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Marco M. Peloso* and Maura Salvatori

On some spaces of holomorphic functions of exponential growth on a half-plane

Abstract: In this paper we study spaces of holomorphic functions on the right half-plane $\mathcal{R}$, that we denote by $\mathcal{M}_\omega^p$, whose growth conditions are given in terms of a translation invariant measure ω on the closed half-plane $\overline{\mathcal{R}}$. Such a measure has the form $\omega = \nu \otimes m$, where m is the Lebesgue measure on $\mathbb{R}$ and ν is a regular Borel measure on $[0, +\infty)$. We call these spaces *generalized Hardy–Bergman* spaces on the half-plane $\mathcal{R}$.

We study in particular the case of ν purely atomic, with point masses on an arithmetic progression on $[0, +\infty)$. We obtain a Paley–Wiener theorem for $\mathcal{M}_\omega^2$, and consequently the expression for its reproducing kernel. We study the growth of functions in such space and in particular show that $\mathcal{M}_\omega^p$ contains functions of order 1. Moreover, we prove that the orthogonal projection from $L^p(\mathcal{R}, d\omega)$ into $\mathcal{M}_\omega^p$ is unbounded for $p \neq 2$.

Furthermore, we compare the spaces $\mathcal{M}_\omega^p$ with the classical Hardy and Bergman spaces, and some other Hardy–Bergman-type spaces introduced more recently.

Keywords: Holomorphic function on half-plane, Reproducing kernel Hilbert space, Hardy spaces, Bergman spaces

MSC: 30H99, 46E22, 30C40

1 Introduction

This paper is concerned with spaces of holomorphic functions on the right half-plane $\mathcal{R}$ whose growth condition is given in terms of a translation invariant regular measure on $\overline{\mathcal{R}}$, and that can be defined as generalized Hardy–Bergman spaces. It is easy to see that a measure ω is translation invariant on $\overline{\mathcal{R}}$ if and only if it has the form $\omega = \nu \otimes m$, where m denotes the Lebesgue measure on $\mathbb{R}$ and ν is a measure on $[0, +\infty)$. We will simply write $dm(y) = dy$. These measures play the same role as the radial measures on the unit disk $\mathbb{D}$.

Holomorphic function spaces on $\mathcal{R}$ with integrability conditions given in terms of this type of measures have been studied by several authors, and here we mention in particular Z. Harper [8, 9], B. Jacob, S. Pott and J. Partington [10], and I. Chalendar and J. Partington [2], and our recent paper [25]. We will come back to the spaces these authors considered and other related function spaces defined on the unit disk by J. A. Peláez and J. Rättyä in Sections 4 and 5.

For $0 < a < b < \infty$, denote by $S_{a,b}$ the vertical strip $\{z = x + iy : a < x < b\}$ and by $H^p(S_{a,b})$ the classical Hardy space

$$H^p(S_{a,b}) = \left\{ f \text{ holomorphic in } S_{a,b} : \sup_{a < x < b} \int_{-\infty}^{+\infty} |f(x+iy)|^p \, dy < \infty \right\}.$$

We simply write S_b to denote $S_{0,b}$.

*Corresponding Author: Marco M. Peloso: Dipartimento di Matematica, Università degli Studi di Milano, Via C. Saldini 50, 20133 Milano, Italy, E-mail: marco.peloso@unimi.it
Maura Salvatori: Dipartimento di Matematica, Università degli Studi di Milano, Via C. Saldini 50, 20133 Milano, Italy, E-mail: maura.salvatori@unimi.it

Definition 1.1. *Let $1 \leq p < \infty$. Let $\omega = \nu \otimes dy$ be a translation invariant measure on $\overline{\mathcal{R}}$.*
When $\nu(\{0\}) = 0$, we define

$$\mathcal{M}^p_\omega(\mathcal{R}) = \left\{ f \in \mathrm{Hol}(\mathcal{R}) : \ f \in H^p(S_{a,b}) \text{ for all } 0 < a < b < \infty, \text{ and } f \in L^p(\mathcal{R}, d\omega) \right\}, \qquad (1)$$

and if $\nu(\{0\}) > 0$, we define

$$\mathcal{M}^p_\omega(\mathcal{R}) = \left\{ f \in \mathrm{Hol}(\mathcal{R}) : \ f \in H^p(S_b) \text{ for all } 0 < b < \infty, \text{ and } f \in L^p(\overline{\mathcal{R}}, d\omega) \right\}. \qquad (2)$$

In both cases we endow $\mathcal{M}^p_\omega(\mathcal{R})$ with the norm

$$\|f\|^p_{\mathcal{M}^p_\omega(\mathcal{R})} = \int\limits_0^{+\infty} \int\limits_{\mathbb{R}} |f(x+iy)|^p \, dy \, d\nu(x).$$

We call the spaces $\mathcal{M}^p_\omega(\mathcal{R})$ generalized Hardy–Bergman spaces on the half-plane $\mathcal{R}$.

We point out that the definition implies that in the case $\nu(\{0\}) > 0$, a function in $\mathcal{M}^p_\omega$, although not initially defined on the imaginary line, admits a boundary value function that in particular is in $L^p(dy)$.

In this paper we consider in particular the measures of the form

$$\omega_{a,\rho} = \nu_{a,\rho}(x) \otimes dy = \sum_{k=0}^{+\infty} \frac{a^n}{n!} \delta_{\rho \frac{n}{2}}(x) \otimes dy, \qquad (3)$$

$a, \rho > 0$ are fixed parameters. Thus, the measure $\omega_{a,\rho}$ is translation invariant in $\overline{\mathcal{R}}$, has purely atomic part $\nu_{a,\rho}$ in the $\mathrm{Re}\, z$-component and moreover, such atomic measure $\nu_{a,\rho}$ has support on an arithmetic progression $\{\frac{\rho}{2}n\}$ with weight $\frac{a^n}{n!}$ at the point-mass $\frac{\rho}{2}n$.

Explicitly, $\mathcal{M}^p_{\omega_{a,\rho}}(\mathcal{R})$ is the space of holomorphic functions on $\mathcal{R}$ that belong to $L^p(\overline{\mathcal{R}}, d\omega_{a,\rho}) \cap \left(\bigcap_{b>0} H^p(S_b) \right)$ with norm

$$\|f\|^p_{\mathcal{M}^p_{\omega_{a,\rho}}(\mathcal{R})} = \sum_{n=0}^{+\infty} \frac{a^n}{n!} \int\limits_{-\infty}^{+\infty} |f(\rho\tfrac{n}{2} + iy)|^p \, dy < \infty, \qquad (4)$$

where f is defined on the imaginary axis as its boundary values as function in $H^p(S_b)$.

We observe that we equivalently could define $\mathcal{M}^p_{\omega_{a,\rho}}(\mathcal{R})$ as the closure in $L^p(\overline{\mathcal{R}}, d\omega_{a,\rho})$ of the space $\mathrm{Hol}(\overline{\mathcal{R}}) \cap \left(\bigcap_{b>0} H^p(S_b) \right)$ with norm given by (4). Moreover, it suffices to require that $f \in \bigcap_{n>0} H^p(S_{\rho \frac{n}{2}})$.

From simplicity of notation, we write $\mathcal{M}^p_{a,\rho}$ in place of $\mathcal{M}^p_{\omega_{a,\rho}}(\mathcal{R})$.

In [25] we introduced and studied the space $\mathcal{M}^2_{\omega_{2,1}} = \mathcal{M}^2_{2,1}$ to give some necessary and some sufficient conditions for the solutions of the Müntz–Szász problem for the Bergman space (on the disk $\Delta = D(1, 1)$). Such problem was first stated and studied by S. Krantz, C. Stoppato and the first named author [14], in connection with the question of completeness in the Bergman space on some special domain in $\mathbb{C}^2$, the so-called *Diedriech–Fornæss worm domain*.

