Topics in the arithmetic of del Pezzo and K3 surfaces

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Samenstelling van de promotiecommissie:

Promotor: Prof. dr. Peter Stevenhagen

Promotor: Prof. dr. Lambertus van Geemen (Università degli studi di Milano)

Copromotor: Dr. Ronald M. van Luijk

Overige leden:

Prof. dr. Adrianus W. van der Vaart

Prof. dr. Anthony Várilly-Alvarado (Rice University)

Prof. dr. Jaap G. Top (Rijksuniversiteit Groningen)

Dr. Martin Bright

Prof. dr. Sebastiaan J. Edixhoven

Prof. dr. Bart de Smit

Dr. Elisa Lorenzo García

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A Rino, Paola, e Gaetano.

The mathematician's patterns, like
the painter's or the poet's, must be
beautiful; the ideas, like the colours or
the words, must fit together in a
harmonious way.
Beauty is the first test:
there is no permanent place in the
world for ugly mathematics.

A mathematician's apology,
G. H. Hardy,
1940.

Contents

Introduction

The present thesis is a collection of results about problems that, during the last four years, have challenged the author. The line connecting the works presented here is the study of the arithmetic of surfaces that are double covers of the projective plane, ramified along a curve of low degree: in particular del Pezzo and K3 surfaces.

In Chapter 1, we recall some preliminary results about lattice theory and algebraic geometry. After giving the definition of a lattice and basic properties of integral lattices, the focus shifts towards algebraic geometry. Namely, the definitions of weighted projective spaces, double covers of surfaces, Picard groups, K3 surfaces, and del Pezzo surfaces are given, together with some properties of these objects that will be of use at a later stage.

The topic of Chapter 2 is the arithmetic of del Pezzo surfaces of degree 2 over finite fields. Del Pezzo surfaces can be classified using their *degree*, that is always an integer between 1 and 9. Morally, the higher the degree the *easier* the surface. For example, the projective plane $\mathbb{P}^2$ is a del Pezzo surface of degree 9; the blow-up of $\mathbb{P}^2$ at one point, and $\mathbb{P}^1 \times \mathbb{P}^1$ are del Pezzo surfaces of degree 8; smooth cubics in $\mathbb{P}^3$ are del Pezzo surfaces of degree 3; double covers of $\mathbb{P}^2$ ramified along a smooth quartic curve give examples of del Pezzo surfaces of degree 2.

It is a fact that every del Pezzo surface over an algebraically closed field is birationally equivalent to $\mathbb{P}^2$ (see [Man86, Theorem IV.24.4]). Over arbitrary fields, the situation is more complicated, and so it is easier to look at weaker notions. Let k be any field and let X be a variety of dimension n over k. The variety X is said to be *unirational* if there exists a dominant rational map $\mathbb{P}^n \dashrightarrow X$, defined over k.

Work of B. Segre, Yu. Manin, J. Kollár, and M. Pieropan prove that every del Pezzo surface of degree $d \geq 3$ defined over k is unirational, provided that the set $X(k)$ of rational points is non-empty. C. Salgado, D. Testa, and A. Várilly-Alvarado prove that all del Pezzo surfaces of degree 2 over a finite field are unirational as well, except possibly for three isomorphism classes of surfaces (see [STVA14, Theorem 1]). In Chapter 2 it is shown that these remaining three cases are also unirational, thus proving the following theorem.

Theorem A. *Every del Pezzo surface of degree 2 over a finite field is unirational.*

A more general criterion for unirationality of del Pezzo surfaces of degree 2 is also given.

Theorem B. *Suppose k is a field of characteristic not equal to 2, and let $\overline{k}$ be an algebraic closure of k. Let X be a del Pezzo surface of degree 2 over k. Let $B \subset \mathbb{P}^2$ be the branch locus of the anti-canonical morphism $\pi\colon X \to \mathbb{P}^2$. Let $C \subset \mathbb{P}^2$ be a projective curve that is birationally equivalent with $\mathbb{P}^1$ over k. Assume that all singular points of C that are contained in B are ordinary singular points. Then the following statements hold.*

1. *Suppose that there is a point $P \in X(k)$ such that $\pi(P) \in C - B$. Suppose that B contains no singular points of C and that all intersection points of B and C have even intersection multiplicity. Then the surface X is unirational.*

2. *Suppose that one of the following two conditions hold.*

   (a) There is a point $Q \in C(k) \cap B(k)$ that is a double or a triple point of C. The curve B contains no other singular points of C, and all intersection points of B and C have even intersection multiplicity.

   (b) There exist two distinct points $Q_1, Q_2 \in C(\overline{k}) \cap B(\overline{k})$ such that B and C intersect with odd multiplicity at Q_1 and Q_2

and with even intersection multiplicity at all other intersection points. Furthermore, the points Q_1 and Q_2 are smooth points or double points on the curve C, and B contains no other singular points of C.

Then there exists a field extension ℓ of k of degree at most 2 for which the preimage $\pi^{-1}(C_\ell)$ is birationally equivalent with $\mathbb{P}^1_\ell$; for each such field ℓ, the surface X_ℓ is unirational.

All these results are part of joint work with Ronald van Luijk; Theorem A has been published in [FvL16]; everything contained in Chapter 2 can also be found in [FvL15].

While Chapter 2 is devoted to the study of the arithmetic of del Pezzo surfaces, Chapter 3 deals with the arithmetic of K3 surfaces. K3 surfaces are a possible 2-dimensional generalisation of elliptic curves, and in the last sixty years they have attracted a growing attention since they are on the boundary between those surface whose geometry and arithmetic we understand pretty well, and those whose geometry and arithmetic is still obscure to us. Smooth quartic surfaces in $\mathbb{P}^3$ are examples of K3 surfaces, as well as double covers of $\mathbb{P}^2$ ramified along a smooth sextic curve.

Let X be a K3 surface. The study of the Picard lattice $\operatorname{Pic} X$ can give information about the arithmetic and the geometry of X. Even though during the last years a range of techniques and theoretical algorithms to compute the Picard lattice have been developed (see Chapter 3 and [PTvL15] for references), we do not know yet of any practical algorithm to compute the Picard lattice of a K3 surface.

In the chapter, the following family of K3 surfaces over $\mathbb{Q}$ is considered:

$$\mathfrak{X}\colon w^2 = x^6 + y^6 + z^6 + tx^2y^2z^2.$$

Let t_0 be an element of $\overline{\mathbb{Q}}$. Then X_{t_0} denotes denotes the member of $\mathfrak{X}$ for $t = t_0$, that is, X_{t_0} is the surface over $\overline{\mathbb{Q}}$ given by the equation $w^2 = x^6 + y^6 + z^6 + t_0x^2y^2z^2$. The main result of the chapter is a description of the Picard lattice of the elements of $\mathfrak{X}$, given by the following theorem.

Theorem C. *Let $t_0 \in \overline{\mathbb{Q}}$ be an algebraic number. Then the surface X_{t_0} has Picard number $\rho(X_{t_0}) \in \{19, 20\}$.*

If $\rho(X_{t_0}) = 19$, then the Picard lattice $\operatorname{Pic} X_{t_0}$ is an even lattice of rank 19, determinant $2^5\, 3^3$, signature $(1, 18)$, and discriminant group isomorphic to $C_6 \times C_{12}^2$.

A more explicit description is given in Theorem 3.1.4. This theorem can be used to rule out information about the geometry and the arithmetic of the elements of the family $\mathfrak{X}$. In the last section of the chapter we give some corollaries in this spirit.

The whole Chapter 3 is part of joint work with Florian Bouyer, Edgar Costa, Christopher Nicholls, and Mckenzie West, and it comes from a problem proposed by Anthony Várilly-Alvarado during the Arizona Winter School 2015 (see [VA15, Project 1]).

In Chapter 4 we continue our study of K3 surfaces. Let k be any field, and let x_0, x_1, x_2, x_3 denote the coordinates of $\mathbb{P}^3_k$. Let $X \subset \mathbb{P}^3$ be a surface. We say that X is *determinantal* if it is defined by an equation of the form

$$X\colon\ \det M = 0,$$

where M is a square matrix whose entries are linear homogeneous polynomials in x_0, x_1, x_2, x_3.

Let $L_{(4,2,-4)}$ be the rank 2 lattice with Gram matrix

$$\begin{pmatrix} 4 & 2 \\ 2 & -4 \end{pmatrix}.$$

In [Ogu15], Oguiso shows that a K3 surface S with Picard lattice isometric to $L_{(4,2,-4)}$ admits a fixed point free automorphism g of positive entropy and can be embedded into $\mathbb{P}^3$ as a quartic surface. In the same paper, Oguiso states that "it seems extremely hard but highly interesting to write down explicitly the equation of S and the action of g in terms of the global homogeneous coordinates of $\mathbb{P}^3$, for at least one of such pairs" (cf. [Ogu15, Remark 4.2]). In [FGvGvL13], it is shown that in fact such surfaces can be embedded as determinantal quartic surfaces. In Chapter 4, as well as in the paper, we provide an explicit example of a determinantal quartic surface over $\mathbb{Q}$ with Picard lattice isometric to $L_{(4,2,-4)}$.

Theorem D. *Let $R = \mathbb{Z}[x_0, x_1, x_2, x_3]$ and let $M \in M_4(R)$ be any 4×4 matrix whose entries are homogeneous polynomials of degree 1 and such that M is congruent modulo 2 to the matrix*

$$M_0 = \begin{pmatrix} x_0 & x_2 & x_1 + x_2 & x_2 + x_3 \\ x_1 & x_2 + x_3 & x_0 + x_1 + x_2 + x_3 & x_0 + x_3 \\ x_0 + x_2 & x_0 + x_1 + x_2 + x_3 & x_0 + x_1 & x_2 \\ x_0 + x_1 + x_3 & x_0 + x_2 & x_3 & x_2 \end{pmatrix}.$$

Denote by X the complex surface in $\mathbb{P}^3$ given by $\det M = 0$. Then X is a K3 surface and its Picard lattice is isometric to $L_{(4,2,-4)}$.

This result is part of joint work with Alice Garbagnati, Bert van Geemen, and Ronald van Luijk; all the results contained in Chapter 4 are also exposed in [FGvGvL13]. In the same paper, an explicit description of the action of the fixed point free automorphism with positive entropy of X is also provided, giving a full answer to Oguiso's remark.

Chapter 1

Background

In this chapter we introduce some basic notions that will come in handy later. In Section 1.1 we introduce lattices, focusing on integral lattices and giving some properties that will be mostly used in Chapter 3; in Section 1.2 we introduce some basic notions of algebraic geometry, together with some well and less well known results that are needed to state and prove the results contained in the next chapters.

1.1 Lattice theory warm up

In this section we introduce the notion of lattices together with some basic results for later use. In the first part we follow [vL05, Section 2.1].

For any two abelian groups A and G, a symmetric bilinear map $A \times A \to G$ is said to be *non-degenerate* if the induced homomorphism $A \to \mathrm{Hom}(A, G)$ is injective.

A *lattice* is a free $\mathbb{Z}$-module L of finite rank endowed with a non-degenerate symmetric, bilinear form $b_L\colon L \times L \to \mathbb{Q}$, called the *pairing* of the lattice. If x, y are two elements of L, the notation $x \cdot y$ may be used instead of $b_L(x, y)$, if no confusion arises.

A lattice is called *integral* if the image of its pairing is contained in $\mathbb{Z}$.

An integral lattice L is called *even* if $b_L(x, x) \in 2\mathbb{Z}$ for every x in L.

A sublattice of L is a submodule L' of L such that b_L is non-degenerate on L'.

A sublattice L' of L is called *primitive* if the quotient L/L' is torsion free.

The *signature* of L is the signature of the vector space $L_{\mathbb{Q}} = L \otimes_{\mathbb{Z}} \mathbb{Q}$ together with the inner product induced by the pairing b_L.

Let E and L be two lattices. We define $E \oplus L$ to be the lattice whose underlying $\mathbb{Z}$-module is $E \times L$ and whose pairing $b_{E \oplus L}$ is defined as follows. Let $(e, l), (e', l')$ be two elements of $E \times L$; then we set

$$b_{E \oplus L}\big((e, l), (e', l')\big) := b_E(e, e') + b_L(l, l').$$

Remark 1.1.1. The natural embeddings of E and L into $E \oplus L$ defined by

$$e \mapsto (e, 0)$$

and

$$l \mapsto (0, l)$$

respectively, both respect the intersection pairings on E, L and $E \oplus L$.

If S is a sublattice of a lattice L, then we define its *orthogonal complement*, denoted by $S^{\perp}$, to be the sublattice of L given by

$$S^{\perp} = \{x \in L \mid \forall y \in S,\ b_L(x, y) = 0\ \}.$$

Lemma 1.1.2. *Let S be a sublattice of a lattice L. The following statements hold.*

1. *The orthogonal complement $S^{\perp}$ of S is a primitive sublattice of L and its rank equals* $\mathrm{rk}(L) - \mathrm{rk}(S)$*;*
2. *$S \oplus S^{\perp}$ is a finite-index sublattice of L;*
3. *$(S^{\perp})^{\perp} = S_{\mathbb{Q}} \cap L$.*

Proof. This is a well known result. For a proof, see for example [vL05, Lemma 2.1.5]. □

Let L be a lattice with pairing b_L. With $L(n)$ we denote the lattice with the same underlying module and pairing given by $n \cdot b_L$.

Let L be a lattice of rank n with pairing b_L and fix a basis $(e_1, ..., e_n)$ of L. Then the *Gram matrix* of L with respect to the basis $(e_1, ..., e_n)$ is the $n \times n$ matrix $[b_L(e_i, e_j)]_{1 \le i,j \le n}$.

The *determinant*, also called *discriminant*, of the lattice L, denoted by $\det L$, is the determinant of any Gram matrix of L. One can easily see that the determinant of a lattice is independent of the choice of the basis, and hence of the Gram matrix.

Remark 1.1.3. Let M be an $r \times r$ symmetric $\mathbb{Q}$-matrix with maximal rank. Then $(\mathbb{Z}^r, M)$ denotes the lattice whose underlying $\mathbb{Z}$-module is $\mathbb{Z}^r$ and whose intersection pairing is defined by

$$e_i \cdot e_j := M[i,j]$$

where $e_1 = (1,0,...,0), ..., e_r = (0,...,0,1)$ is the standard basis of $\mathbb{Z}^r$ and $M[i,j]$ is the (i,j)-th entry of the matrix M.

A lattice L is called *unimodular* if $\det L = \pm 1$.

Lemma 1.1.4. *Let E and L be two lattices of rank m and n, and signature (e_+, e_-) and (l_+, l_-), respectively. Then the lattice $E \oplus L$ has*

1. *rank equal to $m+n$,*
2. *determinant equal to $\det E \cdot \det L$,*
3. *signature equal to $(e_+ + l_+, e_- + l_-)$.*

Proof. Fix the bases $(e_1, ..., e_m)$ and $(l_1, ..., l_n)$ for E and L respectively, and let M and N be the the associated Gram matrices. By the definition of the pairing $b_{E\oplus L}$ it follows that the Gram matrix of $E \oplus L$ with respect to the basis $(e_1, ..., e_m, l_1, ..., l_n)$ is the block matrix

$$\begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix}.$$

The statements follow. □

Lemma 1.1.5. *Let S be a finite-index sublattice of a lattice L. Then the determinant of S equals $[L:S]^2 \cdot \det(L)$.*

Proof. [BHPVdV04, Lemma I.2.1]. □

Let L be an integral lattice. We define the *dual lattice* of L to be the lattice

$$L^* = \{x \in L_{\mathbb{Q}} \mid \forall y \in L,\ b_L(x,y) \in \mathbb{Z}\ \}.$$

The pairing on L^* is given by linearly extending b_L to L^*; we will use b_L to also denote the pairing on L^*.

Remark 1.1.6. Sometimes the dual lattice L^* of an integral lattice L is also defined as $\mathrm{Hom}(L, \mathbb{Z})$. The two definitions are equivalent, in fact L^* and $\mathrm{Hom}(L, \mathbb{Z})$ are isomorphic as abelian groups, and the map $\Psi\colon L^* \to \mathrm{Hom}(L, \mathbb{Z})$ defined by $x \mapsto (x^*\colon y \mapsto b_L(x, y))$ is an isomorphism. In order to see it, let $(e_1, ..., e_r)$ be a basis of L, then there exists a basis $(x_1, ..., x_r)$ of L^* such that $x_i \cdot e_j = \delta_{i,j}$; analogously, there is a basis $(y_1, ..., y_r)$ of $\mathrm{Hom}(L, \mathbb{Z})$ such that $y_i(e_j) = \delta_{i,j}$. Obviously $x_i^* = y_i$, and so it follows that Ψ is an isomorphism.

Given an integral lattice L, it is easy to see that L is a sublattice of the dual lattice L^*; nevertheless, the dual lattice L^* does not need to be integral, since there is no condition on $b_L(x, y)$ to be integral for any x, y inside $L^* - L$.

Lemma 1.1.7. *Let L be an integral lattice. Then L is a finite index sublattice of L^* and $|\det L| = [L^* : L]$.*

Proof. Well known result. For a proof we refer to [vL05, Lemma 2.1.13]. □

Remark 1.1.8. From Lemma 1.1.7 it follows that if L is a unimodular lattice, then L is equal to its dual lattice L^*.

Let L be an integral lattice, let $S \subset L$ be a sublattice and let $T = S^\perp$ be its orthogonal complement inside L. We can naturally embed $S \oplus T$ into L, by sending $(s, t) \in S \oplus T$ to $s + t \in L$.

Let x be an element of L. By Lemma 1.1.2, the lattice $S \oplus T$ has finite-index inside L; let m be the index $[L : S \oplus T]$. Then $mx \in S \oplus T$; write $mx = s + t$, for some $s \in S, t \in T$. Consider the element $s/m \in L_\mathbb{Q}$ and let y be an element of S. Since $t \in T = S^\perp$, one has that $y \cdot s = y \cdot (s + t)$. Then $y \cdot s = y \cdot (s + t) = y \cdot (mx) = m(y \cdot x)$, that is, $y \cdot s$ is divisible by m. It follows that $y \cdot (s/m)$ is an integer and so, by the generality of y, the element $s/m \in S_\mathbb{Q}$ is contained in S^*. The same argument holds to show that $t/m \in T^*$.

Then we define a map $L \to S^* \oplus T^*$ by sending $x \in L$ to the element $(s/m, t/m) \in S^* \oplus T^*$. The next lemma shows that this map is a finite-index embedding.

Lemma 1.1.9. *Let L be an integral lattice, and S a sublattice of L. Let $T = S^\perp$ be the orthogonal complement of S inside L. Then the maps* *defined before are finite-index embeddings.*

$$S \oplus T \hookrightarrow L \hookrightarrow S^* \oplus T^*$$

Proof. The first map is trivially an embedding and, by Lemma 1.1.2, $S \oplus T$ has the same rank as L, so the embedding is finite-index.

Also the second map is trivially injective.

The lattice L has finite index inside $S^* \oplus T^*$ since $S^* \oplus T^*$ has, by Lemma 1.1.7, the same rank as $S \oplus T$, that in turn has the same rank as L, as we have seen before. □

Let L be an even lattice with pairing b_L. We define the *discriminant group* of L to be the quotient

$$A_L := L^*/L.$$

The pairing b_L of L induces a map $q_L\colon A_L \to \mathbb{Q}/2\mathbb{Z}$, called *the discriminant quadratic form of* L, defined by $[x] \mapsto b_L(x, x) + 2\mathbb{Z}$. The discriminant group is a finite group, and the minimal number of generators is denoted by $\ell(A_L)$.

Lemma 1.1.10. *The map q_L is well defined and quadratic. The cardinality of A_L equals $|\det L|$.*

Proof. This is a standard result. For a proof see [vL05, Lemma 2.1.17]. □

Lemma 1.1.11. *Let L be an even lattice of rank r, and let A_L denote its discriminant group. Then $\ell(A_L) \leq r$.*

Proof. The group A_L is generated by the classes of the generators of L^*, and L^* has the same rank as L, namely r. □

Let L be a unimodular lattice, and $S \subset L$ a primitive sublattice of L; let T denote the orthogonal complement $S^\perp$ of S inside L. Recall that $\mathrm{Hom}(L, \mathbb{Z})$ and $\mathrm{Hom}(S, \mathbb{Z})$ are isomorphic to L^* and S^*, respectively (cf. Remark 1.1.6); since L is unimodular, then $L = L^*$ (cf. Remark 1.1.8). The restriction map $\mathrm{Hom}(L, \mathbb{Z}) \to \mathrm{Hom}(S, \mathbb{Z})$ induces a map $L \to A_S$.

$$L = L^* \xrightarrow{\cong} \mathrm{Hom}(L, \mathbb{Z}) \longrightarrow \mathrm{Hom}(S, \mathbb{Z}) \xrightarrow{\cong} S^* \longrightarrow S^*/S = A_S$$

The kernel of this map is $S \oplus T$, and so it induces an isomorphism

$$\psi_S\colon L/(S \oplus T) \to A_S.$$

The analogous construction for L and T induces an isomorphism

$$\psi_T\colon L/(S \oplus T) \to A_T.$$

Let $\delta_S\colon A_S \to A_T$ be the isomorphism given by the composition $\psi_T \circ \psi_S^{-1}$.

Proposition 1.1.12. *Let L, S, T and δ_S be defined as before. Then the following diagram commutes.*

$$\begin{array}{ccc} A_S & \xrightarrow[\delta_S]{\cong} & A_T \\ {\scriptstyle q_S}\downarrow & & \downarrow{\scriptstyle q_{S^T}} \\ \mathbb{Q}/2\mathbb{Z} & \xrightarrow[{[-1]}]{} & \mathbb{Q}/2\mathbb{Z} \end{array}$$

Proof. [Nik79, Proposition 1.6.1] or [BHPVdV04, Lemma I.2.5]. □

Let L be a lattice. With $\mathcal{O}(L)$ we denote the group of isometries of L.

Let S be a sublattice of L. With $\mathcal{O}(L)_S$ we denote the group of isometries of L sending S to itself.

An isometry σ of a lattice L extends by linearity to an isometry of L^*. It therefore induces an automorphism $\bar{\sigma}$ of the discriminant group A_L. In this way we define the map $\rho_L\colon \mathcal{O}(L) \to \mathrm{Aut}(A_L)$.

Corollary 1.1.13. *Let L be an even unimodular lattice and S a primitive sublattice of L. Let $T = S^\perp$ denote the orthogonal complement of S inside L. There is an isomorphism ϱ_S between* $\mathrm{Aut}(A_S)$ *and* $\mathrm{Aut}(A_T)$ *making the following diagram commute.*

$$\begin{array}{ccccc} & & \mathcal{O}(L)_S & & \\ & {\scriptstyle \mathrm{res}_S}\swarrow & & \searrow{\scriptstyle \mathrm{res}_T} & \\ \mathcal{O}(S) & & & & \mathcal{O}(T) \\ {\scriptstyle \rho_S}\downarrow & & & & \downarrow{\scriptstyle \rho_T} \\ \mathrm{Aut}(A_S) & & \xrightarrow[\varrho_S]{\cong} & & \mathrm{Aut}(A_T) \end{array}$$

Proof. Let $\delta_S\colon A_S \to A_T$ be the isomorphism as in Proposition 1.1.12. Define $\varrho_S\colon \operatorname{Aut}(A_S) \to \operatorname{Aut}(A_T)$ by

$$\phi \mapsto \delta_S \circ \phi \circ \delta_S^{-1}.$$

First notice that ϱ is bijective, since the map $\operatorname{Aut}(A_S) \to \operatorname{Aut}(A_T)$ defined by

$$\phi \mapsto \delta_S^{-1} \circ \phi \circ \delta_S$$

serves as its inverse.

The commutativity of the diagram follows from the fact that we use δ_S to identify A_S and A_T. See also [Huy15, Lemma 14.2.5]. □

Lemma 1.1.14. *Let L be a unimodular lattice and S a primitive sublattice of L and keep the notation as in Corollary 1.1.13.*

Let $\operatorname{res}_{S,T}\colon \mathcal{O}(L)_S \to \mathcal{O}(S) \times \mathcal{O}(T)$ be the map defined by

$$\alpha \mapsto (\alpha_{|S}, \alpha_{|T}).$$

Then the map $\operatorname{res}_{S,T}$ is well defined, injective, and its image is

$$\{(\beta, \gamma) \in \mathcal{O}(S) \times \mathcal{O}(T) \mid \varrho_S(\rho_S(\beta)) = \rho_T(\gamma)\}.$$

Proof. See [Huy15, Proposition 14.2.6] or [Nik79, Theorem 1.6.1, Corollary 1.5.2]. □

Proposition 1.1.15. *Let L be an even indefinite lattice of signature (m, n) and rank $m+n$, with discriminant lattice A_L. If $\ell(A_L) \leq m+n-2$, then any other lattice with the same rank, signature and discriminant group is isomorphic to L.*

Proof. See [Nik79, Corollary 1.13.3] or [HT15, Proposition 5]. □

Let L be an even lattice, $S \subseteq L$ a finite-index sublattice, and $\iota\colon S \hookrightarrow L$ the inclusion map.

