that $|V-4x| \neq \emptyset$. Any (-1)-line and any smooth conic with self-intersection 0 of X, passing through x, are fixed components of multiplicity 2 of $|V-4x|$. Let F be the fiber of X containing x.*

(i) *if either F is smooth or x is the singular point of F, then $2F$ is in the fixed part of $|V-4x|$, and the residual part is either $|V-2F-2x|$ or $|V-2F|$ according to the two cases respectively;*

(ii) *let F be a singular fiber, i.e., $F=e_1+e_2$, where e_1 and e_2 are two (-1)-lines, and let x be a smooth point of F, i.e., $x \in e_i \setminus e_j$. Then $F+e_i$ is in the fixed part of $|V-4x|$, the residual part being $|V-F-e_i-2x|$.*

Proof. The first assertion follows by specializing Lemma 2 part (2) to conic fibrations. In particular, this gives (i). Now let $x \in e_1 \setminus e_2$ be a smooth point of the singular fiber $F=e_1+e_2$. By Lemma 2 part (2) we already know that $2e_1$ is in the fixed part of $|V-4x|$, the residual part being $|V-2e_1-2x|$. Note, however, that e_2 is in turn a fixed component of this linear system. Otherwise, for a general element $A \in |V-2e_1-2x|$ we would get $A \cdot e_2 \geq 0$, hence

$$-1 = (\mathcal{L}-2e_1)\cdot e_2 = A \cdot e_2 \geq 0,$$

a contradiction. Note also that $e_2 \not\ni x$; thus

$$|V-4x| = 2e_1 + |V-2e_1-2x| = 2e_1+e_2+|V-2e_1-e_2-2x|.$$

This proves (ii), since $F = e_1 + e_2$. $\square$

Let us stress the use of the expression *residual part* in Propositions 3 and 4. Actually, it may happen that F is not the whole fixed part of $|V - 3x|$.

The above propositions lead to the following inductive general discussion. Let $X \subset \mathbb{P}^N$ be a conic fibration and suppose that $N \geq 3k-1$. Let $x \in X$ and let F be the fiber of X containing x. For a non-negative integer j, consider the vector space $V(-jF)$ of sections $s \in V$ vanishing along F with order $\geq j$, Up to clearing the factor corresponding to jF we can regard $V(-jF)$ as a vector subspace of $H^0(X, \mathcal{L} - jF)$. Let $|V - jF|$ be the linear system defined by its projectivization. The fact that $|V|$ is very ample implies that $\mathrm{codim}_{|V|}(|V - F|) = 3$.

So, assuming that $|V - jF|$ is very ample for all $0 \leq j \leq k-2$ by induction we get

$$\mathrm{codim}_{|V|}(|V - (k-1)F|) = 3(k-1).$$

Suppose that $x \in X \setminus \Sigma$, i.e., F is a smooth fiber of X, then

$$|V - (k+1)x| = (k-1)F + |V - (k-1)F - 2x|,$$

by an iterated application of Proposition 3. Combining this with (2) and the above relation we get

$$\mathrm{rk}(j_{k,x}) = 3(k-1) + h, \tag{8}$$

where h is the number of linearly independent linear conditions to be imposed on the elements of the linear system $|V - (k-1)F|$ in order to have a double point at x. Clearly, $0 \leq h \leq 3$.

1) Suppose furthermore that $|V - (k-1)F|$ is ample and spanned, and let $\mathcal{J}_i = \mathcal{J}_i\big((V(-(k-1)F)\big)$, $i = 1, 2$. In this case, $h \geq 1$ and its precise value can be expressed in terms of the jumping sets $\mathcal{J}_i$, as follows: $h = 3$ if $x \notin \mathcal{J}_1$; $h = 2$ if $x \in \mathcal{J}_1 \setminus \mathcal{J}_2$; $h = 1$ if $x \in \mathcal{J}_2$. In particular, $h = 3$ at every point of X if $|V - (k-1)F|$ is very ample.

For enlightening examples for $k = 3$ we refer to Section 5, Examples 6–10.

2) On the other hand, suppose that $\mathcal{L} - (k-1)F$ is not ample, and let $Y \subset X$ be an irreducible curve such that $(\mathcal{L} - (k-1)F) \cdot Y = 0$, if any. Then $|V - (k-1)F - x| = Y + |V - (k-1)F - Y|$ for every $x \in Y$, hence $h \leq 2$; so $\mathrm{rk}(j_{k,x}) = 3(k-1) + 2 = 3k - 1$ along Y by (8). This means that $Y \subset \Phi_k(X)$ if $N \geq 3k - 1$. An example of this situation will occur in Section 7 when $\mu = 2$ or 3. E. g., let $k = 3$; for $\mu = 2$, we have that $\mathcal{L} - 2F$ is not ample, hence the jumping sets do not enter in the picture. However, (8) is still working and in fact $\mathrm{rk} j_{3,x} = 6 + h = 8$ for a general $x \in \widetilde{\ell_{1,2}}$. Here $\widetilde{\ell_{1,2}}$ is the only curve on which $\mathcal{L} - 2F$ fails to be ample; thus $\widetilde{\ell_{1,2}} \subset \Phi_3(X)$. Similarly, for $\mu = 3$ we get that $\bigcup_{1 \leq i < j \leq 3} \widetilde{\ell_{i,j}} \subset \Phi_3(X)$, see Theorem 22.

In particular, let us specialize the above discussion to analyze case $k = 2$ more in detail. We suppose that $|V - 3x| \neq \emptyset$ for the general (hence for every) point $x \in X$.

We know that $\dim(|V-F|)=N-3$ for any fiber F of X, since its linear span in $\mathbb{P}^N$ is $\langle F\rangle=\mathbb{P}^2$; on the other hand, $\dim(|V-\ell|)=N-2$ if $\ell\subset X$ is a line through x. This, combined with (2) leads to the following conclusions.

a) Suppose that $x\notin\Sigma$; then $|V-3x|=F+|V-F-2x|$, hence $\mathrm{rk}(j_{2,x})=3+h$, where h is the number of linearly independent linear conditions to be imposed on the elements in $|V-F|$ in order to have a double point at x; in particular, $3\leq \mathrm{rk}(j_{2,x})\leq 6$ (since $0\leq h\leq 3$). Moreover, if $\mathcal{L}-F$ is ample and spanned by $V(-F)$, then $\mathrm{rk}(j_{2,x})\leq 5$ if and only if $x\in\mathcal{J}_1(V(-F))$.

b) If $x\in\Sigma\setminus S$, then $|V-3x|=e_i+|V-e_i-2x|$, e_i being the unique component of F containing x. So, $\mathrm{rk}(j_{2,x})=2+h$, where h is the number of linearly independent linear conditions to be imposed on the elements in $|V-e_i|$ in order to have a double point at x; in particular, $3\leq\mathrm{rk}(j_{2,x})\leq 5$ (since $j_{1,x}$ has rank 3 everywhere and $h\leq 3$). A comment analogous to that in a) can be repeated referring to the vector subspace $V(-e_i)$.

c) Let $x\in S$; then $|V-3x|=F+|V-F-x|$; on the other hand, $\mathrm{codim}_{|V|}(|V-F|)=3$. So $\mathrm{rk}(j_{2,x})=3+h$, h being the number of linearly independent linear conditions to be imposed on the elements in $|V-F|$ in order to contain x. Hence $3\leq\mathrm{rk}(j_{2,x})\leq 4$, with equality on the left if and only if x is a base point of $|V-F|$.

This leads to the part of [12, Theorem 11] concerning surfaces, which we restate here for the convenience of the reader.

Theorem 5. *Let $X\subset\mathbb{P}^N=\mathbb{P}(V)$ be a conic fibration over a smooth curve C, and let $\mathcal{L}$, Σ, S, and $V(-F)$ for every fiber F be as above. Suppose that $N\geq s_2$ (or equivalently that $|V-3x|\neq\emptyset$ for every $x\in X$).*

(1) *We have $3\leq\mathrm{rk}(j_{2,x})\leq 6$ for every $x\in X$, and $\mathrm{rk}(j_{2,x})\leq 4$ for every $x\in S$.*
(2) *Furthermore, $\mathrm{rk}(j_{2,x})\leq 5$ at any point x lying on a line of X; in particular, $\Phi_2(X)$ contains the union of all lines of X, hence Σ.*
(3) *If $V(-F)$ spans $\mathcal{L}-F$ for a singular fiber F, then $\mathrm{rk}(j_{2,x})=4$ for $x\in F\cap S$.*
(4) *If $\mathcal{L}-F$ is ample and spanned by $V(-F)$ for a given smooth fiber F, then $\mathrm{rk}(j_{2,x})=6$ for all $x\in F\setminus\mathcal{J}_1(V(-F))$. In particular, if $\mathcal{L}-F$ is ample and $V(-F)$ defines an immersion for any smooth fiber F, then $\mathrm{rk}(j_{2,x})=6$ at all points $x\notin\Sigma$.*

For explicit examples see Section 5, Examples 4–6. In particular, Examples 4 and 5 show that the rank of $j_{2,x}$ at a point $x\in S$ can in fact be either 4 or 3.

4. More on case $k=2$ in the presence of lines or conics transverse to the fibers

As a further progress with respect to Theorem 5 we can refine our analysis of case $k=2$, looking more closely at the points x lying on a line ℓ or on a smooth conic γ non contained in a fiber. Note that in this case the base curve of our conic fibration is $C = \mathbb{P}^1$, as a consequence of the Riemann–Hurwitz theorem. Let us recall a relevant object associated to any conic fibration $X \subset \mathbb{P}^N$ over a smooth curve C. Let $M := \mathbb{P}(\pi_*\mathcal{L})$ be the $\mathbb{P}^2$-bundle over C defined by X, in which X fits as a divisor, and let $\widetilde{\mathcal{L}}$ be the tautological line bundle on M. As shown in [12, Section 3], V can be regarded also as a vector subspace of $H^0(M, \widetilde{\mathcal{L}})$, and, in this perspective, it defines a morphism $\varphi : M \to \mathbb{P}^N$, whose image $R := \varphi(M)$ is the three-dimensional variety ruled by the planes $\langle F\rangle$, which are the linear spans of the fibers F of X. Note that $R \subset \mathbb{P}^N$ is non-degenerate. As shown in [12, Proposition 5], R is a scroll over C if and only if φ is an embedding. This is not always the case, however. For, instance, if $X \subset \mathbb{P}^3$ is a cubic surface, regarded as a rational conic fibration, clearly R is the whole ambient space. By the way, let us note the following fact.

Remark. Let $X \subset \mathbb{P}^3$ be a conic fibration. Then X is a cubic surface. Actually, $K_X = (d-4)\mathcal{L}$, where d is the degree of X. On the other hand, for every fibre F of X we have $\mathcal{L} \cdot F = 2$. Hence the genus formula gives $-2 = K_X \cdot F = 2(d-4)$, i.e., $d = 3$.

The following easy lemma will be very useful.

Lemma 6. *Let $X \subset \mathbb{P}^N$ be a conic fibration, and let F be any fiber.*

(1) *Let $\ell \subset X$ be a line transverse to the fibers. Then either $\ell \cdot F = 1$, or $N = 3$ and $R = \mathbb{P}^3$;*
(2) *Let $\gamma \subset X$ be a smooth conic transverse to the fibers. Then $1 \leq \gamma \cdot F \leq 2$.*

Proof. 1) Clearly $1 \leq \ell \cdot F \leq 2$: suppose that $\ell \cdot F = 2$. Since this happens for every fiber we have that the ruled variety $R \subset \mathbb{P}^N$ consists of a pencil of planes containing ℓ. Therefore $R = \mathbb{P}^3 = \mathbb{P}^N$ (since it is non-degenerate).

2) Clearly $\gamma \cdot F \geq 1$. Suppose that $\gamma \cdot F_0 \geq 3$ for some fiber F_0. Then the two conics γ and F_0 are coplanar, i.e., $\langle\gamma\rangle = \langle F_0\rangle$. But since $\gamma \cdot F = \gamma \cdot F_0 \geq 3$ for every fiber F, we conclude that $\langle F\rangle = \langle F_0\rangle$ for every other fiber F. Thus $F \cap F_0 \neq \emptyset$, the two fibers being coplanar, but this is impossible $\square$

The fact that the intersection index of ℓ or γ with any fiber of X is very low, makes the range of the possible values for $\mathrm{rk}(j_{2,x})$ very restricted at their points (see also Examples 4–6 in Section 5). Of course we have to consider various possibilities, in accordance with points a), b), c) of the discussion in Section 3.

Theorem 7. *Let $X \subset \mathbb{P}^N = \mathbb{P}(V)$ be a conic fibration over $\mathbb{P}^1$ with $N \geq s_2$, let $\mathcal{L}$, Σ, S be as in Section 3. Let $x \in X$, and let F be the fiber through x.*
1) *Suppose that there exists a line $\ell \subset X$, passing through x and transverse to the fibers. Then:*

1a) *$4 \leq \mathrm{rk}(j_{2,x}) \leq 5$ if $x \in X \setminus \Sigma$, with equality on the left if and only if x is in the base locus of $|V - F - \ell|$;*

1b) $3 \leq \mathrm{rk}(j_{2,x}) \leq 4$ *if* $x \in \Sigma \setminus S$, *with equality on the left if and only if* x *is in the base locus of* $|V - e - \ell|$, *where* e *is the irreducible component of* F *containing* x*;*

1c) *it cannot be that* $x \in S$.

2) *Suppose that there exists a smooth conic* $\gamma \subset X$, *passing through* x *and transverse to the fibers. Then:*

2a) $5 \leq \mathrm{rk}(j_{2,x}) \leq 6$ *if* $x \in X \setminus \Sigma$, *with equality on the left if and only if* x *is in the base locus of* $|V - F - \gamma|$*;*

2b) $3 \leq \mathrm{rk}(j_{2,x}) \leq 4$ *or* $4 \leq \mathrm{rk}(j_{2,x}) \leq 5$ *if* $x \in \Sigma \setminus S$, *according to whether* $\gamma \cdot e = 1$ *or* 2, e *being the irreducible component of* F *containing* x*; moreover, equality holds on the left if and only if* x *is in the base locus of* $|V - e - \gamma|$*;*

2c) $\mathrm{rk}(j_{2,x}) = 4$ *if* $x \in S$.