In Section 2 we recall the main results of [25] (without proofs) to motivated the analysis of the slightly more general spaces $\mathcal{M}^2_{a,\rho}$. In Section 3 we prove some of these extensions and, at the same time, show that some of main properties of $\mathcal{M}^2_{2,1}$ can not so easily generalized to the larger class, thus raising some natural questions.

In Section 4 we take a look at other Hardy–Bergman type spaces studied in particular in [8, 9], [10], and [2] and sometimes called *Zen spaces*. These spaces are denoted by A^p_ω. We prove a Paley–Wiener theorem for A^2_ω and, as a consequence, give a description of the reproducing kernel of A^2_ω. We also clarify some properties of A^p_ω, and in particular discuss how functions in such spaces lie in the Hardy space $H^p(\mathcal{R}_a)$, for every $a > 0$, where $\mathcal{R}_a$ is the half-plane $\{\mathrm{Re}\, z > a\}$. In this section we also present some examples of holomorphic functions and norms on $\mathcal{R}$ to illustrate some of the possible behaviors of the average function $a_{f,p}(x) = \int_{\mathbb{R}} |f(x+iy)|^p \, dy$, for $f \in \mathrm{Hol}(\mathcal{R})$.

We conclude the paper with some remarks and open questions.

2 The Müntz–Szász problem for the Bergman space

Goal of this section is to motivate the study of the spaces $\mathcal{M}_{a,\rho}^p$ by recalling the main results from [25], concerning the special case $\mathcal{M}_{2,1}^2$.

Let Δ be the disk $\{\zeta : |\zeta - 1| < 1\}$, denote by dA the Lebesgue measure in $\mathbb{C}$ and consider the (unweighted) Bergman space $A^2(\Delta)$. Then the complex powers $\{\zeta^{\lambda-1}\}$ with $\operatorname{Re}\lambda > 0$ are well defined and in $A^2(\Delta)$.

Following [14], the Müntz–Szász problem for the Bergman space is the question of characterizing the sequences $\{\lambda_j\}$ in $\mathcal{R}$ such that $\{\zeta^{\lambda_j-1}\}$ is a complete set in $A^2(\Delta)$, that is, span$\{\zeta^{\lambda_j-1}\}$ is dense in $A^2(\Delta)$.

The classical Müntz–Szász theorem concerns with the completeness of a set of powers $\{t^{\lambda_j-\frac{1}{2}}\}$ in $L^2([0, 1])$, where $\operatorname{Re}\lambda_j > 0$. The solution was provided in two papers separate by Müntz [20] and by Szász [27] where they show that the set $\{t^{\lambda_j-\frac{1}{2}}\}$ is complete $L^2([0, 1])$ if and only if the sequence $\{\lambda_j\}$ is a set of uniqueness for the Hardy space of the right half-plane $H^2(\mathcal{R})$, that is, if $f \in H^2(\mathcal{R})$ and $f(\lambda_j) = 0$ for every j, then f is identically 0.

As in the classical case, in order to study the Müntz–Szász problem for the Bergman space we transformed the question into characterizing the sets of uniqueness for some (Hilbert) space of holomorphic functions. In [25] we showed that $\{\zeta^{\lambda_j-1}\}$ is complete in $A^2(\Delta)$ if and only if $\{\zeta^{\lambda_j-1}\}$ is a set of uniqueness for $\mathcal{M}_{2,1}^2$ and found some sufficient and some necessary conditions. We now outline the most relevant results of [25].

For $f \in A^2(\Delta)$ and $z \in \mathcal{R}$ we define the Mellin–Bergman transform

$$M_\Delta f(z) = \frac{1}{\pi} \int_\Delta f(\zeta)\bar{\zeta}^{z-1}\, dA(\zeta)\,. \tag{5}$$

The function $\zeta^{\bar{z}-1}$ is well defined and belongs to $A^2(\Delta)$. Then a set $\{\zeta^{\lambda_j-1}\}$ is complete in $A^2(\Delta)$ if and only if $f \in A^2(\Delta)$ and $M_\Delta f(\bar{\lambda}_j) = 0$ for all j implies that f vanishes identically. In order to describe the image of $A^2(\Delta)$ under the Mellin–Bergman transform M_Δ consider the space

$$\mathcal{H} = \left\{g \in \operatorname{Hol}(\mathcal{R}) : \tfrac{\Gamma(1+z)}{2^z}g(z) \in \mathcal{M}_{2,1}^2\right\}, \tag{6}$$

with norm

$$\|g\|_{\mathcal{H}}^2 = \left\|\frac{\Gamma(1+z)}{2^z}g\right\|_{\mathcal{M}_{2,1}^2}^2 = \sum_{n=0}^{+\infty}\int_{-\infty}^{+\infty}|g(\tfrac{n}{2}+iy)|^2\frac{|\Gamma(\tfrac{n}{2}+1+iy)|^2}{\Gamma(n+1)}\, dy\,.$$

Theorem 2.1. *The Mellin–Bergman transform*

$$M_\Delta : A^2(\Delta) \to \mathcal{H}$$

is a surjective isomorphism. The space $\mathcal{H}$ consists of holomorphic functions on $\mathcal{R}$ that are of exponential type at most $\pi/2$ and the polynomials are dense in $\mathcal{H}$. Moreover, it is a Hilbert space with reproducing kernel

$$H(z, w) = \frac{1}{2\pi}\frac{\Gamma(z+\overline{w})}{\Gamma(1+z)\Gamma(1+\overline{w})}\,.$$

We point out that, as corollary of the proof, we obtain a remarkable factorization theorem for functions in $\mathcal{M}_{2,1}^2$. It would be interesting to prove, if it exists, a similar factorization theorem for $\mathcal{M}_{a,\rho}^2$.

Corollary 2.2. *We have that*

$$\mathcal{M}_{2,1}^2 = \frac{\Gamma(1+z)}{2^z}\mathcal{H}\,.$$

Since $\mathcal{H}$ consists of functions of exponential type at most $\pi/2$, using the above factorization, we obtain a formula of Carleman type for functions in $\mathcal{M}_{2,1}^2$, formula (7) below, that now we illustrate.

Recall that the exponent of convergence of a sequence $\{z_j\}$, with $|z_j| \to +\infty$, is $\rho_1 = \inf\{\rho > 0 : \sum_{j=1}^{+\infty} 1/|z_j|^\rho < \infty\}$, the counting function is $n(r) = \#\{z_j : |z_j| \le r\}$ and the upper and lower densities $d^\pm = d_{\{z_j\}}^\pm$ are

$$d^+ = \limsup_{r\to+\infty}\frac{n(r)}{r^{\rho_1}}\,, \quad d^- = \liminf_{r\to+\infty}\frac{n(r)}{r^{\rho_1}}\,.$$

We are now in the position to state a necessary and a sufficient condition for zero-sets of $\mathcal{M}_{2,1}^2$.

Theorem 2.3. *Let* $\{z_j\} \subseteq \mathcal{R}$, $1 \leq |z_j| \to +\infty$. *The following properties hold.*
(i) *If* $\{z_j\}$ *has exponent of convergence 1 and upper density* $d^+ < \frac{1}{2}$, *then* $\{z_j\}$ *is a zero-set for* $\mathcal{M}_{2,1}^2 \cap \text{Hol}(\overline{\mathcal{R}})$.
(ii) *If* $\{z_j\}$ *is a zero-set for* $\mathcal{M}_{2,1}^2 \cap \text{Hol}(\overline{\mathcal{R}})$, *then*

$$\limsup_{R \to +\infty} \frac{1}{\log R} \sum_{|z_j| \leq R} \text{Re}\left(1/z_j\right) \leq \frac{2}{\pi}. \tag{7}$$

The next result gives a partial solution to the Müntz–Szász problem for the Bergman space.