Let $p \in \mathbb{Z}$ be a prime and consider the quotient group L/pL. If x is an element of L, we denote with $[x]_L = x + pL$ its class inside L/pL. The same construction and notation holds if we substitute L with S. When clear from the context, we will drop the subscripts L or S, and we will write simply $[x]$ for $[x]_L$ or $[x]_S$, respectively.

The inclusion map ι induces the homomorphism $\iota_p\colon S/pS \to L/pL$, defined by

$$\iota_p\colon [x]_S \mapsto [x]_L.$$

Remark 1.1.16. Notice that if p is a prime, then S/pS and L/pL are $\mathbb{F}_p$-vector spaces and the homomorphism ι_p is a homomorphism of $\mathbb{F}_p$-vector spaces.

We define S_p to be the kernel of ι_p.

Lemma 1.1.17. *The following equality holds:*

$$S_p = \frac{S \cap pL}{pS}.$$

Proof. The inclusion $\frac{S\cap pL}{pS} \subseteq S_p$ is trivial.

In order to see the other inclusion, let λ be an element of S such that $[\lambda] \in S_p$, that is, $\iota_p([\lambda]) \in pL$. From this it follows that $\lambda = p\lambda'$, for some $\lambda' \in L$. Then $\lambda \in S \cap pL$ and the statement follows. □

Lemma 1.1.18. *Let x, y, x', y' be elements of L such that $[x]_L = [x']_L$ and $[y]_L = [y']_L$. Then $b_L(x, y) \equiv b_L(x', y') \bmod p$.*

Proof. From the hypothesis it follows that there exist two elements $\lambda, \mu \in L$ such that $x' = x + p\lambda$ and $y' = x + p\lambda$. Then

$$\begin{aligned} b_L(x', y') &= b_L(x + p\lambda, y + p\mu) = \\ &= b_L(x, y) + pb_L(x, \mu) + pb_L(\lambda, y) + p^2 b_L(\lambda, \mu) \\ &\equiv b_L(x, y) \bmod p. \end{aligned}$$

□

Using the pairing b_L on L and Lemma 1.1.18, we can define symmetric, bilinear forms on L/pL and S/pS, denoted by

$$b_{L,p} \colon (L/pL)^2 \to \mathbb{Z}/p\mathbb{Z}$$

and

$$b_{S,p} \colon (S/pS)^2 \to \mathbb{Z}/p\mathbb{Z}$$

respectively, both defined by sending $([x], [y])$ to $b_L(x, y) \mod p$.

Lemma 1.1.19. *The following diagram commutes.*

$$\begin{array}{ccc} (S/pS)^2 & \xrightarrow{b_{S,p}} & \mathbb{Z}/p\mathbb{Z} \\ {\scriptstyle \iota_p^2}\downarrow & & \Vert \\ (L/pL)^2 & \xrightarrow[b_{L,p}]{} & \mathbb{Z}/p\mathbb{Z} \end{array}$$

Proof. Let x, y be two elements of S. Then

$$b_{L,p}(\iota_p([x]_S), \iota_p([y]_S)) = b_{L,p}([x]_L, [y]_L) = b_L(x, y) \bmod p.$$

By definition

$$b_{S,p}([x]_S, [y]_S) = b_L(x, y) \bmod p.$$

□

Let $[x]_L$ be an element of L/pL, and define the homomorphism

$$[x]^* \colon S/pS \to \mathbb{Z}/p\mathbb{Z}$$

by sending $[y]_P \in S/pS$ to $b_{L,p}([x], [y])$. In this way we get the morphism

$$\phi_{L,p} \colon L/pL \to \mathrm{Hom}(S/pS, \mathbb{Z}/p\mathbb{Z}),$$

defined by sending $[x]_L$ to $[x]^*$. In the same way, we define the morphism

$$\phi_{S,p} \colon S/pS \to \mathrm{Hom}(S/pS, \mathbb{Z}/p\mathbb{Z}).$$

Let k_p denote the kernel of $\phi_{S,p}$.

Lemma 1.1.20. *The subspace k_p contains S_p and it is fixed by all the isometries of S.*

Proof. First we show $S_p \subseteq k_p$. Let $\mathfrak{x}$ be an element of S_p and fix a representative $x \in S$ of $\mathfrak{x}$, that is $\mathfrak{x} = [x]_S$. By Lemma 1.1.17, there is a $x' \in L$ such that $x = px'$. It follows that

$$[x]^*([y]_S) = [px']^*([y]_S) = b_{L,p}(px', y) = p\, b_{L,p}(x', y) \equiv 0 \bmod p,$$

for any $y \in S$. So $\phi_{S,p}([x]_S) = [x]^* = 0$ and hence $[x]_S \in k_p$.

In order to show that k_p is fixed by the isometries of S, let $[x]_S$ be an element of k_p and σ any isometry of S. Then we have that $[\sigma x]^*([y]_S) = b_L(\sigma x, y) = b_L(x, \sigma^{-1} y)$. Since $[x]_S \in k_p$ we have that $[x]^* = 0$, and so $b_L(x, \sigma^{-1}y) \equiv 0 \mod p$. It follows that, for any $y \in L$, $b_L(\sigma x, y) \equiv 0 \mod p$, and therefore $[\sigma x] \in k_p$. □

Lemma 1.1.21. *The following diagram commutes.*

$$\begin{array}{ccccccc} 0 & \longrightarrow & S_p & \hookrightarrow & S/pS & \xrightarrow{\iota_p} & L/pL \\ & & \downarrow & & \| & & \downarrow \phi_{L,p} \\ 0 & \longrightarrow & k_p & \hookrightarrow & S/pS & \xrightarrow[\phi_{S,p}]{} & \mathrm{Hom}(S/pS, \mathbb{Z}/p\mathbb{Z}) \end{array}$$

Proof. The left square is trivially commutative, since all the maps involved are inclusions.

The right square is also commutative since ι_p preserves the pairing on L/pL (cf. Lemma 1.1.19). □

Remark 1.1.22. Let S be a lattice of rank r and fix a basis $(e_1, ..., e_r)$. Let M be the Gram matrix of S associated to the fixed basis. Then we have that S is isometric to the lattice $(\mathbb{Z}^r, M)$; the isometry is given by sending e_i to the i-th element of the canonical basis of $\mathbb{Z}^r$.

Using this notation, k_p is the subspace of $S/pS \cong (\mathbb{Z}/p\mathbb{Z})^r$ given by the classes of the elements $\underline{x} \in \mathbb{Z}^r$ such that $\underline{x} \cdot M \equiv \underline{0} \mod p$.

Keeping the notation introduced before, let $x \in S$ be such that $[x]_S \in k_p$ and $x^2 \equiv 0 \mod 2p^2$. Let y be another element of S such that $[x]_S = [y]_S$, that is, there is an element $z \in L$ such that $y = x + pz$. It follows that $y^2 = (x+pz)^2 = x^2 + 2px \cdot z + p^2 z^2$. By hypothesis $x^2 \equiv 0 \mod 2p^2$; since $[x]_S \in k_p$, the product $x \cdot z$ is divisible by p, and so $2px \cdot z \equiv 0 \mod 2p^2$; since L, and therefore S, is an even lattice, z^2 is even, and so $p^2 z^2 \equiv 0 \mod 2p^2$; hence $y^2 \equiv 0 \mod 2p^2$. We can then define $k'_p \subset S/pS$ to be the following subset of k_p:

$$k'_p := \{[x]_S \in k_p \mid x^2 \equiv 0 \mod 2p^2\}.$$

Lemma 1.1.23. *The subset $k'_p \subset k_p$ contains S_p and it is invariant under all the isometries of S.*

Proof. First we show that S_p is contained in k'_p. Let $\mathfrak{x}$ be an element of S_p. By Lemma 1.1.17, there is an element $y \in L$ such that $\mathfrak{x} = [py]$. It follows that $\mathfrak{x} = [py + px']$, for any $x' \in S$. Then $\mathfrak{x}^2 = p^2 y^2 + 2p^2 y \cdot x' + p^2 x'^2$. Recall that L is an even lattice, and so $y \cdot x' \in \mathbb{Z}$ and $y^2, x'^2 \in 2\mathbb{Z}$. Then, $\mathfrak{x}^2 \equiv 0 \mod 2p^2$ and thus we have proved $S_p \subseteq k'_p$.

In order to show that k'_p is invariant under the isometries of S consider a class $[x] \in k'_p$ and let σ be an isometry of S. By Lemma 1.1.20 $\sigma[x] \in k_p$. Since σ is an isometry, $(\sigma x)^2 = x^2 \equiv 0 \bmod 2p^2$, and so also σx is an element of k'_p. □

Corollary 1.1.24. *The equality $S = L$ holds if and only if ι_p is injective for every prime p.*

Proof. We only need to prove that if ι_p is injective for every prime p then $S = L$, as the other implication is trivial.

Assume then that ι_p is injective for every prime p. Let λ be an element of L. Since S has finite index inside L, there is a minimal $m \in \mathbb{Z}_{>0}$ such that $m\lambda \in S$.

If $m = 1$ we are done. So assume $m > 1$. Then m can either be a prime or not a prime.

If m is a prime, say q, let $S_q = \frac{S \cap qL}{qS}$ be the kernel of the map $\iota_q \colon S/qS \to L/qL$ (cf. Lemma 1.1.17). Then it follows that $[q\lambda]$ is inside S_q. By assumption, $S_q = \{0\}$. This means that $[q\lambda] = 0$ or, equivalently, that $q\lambda \in qS$. Since S is a torsion-free group (it is a lattice), we can conclude that $\lambda \in S$. But then, by the minimality of m, we get $m = 1$, contradicting the assumption of m to be greater than 1.

If m is not a prime, let p be a prime divisor of m and write $m = pm'$, for some $m' \in \mathbb{Z}$. Using the same argument as before, we show that $m'\lambda$ is in S. In this way we got a $m' < m$ such that $m'\lambda \in S$, contradicting the minimality of m.

This shows that $m = 1$ and so, by generality of λ, we have proved that $S = L$. □

Let L be an integral lattice, and let $S \subset L$ be a finite-index sublattice of L. Let p be a prime, and let e_p denote the dimension of $S_p = \frac{S \cap pL}{pS}$ as $\mathbb{F}_p$-vector space. Let $([y_1], ..., [y_{e_p}])$ be an $\mathbb{F}_p$-basis of S_p. Then there exist $x_1, ..., x_{e_p} \in L - S$ such that $[y_i] = [px_i]$, for $i = 1, ..., e_p$. Let S' be the sublattice of L generated by $S \cup \{x_1, ..., x_{e_p}\}$. Obviously S is a finite-index sublattice of S' and, by construction, we have that pS' is contained in S.

Lemma 1.1.25. *Let L, S, S', e_p and $x_1, ..., x_{e_p} \in L - S$ be defined as before. Then S'/S is an $\mathbb{F}_p$ vector space of dimension e_p.*

Proof. Since pS' is contained in S, the quotient S'/S is an $\mathbb{F}_p$-vector space. We claim that the classes $[x_1], ..., [x_{e_p}]$ form an $\mathbb{F}_p$-basis for S'/S. Clearly, they generate it, since they are the only generators of S' not contained in S. To show that they are linearly independent, assume by contradiction that there are $a_1, ..., a_{e_p} \in \mathbb{F}_p$ such that $a_1[x_1] + ... + a_{e_p}[x_{e_p}] = 0$. This means that if we lift the classes $a_1, ..., a_{e_p} \in \mathbb{F}_p$ to the integers $b_1, ..., b_{e_p} \in \mathbb{Z}$, then $b_1x_1 + ... + b_{e_p}x_{e_p}$ is inside S; so, multiplying by p, it follows that $b_1y_1 + ... + b_{e_p}y_{e_p} \in pS$. This last statement implies that $a_1[y_1] + ... + a_{e_p}[y_{e_p}] = 0 \in S/pS$, contradicting the hypothesis on $([y_1], ..., [y_{e_p}])$ to be an $\mathbb{F}_p$-basis of S_p. Then $([x_1], ..., [x_{e_p}])$ is an $\mathbb{F}_p$-basis for S'/S and the statement follows. ☐

Corollary 1.1.26. *S_p and S'/S are isomorphic as $\mathbb{F}_p$-vector spaces.*

Proof. By Lemma 1.1.25, S'/S is an $\mathbb{F}_p$-vector space of dimension e_p; the $\mathbb{F}_p$-vector space S_p has dimension e_p by definition. So S_p and S'/S are two $\mathbb{F}_p$-vector spaces of the same dimension, hence they are isomorphic. ☐

Remark 1.1.27. A more direct way to show that S_p and S'/S are isomorphic is given by considering the following commutative diagram with exact rows.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & S_p & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S & \xrightarrow{[p]} & S & \longrightarrow & S/pS & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S' & \xrightarrow{[p]} & S' & \longrightarrow & S'/pS' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & S'/S & \xrightarrow{[p]} & S'/S & & & &
\end{array}
$$

Then, applying the snake lemma, we have the exact sequence

$$0 \longrightarrow S_p \longrightarrow S'/S \xrightarrow{[p]} S'/S.$$

Since $pS' \subseteq S$, the map $[p]$ given by the multiplication by p is the zero map. The map $S_p \to S'/S$ is then an isomorphism.

Proposition 1.1.28. *Let p be a prime, and let L, S, S' and e_p be defined as before. Then* $\det S' = p^{-2e_p} \det S$.

Proof. Since S is a finite-index sublattice of S', it follows that the index $[S' : S]$ equals the cardinality of S'/S; by Lemma 1.1.25, the $\mathbb{F}_p$-vector space S'/S has dimension e_p, and so

$$[S' : S] = \#(S'/S) = p^{e_p}.$$

Then, by Lemma 1.1.5, we have that $\det S = p^{2e_p} \det S'$ or, equivalently, $\det S' = p^{-2e_p} \det S$. □

Remark 1.1.29. Since L is an integral lattice, so are S and S', and therefore $\det S$ and $\det S'$ are both integers. It follows that, for any prime p, if p^m is the maximal power of p dividing $\det S$, then $2e_p \leq m$.

As immediate consequence, we have that the map ι_p is injective for all the primes p whose square does not divide $\det S$.

Remark 1.1.30 (Some classic lattices). Here we introduce the notation for some notable lattices. These lattices will be useful later.

With U we denote the lattice of rank 2 and Gram matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Let n be a positive integer.

With A_n we denote the lattice associated to the root system A_n. It is an even, positive definite lattice of rank n and determinant $n+1$. See [CS99, Section 4.6.1] for more information.

With E_8 we denote the lattice associated to the root system E_8. It is an even, positive definite lattice of rank 8 and determinant 1. See [CS99, Section 4.8.1] for more information.

With Λ_{K3} we denote the lattice given by

$$\Lambda_{K3} := U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}.$$

One can immediately notice that Λ_{K3} is an even unimodular lattice of rank 22, determinant -1, and signature $(3, 19)$.

1.2 Geometric background

In this section we give some general definitions and results in algebraic geometry. We focus on the study of surfaces. After giving the definition of surface, we present some well-known results about the Picard group of a surface, double covers, K3 surfaces, and del Pezzo surfaces.

Let k be a field. A *variety* over k is a separated, geometrically reduced scheme X that is of finite type over $\operatorname{Spec} k$.

We say that a variety X is *smooth* if the morphism $X \to \operatorname{Spec} k$ is smooth.

A variety has *pure dimension d* if all its irreducible components have dimension d.
A *curve* is a variety of pure dimension 1.
A *surface* is a variety of pure dimension 2.
A *three-fold* is a variety of pure dimension 3.

Let X be a variety over a field k, and let K be any extension of k. Then we denote by X_K the base-change of X to K. Let $\overline{k}$ be a fixed algebraic closure of k. Then we denote by $\overline{X} := X_{\overline{k}}$ the base-change of X to $\overline{k}$.

1.2.1 The Picard lattice

In this subsection we introduce the notion of *Picard lattice* of a surface. In doing so we basically follow [Har77, Section II.6] and [vL05, Section 2.2].

Let X be a scheme. We define the *Picard group* of X, denoted by $\operatorname{Pic} X$, to be the group of isomorphism classes of invertible sheaves of X (see [Har77, p.143]).

Remark 1.2.1. Equivalently, one can define the Picard group of X as the group $H^1(X, \mathcal{O}^*)$. In fact [Har77, Exercise III.4.5] shows that $\operatorname{Pic} X \cong H^1(X, \mathcal{O}^*)$.

Let X be an irreducible variety over a field k. We define the *Cartier divisor group*, denoted by $\operatorname{CaDiv} X$ to be the group $H^0(X, \mathcal{K}^*/\mathcal{O}^*)$, where $\mathcal{K}$ is the sheaf of total quotient rings of $\mathcal{O}$. A Cartier divisor is *principal* if it is in the image $\operatorname{PCaDiv} X$ of the natural map $H^0(X, \mathcal{K}^*) \to H^0(X, \mathcal{K}^*/\mathcal{O}^*)$. We define the *Cartier divisor class group*, denoted by $\operatorname{CaCl} X$, to be the quotient $\operatorname{CaDiv} X / \operatorname{PCaDiv} X$. For more details about these definitions, see [Har77, p.141], or also [HS00, A.2.2].

Assume X to be smooth, and let $K(X)$ denote the function field of X. We define the *(Weil) divisor group*, denoted by $\operatorname{Div} X$, to be free abelian group generated by all the prime Weil divisors of X. The *group of principal divisors* of X, denoted by $\operatorname{PDiv} X$, is the image of the map $K(X)^* \to \operatorname{Div} X$, defined by sending a function f to the divisor $(f) = \sum_Y v_Y(f)Y$, where the sum is over all the prime Weil divisors Y and $v_Y(f)$ is the valuation of f in the discrete valuation ring associated to the generic point of Y. We define the *(Weil) divisor class group*, denoted by $\operatorname{Cl} X$, to be the quotient $\operatorname{Div} X / \operatorname{PDiv} X$. For more details about these definitions, see [Har77, p.130], or also [HS00, A.2.1].

Proposition 1.2.2. *Let X be an irreducible, smooth variety over a field k. Then there are natural isomorphisms*

$$\operatorname{Div} X \cong \operatorname{CaDiv} X,$$

and

$$\operatorname{Pic} X \cong \operatorname{CaCl} X \cong \operatorname{Cl} X.$$

Proof. See [Har77, Proposition II.6.11] for the proof of $\operatorname{Div} X \cong \operatorname{CaDiv} X$.

See [Har77, Proposition II.6.15] for the proof of $\operatorname{Pic} X \cong \operatorname{CaCl} X$.

See [Har77, Corollary II.6.16] for the proof of $\operatorname{Pic} X \cong \operatorname{Cl} X$. □

Remark 1.2.3. If X is a smooth, irreducible variety, then we can identify Weil divisors and Cartier divisors. We will then simply talk about divisors, without specifying 'Weil' or 'Cartier'. In general, if we leave out this specification, a divisor is intended to be a Weil divisor.

From now on, let X be a projective, smooth, geometrically irreducible surface over a field k. Fix an algebraic closure $\overline{k}$ of k and let $\overline{X} = X_{\overline{k}}$ denote the base-change of X to $\overline{k}$.

Theorem 1.2.4. *There is a unique pairing* $\operatorname{Div} \overline{X} \times \operatorname{Div} \overline{X} \to \mathbb{Z}$, *denoted by $C \cdot D$ for any two divisors C, D, such that*

1. *if C and D are nonsingular curves meeting transversally, then $C \cdot D = \#(C \cap D)$, the number of points of $C \cap D$;*
2. *$C \cdot D = D \cdot C$;*
3. *$(C_1 + C_2) \cdot D = C_1 \cdot D + C_2 \cdot D$;*
4. *if D is a principal divisor then $D \cdot C = 0$, for any divisor C.*

Proof. [Har77, Theorem V.1.1]. □

We call this unique pairing on $\operatorname{Div} \overline{X}$ the *intersection pairing* of X. Let k_1 be an extension of k such that $k \subseteq k_1 \subseteq \overline{k}$. Then the intersection pairing of X restricts to a pairing on $\operatorname{Div} X_{k_1}$; in particular, it restricts to a pairing on $\operatorname{Div} X$.

Remark 1.2.5. From Theorem 1.2.4.(4), it immediately follows that the intersection pairing of X induces a pairing on $\operatorname{Cl} X \cong \operatorname{Pic} X$.

Let $D, E \in \operatorname{Div} X$ be two divisors of X. We say that D and E are *linearly equivalent*, denoted by $D \sim_{\text{lin}} E$, if and only if they have the same class inside $\operatorname{Cl} X \cong \operatorname{Pic} X$.

Remark 1.2.6. Trivially, $\operatorname{Div} X / \sim_{\text{lin}} = \operatorname{Cl} X$.

Let T be a non-singular curve. We define an *algebraic family of effective divisors on X parametrised by T* to be an effective Cartier divisor $\mathcal{D}$ on $X \times T$, flat over T (cf. [Har77, Example III.9.8.5]).

Let D and E be two divisors of X. We say that D and E are *prealgebraically equivalent* if and only if there are two non singular curves T_1, T_2 defined over $\overline{k}$, two algebraic families $\mathcal{D}_1$ and $\mathcal{D}_2$ of effective divisors on $\overline{X}$ parametrised by T_1 and T_2 respectively, two closed fibers D_1, E_1 of $\mathcal{D}_1$ and two closed fibers D_2, E_2 of $\mathcal{D}_2$, such that $D = D_1 - D_2,\ E = E_1 - E_2$. We say that D and E are *algebraically equivalent*, denoted by $D \sim_{\text{alg}} E$, if there is a chain of divisors $D = C_0, C_1, ..., C_n = E$ in $\operatorname{Div} \overline{X}$ such that C_i and C_{i+1} are prealgebraically equivalent, for $i = 0, ..., n-1$. Let $\operatorname{Div}^0_{\text{alg}} X$ be the group of divisors of X that are algebraically equivalent to 0, and let $\operatorname{Pic}^0_{\text{alg}} X$ be its image inside $\operatorname{Pic} X$. We define the *Néron–Severi group* of X, denoted by $\operatorname{NS} X$, to be the quotient $\operatorname{Div} X / \operatorname{Div}^0_{\text{alg}} X$. For more details about these definitions see [Har77, Exercise V.1.7].

Theorem 1.2.7 (Néron–Severi)**.** *Let X be defined as before. Then $\operatorname{NS} X$ is a finitely generated abelian group.*

Proof. See [LN59] or [Nér52] for a proof with k arbitrary. See [Har77, Appendix B.5] for a proof with $k = \mathbb{C}$. □

Remark 1.2.8. By Theorem 1.2.7, we have that $\operatorname{NS} X \cong \mathbb{Z}^\rho \oplus (\operatorname{NS} X)_{\text{tors}}$, for some integer $\rho \in \mathbb{Z}_{\geq 0}$. We define this $\rho = \rho(X)$ to be the *Picard number* of X. Note that $\rho = \dim_{\mathbb{Q}} \operatorname{NS}(X) \otimes \mathbb{Q}$.

We say that D and E are *numerically equivalent*, using the notation $D \sim_{\text{num}} E$, if and only if $D \cdot C = E \cdot C$ for every divisor $C \in \operatorname{Div} X$. Let $\operatorname{Div}^0_{\text{num}} X$ be the group of divisors of X that are numerically equivalent to 0, and let $\operatorname{Pic}^0_{\text{num}} X$ be its image inside $\operatorname{Pic} X$. We define $\operatorname{Num} X$ to be the quotient $\operatorname{Div} X / \operatorname{Div}^0_{\text{num}}$.

Remark 1.2.9. Let D and E two divisors of X. From Theorem 1.2.4.(4) it immediately follows that if D and E are linearly equivalent, they are numerically equivalent too (cf. Proposition 1.2.11).

Proposition 1.2.10. *The group* $\operatorname{Num} X$ *is a torsion free abelian group.*

Proof. The group $\operatorname{Num} X$ is abelian since it is a quotient of $\operatorname{Div} X$, which is abelian by definition.

In order to see that $\operatorname{Num} X$ is torsion free let D be a divisor of X and let $[D]_{\text{num}}$ its class inside $\operatorname{Num} X$. Assume $m[D]_{\text{num}} = [mD]_{\text{num}} = 0$. This means that $(mD) \cdot C = 0$, for every divisor $C \in \operatorname{Div} X$. It follows that, for every divisor $C \in \operatorname{Div} X$

$$0 = (mD) \cdot C = m(D \cdot C),$$

and so either $m = 0$, or $(D \cdot C) = 0$ for every $C \in \operatorname{Div} X$, i.e., $[D]_{\text{num}} = 0$. Hence $\operatorname{Num} X$ is torsion free. □

Proposition 1.2.11. *Let* D, E *be two divisors of* X*. If* $D \sim_{\text{lin}} E$*, then* $D \sim_{\text{alg}} E$*. If* $D \sim_{\text{alg}} E$*, then* $D \sim_{\text{num}} E$*.*

Proof. See [Har77, Exercise V.1.7.(b) and (c)]. □

Remark 1.2.12. The previous proposition tells us that there are two natural surjections:

$$\operatorname{Pic} X \to \operatorname{NS} X \to \operatorname{Num} X.$$

Remark 1.2.13. From Proposition 1.2.11, we trivially get that:

$$\operatorname{Pic}^0_{\text{alg}} X \subseteq \operatorname{Pic}^0_{\text{num}} X,$$

and that

$$\operatorname{Pic} X / \operatorname{Pic}^0_{\text{alg}} X \cong \operatorname{NS} X,$$
$$\operatorname{Pic} X / \operatorname{Pic}^0_{\text{num}} X \cong \operatorname{Num} X.$$

Proposition 1.2.14. *The natural map* $\operatorname{Num} X \to \operatorname{NS} X/(\operatorname{NS} X)_{\mathrm{tors}}$ *is an isomorphism.*

Proof. It follows from [vL05, Proposition 2.2.17] and Remark 1.2.13. □

Remark 1.2.15. From Proposition 1.2.14, it follows that $\operatorname{Num} X$ is a free $\mathbb{Z}$-module of rank $\rho(X)$; also note that the intersection pairing of X naturally induces a pairing on $\operatorname{Num} X$. Then $\operatorname{Num} X$, endowed with the pairing induced by the intersection pairing, is a lattice of rank $\rho(X)$.