Proof. 1a) Let $x \in X \setminus \Sigma$. We know that $\ell \cdot F = 1$, by Lemma 6(1) (since $N > 3$). Hence ℓ is a section of X. According to Proposition 3 we know that $|V - 3x| = F + \ell + |V - F - \ell - x|$, since both F and ℓ are smooth. We have $\mathrm{codim}_{|V|}(|V - F - \ell|) = 4$. In conclusion, $4 \leq \mathrm{rk}(j_{2,x}) \leq 5$. Moreover equality holds on the left if and only if x is in the base locus of $|V - F - \ell|$.

2a) We know that $1 \leq \gamma \cdot F \leq 2$ by Lemma 6(2). According to Proposition 3 we have $|V - 3x| = F + \gamma + |V - F - \gamma - x|$ since both F and γ are smooth. Then $\mathrm{codim}_{|V|}(|V - F - \gamma|) = 5$. In conclusion, $\mathrm{rk}(j_{2,x}) = 6$ unless x is in the base locus of $|V - F - \gamma|$.

1b) Let $x \in \Sigma \setminus S$ and let e be the component of F containing x. Clearly $\ell \cdot e = 1$, since $\ell \ni x$, hence $1 \leq \ell \cdot F \leq 2$. However, equality on the right would imply $N = 3$ by Lemma 6(1), which is not the case. Therefore $\ell \cdot F = 1$ and then ℓ is a section of X. We know that $|V - 3x| = e + \ell + |V - e - \ell - x|$, by Proposition 3, since both e and ℓ are smooth. We have $\mathrm{codim}_{|V|}(|V - e - \ell|) = 3$. In conclusion, $3 \leq \mathrm{rk}(j_{2,x}) \leq 4$, and equality holds on the left if and only if x is in the base locus of $|V - e - \ell|$.

2b) Clearly, $1 \leq \gamma \cdot e \leq 2$, since e is a line and γ is a conic, both containing x. So we have two possibilities. First assume that $\gamma \cdot e = 1$. We know that $|V - 3x| = e + \gamma + |V - e - \gamma - x|$, by Proposition 3, since both e and γ are smooth, We have $\mathrm{codim}_{|V|}(|V - e - \gamma|) = 4$. In conclusion, $4 \leq \mathrm{rk}(j_{2,x}) \leq 5$, and equality holds on the left if and only if x is in the base locus of $|V - e - \gamma|$. Next, assume that $\gamma \cdot e = 2$. In this case, $|V - 3x| = e + \gamma + |V - e - \gamma - x|$, as before. However, $\mathrm{codim}_{|V|}(|V - e - \gamma|) = 3$. In conclusion, $3 \leq \mathrm{rk}(j_{2,x}) \leq 4$, and equality holds on the left if and only if x is in the base locus of $|V - e - \gamma|$.

1c) If $x \in S$ we can write $F = e_1 + e_2$, the two lines e_1, e_2 meeting at x. Then $\ell \cdot (e_1 + e_2) = 2$, which implies $N = 3$ by Lemma 6(1). But this is impossible.

2c) As before, $F = e_1 + e_2$, the two lines e_1, e_2 meeting at x. By Proposition 3, e_1, e_2 and the smooth conic γ are fixed components of $|V - 3x|$, and $|V - 3x| = F + \gamma + |V - F - \gamma|$. Thus $\mathrm{rk}(j_{2,x}) = \mathrm{codim}_{|V|}(|V - F - \gamma|)$.

To impose on the elements of $|V|$ to contain F requires three conditions. Clearly, $\gamma \cdot e_i \geq 1$ for $i = 1, 2$; on the other hand, $\gamma \cdot F \leq 2$ by Lemma 6(2), hence $\gamma \cdot e_i = 1$ for $i = 1, 2$. Thus x is a ramification point of the double cover $\pi|_\gamma : \gamma \to \mathbb{P}^1$, hence, to impose on the elements of $|V - F|$ to contain γ simply requires a single condition. Therefore $\mathrm{rk}(j_{2,x}) = 4$. □

Remarks. i) The argument used in proving 2a) shows, in particular, that a point at which two smooth conics on X meet is not an inflectional point for $k = 2$, in general. For an illustration of this case see Section 5, Example 4.

ii) Consider 1b): for an example in which equality on the right holds at some points, see Example 4 in Section 5. In fact we have $\mathrm{rk}(j_{2,x}) = 4$ at four points: namely, the points of F_j $(j = 1, 2)$ lying on e_0 or on $\widetilde{\ell_{1,2}}$.

iii) A point x where equality $\mathrm{rk}(j_{2,x}) = 3$ holds in 1b) and 2b) represents a very rare circumstance. Actually, at such a point x, $\mathrm{Osc}^2_x(X)$ is just the projective tangent plane to X. Note that the same four points mentioned in ii), when regarded as points on the projected surface Y (see Example 5 in Section 5), satisfy equality on the left in 1b). Unfortunately we have no examples where $\mathrm{rk}(j_{2,x}) = 3$ for 2b).

iv) In Theorem 7 we assumed that N is large enough. Note however that 1c) can occur allowing $N = 3$. For instance, with the same notation as in Section 2, let X be the cubic surface obtained by blowing-up $\mathbb{P}^2$ at points $p_0, \dots, p_5$, in general position but such that the line $\ell_1 = \langle p_0, p_1 \rangle$ is tangent to the conic c passing through $p_1, \dots, p_5$ at p_1, and let $x \in X$ be the point corresponding to the direction of ℓ_1 at p_1. Then x is an Eckardt point of X and $|\mathcal{L} - 3x|$ consists of the following three coplanar lines meeting at x: $e_1, \widetilde{\ell_1}$ and $\widetilde{c}$, the proper transform of the conic c.

v) In the previous analysis all fixed components arising in $|V - 3x|$ contain x. We want to stress that in $|V - 4x|$ there can be fixed components not containing x, as Proposition 4 (ii) shows. See also Example 10 in Section 5.

To give a precise idea of the range of applicability of Theorem 7, here we characterize conic fibrations containing a line (a smooth conic, respectively) transverse to the fibers. Let $\mathbb{F}_e$, e, s and f be as in Section 2. We denote by $[D]$ the line bundle corresponding to a divisor D on $\mathbb{F}_e$. We start with lines.

Proposition 8. *Let $X \subset \mathbb{P}^N$ be a conic fibration, let $\mathcal{L}$ be the hyperplane bundle, and let μ be the number of reducible fibers. Suppose that there exists a line $\ell \subset X$, not contained in a fiber. Then there exist a birational morphism $\eta : X \to \mathbb{F}_e$ contracting a component e_i of each reducible fiber F_i to a point p_i $(i = 1, \dots \mu)$ and an ample line bundle $\mathcal{L}_0$ on $\mathbb{F}_e$ such that $\mathcal{L} = \eta^*\mathcal{L}_0 - \sum_{i=1}^{\mu} e_i$. Moreover, the image $\ell_0 := \eta(\ell)$ is a smooth rational curve and one of the following holds:*

1. *$e \leq 1$, μ is odd and $\geq 3 + 2e$, ℓ_0 is linearly equivalent to $2s + (e+1)f$, $p_i \in \ell_0$ for every $i = 1, \dots, \mu$, and $\mathcal{L}_0 = [2s + (e + \frac{\mu-1}{2})f]$;*
2. *no conditions on e, $\ell_0 = s$, $p_i \notin \ell_0$ for every $i = 1, \dots, \mu$, and $\mathcal{L}_0 = [2s + (2e + 1)f]$;*

Proof. As observed, X is a rational conic fibration; Let $\pi : X \to \mathbb{P}^1$ be the fibration map. Every singular fiber F_i $(i = 1, \dots, \mu)$ of X is reducible, so we can write $F_i = e_i + e_i'$, where both e_i and e_i' are (-1)-curves, $e_i \cdot e_i' = 1$ and $\mathcal{L} \cdot e_i = \mathcal{L} \cdot e_i' = 1$. Since e_i and e_i' are lines, like ℓ, we have $\ell \cdot e_i \leq 1$ and $\ell \cdot e_i' \leq 1$. Let F be a general fiber of X; since ℓ is not in a fiber, we have $0 < \ell \cdot F = \ell \cdot F_i = \ell \cdot e_i + \ell \cdot e_i'$ for every i. Hence there are two possibilities: either

a) $\ell \cdot e_i = \ell \cdot e_i' = 1$ for every i (in which case $\ell \cdot F = 2$), or
b) up to exchanging e_i with e_i' for some index i, we can suppose that $\ell \cdot e_i = 0$ and $\ell \cdot e_i' = 1$ for every i (in which case $\ell \cdot F = 1$).

Let $\eta : X \to X_0$ be the birational morphism contracting the μ exceptional curves e_i. Then the smooth surface X_0 has a $\mathbb{P}^1$-bundle structure induced by π, hence $X_0 = \mathbb{F}_e$ for some e, since X is rational. Let $p_i = \eta(e_i)$; then $F = \eta^*(f)$ if no p_i belongs to the fiber f of $\mathbb{F}_e$. Note that the curve $\ell_0 := \eta(\ell)$ is a smooth rational curve, since ℓ is a line and $\ell \cdot e_i \leq 1$ for every i. Moreover, ℓ_0 contains either all p_i's or no one of them according to cases a) and b). So, we have $\eta^*\ell_0 = \ell + \sum_{i=1}^{\mu} e_i$ in case a) while $\eta^*\ell_0 = \ell$ in case b). Now consider on $\mathbb{F}_e$ the line bundle $\mathcal{L}_0 := (\eta_*\mathcal{L})^{\vee\vee}$ (double dual). Then $\mathcal{L} = \eta^*\mathcal{L}_0 - \sum_{i=1}^{\mu} e_i$ and the Nakai–Moishezon criterion shows that $\mathcal{L}_0$ is ample. Furthermore, the condition

$$2 = \mathcal{L} \cdot F = (\eta^*\mathcal{L}_0 - \sum_{i=1}^{\mu} e_i) \cdot \eta^* f = \mathcal{L}_0 \cdot f$$

says that $(\mathbb{F}_e, \mathcal{L}_0)$ is a conic bundle. Therefore, $\mathcal{L}_0 = [2s + \beta f]$, where β is an integer satisfying the inequality $\beta > 2e$, due to the ampleness. On the other hand, from the condition $1 = \mathcal{L} \cdot \ell = (\eta^*\mathcal{L}_0 - \sum_{i=1}^{\mu} e_i) \cdot \ell$ we obtain that

$$\mathcal{L}_0 \cdot \ell_0 = \begin{cases} \mu + 1 & \text{in case a)} \\ 1 & \text{in case b).} \end{cases}$$

Moreover, since $F \cdot \ell$ is 2 or 1 according to cases a) and b), we argue that ℓ_0 is a bisection of $\mathbb{F}_e$ in case a) and a section in case b). So, in case a), $\ell_0 \sim 2s + yf$ (linearly equivalent), for some integer $y \geq 2e$, with strict inequality if $e = 0$ (e. g., see [7, Corollary 2.18 (b), p. 380]). Recalling that $K_{\mathbb{F}_e} = -2s - (2 + e)f$, the genus formula gives

$$-2 = 2g(\ell_0) - 2 = \ell_0 \cdot (\ell_0 + K_{\mathbb{F}_e}) = (2s + yf) \cdot (y - 2 - e)f = 2(y - 2 - e).$$

Therefore, $2e \leq y = e + 1$, which implies $e \leq 1$ and $\ell_0 \sim 2s + (e + 1)f$. On the other hand,

$$\mu + 1 = \mathcal{L}_0 \cdot \ell_0 = (2s + \beta f) \cdot (2s + (e + 1)f) = 2\beta - 2e + 2.$$

In conclusion, $\mu = 1 + 2(\beta - e) \geq 1 + 2(e + 1) = 3 + 2e$ and $\beta = e + \frac{1}{2}(\mu - 1)$, where $e = 0$ or 1. This gives (1) in the statement. Now consider case b). Here ℓ_0 is a section, hence either b1) $\ell_0 = s$ or b2) $\ell_0 \sim s + yf$, for some integer $y \geq e$, with strict inequality if $e = 0$. Recall that $\mathcal{L}_0 \cdot \ell_0 = 1$. In subcase b1) this gives $\beta = 2e+1$, which leads to (2) in the statement. In subcase b2) we get

$$1 = \mathcal{L}_0 \cdot \ell_0 = (2s+\beta f)\cdot(s+yf) = 2(y-e)+\beta \geq \beta > 0.$$

Therefore, $\beta = 1$ and $y = e$, hence $e > 0$. On the other hand, $1 = \beta > 2e$, due to the ampleness condition, which implies $e = 0$, a contradiction □

In the same vein we can prove an analogue of Proposition 8 for conic fibrations containing a smooth conic not in the fibers.