Theorem 2.4. *A sequence* $\{z_j\}$ *of points in* $\mathcal{R}$ *such that* $\text{Re}\, z_j \geq \varepsilon_0$, *for some* $\varepsilon_0 > 0$ *and that violates condition* (7), *is a set of uniqueness for* $\mathcal{M}_{2,1}^2(\mathcal{R})$.
As a consequence, if $\{z_j\}$ *is a sequence as above, the set of powers* $\{\zeta^{z_j-1}\}$ *is a complete set in* $A^2(\Delta)$.

3 The spaces $\mathcal{M}_{a,\rho}^2$

In this section we study the basic properties of $\mathcal{M}_{a,\rho}^2$. In particular we prove a Paley–Wiener type theorem that allows us to compute its reproducing kernel. We also prove that the Mellin transform is a a surjective isometry between a suitable L^2 space on the positive half-line and $\mathcal{M}_{a,\rho}^2$.

Notice that trivially $H^p(\mathcal{R})$ is a closed subset of $\mathcal{M}_{a,\rho}^p$ and that $\mathcal{M}_{a,\rho}^p$ is closed in $L^p(\overline{\mathcal{R}}, d\omega)$.

3.1 The Paley–Wiener type theorem and its consequences

We being by proving a characterization of $\mathcal{M}_{a,\rho}^2$ in terms of the Fourier transform of its boundary values, as in the spirit of the classical Paley–Wiener theorem.

The Fourier transform of a function $\psi \in L^1(\mathbb{R})$ is

$$\widehat{\psi}(\xi) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} \psi(x) e^{-ix\xi} \, dx \,.$$

For $f \in \mathcal{M}_{a,\rho}^2$, we write $f_0 = f(0 + i\cdot)$ to denote its boundary values on the imaginary axis. We recall that the classical Paley–Wiener theorem for $\mathcal{H}^2(\mathcal{R})$ establishes a surjective isomorphisms between $\mathcal{H}^2(\mathcal{R})$ and $L^2((-\infty, 0), d\xi))$.

Theorem 3.1. *Let* $f \in \mathcal{M}_{a,\rho}^2$. *Then* $\widehat{f}_0 \in L^2(\mathbb{R}, e^{ae^{\rho\xi}} \, d\xi)$ *and*

$$\|f\|_{\mathcal{M}_{a,\rho}^2} = \|\widehat{f}_0\|_{L^2(\mathbb{R}, e^{ae^{\rho\xi}})} \,. \tag{8}$$

Conversely, if $\psi \in L^2(\mathbb{R}, e^{ae^{\rho\xi}} \, d\xi)$ *and for* $z \in \mathcal{R}$ *we set*

$$f(z) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} \psi(\xi) e^{z\xi} \, d\xi \,, \tag{9}$$

then $f \in \mathcal{M}_{a,\rho}^2$, *equality* (8) *holds and* $\psi = \widehat{f}_0$.

Proof. Since $f \in H^2(S_{\rho\frac{n}{2}})$ for every n, by the classical Paley–Wiener theorem on a strip ([21], see also [25, Thm. 1.1]), we see that

$$\|f\|_{\mathcal{M}_{a,\rho}^2}^2 = \sum_{n=0}^{+\infty} \frac{a^n}{n!} \|f(\rho\tfrac{n}{2} + i\cdot)\|_{L^2(\mathbb{R})}^2$$

$$= \sum_{n=0}^{+\infty} \frac{a^n}{n!} \big\| [f(\rho\tfrac{n}{2} + i\cdot)]\hat{} \big\|^2_{L^2(\mathbb{R})}$$

$$= \sum_{n=0}^{+\infty} \frac{a^n}{n!} \int_{-\infty}^{+\infty} e^{n\rho\xi} |\widehat{f}_0(\xi)|^2 \, d\xi$$

$$= \int_{-\infty}^{+\infty} |\widehat{f}_0(\xi)|^2 e^{ae^{\rho\xi}} \, d\xi . \tag{10}$$

Conversely, given $\psi \in L^2(\mathbb{R}, e^{ae^{\rho\xi}} d\xi)$ observe that the integral in (9) is absolutely convergent for $z \in \mathcal{R}$:

$$\int_{-\infty}^{+\infty} |\psi(\xi)e^{z\xi}| \, d\xi \leq \|\psi\|_{L^2(\mathbb{R}, e^{ae^{\rho\xi}} d\xi)} \left(\int_{-\infty}^{+\infty} e^{2x\xi} e^{-ae^{\rho\xi}} \, d\xi \right)^{1/2} < \infty . \tag{11}$$

Therefore, if f is given by (9) it is holomorphic in $\mathcal{R}$ and $f \in H^2(S_{\rho\frac{n}{2}})$ for every n, since $e^{\rho\frac{n}{2}(\cdot)}\psi \in L^2(\mathbb{R})$. It is also clear that $\widehat{f}_0 = \psi$ and arguing as for (10) we obtain (8). □

Corollary 3.2. *The space $\mathcal{M}^2_{a,\rho}$ is a reproducing kernel Hilbert space and its reproducing kernel is*

$$K(z, w) = \frac{1}{2\pi\rho} a^{-\frac{z+\overline{w}}{\rho}} \Gamma\Big(\frac{z + \overline{w}}{\rho}\Big) .$$

Proof. By (11) it follows that point evaluations are continuous in $\mathcal{M}^2_{a,\rho}$ and it is elementary to see that it is a Hilbert space.

Let $K_z \in \mathcal{M}^2_{a,\rho}$ be such that $\langle f, K_z \rangle_{\mathcal{M}^2_{a,\rho}} = f(z)$ for every $z \in \mathcal{R}$ and every $f \in \mathcal{M}^2_{a,\rho}$. By Theorem 3.1 and (9) we obtain

$$f(z) = \sum_{n=0}^{+\infty} \frac{a^n}{n!} \langle f(\rho\tfrac{n}{2} + i\cdot), K_z(\rho\tfrac{n}{2} + i\cdot) \rangle_{L^2}$$

$$= \sum_{n=0}^{+\infty} \frac{a^n}{n!} \langle [f(\rho\tfrac{n}{2} + i\cdot)]\hat{}, [K_z(\rho\tfrac{n}{2} + i\cdot)]\hat{} \rangle_{L^2}$$

$$= \sum_{n=0}^{+\infty} \frac{a^n}{n!} \langle e^{\rho\frac{n}{2}\xi} \widehat{f}_0, e^{\rho\frac{n}{2}\xi} \widehat{K}_{z,0} \rangle_{L^2}$$

$$= \int_{-\infty}^{+\infty} \widehat{f}_0(\xi) \overline{\widehat{K_{z,0}}(\xi)} \, e^{ae^{\rho\xi}} \, d\xi ,$$

where switching the integral with the sum is justified since the last integral converges absolutely.