Also, using the surjection $\operatorname{NS} X \to \operatorname{Num} X$, the pairing on $\operatorname{Num} X$ induces a pairing on $\operatorname{NS} X$.

We can summarize the previous definitions and results with the following commutative diagrams with exact rows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{PDiv} X & \hookrightarrow & \operatorname{Div} X & \longrightarrow & \operatorname{Pic} X & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Div}^0_{\mathrm{alg}} X & \hookrightarrow & \operatorname{Div} X & \longrightarrow & \operatorname{NS} X & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Div}^0_{\mathrm{num}} X & \hookrightarrow & \operatorname{Div} X & \longrightarrow & \operatorname{Num} X & \longrightarrow & 0
\end{array}$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Pic}^0_{\mathrm{alg}} X & \hookrightarrow & \operatorname{Pic} X & \longrightarrow & \operatorname{NS} X & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Pic}^0_{\mathrm{num}} X & \hookrightarrow & \operatorname{Pic} X & \longrightarrow & \operatorname{Num} X & \longrightarrow & 0
\end{array}$$

Remark 1.2.16. If the adjective *'geometric'* precedes any of the operators of this subsection introduced so far, then we mean the operator acting on $\overline{X}$ instead of X. For example, the *geometric Picard group* of X is the Picard group of $\overline{X}$, that is, $\operatorname{Pic} \overline{X}$.

Assume k is perfect and let $G_k := \operatorname{Gal}(\overline{k}/k)$ be the absolute Galois group of k, and fix an embedding of $Xbar$ inside a projective space over $\overline{k}$; then G_k acts on the set of prime divisors of $\overline{X}$, by acting on the coefficients of the equations defining them. This induces an action of

G_k on $\operatorname{Div}\overline{X}$ and, since G_k sends principal divisors to principal divisors, it also induces an action of G_k on $\operatorname{Pic}\overline{X}$.

Let $k_1 \subset \overline{k}$ be an algebraic extension of k. Let D be an element of $\operatorname{Div}\overline{X}$. We say that k_1 is *the field of definition of* D if $\operatorname{Gal}(\overline{k}/k_1)$ is the stabilizer of D inside G_k; we say that D *can be defined* over k_1 if $\operatorname{Gal}(\overline{k}/k_1)$ is contained in the stabilizer of D inside G_k.

Analogously, if $[D]$ is an element of $\operatorname{Pic}\overline{X}$, we say that k_1 is *the field of definition of* $[D]$ if $\operatorname{Gal}(\overline{k}/k_1)$ is the stabilizer of $[D]$ inside G_k; we say that $[D]$ *can be defined* over k_1 if $\operatorname{Gal}(\overline{k}/k_1)$ is contained in the stabilizer of $[D]$ inside G_k.

Remark 1.2.17. Let $k_1 \subset \overline{k}$ an algebraic extension of k. Let D be an element of $\operatorname{Div}\overline{X}$ and let $[D]$ denote its class inside $\operatorname{Pic}\overline{X}$. The fact k_1 is the field of definition of $[D]$ does not imply that k_1 is the field of definition of D: there might be an element $\sigma \in \operatorname{Gal}(\overline{k}/k)$ sending D to $D' = \sigma D$, such that $D' \neq D$ but $[D] = [D']$. For the same reason, the fact that $[D]$ can be defined over k_1 does not imply that D can be defined over k_1.

Let X be a surface over $k = \mathbb{C}$. Then we can consider the complex analytic space X_h associated to X. The topological space of X_h has $X(\mathbb{C})$ as underlying set. Let $\mathcal{O}_{X_h}$ denote structure sheaf of X_h. The exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_{X_h} \to \mathcal{O}^*_{X_h} \to 0$$

of sheaves induces an exact sequence of (cohomology) groups

$$0 \to H^1(X_h, \mathbb{Z}) \to H^1(X_h, \mathcal{O}_{X_h}) \to H^1(X_h, \mathcal{O}^*_{X_h}) \to H^2(X_h, \mathbb{Z}) \to \dots .$$

Serre, in [Ser56], showed that $H^i(X_h, \mathcal{O}_{X_h}) \cong H^i(X, \mathcal{O}_X)$ for every i. Since $\operatorname{Pic} X \cong H^1(X, \mathcal{O}^*_X)$ (cf. Remark 1.2.1), we have the following exact sequence of groups.

$$0 \to H^1(X_h, \mathbb{Z}) \to H^1(X, \mathcal{O}_X) \to \operatorname{Pic} X \to H^2(X_h, \mathbb{Z}) \to \dots . \qquad (1.1)$$

Proposition 1.2.18. *The Néron–Severi group* $\operatorname{NS} X$ *is isomorphic to a subgroup of* $H^2(X_h, \mathbb{Z})$ *and the second Betti number* $b_2 = \dim H^2(X_h, \mathbb{C})$ *is an upper bound for the Picard number of* X.

Proof. The image of $H^1(X, \mathcal{O}_X)$ inside $\operatorname{Pic} X$ is exactly $\operatorname{Pic}^0_{\text{alg}} X$ (see [Har77, Appendix B, p. 447]). Recalling that $\operatorname{NS} X \equiv \operatorname{Pic} X / \operatorname{Pic}^0_{\text{alg}} X$ (cf. Remark 1.2.13), the statement immediately follows from the exact sequence (1.1). □

Remark 1.2.19. The pairing of NS X induced by the intersection pairing of X (cf. Remark 1.2.15) corresponds to the cup-product of $H^2(X_h, \mathbb{Z})$.

1.2.2 Weighted projective spaces

In the next sections, we will use the notion of *weighted projective space.* In introducing it we follow [Dol82].

Let $Q = (q_0, ..., q_r)$ be a $r+1$-tuple of positive integers. Let k be any field and let $S(Q)$ be the polynomial algebra $k[T_0, ..., T_r]$ over the field k graded by the conditions

$$\deg T_i = q_i,$$

for $i = 0, ..., r$. We define the *weighted projective space of type* Q, or *weighted projective space with weights* Q, the projective scheme given by $\operatorname{Proj}(S(Q))$, denoted by $\mathbb{P}_k(Q)$.

If $k = \mathbb{Q}$, we might drop the subscript and write $\mathbb{P}(Q)$ for $\mathbb{P}_{\mathbb{Q}}(Q)$.

Example 1.2.20. If $Q = (\underbrace{1, ..., 1}_{r+1})$, then the weighted projective space with weights Q is simply the projective space $\mathbb{P}^r$.

Let $f(T_0, ..., T_1)$ be a homogeneous polynomial of weighted degree d. Then the equation $f(T_0, ..., T_1) = 0$ defines an *hypersurface* of degree d in $\mathbb{P}_k(Q)$. We say that an hypersurface of $\mathbb{P}_k(Q)$ is an *hyperplane* if it has degree $d = 1$.

Example 1.2.21. The equation $T_i = 0$ defines an hyperplane if and only if $q_i = 1$.

Remark 1.2.22. For more theory and results about weighted projective spaces we refer to [Dol82] and [Kol96, V.1.3].

1.2.3 Double covers

In this subsection we introduce the notion of *double cover* of a surface, focusing on double covers of the projective plane. Given a double cover

X of the projective plane, the main goal of the subsection is to give a characterization of the plane curves whose pull-back on X splits into two irreducible components. A large deal of what is described in this subsection is a part of a joint work with Ronald van Luijk, and can be found in [FvL15, Section 5].

Let k be a field with characteristic different from 2, and fix an algebraic closure $\overline{k}$ of k.

Let X, Y be two smooth, projective surfaces defined over k. We say that X is a *double cover* of Y if there is a morphism $f\colon X \to Y$ that is surjective, finite and of degree 2 (see [Har77, Section II.3] for these definitions).

Let $\pi\colon X \to Y$ be a double cover of Y.

By definition of the double cover, the pre-image inside $X(\overline{k})$ of a point of $Y(\overline{k})$ has at most 2 elements. We define the *branch locus* of π, denoted by $B \subset Y(\overline{k})$, to be the subset of $Y(\overline{k})$ defined by

$$\{x \in Y(\overline{k}) \mid \#\pi^{-1}(x) = 1 \}.$$

Proposition 1.2.23. *The branch locus of* π *is a divisor of* Y.

Proof. It follows from [Zar58]. □

We define the *ramification locus* of π, denoted by $R \subset X$, to be the preimage $\pi^{-1}(B)$ on X of the branch locus B.

The double cover $\pi\colon X \to Y$ induces the involution ι_X on X, defined by sending each point $P \in X$ to the unique other point of the fiber $\pi^{-1}(\pi(P))$, unless $P \in R$; if $P \in R$, then ι_X fixes P.

The following definitions are given as in [FvL15, Section 5.1]. Given a curve C over k, the normalisation map $\vartheta\colon \tilde{C} \to C$ is unique up to isomorphism; the curve $\tilde{C}$ is regular and both $\tilde{C}$ and ϑ are called the normalisation of C. For more details, see [Mum99, Theorem III.8.3] for the case that C is irreducible; for the general case, take the disjoint unions of the normalisations of the irreducible components. If P is a singular point of C, we say that P is an *ordinary singular point* if, when we consider the blow up of C at P, all the points above P are smooth.

Recall that the geometric genus $g(C)$ of a geometrically integral curve C over k is defined to be the geometric genus of the unique regular projective geometrically integral model of C; for the definition of geometric genus, see [Har77, p. 181]. If C is itself projective, then this model is the normalisation $\tilde{C}$ of C. Note that we have $g(C_{\overline{k}}) \leq g(C)$ with equality if and only if $\tilde{C}$ is smooth (see [Tat52]). In particular, we have $g(C) = 0$ if and only if $\tilde{C}$ is smooth and C is geometrically rational.

Let $C \subset \mathbb{P}^2_k$ be a curve over k and $S \in C$ a closed point of C with local ring $\mathcal{O}_{S,C}$; let $V \subset \mathbb{P}^2$ be an open neighbourhood of S, and let $C' \subset \mathbb{P}^2_k$ be a curve that in V is given by $h = 0$ for some $h \in \mathcal{O}_{\mathbb{P}^2}(V)$; assume that S does not lie on a common component of C and C'; then the intersection multiplicity $\mu_S(C, C')$ of C and C' at S is the length of the $\mathcal{O}_{S,C}$-module $\mathcal{O}_{S,C}/(h)$. If S is a smooth point of C, then the local ring $\mathcal{O}_{S,C}$ is a discrete valuation ring, say with valuation v_S, and $\mu_S(C, C')$ equals $v_S(h)$.

We extend the notion of intersection multiplicity, replacing the point S on the curve C by a *branch* of C, that is, a point of the normalisation of C.

Let $C \subset \mathbb{P}^2$ be a curve and let $\vartheta\colon \tilde{C} \to C$ be the normalisation of C. Let $T \in \tilde{C}$ be a closed point with local ring $\mathcal{O}_{T,\tilde{C}}$. Let $C' \subset \mathbb{P}^2_k$ be a curve that is given in an open neighbourhood $V \subset \mathbb{P}^2$ of $\vartheta(T)$ by $h = 0$ for some $h \in \mathcal{O}_{\mathbb{P}^2_k}(V)$. If the curves C and C' have no irreducible components in common, then the intersection multiplicity $\mu_T(\tilde{C}, C')$ of $\tilde{C}$ and C' at T is defined to be the length of the $\mathcal{O}_{T,\tilde{C}}$-module $\mathcal{O}_{T,\tilde{C}}/(\vartheta^* h)$.

With the same notation as above, the quantity $\mu_T(\tilde{C}, C')$ is the same as $\mathrm{ord}_T(h)$ as defined in [Ful98, Section 1.2]. Since $\tilde{C}$ is regular, the local ring $\mathcal{O}_{T,\tilde{C}}$ is a discrete valuation ring, say with valuation v_T, and we have $\mu_T(\tilde{C}, C') = v_T(\vartheta^* h)$. If k is algebraically closed, then we have $\mu_T(\tilde{C}, C') = \dim_k \mathcal{O}_{T,\tilde{C}}/(\vartheta^* h)$.

Lemma 1.2.24. *Let $C, C' \subset \mathbb{P}^2_k$ be curves with no common irreducible components, and let $\vartheta\colon \tilde{C} \to C$ be the normalisation of C. Then for every $S \in C$ we have*

$$\mu_S(C, C') = \sum_{T \mapsto S} \mu_T(\tilde{C}, C') \cdot [k(T) : k(S)],$$

where the summation runs over all closed points $T \in \tilde{C}$ with $\vartheta(T) = S$ and where $[k(T) : k(S)]$ denotes the degree of the residue field extension.

Proof. This follows immediately from [Ful98, Example 1.2.3]. □

Let $C, C' \subset \mathbb{P}^2_k$ be curves over k that do not have any components in common. Let Γ denote either C or its normalisation $\tilde{C}$. Then we define the subset $b(\Gamma, C')$ of $\Gamma(\overline{k})$ as

$$b(\Gamma, C') = \{T \in \Gamma(\overline{k}) \;:\; \mu_T(\Gamma, C') \text{ is odd}\}.$$

From now on let X be a smooth, projective, irreducible surface over k, and let $\pi\colon X \to \mathbb{P}^2_k$ be a double cover of the projective plane.

Lemma 1.2.25. *Let D be a geometrically integral curve on X, let $C = \pi(D)$ be its image under π, and assume C is not equal to the branch locus B. Let $\tilde{D}_{\overline{k}}, \tilde{C}_{\overline{k}}$ be the normalisations of $D_{\overline{k}}$ and $C_{\overline{k}}$ respectively. The restriction of π to D induces a morphism $\tilde{\pi}\colon \tilde{D}_{\overline{k}} \to \tilde{C}_{\overline{k}}$. The branch locus of $\tilde{\pi}$ is exactly $b(\tilde{C}, B) \subset \tilde{C}(\overline{k})$.*

Proof. We present the proof as in [FvL15, Lemma 5.4]. Without loss of generality, we assume $k = \overline{k}$. Let ϑ denote the normalisation map $\tilde{C} \to C$. Let $T \in \tilde{C}(k)$ be a point. Since $\tilde{C}$ is regular, the local ring $\mathcal{O}_{T,\tilde{C}}$ is a discrete valuation ring, say with valuation v_T. As the characteristic of k is not equal to 2, there is an open neighbourhood $V \subset \mathbb{P}^2$ of $\vartheta(T)$ and an element $h \in \mathcal{O}_{\mathbb{P}^2}(V)$ such that the double cover $\pi^{-1}(V)$ of V is isomorphic to the subvariety of $V \times \mathbb{A}^1(u)$ given by $u^2 = h$. We denote the image of h in the local ring $\mathcal{O}_{T,\tilde{C}}$ and the function field $k(\tilde{C}) = k(C)$ by h as well. The extension $k(C) \subset k(D)$ of function fields is obtained by adjoining a square root $\eta \in k(D)$ of h to $k(C)$. Note that the degree of the restriction of π to D is 1 if and only if this extension is trivial, i.e., h is a square in $k(C)$. The intersection $B \cap V$ is given by $h = 0$, so we have $\mu_T(\tilde{C}, B) = v_T(h)$. Suppose $T' \in \tilde{D}(k)$ is a point with $\tilde{\pi}(T') = T$. Since the characteristic of k is not equal to 2, the extension $\mathcal{O}_{T,\tilde{C}} \subset \mathcal{O}_{T',\tilde{D}}$ of discrete valuation rings of $k(C)$ and $k(D) = k(C)(\eta)$, respectively, is ramified if and only if $v_T(h)$ is odd, that is, T is contained in $b(\tilde{C}, B)$, which proves the lemma. □

Proposition 1.2.26. *Let D be a geometrically integral projective curve on X, let $C = \pi(D)$ be its image under π, and assume $g(C) = 0$. Assume also that C is not equal to the branch locus B. Let $\tilde{C}$ denote the normalisation of C and set $n = \#b(\tilde{C}, B)$. The following statements hold.*

1. *If $n = 0$, then π restricts to a birational morphism $D \to C$ and $g(D) = 0$.*

2. *If $n > 0$, then π restricts to a double cover $D \to C$ and we have that $g(D) = g(D_{\overline{k}}) = \frac{1}{2}n - 1$.*

Proof. From $g(C) = 0$, we find that the normalisation $\tilde{C}$ is smooth. Since the characteristic of k is not 2 and for any finite separable field extension ℓ of k we have $g(D_\ell) = g(D)$ (see [Tat52, Corollary 2]), we may (and do) replace k, without loss of generality, by a quadratic extension ℓ for which $\tilde{C}(\ell) \neq \emptyset$. Then $\tilde{C}$ is isomorphic to $\mathbb{P}^1$. Let $\tilde{D}$ denote the normalisation of D. The morphism π induces a morphism $\tilde{\pi}\colon \tilde{D} \to \tilde{C} \cong \mathbb{P}^1$ of degree at most 2. We claim that $\tilde{D}$ is smooth. Indeed, if $\deg(\tilde{\pi}) = 1$, then this is clear. If $\deg(\tilde{\pi}) = 2$, then because the characteristic of k is not 2, the curve $\tilde{D}$ can be covered by open affine curves that are given by $y^2 = f(x)$ for some polynomial $f \in k[x]$; the regularity of $\tilde{D}$ implies that each polynomial f is separable, which implies that $\tilde{D}$ is smooth. This shows that $g(D) = g(D_{\overline{k}})$, so we may (and do) replace k, without loss of generality, by $\overline{k}$.

By hypotheses, C does not equal the branch locus B, so we may apply Lemma 1.2.25. The Riemann-Hurwitz formula then yields

$$2g(D) - 2 = 2g(\tilde{D}) - 2 = \deg(\tilde{\pi}) \cdot (2g(\tilde{C}) - 2) + n = n - 2\deg(\tilde{\pi}).$$

If $n = 0$, then we find $\deg(\tilde{\pi}) = 1$ and $g(D) = 0$. If $n > 0$, then $\tilde{\pi}$ is not unramified, so $\deg(\tilde{\pi}) = 2$ and we obtain $g(D) = \frac{1}{2}n - 1$. $\square$

Corollary 1.2.27. *Let $C \subset \mathbb{P}^2$ be a geometrically integral projective curve with $g(C) = 0$ that is not equal to the branch locus B. Let $\tilde{C}$ denote its normalisation and set $n = \#b(\tilde{C}, B)$. The following statements hold.*

1. *If $n = 0$, then there exists a field extension ℓ of k of degree at most 2 such that the preimage $\pi^{-1}(C_\ell)$ consists of two irreducible components that are birationally equivalent with C_ℓ.*

2. *If $n > 0$, then the preimage $\pi^{-1}(C)$ is geometrically integral and has geometric genus $\frac{1}{2}n - 1$.*

Proof. Let $A = \pi^*(C)$ the pullback of the curve C on the surface X. Since C is geometrically integral and $C \neq B$, the curve A is geometrically reduced. The morphism $A \to C$ induced by π has degree 2, and so $\overline{A} = A \times_k \overline{k}$ consists of at most two components. Then there is an extension ℓ of k of degree at most 2 such that the components of A_ℓ are geometrically irreducible. Let ℓ be such an extension and let D be an irreducible component of A_ℓ.

Suppose $n = 0$. Applying Proposition 1.2.26 to D_ℓ and $C_\ell = \pi(D_\ell)$ shows that the morphism $D_\ell \to C_\ell$ induced by π is a birational map. Since $D_\ell \to C_\ell$ has degree 2, there is a unique second component of A_ℓ, which equals $\iota(D_\ell)$. This proves the first statement.

Suppose $n > 0$. By Proposition 1.2.26, the morphism $D_{\overline{k}} \to C_{\overline{k}}$ induced by π has degree 2, so $D_{\overline{k}}$ is the only component of $A_{\overline{k}}$ and therefore A is geometrically integral. Its genus follows from Proposition 1.2.26. □

Remark 1.2.28. In Corollary 1.2.27 the hypotheses do not involve only the curve C, but also its normalisation $\tilde{C}$. In particular, in (1) we assume that $\#b(\tilde{C}, B) = 0$. Even though the cardinalities of $b(\tilde{C}, B)$ and $b(C, B)$ are not always the same, in some cases the equality holds: for example, if C is smooth, then $C \cong \tilde{C}$, and so $\#b(\tilde{C}, B) = \#b(C, B)$; if C is singular, but all the singularities lie outside $C \cap B$, then again the equality holds. For more details about the relation between $\#b(\tilde{C}, B)$ and $\#b(C, B)$, see Propositions 1.2.29 and 1.2.31.

In the previous results, we described the preimage $\pi^{-1}(C) \subset X$ of a curve $C \subset \mathbb{P}^2$, by looking at the intersection points of the branch locus B with the normalisation $\tilde{C}$ of C. It is possible to give an analogous description of $\pi^{-1}(C)$ by looking at the intersection points of B and C itself, if we assume that all singular points of C that lie on B are ordinary singular points.

The following proposition describes the integer n used in Proposition 1.2.26 in terms of C directly.

Proposition 1.2.29. *Let $C, C' \subset \mathbb{P}^2$ be two projective plane curves with no components in common. Let $\tilde{C}$ be the normalisation of C. Assume also that C' is smooth and that all singular points of C that lie on C' are ordinary singularities of C. For each point $S \in C(\overline{k})$, let m_S denote* *the multiplicity of S on C. Then we have*

$$\#b(\tilde{C}, C') = \sum_{S \in C(\overline{k}) \cap C'(\overline{k})} c_S(C, C')$$

with

$$c_S(C, C') = \begin{cases} m_S & \text{if } m_S \equiv \mu_S(C, C') \pmod 2, \\ m_S - 1 & \text{if } m_S \not\equiv \mu_S(C, C') \pmod 2. \end{cases}$$

Proof. Let $\vartheta \colon \tilde{C} \to C$ be the normalisation map. Then we may write $b(\tilde{C}, C') = \bigcup_S b_S(\tilde{C}, C')$ with

$$b_S(\tilde{C}, C') = \{T \in \vartheta^{-1}(S) \ : \ \mu_T(\tilde{C}, C') \text{ is odd}\}$$

and where the disjoint union runs over all $S \in C(\overline{k}) \cap C'(\overline{k})$. Suppose $S \in C(\overline{k}) \cap C'(\overline{k})$. Since C' is smooth and the point S is either smooth or an ordinary singularity on C, at most one of the m_S points $T \in \vartheta^{-1}(S)$ satisfies $\mu_T(\tilde{C}, C') > 1$. Hence, there is a point $T_0 \in \vartheta^{-1}(S)$ such that for all $T \in \vartheta^{-1}(S)$ with $T \neq T_0$ we have $\mu_T(\tilde{C}, C') = 1$ and thus $T \in b_S(\tilde{C}, C')$. Since we are working over an algebraically closed field, Lemma 1.2.24 yields $\mu_S(C, C') = \mu_{T_0}(\tilde{C}, C') + m_S - 1$. Hence, we have $T_0 \in b_S(\tilde{C}, C')$ if and only if m_S and $\mu_S(C, C')$ have the same parity. It follows that $\#b_S(\tilde{C}, C') = c_S(C, C')$. The proposition follows. □

We will continue to use the notation $c_S(C, C')$ of Proposition 1.2.29, which we call the *contribution* of S with respect to C'. We set $c_S(C, C')$ equal to 0 for $S \in C(\overline{k})$ with $S \notin C'$.

Remark 1.2.30. Let $C \subset \mathbb{P}^2$ be a geometrically integral projective curve. The points of contribution 0 with respect to C' are the points of $C(\overline{k})$ that are not on C', together with the smooth points $S \in C(\overline{k})$ for which $\mu_S(C, C')$ is even. The points of contribution 1 are the smooth and double points S of $C(\overline{k})$ with $S \in C'$ for which $\mu_S(C, C')$ is odd. The points of type $m > 1$ are the ordinary singular points S of $C(\overline{k})$ of multiplicity m or $m+1$ with $S \in C'$ for which $\mu_S(C, C') \equiv m \pmod 2$.