Proposition 9. *Let $X \subset \mathbb{P}^N$ be a conic fibration, let $\mathcal{L}$ be the hyperplane bundle, and let μ be the number of reducible fibers. Suppose that there exists a smooth conic $\gamma \subset X$, not contained in a fiber. Then there exist a birational morphism $\eta : X \to \mathbb{F}_e$ contracting a component e_i of each reducible fiber F_i to a point p_i ($i = 1, \dots \mu$) and an ample line bundle $\mathcal{L}_0$ on $\mathbb{F}_e$ such that $\mathcal{L} = \eta^*\mathcal{L}_0 - \sum_{i=1}^{\mu} e_i$. Moreover, the image $\gamma_0 := \eta(\gamma)$ is a smooth rational curve containing p_i only for $i = 1, \dots, \nu$ up to renaming, where $0 \leq \nu \leq \mu$, and one of the following holds:*

1. *$e \leq 1$, ν is even and $\geq 2+2e$, γ_0 is linearly equivalent to $2s+(e+1)f$, and $\mathcal{L}_0 = [2s+(e+\frac{\nu}{2})f]$;*
2. *no conditions on e, $\gamma_0 = s$, $\nu = 0$, and $\mathcal{L}_0 = [2s+(2e+2)f]$.*

Proof. As we already said, X is a rational conic fibration. Moreover, $1 \leq \gamma\cdot F \leq 2$ by Lemma 6(2). Now we can proceed as in the proof of Proposition 8, and we have to consider two possibilities according to whether $\gamma \cdot F = 2$ or 1: either

a) up to reordering the reducible fibers and up to exchanging the components of some of them, $\gamma \cdot e_i = \gamma \cdot e_i' = 1$ for $i = 1, \dots, \nu$ while $\gamma \cdot e_j = 0$ and $\gamma \cdot e_j' = 2$ for $j = \nu+1, \dots, \mu$; or

b) up to exchanging the components of some reducible fibers, $\gamma \cdot e_i = 0$ and $\gamma \cdot e_i' = 1$ for every $i = 1, \dots, \mu$ (in this case set $\nu = 0$).

As in the proof of Proposition 8, consider the contraction $\eta : X \to X_0$ of $e_1, \dots, e_\mu$. Then, $X_0 = \mathbb{F}_e$ for some e, there exists an ample line bundle $\mathcal{L}_0$ on $\mathbb{F}_e$ such that $\mathcal{L} = \eta^*\mathcal{L}_0 - \sum_{i=1}^{\mu} e_i$, and $\gamma_0 := \eta(\gamma)$ is a smooth rational curve, since γ is a smooth conic and $\gamma\cdot e_i \leq 1$ for every i. Moreover, γ_0 contains only $p_1, \dots, p_\nu$ in case a) and no p_i in case b). Therefore, $\eta^*(\gamma_0) = \gamma + \sum_{i=1}^{\nu} e_i$ in case a) while $\eta^*(\gamma_0) = \gamma$ in case b). We have $\gamma \cdot \mathcal{L} = 2$, since γ is a conic, and this gives

$$\mathcal{L}_0 \cdot \gamma_0 = \begin{cases} \nu+2 & \text{in case a)} \\ 2 & \text{in case b).} \end{cases}$$

Moreover, as before, condition $2 = \mathcal{L}\cdot F = \mathcal{L}_0 \cdot f$ allow us to write $\mathcal{L}_0 = [2s+\beta f]$, with $\beta > 2e$ due to the ampleness. Finally, note that $\gamma \cdot F = \gamma_0 \cdot f$. Now consider case a). Here γ_0 is a bisection, hence it is linearly equivalent to $2s+yf$, for some integer $y \geq 2e$ (strict inequality if $e = 0$). The genus formula shows that $y = e+1$, which implies $e \leq 1$. On the other hand,

$$\nu + 2 = \mathcal{L}_0 \cdot \gamma_0 = (2s+\beta f)\cdot\big(2s+(e+1)f\big) = 2(\beta - e)+2.$$

This shows that ν has to be even, $\beta = e + \frac{\nu}{2}$, and thus the inequality $\beta > 2e$ implies $\nu \geq 2e + 2$, due to the parity. This gives (1) in the statement. Next consider case b). Here γ_0 is a section, hence either $\gamma_0 = s$ or $\gamma_0 \sim s + yf$, for some integer $y \geq e$ (strict inequality if $e = 0$). Using condition $\mathcal{L}_0 \cdot \gamma_0 = 2$ we thus see that the former possibility leads to (2) in the statement, while the latter gives a contradiction. $\square$

Castelnuovo surfaces introduced in Section 2 fit into case (2) of Proposition 9. Actually, factoring $\theta : X \to \mathbb{P}^2$ as $\theta = \eta \circ \sigma$ where $\sigma : \mathbb{F}_1 \to \mathbb{P}^2$ is the blowing-up at p_0 and $\eta : X \to \mathbb{F}_1$ is the blowing-up at the μ points corresponding via σ to $p_1, \dots, p_\mu$, we have seen that $\mathcal{L} = \eta^*\mathcal{L}_0 - \sum_{i=1}^{\mu} e_i$, where $\mathcal{L}_0$ is the line bundle on $\mathbb{F}_1$ given by $[2s+4f]$, namely, the line bundle which appears in Proposition 9, case (2), since $e = 1$.

5. EXAMPLES

Here we collect several examples to illustrate the various situations we met in the previous discussion. In general, to evaluate $\mathrm{rk}(j_{k,x})$ we identify a suitable linear system, say $\mathcal{S}$ $(= \mathcal{S}_{k,x})$, such that: 1) $\mathcal{S}$ is projectively equivalent to the residual part of $|V - (k+1)x|$ with respect to the largest fixed part we are able to recognize, and 2) $\dim(\mathcal{S})$ can be easily computed. Then, (2) implies

$$\mathrm{rk}(j_{k,x}) = \dim(|V|) - \dim(\mathcal{S}).$$

Let us just note that $\mathcal{S}$ is not necessarily a linear system on X itself; sometimes it is convenient to look at it as a linear system on another surface related to X.

Example 1. Let X be $\mathbb{F}_1$ embedded in $\mathbb{P}^8$ by $\mathcal{L} = -K_X = [2s+3f]$. Let $x \in X$ be any point and let F be the fiber through x. By Proposition 3 we have $|\mathcal{L} - 3x| = F + |\mathcal{L} - F - 2x|$. Note that $\mathcal{M} := \mathcal{L} - F = [2s+2f]$ is spanned, $\mathcal{M}^2 = 4$, and $h^0(\mathcal{M}) = 6$, by (5). Moreover, the only irreducible curve in X having intersection zero with $\mathcal{M}$ is s. Therefore, $\mathcal{M}$ defines a morphism $\phi : X \to \mathbb{P}^5$, contracting s, whose image is the Veronese surface, as one can easily see; moreover, ϕ is birational. Suppose that $x \notin s$. Then letting $\mathcal{S} = |\mathcal{M} - 2x|$ we get $\mathrm{rk}(j_{2,x}) = 8 - 2 = 6$. On the other hand, let $x \in s$; then $|\mathcal{L} - F - x| = s + |\mathcal{L} - F - s|$, because $s \cdot \mathcal{M} = 0$, and so $|\mathcal{L} - F - 2x| = s + |\mathcal{L} - F - s - x|$. But $\mathcal{L} - F - s = \mathcal{M} - s = [s+2f]$ is very ample and $\dim(|s+2f|) = 4$. Hence letting $\mathcal{S} = |s + 2f - x|$, we get $\mathrm{rk}(j_{2,x}) = 8-3 = 5$ if $x \in s$. In conclusion, $\Phi_2(X) = s$. This agrees with [11, Theorem 2.1], since $(X, \mathcal{L})$ is the non-minimal del Pezzo surface of degree 8. Now let us determine $\Phi_3(X)$. Fix $x \in X$ and let F be the fiber through x, as before. Then $|\mathcal{L} - 4x| = 2F + |\mathcal{L} - 2F - 2x| = 2F + |2s + f - 2x|$ by Proposition 4. Note that $|2s+f| = s + |s+f|$. Thus

$$|\mathcal{L} - 4x| = 2F + s + |\mathcal{L} - 2F - s - rx| = 2F + s + |s + f - rx|,$$

where $r = 1$ or 2 according to whether $x \in s$ or $x \notin s$. The linear system $|s+f|$ is a base-point-free net, $(s+f) \cdot s = 0$ and $(s+f)^2 = 1$. In other words, $|s+f|$ defines a morphism $\psi : X \to \mathbb{P}^2$, which is birational and contracts s to a point. So, $\dim(|s+f-rx|) = \dim(|\mathcal{O}_{\mathbb{P}^2}(1) - rx'|)$, where $x' = \psi(x)$. Call $\mathcal{S}$ this last linear system: $\mathcal{S}$ is either empty or a pencil according to whether $r = 2$ or 1, respectively. So, if $x \notin s$ then $r = 2$, hence $\mathrm{rk}(j_{3,x}) = 8-(-1) = 9$, while if $x \in s$ then $r = 1$, hence $\mathrm{rk}(j_{3,x}) = 8-1 = 7$. In conclusion, $s_3 = 9$, and $\Phi_3(X) = s$.

Example 2. Let $X \subset \mathbb{P}^8$ be the del Pezzo surface of degree 8 isomorphic to $\mathbb{F}_0$, namely, $(X, \mathcal{L}) = (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(2,2))$. Here $|V| = |\mathcal{L}|$. Let $x \in X$ and let F and E be the fibers through x of the two projections of X. Both E and F are fibers of two distinct conic bundle structures of X. Thus, according to Propositions 3 and 4 we have $|\mathcal{L}-3x| = E+F+|\mathcal{O}(1,1)-x|$, while $|\mathcal{L}-4x| = \{2E+2F\}$, respectively. Hence $\dim(|\mathcal{L}-3x|) = \dim(|\mathcal{O}(1,1)|)-1 = 2$, while $\dim(|\mathcal{L}-4x|) = 0$. Therefore, for every point $x \in X$, $\mathrm{rk}(j_{2,x}) = 8-2 = 6$ (in accordance with the fact that $(X, \mathcal{L})$ is 2-regular [10, Remark 2.2]), and $\mathrm{rk}(j_{3,x}) = 8$. In other words, $s_3 = 8$, and our $X \subset \mathbb{P}^8$ is perfectly hypo-osculating for $k = 3$.

Example 3. Let X be the del Pezzo surface of degree 7, linearly normally embedded in $\mathbb{P}^7$. Here $\mathcal{L} = -K_X = \sigma^*\mathcal{O}_{\mathbb{P}^2}(3) - e_0 - e_1$, e_0 is a section of π and there is a single singular fiber, namely $F_1 = e_1 + \widetilde{\ell}_1$. By [11, Theorem 2.1], $\mathrm{rk}(j_{2,x}) = 6$ if $x \notin F_1 \cup e_0$, while $\mathrm{rk}(j_{2,x}) = 5$ for $x \in F_1 \cup e_0$, except at the two points $x_i := e_i \cap \widetilde{\ell}_1$, $i = 0,1$, where $\mathrm{rk}(j_{2,x}) = 4$. In particular, $\Phi_2(X) = F_1 \cup e_0$. The same results can be obtained by looking at $|\mathcal{L}-3x|$ and taking into account Proposition 3. Now look at $|\mathcal{L}-4x|$. Obviously, $|\mathcal{L}-4x| = \emptyset$ if $x \notin F_1 \cup e_0$, since there are no plane cubics with a point of multiplicity 4. In fact X is hypo-osculating for $k = 3$. Let $x \in e_0 \setminus F_1$ and let F be the fiber through x. By Proposition 4 and Lemma 2 we see that $|\mathcal{L}-4x| = 2F + e_0 + |\mathcal{L} - 2F - e_0 - x|$. On the other hand, $\mathcal{L} - 2F - e_0 = \sigma^*\mathcal{O}_{\mathbb{P}^2}(1) - e_1$, hence letting $\mathcal{S} = |\mathcal{O}_{\mathbb{P}^2}(1) - p_1 - x'|$, where $x' = \sigma(x)$, we get $\mathrm{rk}(j_{3,x}) = 7$ for $x \in e_0 \setminus F_1$. Next suppose that $x \in \widetilde{\ell}_1 \setminus \{e_0 \cup e_1\}$. In this case, by Proposition 4 and Lemma 2 again, we have $|\mathcal{L}-4x| = F_1 + \widetilde{\ell}_1 + e_0 + |\mathcal{L} - 2\widetilde{\ell}_1 - e_0 - e_1 - 2x|$, since x is only a double point for the fixed part. Moreover, $\mathcal{L} - 2\widetilde{\ell}_1 - e_0 - e_1 = \sigma^*\mathcal{O}_{\mathbb{P}^2}(1)$, hence $\dim(|\mathcal{L}-4x|) = \dim(|\mathcal{O}_{\mathbb{P}^2}(1) - 2x'|)$. So, $|\mathcal{L}-4x| = \emptyset$ and then $\mathrm{rk}(j_{3,x}) = 8$ at these points. Finally, consider the point $x = e_0 \cap \widetilde{\ell}_1$. Then $|\mathcal{L}-4x| = F_1 + \widetilde{\ell}_1 + e_0 + |\mathcal{L} - 2\widetilde{\ell}_1 - e_0 - e_1 - x|$. Moreover, $\mathcal{L} - 2\widetilde{\ell}_1 - e_0 - e_1 = \sigma^*\mathcal{O}_{\mathbb{P}^2}(1)$, hence letting $\mathcal{S} = |\mathcal{O}_{\mathbb{P}^2}(1) - x'|$, we get $\mathrm{rk}(j_{3,x}) = 6$. Up to replacing π with the conic fibration structure given by the projection from p_1 we get the same conclusions for points lying on e_1. Thus, $\Phi_3(X) = e_0 \cup e_1$. The fact that $\Phi_2(X) \not\subset \Phi_3(X)$ is not surprising, since our surface $X \subset \mathbb{P}^7$ is hypo-osculating for $k = 3$.