On the other hand,

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{z\xi} \widehat{f}_0(\xi) \, d\xi ,$$

so that

$$\widehat{K_{z,0}}(\xi) = \frac{1}{\sqrt{2\pi}} e^{-ae^{\rho\xi}} e^{\overline{z}\xi} ,$$

and

$$K_z(w) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{w\xi} e^{-ae^{\rho\xi}} e^{\overline{z}\xi} \, d\xi$$

$$= \frac{1}{2\pi\rho} a^{-\frac{w+\overline{z}}{\rho}} \int_0^{+\infty} t^{w+\overline{z}-1} e^{-t} \, dt$$

$$= \frac{1}{2\pi\rho} a^{-\frac{w+\overline{z}}{\rho}} \Gamma\Big(\frac{w+\overline{z}}{\rho}\Big). \qquad \square$$

It is now clear that $\mathcal{M}^2_{a,\rho}$ contains functions of order 1 in the right half-plane, namely, for $w \in \mathcal{R}$

$$K_w(z) = \frac{1}{2\pi\rho} a^{-\frac{z+\overline{w}}{\rho}} \Gamma\Big(\frac{z+\overline{w}}{\rho}\Big).$$

The next result shows that the growth of functions in $\mathcal{M}^2_{a,\rho}$ is at most of order 1. Precisely,

Corollary 3.3. *Let be $a, \rho > 0$ given. The functions in $\mathcal{M}^2_{a,\rho}$ satisfy the growth condition*

$$|f(z)| \le C(\operatorname{Re} z)^{1/4}\Big(\frac{2}{a}\Big)^{(\operatorname{Re} z)/\rho} \Gamma\Big(\frac{\operatorname{Re} z}{\rho}\Big).$$

Proof. We have

$$
\begin{aligned}
|f(z)| = \big|\langle \widehat{f}_0, \widehat{K_{z,0}}\rangle_{L^2(\mathbb{R}, e^{ae^{\rho\xi}})}\big| &\le C\big\|\widehat{K_{z,0}}\big\|_{L^2(\mathbb{R}, e^{ae^{\rho\xi}})} \\
&\le C\Big(\int_{\mathbb{R}} e^{2\operatorname{Re} z\xi} e^{-ae^{\rho\xi}}\, d\xi\Big)^{1/2} = C\frac{1}{a^{(\operatorname{Re} z)/\rho}}\Gamma\Big(\frac{2\operatorname{Re} z}{\rho}\Big)^{1/2} \\
&\le C(\operatorname{Re} z)^{1/4}\Big(\frac{2}{a}\Big)^{(\operatorname{Re} z)/\rho}\Gamma\Big(\frac{\operatorname{Re} z}{\rho}\Big),
\end{aligned}
$$

using the standard asymptotics of the Gamma function. $\qquad \square$

The next result is obvious.

Corollary 3.4. *Given $a, a' > 0$ and $\rho, \rho' > 0$ we have the continuous embedding $\mathcal{M}^2_{a,\rho} \subseteq \mathcal{M}^2_{a',\rho'}$ if and only if $\rho > \rho'$ for any a, a' or if $\rho = \rho'$ and $a > a'$.*

There is one more consequence of the Payley–Wiener type theorem about a density result in $\mathcal{M}^2_{a,\rho}$. To state it we first need to introduce some further notation.

For every $\varepsilon > 0$ we denote by $\mathcal{M}^p_{a,\rho}(\mathcal{R}_{-\varepsilon})$ the subspace of $\mathcal{M}^p_{a,\rho}$ of functions that are holomorphic for $\operatorname{Re} z > -\varepsilon$ and that are in $H^p(S_{(-\varepsilon,b)})$ for every $b > 0$.

The following proposition is proved as [25, Prop. 3.4] and it is used in the next section.

Proposition 3.5. *For $1 \le p, q < \infty$, the space $\bigcap_{\varepsilon > 0} \mathcal{M}^p_{a,\rho}(\mathcal{R}_{-\varepsilon}) \cap \mathcal{M}^q_{a,\rho}(\mathcal{R}_{-\varepsilon})$ is dense in $\mathcal{M}^p_{a,\rho}$.*

3.2 The Mellin transform

We want to show that the Mellin transform is a surjective isometry between $\mathcal{M}^2_{a,\rho}$ and some suitable L^2 space of the positive half-line.

More precisely, if φ is a function defined on $(0, +\infty)$ we consider the (re-normalized) Mellin transform, that is

$$M\varphi(z) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \varphi(t) t^{z-1}\, dt. \qquad (12)$$

Theorem 3.6. *The mapping*

$$M : L^2\Big((0, +\infty),\, e^{a\xi^\rho}\, \frac{d\xi}{\xi}\Big) \to \mathcal{M}^2_{a,\rho}$$

is a surjective isometry.

Clearly, the main point of this result is the surjectivity of M.

Proof. Suppose that $\varphi \in L^2\big((0, +\infty), e^{a\xi^\rho}\,\frac{d\xi}{\xi}\big)$, then for every $z = x + iy \in \mathcal{R}$

$$|M\varphi(z)| \le \frac{1}{\sqrt{2\pi}} \int\limits_0^{+\infty} |\varphi(\xi)|\xi^{x-1}\,d\xi$$

$$\le \frac{1}{\sqrt{2\pi}} \|\varphi\|_{L^2((0,+\infty),e^{a\xi^\rho}\,\frac{d\xi}{\xi})} \Big(\int\limits_0^{+\infty} e^{-a\xi^\rho}\xi^{2x-1}\,d\xi \Big)^{1/2}$$

$$= \frac{1}{\sqrt{2\pi\rho}} \|\varphi\|_{L^2((0,+\infty),e^{a\xi^\rho}\,\frac{d\xi}{\xi})} a^{-\frac{x}{\rho}}\, \Gamma\big(\tfrac{2x}{\rho}\big)^{1/2}.$$

This shows that $M\varphi$ is a well defined holomorphic function in $\mathcal{R}$. Next, we observe that

$$M\varphi(z) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} (\varphi \circ \exp)(s)e^{zs}\,ds$$

$$= \mathcal{F}^{-1}\big((\varphi \circ \exp)e^{x(\cdot)}\big)(y),$$

so that we obtain

$$\big\|M\varphi(x + i\cdot)\big\|_{L^2(\mathbb{R})}^2 = \|\varphi\|_{L^2((0,+\infty),\xi^{2x-1}d\xi)}^2$$

$$\le C_x \|\varphi\|_{L^2((0,+\infty),e^{a\xi^\rho}\,\frac{d\xi}{\xi})}^2,$$

uniformly in $x \in (0, b]$. Hence, $M\varphi \in H^2(S_b)$ for every $b > 0$ if $\varphi \in L^2((0, +\infty), e^{a\xi^\rho}\,\frac{d\xi}{\xi})$.

Next we consider $\varphi, \psi \in L^2((0, +\infty), e^{a\xi^\rho}\,\frac{d\xi}{\xi})$ and we first assume that both have compact support. Then,

$$\langle M\varphi, M\psi \rangle_{\mathcal{M}^2_{a,\rho}} = \frac{1}{2\pi} \sum_{n=0}^{+\infty} \frac{a^n}{n!} \int\limits_{-\infty}^{+\infty} M\varphi(\tfrac{n}{2} + iy)\overline{M\psi(\tfrac{n}{2} + iy)}\,dy$$

$$= \sum_{n=0}^{+\infty} \frac{a^n}{n!} \int\limits_{-\infty}^{+\infty} \mathcal{F}^{-1}\big((\varphi \circ \exp)e^{\rho\frac{n}{2}(\cdot)}\big)(y)\overline{\mathcal{F}^{-1}\big((\psi \circ \exp)e^{\rho\frac{n}{2}(\cdot)}\big)(y)}\,dy$$

$$= \sum_{n=0}^{+\infty} \frac{a^n}{n!} \int\limits_{-\infty}^{+\infty} \varphi\big(e^y\big)e^{\rho\frac{n}{2}y}\overline{\psi\big(e^y\big)}e^{\rho\frac{n}{2}y}\,dy$$

$$= \int\limits_{-\infty}^{+\infty} \varphi\big(e^y\big)\overline{\psi\big(e^y\big)}e^{2e^y}\,dy$$

$$= \int\limits_0^{+\infty} \varphi(t)\overline{\psi(t)}\,e^{at^\rho}\frac{dt}{t}$$

$$= \langle \varphi, \psi \rangle_{L^2((0,+\infty),e^{a\xi^\rho}\,\frac{d\xi}{\xi})}.$$