Proposition 1.2.31. *Let C and C' be two geometrically integral projective curves in $\mathbb{P}^2$. Let $\tilde{C}$ denote the normalisation of C and let $C^{\mathrm{s}} \subset C(\overline{k})$ denote the set of singular points of C. Assume that C' is smooth and that all singular points of C that lie on C' are ordinary. Then the following statements hold.*

1. *The set $b(\tilde{C}, C')$ is empty if and only if the sets $b(C, C')$ and $C^{\mathrm{s}} \cap C'$ are.*

2. *We have $\#b(\tilde{C}, C') = 2$ if and only if either*

   (a) *$b(C, C') = \emptyset$ and there exists a point $S \in C(\overline{k})$ such that $m_S \in \{2, 3\}$ and $C^{\mathrm{s}} \cap C' = \{S\}$, or*

   (b) *there exist two points of C, say $S_1, S_2 \in C(\overline{k})$, with $S_1 \neq S_2$, such that $b(C, C') = \{S_1, S_2\}$ and $m_{S_1}, m_{S_2} \in \{1, 2\}$ and $C^{\mathrm{s}} \cap C' \subset \{S_1, S_2\}$.*

Proof. Given that the contributions $c_S(C, C')$ are nonnegative, this follows easily from Proposition 1.2.29 and Remark 1.2.30. □

1.2.4 K3 surfaces

In this subsection we briefly introduce the notion of $K3$ surface, giving the definition, some basic properties and some results that will be needed in the following of the thesis. For an extensive study of the topic, see [Huy15]; for more details about K3 surfaces over $\mathbb{C}$, see [BHPVdV04].

Let k be any field. A *K3 surface* over k is a smooth, projective, geometrically irreducible surface X with canonical divisor $K_X \sim_{\mathrm{lin}} 0$ and $H^1(X, \mathcal{O}_X) = 0$. A *complex* K3 surface is a K3 surface defined over $k = \mathbb{C}$.

Let X be a K3 surface over a field k. For $p, q \in \{0, 1, 2\}$, we define the (p, q)-Hodge number as

$$h^{p,q} := \dim H^q(X, \Omega_X^p),$$

where $\Omega_X^q = \bigwedge^q \Omega_X$ is the sheaf of regular q-forms on X.

Remark 1.2.32. Let X be a complex K3 surface. Then one can consider the Hodge structure on $H^i(X, \mathbb{C}) = \bigoplus_{p+q=i} H^{p,q}(X)$, where $H^{p,q}(X)$ denotes the group $H^p(X, \Omega_X^q)$.

For an introduction to Hodge theory on complex surfaces, we refer to [BHPVdV04, Section IV.2]; for an extensive study of Hodge theory on complex manifolds (and not only), see [Voi07].

We present now some basic results about K3 surfaces over any field k first, and then for complex K3 surfaces in particular.

Proposition 1.2.33. *Let X be a K3 over a field k. The the following statements hold.*

1. *Linear, algebraic and numeric equivalences are all equivalent, that is,* $\operatorname{Pic} X \cong \operatorname{NS} X \cong \operatorname{Num} X$.

2. *The Hodge diamond of X is the following.*

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

3. *The Picard number of X is at most 22, that is,*

$$\rho(X) \leq 22.$$

4. *The arithmetic genus p_a of X is*

$$p_a(X) = 1.$$

Proof. 1. [Huy15, Proposition 1.2.4].

2. [Huy15, Subsection 1.2.4].

3. [Huy15, Remark 1.3.7].

4. By definition $p_a = \dim H^2(X, \mathcal{O}_X) - \dim H^1(X, \mathcal{O}_X)$. Since X is a K3 surface, $H^1(X, \mathcal{O}_X) = 0$ and $\mathcal{O}_X \cong \omega_X = \Omega_X^2$, and so it follows that $\dim H^1(X, \mathcal{O}_X) = 0$ and

$$\dim H^2(X, \mathcal{O}_X) = \dim H^2(X, \Omega_X^2) = h^{2,2} = 1,$$

using point (2). Then $p_a = 1 - 0 = 1$. See also [Har77, Exercise III.5.3] and [Huy15, Subsection 1.2.3].

□

From Proposition 1.2.33.(1) it follows that $\operatorname{Pic} X$, endowed with the pairing induced by the intersection pairing of X, is a lattice of rank $\rho(X)$ (cf. Remark 1.2.15), called the *Picard lattice* of X.

Proposition 1.2.34. *The lattice* Pic X *is an even lattice of signature* $(1, \rho(X) - 1)$.

Proof. The parity of Pic X follows from the adjunction formula for surfaces (see [Har77, Proposition V.1.5]), recalling that X is a K3 surface and so $K = 0$.

The signature immediately follows from the Hodge index theorem (cf. [Har77, Theorem V.1.9]). □

Lemma 1.2.35. *Let X be a K3 surface over a field k, and let $D \in \operatorname{Div} X$ be such that $D^2 = -2$. Then either D or $-D$ is linearly equivalent to an effective divisor.*

Proof. Let $\mathcal{L}(D)$ be the sheaf associated to the divisor D, and set $h^i(D) = \dim H^i(X, \mathcal{L}(D))$. By the Riemann–Roch formula we have

$$h^0(D) - h^1(D) + h^2(D) = \frac{1}{2} D \cdot (D - K) + 1 + p_a,$$

(cf. [Har77, Theorem V.1.6]). By Serre duality (cf. [Har77, Theorem III.7.7]), $h^2(D) = h^0(K - D)$. Since X is a K3 surface, $K = 0$ (by definition) and $p_a = 1$ (cf. Proposition 1.2.33); since, by initial assumption, $D^2 = -2$, we have

$$h^0(D) - h^1(D) + h^0(-D) = 1.$$

Since the terms on the left-hand side of the equation are all non-negative integers,

$$h^0(D) + h^0(-D) \geq 1.$$

It follows that $h^0(D) \geq 1$ or $h^0(-D) \geq 1$, that is, D or $-D$ is linearly equivalent to an effective divisor, respectively. □

If X is a K3 surface over $k = \mathbb{C}$, then we have some more results.

Proposition 1.2.36. *Let X be a complex K3 surface. Then the following statements hold.*

1. *The cohomology group $H^2(X, \mathbb{Z})$, endowed with the cup product, is a lattice isomorphic to the lattice Λ_{K3} (cf. Remark 1.1.30).*

2. *There is a primitive embedding of lattices* $\operatorname{Pic} X \hookrightarrow H^2(X, \mathbb{Z})$*. The image of the embedding is* $H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$*.*

3. $\rho(X) \leq 20$*.*

4. *Let* X' *be another complex K3 surface, and assume there is a dominant rational map* $X' \dashrightarrow X$*. Then* $\rho(X) = \rho(X')$*.*

Proof. 1. [BHPVdV04, Proposition VIII.3.3.(ii)].

2. It follows from Lefschetz $(1, 1)$ Theorem (cf. [BHPVdV04, Theorem IV.2.13]).

3. It directly follows from point (2) of this proposition, Remark 1.2.32, and Proposition 1.2.33.(2).

4. [Sch13, Proposition 10.2].

□

Remark 1.2.37. If X is a K3 surface over $\mathbb{C}$, Proposition 1.2.36.(2) tells us that there is a primitive embedding of lattices $\operatorname{Pic} X \hookrightarrow H^2(X, \mathbb{Z})$. If we consider the étale cohomology instead of the singular one, a similar statement holds also for K3 surfaces defined over finite fields, as follows.

Let X be a K3 surface over a finite field k of characteristic p. Let ℓ be a prime different from p and define the étale cohomology groups $H^i_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell)$ and the *Tate twist* $H^i_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(1))$ as in [Mil80].

It turns out that $H^i_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(1))$ is a $\mathbb{Z}_\ell$-module of rank $1, 0, 22, 0, 1$ for $i = 0, 1, 2, 3, 4$ (cf. [Băd01, Section 8.4 and Theorem 10.3]). In particular, $H^2_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(1))$ has rank 22, it is endowed with a perfect pairing with values in $\mathbb{Z}_\ell$, and there is a primitive embedding of lattices $\operatorname{Pic} \overline{X} \otimes \mathbb{Z}_\ell \hookrightarrow H^2_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(1))$, respecting the given pairings (see [Mil80, Remark V.3.29.(d)]).

Remark 1.2.38. The lattice isomorphism $H^2(X, \mathbb{Z}) \cong \Lambda_{K3}$ in Proposition 1.2.36.(1) is not unique, nor canonical. Fixing such an isomorphism ϕ is called a *marking* of X. The pair (X, ϕ) is called a *marked K3 surface.*

Remark 1.2.39. Let X be a K3 surface over a field k. If $\rho(\overline{X}) = 22$, then X is said to be *supersingular.*

If $k \hookrightarrow \mathbb{C}$ and $\rho(\overline{X}) = 20$, then X is said to be *singular.*

Remark 1.2.40. If X is a complex K3 surface, then we define the *transcendental lattice* of X, denoted by $T = T(X)$, to be the orthogonal complement of the image of $\operatorname{Pic} X$ inside $H^2(X, \mathbb{Z})$. Note that from Proposition 1.2.36.(2) one has $H^{2,0}(X) \oplus H^{0,2}(X) \subseteq T(X) \otimes \mathbb{C}$.

In what follows, if X is a complex K3 surface, we will identify $\operatorname{Pic} X$ with its image inside $H^2(X, \mathbb{Z})$.

After giving some basic definitions, we state the Global Torelli Theorem for K3 surfaces, and we show how it can be used to obtain some information about the automorphism group of a complex K3 surface.

Let X and Y be two complex K3 surfaces. A lattice homomorphism ϕ between $H^2(Y, \mathbb{Z})$ and $H^2(X, \mathbb{Z})$ is called a *Hodge isometry* if it preserves the lattice pairing and its $\mathbb{C}$-linear extension $\phi_{\mathbb{C}}$ preserves the Hodge structure, that is, $\phi_{\mathbb{C}}(H^{p,q}(Y)) = H^{p,q}(X)$.

A Hodge isometry is called *effective* if it sends ample classes to ample classes.

Proposition 1.2.41. *Let $f\colon X \to Y$ be an isomorphism between two K3 surfaces. The isomorphism f induces, by pull-back, a lattice homomorphism $f^*\colon H^2(Y, \mathbb{Z}) \to H^2(X, \mathbb{Z})$. The homomorphism f^* is an effective Hodge isometry.*

Proof. Let $f^*\colon H^2(Y, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ be the homomorphism induced by f, by pull-back. Since f is an isomorphism, f^* is an isometry of lattices. The pull-back of a holomorphic 2-form of Y is a holomorphic form of X, hence the $\mathbb{C}$-linear extension $f^*_{\mathbb{C}}$ of f^* sends $H^{2,0}(Y)$ to $H^{2,0}(X)$; since $H^{0,2}(Y) = \overline{H^{2,0}(Y)}$, we also have that $f^*_{\mathbb{C}}$ sends $H^{0,2}(Y)$ to $H^{0,2}(X)$; hence $f^*_{\mathbb{C}}$ sends $H^{2,0}(Y) \oplus H^{0,2}(Y)$ to $H^{2,0}(X) \oplus H^{0,2}(X)$ and therefore also $H^{1,1}(Y)$ to $H^{1,1}(X)$. Thus, f^* is an Hodge isometry.

To show that f^* is also effective, let $D \in \operatorname{Pic} Y$ be a very ample class. Then D gives an embedding $\phi_D\colon Y \to \mathbb{P}^n$, for some integer n, determined by a basis $(s_0, ..., s_n)$ of $H^0(Y, D)$. If f is an isomorphism, then the composition $f \circ \phi_D$ is an embedding of X, given by the elements $f \circ s_i = f^* s_i \in H^0(X, f^* D)$. Thus, also $f^* D$ is a very ample class. Using the linearity of f^*, it follows that f^* sends ample divisor classes to ample divisor classes, proving the statement. □

The previous proposition states that every isomorphism $X \to Y$ of complex K3 surfaces gives an effective Hodge isometry from $H^2(Y, \mathbb{Z})$

to $H^2(X, \mathbb{Z})$. The converse is also true, as shown by the following theorem.

Theorem 1.2.42 (Torelli theorem for K3 surfaces)**.** *Let X, Y be two complex K3 surfaces and let $\phi\colon H^2(Y, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ be an effective Hodge isometry. Then there is a (unique) isomorphism $f\colon X \to Y$ such that $\phi = f^*$.*

Proof. It follows from [BHPVdV04, Theorem VIII.11.1] and [BHPVdV04, Corollary VIII.11.4]. □

Let X be a K3 surface and let f be an automorphism of X. Then f induces, by pull-back, an isometry, say ϕ, of $\operatorname{Pic} X$. Define the map $(\cdot)^*_{\mathrm{Pic}}\colon \operatorname{Aut}(X) \to \mathcal{O}(\operatorname{Pic} X)$ by sending any f to the corresponding ϕ. In general, $(\cdot)^*_{\mathrm{Pic}}$ does not need to be injective, but we will show that in some cases it is so (cf. Proposition 1.2.47).

Remark 1.2.43*.* We have seen that every automorphism f of X induces an effective Hodge isometry f^* of $H^2(X, \mathbb{Z})$ (cf. Proposition 1.2.41). Let $\mathcal{O}_H(H^2(X, \mathbb{Z}))$ denote the subgroup of effective Hodge isometries of $H^2(X, \mathbb{Z})$, and let us identify $\operatorname{Pic} X$ with its image inside $H^2(X, \mathbb{Z})$. Then, by Proposition 1.2.36.(2), $\operatorname{Pic} X = H^{1,1}(X) \cap H^2(X, \mathbb{Z})$, and so an effective Hodge isometry sends the Picard lattice to itself. Then, we can define the restriction map

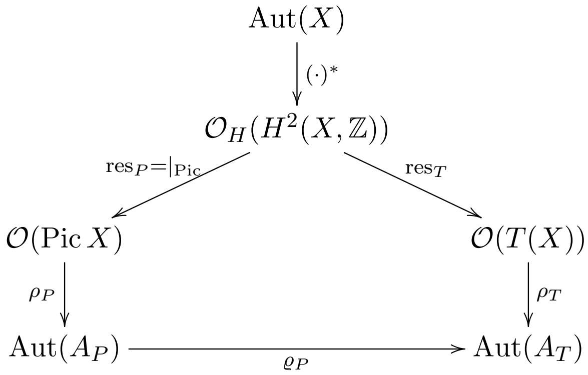
$$|_{\mathrm{Pic}}\colon \mathcal{O}_H(H^2(X, \mathbb{Z})) \to \mathcal{O}(\operatorname{Pic} X)$$

sending an effective Hodge isometry of $H^2(X, \mathbb{Z})$ to the isometry it induces on $\operatorname{Pic} X$. Note that if f is an automorphism of X, then the isometry of $\operatorname{Pic} X$ it induces equals the map $(f^*)|_{\mathrm{Pic}}$. In other words, the following diagram is commutative.

$$\begin{array}{ccc}
\operatorname{Aut}(X) & \xrightarrow{\quad (\cdot)^*_{\mathrm{Pic}} \quad} & \mathcal{O}(\operatorname{Pic} X) \\
{\scriptstyle (\cdot)^*} \searrow & & \nearrow {\scriptstyle |_{\mathrm{Pic}}} \\
& \mathcal{O}_H(H^2(X, \mathbb{Z})) &
\end{array}$$

Thanks to Theorem 1.2.42, we know that the automorphisms of X are in a 1-to-1 correspondence with the effective Hodge isometries of $H^2(X, \mathbb{Z})$.

Let $T(X)$ be the transcendental lattice of X. By Corollary 1.1.13, there is an isomorphism $\varrho\colon \operatorname{Aut}(A_P) \to \operatorname{Aut}(A_T)$ between the automorphism groups of the discriminant groups $A_P = A_{\operatorname{Pic}(X)}$ and $A_T = A_{T(X)}$, making the following diagram commute.



Proposition 1.2.44. *The group $\mathcal{O}_H(H^2(X,\mathbb{Z}))$ is isomorphic to a subgroup of the group*

$$\{(\beta,\gamma) \in \mathcal{O}(\operatorname{Pic} X) \times \mathcal{O}(T(X)) \mid \varrho_P(\rho_P(\beta)) = \rho_T(\gamma)\}. \tag{1.2}$$

Proof. Let $\mathcal{O}_{\operatorname{Pic}}(H^2(X,\mathbb{Z}))$ be the subgroup of $\mathcal{O}(H^2(X,\mathbb{Z}))$ given by all the isometries sending $\operatorname{Pic} X$ to itself. Then, by definition, $\mathcal{O}_H(H^2(X,\mathbb{Z}))$ is contained in $\mathcal{O}_{\operatorname{Pic}}(H^2(X,\mathbb{Z}))$. Also, from Lemma 1.1.14, we have that $\mathcal{O}_{\operatorname{Pic}}(H^2(X,\mathbb{Z}))$ is isomorphic to the group (1.2). The statement follows. $\square$

Corollary 1.2.45. *The group* $\operatorname{Aut}(X)$ *embeds into the group* (1.2) *in Proposition 1.2.44.*

Proof. By the Torelli theorem for K3 surfaces (cf. Theorem 1.2.42), the group $\operatorname{Aut}(X)$ is in 1-to-1 correspondence with $\mathcal{O}_H(H^2(X,\mathbb{Z}))$. By Proposition 1.2.44, $\mathcal{O}_H(H^2(X,\mathbb{Z}))$ is isomorphic to a subgroup of the group (1.2). $\square$

Proposition 1.2.46. *Let X be a K3 surface with odd Picard number. Then $\mathcal{O}(T(X)) = \{\pm \operatorname{id}_T\}$.*

Proof. See Corollary [Huy15, 3.3.5]. $\square$

Proposition 1.2.47. *Let X be a complex K3 surface, and assume that its Picard lattice has odd rank and discriminant not a power of* 2*. Then the map* $(\cdot)^*_{\mathrm{Pic}}\colon \operatorname{Aut}(X) \to \mathcal{O}(\operatorname{Pic} X)$ *is injective.*

Proof. We have seen that the map $(\cdot)^*_{\mathrm{Pic}}$ equals the composition of the pull-back $(\cdot)^*\colon \operatorname{Aut}(X) \to \mathcal{O}_H(H^2(X,\mathbb{Z}))$ and the restriction map $|_{\mathrm{Pic}}\colon \mathcal{O}_H(H^2(X,\mathbb{Z})) \to \mathcal{O}(\operatorname{Pic} X)$ (cf. Remark 1.2.43). It follows that if $\phi \in \mathcal{O}(\operatorname{Pic} X)$ is an element in the image of $\operatorname{Aut}(X)$, then there is an automorphism f of X such that $\phi = f^*_{|\operatorname{Pic} X}$. Now assume that there is also another such automorphism, say f', such that $\phi = (f')^*_{\operatorname{Pic} X}$ or, equivalently, such that $\phi = f'^*_{|\operatorname{Pic} X}$. By Proposition 1.2.44 we have that the automorphisms f and f' respectively correspond to the elements (ϕ, ρ) and (ϕ, ρ') in $\mathcal{O}(\operatorname{Pic} X) \times \mathcal{O}(T(X))$, with ϕ inducing the same automorphism on $A_P = A_T$ as ρ and ρ' respectively.

By Proposition 1.2.46 we have that $\rho, \rho' \in \{\pm \mathrm{id}_{T(X)}\}$. If $\rho = \rho'$, then f and f' correspond to the same element in $\mathcal{O}(\operatorname{Pic} X) \times \mathcal{O}(T(X))$ and therefore they must be equal (cf. Corollary 1.2.45). Then assume, without any loss of generality, that $\rho = \mathrm{id}_T$ and $\rho' = -\mathrm{id}_T$. It follows that ρ induces the identity on A_P and ρ' the multiplication by -1, and they both must be equal to the morphism induced by ϕ. The identity and the multiplication by -1 can be the same map only if A_P is isomorphic to a power of the group $\mathbb{Z}/2\mathbb{Z}$. Since the cardinality of the discriminant group of a lattice equals the determinant of the lattice, and by the initial hypothesis the determinant of $\operatorname{Pic} X$ is not a power of 2, then A_P cannot be isomorphic to a power of $\mathbb{Z}/2\mathbb{Z}$ and therefore ρ and ρ' do not induce the same automorphism of A_P. This way we get a contradiction, coming from the assumption that $\rho \neq \rho'$. Hence, $\rho = \rho'$ and this concludes the argument. □

After talking about automorphism of K3 surfaces in general, we introduce the notion of symplectic automorphisms. Let X be a K3 surface and let f be an automorphism of X. We say that f is *symplectic* if the induced action on $H^0(X, \Omega^2) = H^{2,0}(X)$ is the identity. The symplectic automorphisms of X form a subgroup of $\operatorname{Aut}(X)$, denoted by

$$\operatorname{Aut}_s(X) \subset \operatorname{Aut}(X).$$

Lemma 1.2.48. *Let X be a K3 surface and let f be an automorphism of X. Then f is symplectic if and only if f^* acts as the identity on $T(X)$.*

Proof. See [Huy15, Remark 15.1.2]. □

Proposition 1.2.49. *Let X be a complex K3 surface. Let f be a symplectic automorphism of X, and assume f has finite order n. Then f fixes a finite number of points of X. In particular, if $\#\operatorname{Fix}(f)$ denotes the number of points fixed by f, we have that only the following tuples $(n, \#\operatorname{Fix}(f))$ can and do occur.*

n	2	3	4	5	6	7	8
$\#\operatorname{Fix}(f)$	8	6	4	4	2	3	2
$\rho(X) \geq$	9	13	15	17	17	19	19

The table has been completed by a lower bound for $\rho(X)$ coming from the existence of a symplectic automorphism of order n.

Proof. See [Huy15, Section 15.1.2]. □

We conclude the section by giving some results about families of K3 surfaces.

Theorem 1.2.50. *The family of marked complex K3 surfaces with Picard number at least ρ is parametrised by the union of countably many complex manifolds of dimension $20 - \rho$.*

Proof. It follows from [Dol96, Corollary 3.2]. □

Lemma 1.2.51. *Let $\mathfrak{X} \to \mathbb{A}^1_k$ be a flat proper morphism over a field k, such that its fibers are K3 surfaces. Assume the characteristic of k to be 0. Let η and t be the generic point and a closed point of $\mathbb{A}^1$, respectively, and let X_η and X_t denote the fibers above η and t respectively. Then the specialization map*

$$\operatorname{sp}_t \colon \operatorname{Pic} X_\eta \to \operatorname{Pic} X_t$$

preserves the intersection pairing, is injective and has torsion-free cokernel.

The same holds also if we base-extend X_η and X_t to an algebraic closure of their field of definition.

Proof. It follows from [MP12, Proposition 3.6]. See also [Huy15, Proposition 17.2.10]. □

Lemma 1.2.52. *Let k be a number field and let $\mathcal{O}_k$ be its ring of integers. Let X be a K3 surface over k, and let $\mathfrak{X} \to \operatorname{Spec}(\mathcal{O}_k)$ be an integral model of X. Let $\mathfrak{p}$ be a prime of good reduction for $\mathfrak{X}$, that is, the fiber $\mathfrak{X}_\mathfrak{p}$ is a K3 surface. Then the reduction map*

$$\mathrm{sp}_\mathfrak{p} \colon \operatorname{Pic} X \to \operatorname{Pic} \mathfrak{X}_\mathfrak{p}$$

preserves the intersection pairing, is injective and has torsion-free cokernel.

The same holds also if we base-extend X and $\mathfrak{X}_\mathfrak{p}$ to an algebraic closure of their field of definition.

Proof. Just note that if η is the generic point of $\operatorname{Spec}(\mathcal{O}_k)$, and $\mathfrak{X}_\eta$ denotes the fiber of $\mathfrak{X}$ above η, then $\mathfrak{X}_\eta \cong X$. The result then follows from [MP12, Proposition 3.6]. See also [EJ11, Theorem 3.4] and [Huy15, Remark 17.2.11]. □

1.2.5 Del Pezzo surfaces

In this section we introduce the notion of del Pezzo surface and some basic results about these surfaces, focusing on del Pezzo surfaces of degree 1 and 2. For a general introduction to del Pezzo surfaces we refer to [Man86, Sections IV.24-26] and [Kol96, Section III.3]; another standard reference is also [Dem80].

Let X be a smooth, projective, geometrically irreducible surface over a field k. We say that X is a *del Pezzo surface* if its anti-canonical divisor $-K_X$ is ample. We define the *degree* of X to be the self intersection K_X^2 of its (anti-)canonical divisor.

From now until the end of the subsection, let X denote a del Pezzo surface over k, and let d denote the the degree of X.

Lemma 1.2.53. *Keeping the notation introduced before, one has the following inequality: $1 \leq d \leq 9$.*

Proof. [Man86, Theorem IV.24.3.(i)]. □

Remark 1.2.54. A set of closed points on the plane is said to be in *general position* if no three points lie on a line; no six points lie on a conic; no eight points lie on a singular cubic, with one of the points at the singularity.

Theorem 1.2.55. *Keeping the notation as before, the following statements hold, under the assumption that k is algebraically closed.*

1. *If $d = 9$, then X is isomorphic to $\mathbb{P}^2$.*
2. *If $d = 8$, then X is isomorphic to either $\mathbb{P}^1 \times \mathbb{P}^1$ or to the blow-up of $\mathbb{P}^2$ at one point.*
3. *If $7 \geq d \geq 1$, then X is isomorphic to the blow-up of $\mathbb{P}^2$ at $9 - d$ points in general position.*

If $d \geq 3$, then the converse of the above statements is also true, that is, the blow up of $\mathbb{P}^2$ at $9 - d$ points in general position is a del Pezzo surface of degree d.

Proof. [Man86, Theorem IV.24.4]. □

Remark 1.2.56. For $d \in \{1, 2\}$ stricter conditions on the points are required in order for the converse of Theorem 1.2.55.(3) to hold. See [Man86, Theorem IV.26.2].