Example 4. Let X be the del Pezzo surface of degree 6 linearly normally embedded in $\mathbb{P}^6$. Here $\mathcal{L} = -K_X = \sigma^*\mathcal{O}_{\mathbb{P}^2}(3) - e_0 - e_1 - e_2$ and the singular fibers of π are $F_1 = e_1 + \widetilde{\ell}_1$ and $F_2 = e_2 + \widetilde{\ell}_2$. So $\Sigma = F_1 \cup F_2$ and, clearly, S

consists of the two points $x_j := e_j \cap \widetilde{\ell_j}$, $j = 1, 2$. Let x be a point of $X \setminus \Sigma$ and set $x' := \sigma(x)$. Then the smooth fiber F containing x is the proper transform of the line $\langle p_0, x' \rangle$ via σ. Recall that the only lines of $(X, \mathcal{L})$ are the six edges of the hexagon $\mathcal{H}$ on X arising from the trilateral determined by p_0, p_1, p_2, namely $e_0, \widetilde{\ell_1}, e_1, \widetilde{\ell_{1,2}}, e_2, \widetilde{\ell_2}$. On the other hand, it is easy to see that any smooth conic γ of $(X, \mathcal{L})$ which is not a fiber of π necessarily must be a fiber of one of the two other conic fibrations $\pi_j : X \to \mathbb{P}^1$ induced by the projection of $\mathbb{P}^2 \setminus \{p_j\}$ onto a general line, for $j = 1, 2$ respectively. So, if $x \in X$ is general, then there are no lines of $(X, \mathcal{L})$ passing through x, but there are exactly three smooth conics: the proper transforms of the lines $\langle p_i, x' \rangle$, $i = 0, 1, 2$. In particular, if $x \in X \setminus \Sigma$ and $x \in F$, then taking as γ the proper transform of $\langle p_1, x' \rangle$ via σ we get $|V - 3x| = F + \gamma + |V - F - \gamma - x|$. But the linear system $|V - F - \gamma|$ corresponds to the pencil of lines in $\mathbb{P}^2$ passing through p_2. Therefore $\dim(|V - 3x|) = 0$, i. e., $\mathrm{rk}(j_{2,x}) = 6$. Clearly, this holds for every $x \in X$ whose corresponding point $x' = \sigma(x)$ is not on the trilateral of vertices p_0, p_1, p_2. This is in accordance with [11, Theorem 2.1], where we showed that $\Phi_2(X)$ is the hexagon $\mathcal{H}$. Now let $x \in \Sigma$; for instance, $x \in F_1$. Then [11, Theorem 2.1] tells us that $\mathrm{rk}(j_{2,x}) = 5$ if x is general, whilst $\mathrm{rk}(j_{2,x}) = 4$ at the three points x_1, $e_0 \cap \widetilde{\ell_1}$, and $e_1 \cap \widetilde{\ell_{1,2}}$. This is obvious for x_1, since it lies on S. The reason why $\mathrm{rk}(j_{2,x}) = 4$ at $x = e_0 \cap \widetilde{\ell_1}$. is that e_0 is a line transverse to the fibers, hence it is a further fixed component of $|\mathcal{L} - 3x|$. Thus $|\mathcal{L} - 3x| = \widetilde{\ell_1} + e_0 + |\mathcal{L} - \widetilde{\ell_1} - e_0 - x|$. We thus see that $\mathrm{rk}(j_{2,x}) = 6 - \dim(\mathcal{S})$, $\mathcal{S}$ being the net of conics tangent to ℓ_1 at p_0 and passing through p_2. Of course, what we said for F_1 can be repeated verbatim for F_2. Clearly X is hypo-osculating for $k = 3$.

Example 5. Let X and $\mathcal{L}$ be as in Example 4. A discovery of Togliatti [16] (see also [11, Section 4]) is that there exists a codimension 1 vector subspace W of $H^0(X, \mathcal{L})$ such that the projection $\mathbb{P}^6 - - \to \mathbb{P}^5 = \mathbb{P}(W)$ maps X isomorphically to a smooth surface $Y \subset \mathbb{P}^5$ which is hypo-osculating for $k = 2$. Clearly we can identify the conic fibration induced on Y by $\pi : X \to \mathbb{P}^1$ with that of X and use the same letters as before to denote curves on Y obtained via the isomorphic projection from those on X. Then, according to [11, Proposition 4.3] we have that $\mathrm{rk}(j_{2,y}) = 5$ at all points of Y except at the six vertices of the hexagon $\mathcal{H}$, where $\mathrm{rk}(j_{2,y}) = 3$. In particular, on the singular fiber F_j ($j = 1, 2$) $\mathrm{rk}(j_{2,y}) = 3$ only at three points, namely x_j (i.e., the point in S), and the two points lying on lines of $(X, \mathcal{L})$ not contained in a fiber of π, i.e., e_0 and $\widetilde{\ell_{1,2}}$.

For the del Pezzo surfaces of degree 5 and 4, linearly normally embedded in $\mathbb{P}^5$ and $\mathbb{P}^4$, we refer to examples a and b in Section 6.

Example 6. Let X be the del Pezzo surface with $K_X^2 = 2$. Consider the conic fibration $\pi : X \to \mathbb{P}^1$ and recall that $-K_X = \sigma^* \mathcal{O}_{\mathbb{P}^2}(3) - \sum_{i=0}^{6} e_i$ is ample and spanned but not very ample; here $\mu = 6$. Let F be a general fiber of π, and set $\mathcal{L}_m := -K_X + mF$ for every integer $m \geq 0$. By using

Reider's theorem [14] one can easily prove that $\mathcal{L}_m$ is very ample for $m \geq 1$. Note that $\mathcal{L}_m|_F = (-K_X)|_F = \mathcal{O}_{\mathbb{P}^1}(2)$. So, from the exact sequence

$$0 \to \mathcal{L}_{m-1} \to \mathcal{L}_m \to \mathcal{L}_m|_F = \mathcal{O}_{\mathbb{P}^1}(2) \to 0$$

for $m \geq 1$, computing cohomology inductively, we get $h^1(\mathcal{L}_m) = 0$ and $h^0(\mathcal{L}_m) = h^0(\mathcal{L}_0) + 3m = h^0(-K_X) + 3m = 3(1+m)$. On the other hand, $\mathcal{L}_m^2 = K_X^2 + 2m(-K_X) \cdot F = 2 + 4m$. For instance, let $m = 2$. Then $\mathcal{L} := \mathcal{L}_2$ embeds X in $\mathbb{P}^8$ as a conic fibration of degree $d = 10$. As to the plane model of $(X, \mathcal{L})$, it is immediate to see that $|\mathcal{L}|$ corresponds to a linear system of plane quintics having a triple point and six further base points. Then $(X, \mathcal{L})$ has sectional genus 3. Take $x \in F$ on a smooth fiber. Since $X \subset \mathbb{P}^8$ and $8 \geq 3k - 1$ all the assumptions of the general discussion in Section 3 are satisfied for $k \leq 3$, with $V = H^0(X, \mathcal{L})$, since $\mathcal{L}$ and $\mathcal{L} - F$ are both very ample and $\mathcal{L} - 2F$ is ample and spanned. Recall that here $\mathcal{J}_1$ is the ramification divisor $\mathfrak{R}$ of the double cover $X \to \mathbb{P}^2$ defined by $|-K_X| = |\mathcal{L} - 2F|$, while $\mathcal{J}_2 = \emptyset$. In conclusion, $\mathrm{rk}(j_{2,x}) = 6$ for every $x \notin \Sigma$. So, $\Phi_2(X) = \Sigma$. On the other hand, $\mathrm{rk}(j_{3,x}) = 8$ or 9 according to whether x is or is not on $\mathfrak{R}$. Thus $s_3 = 9$ and $\Phi_3(X) = \Sigma \cup \mathfrak{R}$.

Example 7. Let X be the del Pezzo surface with $K_X^2 = 1$. The morphism $\pi : X \to \mathbb{P}^1$ makes X a conic fibration with respect to $-K_X$, which here is only ample, and spanned just outside a single point x_0 (see Section 2); moreover, $\mu = 7$. Let F be a general fiber of π. We claim that $\mathcal{L}_m := -K_X + mF$ is very ample for every $m \geq 1$. This follows from Reider's theorem for $m \geq 2$, while for $m = 1$, $-K_X + F = \sigma^*\mathcal{O}_{\mathbb{P}^2}(4) - 2e_0 - \sum_{j=1}^7 e_j$, hence the very ampleness follows from [2, Remark 3.4.1, fifth case in Table I], since the points p_i's are in general position. Arguing as in Example 6, we get $h^0(\mathcal{L}_m) = 2 + 3m$. On the other hand, $\mathcal{L}_m^2 = K_X^2 + 2m(-K_X)F = 1 + 4m$. Set $m = 3$; then $\mathcal{L} := \mathcal{L}_3$ embeds X in $\mathbb{P}^{10}$ as a conic fibration of degree $d = 13$. As to the plane model of $(X, \mathcal{L})$, since $\mathcal{L} = -K_X + 3F = \sigma^*\mathcal{O}_{\mathbb{P}^2}(6) - 4e_0 - \sum_{j=1}^7 e_j$, $|\mathcal{L}|$ corresponds to a linear system of plane sextics having a 4-tuple point and seven further base points, and then $(X, \mathcal{L})$ has sectional genus 4. Take $x \in F$ on a smooth fiber. The general discussion in Section 3 applies with $V = H^0(X, \mathcal{L})$, since $\mathcal{L}$, $\mathcal{L} - F$ and $\mathcal{L} - 2F$ are all very ample, and $3k - 1 \leq 10$ for $k = 3$. Thus $\mathrm{rk}(j_{2,x}) = 6$ for every $x \notin \Sigma$ and then $\Phi_2(X) = \Sigma$. On the other hand, by Proposition 4, $|\mathcal{L} - 4x| = 2F + |\mathcal{L} - 2F - 2x| = 2F + |\mathcal{L}_1 - 2x|$. Since $\mathcal{L}_1 = -K_X + F$ is very ample (8) applies and then $\mathrm{rk}(j_{3,x}) = 9$ for every $x \notin \Sigma$. Therefore $\Phi_3(X) = \Sigma$. Let $k = 4$. Since the inequality $3k \leq N + 1 = 11$ is not satisfied, X is hypo-osculating. Clearly, $\Sigma \cup x_0 \subseteq \Phi_4(X)$; moreover, pushing the discussion in Section 3 one step further, for $x \notin (\Sigma \cup x_0)$ we see that $|\mathcal{L} - 5x| = \emptyset$, i.e., $\mathrm{rk}(j_{3,x}) = 11$, except at the singular point x of any singular element of $|-K_X|$, where, obviously, $\dim(|-K_X - 2x|) = 0$. So $\Phi_4(X) \setminus (\Sigma \cup x_0)$ is a finite set, and $\mathrm{rk}(j_{4,x}) = 10$ at each of its points.

Example 8. Let $X = \mathbb{F}_e$ and let $\mathcal{L} = [2s + \beta f]$, for some integer $\beta \geq 2e + 3$. This condition ensures that $\mathcal{L}$, $\mathcal{L} - f$, and $\mathcal{L} - 2f$ are very ample, for any fiber f. Consider the rational conic bundle $X \subset \mathbb{P}^N$ embedded by $|\mathcal{L}|$. Since $h^0(\mathcal{L}) = 3(\beta - e + 1)$ by (5), we have $N \geq 3(e+3)+2 = 11+3e \geq 11$. So, the simplest case is that of $(X, \mathcal{L}) = (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(2,3))$, in which, $X \subset \mathbb{P}^{11}$. In particular, for $k \leq 3$ the general discussion in Section 3 applies. Fix $x \in X$ and let F be the fiber through x. Since $|\mathcal{L} - 3x| = F + |\mathcal{L} - F - 2x|$ by Proposition 3 and $\mathcal{L} - F$ is very ample we get $\mathrm{rk}(j_{2,x}) = 6$ at every point $x \in X$ by (8). Moreover, $|\mathcal{L} - 4x| = 2F + |\mathcal{L} - 2F - 2x|$, by Proposition 4, and then $\mathrm{rk}(j_{3,x}) = 9$ at every point $x \in X$, since $\mathcal{L} - 2F$ is very ample. In conclusion, X is uninflected for both $k = 2$ and 3. Since $N \geq 11$, there is room enough to consider also case $k = 4$. We have $|\mathcal{L} - 5x| = 3F + |\mathcal{L} - 3F - 2x|$. Note that $\mathcal{L} - 3F = [2s + (\beta - 3)f]$ is very ample as well, provided that $\beta > 2e + 3$, hence $\mathrm{rk}(j_{4,x}) = 12$ for every $x \in X$ by (8). Therefore $\Phi_4(X) = \emptyset$. Now let $\beta = 2e + 3$. To exhibit different behaviors, let us consider cases, $e = 0$ and $e = 1$. First, suppose that $e = 0$. Then $X = \mathbb{P}^1 \times \mathbb{P}^1$ embedded in $\mathbb{P}^{11}$ by $|\mathcal{O}(2,3)|$. In this case, $\mathcal{L} - 3F = [2s] = \mathcal{O}(2,0)$, hence $\dim(|\mathcal{L} - 5x|) = \dim(|2s - 2x|) = 0$ for every $x \in X$. This is equivalent to $\mathrm{rk}(j_{4,x}) = 11$, which means that $X \subset \mathbb{P}^{11}$ is perfectly hypo-osculating. Next suppose that $e = 1$; then $(X, \mathcal{L}) = (\mathbb{F}_1, [2s + 5f])$ and $|\mathcal{L}|$ embeds X in $\mathbb{P}^{14}$. We have $|\mathcal{L} - 5x| = 3F + |\mathcal{L} - 3F - 2x|$ even in this case, but now, $\mathcal{L} - 3F = [2s + 2f]$ is the line bundle we denoted by $\mathcal{M}$ in Example 1. Thus $|\mathcal{L} - 5x| = 3F + |\mathcal{M} - 2x|$, and by what we proved there, $\dim(|\mathcal{M} - 2x|) = 3$ or 2 according to whether $x \in s$ or $x \notin s$, respectively. Then $\mathrm{rk}(j_{4,x}) = 14 - \dim(|\mathcal{M} - 2x|)$, since $\dim(|\mathcal{L}|) = 14$. Therefore, $s_4 = 12$, and $\Phi_4(X) = s$, with $\mathrm{rk}(j_{4,x}) = 11$ for every $x \in s$.

Finally let us discuss some irrational conic bundles.