Therefore, the Mellin transform M is a partial isometry. In order to prove that M is onto we need to recall some well-known facts about the inversion of the Mellin. If, for every c in some interval I it is $g(c + i\cdot) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ and $g(c + iy) \to 0$ as $|y| \to +\infty$ uniformly in $c \in I$, then

$$M_c^{-1}g(\xi) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} g(c + it)\xi^{-c-it}\,dt \tag{13}$$

is well defined for every $\xi > 0$, independent of $c \in I$ and $MM_c^{-1}g = g$. Fix now $\varepsilon > 0$ and suppose $f \in \mathcal{M}^2_{a,\rho}(\mathcal{R}_{-\varepsilon}) \cap \mathcal{M}^1_{a,\rho}(\mathcal{R}_{-\varepsilon})$, satisfying $f(c + iy) \to 0$ as $|y| \to +\infty$ uniformly in $c \in I$. Therefore, $M_{\rho\frac{n}{2}}^{-1}f$ is

independent of $n \geq 0$ and $MM_{\rho \frac{n}{2}}^{-1} f = f$. For every such function f, we define

$$M^{-1} f = M_{\frac{n}{2}}^{-1} f \tag{14}$$

and, by density, it is enough to show that

$$\left\| M^{-1} f \right\|_{L^2\left((0,+\infty), \, e^{a\xi\rho} \frac{d\xi}{\xi}\right)} = \left\| f \right\|_{\mathcal{M}_{a,\rho}^2}. \tag{15}$$

If $f(c + i\cdot) \in L^2(\mathbb{R})$ we put $\varphi_c(t) = t^{-c} f(c - i \log t)$ and claim that if $\xi > 0$ we have

$$\xi^c M_c^{-1} f(\xi) = M\varphi_c(c + i \log \xi). \tag{16}$$

For,

$$
\begin{aligned}
(M\varphi_c)(c + i \log \xi) &= \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} t^{-c} f(c - i \log t) t^{c + i\xi - 1} \, dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(c + is) \xi^{-is} \, ds \\
&= \xi^c \left(M_c^{-1} f \right)(\xi)
\end{aligned}
$$

as we claimed. Now, [1, Lemma 2.3] shows that, for every function g such that $g(t)t^c \in L^2\left((0,+\infty), \frac{dt}{t}\right)$ it is

$$\int_0^{+\infty} t^{2c} |g(t)|^2 \frac{dt}{t} = \int_{-\infty}^{+\infty} |Mg(c + iy)|^2 \, dy.$$

Using all the above, arguing as in [25, Lemma 3.2] we obtain

$$\left\| f \right\|_{\mathcal{M}_{a,\rho}^2}^2 = \left\| M^{-1} f \right\|_{L^2\left((0,+\infty), \, e^{a\xi\rho} \frac{d\xi}{\xi}\right)}^2$$

that is, (15) holds, and we are done. $\qquad\qquad\square$

3.3 Unboundedness of the orthogonal projection

Next we consider the question of the boundedness of the orthogonal projection. We find it quite interesting that the orthogonal projection does not extend to a bounded operator on $\mathcal{M}_{a,\rho}^p$ for any $p \neq 2$. Analogous result was proved by M. Dostanić in the case of weighted Bergman spaces on the unit disk when the weights are exponentially decreasing at the boundary [3].

Theorem 3.7. *The orthogonal projection operator* $P : L^2(\overline{\mathcal{R}}, d\omega_{a,\rho}) \to \mathcal{M}_{a,\rho}^2$, *is unbounded as operator*

$$P : L^p(\overline{\mathcal{R}}, d\omega_{a,\rho}) \cap L^2(\overline{\mathcal{R}}, d\omega_{a,\rho}) \to \mathcal{M}_{a,\rho}^p$$

for every $p \neq 2$.

Proof. The proof follows the same lines of the one for the spaces $\mathcal{M}_{2,1}^p$ (see [25, Thm. 8]) so that we simply sketch it. A necessary condition for the projections to be bounded on L^p is that the kernels K_w belong to $\mathcal{M}^{p'}(\overline{\mathcal{R}}, d\omega_{a,\rho})$ (with $1/p + 1/p' = 1$). By duality and since P is self-adjoint it is enough to show that $K_w \notin L^p(\overline{\mathcal{R}}, d\omega_{a,\rho})$ for any $p > 2$.

First we observe that, there exists $C > 0$ such that

$$\left| \Gamma\left(\frac{n}{2} + \frac{u}{\rho} + iy\right) \right|^p \geq C e^{-p\frac{u}{\rho}} e^{-p\frac{\pi}{2}|y|} \exp\left\{ \frac{p(n-1)}{2} \log\left(\frac{n}{2}\right) - \frac{pn}{2} \right\}.$$

Therefore,

$$\|K_w\|_{\mathcal{M}^p}^p = C \frac{1}{a^{\frac{pu}{\rho}}} \sum_{n=0}^{+\infty} \frac{a^{n(1-\frac{p}{2})}}{n!} \int_{-\infty}^{+\infty} \left|\Gamma(\tfrac{n}{2} + \tfrac{u}{\rho} + iy)\right|^p dy$$

$$\geq C_u \sum_{n=0}^{+\infty} \frac{e^{n[(1-\frac{p}{2})\log a - \frac{p}{2}(1+\log 2)]}}{n! \, n^{\frac{p}{2}}} e^{\frac{p}{2} n \log n},$$

which clearly diverges when $p > 2$. $\hfill\square$

3.4 Some open questions

We collect here some properties of $\mathcal{M}_{2,1}^2$ that did not easily carry over to the more general case of $\mathcal{M}_{a,\rho}^2$.

1. Corollary 2.2 shows that $\mathcal{M}_{2,1}^2 = \frac{\Gamma(1+z)}{2^z}\mathcal{H}$, where $\mathcal{H}$ consists of functions of exponential type $\pi/2$ in $\mathcal{R}$. Thus, functions in $\mathcal{M}_{2,1}^2$ are of order 1 in $\mathcal{R}$, but can be factored as product of a non-vanishing term times a function in the space $\mathcal{H}$. It is interesting to notice that the space $\mathcal{H}$, that appears in [25] as $M_\Delta(A^2(\Delta))$, had already appeared in the literature, in a different context [15, 16].

   It would certainly be of interest to extend this factorization result to the case $\mathcal{M}_{a,\rho}^2$, and possibly also the description of the analogous of the factor $\mathcal{H} = M_\Delta(A^2(\Delta))$.

2. Along the same lines as above, we mention that we do not know whether the Carleman formula, Theorem 2.3 (ii), holds in the general case of $\mathcal{M}_{a,\rho}^2$. Again, proving a factorization theorem for $\mathcal{M}_{a,\rho}^2$ would give us a tool for describing its zero-sets, as we obtained in the case of $\mathcal{M}_{2,1}^2$.

3. It would also be of interest to obtain a description of the Müntz–Szász problem for *weighted* Bergman spaces on Δ. Here we have two most natural choices of weights, namely $v(\zeta) = (1 - |\zeta - 1|^2)^\alpha$, and $\tilde{v}(\zeta) = |\zeta|^\alpha$, $\alpha > -1$. The weight v is radially symmetric in the disk Δ, while $\tilde{v}$ is radial in $\mathbb{C}$.

4 Comparison with other function spaces

In this section we compare the space $\mathcal{M}_{a,\rho}^p$ with other Hilbert spaces of holomorphic functions on the right half-plane.

4.1 Other spaces of Hardy–Bergman type: Zen spaces

Let

$$A_\omega^p(\mathcal{R}) = \left\{ f \in \mathrm{Hol}(\mathcal{R}) : \sup_{r>0} \int_0^{+\infty} \int_{\mathbb{R}} |f(x + r + iy)|^p \, dy \, dv(x) < +\infty \right\}, \tag{17}$$

where $\omega = v(x) \otimes dy$ is a regular Borel measure on $\mathcal{R}$ and v is such that there exists $R > 0$ such that

$$\sup_{t>0} \frac{v([0, 2t))}{v([0, t))} \leq R, \tag{18}$$

which is a doubling-type condition at the origin.