Corollary 1.2.57. *Let X be a del Pezzo surface over an algebraically closed field k. Then X is birational to $\mathbb{P}^2_k$.*

Proof. Trivial using Theorem 1.2.55. □

Corollary 1.2.58. *Let X be a del Pezzo surface over k, assume that X is not birational to $\mathbb{P}^1 \times \mathbb{P}^1$ and that k is algebraically closed. Set $r := 9 - d$. Then $\operatorname{Pic} X$, endowed with the intersection pairing of X, is a lattice of rank $r + 1$, admitting a basis $(E_0, E_1, ..., E_r)$ such that*

- $E_0^2 = 1$;
- $E_i^2 = -1$, *for* $i = 1, ..., r$;
- $E_i \cdot E_j = 0$, *for every* $i \neq j$.

Proof. [Man86, Proposition IV.25.1]. □

Proposition 1.2.59. *Let X be a del Pezzo surface of degree d over k.*

If $d = 1$, then X is isomorphic to a hypersurface of degree 6 inside $\mathbb{P}_k(1, 1, 2, 3)$. Conversely, any smooth hypersurface of degree 6 inside $\mathbb{P}_k(1, 1, 2, 3)$ is a del Pezzo surface of degree 1.

If $d = 2$ then X is isomorphic to a hypersurface of degree 4 inside $\mathbb{P}_k(1, 1, 1, 2)$. Conversely, any smooth hypersurface of degree 4 inside $\mathbb{P}_k(1, 1, 1, 2)$ is a del Pezzo surface of degree 2.

Proof. See [Kol96, Theorem III.3.5]. □

Chapter 2

Unirationality of del Pezzo surfaces of degree 2

In this section we will present some results about unirationality of del Pezzo surfaces of degree 2. In particular, we will show that all del Pezzo surfaces of degree 2 over a finite field are unirational. All the material presented in this chapter is part of joint work with Ronald van Luijk, and it can be found in [FvL15]; many of these results have already been published in [FvL16].

2.1 The main results

In Chapter 1 we have already seen that every del Pezzo surface, and so in particular every del Pezzo surface of degree 2, over an algebraically closed field is birational to the projective plane.

The same statement does not need to hold if the field is not algebraically closed, and so we look at weaker notions. Let k be any field and let X be a variety of dimension n over k. We say that X is *unirational* if there exists a dominant rational map $\mathbb{P}^n \dashrightarrow X$, defined over k.

Work of B. Segre, Yu. Manin, A. Knecht, J. Kollár, and M. Pieropan prove that every del Pezzo surface of degree $d \geq 3$ defined over k is unirational, provided that the set $X(k)$ of rational points is non-empty. For references, see [Seg43, Seg51] for $k = \mathbb{Q}$ and $d = 3$, see [Man86, Theorem 29.4 and 30.1] for $d \geq 3$ with the assumption that k is large enough for $d \in \{3, 4\}$. See [Kol02, Theorem 1.1] for $d = 3$ in general. See [Pie12, Proposition 5.19] and, independently, [Kne15, Theorem 2.1] for $d = 4$ in general. Since all del Pezzo surfaces over finite fields have a rational point (see [Man86, Corollary 27.1.1]), this implies that every del Pezzo surface of degree at least 3 over a finite field is unirational.

Building on work by Manin (see [Man86, Theorem 29.4]), C. Salgado, D. Testa, and A. Várilly-Alvarado prove that all del Pezzo surfaces of degree 2 over a finite field are unirational as well, except possibly for three isomorphism classes of surfaces (see [STVA14, Theorem 1]). In this chapter, we show that these remaining three cases are also unirational, thus proving our first main theorem.

Theorem 2.1.1. *Every del Pezzo surface of degree 2 over a finite field is unirational.*

More generally, we give some sufficient conditions for a del Pezzo surface of degree 2 to be unirational.

Theorem 2.1.2. *Suppose k is a field of characteristic not equal to 2, and let $\overline{k}$ be an algebraic closure of k. Let X be a del Pezzo surface of degree 2 over k. Let $B \subset \mathbb{P}^2$ be the branch locus of the anti-canonical morphism $\pi\colon X \to \mathbb{P}^2$. Let $C \subset \mathbb{P}^2$ be a projective curve that is birationally equivalent to $\mathbb{P}^1$ over k. Assume that all singular points of C that are contained in B are ordinary singular points. Then the following statements hold.*

1. *Suppose that there is a point $P \in X(k)$ such that $\pi(P) \in C - B$. Suppose that B contains no singular points of C and that all intersection points of B and C have even intersection multiplicity. Then the surface X is unirational.*

2. *Suppose that one of the following two conditions hold.*

   (a) *There is a point $Q \in C(k) \cap B(k)$ that is a double or a triple point of C. The curve B contains no other singular points of C, and all intersection points of B and C have even intersection multiplicity.*

(b) There exist two distinct points $Q_1, Q_2 \in C(\overline{k}) \cap B(\overline{k})$ such that B and C intersect with odd multiplicity at Q_1 and Q_2 and with even intersection multiplicity at all other intersection points. Furthermore, the points Q_1 and Q_2 are smooth points or double points on the curve C, and B contains no other singular points of C.

Then there exists a field extension ℓ of k of degree at most 2 for which the preimage $\pi^{-1}(C_\ell)$ is birationally equivalent with $\mathbb{P}^1_\ell$; for each such field ℓ, the surface X_ℓ is unirational.

Corollary 2.1.3. *Suppose k is a field of characteristic not equal to 2. Let X be a del Pezzo surface of degree 2 over k. Assume that X has a k-rational point, say P. Let $C \subset \mathbb{P}^2$ be a geometrically integral curve over k of degree $d \geq 2$ and suppose that $\pi(P)$ is a point of multiplicity $d-1$ on C. Suppose, moreover, that C intersects the branch locus B of the anti-canonical morphism $\pi\colon X \to \mathbb{P}^2$ with even multiplicity everywhere. Then the following statements hold.*

1. *If $\pi(P)$ is not contained in B, then X is unirational.*

2. *If $\pi(P)$ is contained in B, it is an ordinary singular point on C and we have $d \in \{3, 4\}$, then there exists a field extension ℓ of k of degree at most 2 for which the preimage $\pi^{-1}(C_\ell)$ is birationally equivalent with $\mathbb{P}^1_\ell$; for each such field ℓ, the surface X_ℓ is unirational.*

In the next section, we will present the three difficult surfaces and prove Theorem 2.1.1. The main tool is Lemma 2.2.2, which states that it suffices to construct a rational curve on each of the three del Pezzo surfaces.

Recall that if X is a del Pezzo surface of degree 2, then X admits 56 exceptional curves (cf. [Man86, Theorem IV.26.2]). A point on X is called a *generalised Eckardt point* if it lies on four of the 56 exceptional curves.

If a point P on a del Pezzo is not a generalised Eckardt point, and it does not lie on the ramification locus of the anti-canonical morphism, then Manin's construction, extended by C. Salgado, D. Testa, and A. Várilly-Alvarado, yields a rational curve that satisfies the assumptions of case (1) of Corollary 2.1.3 with the degree d being such that there are $4 - d$ exceptional curves through P (cf. Example 2.3.7).

The three difficult surfaces do not contain such a point. The proofs of unirationality of these three cases use a rational curve that is an example of case (2) of Corollary 2.1.3 instead (cf. Remark 2.2.4 and Example 2.3.9). Here we benefit from the fact that if k is a finite field, then any curve that becomes birationally equivalent with $\mathbb{P}^1$ over an extension of k, already is birationally equivalent with $\mathbb{P}^1$ over k itself. For two of the three cases, the rational curve we use has degree 4. For the last case, the curve we use has degree 3, but there also exist quartic curves satisfying the hypotheses of case (2) of Corollary 2.1.3. This raises the following question (cf. Question 2.4.6, Remark 2.4.8, and Example 2.4.9), which together with case (2) of Corollary 2.1.3 could help proving unirationality of del Pezzo surfaces of degree 2 over any field of characteristic not equal to 2.

Question 2.1.4. *Let $d \in \{3, 4\}$ be an integer. Let X be a del Pezzo surface of degree two over a field of characteristic not equal to* 2*, and let $P \in X(k)$ be a point on the ramification locus of the anti-canonical map $\pi\colon X \to \mathbb{P}^2$. Does there exist a geometrically integral curve of degree d in $\mathbb{P}^2$ over k that has an ordinary singular point of multiplicity $d-1$ at $\pi(P)$, and that intersects the branch locus of π with even multiplicity everywhere?*

For some d, X, and P, the answer to this question is negative (see Example 2.4.9), but in all cases we know of (all over finite fields), there do exist singular curves of degree d with a point of multiplicity at least $d-1$ at $\pi(P)$. Hence, it may be true that the answer to Question 2.1.4 is positive for X and P general enough.

In line with case (1) of Corollary 2.1.3, we can ask, in fact for any integer $d \geq 1$, an analogous question for points P that do not lie on the ramification locus, where we do not require the singular point to be ordinary. In this case, if P lies on $r \leq 3$ exceptional curves, then Manin's construction shows that the answer is positive for degree $d = 4 - r$. Therefore, this analogous question is especially interesting when P lies on four exceptional curves (cf. Remark 2.3.5 and Example 2.3.8).

In Section 2.3 we prove Theorem 2.1.2 and a generalisation, Corollary 2.1.3. In Section 2.4 we discuss how to search for curves satisfying the assumptions of Theorem 2.1.2 and in particular of Corollary 2.1.3.

2.2 Proof of the first main theorem

Set $k_1 = k_2 = \mathbb{F}_3$ and $k_3 = \mathbb{F}_9$. Let $\gamma \in k_3$ denote an element satisfying $\gamma^2 = \gamma + 1$. Note that γ is not a square in k_3. For $i \in \{1, 2, 3\}$, we define the surface X_i in $\mathbb{P} = \mathbb{P}(1, 1, 1, 2)$ with coordinates x, y, z, w over k_i by

$$\begin{aligned} X_1 :\ -w^2 &= (x^2 + y^2)^2 + y^3z - yz^3, \\ X_2 :\ -w^2 &= x^4 + y^3z - yz^3, \\ X_3 :\ \gamma w^2 &= x^4 + y^4 + z^4. \end{aligned}$$

These surfaces are smooth, so they are del Pezzo surfaces of degree 2. C. Salgado, D. Testa, and A. Várilly-Alvarado proved the following result.

Theorem 2.2.1. *Let X be a del Pezzo surface of degree 2 over a finite field. If X is not isomorphic to X_1, X_2, or X_3, then X is unirational.*

Proof. See [STVA14, Theorem 1]. □

We will use the following lemma to prove the complementary statement, namely that X_1, X_2, and X_3 are unirational as well.

Lemma 2.2.2. *Let X be a del Pezzo surface of degree 2 over a field k. Suppose that $\rho \colon \mathbb{P}^1 \to X$ is a non-constant morphism; if the characteristic of k is 2 and the image of ρ is contained in the ramification divisor R_X, then assume also that the field k is perfect. Then X is unirational.*

Proof. See [STVA14, Theorem 17]. □

For $i \in \{1, 2, 3\}$, we define a morphism $\rho_i \colon \mathbb{P}^1 \to X_i$ by extending the map $\mathbb{A}^1(t) \to X_i$ given by

$$t \mapsto (x_i(t) : y_i(t) : z_i(t) : w_i(t)),$$

where

$$\begin{aligned} x_1(t) &= t^2(t^2-1), & x_2(t) &= t(t^2+1)(t^4-1), \\ y_1(t) &= t^2(t^2-1)^2, & y_2(t) &= -t^4, \\ z_1(t) &= t^8-t^2+1, & z_2(t) &= t^8+1, \\ w_1(t) &= t(t^2-1)(t^4+1)(t^8+1), & w_2(t) &= t^2(t^2+1)(t^{10}-1), \end{aligned}$$

$$\begin{aligned} x_3(t) &= (t^4+1)(t^2-\gamma^3), \\ y_3(t) &= (t^4-1)(t^2+\gamma^3), \\ z_3(t) &= (t^4+\gamma^2)(t^2-\gamma), \\ w_3(t) &= \gamma^2 t(t^8-1)(t^2+\gamma). \end{aligned}$$

It is easy to check for each i that the morphism ρ_i is well defined, that is, the polynomials x_i, y_i, z_i, and w_i satisfy the equation of X_i, and that ρ_i is non-constant. The methods used to find these curves are exposed in Section 2.4.

Theorem 2.2.3. *The del Pezzo surfaces X_1, X_2, and X_3 are unirational.*

Proof. By Lemma 2.2.2, the existence of ρ_1, ρ_2, and ρ_3 implies that X_1, X_2, and X_3 are unirational. □

Proof of Theorem 2.1.1. This follows from Theorems 2.2.1 and 2.2.3. □

Remark 2.2.4. Take any $i \in \{1,2,3\}$. Set $A_i = \rho_i(\mathbb{P}^1)$ and $C_i = \pi_i(A_i)$, where $\pi_i = \pi_{X_i} \colon X_i \to \mathbb{P}^2$ is as described in the previous section. By Remark 2 of [STVA14], the surface X_i is minimal, and the Picard group $\operatorname{Pic} X_i$ is generated by the class of the anti-canonical divisor $-K_{X_i}$. The same remark states that the linear system $|-nK_{X_i}|$ does not contain a geometrically integral curve of geometric genus zero for $n \leq 3$ if $i \in \{1,2\}$, nor for $n \leq 2$ if $i = 3$. For $i \in \{1,2\}$, the curve A_i has degree 8, so it is contained in the linear system $|-4K_{X_i}|$. The curve A_3 has degree 6, so it is contained in the linear system $|-3K_{X_i}|$. This means that the curve C_i has minimal degree among all rational curves on X_i. The restriction of π_i to A_i is a double cover $A_i \to C_i$. The curve

$C_i \subset \mathbb{P}^2$ has degree 4 for $i \in \{1,2\}$ and degree 3 for $i = 3$, and C_i is given by the vanishing of h_i, with

$$\begin{aligned} h_1 =& x^4 + xy^3 + y^4 - x^2yz - xy^2z, \\ h_2 =& x^4 - x^2y^2 - y^4 + x^2yz + yz^3, \\ h_3 =& x^2y + xy^2 + x^2z - xyz + y^2z - xz^2 - yz^2 - z^3. \end{aligned}$$

For $i \in \{1,2\}$, the curve C_i has an ordinary triple point Q_i, with $Q_1 = (0:0:1)$, $Q_2 = (0:1:1)$. The curve C_3 has an ordinary double point at $Q_3 = (1:1:1)$. For all i, the point Q_i lies on the branch locus $B_i = B_{X_i}$.

We will see later that the curve C_i intersects the branch locus B_i with even multiplicity everywhere. Of course, one could check this directly as well using the polynomial h_i. In fact, had we *defined* C_i by the vanishing of h_i, then one would easily check that C_i satisfies the conditions of part (2) of Corollary 2.1.3, which gives an alternative proof unirationality of X_i without the need of the explicit morphism ρ_i (see Example 2.3.9). Indeed, in practice we first found the curves C_1, C_2, and C_3, and then constructed the parametrisations ρ_1, ρ_2, ρ_3, which allow for the more direct proof that we gave of Theorem 2.2.3.

2.3 Proof of the second main theorem

Let k be a field of characteristic different from 2 and recall the notation introduced in Section 1.2.3. In what follows X denotes a del Pezzo surface of degree 2 over k, the map $\pi\colon X \to \mathbb{P}^2$ is its associated double covering map, with branch locus $B \subset \mathbb{P}^2$ and ramification locus $R \subseteq X$. The map $\iota\colon X \to X$ is the involution of X induced by the double covering map π. Let P be a point inside $X(k)$.

Combining Lemma 2.2.2 and Corollary 1.2.27 it is possible to relate the existence of some particular plane curves with the unirationality of a del Pezzo surface of degree 2.

Proposition 2.3.1. *Let $C \subset \mathbb{P}^2$ be a geometrically integral projective curve with $g(C) = 0$. Let $\tilde{C}$ denote its normalisation and set $n = \#b(\tilde{C}, B)$. The following statements hold.*

1. *If $n = 0$, then there exists a field extension ℓ of k of degree at most 2 such that the preimage $\pi^{-1}(C_\ell)$ consists of two irreducible components that are birationally equivalent to C_ℓ. For each such ℓ for which C_ℓ is rational, the surface X_ℓ is unirational.*

2. *If $n = 0$ and C is rational and there exists a rational point $P \in X(k)$ with $\pi(P) \in C - B$, then the preimage $\pi^{-1}(C)$ consists of two rational components and X is unirational.*

3. *If $n = 2$ and the preimage $\pi^{-1}(C)$ is rational, then the surface X is unirational.*

Proof. First note that since B is a smooth quartic, it has genus 3, then by the initial hypothesis $g(C) = 0$ it follows that $C \neq B$. Let $A = \pi^*(C)$ the pull back of the curve C on the surface X. Since C is geometrically integral and $C \neq B$, the curve A is geometrically reduced. The morphism $A \to C$ induced by π has degree 2, and so $\overline{A} = A \times_k \overline{k}$ consists of at most two components. Then there is an extension ℓ of k of degree at most 2 such that the components of A_ℓ are geometrically irreducible. Let ℓ be such an extension and let D be an irreducible component of A_ℓ.

Suppose $n = 0$. Then, from Corollary 1.2.27.(1), the preimage $\pi^{-1}(C_\ell)$ consists of two irreducible components that are birationally equivalent to C_ℓ. If, moreover, C_ℓ is rational, then Lemma 2.2.2 implies the unirationality of X_ℓ, proving statement *(1)*.

Assume C is itself rational and there is a rational point $P \in X(k)$ such that $\pi(P) \in C - B$. Then we have that $P \neq \iota(P)$ and the points P and $\iota(P)$ lie in different components of $A_\ell = D_\ell \cup \iota(D_\ell)$. Since the Galois group $G = G(\ell/k)$ fixes the points P and $\iota(P)$, it follows that G also fixes D_ℓ and $\iota(D_\ell)$, so these components are defined over k. Then statement *(2)* follows from *(1)* taking $\ell = k$.

Statement *(3)* follows immediately from Corollary 1.2.27.(2) and Lemma 2.2.2. □

Remark 2.3.2. Note that statement (1) of Proposition 2.3.1 is consistent with [STVA14, Corollary 1.3], in which it is stated that if X is a del Pezzo surface of degree 2 over a finite field k, then there is a quadratic extension k'/k such that $X_{k'}$ is unirational.

Remark 2.3.3. Let D be a geometrically integral curve over a field k with $g(D) = 0$. Then there exists a field extension ℓ of k of degree at most 2 such that D_ℓ is rational. In fact, if k is a finite field, then D is rational over k. Therefore, if k is finite in Proposition 2.3.1, then C is rational; moreover, by case *(3)* we conclude that if $n = 2$, then X is unirational over k.

Remark 2.3.4. Propositions 1.2.26 and 2.3.1 imply that the geometrically integral projective curves $D \subset X$ with $g(D) = 0$ are exactly the geometrically irreducible components above geometrically integral projective curves $C \subset \mathbb{P}^2$ with $g(C) = 0$ and $\#b(\tilde{C}, B) \in \{0, 2\}$, where $\tilde{C}$ denotes the normalisation of C.

Remark 2.3.5. Suppose $P \in X(k)$ is a rational point that does not lie on the ramification curve, so $\pi(P) \notin B$. Suppose C is a geometrically integral curve of degree d that has a singular point of multiplicity $d-1$ at $\pi(P)$, and that intersects B with even multiplicity everywhere. Then Proposition 1.2.31 shows that $b(\tilde{C}, B)$ is empty, so, by Corollary 1.2.27, the pull back $\pi^*(C)$ splits into two components.

If X is general enough, then the Picard group $\operatorname{Pic} X$ of X is generated by the canonical divisor K_X, and the automorphism group of X acts trivially on $\operatorname{Pic} X$, so these two components would be linearly equivalent to the same multiple of K_X; as their union is linearly equivalent to $-dK_X$, we find that d is even. Hence, for odd d, the answer to the analogous question mentioned below Question 2.1.4 is negative for X general enough.

It is possible, however, that, even for odd d, a variation of this analogous question still has a positive answer. If we forget the del Pezzo surface, and only consider the quartic curve $B \subset \mathbb{P}^2$ with a point $Q \in \mathbb{P}^2$ that does not lie on B, we could ask for the existence of a curve of degree d that intersects B with even multiplicity everywhere, and on which Q is a point of multiplicity $d-1$. The argument above merely shows that if such a curve exists for odd d and Q lifts to a rational point on the del Pezzo surface, then the surface does not have Picard number one.

Proof of Theorem 2.1.2. Assume that the assumptions of statement (1) hold. This implies that $C^s \cap B = \emptyset$ and $b(C, B) = \emptyset$. Therefore, by

Proposition 1.2.31, we have $\#b(\tilde{C}, B) = 0$. Statement *(1)* follows from applying part (2) of Proposition 2.3.1.

Assume statement *(2a)* holds. This means that $C^s \cap B = \{Q\}$ and $b(C, B) = \emptyset$. Since Q is a double or triple point of C, Proposition 1.2.31 implies that $\#b(\tilde{C}, B) = 2$. The conclusion of statement *(2)* follows from applying part (3) of Proposition 2.3.1 and Remark 2.3.3.

Assume statement *(2b)* holds. It means that $b(C, B) = \{Q_1, Q_2\}$ and $C^s \cap B \subseteq \{Q_1, Q_2\}$. Since the points Q_1 and Q_2 are distinct, Proposition 1.2.31 implies that $\#b(\tilde{C}, B) = 2$. As before, the conclusion of statement *(2)* follows from part (3) of Proposition 2.3.1 and Remark 2.3.3. This concludes the proof of the theorem. □

Proof of Corollary 2.1.3. Set $Q = \pi(P)$. Let $\mathfrak{L}_Q$ denote the line in the dual of $\mathbb{P}^2$ consisting of all lines $L \subset \mathbb{P}^2$ going through Q, and note that $\mathfrak{L}_Q$ is isomorphic to $\mathbb{P}^1$. Since C has degree d and $\pi(P)$ is a point of multiplicity $d-1$, each line in $\mathfrak{L}_Q$ intersects C in a unique d-th point, counting with multiplicity. It follows that C is smooth at all points $T \neq Q$. It also follows that the rational map $C \to \mathfrak{L}_Q$ that sends a point $T \in C$ to the line through T and Q is birational, so C is birationally equivalent with $\mathbb{P}^1$. By hypothesis, all intersection points of B and C have even intersection multiplicity.

Assume that Q is not contained in B. Since C is smooth away from Q, the curve B contains no singular points of C. Then X is unirational by part (1) of Theorem 2.1.2. This proves part (1).

Assume that Q is contained in B, that Q is an ordinary singularity of C, and $d \in \{3, 4\}$. Then Q is a double or a triple point of C. Since Q is the only singularity of C, the curve B contains no other singular points of C. Then X is unirational by part (2) of Theorem 2.1.2. This proves part (2). □

We now give some examples of curves that satisfy the conditions of Theorem 2.1.2 or Corollary 2.1.3.

Example 2.3.6. If C is a bitangent to the branch curve B that is defined over k, and $C(k)$ contains a point $Q \notin B$ that lifts to a k-rational point on X, then Theorem 2.1.2 implies that X is unirational. We can also prove this directly. Indeed, in this case the pull back $\pi^{-1}(C)$ consists of two exceptional curves that are defined over k, so X is not minimal.

Blowing down one of these exceptional curves yields a del Pezzo surface Y of degree 3 with a rational point. This implies that Y, and therefore also X, is unirational.

Example 2.3.7. Suppose the point $P \in X(k)$ is not a generalised Eckardt point and P is not on the ramification curve. Set $Q = \pi(P)$, let $\rho\colon \mathfrak{L}_Q \to X$ be as in [FvL15, Section 4, p.6], and set $C = \pi(\rho(\mathfrak{L}_Q))$. Then by [FvL15, Proposition 4.14], the map $\rho(\mathfrak{L}_Q) \to C$ has degree 1, so by Propositions 1.2.26 and 1.2.31, the intersection multiplicity of C and the branch curve B is even at all intersection points. Also by [FvL15, Proposition 4.14], the curve C has a point Q off the branch curve B of multiplicity $\deg C - 1$, so the curves of Manin's construction are examples of the curves described in Corollary 2.1.3. For further discussion on this see [FvL15, Remark 4.15].

Example 2.3.8. Consider the surface $X \subset \mathbb{P}(1,1,1,2)$ over $\mathbb{F}_3$, defined by the equation

$$w^2 = x^4 + y^4 + z^4.$$

The surface X is a del Pezzo surface of degree 2. All its rational points either are on the ramification curve, or they are generalised Eckardt points. In fact, the surface X has 154 rational points over $\mathbb{F}_9$, with 28 of those lying on the ramification locus. The remaining 126 are generalised Eckardt points, which is also the maximum number of generalised Eckardt points a del Pezzo surface of degree two can have (see [STVA14, before Example 7]). It follows that Manin's method does not apply to this surface. Let P be the point $(0:0:1:1)$ on X. Then P is a generalised Eckardt point and its image $Q = \pi(P) = (0:0:1) \in \mathbb{P}^2$ does not lie on the branch locus B, which is given by $x^4 + y^4 + z^4 = 0$. Consider the curve $C \subset \mathbb{P}^2$ given by $x^3y + xy^3 = z(x+y)^2(y-x)$. The curve C is a geometrically integral quartic plane curve that has a triple point at Q and that intersects B with even multiplicity everywhere. Therefore, by case (1) of Corollary 2.1.3, the surface X is unirational.