Example 9 (an indecomposable elliptic conic bundle in $\mathbb{P}^8$). Let C be a smooth curve of genus 1 and let $\mathcal{U}$ be the rank-2 vector bundle on C arising as a non split extension

$$0 \to \mathcal{O}_C \to \mathcal{U} \to \mathcal{O}_C(z) \to 0, \tag{9}$$

for some $z \in C$. Recall that $\mathcal{U}$ is ample. Consider the $\mathbb{P}^1$-bundle over C, $X := \mathbb{P}(\mathcal{U})$, with projection π, denote by σ the tautological section and set $F_y = \pi^{-1}(y)$ for any $y \in C$. The line bundle $\mathcal{L} := 2\sigma + F_y + F_{y'}$ is very ample for any $y, y' \in C$, by [3, Theorem 6.3]. Moreover, as $h^0(\mathcal{L}) = h^0(S^2\mathcal{U} \otimes \mathcal{O}_C(y + y')) = 3 + 3\deg(y + y') = 9$, $|\mathcal{L}|$ embeds X in $\mathbb{P}^8$ as a conic bundle of degree $\mathcal{L}^2 = (2\sigma + 2F_y)^2 = 12$. Fix any point $x \in X$ and let F be the fiber containing x. We have $|\mathcal{L} - 3x| = F + |\mathcal{L} - F - 2x|$, by Proposition 3. Take $x' \in C$ such that $x' + \pi(x) \sim y + y'$. Then $\mathcal{L} - F = 2\sigma + F_{x'}$ and this line bundle is very ample, as Reider's theorem shows. Thus $\mathcal{S} = |\mathcal{L} - F - 2x|$ has dimension 2 and then $\mathrm{rk}(j_{2,x}) = 6$. Therefore $\Phi_2(X) = \emptyset$. Now let us determine $\Phi_3(X)$. By using Proposition 4 and arguing as before, we are reduced to consider $\mathcal{S} = |\mathcal{L} - 2F - 2x|$. Note that $\mathcal{L} - 2F$ is numerically equivalent to 2σ, hence it is ample and spanned, as Reider's theorem immediately shows. As $h^0(\mathcal{L} - 2F) = h^0(S^2\mathcal{U} \otimes \mathcal{O}_C(y + y' - 2\pi(x))) = \deg(S^2\mathcal{U}) = 3$, it defines a finite morphism $\varphi : X \to \mathbb{P}^2$ of degree $(\mathcal{L} - 2F)^2 = 4$. Let $\mathcal{J}_i$ denote the i-th jumping set of $\mathcal{L} - 2F$ for $i = 1, 2$. Thus (8) implies that $\mathrm{rk}(j_{3,x}) = 9, 8,$ or 7, according to whether $x \notin \mathcal{J}_1$, $x \in \mathcal{J}_1 \setminus \mathcal{J}_2$, or $x \in \mathcal{J}_2$, respectively. In conclusion, $s_3 = 9$ and $\Phi_3(X) \cap F = \mathcal{J}_1 \cap F$, for every fiber F.

Example 10 (Segre product in $\mathbb{P}^8$). Let $X \subset \mathbb{P}^8$ be the Segre product of a smooth plane cubic $C \subset \mathbb{P}^2$, with hyperplane bundle $H = \mathcal{O}_C(y_1 + y_2 + y_3)$, $(y_i \in C)$, and of a smooth conic, namely the image of $\mathbb{P}^1$ embedded in $\mathbb{P}^2$ via $\mathcal{O}_{\mathbb{P}^1}(2)$. Here $\mathcal{L} = \pi^* H \otimes \rho^* \mathcal{O}_{\mathbb{P}^1}(2)$, where π and ρ are the projections of X onto the factors. Clearly, X is a conic bundle via π, and it is linearly normally embedded in $\mathbb{P}^8$ by $\mathcal{L}$. Let $x \in X$ and let F be the fiber of π through x. As to $\Phi_2(X)$ we can prove the following: $\mathrm{rk}(j_{2,x}) = 6$ or 5, according to whether $x \in X \setminus \mathcal{J}_1$, or $x \in \mathcal{J}_1$ respectively, where $\mathcal{J}_1$ is the first jumping set of $\mathcal{L} - F$. In particular $\Phi_2(X) = \mathcal{J}_1$. As to 3-osculation, we get that $\mathrm{rk}(j_{3,x}) = 7$ or 8, according to whether $\pi(x)$ is a flex of C or not, as expected. In conclusion, X is hypo-osculating for $k = 3$, and $\Phi_3(X)$ consists of the nine fibers $\pi^{-1}(y)$ such that $3y \in |H|$. We omit details for brevity.

6. Conic fibrations with inflectional locus of expected codimension

Let $X \subset \mathbb{P}^N = \mathbb{P}(V)$ be a conic fibration over a smooth curve C. Here we focus on the inflectional locus $\Phi_k(X)$, k being the largest integer such that $3k \le N + 1$, except for $k = 2$, in which case, for sake of completeness, we allow the possibility $N = 4$. Consider the sheaf homomorphism $j_k : V_X \to \mathcal{P}^k_X(\mathcal{L})$ and recall that V_X and $\mathcal{P}^k_X(\mathcal{L})$ are locally free sheaves with ranks $\dim(V) = N + 1$ and $\binom{k+2}{2}$, respectively. We set

$$\ell := \Big(N + 1 - (s_k - 1)\Big)\left(\binom{k+2}{2} - (s_k - 1)\right). \tag{10}$$

Clearly $\Phi_k(X)$ has codimension $\le \ell$: as usual, if equality holds, we say that $\Phi_k(X)$ has the expected codimension. It is natural to ask when ℓ coincides with the true codimension of $\Phi_k(X)$. Clearly, this can happen very rarely, since X has dimension 2. In fact we have the following answer.

Proposition 10. *Let $X \subset \mathbb{P}^N$ be a conic fibration and suppose that $\Phi_k(X)$ has the expected codimension. Then $k \le 3$. Moreover, (k, s_k, N, ℓ) can only be one of the following 4-tuples:*

(a) $(2, 6, 5, 1)$;
(b) $(2, 5, 4, 2)$;
(c) $(2, 6, 6, 2)$;
(d) $(3, 9, 8, 2)$.

Proof. By assumption $\Phi_k(X)$ has codimension ℓ, given by (10). Since $\ell \leq \dim(X) = 2$ we get

$$N \leq s_k - 2 + \frac{2}{\binom{k+2}{2} - (s_k - 1)}.$$

On the other hand, $N \geq s_k - 1$, since $X \subset \mathbb{P}^N$. Hence

$$s_k - 1 \leq N \leq s_k - 2 + \frac{2}{\binom{k+2}{2} - (s_k - 1)}. \tag{11}$$

In particular, this says that $\binom{k+2}{2} - (s_k - 1) \geq 2$. Equivalently,

$$\binom{k+2}{2} - 1 \leq s_k,$$

and recalling (3) we conclude that $k \leq 3$. Now, letting $k = 3$ and going over the above inequalities we see that $s_3 \geq 9$, hence $s_3 = 9$ by (3), and then $N = 8$ by (11). Similarly, letting $k = 2$ we get $s_2 \geq 5$. On the other hand, $s_2 \leq 6$ by (3). Thus (11) shows that $5 \leq N \leq 6$ if $s_2 = 6$ (the maximum), while $N = 4$ if $s_2 = 5$. The corresponding values of ℓ are given by (10). $\square$

The following result generalizes [12, Theorem 17].

Theorem 11. *Let $X \subset \mathbb{P}^N = \mathbb{P}(V)$ be a conic fibration over a smooth curve, and let s_k be the generic rank of j_k. Suppose that $\Phi_k(X)$ has the expected codimension ℓ expressed by* (10) *(or, possibly, is empty). Then the following holds according to the cases listed in Proposition 10: in case* (a) *$\Phi_2(X)$ is a 1-dimensional cycle with class*

$$4K_X + 6L. \tag{12}$$

In cases (b) *and* (c) *$\Phi_2(X)$ is a 0-dimensional cycle of degree ι, and*

$$\iota = 5K_X^2 + 5c_2(X) + 20K_XL + 15L^2 \tag{13}$$

in case (b)*, while*

$$\iota = 11K_X^2 - 5c_2(X) + 28K_XL + 21L^2 \tag{14}$$

in case (c)*. Finally, in case* (d)*, $\Phi_3(X)$ is a 0-dimensional cycle of degree*

$$\iota = 40K_X^2 + 15c_2(X) + 90K_XL + 45L^2. \tag{15}$$

Proof. Since s_k is the generic rank of the vector bundle map $j_k : V_X \to \mathcal{P}^k_X(\mathcal{L})$, the inflectional locus $\Phi_k(X)$ can be regarded as the degeneracy locus $D_{s_k-1}(j_k)$ of j_k. By assumption, it has the expected codimension (or, possibly, is empty), hence its cohomology class is given by the Porteous formula [6, Thm. 14.4, p. 254]:

$$\det\left[c_{\binom{k+2}{2}-(s_k-1)-i+j}\big(\mathcal{P}^k_X(\mathcal{L}) \otimes V_X^\vee\big)\right], \qquad 1 \leq i,j \leq \dim(V) - (s_k - 1).$$

Therefore, since V_X is the trivial bundle we get: $[\Phi_2(X)] = c_1\big(\mathcal{P}^2_X(\mathcal{L})\big)$ in case (a), $c_2\big(\mathcal{P}^2_X(\mathcal{L})\big)$ in case (b), and $c_1\big(\mathcal{P}^2_X(\mathcal{L})\big)^2 - c_2\big(\mathcal{P}^2_X(\mathcal{L})\big)$ in case (c);

$[\Phi_3(X)] = c_2\big(\mathcal{P}^3_X(\mathcal{L})\big)$, in case (d). Then the expressions in the statement follow taking into account Lemma 1. $\square$

Not all cases in Proposition 10 are "a priori" effective.

Example a. (12) is the expression found by Shifrin [15, Proposition 0.3] for the class of $\Phi_2(X)$ for any not hypo-osculating smooth surface $X \subset \mathbb{P}^5$. In particular, if $X \subset \mathbb{P}^5$ is the del Pezzo surface of degree 5, then by [11, Theorem 2.1], $\Phi_2(X)$ consists of the proper transforms via $\sigma : X \to \mathbb{P}^2$ of the lines joining the points $p_0, \dots, p_3$ in pairs plus the four exceptional curves e_i, $(i = 0, \dots, 3)$. An immediate check shows that the sum of all these curves belongs to $|-2K_X| = |4K_X + 6\mathcal{L}|$, in accordance with (12).

Example b. Recall that there are just three types of conic fibrations in $\mathbb{P}^4$ (see [12, Section 8] and references therein). One of them is a conic fibration over a smooth curve of genus 1 [1]; it has 8 singular fibers, degree 8 and sectional genus 5, hence formula (13) gives $\iota = 120$. The remaining types are both rational: they are the del Pezzo surface of degree 4 and the Castelnuovo surface of degree 5. Formula (13) gives $\iota = 40$ and $\iota = 75$ in these two cases, respectively. For the explicit description of $\Phi_2(X)$ in the latter case we refer to Section 7. Here we describe $\Phi_2(X)$ for the quartic del Pezzo surface $X \subset \mathbb{P}^4$, taking this opportunity to amend a wrong assertion in [11, beginning of p. 351] (where five inflectional points are missed). Notation as in Section 2. First consider points $x \in X \setminus (\cup_{i=0}^4 e_i)$ and let $x' = \sigma(x)$. Then $|\mathcal{L} - 3x| = \emptyset$ unless $x' = \ell_{i,j} \cap \ell_{h,k}$, with $i, j \notin \{h, k\}$. In this case, let p_m be the fifth point, i. e., $\{p_0, \dots, p_4\} \setminus (\ell_{i,j} \cup \ell_{h,k}) = \{p_m\}$. Then $|\mathcal{L} - 3x|$ consists of the single divisor $\widetilde{\ell_{i,j}} + \widetilde{\ell_{h,k}} + \widetilde{\ell}$, where $\widetilde{\ell}$ is the proper transform via σ of the line $\langle x', p_m\rangle$, hence $\mathrm{rk}(j_{2,x}) = 4$. Since for any $m \in \{0, \dots, 4\}$ there are three pairs of lines $\ell_{i,j}, \ell_{h,k}$ as above, we get a finite subset $T \subset X \setminus (\cup_{i=0}^4 e_i)$ consisting of 15 inflectional points of this type. Next, consider points $x \in e_i$ for some $i = 0, \dots, 4$, and, for simplicity, suppose that $i = 0$. Let C be the unique conic through $p_0, \dots, p_4$, let ℓ_j be the line $\langle p_0, p_j\rangle$ if $j \geq 1$ and the tangent line to C at p_0 if $j = 0$, and consider their proper transforms $\widetilde{C}$ and $\widetilde{\ell_j}$, respectively. Let $j \geq 1$. We have $\sigma^*C + \sigma^*\ell_j = \widetilde{C} + \sum_{h=0}^4 e_h + \widetilde{\ell_j} + e_0 + e_j$, hence $\widetilde{C} + \widetilde{\ell_j} + e_0 + e_j$ is an element of $|\mathcal{L}| = |\sigma^*\mathcal{O}_{\mathbb{P}^2}(3) - \sum_{h=0}^4 e_h|$, endowed with a triple point at $x_j \in e_0$, the point corresponding to the direction of ℓ_j at p_0 $(j = 1, \dots, 4)$. Now let $j = 0$; then $\sigma^*C + \sigma^*\ell_0 = \widetilde{C} + \sum_{h=0}^4 e_h + \widetilde{\ell_0} + e_0$, hence $\widetilde{C} + \widetilde{\ell_0} + e_0 \in |\mathcal{L}|$ and has a triple point at $x_0 \in e_0$, the point corresponding to the direction of ℓ_0 at p_0. Moreover, we can see that $|\mathcal{L} - 3x| = \emptyset$ for any other point $x \in e_0$. It thus follows that $\Phi_2(X) \cap e_0$ consists of the five points $x_0, \dots, x_4$ described above. This discussion can be repeated verbatim for every e_i, $i = 0, \dots, 4$, and this leads to 25 inflectional points lying on $\cup_{i=0}^4 e_i$. So we get 40 inflectional points in total, the number predicted by (13).

A conic fibration $X \subset \mathbb{P}^6$ as in case (c) must be a conic bundle. Otherwise $\Phi_2(X) \supseteq \Sigma$, which is impossible. For an example see [12, p. 393].