Such spaces have been studied by several authors; here we mention Z. Harper [8, 9], B. Jacob, S. Pott and J. Partington [10], and by I. Chalendar and J. Partington [2], and they are sometimes called *Zen spaces*. We mention here that previously, this kind of spaces had been considered in [7] by G. Garrigós, in the much more general case of tube domains over cones, although only for quasi-invariant measures dv.

In [10] two facts about the spaces A_ω^p are stated without proof, namely:

(i) A_ω^2 is a Hilbert space;

(ii) $f \in A_\omega^p$ implies that $f \in H^p(\mathcal{R}_a)$ for every $a > 0$, where $H^p(\mathcal{R}_a)$ denotes the Hardy space of the half-plane $\{\operatorname{Re} z > a\}$.

We believe that both statements require some proof. The reason being that, from the definition (17) it follows that for $f \in A_\omega^p$, the avarages $a_{f_r,p} = \int_\mathbb{R} |f(x+iy)|^p \, dy$ are ν-a.e. finite, for every $r > 0$, where $f_r = f(\cdot + r)$. However, this condition is not easily exploited.

We remark that, for $1 \le p < \infty$, the requirement $f \in H^p(S_{a,b})$ for all $0 < a < b < \infty$, implies that the average function $a_{f,p}(x)$ is finite everywhere in $(0, +\infty)$ and it is convex. For, if $x = \delta a + (1 - \delta)b$, for some $0 < \delta < 1$, then it is well known that

$$ a_{f,p}(x) \le a_{f,p}(a)^\delta a_{f,p}(b)^{1-\delta} \le \delta a_{f,p}(a) + (1 - \delta) a_{f,p}(b) \, . $$

This implies that $a_{f,p}$ must attains its supremum value as either limit as $x \to 0^+$ or as $x \to +\infty$.

Hence, once we know that $f \in A_\omega^p$ implies that $f \in H^p(S_{a,b})$, then the "sup"-condition in the definition of the norm of A_ω^p forces $a_{f,p}(x) \to 0^+$ as $x \to +\infty$. Thus, in particular $f \in H^p(\mathcal{R}_a)$, for every $a > 0$.

However, if $f \in \operatorname{Hol}(\mathcal{R})$ the average function $a_{f,p}$ can have some extreme behaviors, as one could see from [9]. For example, it follows easily from [9, Sec. 4] that given any sequence of separated intervals $I_n \subset (0, +\infty)$, that is such that $\overline{I}_n \cap \overline{I}_m = \emptyset$ for $n \ne m$, there exists $f \in \operatorname{Hol}(\mathcal{R})$ such that $f \in H^p(S_{I_n})$, for all n, but f is not in H^p of any larger strip.

In the remainder of this section we are going to prove that functions in A_ω^p can be bounded uniformly on compact subsets of $\mathcal{R}$ only in terms of the A_ω^p-norm and of some constant depending only the compact set. This property will (easily) imply that A_ω^2 is a Hilbert space, and also provide a tool to prove statement (ii) above.

Proposition 4.1. *Let $1 \le p < \infty$ and A_ω^p and ν be as in (17) and (18), resp. Then, for each compact set $E \subset \mathcal{R}$ there exists $C_E > 0$ such that*

$$ \sup_{\lambda \in E} |F(\lambda)| \le C_E \|F\|_{A_\omega^p} \, . $$

Proof. Assume first that the measure ν has an atomic part ν_{at}. If $\nu(\{0\}) > 0$ then A_ω^p embeds continuously in $H^p(\mathcal{R})$, and the statement follows. If $\nu(\{0\}) = 0$, but ν_{at} has support $\{a_n\}_{n\in\mathbb{Z}}$, with $a_n \to 0^+$ as $n \to -\infty$. Arguing as before, we obtain that $f \in A_\omega^p$ belongs to $H^p(\mathcal{R}_{a_n})$ for every n, and the conclusion follows again.

Suppose now that ν has no atomic part that accumulates at the orgin. Let $a > 0$. By condition (18) we can find $\varepsilon < \frac{a}{4}$ such that $\nu([\frac{3}{4}\varepsilon, \frac{5}{4}\varepsilon)) > 0$. Consider the function $g(t) = \nu([t - \frac{\varepsilon}{2}, t + \frac{\varepsilon}{2}))$. Notice that $g(\varepsilon) > 0$, and since ν has no atomic part that accumulates at the origin, g is continuous, for ε suitably small. Then, g has a positive minimum in an interval $[\varepsilon - \delta, \varepsilon + \delta]$.

For $\lambda \in \overline{S}_{a-\delta,a+\delta}$, let $\lambda_0 = \lambda - (a - \varepsilon)$, so that $\operatorname{Re} \lambda_0 \in [\varepsilon - \delta, \varepsilon + \delta]$. Denote also $Q(\lambda_0, \varepsilon)$ the square centered in λ_0 with sides of lenght 2ε. We then have,

$$
\begin{aligned}
|F(\lambda)|^p &= \frac{1}{\omega(D(\lambda_0, \varepsilon))} \int_{D(\lambda_0,\varepsilon)} |F(\lambda_0 + a - \varepsilon)|^p \, d\omega(z) \\
&\le \frac{C}{\varepsilon^2 \omega(D(\lambda_0,\varepsilon))} \int_{D(\lambda_0,\varepsilon)} \int_{D(\lambda_0,\varepsilon)} |F(w + a - \varepsilon)|^p \, dA(w) \, d\omega(z) \\
&\le \frac{C}{\varepsilon^2 \omega(D(\lambda_0,\varepsilon))} \int_{D(\lambda_0,\varepsilon)} \int_{D(z,2\varepsilon)} |F(w + a - \varepsilon)|^p \, dA(w) \, d\omega(z) \\
&= \frac{C}{\varepsilon^2 \omega(D(\lambda_0,\varepsilon))} \int_{D(\lambda_0,\varepsilon)} \int_{D(0,2\varepsilon)} |F(z + \zeta + a - \varepsilon)|^p \, dA(\zeta) \, d\omega(z) \\
&= \frac{C}{\varepsilon^2 \omega(D(\lambda_0,\varepsilon))} \int_{D(0,2\varepsilon)} \int_{D(\lambda_0,\varepsilon)} |F(z + \zeta + a - \varepsilon)|^p \, d\omega(z) \, dA(\zeta)
\end{aligned}
$$

$$\leq \frac{C}{\varepsilon^2 \omega(D(\lambda_0, \varepsilon))} \int\limits_{D(0,2\varepsilon)} \int\limits_{Q(\lambda_0,\varepsilon)} |F(x + \xi + a - \varepsilon + i(y + \eta))|^p \, dv(x) dy \, dA(\zeta)$$

$$\leq C_\varepsilon \frac{1}{\nu([\operatorname{Re}\lambda_0 - \frac{\varepsilon}{2}, \operatorname{Re}\lambda_0 + \frac{\varepsilon}{2}))} \|F\|_{A_\omega^p}^p$$

$$\leq C_\varepsilon \|F\|_{A_\omega^p}^p. \tag{19}$$

because of our construction of the function g. □

An easy argument now shows that A_ω^2 is a Hilbert space. We do not know of any proof of such fact that does not use the fact that norm convergence implies uniform convergence on compact subsets.