Of course, unirationality of X was already known: it follows for instance from Lemma 20 in [STVA14] (cf. Example 2.3.10 below). It is nice to see, though, that, even though Manin's construction and the generalisation in [STVA14] do not produce a curve in $\mathbb{P}^2$ of some degree d with a point of multiplicity $d-1$ at Q, and even intersection multiplicity with B everywhere, such curves do still exist, and then case (1) of

Corollary 2.1.3 implies unirationality of X. This gives a positive answer to the question below Question 2.1.4 for $d = 4$ and this particular surface X and this generalised Eckardt point P.

One might ask whether there are curves of lower degree satisfying the hypotheses of case (1) of Corollary 2.1.3. Indeed, there are conics that do, for example the one given by $y^2 = xz$. An exhaustive computer search, based on Proposition 2.3.1.(2), and Corollary 2.4.2, shows that there are no cubic curves with a double point at Q satisfying the hypotheses of Corollary 2.1.3 and its case (1).

Example 2.3.9. Let X_1, X_2, X_3 be the three del Pezzo surfaces defined as in Section 2.2 and let B_i be their branch locus, for $i = 1, 2, 3$. For $i = 1, 2, 3$, all rational points of the surface X_i lie on the ramification locus. Consider the rational points $P_1 = (0 : 0 : 1 : 0) \in X_1$, $P_2 = (0 : 1 : 1 : 0) \in X_2$, and $P_3 = (1 : 1 : 1 : 0) \in X_3$, and set $Q_i = \pi(P_i)$. Clearly, we have $Q_i \in B_i$. Set $d_1 = d_2 = 4$ and $d_3 = 3$. Let $C_i \subset \mathbb{P}^2$ be the projective plane curve of degree d_i given by the polynomial h_i defined as in Remark 2.2.4. The curve C_i is geometrically irreducible and it has an ordinary singular point at Q_i of multiplicity $d_i - 1$. Given that the curve C_i pulls back to the geometrically irreducible rational curve A_i of Remark 2.2.4, we find from Corollary 1.2.27 and Proposition 1.2.31 that C_i intersects B_i with even multiplicity everywhere.

Of course, one could also check directly that C_i intersects B_i with even multiplicity everywhere. Then Corollary 2.1.3 and Remark 2.3.3 give an alternative proof that the surface X_i is unirational (cf. Remark 2.2.4). There is a quartic alternative for C_3 as well. The curve $C_3' \subset \mathbb{P}^2$ given by the vanishing of

$$\begin{aligned} h_3' =& \gamma^2x^4 + x^3y + \gamma x^2y^2 + \gamma^3xy^3 - y^4 + x^3z + \gamma x^2yz + xy^2z \\ & - \gamma y^3z + \gamma x^2z^2 + xyz^2 + \gamma^3y^2z^2 + \gamma^3xz^3 - \gamma yz^3 - z^4 \end{aligned}$$

is geometrically integral, has an ordinary triple point at $(-1 : 1 : 1)$, and intersects B with even multiplicity everywhere.

Example 2.3.10. Let k be a field with characteristic different from 2. Let $a_1, \ldots, a_6 \in k$ be such that the variety X in the weighted projective space $\mathbb{P} = \mathbb{P}(1, 1, 1, 2)$ defined by

$$w^2 = a_1^2x^4 + a_2^2y^4 + a_3^2z^4 + a_4x^2y^2 + a_5x^2z^2 + a_6y^2z^2$$

is a del Pezzo surface of degree 2. This is the surface of Lemma 20 in [STVA14], where it is noted that the surface in $\mathbb{P}$ given by the equation $w = a_1x^2 + a_2y^2 + a_3z^2$ intersects the surface X in a curve D, which the anti-canonical map $\pi\colon X \to \mathbb{P}^2$ sends isomorphically to the plane quartic curve $C \subset \mathbb{P}^2$ given by

$$(a_4 - 2a_1a_2)x^2y^2 + (a_5 - 2a_1a_3)x^2z^2 + (a_6 - 2a_2a_3)y^2z^2 = 0.$$

They also note that this curve C is birationally equivalent to a conic under the standard Cremona transformation, so C and D are rational over an extension of k of degree at most 2. If they are rational over k, then X is unirational.

Indeed, one checks that the curve C satisfies the conditions of part (1) of Proposition 2.3.1, and if C is rational over k, then it also satisfies the conditions of part (1) of Theorem 2.1.2, where one can take P to be any of the points on X above any of the singular points $(0:0:1)$, $(0:1:0)$, and $(1:0:0)$ of C.

2.4 Finding appropriate curves

In this section, we assume that the characteristic of k is not 2, and we give sufficient easily-verifiable conditions for a curve C to satisfy the hypotheses of Corollary 2.1.3. This is also how we found the three curves, C_1, C_2, and C_3 of Remark 2.2.4, whose existence implies unirationality of the three difficult surfaces X_1, X_2, X_3 (see Example 2.3.9 and Remark 2.4.7).

Let $X \subset \mathbb{P}(1,1,1,2)$ be a del Pezzo surface of degree 2, given by $w^2 = g$ with $g \in k[x,y,z]$ homogeneous of degree 4. Let $B \subset \mathbb{P}^2(x,y,z)$ be the branch curve of the projection $\pi\colon X \to \mathbb{P}^2$. Then B is given by $g = 0$. Let $P \in X(k)$ be a rational point and set $Q = \pi(P)$. Without loss of generality, we assume $Q = (0:0:1)$. Let $C \subset \mathbb{P}^2$ be a geometrically irreducible curve of degree $d \geq 2$ on which Q is a point of multiplicity $d-1$.

There are coprime homogeneous polynomials $f_{d-1}, f_d \in k[x,y]$ of degree $d-1$ and d, respectively, such that C is given by $zf_{d-1} = f_d$. The projection away from Q induces a birational map from C to the family $\mathfrak{L}_Q$ of lines in $\mathbb{P}^2$ through Q. Its inverse is a morphism ϑ that sends a line $L \in \mathfrak{L}_Q$ to the d-th intersection point of L with C. If we identify $\mathfrak{L}_Q$ with $\mathbb{P}^1$, where $(s:t) \in \mathbb{P}^1$ corresponds to the line given by $sy = tx$, then $\vartheta\colon \mathbb{P}^1 \to C$ sends $(s:t)$ to

$$(sf_{d-1}(s,t) : tf_{d-1}(s,t) : f_d(s,t)).$$

The curve C has no singularities outside Q, and we may identify the morphism $\vartheta\colon \mathbb{P}^1 \to C$ with the normalisation of C. The points on $\mathbb{P}^1$ above the point Q are exactly the points where $f_{d-1}(s,t)$ vanishes. The curve C has an ordinary singularity at Q if and only if $d > 2$ and $f_{d-1}(s,t)$ vanishes at $d-1$ distinct $\overline{k}$-points of $\mathbb{P}^1(s,t)$.

The pull back $\pi^*(C)$ is birationally equivalent with the curve given by $w^2 = G$ in the weighted projective space $\mathbb{P}(1,1,2d)$ with coordinates s, t, w, and with

$$G = g\big(sf_{d-1}(s,t), tf_{d-1}(s,t), f_d(s,t)\big) \in k[s,t].$$

Proposition 2.4.1. *For any point $T \in \mathbb{P}^1(\overline{k})$, the intersection multiplicity $\mu_T(\mathbb{P}^1, B)$ equals the order of vanishing of G at T.*

Proof. Since C either has degree 2 or it is singular, it is not equal to B. As C is irreducible, it has no irreducible components in common with B. By symmetry between s and t, we may assume $T = (\alpha : 1)$ for some $\alpha \in \overline{k}$. Then the local ring $\mathcal{O}_{T,\mathbb{P}^1}$ is isomorphic to the localisation of $\overline{k}[s]$ at the maximal ideal $(s-\alpha)$. Let $\ell \in k[x,y,z]$ be a linear form that does not vanish at $\vartheta(T)$. Then locally around $\vartheta(T) \in \mathbb{P}^2$, the curve B is given by the vanishing of the element g/ℓ^4, whose image in $\mathcal{O}_{T,\mathbb{P}^1}$ is $G(s,1)/L(s,1)^4$ with $L(s,t) = \ell\big(sf_{d-1}(s,t), tf_{d-1}(s,t), f_d(s,t)\big)$. Since $L(s,1)$ does not vanish at α, we find that $\mu_T(\mathbb{P}^1, B)$ equals the order of vanishing of $G(s,1)$ at α, which equals the order of vanishing of G at T. $\square$

Corollary 2.4.2. *We have $b(\mathbb{P}^1, B) = \emptyset$ if and only if G is a square in $\overline{k}[s,t]$.*

Proof. By Proposition 2.4.1, we have $b(\mathbb{P}^1, B) = \emptyset$ if and only if the order of vanishing of G is even at every point $T \in \mathbb{P}^1(\overline{k})$. This is equivalent with G being a square in $\overline{k}[s,t]$. $\square$

If B does not contain the unique singular point Q of C, then ϑ induces a bijection $b(\mathbb{P}^1, B) \to b(C, B)$, so in this case we also have $b(C, B) = \emptyset$ if and only if G is a square in $\overline{k}[s,t]$. The following proposition gives an analogue of this statement when Q is contained in B.

Proposition 2.4.3. *Suppose that Q is contained in B. Then the rational polynomial $H = G/f_{d-1}(s,t)$ is in fact contained in $k[s,t]$. Suppose, furthermore, that the tangent line to B at Q is given by $h = 0$ with $h \in k[x,y]$, and that Q is an ordinary singular point on the curve C. Then the following statements hold.*

1. *Suppose d is odd. Then the set $b(C,B)$ is empty if and only if H is a square in $\overline{k}[s,t]$.*

2. *Suppose d is even. If h divides f_{d-1}, then $H/h(s,t)$ is contained in $k[s,t]$. The set $b(C,B)$ is empty if and only if h divides f_{d-1} and $H/h(s,t)$ is a square in $\overline{k}[s,t]$.*

Proof. Write $g = \sum_{i=0}^{4} g_i z^{4-i}$, where $g_i \in k[x,y]$ is homogeneous of degree i for all $0 \leq i \leq 4$. If $g(Q) = g_0$ vanishes, then each monomial of g is divisible by x or y, which implies that G is divisible by f_{d-1}, which in turn shows $H \in k[s,t]$. Suppose that all hypotheses hold. By $g(Q) = 0$ we find $g_0 = 0$. The tangent line to B at Q is given by $g_1 = 0$, so h is a scalar multiple of g_1. Note that all statements are invariant under the action of $\mathrm{GL}_2(k)$ on $\mathbb{P}^1$ and $\mathbb{P}^2$ given on their respective homogeneous coordinate rings $k[s,t]$ and $k[x,y,z]$ by $\gamma(s) = as + bt, \gamma(t) = cs + dt$ and $\gamma(x) = ax + by, \gamma(y) = cx + dy, \gamma(z) = z$ for

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

After applying an appropriate element $\gamma \in \mathrm{GL}_2(k)$ and rescaling h, we assume, without loss of generality, that $h = g_1 = y$.

If y divides f_{d-1}, then t divides $f_{d-1}(s,t)$; since all monomials in g besides y are divisible by x^2, xy, or y^2, it follows that in this case G is divisible by $tf_{d-1}(s,t)$, so H/t is contained in $k[s,t]$. This does not depend on d being even.

In the open neighbourhood of Q given by $z \neq 0$, the curve B is given by the vanishing of $g/z^4 = g(x/z, y/z, 1) = \sum_i g_i(x/z, y/z)$. The maximal ideal $\mathfrak{m}$ of the local ring $\mathcal{O}_{Q,C}$ is generated by x/z and y/z, so the image of g/z^4 in $\mathcal{O}_{Q,C}/\mathfrak{m}^2$ is $g_1(x/z, y/z) = y/z$. Let $T \in \mathbb{P}^1(\overline{k})$ be a point with $\vartheta(T) = Q$, and let $\mathfrak{n}$ be the maximal ideal of the local ring $\mathcal{O}_{T,\mathbb{P}^1}$. Then the image of g in $\mathcal{O}_{T,\mathbb{P}^1}/\mathfrak{n}^2$ equals the image of y/z, which is $tf_{d-1}(s,t)/f_d(s,t)$. The point T corresponds to a linear factor of $f_{d-1}(s,t)$. Since $f_d(s,t)$ does not vanish at T, we find that the valuation $v_T(g)$ of g in $\mathcal{O}_{T,\mathbb{P}^1}$ is at least 2 if t vanishes at T, that is, $\mu_T(\mathbb{P}^1, B) \geq 2$ if $T = (1:0)$. We have $\mu_T(\mathbb{P}^1, B) = v(g) = 1$ if $T \neq (1:0)$. From Lemma 1.2.24 we conclude

$$\mu_Q(C,B) = \begin{cases} d - 2 + \mu_{(1:0)}(\mathbb{P}^1, B) & \text{if } y \text{ divides } f_{d-1}, \\ d-1 & \text{otherwise.} \end{cases} \tag{2.1}$$

We now consider the two cases.

1. Suppose d is odd. From (2.1) it follows that $\mu_Q(C,B)$ is even if and only if either y divides f_{d-1} and $\mu_{(1:0)}(\mathbb{P}^1, B)$ is odd, or y does not divide f_{d-1}. This happens if and only if $\mu_T(\mathbb{P}^1, B)$ is odd for all $T \in \mathbb{P}^1$ at which $f_{d-1}(s,t)$ vanishes. For all other points $R \in C$ with $R \neq Q$, the multiplicity $\mu_R(C,B)$ is even if and only if $\mu_{\vartheta^{-1}(R)}(\mathbb{P}^1, B)$ is even. From Proposition 2.4.1, we conclude that $b(C,B)$ is empty if and only if the order of vanishing of G is odd at all points $T \in \mathbb{P}^1$ at which $f_{d-1}(s,t)$ vanishes and even at all other points. This is equivalent with H being a square in $\overline{k}[s,t]$.

2. Suppose d is even. From (2.1) it follows that $\mu_Q(C,B)$ is even if and only if y divides f_{d-1} and $\mu_{(1:0)}(\mathbb{P}^1, B)$ is even. As in the case for odd d, this implies that $b(C,B)$ is empty if and only if the order of vanishing of G is odd at all points $T \neq (1:0)$ at which $f_{d-1}(s,t)$ vanishes, and even at all other points, including $(1:0)$. Since the order of vanishing of $tf_{d-1}(s,t)$ at $(1:0)$ is 2, this is equivalent to $G/(tf_{d-1}(s,t)) = H/t$ being a square in $\overline{k}[s,t]$.

This finishes the proof. □

We have already seen that the pull back $\pi^*(C)$ is birationally equivalent with the curve given by $w^2 = G$ in $\mathbb{P}(1,1,2d)$. This curve splits into two k-rational components if and only if G is a square in $k[s,t]$. If Q is an ordinary singular point of C that lies on B, then this never happens. However, the curve $\pi^*(C)$ may itself be k-rational, in which case G factors as a square times a quadric.

We will now focus on the case $d = 4$, so Q is a triple point. The following corollary says that if Q is an ordinary triple point, then we do not need to factorise G, as we know exactly which part should be the square, and which the quadric.

Corollary 2.4.4. *Suppose that Q is an ordinary singular point of C that lies on B. If the pull back $\pi^*(C) \subset X$ is k-rational, then we have $d \leq 4$.*

Moreover, suppose $d = 4$, and let the tangent line to B at Q be given by $h = 0$ with $h \in k[x, y]$. Then the pull back $\pi^*(C) \subset X$ is k-rational if and only if there is a constant $c \in k^*$ such that the following statements hold:

1. *the polynomial h divides f_3;*
2. *the polynomial $cH(s,t)/h(s,t)$ is a square in $k[s,t]$;*
3. *the conic given by $cw^2 = f_3(s,t)/h(s,t)$ in $\mathbb{P}^2(s,t,w)$ is k-rational.*

Proof. Suppose $\pi^*(C)$ is k-rational. Then $\pi^*(C)$ is geometrically integral and has genus $g(\pi^*(C)) = 0$. From Proposition 1.2.26 we obtain $b(\mathbb{P}^1, B) = 2$. From Proposition 1.2.29 we conclude that the contribution $c_Q(C, B)$ is at most 2. Moreover, this proposition also gives $c_Q(C, B) \geq d - 2$ with equality if and only if $\mu_Q(C, B)$ is even. We conclude $d \leq 4$.

Suppose $d = 4$. Then we have equality $c_Q(C, B) = 2 = \#b(\mathbb{P}^1, B)$, so $\mu_Q(C, B)$ is even, and we find that $b(C, B)$ is empty. From Proposition 2.4.3 we find that h divides f_3, and $m = H(s,t)/h(s,t)$ is a square in $\overline{k}[s,t]$. Let c be the main coefficient of $m(s,1)$. Then cm is a square in $k[s,t]$. Therefore, the k-rational curve given by $w^2 = G$ with

$$G = cm \cdot h^2(s,t) \cdot c^{-1} f_3(s,t)/h(s,t) \tag{2.2}$$

in $\mathbb{P}(1, 1, 2d)$ is birationally equivalent with the conic given by the equation $cw^2 = f_3(s,t)/h(s,t)$ in $\mathbb{P}^2(s,t,w)$, which is therefore also k-rational.

Conversely, if there is a constant c such that $cH(s,t)/h(s,t)$ is a square in $k[s,t]$, then it follows from (2.2) that the conic given by

$cw^2 = f_3(s,t)/h(s,t)$ in $\mathbb{P}^2(s,t,w)$ is birationally equivalent with the curve in $\mathbb{P}(1,1,2d)$ given by $w^2 = G$, which is birationally equivalent with $\pi^*(C)$. Hence, if this conic is k-rational, then so is $\pi^*(C)$. □

Remark 2.4.5. Corollary 2.4.4 helps us in finding all curves C of degree $d = 4$ that satisfy the conditions of case (2) of Corollary 2.1.3 with $\ell = k$. More explicitly, after a linear transformation of $\mathbb{P}^2$, we may assume that $Q = (0:0:1)$, and the tangent line to B at Q is given by $y = 0$. Then we claim that every curve C of degree $d = 4$ that satisfies the conditions of case (2) of Corollary 2.1.3 with $\ell = k$ is given by

$$yz\phi_2 = x^4 + y\phi_3$$

for some homogeneous $\phi_2, \phi_3 \in k[x,y]$ of degree 2 and 3, respectively, with ϕ_2 squarefree and not divisible by y. Indeed, we find that f_3 is divisible by y, so there is a $\phi_2 \in k[x,y]$ such that $f_3 = y\phi_2$; since C is irreducible, the polynomial f_4 is not divisible by y, so the coefficient of x^4 in f_4 is nonzero, and after scaling ϕ_2, f_3, and f_4, we may assume that there exists a $\phi_3 \in k[x,y]$ such that $f_4 = x^4 + y\phi_3$. Moreover, Q is an ordinary singularity if and only if ϕ_2 is squarefree and not divisible by y.

Hence, to find all such curves C, we are looking for all pairs (ϕ_2, ϕ_3) with $\phi_i \in k[x,y]$ homogeneous of degree i, such that

1. the polynomial ϕ_2 is squarefree and y does not divide ϕ_2,
2. the curve given by $yz\phi_2 = x^4 + y\phi_3$ is geometrically integral,
3. there is a constant $c \in k^*$ such that polynomial $c \cdot G(s,t)/(t^2\phi_2(s,t))$ with
$$G = g\big(st\phi_2(s,t),\ t^2\phi_2(s,t),\ s^4 + t\phi_3(s,t)\big)$$
is a square,
4. the conic given by $cw^2 = \phi_2(s,t)$ in $\mathbb{P}^2(s,t,w)$, with c as in (3), is k-rational.

Note for (3) that, because the characteristic is not 2, a homogeneous polynomial $H \in k[s,t]$ of even degree is a square in $\overline{k}[s,t]$ if and only if there is a constant $c \in k^*$ such that cH is a square in $k[s,t]$, which happens if and only if $\gamma^{-1}H(s,1)$ is a square in $k[s]$, where γ is the main coefficient of $H(s,1)$. This follows from the fact that a monic polynomial in $k[s]$ is a square in $k[s]$ if and only if it is a square in $\overline{k}[s]$. Moreover, the $c \in k^*$ for which cH is a square, form a coset in k^*/k^{*2}, so whether or not (4) holds does not depend on the choice of c.

Question 2.1.4 for $d = 4$ can be rephrased using Remark 2.4.5. It is equivalent to the following question.

Question 2.4.6. *Let k be a field of characteristic not equal to 2, and $g \in k[x,y,z]$ a homogeneous polynomial of degree 4 such that the curve $B \subset \mathbb{P}^2(x,y,z)$ given by the equation $g = 0$ is smooth, it contains the point $Q = (0:0:1)$, and the tangent line to B at Q is given by $y = 0$. Do there exist homogeneous polynomials $\phi_2, \phi_3 \in k[x,y]$ of degree 2 and 3, such that conditions (1)–(3) of Remark 2.4.5 are satisfied?*

Remark 2.4.7. If k is a ("small") finite field, then we can list all pairs (ϕ_2, ϕ_3) with $\phi_i \in k[x,y]$ homogeneous of degree i, and check for each whether the conditions (1)–(4) of Remark 2.4.5 are satisfied. In fact, condition (4) is automatically satisfied over finite fields. Indeed, this is how we found the curves C_1, C_2 given in Remark 2.2.4, whose existence implies unirationality of the three difficult surfaces X_1, X_2 (see Example 2.3.9). Finding the rational cubic curve C_3 on X_3, as given in Remark 2.2.4, was easier, based on part (1) of Proposition 2.4.3.

Remark 2.4.8. For any integer i, let $k[x,y]_i$ denote the $(i+1)$-dimensional space of homogeneous polynomials of degree i. In general, over any field, we can describe the set of pairs $(\phi_2, \phi_3) \in k[x,y]_2 \times k[x,y]_3$ satisfying condition (3) of Remark 2.4.5 as follows.

Identify $k[x,y]_2 \times k[x,y]_3$ with the affine space $\mathbb{A}^7$ and let R denote the coordinate ring of $\mathbb{A}^7$, that is, R is the polynomial ring in the $3 + 4 = 7$ coefficients of ϕ_2 and ϕ_3. Let $Z \subset \mathbb{A}^7$ be the locus of all (ϕ_2, ϕ_3) that satisfy condition (3).

For *generic* ϕ_2, ϕ_3, that is, with the variables of R as coefficients, the coefficients of the polynomial

$$G' = G(s,t)/(t^2\phi_2(s,t))$$

of condition (3) of Remark 2.4.5 lie in R. For general enough g, the coefficient $c \in R$ of s^{12} in the polynomial $G' \in R[s,t]$ is nonzero. On the open set U of $\mathbb{A}^7$ given by $c \neq 0$, we may complete $G'(s,1)$ to a square in the sense that there are polynomials $G_1, G_2 \in R[c^{-1}][s]$ with G_1 monic of degree 6 in s and G_2 of degree at most 5 in s such that $G'(s,1) = cG_1^2 - G_2$. The vanishing of the six coefficients in R of G_2 determines the locus $Z \cap U$ inside U of all pairs (ϕ_2, ϕ_3) at which cG' is a square. Note that we have $c = G'(1,0)$. For each point $(s_0 : t_0) \in \mathbb{P}^1$, we can use an automorphism of $\mathbb{P}^1$ that sends $(s_0 : t_0)$ to $(1 : 0)$, to similarly describe the intersection of Z with the open subset of $\mathbb{A}^7$ where $G'(s_0, t_0)$ is nonzero; it is also given by the vanishing of six polynomials in R. We can cover $\mathbb{A}^7$ with open subsets of this form, thus describing Z completely.

A naive dimension count suggests that the locus Z has dimension $7 - 6 = 1$. This is consistent with the following, similarly naive, dimension count. The family of quartic curves in $\mathbb{P}^2$ is 14-dimensional, as it is the projective space $\mathbb{P}(k[x,y,z]_4)$, where $k[x,y,z]_4$ is the 15-dimensional vector space of polynomials of degree 4. The codimension of the subset of those curves having a triple point at Q is 6, and demanding that the intersection multiplicity $\mu_Q(C, B)$ is at least 4 cuts down another dimension. Since B is also a quartic curve, by Bezout's theorem it follows that B and C have 16 intersection points, counted with multiplicity. Hence, generically, the curves in the remaining 7-dimensional family intersect B, besides in Q, in $16 - 4 = 12$ more points. One might expect the subfamily of those curves where this degenerates to six points with multiplicity 2 to have codimension 6, in which case this would leave a 1-dimensional family of quartic curves with a triple point at $Q \in B$ and intersecting B with even multiplicity everywhere.