As to case (d), note that conic fibrations as in Examples 1, 6, 7, 9 in Section 5 are in $\mathbb{P}^8$ and satisfy $s_3 = 9$; however, $\Phi_3(X)$ has not the expected codimension. In fact we have

Theorem 12. *Case* (d) *of Proposition 10 does not occur.*

This will be proved in three steps.

Step 1. Suppose that $X \subset \mathbb{P}^8$ is a conic fibration as in case (d) of Proposition 10. Then X must be a conic bundle; otherwise Σ would be contained in $\Phi_3(X)$, contradicting $\ell = 2$. Thus $\mu = 0$. Let d and g be the degree and the sectional genus of X and let q be the genus of the base curve C of X. Then, recalling Theorem 11, (15) can be rewritten as

$$\iota = 180(g-1) - 45d - 380(q-1).$$

Since X is a $\mathbb{P}^1$-bundle over C, expressing the numerical classes of $\mathcal{L}$ and K_X in terms of the tautological line bundle and a fiber, genus formula immediately leads to the relation $d = 4(g-1) - 8(q-1)$. Substituting in the previous formula, we thus get:

$$\iota = 20(1-q).$$

In particular, for the time being, this leads to the following conclusion.

Corollary 13. *Let $X \subset \mathbb{P}^8$ be a conic bundle over a smooth curve C of genus q, with $s_3 = 9$.*

(1) *If $q \geq 2$ then $\Phi_3(X)$ is* 1*-dimensional;*
(2) *If $q = 1$ and $\ell = 2$ then X is uninflected (i.e., $\Phi_3(X) = \emptyset$).*

So, in case (d) of Proposition 10 X is a conic bundle and $q = 0$ or 1.

Step 2. Suppose that X is linearly normally embedded in $\mathbb{P}^8$.

Proposition 14. *Let X be either a rational or an elliptic conic bundle, linearly normally embedded in $\mathbb{P}^N$, with $8 \leq N \leq 10$; then $N = 8$. Moreover, if $s_3 = 9$, then $\Phi_3(X)$ is* 1*-dimensional.*

Proof. We have $h^0(\mathcal{L}) = N + 1$, due to the linear normality. In both cases $q = 0$ or 1, set $X = \mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is a vector bundle of rank 2 on C, normalized as in [7, p. 373]; then $\mathcal{L} = [2s + \pi^*\mathfrak{b}]$ for some divisor $\mathfrak{b}$ on C of degree β. Here s stands for the tautological section of $\mathcal{E}$, hence $s^2 = \deg \mathcal{E}$. Set $e = -\deg \mathcal{E}$, then $s^2 = -e$ and the degree of X is $d = \mathcal{L}^2 = 4(\beta - e)$. In particular, this says that $\beta - e > 0$. Let $\pi : X \to C$ be the bundle projection. By the projection formula we have $h^0(\mathcal{L}) = h^0(\pi_*\mathcal{L}) = h^0\big(S^2\mathcal{E} \otimes \mathcal{O}_C(\mathfrak{b})\big)$.

Let $q = 0$; then $X = \mathbb{F}_e$, and $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)$. Moreover, $\beta > 2e$, due to the (very) ampleness of $\mathcal{L}$. Thus, $N + 1 = h^0(\mathcal{L}) = 3(\beta - e + 1)$, by (5). Since $8 \leq N \leq 10$, this shows that $N = 8$, hence $\beta = e + 2$. So, recalling that $\beta > 2e$, we obtain only two possibilities: $(X, \mathcal{L}) = (\mathbb{F}_0, [2s + 2f])$ or $(\mathbb{F}_1, [2s + 3f])$. Examples 1 and 6 in Section 5 show that X is perfectly hypo-osculating for $k = 3$, i.e. $s_3 < 9$, in the former case, while X has a 1-dimensional $\Phi_3(X)$ in the latter.

Next, let $q = 1$; then $\mathcal{E}$ can be either decomposable or indecomposable. In the former case, $\mathcal{E} = \mathcal{O}_C \oplus \mathcal{O}_C(-\mathfrak{e})$, where $\mathfrak{e}$ is a divisor on C of degree $e \geq 0$ [7, Theorem 2.12 p. 376]. On the other hand, if $\mathcal{E}$ is indecomposable, then either $\deg \mathcal{E} = 0$ and $\mathcal{E}$ is a non-split extension of $\mathcal{O}_C$ by $\mathcal{O}_C$, or $\deg \mathcal{E} = 1$ and $\mathcal{E} = \mathcal{U}$ is as in (9) [7, Theorem 2.15 p. 377]. In every case, $\deg S^2\mathcal{E} = 3 \deg \mathcal{E} = -3e$ and then $\deg\big(S^2\mathcal{E} \otimes \mathcal{O}_C(\mathfrak{b})\big) = 3(\beta - e)$, which is positive. Therefore, $S^2\mathcal{E} \otimes \mathcal{O}_C(\mathfrak{b})$ is non-special, hence $N + 1 = h^0(\mathcal{L}) = h^0\big(S^2\mathcal{E} \otimes \mathcal{O}_C(\mathfrak{b})\big) = 3(\beta - e)$, by the Riemann–Roch theorem. Since $8 \leq N \leq 10$, this shows that $N = 8$, hence $\beta = e + 3$. The minimum value of β for $\mathcal{L}$ being very ample is $2e + 3$ by [3, Sections 5 and 6], hence $e \leq 0$. In conclusion we get only the following possibilities: $(e, \beta) = (0, 3)$, with $\mathcal{E}$ being either decomposable or indecomposable, and $(e, \beta) = (-1, 2)$ ($\mathcal{E}$ indecomposable).

First suppose that $(e, \beta) = (-1, 2)$; then we can write $\mathfrak{b} = y + y'$ for some $y, y' \in C$. So, $\mathcal{L} = [2s + F_y + F_{y'}]$ and then $(X, \mathcal{L})$ is as in Example 9 in Section 5 (up to renaming σ with s). According to what we proved there, $s_3 = 9$ and $\Phi_3(X)$ is 1-dimensional, since for every fiber F of X, $\Phi_3(X) \cap F$ consists of the (four) points cut out on F by the ramification divisor of the 4-tuple cover $\varphi : X \to \mathbb{P}^2$ defined by $|\mathcal{L} - 2F|$.

Suppose now that $(e, \beta) = (0, 3)$. First note that if $\mathcal{E}$ is trivial, then X is as in Example 10 in Section 5. As we have seen, this X is hypo-osculating for $k = 3$, but this contradicts condition $s_3 = 9$. So we can suppose that $\mathcal{E}$ is nontrivial. Regardless whether $\mathcal{E}$ is decomposable or not, write $\mathfrak{b} = y_1 + y_2 + y_3$ for some points $y_i \in C$. We can find points $y_0, y' \in C$ such that $2y_0 + y' \in |\mathfrak{b}|$. So, letting F_0 and F' denote the corresponding fibers, we have that $\mathcal{L} = [2s + 2F_0 + F']$. Let x be a point on F_0. Then, $|\mathcal{L} - 4x| = 2F_0 + |\mathcal{L} - 2F_0 - 2x|$, by Proposition 4. Moreover, $\mathcal{L} - 2F_0 = [2s + F']$. Hence, $(\mathcal{L} - 2F_0) \cdot s = 1$. So, if we choose $x = s \cap F_0$, then s must be a fixed component of $|\mathcal{L} - 2F_0 - 2x|$, i. e., $|\mathcal{L} - 2F_0 - 2x| = s + |\mathcal{L} - 2F_0 - s - x|$. Note that $s + F'$ is a divisor in the linear system $|\mathcal{L} - 2F_0 - s - x|$. Therefore $\dim(|\mathcal{L} - 4x|) \geq 0$, i.e., $\mathrm{rk}(j_{3,x}) \leq 8$ at $x = s \cap F_0$. This means that either our $X \subset \mathbb{P}^8$ is hypo-osculating for $k = 3$, or $s_3 = 9$ and $x \in \Phi_3(X)$, which contradicts Corollary 13,(2). This concludes the proof. □

Step 3. The linearly normal case being settled, we can suppose that $X \subset \mathbb{P}^8$ is not linearly normal. Then our X arises from a rational or elliptic conic bundle $Y \subset \mathbb{P}^9$ via an isomorphic projection $\pi_c : \mathbb{P}^9 - - \to \mathbb{P}^8$ from a point $c \in \mathbb{P}^9 \setminus Y$ (and not even Y is linearly normal, by Proposition 14). Since $s_3 = 9$ for X, we have that $s_3 = 9$ also for Y. Hence there is a dense Zariski open subset $\mathcal{U} \subseteq Y$, where $\mathrm{rk}(j_{3,y}) = 9$, and then the 3-osculating spaces to Y at these points $y \in \mathcal{U}$ are hyperplanes. Set $H_y := \mathrm{Osc}^3_y(Y)$ for $y \in \mathcal{U}$; then $H_y = \mathbb{P}^8$. Via the projection π_c we have $\pi_c(H_y) = \mathrm{Osc}^3_x(X)$ where $x = \pi_c(y)$: clearly, this is the whole $\mathbb{P}^8$ if $c \notin H_y$, while it is a $\mathbb{P}^7$ if $H_y \ni c$, and in this case, $x \in \Phi_3(X)$. Note that it cannot be $c \in H_y$ for all $y \in \mathcal{U}$; otherwise, s_3 could not be 9 for our X. Therefore, imposing to H_y

the condition of containing c defines a divisor $\mathcal{D}$ in $\mathcal{U}$. Thus, for every $y \in \mathcal{D}$ we have that $\mathrm{Osc}^3_x(X)$ is a $\mathbb{P}^7$, hence $x = \pi_c(y) \in \Phi_3(X)$. Identifying Y with X via π_c, this shows that $\Phi_3(X) \cap \mathcal{U}$ is 1-dimensional. This concludes the proof of Theorem 12.

As a consequence of Theorem 12 we have the following result.

Corollary 15. *Let $X \subset \mathbb{P}^8$ be any conic fibration. If X is not hypo-osculating for $k = 3$, then* $\dim(\Phi_3(X)) = 1$.

7. The osculatory behavior of Castelnuovo surfaces

Here we discuss the osculatory behavior of Castelnuovo surfaces for $k = 2$ and 3. Thanks to the plane model of such surfaces, our approach is similar to that used in [11] for del Pezzo surfaces, but taking into account the conic fibration structure, the results in Section 3 enhance the efficiency of our computations both for $k = 2$ and 3.

Let $X \subset \mathbb{P}^N$ be a Castelnuovo surface. Referring to Section 2 we can write $\mathcal{L} = \theta^*\mathcal{O}_{\mathbb{P}^2}(4) - 2e_0 - e_1 - \dots - e_\mu$, since $r = \mu$ (the number of singular fibers of the conic fibration $\pi : X \to \mathbb{P}^1$). Everywhere in this Section we assume that X is linearly normally embedded. This is equivalent to $N = d - 1 = 11 - \mu$, where d is the degree. All results will be stated in terms of μ: the conversion in terms of d is immediate, since $d + \mu = 12$.

7.1. The second inflectional locus of Castelnuovo surfaces.

First of all, since $\mu + 1 \le 8$ and $\mathcal{P}$ satisfies condition (6), X is a del Pezzo surface for $\mu \le 6$, but not necessarily if $\mu = 7$. In this subsection we consider case $k = 2$. So we have to look at $|\mathcal{L} - 3x|$ for every $x \in X$.

Lemma 16. *Let $x \in X$ and let F be the fiber of X containing x. Then*

$$|\mathcal{L} - 3x| = \begin{cases} F + |-K_X - 2x| & \text{if } F \text{ is smooth,} \\ \widetilde{\ell}_i + |-K_X + e_i - 2x| & \text{if } F = \widetilde{\ell}_i + e_i \text{ and } x \in \widetilde{\ell}_i \setminus e_i, \\ e_i + |-K_X + \widetilde{\ell}_i - 2x| & \text{if } F = \widetilde{\ell}_i + e_i \text{ and } x \in e_i \setminus \widetilde{\ell}_i, \\ F + |-K_X - x| & \text{if } F = \widetilde{\ell}_i + e_i \text{ and } x = \widetilde{\ell}_i \cap e_i. \end{cases}$$

Proof. If F is smooth, then $F = \theta^*\mathcal{O}_{\mathbb{P}^2}(1) - e_0$, while, if F is singular, i.e., $F = \widetilde{\ell}_i + e_i$ for some $i = 1, \dots, \mu$, then $\widetilde{\ell}_i = \theta^*\mathcal{O}_{\mathbb{P}^2}(1) - e_0 - e_i$, hence $F = \theta^*\mathcal{O}_{\mathbb{P}^2}(1) - e_0$, again. Therefore $\mathcal{L} - F = -K_X$ for every fiber F. This, combined with Proposition 3, proves the assertion in all cases: for instance, if $x \in \widetilde{\ell}_i + e_i$, we have $\mathcal{L} - \widetilde{\ell}_i = \mathcal{L} - F + e_i = -K_X + e_i$. □

Lemma 16 allows us to evaluate $\mathrm{rank}(j_{2,x})$ at every point $x \in X$. First suppose that $x \in X \setminus \Sigma$.

Proposition 17. *Let $X \subset \mathbb{P}^N$ be a Castelnuovo surface, let Σ, μ be as before, and let $x \in X \setminus \Sigma$.*

(1) *If $\mu \le 5$, then* $\mathrm{rk}(j_{2,x}) = 6$*;*

(2) *if* $\mu = 6$, *then* $\mathrm{rk}(j_{2,x}) = 6$ *or* 5 *according to whether* x *is general or* $x \in \mathcal{J}_1(-K_X)$*;*
(3) *if* $\mu = 7$, *then* $\mathrm{rk}(j_{2,x}) = 5$ *or* 4 *according to whether* x *is general or a singular point of any singular element of the pencil* $|-K_X|$.