It also follows that $f \in A_\omega^p$ is bounded on every closed strip $\overline{S_{a,b}} \subset \mathcal{R}$. Therefore, if $f(x + i\cdot) \in L^p(\mathbb{R})$ for $x = a, b$, it follows that $f \in H^p(S_{a,b})$. Hence we have,

Corollary 4.2. (1) *If $f \in A_\omega^p$ then $f \in H^p(\mathcal{R}_a)$ for any $a > 0$ and*

$$\|f\|_{A_\omega^p}^p = \int\limits_0^{+\infty} \int\limits_{\mathbb{R}} |f(x + iy)|^p \, dy \, dv(x).$$

(2) *When $\nu(\{0\}) = 0$, we have the equality*

$$A_\omega^p = \left\{ f \in \operatorname{Hol}(\mathcal{R}) : \ f \in H^p(\mathcal{R}_a) \text{ for all } a > 0, \text{ and } f \in L^p(\mathcal{R}, d\omega) \right\},$$

and if $\nu(\{0\}) > 0$ we have the equality

$$A_\omega^p = \left\{ f \in \operatorname{Hol}(\mathcal{R}) : \ f \in H^p(\mathcal{R}) \text{ and } f \in L^p(\overline{\mathcal{R}}, d\omega) \right\}.$$

In both cases we have

$$\|f\|_{A_\omega^p}^p = \int\limits_0^{+\infty} \int\limits_{\mathbb{R}} |f(x + iy)|^p \, dy \, dv(x).$$

It is now clear that A_ω^p is a closed subspace of $\mathcal{M}_\omega^p$. It is worth to notice that while functions in A_ω^p are bounded in $\mathcal{R}_a$ for every $a > 0$, functions in $\mathcal{M}_\omega^p$ are, in general, allow to grow at infinity, as the case of $\mathcal{M}_{a,\rho}^p$ shows.

A significant consequence of Prop. 4.1 is a Paley–Wiener theorem type theorem for the space and in $f \in A_\omega^2$. In particular it proves that isometry considered in [10, Prop. 2.3], while its proof is inspired by the one of [9, Thm. 2.1].

We need a couple of definitions. For $\xi < 0$ we set let

$$v(\xi) = \int\limits_0^{+\infty} e^{2\xi x} \, dv(x).$$

It was already observed in [10, Prop. 2.3] that the condition (18) implies that the integral above converges. For $\varphi \in L^2((-\infty, 0), v(\xi)d\xi)$ and $z \in \mathcal{R}$ define

$$T\varphi(z) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{0} e^{z\xi} \varphi(\xi) \, d\xi = \mathcal{F}^{-1}\left(e^{x\xi}\varphi(\xi)\right)(y).$$

Theorem 4.3. (1) *If $\varphi \in L^2((-\infty, 0), v(\xi)d\xi)$ then $T\varphi \in A_\omega^2$ and*

$$\|T\varphi\|_{A_\omega^2} = \|\varphi\|_{L^2((-\infty,0),v(\xi)d\xi)}.$$

(2) *Conversely, if $F \in A_\omega^2$, then there exists $\varphi \in L^2((-\infty, 0), v(\xi)d\xi)$ such that $F = T\varphi$ and $\|F\|_{A_\omega^2} = \|\varphi\|_{L^2((-\infty,0),v(\xi)d\xi)}.$*

Proof. We remark again that (1) is just [10, Prop. 2.3].

For (2), let $0 < a < b < \infty$. We know that $\int_{\mathbb{R}} |F(x+iy)|^2 \, dy < +\infty$, for all $x > 0$. Let $Y > 0$ be fixed and let R_Y be the rectangle of vertices $(a, -Y)$, $(b, -Y)$, (b, Y), and (a, Y). By Cauchy's integral formula, for $\lambda \in R_Y$ we have

$$F(\lambda) = \frac{1}{2\pi i} \int_{\partial R_Y} \frac{F(z)}{z - \lambda} \, dz \,.$$

Letting $Y \to +\infty$, using the fact that F is bounded in every closed strip, for $a < \operatorname{Re} \lambda < b$ we obtain

$$
\begin{aligned}
F(\lambda) &= \frac{1}{2\pi i} \left(\int_{b+i\mathbb{R}} \frac{F(z)}{z - \lambda} \, dz - \int_{a+i\mathbb{R}} \frac{F(z)}{z - \lambda} \, dz \right) \\
&= \frac{1}{2\pi} \left(\int_{\mathbb{R}} \frac{F(b + iy)}{iy - (\lambda - b)} \, dy - \int_{\mathbb{R}} \frac{F(a + iy)}{iy - (\lambda - a)} \, dy \right).
\end{aligned}
$$

Now notice that

$$\left| \int_{\mathbb{R}} \frac{F(b + iy)}{iy - (\lambda - b)} \, dy \right|^2 \le \| F_b \|_{L^2(\mathbb{R})}^2 \int_{\mathbb{R}} \frac{1}{y^2 + |\operatorname{Re}\lambda - b|^2} \, dy \to +\infty$$

as $b \to +\infty$, since $F \in H^2(\mathcal{R}_{-\varepsilon})$ for every $\varepsilon > 0$. Therefore, observing that $\lambda - a \in \mathcal{R}$,

$$F(\lambda) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{F(a + iy)}{(\lambda - a) - iy} \, dy = \mathcal{S}(F_a)(\lambda - a) \,, \tag{20}$$

where $\mathcal{S}$ denotes the Szegö projection on $H^2(\mathcal{R})$. By the classical Paley–Wiener theorem there exists $g_a \in L^2((-\infty, 0), d\xi)$ such that

$$F(\lambda) = \mathcal{S}(F_a)(\lambda - a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{(\lambda - a)\xi} g_a(\xi) \, d\xi \,,$$

for $\operatorname{Re} \lambda > a$. By the uniqueness of the Fourier transform, $e^{-a\xi} g_a = e^{-a'\xi} g_{a'}$ for every $a, a' > 0$.

Hence,

$$F(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{\lambda \xi} \varphi(\xi) \, d\xi \,, \tag{21}$$

where, for $\xi < 0$, $\varphi(\xi) = e^{-a\xi} g_a(\xi)$, any $a > 0$, so that $e^{a\xi} \varphi \in L^2((-\infty, 0), d\xi)$ for every $a > 0$. Thus the integral in (21) is indeed absolutely convergent and by Plancherel theorem again,

$$
\begin{aligned}
\| F \|_{A_\omega^2}^2 &= \sup_{r>0} \int_{0}^{+\infty} \int_{\mathbb{R}} |\widehat{F_{x+r}}(\xi)|^2 \, d\xi \, d\nu(x) \\
&= \sup_{r>0} \int_{0}^{+\infty} \int_{-\infty}^{0} e^{2(x+r)\xi} |\varphi(\xi)|^2 \, d\xi \, d\nu(x) \\
&= \sup_{r>0} \int_{-\infty}^{0} \left(\int_{0}^{+\infty} e^{2x\xi} \, d\nu(x) \right) e^{2r\xi} |\varphi(\xi)|^2 \, d\xi \\
&= \int_{-\infty}^{0} |\varphi(\xi)|^2 v(\xi) \, d\xi \,,
\end{aligned}
$$

as we wished to show. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

As usual, given a Paley–Wiener theorem for A_ω^2, it is possible to obtain its reproducing kernel. Indeed,

Theorem 4.4. *The reproducing kernel for A_ω^2 is*

$$K(z, w) = \frac{1}{2\pi} \int_{-\infty}^{0} e^{(z+\overline{w})\xi} \frac{d\xi}{v(\xi)}.$$

Proof. With the notation as in Theorem 4.3, to every function G in A_ω^2 we associate the function φ_G in $L^2((-\infty, 0), v(\xi)d\xi)$ such that $T\varphi_G = G$.

From Proposition 4.1 we know that A_ω^2 is a reproducing kernel Hilbert space. Let K_z be the reproducing kernel in A_ω^2, that is, for $F \in A_\omega^2$ and $z \in \mathcal{R}$ it holds that $F(z) = \langle F, K_z \rangle_{A_\omega^2}$.