However, the locus Z also contains some degenerate components that we are not interested in. For example, the locus of all $(0, \phi_3)$ for which $f_4 = x^4 + y\phi_3$ is a square is contained in Z and has dimension 2. Also, for any smooth conic Γ that contains Q, that has its tangent line at Q given by $y = 0$, and that has even intersection multiplicity with B everywhere, we get a 1-dimensional subset of Z consisting of pairs (ϕ_2, ϕ_3) that correspond with the union of Γ with any double line through Q (these lines are parametrised by $\mathbb{P}^1$). Note that in all these degenerate cases the curve C is reducible. Another degenerate case is the limit of Manin's construction. By [FvL15, Remark 4.11], this limit curve is the non-reduced curve $\pi_*\pi^*(2L) = 4L$, where L is the tangent line to B at Q, given by $y = 0$. Hence, this quartic curve is given by $y^4 = 0$, which does not correspond to a point on the affine set Z, as the coefficient of x^4 is zero.

Let Z_0 denote the affine subset of Z corresponding to curves C that are geometrically integral and on which Q is an ordinary triple point. Then Questions 2.1.4 (for $d = 4$) and 2.4.6 can be rephrased by asking whether the subset Z_0 contains a k-rational point.

Example 2.4.9. Let $B \subset \mathbb{P}^2$ be the smooth curve given by

$$y^4 - x^4 - x^3y - xy^3 + y^3z + yz^3 = 0$$

over $k = \mathbb{F}_3$, and let Q be the point $(0 : 0 : 1) \in B(k)$. The tangent line to B at Q is given by $y = 0$. Running through all the homogeneous polynomials in x, y of degree 2 and 3 over k one can find that there is no pair (ϕ_2, ϕ_3) of polynomials satisfying conditions (1)–(3) of Remark 2.4.5; there do exist pairs satisfying only conditions (2) and (3). This means that Questions 2.1.4 (for $d = 4$) and 2.4.6 have negative answer in this specific case. It could, however, still be true that the answer is positive for X and P general enough.

Notice that the curve B is isomorphic to the Fermat curve of degree four, that is, the curve given by $x^4 + y^4 + z^4 = 0$, via the following linear change of variables:

$$(x : y : z) \mapsto (x - z : x + y + z : z).$$

Chapter 3

The geometric Picard lattice of the K3 surfaces in a family

The geometric Picard lattice of a K3 surface can give information about the geometry as well as the arithmetic of the surface. A large literature is devoted to the computation of the Picard lattice of a K3 surface. In [PTvL15], Bjorn Poonen, Damiano Testa, and Ronald van Luijk give an algorithm to compute the geometric Néron-Severi group of any smooth, projective, geometrically integral variety X. The algorithm works under the assumption that it is possible to explicitly compute the Galois modules of X with finite coefficients, and it terminates if and only if the Tate conjecture holds for X. In [HKT13], Hassett, Kresch, and Tschinkel give an effective algorithm to compute the Picard lattice of a K3 surface of degree two. The algorithm is "effective" in the sense that given the equations defining the surface, it returns the Galois module structure of the geometric Picard lattice of the surface. Even though these algorithms show that in principle it is possible to compute the geometric Picard lattice of a K3 surface, in practice the computations involved are very hard to perform, making the algorithms highly impractical. The main problem in the task of computing the geometric Picard lattice is to find enough divisors to generate the whole geometric Picard lattice. This remains the main issue even if we are only interested in the geometric Picard number, that is, the rank of the geometric Picard lattice. Work of Charles, Elsenhans, Jahnel, Kloosterman, Kuwata, van Luijk, and others, show that there are different ways to provide lower and upper bounds for the geometric Picard number. See [vL05] and [vL07] for a method to give an upper bound of the geometric Picard number by looking at the reduction of the surface over different finite fields; this method is later applied by Stephan Elsenhans and Jörg Jahnel in [EJ08b] and [EJ08a]. Kuwata and Kloosterman, in [Kuw00] and [Klo07], provide explicit examples of elliptic K3 surfaces with geometric Picard number $\rho \geq r$, for $r = 0, 1, ..., 18$. In [Cha14], François Charles provides a non-deterministic algorithm to compute the geometric Picard number of a K3 surface. We suggest to consult [PTvL15] for a more accurate summary on this topic. All these methods, as well as the algorithm given by Charles, rely on the ability to explicitly find enough divisors on the surface. We are not aware of the existence of any practical algorithm that, given a surface X as input, returns a set of divisors on X generating the geometric Picard lattice of X.

In this chapter we consider a 1-dimensional family of K3 surfaces, and we give an explicit description of the geometric Picard lattice of the generic member of the family, providing also an explicit set of divisors generating the Picard lattice. This information can then be used to describe the geometric Picard lattice of every member of the family.

This chapter is part of joint work with Florian Bouyer, Edgar Costa, Christopher Nicholls, and Mckenzie West. The joint work has its roots in a question proposed by Anthony Várilly-Alvarado during the Arizona Winter School 2015 (see [VA15, Project 1]). We are also indebted with Alice Garbagnati for the comments that led to Proposition 3.7.6.

3.1 The main result

Let k be any field; recalling the notation introduced in Subsection 1.2.2, we will use $\mathbb{P}_k$ to denote the weighted projective space $\mathbb{P}_k(1,1,1,3)$; also, let $\mathbb{A}^1_k$ denote the affine line over k. Sometimes we might drop the index k in $\mathbb{P}_k$ and $\mathbb{A}^1_k$, if no confusion arises.

Let $\mathbb{Q}$ be the field of rational numbers and fix an algebraic closure $\overline{\mathbb{Q}}$. Let t and x, y, z, w be the coordinates of $\mathbb{A}^1_{\mathbb{Q}}$ and $\mathbb{P}_{\mathbb{Q}}$, respectively.

Let $\mathfrak{X} \subset \mathbb{A}^1 \times \mathbb{P}$ be the threefold over $\mathbb{Q}$ defined by

$$\mathfrak{X}\colon w^2 = x^6 + y^6 + z^6 + tx^2y^2z^2. \tag{3.1}$$

Let $p\colon \mathfrak{X} \to \mathbb{A}^1$ be the projection of $\mathfrak{X}$ to $\mathbb{A}^1$, that is, the map defined by sending the point $(t_0, (x_0 : y_0 : z_0 : w_0)) \in \mathfrak{X}$ to the point $t_0 \in \mathbb{A}^1$.

Let t_0 be a point in $\mathbb{A}^1$. The fiber $p^{-1}(t_0) \subset \mathbb{A}^1 \times \mathbb{P}$ of $\mathfrak{X}$ over t_0 is given by the following equations

$$p^{-1}(t_0)\colon \begin{cases} w^2 &= x^6 + y^6 + z^6 + t_0x^2y^2z^2 \\ t &= t_0 \end{cases}$$

The fiber $p^{-1}(t_0)$ naturally embeds into $\mathbb{P}$, and we denote its image inside $\mathbb{P}$ by X_{t_0}; we also denote by B_{t_0} the plane sextic curve defined by

$$B_{t_0}\colon x^6 + y^6 + z^6 + t_0x^2y^2z^2 = 0. \tag{3.2}$$

Proposition 3.1.1. *Let t_0 be a point of $\mathbb{A}^1_{\mathbb{Q}} \setminus \{-3, -3\zeta_3, -3\zeta_3^2\}$, where ζ_3 is a primitive third root of unity. Then X_{t_0} is a K3 surface.*

Proof. The surface X_{t_0} is defined by the equation

$$X_{t_0}\colon w^2 = x^6 + y^6 + z^6 + t_0x^2y^2z^2,$$

and it is a double cover of $\mathbb{P}^2$ ramified above the sextic curve $B_{t_0} \subset \mathbb{P}^2$.

The curve B_{t_0} admits singular points if following system of equations admits solutions.

$$\begin{cases} 3x^5 + t_0xy^2z^2 &= 0 \\ 3y^5 + t_0x^2yz^2 &= 0 \\ 3z^5 + t_0x^2y^2z &= 0 \end{cases}$$

One can see that this happens if and only if $t_0 = -3, -3\zeta_3, -3\zeta_3^2$. So, for $t_0 \neq -3, -3\zeta_3, -3\zeta_3^2$, the curve B_{t_0} is smooth and, therefore, X_{t_0} is a K3 surface. □

Remark 3.1.2. Define $t_i := -3\zeta_3^i$, for $i \in \{0, 1, 2\}$. We claim that the surfaces X_{t_i}, for $i = 0, 1, 2$, are non-smooth and, therefore, are not K3 surfaces.

One can easily see that $X_{t_0} = X_{-3}$ has four ordinary double points: $(1 : \pm 1 : \pm 1 : 0)$.

For $i = 1, 2$, the map $(x : y : z : w) \mapsto (\zeta_3^i x : y : z : w)$ gives an isomorphism $X_{-3} \to X_{t_i}$. So also X_{t_i} has four ordinary double points, namely the points $(\zeta_3^i : \pm 1 : \pm 1 : 0)$, for $i = 1, 2$.

Nevertheless, for $i = 0, 1, 2$, blowing up X_{t_i} at its singular points, we do obtain a K3 surface.

Let η be the generic point of $\mathbb{A}^1$ and let $K = \kappa(\eta)$ denote the residue field of η, that is, the function field $\mathbb{Q}(t)$. Fix an algebraic closure $\overline{K}$ of K such that $\overline{\mathbb{Q}} \subset \overline{K}$. Consider the fiber $p^{-1}(\eta)/K$ of $\mathfrak{X} \subset \mathbb{A}^1 \times \mathbb{P}$ above η. The fiber $p^{-1}(\eta)$ naturally embeds into $\mathbb{P}_K$. We denote by X_η the image of $p^{-1}(\eta)$ inside $\mathbb{P}_K$. Then X_η is the surface over K given by the equation

$$X_\eta \colon w^2 = x^6 + y^6 + z^6 + tx^2y^2z^2. \tag{3.3}$$

By Proposition 3.1.1, the surface $X_\eta \subset \mathbb{P}_K$ is a K3 surface.

The main goal of this chapter is to give a description of the geometric Picard lattice of X_η; using this we can get information about the geometric Picard lattice of any fiber of $\mathfrak{X}$.

The first step in order to achieve the description of $\operatorname{Pic} \overline{X_\eta}$, is to compute the geometric Picard number of X_η.

Proposition 3.1.3. *The geometric Picard lattice of X_η has rank 19, that is, $\rho(\overline{X_\eta}) = 19$.*

Proof. See Subsection 3.3.3. □

Using some explicit divisors of $\overline{X_\eta}$ it is then possible to give a complete description of $\operatorname{Pic} \overline{X_\eta}$, as shown by the main theorem below.

Theorem 3.1.4. *Let η be the generic point of $\mathbb{A}^1$. Then the generic fiber $X_\eta = p^{-1}(\eta)$ of $\mathfrak{X}$ is a K3 surface with geometric Picard lattice isometric to the lattice*

$$U \oplus E_8(-1) \oplus A_5(-1) \oplus A_2(-1) \oplus A_2(-4). \tag{3.4}$$

The proof of the theorem is given in two steps: first finding some divisors on the surface and computing the lattice Λ they generate, then proving that Λ is the full geometric Picard lattice.

3.2 An automorphism subgroup of the Picard lattice

Isometries of the Picard lattice of a K3 surface can be very useful in order to find divisors (cf. Section 3.3). We have seen that an automorphism of a K3 surface induces an (effective) isometry of the Picard lattice. In this section, using the symmetries of the equation defining X_η, we provide some automorphisms of X_η, and hence some (effective Hodge) isometries of $\operatorname{Pic} X_\eta$.

Let K and $\overline{K}$ be defined as before. Let $\zeta_{12} \in \overline{\mathbb{Q}} \subset \overline{K}$ be a primitive 12-th root of unity and define $\zeta_6 := \zeta_{12}^2$, $\zeta_4 := \zeta_{12}^3$, and $\zeta_3 := \zeta_{12}^4$.

Remark 3.2.1. Note that ζ_i is a primitive i-th root of unity, for $i \in \{3, 4, 6\}$.

Let $\mathbb{Q}(\zeta_3)$ be the number field obtained by adjoining ζ_3 to $\mathbb{Q}$, i.e., the 3rd cyclotomic field. Since $\zeta_6 = \zeta_3 + 1$, we have that $\zeta_6 \in \mathbb{Q}(\zeta_3)$.

Throughout this section, and also in the following ones, we will use the notation μ_n, C_n, D_n, and S_n to denote respectively the group of n-th roots of unity inside $\overline{\mathbb{Q}} \subset \overline{K}$, the cyclic group of order n, the group of symmetries of the regular n-polygon (that is, the dihedral group of order $2n$), and the permutation group of a set with n elements, for any positive integer n.

Consider the following automorphisms of $\mathbb{P}_{\mathbb{Q}(\zeta_3)}$:

- For any permutation σ of the set $\{x, y, z\}$ of coordinate functions of $\mathbb{P}_{\mathbb{Q}(\zeta_3)}$, consider the induced automorphism $\bar{\sigma}\colon \mathbb{P}_{\mathbb{Q}(\zeta_3)} \to \mathbb{P}_{\mathbb{Q}(\zeta_3)}$ defined by

$$\bar{\sigma}\colon P \mapsto (\sigma(x)(P) : \sigma(y)(P) : \sigma(z)(P) : w(P)).$$

- For any triple $(i, j, k) \in (\mathbb{Z}/6\mathbb{Z})^3$ such that $2(i+j+k) \equiv 0 \mod 6$ consider the automorphism $\psi_{i,j,k}\colon \mathbb{P}_{\mathbb{Q}(\zeta_3)} \to \mathbb{P}_{\mathbb{Q}(\zeta_3)}$ defined by

$$\psi_{i,j,k}\colon (x : y : z : w) \mapsto (\zeta_6^i x : \zeta_6^j y : \zeta_6^k z : w).$$

Remark 3.2.2. Since $\mathbb{P}_{\mathbb{Q}(\zeta_3)}$ is a weighted projective space with weights $(1, 1, 1, 3)$, we have that

$$(x : y : z : w) = (\zeta_3 x : \zeta_3 y : \zeta_3 z : w) = (\zeta_3^2 x : \zeta_3^2 y : \zeta_3^2 z : w)$$

and therefore $\psi_{4,4,4} = \psi_{2,2,2} = \mathrm{id}$. One can easily check that no other automorphism $\psi_{i,j,k}$ equals the identity.

The above automorphisms of $\mathbb{P}_{\mathbb{Q}(\zeta_3)}$ can be extended to automorphisms of $\mathbb{P}_{\overline{K}}$. Let $G_1, G_2 \subset \operatorname{Aut}(\mathbb{P}_{\overline{K}})$ be the subgroups of $\operatorname{Aut}(\mathbb{P}_{\overline{K}})$ generated by the extension to $\mathbb{P}_{\overline{K}}$ of the automorphisms $\bar{\sigma}$ and $\psi_{i,j,k}$, respectively. We define $G = \langle G_1, G_2 \rangle$ the subgroup of $\operatorname{Aut}(\mathbb{P}_{\overline{K}})$ generated by the elements of G_1 and G_2.

Let ψ and ς be two elements of G_1 and G_2, respectively. One can easily see that the automorphism given by $\varsigma^{-1}\psi\varsigma$ is an element of G_2. We can then define an action of G_1 on G_2, by sending $(\varsigma, \psi) \in G_1 \times G_2$ to $\varsigma^{-1}\psi\varsigma \in G_2$. Let $G_1 \ltimes G_2$ denote the semidirect product of G_1 and G_2, with G_1 acting on G_2 as described above. It is easy to see that $G = G_1 \ltimes G_2$.

With the following results we give a description of G_1, G_2, and G as abstract groups. Let $\Sigma \subset \mu_6^3$ be the subgroup of $\mu_6^3 = \mu_6 \times \mu_6 \times \mu_6$ defined by

$$\Sigma := \{(\zeta, \xi, \theta) \in \mu_6^3 \ : \ \zeta\xi\theta = \pm 1\}.$$

Remark 3.2.3. The group Σ is isomorphic to $\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \times \{0, 3\}$. To see this, let (ζ, ξ, θ) be an element of Σ. Since $\zeta, \xi, \theta \in \mu_6$, there are $i, j, k \in \{0, 1, ..., 5\}$ such that $\zeta = \zeta_6^i, \xi = \zeta_6^j, \theta = \zeta_6^k$; since $\zeta\xi\theta = \pm 1$, we have that $i + j + k \in \{0, 3\}$. Then the map $\Sigma \to \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \times \{0, 3\}$ given by

$$(\zeta, \xi, \theta) \to (i, j, i + j + k)$$

is well defined and in fact it is an isomorphism of groups.

Let $\Delta \colon \mu_3 \hookrightarrow \mu_6^3$ be the embedding defined by

$$\Delta \colon \zeta \to (\zeta, \zeta, \zeta).$$

It is easy to see that the image of Δ is a normal subgroup of Σ. Let H denote the quotient group

$$H := \Sigma / \operatorname{im}(\Delta). \tag{3.5}$$

Remark 3.2.4. As an easy exercise in group theory, one can show that the group H is isomorphic to the group $C_2^2 \times C_6$.

Lemma 3.2.5. *The following statements hold:*

i) G_1 *is isomorphic to the symmetric group* S_3*;*

ii) G_2 *is isomorphic to the group* H *defined in* (3.5)*;*

iii) G *is isomorphic to* $S_3 \ltimes H$*, where the action of* S_3 *on* H *is given by permuting the coordinates of the elements of* H*.*

Proof. i) Trivial from the definition of G_1. In fact, recalling the definition of $\bar{\sigma}$, the map

$$\bar{\sigma} \mapsto \sigma$$

gives an isomorphism between G_1 and S_3.

ii) Let (ζ, ξ, θ) be an element of Σ and let i, j, k be defined as in Remark 3.2.3. Then $i+j+k \in \{0, 3\}$ or, equivalently, $2(i+j+k) \equiv 0 \mod 6$. We can then consider the map $\Sigma \to G_2$ given by

$$(\zeta, \xi, \theta) \mapsto \psi_{i,j,k}.$$

The map is clearly surjective; by Remark 3.2.2, it follows that the kernel is the subgroup $\{(0,0,0),(2,2,2),(4,4,4)\}$; so

$$G_2 \cong \Sigma/\{(0,0,0),(2,2,2),(4,4,4)\} = H,$$

concluding the proof.

iii) The statement trivially follows by recalling that $G = G_1 \ltimes G_2$ and then applying the isomorphisms used to prove points *i)* and *ii)*.

□

Corollary 3.2.6. *The group* G *has cardinality* $2^4\,3^2$.

Proof. By Lemma 3.2.5.(i), $G_1 \cong S_3$ and so $\#G_1 = 3! = 6$.

By Lemma 3.2.5.(ii), $G_2 \cong H$, with $H = \Sigma/\operatorname{im}(\Delta)$. The group Σ has cardinality $6^2\,2$ (cf. Remark 3.2.3); H is a quotient of Σ by a subgroup of order 3, hence $\#H = 6^2\,2/3 = 6 \cdot 2^2$. Alternatively, one can use Remark 3.2.4.

Since $G = G_1 \ltimes G_2$, it follows that $\#G = \#G_1 \cdot \#G_2 = 6 \cdot (6 \cdot 2^2)$, proving the statement. □

Lemma 3.2.7. *All the elements of* G *fix the surface* X_η.

Proof. To prove the statement it is enough to check that the automorphisms of G fix the equation defining X_η. □

Using Lemma 3.2.7, we can define the map $\mathrm{res}_\eta\colon G \to \mathrm{Aut}(X_\eta)$, sending an element of G to the automorphism of X_η it induces.

Lemma 3.2.8. *The map* $\mathrm{res}_\eta\colon G \to \mathrm{Aut}(X_\eta)$ *is injective.*

Proof. The statement is equivalent to saying that every element of G induces a non-trivial automorphism of X_η. The fixed subspace of a non-trivial element of G is a subspace defined by n linear equations, with $n \in \{1, 2, 3\}$, and therefore it cannot contain the surface X_η. □

With abuse of notation, we will use the symbols $\bar{\sigma}, \psi_{i,j,k}$ both for the automorphisms of $\mathbb{P}_{\overline{K}}$ and X_η; we will also use G to indicate both the subgroup of $\mathrm{Aut}(\mathbb{P}_{\overline{K}})$ and the image of res_η.

Remark 3.2.9. Since $\pi\colon X_\eta \to \mathbb{P}^2$ is a double cover of $\mathbb{P}^2$, one can consider the involution ι of X_η given by switching the elements inside the fibers of π. Keeping in mind the equation of X_η, we have that ι is given by

$$\iota\colon (x : y : z : w) \mapsto (x : y : z : -w).$$

Then it follows that $\iota = \psi_{3,3,3} \in G$.

Corollary 3.2.10. *Let P be a (not necessarily closed) point of $\mathbb{A}^1_{\mathbb{Q}}$, and let X_P be the K3 surface corresponding to the fiber of $\mathfrak{X}$ over P. Assume that its geometric Picard group has odd rank and discriminant not a power of 2. Then the group G acts faithfully on* $\mathrm{Pic}\,\overline{X_P}$.

Proof. From Proposition 1.2.47, it follows that G embeds into $\mathcal{O}(\mathrm{Pic}\,\overline{X_P})$. □

Let $G_s \subset G$ denote the subgroup of G given by the symplectic automorphisms of X_η in G.

Lemma 3.2.11. *The subgroup G_s has cardinality* 72 *and it is generated by* $\psi_{3,3,3} \circ \overline{\sigma}_{(12)}$, $\overline{\sigma}_{(123)}$, $\psi_{2,4,0}$, $\psi_{0,3,3}$, *and* $\psi_{3,0,3}$.

Proof. First notice that the involution $\psi_{3,3,3}$ fixes infinitely many points of X (in fact, it fixes the ramification locus R), and so, by Proposition 1.2.49, it is not symplectic; it follows that G_s has index at least 2 inside G, and so $\#G_s \leq 72$.

Again using Proposition 1.2.49, one can check that all the automorphisms listed in the statement are symplectic. Then, by easy computations, one sees that they generate a subgroup of order 72. □

Remark 3.2.12. Using `MAGMA`, one can easily check that G_s is isomorphic to the group $\mathfrak{A}_{4,3}$; this group is called `SmallGroup(72,43)` in `MAGMA` and GAP. See [Fes16] for these computations. We refer to [Has12, Appendix: computations using GAP] and [GAP16] for more details about groups database in GAP.

Remark 3.2.13. The elements of G_s have either order 2 or 3. So, by Proposition 1.2.49, it follows that $\rho(\overline{X_\eta}) \geq 13$.

3.3 Some divisors on X_η

In this section we will explain how we found some divisors on X_η generating a rank 19 sublattice of $\operatorname{Pic} \overline{X_\eta}$.

We have three main tools to find divisors on X_η:

1. the structure on X_η of double cover of $\mathbb{P}^2$ (cf. Subsection 3.3.1);
2. the structure on X_η of double cover of a del Pezzo surface of degree 1 (cf. Subsection 3.3.2);
3. the automorphisms of X_η (cf. Section 3.2).

3.3.1 X_η as double cover of $\mathbb{P}^2$

Let $\pi\colon X_\eta \to \mathbb{P}^2_K$ be the map defined by

$$\pi\colon (x : y : z : w) \to (x : y : z).$$

Let $B_\eta \subset \mathbb{P}^2_K$ be the smooth sextic plane curve defined by

$$x^6 + y^6 + z^6 + tx^2y^2z^2 = 0. \tag{3.6}$$

Lemma 3.3.1. *The map π is a 2-to-1 map ramified above B_η.*

Proof. Let $P = (x_0 : y_0 : z_0)$ be a point of $\mathbb{P}^2_K$. Then it is easy to see that $\pi^{-1}(P) = \{(x_0 : y_0 : z_0 : w_0), (x_0 : y_0 : z_0 : -w_0)\}$, where $w_0 \in \overline{K}$ is a square root of the quantity $x_0^6 + y_0^6 + z_0^6 + tx_0^2y_0^2z_0^2$.

The ramification points of π are then the points whose coordinates make that quantity vanish, that is, the points $(x : y : z) \in \mathbb{P}^2$ lying on the curve defined in (3.6). This concludes the proof. $\square$

Recalling the notation introduced in Subsection 1.2.3, the curve B_η is the branch locus of π, and its pre-image R_η on X_η is the ramification locus. If no confusion arises, later we might drop the index η to denote B_η and R_η, writing just B and R.

Proposition 3.3.2. *Let $C \subset \mathbb{P}^2$ be an irreducible plane curve of degree $d \neq 6$, and let $D = \pi^*(C)$ be its pull-back via π. Assume that D splits into two irreducible components, say $D = D_1 + D_2$. Then neither D_1 nor D_2 is equal to a multiple of the hyperplane section in* $\operatorname{Pic} \overline{X_\eta}$.

Proof. Since π is a 2-to-1 map, the components D_1 and D_2 are both isomorphic to C, and they are switched by the involution $\psi_{3,3,3}$ (cf. Remark 3.2.9). This means that $D_1^2 = D_2^2$ and $D_1 \cdot H = D_2 \cdot H$, with H being the hyperplane section class. Since C has degree d and D is a double cover of C, we have that $D \cdot H = 2d$ and, by $D_1 \cdot H = D_2 \cdot H$, it follows that $D_1 \cdot H = D_2 \cdot H = d$.

The intersection $D_1 \cdot D_2$ is given by the points lying above the points of $C \cap B$. Recall that B is the branch locus, it has degree 6, and C intersects B with even multiplicity everywhere. Then $D_1 \cdot D_2 = 3d$. Combining this with

$$\begin{aligned} 2d^2 &= 2C^2 \\ &= \pi_*\pi^*(C)^2 \\ &= \pi_*(D)^2 \\ &= D^2 \\ &= (D_1 + D_2)^2 \\ &= D_1^2 + 2D_1 \cdot D_2 + D_2^2 \\ &= 2D_1^2 + 6d, \end{aligned}$$

and, therefore, $D_2^2 = D_1^2 = d^2 - 3d$.