Proof. Let F be the smooth fiber of X containing x. By Lemma 16, if $|-K_X - 2x| = \emptyset$ then $\mathrm{rk}(j_{2,x}) = N + 1 = 12 - \mu$ by (2). Since $s_2 \leq 6$, this can occur only for $\mu \geq 6$. On the other hand, if $|-K_X - 2x| \neq \emptyset$ then

$$\mathrm{rk}(j_{2,x}) = N - \big(\dim(|-K_X|) - h\big), \tag{16}$$

by (2), where h is the number of linearly independent linear conditions to be imposed on an element of $|-K_X|$ in order to have a double point at x. Hence $\mathrm{rk}(j_{2,x}) = 11 - \mu - (8 - \mu) + h = 3 + h$. Clearly, $h = 3$ when $-K_X$ is very ample, and this proves (1). If $\mu = 6$, then $h = 3$ except for the points $x \in \mathcal{J}_1(-K_X)$, where $h = 2$. Recall that $\mathcal{J}_2(-K_X) = \emptyset$ for the del Pezzo surface of degree 2; so there are no points where $h = 1$. This gives (2). Finally, let $\mu = 7$. According to what we said in Section 2, $|-K_X|$ is a pencil, hence $|-K_X - 2x| = \emptyset$ for the general point x, including x_0, which implies that $\mathrm{rk}(j_{2,x}) = N + 1 = 5$. On the other hand, $h = 1$ tautologically, at any singular point of every singular element of $|-K_X|$, hence $\mathrm{rk}(j_{2,x}) = 4$ at those special points. Clearly, if $-K_X$ is not ample, these points include those constituting $G \cap e_{i_0}$, where G is the (-2)-curve preventing $-K_X$ from being ample. As we said, they cannot be in $X \setminus \Sigma$ unless $i_0 = 0$. This completes the proof. □

Remarks. i) Case (2) provides an example for point (4) of Theorem 5.

ii) In case $\mu = 6$, $\mathcal{J}_1(-K_X)$ is the ramification divisor of the double plane $\varphi : X \to \mathbb{P}^2$ defined by $|-K_X|$. By the ramification formula, $\mathcal{J}_1(-K_X) \in |K_X + 3\varphi^*\mathcal{O}_{\mathbb{P}^2}(1)| = |-2K_X|$. Therefore, $\mathcal{J}_1(-K_X) \cdot F = -2K_X \cdot F = 4$. This means that $\Phi_2(X)$ is a 4-section of X, outside of Σ.

iii) Clearly Proposition 17 includes the case of $x \in e_0 \setminus \Sigma$. So, if $\mu = 0$ then $\Sigma = \emptyset$ and the discussion is concluded: $X \subset \mathbb{P}^{11}$ is uninflected. On the other hand, if $x \in e_0 \cap \Sigma$ then $x \in \widetilde{\ell}_i$ for some $i = 1, \dots, \mu$, and this case will be settled by the subsequent discussion.

Now consider points $x \in \Sigma$. We have to analyze the following three possibilities: 1) $x \in \widetilde{\ell}_i \setminus e_i$ for some $i = 1, \dots, \mu$ (including the case $x = e_0 \cap \widetilde{\ell}_i$); 2) $x \in e_i \setminus \widetilde{\ell}_i$ for some $i = 1, \dots, \mu$; 3) $x \in S$.

1) Let $x \in \widetilde{\ell}_i \setminus e_i$, and for simplicity set $i = 1$. Lemma 16 implies that $\dim(|\mathcal{L} - 3x|) = \dim(|-K_X + e_1 - 2x|)$. Moreover, $\dim(|-K_X + e_1|) = \dim(|-K_X|) + 1$. Note that this is the dimension of $|-K_{X_1}|$, the surface X_1 being obtained by blowing up $\mathbb{P}^2$ at $\mathcal{P} \setminus \{p_1\}$. Clearly, X_1 is a del Pezzo surface with $K_{X_1}^2 = K_X^2 + 1 = 9 - \mu$, and $\dim(|-K_{X_1}|) = 9 - \mu$. Due to the fact that $\mathcal{P}$ consists of distinct points, we can factor $\theta : X \to \mathbb{P}^2$ as $\theta = \beta \circ \alpha$, where $\alpha : X_1 \to \mathbb{P}^2$ is the blowing up of $\mathbb{P}^2$ at $\mathcal{P} \setminus \{p_1\}$ and $\beta : X \to X_1$ is the blowing up of X_1 at $\alpha^{-1}(p_1)$. Since $K_X = \beta^* K_{X_1} + e_1$, we thus see that

$|-K_X+e_1-2x|) = \beta^*|-K_{X_1}-2x_1|$, where $x_1 = \beta(x)$. Arguing as in the proof of Proposition 17, we get the following. If $|-K_{X_1}-2x_1| = \emptyset$, then $\mathrm{rk}(j_{2,x}) = N+1 = 12-\mu$ (which can happen only for $\mu \geq 6$). On the other hand, if $|-K_{X_1}-2x_1| \neq \emptyset$ then

$$\mathrm{rk}(j_{2,x}) = N - \big(\dim(|-K_{X_1}|) - h\big) \tag{17}$$

where h is the number of linearly independent linear conditions to be imposed on an element of $|-K_{X_1}|$ in order to have a double point at x_1. Hence $\mathrm{rk}(j_{2,x}) = 11-\mu-(9-\mu)+h = 2+h$. Now, let $\mu \leq 6$; then $-K_{X_1}$ is very ample, hence $\mathrm{rk}(j_{2,x}) = 5$ for every $x \in \widetilde{\ell_1}$. On the other hand, if $\mu = 7$, then $\mathrm{rk}(j_{2,x}) = 5$ except at points $x \in \widetilde{\ell_1} \cap \beta^*(\mathcal{J}_1(-K_{X_1}))$, where $\mathrm{rk}(j_{2,x}) = 4$. Recall that $\mathcal{J}_2(-K_{X_1}) = \emptyset$, X_1 being a del Pezzo surface of degree 2, so there are no points where $h = 1$. Clearly what we obtained holds for any $i = 1, \dots, \mu$.

Remark. We stress the following fact concerning case $\mu = 7$. Let $F_1 = \beta(\widetilde{\ell_1})$. Then F_1 is a smooth fiber of X_1, and $\widetilde{\ell_1} = \beta^*F_1 - e_1$. As $K_X = \beta^*K_{X_1} + e_1$ this gives $\widetilde{\ell_1} \cdot \beta^*(\mathcal{J}_1(-K_{X_1})) = \beta^*F_1 \cdot \beta^*(-2K_{X_1}) = -2F_1 \cdot K_{X_1} = 4$. So, there are four points x on $\widetilde{\ell_1} \setminus e_1$ where $\mathrm{rk}(j_{2,x}) = 4$. Clearly the same holds for every $i = 1, \dots, \mu$.

2) Points $x \in e_i \setminus \widetilde{\ell_i}$ for some $i = 1, \dots, \mu$ can be discussed in a symmetric way, considering the surface X' obtained by contracting the (-1)-curve $\widetilde{\ell_1}$ and we get $\mathrm{rk}(j_{2,x}) = N+1 = 5$ at the general point of e_i, while $\mathrm{rk}(j_{2,x}) = 4$ at four points. We omit details for brevity.

3) Finally, let $x \in S$, i. e., $x = \widetilde{\ell_i} \cap e_i$ for some $i = 1 \dots, \mu$; let $i = 1$. Lemma 16 says that $\dim(|\mathcal{L}-3x|) = \dim(|-K_X-x|)$. Note that $|-K_X-x| \neq \emptyset$, since $|-K_X|$ has dimension $8-\mu$. So, (2) gives (16) again, where, now, h is the number of linearly independent linear conditions to be imposed on an element of $|-K_X|$ in order to contain x. By what we said on $-K_X$, $h = 1$ except when $\mu = 7$ and $x = x_0$, the unique base point of the pencil $|-K_X|$. Clearly, what we said holds for any $i = 1, \dots, \mu$. Therefore, $\mathrm{rk}(j_{2,x}) = 11-\mu-(8-\mu)+1 = 4$ at every point of S, unless $\mu = 7$ and $x = x_0$ in which case $\mathrm{rk}(j_{2,x}) = 3$. This last circumstance is equivalent to saying that the unassigned base point of the pencil of plane cubics defined by $\mathcal{P}$ is the infinitely near point to p_i corresponding to the line ℓ_i, for some $i \in 1, \dots, \mu$, which makes the configuration $\mathcal{P}$ rather special.

The above conclusions are summarized as follows.

Theorem 18. *Let $X \subset \mathbb{P}^N$ be a Castelnuovo surface, let Σ, S, μ be as before, and let $x \in X$.*

(1) *If $\mu \leq 5$ then $\Phi_2(X) = \Sigma$, and there* $\mathrm{rk}(j_{2,x}) = 5$ *or* 4 *according to whether x is general or $x \in S$;*
(2) *if $\mu = 6$, then $\Phi_2(X) = \Sigma \cup \mathcal{J}_1(-K_X)$, and there* $\mathrm{rk}(j_{2,x}) = 5$ *or* 4 *according to whether x is general or $x \in S$;*

(3) *if* $\mu = 7$, *then* X *is hypo-osculating, and* $\Phi_2(X)$ *consists of nine points on every singular fiber, including* S, *and of the singular points of the singular elements of* $|-K_X|$. *We have* $\mathrm{rk}(j_{2,x}) = 4$ *at each of them unless* $x = x_0$ *(the base point of* $|-K_X|$*) belongs to* S, *in which case* $\mathrm{rk}(j_{2,x_0}) = 3$.

Remarks. i) Point (3) generalizes the last assertion in [12, Section 8], since there points $p_0, \dots, p_7$ were assumed to satisfy also (7). In particular, for the Castelnuovo surface $X \subset \mathbb{P}^4$ of degree 5 it turns out that $\Phi_2(X)$ consists of $63 + 12 = 75$ points, in accordance with (13).

ii) Condition $\mathrm{rk}(j_{2,x}) = 3$ at some point $x \in X$ is equivalent to $|\mathcal{L} - 2x| = |\mathcal{L} - 3x|$; in other words, $\mathrm{Osc}^2_x(X)$ is just the projective tangent plane to X at x. E. g., if $\mu = 7$ this happens at x_0, provided that $x_0 \in S$.

7.2. **The third inflectional locus of Castelnuovo surfaces.** Now let us come to case $k = 3$. Here we are interested in $|\mathcal{L} - 4x|$ for $x \in X$. According to the expression of $\mathcal{L}$, any element of $|\mathcal{L} - 4x|$ is of the form $\theta^* D - 2e_0 - \sum_{i=1}^{\mu} e_i$, where D is a plane quartic containing $P(= \mathcal{P} \setminus \{p_0\})$, having a double point at p_0, and satisfying the further conditions necessary to produce a singular point of multiplicity 4 at x. For any $x \in X$, set $x' = \theta(x)$, where x' is either a point of $\mathbb{P}^2$ distinct from the p_j's, or an infinitely near point to p_j (if $x \in e_j$ for some $j = 0, 1, \dots, \mu$). We denote by ℓ_x the line $\langle p_0, x' \rangle$ if $x \notin \cup_{i=1}^{\mu} e_i$. Then $\theta^* \ell_x = F + e_0$, F being the (either smooth or singular) fiber of X containing x. By Proposition 4 we know that F is a fixed component with multiplicity 2 of $|\mathcal{L} - 4x|$, if $x \in X \setminus \Sigma$. First suppose that $x \in X \setminus (\Sigma \cup e_0)$. Let D be any plane quartic with a double point at p_0, containing P, and having a singular point of multiplicity 4 at x', so that $\theta^* D - 2e_0 - \sum_{i=1}^{\mu} e_i$ defines an element in $|\mathcal{L} - 4x|$. The above assertion is equivalent to claiming that D is reducible and contains the line ℓ_x as a component of multiplicity two. Furthermore, the residual component, which is a conic, must be in turn reducible into two lines ℓ', ℓ'' meeting at x', in order to have a singular point at x'. Finally, the points $p_1, \dots, p_\mu$, if any, must lie on $\ell' \cup \ell''$. So, letting $\mathcal{S} := |\mathcal{O}_{\mathbb{P}^2}(2) - P - 2x'|$ denote the linear system of conics containing P and having a singular point at x', what we said at the beginning of Section 5 implies

$$\mathrm{rk}(j_{3,x}) = N - \dim(\mathcal{S}). \tag{18}$$

Since no three points of $\mathcal{P}$ are collinear, $\ell' \cup \ell''$ contains four of them at most. So, if $\mu \geq 5$, then $\mathcal{S} = \emptyset$, hence $\mathrm{rk}(j_{3,x}) = N + 1$ on $X \setminus (\Sigma \cup e_0)$ by (18). For small values of μ we have the following: if $\mu \leq 2$ then $\dim(\mathcal{S}) = 2 - \mu$, unless $\mu = 2$ and $x' \in \ell_{1,2}$, in which case, $\dim(\mathcal{S}) = 1$. Thus (18) gives $\mathrm{rk}(j_{3,x}) = 9$ or 8, according to the two cases respectively. If $\mu = 3$, then $\mathcal{S} = |\mathcal{O}_{\mathbb{P}^2}(2) - p_1 - p_2 - p_3 - 2x'|$ is empty in general, while $\dim(\mathcal{S}) = 0$ if $x' \in \ell_{i,j}$ for some $1 \leq 1 < j \leq 3$. By (18) we thus get $\mathrm{rk}(j_{3,x}) = N + 1 = 9$ or $= 8$, accordingly. Finally, let $\mu = 4$: then $\mathcal{S} = \emptyset$ in general, while $\dim(\mathcal{S}) = 0$

if x' is an intersection point of two edges of the quadrilateral with vertices at P. Thus $\mathrm{rk}(j_{3,x}) = N+1 = 8$ or 7, according to cases. This proves

Lemma 19. *Let $X \subset \mathbb{P}^N$ be a Castelnuovo surface and let Σ, P, θ, μ be as before. Consider any point $x \in X \setminus (\Sigma \cup e_0)$ and let $x' = \theta(x)$. Then $\mathrm{rk}(j_{3,x}) = N+1$ except in the following cases:*