Then,

$$
\begin{aligned}
F(z) &= \int_{0}^{+\infty} \langle F(u + i\cdot), K_z(u + i\cdot) \rangle_{L^2(\mathbb{R})} d\nu(u) \\
&= \int_{0}^{+\infty} \langle [F(u + i\cdot)]\widehat{\,}, [K_z(u + i\cdot)]\widehat{\,} \rangle_{L^2(\mathbb{R})} d\nu(u) \\
&= \int_{0}^{+\infty} \langle e^{u\xi}\varphi_F, e^{u\xi}\varphi_{K_z} \rangle_{L^2(\mathbb{R})} d\nu(u) \\
&= \int_{-\infty}^{0} \varphi_F(\xi)\overline{\varphi_{K_z}(\xi)} v(\xi)\, d\xi,
\end{aligned}
$$

where switching the integral with the sum is justisfied since the last integral converges absolutely.

On the other hand,

$$F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{z\xi} \varphi_F(\xi)\, d\xi,$$

so that

$$\varphi_{K_z}(\xi) = \frac{1}{\sqrt{2\pi}} \frac{1}{v(\xi)} e^{\overline{z}\xi},$$

and

$$K_z(w) = \frac{1}{2\pi} \int_{-\infty}^{0} e^{w\xi} e^{\overline{z}\xi} \frac{1}{v(\xi)}\, d\xi$$

as we wished to show. □

4.2 The space $L^2(\overline{\mathcal{R}}, d\omega) \cap \mathrm{Hol}(\overline{\mathcal{R}})$

We take the opportunity to discuss the naive definition of Bergman space with respect to a general measure such as $\omega_{a,\rho}$. Define $\mathcal{X}_{a,\rho}^2$ as the closure of the holomorphic functions on $\mathcal{R}$ that extend continuously to the boundary and in the norm of $L^2(\overline{\mathcal{R}}, d\omega_{a,\rho})$.

We show that such definition is quite inadequate from the point of view of complex analysis. For, we prove the following

Proposition 4.5. *The following properties hold:*

(1) the set of points $z \in \mathcal{R}$ such that $\mathcal{X}_{a,\rho}^2 \ni f \mapsto f(z)$ is unbounded is dense in $\overline{\mathcal{R}}$;

(2) $\mathcal{X}_{a,\rho}^2$ contains functions that are not holomorphic in $\mathcal{R}$.

Proof. We prove the statement for the space $\mathcal{X}_{2,1}^2$, the proof in the general case being completly analogous.

(1) Let $h(z) = (1 + z)^{-1} \exp\{ie^{2\pi iz}\}$ and set $f_k(z) = h(kz)$. Observe that $|\exp\{ie^{2\pi ik(\frac{n}{2}+iy)}\}| = |\exp\{ie^{-2k\pi y}\cos(kn\pi)\}| = 1$. Then

- $f_k(\frac{n}{2} + iy) \to 0$ a.e. *dy* as $k \to +\infty$;

- $\|f_k\|_{L^2(\mathcal{R},d\omega_{2,1})}^2 = \pi \sum_{n=0}^{+\infty} \frac{2^n}{n!}\left(1 + k\frac{n}{2}\right)^{-2} \to 0$, as $k \to +\infty$,

as an easy calculation shows. Now let $z \in \mathcal{R}$ with Re $z > 0$ rational, equal to p/q with p, q relatively prime and $q \neq 2$. Such points are dense in $\overline{\mathcal{R}}$ and

$$f_k(z) = \frac{1}{1 + \frac{kp}{q} + iky} \exp\{ie^{-2k\pi y}\left(\cos(kp\pi/q) + i\sin(kp\pi/q)\right)\}.$$

If we choose $k = \ell_0 + 2q\ell, \ell = 1, 2, \ldots$ we see that

$$|f_k(z)| \approx \ell^{-1}e^{-2(\ell_0+2q\ell)y}\sin(\ell_0 p\pi/q).$$

Choosing ℓ_0 such that $\sin(\ell_0 p\pi/q)$ we see that $|f_k(z)| \to +\infty$ if $y < 0$. Thus, the point evalution at $z = \frac{p}{q} + iy$ with $y < 0$ are not bounded on $\mathcal{X}_{2,1}^2$.

To deal with the case $y > 0$ it suffices to replace $\exp\{ie^{2\pi iz}\}$ with $\exp\{ie^{-2\pi iz}\}$ in the definition of f_k.

(2) Let

$$f_k(z) = \frac{1}{1+z}\frac{\exp\{ie^{4k\pi iz} - 1\}}{e^{4k\pi iz}}.$$

Since

$$f_k(\tfrac{n}{2} + iy) = \frac{1}{1 + \frac{n}{2} + iy}\frac{\exp\{ie^{-4k\pi y} - 1\}}{e^{-4k\pi y}}$$

and for $t > 0$, $\left|\frac{e^{it}-t}{t}\right| \leq C$, we have

$$\|f_k\|_{L^2(\mathcal{R},d\omega 2,1)}^2 = \sum_{n=0}^{+\infty}\frac{2^n}{n!}\int_{\mathbb{R}}|f_k(\tfrac{n}{2} + iy)|^2\,dy$$

$$\leq C\sum_{n=0}^{+\infty}\frac{2^n}{n!}\int_{\mathbb{R}}\frac{1}{\left(1 + \frac{n}{2}\right)^2 + y^2}\,dy$$

$$\leq C.$$

Then, $f_k \in \mathcal{H}$ for $k = 1, 2, \ldots$. Moreover, $f_k \to g$ ω-a.e. where,

$$g(\tfrac{n}{2} + iy) = \begin{cases} i(1 + \frac{n}{2} + iy)^{-1} & y > 0 \\ 0 & y < 0. \end{cases}$$

It is now easy to see that $f_k \to g$ in $L^2(\mathbb{R}, d\omega)$. Since g cannot be extended to a holomorphic function on $\mathcal{R}$, this concludes the proof. □

5 Final remarks

It is interesting to notice that the space $\mathcal{H}$ defined in (6), and that in [25] we showed to be equal to $M_\Delta(A^2\Delta)$, had already appeared in the literature, in a different context [15, 16]. However, $\mathcal{H}$ can be described as the closure of polynomials in the $L^2(\mathcal{R}, d\mu)$-norm where $d\mu(z) = 2^{-x}|\Gamma(1 + z)|^2 dA(z)$, which is not translation invariant in $\mathcal{R}$. In [15] the authors also discussed a Müntz–Szász-type question, concerning the completeness of the powers $\{(1 - z)^{\lambda_n}\}$ in $H^2(\mathbb{D})$, for $\lambda_n > 0$ and $\lambda_{n+1} - \lambda_n > \delta$; their results however have no (obvious) connection with the Müntz–Szász problem for the Bergman space.

Of course, there exist other papers dealing with Müntz–Szász-type questions. In [26] A. Sedletkskii studied the completeness of sets of exponentials in weighted L^p spaces on $(0, +\infty)$ in terms of zeros of functions in the classical Bergman space on a half-plane.

In the recent years, in a series of interesting papers [22–24] Peláez and Rättyä studied weighted Bergman spaces on the unit disk, with respect to radial weights satisfying a doubling condition, and therefore are allowed to vanish at the boundary of finite order. In contrast, our spaces $\mathcal{M}_\omega^2$ allow exponential growth, as we have pointed out.

In [18, 19] E. Lukacs studied positive measures ν_L on the real line that are Fourier transform of restriction of entire functions. Then, we can consider the measures on $\overline{\mathcal{R}}$ of the form $\nu_L \otimes dy$, (with ν_L restricted to $[0, +\infty)$). We believe these measures constitute an interesting class of measures for which studying the properties of the function spaces $\mathcal{M}_\omega^p$. We wish to come back to this, and the other open problems we mentioned, in a future work.

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