1. *$\mu = 4$ and x' is an intersection point of two edges of the quadrilateral with vertices at P: there $\mathrm{rk}(j_{3,x}) = 7$;*
2. *$\mu = 3$ and $x' \in \ell_{i,j}$ for some $1 \leq i < j \leq 3$: there $\mathrm{rk}(j_{3,x}) = 8$;*
3. *$\mu = 2$, in which case $\mathrm{rk}(j_{3,x}) = 9$ or 8, according to whether x is general or $x' \in \ell_{1,2}$;*
4. *$\mu \leq 1$, in which case $\mathrm{rk}(j_{3,x}) = 9$.*

Now consider points $x \in e_0 \setminus \Sigma$. In this case even the smooth conic e_0 is a fixed component of $|\mathcal{L} - 4x|$, by Lemma 2, part (1), hence $|\mathcal{L} - 4x| = 2F + e_0 + |\mathcal{L} - 2F - e_0 - x|$. Recalling the expression of $\mathcal{L}$ and the fact that $\theta^*\ell_x = F + e_0$, we have that $\mathcal{L} - 2F - e_0 = \theta^*\mathcal{O}_{\mathbb{P}^2}(2) - \sum_{j=0}^{\mu} e_j$, hence $\dim(|\mathcal{L} - 4x|) = \dim(\mathcal{S})$, where $\mathcal{S} := |\mathcal{O}_{\mathbb{P}^2}(2) - \mathcal{P} - x'|$ is the linear system of conics through $p_0, p_1, \dots, p_\mu$ and tangent at p_0 to the direction represented by x'. Then $\mathcal{S} = \emptyset$ if $\mu \geq 5$. If $\mu = 4$ there is just one conic through $\mathcal{P}$, so $\dim(\mathcal{S}) = 0$ only for the point x of e_0 representing the tangent direction to that conic at p_0. If $\mu \leq 3$, then $|\mathcal{O}_{\mathbb{P}^2}(2) - \mathcal{P}|$ has dimension $5-(\mu+1) = 4-\mu$, so that imposing the further condition about the tangency to x' at p_0 we get $\dim(\mathcal{S}) = 3 - \mu$. Therefore, by (18) we get the following conclusion.

Lemma 20. *Let $X \subset \mathbb{P}^N$ be a Castelnuovo surface, let Σ, $\mathcal{P}$, μ be as before, and let $x \in e_0 \setminus \Sigma$. Then $\mathrm{rk}(j_{3,x}) = N+1$ except in the following cases:*

1. *$\mu = 4$, in which case $\mathrm{rk}(j_{3,x}) = 8$ or 7 according to whether x is general or it is the point representing the tangent direction at p_0 to the smooth conic containing $\mathcal{P}$;*
2. *$\mu \leq 3$, in which case $\mathrm{rk}(j_{3,x}) = 8$ at every point of $e_0 \setminus \Sigma$.*

Clearly, if $\mu = 0$ the discussion is complete: for $X \subset \mathbb{P}^{11}$ we have $\Phi_3(X) = e_0$, and $\mathrm{rk}(j_{3,x}) = 8$ at every inflectional point.

Now let $x \in \Sigma$. As we did for $k = 2$, we have to analyze three possibilities.

1) Suppose that $x \in \widetilde{\ell_i} \setminus e_i$ for some i, and let $i = 1$ for simplicity. So, $\mu \geq 1$, and $\ell_x = \ell_1 \ni x'$. Moreover, $\dim(|\mathcal{L} - 4x|) = \dim(|\mathcal{L} - (2\widetilde{\ell_1} + e_1) - 2x|)$ by Proposition 4. The expression of $\mathcal{L}$ and the fact that $\theta^*\ell_1 = \widetilde{\ell_1} + e_0 + e_1$ give $\mathcal{L} - (2\widetilde{\ell_1} + e_1) = \theta^*\mathcal{O}_{\mathbb{P}^2}(2) - \sum_{j=2}^{\mu} e_j$. Hence, $\dim(|\mathcal{L} - 4x|) = \dim(\mathcal{S})$, where $\mathcal{S} = |\mathcal{O}_{\mathbb{P}^2}(2) - p_2 - \cdots - p_\mu - 2x'|$. Therefore what we said before Lemma 19 when $x' \notin \cup_{i=1}^{\mu} \ell_i$ can be repeated here by replacing μ with $\mu - 1$. In conclusion, $\mathcal{S} = \emptyset$ if $\mu \geq 6$, hence $\mathrm{rk}(j_{3,x}) = N+1$ on $\ell_1 \setminus e_1$ by (18). As to the smaller values of μ, if $1 \leq \mu \leq 3$ then $\dim(\mathcal{S}) = 3-\mu$, except when $\mu = 3$ and $x' = \ell_1 \cap \ell_{2,3}$, in which case $\dim(\mathcal{S}) = 1$. Thus (18) gives $\mathrm{rk}(j_{3,x}) = 8$ or 7, according to the two cases respectively. If $\mu = 4$ then $\mathcal{S} = \emptyset$ in general,

while $\dim(\mathcal{S}) = 0$ if $x' = \ell_1 \cap \ell_{i,j}$ for some $2 \le i < j \le 4$, and (18) gives $\mathrm{rk}(j_{3,x}) = N + 1 = 8$ or 7, accordingly. Finally, let $\mu = 5$: then $\mathcal{S} = \emptyset$ in general, while $\dim(\mathcal{S}) = 0$ if x' is the intersection point of two edges of the quadrilateral defined by p_2, p_3, p_4, p_5. Hence $\mathrm{rk}(j_{3,x}) = N + 1 = 7$, except for this special point x, where $\mathrm{rk}(j_{3,x}) = 6$.

2) Let $x \in e_i \setminus \widetilde{\ell_i}$ for some $i = 1, \dots, \mu$, and let $i = 1$ for simplicity. Consider $\ell_{1,x} = \langle p_1, x' \rangle$, the line through p_1 whose direction corresponds to x and let C be its proper transform on X. Then $\theta^* \ell_{1,x} = C + e_1$. Note that $\ell_{1,x} \neq \ell_1$, because $x \notin S$.

Lemma 21. *Let D be any plane nodal quartic such that $\theta^* D - 2e_0 - \sum_{i=1}^{\mu} e_i \in |\mathcal{L} - 4x|$. Then $D = \ell_1 + \ell_{1,x} + G$, where $G \in |\mathcal{O}_{\mathbb{P}^2}(2) - \mathcal{P} - x'|$ is a conic through $\mathcal{P}$ and tangent at p_1 to the direction x', corresponding to x.*

Proof. $2e_1 + \widetilde{\ell_1}$ is in the fixed part of $|\mathcal{L} - 4x|$, by Proposition 4. We have $C \cdot \mathcal{L} = (\theta^* \mathcal{O}_{\mathbb{P}^2}(1) - e_1) \cdot (\mathcal{O}_{\mathbb{P}^2}(4) - 2e_0 - \sum_{i=1}^{\mu} e_i) = 3$, hence C is another fixed component of $|\mathcal{L} - 4x|$, by Lemma 2, part (1). Note that $C \ni x$. Therefore, any element of $|\mathcal{L} - 4x|$ can be written as $\widetilde{\ell_1} + 2e_1 + C + R$, where R is an element of $|\mathcal{L} - (\widetilde{\ell_1} + 2e_1 + C)| = |\mathcal{L} - (\theta^* \mathcal{O}_{\mathbb{P}^2}(2) - e_0)| = |\theta^* \mathcal{O}_{\mathbb{P}^2}(2) - \sum_{j=0}^{\mu} e_j|$, passing through x (since this condition is enough to guarantee a four-tuple point at x). This concludes the proof. □

By Lemma 21 $\dim(|\mathcal{L} - 4x|) = \dim(\mathcal{S})$ where now $\mathcal{S} = |\mathcal{O}_{\mathbb{P}^2}(2) - \mathcal{P} - x'|$. So, $\mathcal{S} = \emptyset$ for $\mu \ge 4$ and then $\mathrm{rk}(j_{3,x}) = N + 1$, unless $\mu = 4$ and x' is the tangent direction at p_1 to the unique conic containing $\mathcal{P}$: in this case, $\mathrm{rk}(j_{3,x}) = N = 7$ at the corresponding point x, by (18). For $\mu \le 3$ we have that $\dim(\mathcal{S}) = 3 - \mu$, hence $\mathrm{rk}(j_{3,x}) = N - \dim(\mathcal{S}) = (11 - \mu) - (3 - \mu) = 8$ by (18); actually no four of the base points of $\mathcal{S}$ can be collinear, even if x' is in some special position with respect to $\mathcal{P}$. Clearly, the previous discussion applies to any $i = 1, \dots \mu$.

3) Finally, let $x \in S$, i.e., $x = e_i \cap \widetilde{\ell_i}$ for some $i = 1, \dots, \mu$, and set $i = 1$ again. We have $|\mathcal{L} - 4x| = 2(\widetilde{\ell_1} + e_1) + |\mathcal{L} - 2(\widetilde{\ell_1} + e_1)|$, by Proposition 4, and $\mathcal{L} - 2(\widetilde{\ell_1} + e_1) = \theta^* \mathcal{O}_{\mathbb{P}^2}(2) - \sum_{i=1}^{\mu} e_i$. Hence $\dim(|\mathcal{L} - 4x|) = \dim(\mathcal{S})$, where now $\mathcal{S} = |\mathcal{O}_{\mathbb{P}^2}(2) - P|$ is the linear system of conics containing P. Thus $\mathcal{S} = \emptyset$ if $\mu \ge 6$, while $\dim(\mathcal{S}) = 5 - \mu$ if $\mu \le 5$. What we said can be repeated for every $i = 1, \dots, \mu$. So, (18) gives the following. Let $x \in S$; then $\mathrm{rk}(j_{3,x}) = N + 1$ if $\mu \ge 6$, while $\mathrm{rk}(j_{3,x}) = 6$ if $\mu \le 5$.

The above conclusions can be summarized as follows.

Theorem 22. *Let $X \subset \mathbb{P}^N$ be a Castelnuovo surface, let $\mathcal{P}$, θ, Σ, S, μ be as before; set $\widetilde{\ell_{i,k}} = \theta^{-1}(\ell_{i,k})$ for $1 \le i < k \le \mu$, and let $x \in X$.*

(1) *If $\mu \le 2$, then*

$$\Phi_3(X) = \Sigma \cup e_0 \cup \widetilde{\ell_{1,2}}$$

($\Sigma \cup e_0$ if $\mu = 1$, and simply e_0 if $\mu = 0$), and there $\mathrm{rk}(j_{3,x}) = 8$ or 6 according to whether x is general or $x \in S$;

(2) *if* $\mu = 3$, *then*

$$\Phi_3(X) = \Sigma \cup e_0 \cup \big(\bigcup_{1 \leq i < j \leq 3} \widetilde{\ell_{i,j}} \big)$$

and there, $\mathrm{rk}(j_{3,x}) = 8$ *except at* $x = \widetilde{\ell_1} \cap \widetilde{\ell_{2,3}}$, *where* $\mathrm{rk}(j_{3,x}) = 7$ *and on* S, *where* $\mathrm{rk}(j_{3,x}) = 6$*;*

(3) *if* $\mu = 4$, *then* X *is hypo-osculating and* $\Phi_3(X)$ *consists of the points corresponding to the intersections of pairs of lines joining couples of points of* $\mathcal{P}$, *plus the five points representing the tangent direction at* p_j *to the conic containing* $\mathcal{P}$ $(j = 0, \dots, 4)$, *plus* S*; moreover,* $\mathrm{rk}(j_{3,x}) = 7$ *at each of these points except on* S, *where* $\mathrm{rk}(j_{3,x}) = 6$*;*

(4) *if* $\mu = 5$, *then* X *is hypo-osculating and* $\Phi_3(X)$ *consists of* S *and of the points corresponding to the common points in* $\mathbb{P}^2$, *if any, of three lines joining the six points of* $\mathcal{P}$ *in pairs; furthermore,* $\mathrm{rk}(j_{3,x}) = 6$ *at all these points.*

(5) *if* $\mu \geq 6$, *then* X *is perfectly hypo-osculating, and* $\mathrm{rk}(j_{3,x}) = N + 1$ *at every point.*

Remark. In case $\mu = 5$ the existence of a common point x' to the three lines containing the six points of $\mathcal{P}$ in pairs makes the configuration rather special, though it does not contradict condition (6). Actually, referring to the linear system of cubics through $\mathcal{P}$, this simply says that x is an Eckardt point for the cubic surface obtained by embedding X in $\mathbb{P}^3$ via $|-K_X|$.

Finally, let us spend few words on $k = 4$. Since k must satisfy the condition $3k \leq N + 1$, either X is hypo-osculating for $k = 4$, or $N = 11$, in which case, $(X, \mathcal{L}) = (\mathbb{F}_1, \theta^*\mathcal{O}_{\mathbb{P}^2}(4) - 2e_0)$. Let $x \in X$; elements of $|\mathcal{L} - 5x|$, if any, are of the form $\theta^*D - 2e_0$, for D a suitable plane quartic. Hence $|\mathcal{L} - 5x| = \emptyset$ unless $x \in e_0$. So, let $x \in e_0$; then the fiber F through x is a fixed component of multiplicity 3 of $|\mathcal{L} - 5x|$, hence $D = 3\ell_x + \ell'$, ℓ' being a line. Since $\theta^*\ell_x = F + e_0$, we see that $\theta^*D - 2e_0$ has a singular point of multiplicity ≥ 5 at x if and only if $\ell' \ni p_0$. Thus $\dim(|\mathcal{L} - 5x|) = 1$, since ℓ' can vary in the pencil $|\mathcal{O}_{\mathbb{P}^2}(1) - p_0|$. In conclusion, $\Phi_4(X) = e_0$ and $\mathrm{rk}(j_{4,x}) = 10$, according to (2), for every $x \in e_0$.

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