Finally, recall that the hyperplane class H is the pull-back of the class of the line, and therefore $H^2 = 2$.

Then we can see that H and D_i, for any $i = 1, 2$, generate a lattice whose intersection matrix is

$$\begin{pmatrix} 2 & d \\ d & d^2 - 3d \end{pmatrix}. \tag{3.7}$$

The discriminant of the intersection matrix is $d(d-6)$. The integer d is the degree of a curve, so $d > 0$, and by hypotheses $d \neq 6$; therefore the discriminant is different from 0 and this proves that D_i, for $i = 1, 2$, is not linearly equivalent to any multiple of H. □

Remark 3.3.3. With the computations used to prove Proposition 3.3.2 one can also show that D_1 and D_2 are linearly independent: in fact, they generate a sublattice of $\operatorname{Pic} \overline{X_\eta}$ with Gram matrix

$$\begin{pmatrix} d^2 - 3d & 3d \\ 3d & d^2 - 3d \end{pmatrix}. \tag{3.8}$$

The determinant of (3.8) is $d^3(d-6)$, and so, if $d \neq 0, 6$, it is non-zero and it shows that D_1 and D_2 are linearly independent.

A sublattice of rank 2 is the most we can get from D_1, D_2 and H, even though these three divisor are pairwise linearly independent: recall that $D_1 + D_2 = D = dH$.

Remark 3.3.4. Combining Corollary 1.2.27 and Proposition 3.3.2 we have a useful criterion to find irreducible plane curves C such that the irreducible components C_1, C_2 of its pull-back on X_η are not linearly equivalent to the hyperplane section, and that therefore generate a sublattice of the geometric Picard lattice of rank 2. In order to find such a curve, we look for genus 0 plane curves intersecting B_η with even multiplicity everywhere.

The first try was given by looking for tri-tangent lines. We found that such lines do not exist. Then we started looking for plane conics. Looking for *all* the plane conics intersecting B_η with even intersection everywhere is complicated so, using the fact that B_η is given by a symmetric equation, we first looked for conics with symmetric equations too; in particular, we looked for diagonal conics and we found that all the diagonal conics with third roots of unity as coefficients intersect B_η with even multiplicity everywhere (cf. Proposition 3.3.13).

Remark 3.3.5. Even though it turned out that there exist no plane lines that are tri-tangent to B_η, it might happen that such lines exist for some special value of t. Indeed, we found that the branch locus B_{t_0} admits a tri-tangent line if and only if

$$t_0 = 0, -\frac{33}{2}\zeta, -5\zeta, \tag{3.9}$$

with $\zeta \in \mu_3$. For the computations see [Fes16].

3.3.2 X_η as double cover of a del Pezzo surface of degree 1

Let $\mathbb{P}_K(1,1,2,3)$ be the weighted projective space over $K = \mathbb{Q}(t)$ with coordinates x', y', z' and w'.

Let $\pi_z \colon \mathbb{P}_K \to \mathbb{P}_K(1,1,2,3)$ be the map defined by

$$\pi_z \colon (x:y:z:w) \mapsto (x:y:z^2:w).$$

It is easy to see that the map π_z is a 2-to-1 map ramified along the plane $\{z=0\} \subseteq \mathbb{P}_K$.

Let $X'_\eta \subset \mathbb{P}_K(1,1,2,3)$ be the surface defined by

$$X'_\eta \colon w'^2 = x'^6 + y'^6 + z'^3 + tx'^2y'^2z'.$$

Lemma 3.3.6. *The surface $X'_\eta \subset \mathbb{P}_K(1,1,2,3)$ is a del Pezzo surface of degree* 1.

Proof. From Proposition 1.2.59. □

Proposition 3.3.7. *The map $\pi_z \colon \mathbb{P}_K \to \mathbb{P}_K(1,1,2,3)$ induces a* 2*-to-*1 *morphism $X_\eta \to X'_\eta$, that is, $\pi_{z|X_\eta} \colon X_\eta \to X'_\eta$ is a double cover of X'_η.*

Proof. First notice that the map $\pi_z \colon (x:y:z:w) \to (x:y:z^2:w)$ sends points of X_η to points of X'_η. Then notice that π_z is defined everywhere on $\mathbb{P}$, hence it is defined everywhere on X_η. Let $(x':y':z':w')$ be a point of X'_η, and denote it by Q. It is easy to see that its preimage $\pi_z^{-1}(Q)$ in X_η is the set $\{(x':y':\pm\zeta:w')\}$, where ζ is an element in $\overline{K}$ such that $\zeta^2 = z'$. □

Remark 3.3.8. In fact, X'_η is not the only del Pezzo doubly covered by X_η. Exploiting the symmetry of X_η it easy to see, using the same argument as for X'_η, that the morphisms

$$\pi_x \colon (x:y:z:w) \mapsto (x^2:y:z:w),$$
$$\pi_y \colon (x:y:z:w) \mapsto (x:y^2:z:w),$$

from $\mathbb{P}_K$ to $\mathbb{P}(2,1,1,3)_K$ and $\mathbb{P}(1,2,1,3)_K$ respectively, induce on X_η a double cover structure of the del Pezzo surfaces of degree 1

$$X''_\eta\colon w''^2 = x''^3 + y''^6 + z''^6 + tx''y''^2z''^2$$

and

$$X'''_\eta\colon w'''^2 = x'''^6 + y'''^3 + z'''^6 + tx'''^2y'''z'''^2.$$

Remark 3.3.9. The structure of double cover of a del Pezzo surface of degree 1 on X_η can be used to obtain more divisors that are linearly independent. In fact, the Picard lattice of a del Pezzo surface of degree 1 over an algebraically closed field has rank 9 (cf. Corollary 1.2.58). Let $E_1, ..., E_9$ be generators of $\operatorname{Pic}\overline{X'_\eta}$. Then their pullbacks $\pi^*(E_1), ..., \pi^*(E_9)$ are nine linearly independent divisors on $\overline{X_\eta}$. Pulling back also nine generators of $\operatorname{Pic}\overline{X''_\eta}$ and $\operatorname{Pic}\overline{X'''_\eta}$ (see the definition of these surfaces in Remark 3.3.8) or, equivalently, considering the orbits of $\pi^*(E_1), ..., \pi^*(E_9)$ under the action of G_1, one gets $9 \times 3 = 27$ divisors on $\overline{X_\eta}$, generating a sublattice of $\operatorname{Pic}\overline{X_\eta}$ of rank 13.

3.3.3 Explicit divisors

We have seen that X_η can be endowed with two structures: the structure of double cover of the plane, and the structure of double cover of a del Pezzo surface of degree 1. Using these two structures we have been able to explicitly compute some divisors on X_η. Some of these divisors are not defined over $K = \mathbb{Q}(t)$, but only over some algebraic extension of K. In order to define them, we need to introduce some elements of $\overline{K}$.

Let ζ_{12}, ζ_6 and ζ_3 be defined as before (cf. 3.2), and define $\zeta_4 := \zeta_{12}^3$.

Remark 3.3.10. The element $\zeta_4 \in \overline{K}$ is a primitive 4-th root of unity.

Consider the elements $t + 3\zeta_3^i \in K(\zeta_3)$ for $i \in \{0,1,2\}$, and let $\beta_i \in \overline{K}$ be a square root of $t + 3\zeta_3^i$, i.e. $\beta_i^2 = t + 3\zeta_3^i$ for $i \in \{0,1,2\}$.

We denote by K_1 the field $K(\zeta_{12}, \beta_0, \beta_1, \beta_2)$.

Let $h(v) \in K[v]$ be the polynomial $h := v^3 + tv^2 + 4$ and let c_0, c_1 and c_2 be its roots in $\overline{K}$. The polynomial h has discriminant $\Delta = -16(t^3 + 27) = (4\zeta_4\beta_0\beta_1\beta_2)^2$. Let δ denote the element $4\zeta_4\beta_0\beta_1\beta_2$ inside K_1; one can easily check that δ is a square root of Δ. Let K_2 be the field obtained by adjoining c_0 to K_1, that is, K_2 is the field

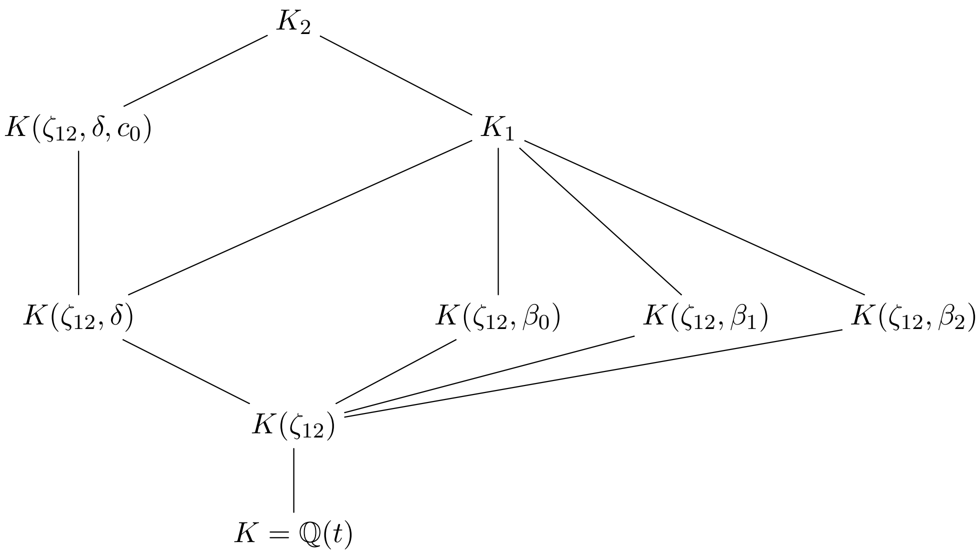


Figure 3.1: The field-diagram showing the construction of K_2 stated above.

$K(\zeta_{12}, \beta_0, \beta_1, \beta_2, c_0)$. We will see later (cf. Lemma 3.4.2) that also c_1 and c_2 are contained in K_2.

Let $\mathcal{D}' = \{D'_1, ..., D'_4\}$ be the set of divisors on X_η given by

$$D'_1\colon \begin{cases} x^2 + y^2 + \zeta_3 z^2 &= 0 \\ w - \beta_1 xyz &= 0 \end{cases}$$

$$D'_2\colon \begin{cases} x^2 + \zeta_3 y^2 + \zeta_3^2 z^2 &= 0 \\ w - \beta_0 xyz &= 0 \end{cases}$$

$$D'_3\colon \begin{cases} c_0 \delta x^2 - 2(9c_0^2 + 3tc_0 - 2t^2)xy + 2\delta y^2 - \delta z^2 &= 0 \\ (x^3 + a_3 x^2 y + b_3 xy^2 + c_3 y^3)(c_0^2 c_1 + 2) - 2w &= 0 \end{cases}$$

$$D'_4\colon \begin{cases} x^2 + y^2 + \zeta_3^2 z^2 &= 0 \\ w - \beta_2 xyz &= 0 \end{cases}$$

where

$$\begin{aligned} a_3 &= \frac{9c_0 + 6t}{4(t^3+27)}\delta, \\ b_3 &= -c_0^2 - tc_0, \\ c_3 &= \frac{18 - 3t^2c_0 - 3tc_0^2}{8(t^3+27)}. \end{aligned}$$

Remark 3.3.11. One can easily check that the curves $D_1', \ldots, D_4' \subseteq \mathbb{P}$ lie on X_η.

Remark 3.3.12. Although all the divisors listed above look like the pull-back of a plane conic, divisor D_3' was originally found as the pull-back of a generator of $\operatorname{Pic} \overline{X_\eta'}$.

For every $i = 1, \ldots, 4$, the divisor $D_i' \subset X_\eta$ is defined by two equations, namely $f_i = w - g_i = 0$, where f_i and g_i are two homogeneous polynomials in x, y, z of degree 2 and 3 respectively. Since the polynomial f_i has no w-term, we denote by C_i the conic of $\mathbb{P}^2_K$ it defines.

Proposition 3.3.13. *For every $i \in \{1, \ldots, 4\}$, the following statements hold:*

1. *the conic $C_i \subset \mathbb{P}^2_K$ intersects the branch locus B_η of π with even multiplicity everywhere;*
2. *the divisor D_i' of $\overline{X_\eta}$ is an irreducible component of the pull-back of C_i via π;*
3. *the curve $D_i' \subset \overline{X_\eta}$ is isomorphic to the conic C_i.*

Proof.

3. The restriction of π to D_i' induces an isomorphism to C_i. The inverse is given by the map $C_i \to D_i'$ sending $(x : y : z)$ to $(x : y : z : g_i(x, y, z))$.
2. It follows from 3.
1. The curve D_i' maps 1-to-1 to C_i, so it is not the only component. The statement follows from Corollary 1.2.27.

□

Let $G\mathcal{D}'$ denote the set $\{\, sD'_i \;:\; s \in G,\; i \in \{1,2,3,4\}\,\}$, obtained by letting the automorphisms of G act on the elements of $\mathcal{D}'$.

Let Λ' be the sublattice of $\operatorname{Pic} \overline{X_\eta}$ generated by the elements of $G\mathcal{D}'$.

Proposition 3.3.14. *The lattice Λ' is an even lattice of rank 19, signature $(1, 19)$, discriminant $2^{21}3^3$ and discriminant group isomorphic to $C_2^{16} \times C_6 \times C_{12}^2$.*

Before presenting the proof, we introduce some notations that will be useful in the proof and later in this chapter too.

Let k be any field, let A be the polynomial ring $k[v]$ and let F be the field of fractions of A, that is, $F = \operatorname{Frac} A = k(v)$. Fix an algebraic closure $\overline{F}$ of F, and let v_0 be an element inside $\overline{F}$. We define the *specialization of the field F to v_0*, denoted by F_{v_0}, the field $\operatorname{Frac} k[v_0]$. Note that F_{v_0} is a finite algebraic extension of k.

Example 3.3.15. Let t_0 be an element of $\overline{\mathbb{Q}}$. Then the specialization of $K = \mathbb{Q}(t)$ at t_0 is the number field $\mathbb{Q}(t_0)$.

Let $t_0 \in \mathbb{Z}$ be an integer, fix an integral model Ξ_{t_0} for the surface X_{t_0}. Let $p \in \mathbb{Z}$ be a prime of good reduction for Ξ_{t_0}, and let $\mathbb{F}_p$ denote the field with p elements.

Let K_{2,t_0} be the number field obtained by specializing

$$K_2 = \mathbb{Q}(\zeta_{12}, \beta_0, \beta_1, \beta_2, c_0)(t)$$

to $t = t_0$, let $\mathcal{O}_{t_0}$ denote the ring of integers of K_{2,t_0}, and let $\mathfrak{p}$ be a prime of $\mathcal{O}_{t_0}$ lying above p. Let $\kappa(\mathfrak{p})$ be the residue field $\mathcal{O}_{t_0}/\mathfrak{p}$. The field $\kappa(\mathfrak{p})$ is isomorphic to $\mathbb{F}_{p^m}$, for some $m \in \mathbb{Z}_{>0}$.

Let $X_{t_0,p}/\mathbb{F}_p$ be denote the reduction of Ξ_{t_0} modulo p. Let $B_{t_0,p} \subseteq \mathbb{P}^2_{\mathbb{F}_p}$ denote the branch locus of $X_{t_0,p}$.

Let D be one of the divisors of X_η in $G\mathcal{D}'$, and let $\overline{D}$ denote its Zariski closure inside $\mathfrak{X}$. Then $\overline{D}$ is a divisor of $\mathfrak{X}$. We define D_{t_0} to be the *specialization of $\overline{D}$ at* t_0, that is, the divisor on X_{t_0} obtained by taking the fiber of $\overline{D}$ above t_0. Note that not all the divisors of $G\mathcal{D}'$ can be specialized to any $t_0 \in \overline{\mathbb{Q}}$: in fact, for example, D'_3 cannot be specialized to $t = -3$. Assume that D can be specialized to t_0 and that $\mathfrak{p} \in \mathcal{O}_{t_0}$ is a prime of good reduction for Ξ_{t_0}. Then let $\overline{D_{t_0}}$ be the Zariski closure of D_{t_0} inside Ξ_{t_0}. We define $D_{t_0,\mathfrak{p}}$ to be the reduction modulo $\mathfrak{p}$ of $\overline{D_{t_0}}$. The curve $D_{t_0,\mathfrak{p}}$ is a divisor on $X_{t_0,p}/\mathbb{F}_p$ that can be defined over $\mathbb{F}_{p^m}$. Notice that the procedure of going from a divisor of X_η to a divisor of $X_{t_0,p}$ consists of the same step, repeated twice: taking the closure of a divisor of the generic fiber of a family and specialising it to a special fiber.

Proof of Proposition 3.3.14. The main step in order to prove the statement is to compute the intersection matrix $[D\cdot D']_{D,D'\in G\mathcal{D}'}$, that is, the intersection numbers $D\cdot D'$ for all the elements D, D' in the set $G\mathcal{D}'$. In doing so, it is helpful to recall that: the intersection form is symmetric, and so $D\cdot D' = D'\cdot D$; the surface X_η is a K3 surface and then, from the adjunction formula, it follows that if D is the divisor given by an irreducible curve with arithmetic genus g then $D^2 = 2g-2$. Let D be any divisor in $G\mathcal{D}'$. From Proposition 3.3.13, D is isomorphic to a plane conic C and, therefore, it has genus $g=0$. From this it follows that $D^2=-2$, that is, all the divisors in $G\mathcal{D}'$ have self intersection -2.

Computing the intersection number of two divisors defined over a function field is an expensive computation for a computer, this is why we reduce our computations to computations over finite fields. Fix an integer $t_0\in\mathbb{Z}$, and an integral model for X_{t_0}. Let p be a prime of good reduction for the fixed integral model of X_{t_0} and, recalling the notation introduced before starting the proof, let K_{2,t_0} be the specialization of K_2 to t_0, $\mathcal{O}_{t_0}$ be the ring of integers of K_{2,t_0} and $\mathfrak{p}$ be a prime of $\mathcal{O}_{t_0}$ lying above p. Using lemmas 1.2.51 and 1.2.52, if D, D' are two divisors on X_η, then $D\cdot D' = D_{t_0,\mathfrak{p}}\cdot D'_{t_0,\mathfrak{p}}$. Since all divisors $D\in G\mathcal{D}'$ are defined over K_2, all the divisors $D_{t_0,p}$ are defined over the finite field $\mathbb{F}_{p^m}$, for some $m\in\mathbb{Z}_{>0}$.

If $D_{t_0,p}$ and $D'_{t_0,p}$ have no components in common, then the intersection $D_{t_0,p}\cap D'_{t_0,p}$ is a zero-dimensional scheme over $\mathbb{F}_{p^m}$. Using `MAGMA` (cf. [BCP97]) it is possible to compute its degree. Since we are considering divisors on a smooth surface, the degree of the zero-dimensional scheme given by the intersection of the two divisors equals the sum of the intersection multiplicities of the points of intersection of the two divisors (see [HS00, A.2.3]), and so the degree of $D_{t_0,p}\cap D'_{t_0,p}$ is the intersection number $D_{t_0,p}\cdot D'_{t_0,p} = D\cdot D'$. In this way we get the intersection matrix of the lattice Λ' generated by $D\in G\mathcal{D}'$. Using the intersection matrix of the generators of a lattice, one is able to compute the rank, the signature, the determinant, and the discriminant group of the lattice. One can find the MAGMA code used to perform these computations, and that led to the results in the statement, in [Fes16]. □

Remark 3.3.16. Let X be a surface over a field k. In Theorem 1.2.4, we state that there is a unique integral pairing of $\operatorname{Div}\overline{X}$ satisfying the intuitive conditions that an intersection pairing should satisfy. Such intersection pairing can be explicitly defined as the alternating sum of the length of the Tor groups of the two divisors. On smooth surfaces, the only non-zero term of this sum is the first term, that coincides with the degree of zero-dimensional scheme defined by the intersection of two divisors; this is what we used in proving Proposition 3.3.14, in order to compute the intersection numbers of the divisors.

The above definition of the intersection pairing can be generalised to schemes of higher dimension, and in general it is not true that the intersection number equals the degree of scheme defined by the intersection of the two divisors. Also notice that in higher dimension, the intersection of the two divisors does not need to be zero-dimensional.

For the explicit definition of the intersection pairing and more details about this topic, see [Har77, Appendix A].

Remark 3.3.17. The divisors $D'_1, ..., D'_4$ are only some of the divisors we found using the methods described in subsections 3.3.1 and 3.3.2. They have been presented here because they form a minimal set of independent divisors such that their orbits under the action of G generate a rank 19 sublattice of $\operatorname{Pic}\overline{X_\eta}$. In fact, for any $j \in \{1, ..., 4\}$ the set

$$G(\mathcal{D}' - \{D'_j\}) = \{sD_i : s \in G,\ i \in \{1, ..., 4\} - \{j\}\}$$

generates a sublattice of $\operatorname{Pic}\overline{X_\eta}$ of rank at most 17 (see [Fes16] for the computations).

Having a finite-index sublattice has been very important in order to saturate Λ' and obtain the full geometric Picard lattice: in fact, it tells us which field all the classes of $\operatorname{Pic}\overline{X_\eta}$ are defined over (cf. Proposition 3.4.8), helping us in finding more divisors with computational methods (see Remark 3.5.1).

Proposition 3.3.14 tells us that the geometric Picard number of the generic fiber of $\mathfrak{X}$ is at least 19. A priori the rank of $\operatorname{Pic}\overline{X_\eta}$ could be also 20. In order to prove that this is not the case, that is, in order to prove Proposition 3.1.3, we need to show that the family $\mathfrak{X}$ is non-isotrivial. So we will show that there are two smooth fibers with different geometric Picard number (cf. Lemma 3.3.21). In fact, on the one hand it is possible to show that 19 is an upper bound for $\rho(\overline{X_{t_0}})$, for several values of $t_0 \in \mathbb{Q}$ (cf. Remark 3.3.18); on the other hand, it is possible to exhibit a concrete example of a fiber X_{t_0} with geometric Picard number equal to 20 (cf. Example 3.3.20).

Remark 3.3.18. As we have seen in the introduction of this section, there are several methods to give an upper-bound for the Picard number of a $K3$ surface. During the Arizona Winter school 2015, using methods described in [vL07], [EJ08b], [EJ08a], and [Har15], Stephan Elsenhans computed an upper bound for $\rho(\overline{X_{t_0}})$, for every $t_0 \in \mathbb{Q}$ with naïve height at most 10^4. For such a $t_0 \in \mathbb{Q}$, let $\ell(t_0)$ denote the upper bound computed by Elsenhans. Then $\ell(t_0) = 19$ for all the values considered, except for

$$t_0 = -255/4, -33/2, -5, 0, 8, 15/4, 24, 240, 1320. \tag{3.10}$$

For these values of t_0 the upper-bound trivially turns out to be 20.

These computations, together with Proposition 3.3.14, show that if $t_0 \in \mathbb{Q}$ is a number for which $\ell(t_0)$ has been computed and equals 19, then $\rho(\overline{X_{t_0}}) = 19$.

Remark 3.3.19. Notice that the rational values of t_0 for which B_{t_0} admits a tri-tangent line, i.e., the real values listed in (3.9), Remark 3.3.5, are contained in the values listed in (3.10).

Example 3.3.20. Let $t_0 = 0$ and consider the K3 surface $X_0 = X_{t_0}$. The ramification locus

$$B_0 \colon x^6 + y^6 + z^6 = 0$$

of X_0 admits tri-tangent lines, for example the line $L\colon y = \zeta_{12} z$. The pull-back $\pi^* L$ of L on X_0 via π splits into two irreducible components:

$$L_i = \begin{cases} y - \zeta_{12} z & = 0 \\ w + (-1)^i x^3 & = 0 \end{cases}$$

for $i = 1, 2$.

Let $\mathcal{D}_0 \subset \operatorname{Pic} \overline{X_0}$ be the set of classes of divisors of $\overline{X_0}$ given by the specialisation of the classes inside $\mathcal{D}'$ to $t_0 = 0$, and the class $[L_1]$. Let $G\mathcal{D}_0$ be the union of the G-orbits of the elements of $\mathcal{D}_0$.

Using the same technique used to prove Proposition 3.3.14, one can prove that the elements of $G\mathcal{D}_0$ generate a sublattice of $\operatorname{Pic}\overline{X_0}$ of rank 20. See [Fes16] for the explicit computations. To our knowledge, Masahiro Nakahara, a graduate student of Anthony Várilly-Alvarado, has been the first to point out that X_0 is a singular K3 surface.

Lemma 3.3.21. *The family $\mathfrak{X}$ is not isotrivial, that is, not all the smooth fibers are isomorphic.*

Proof. Remark 3.3.18 shows that there are many smooth fibers with geometric Picard number 19; Example 3.3.20 shows that the fiber $p^{-1}(0)$ is smooth and has geometric Picard number equal to 20. Then $\mathfrak{X}$ has at least two smooth fibers that are not isomorphic. □

We are now able to prove Proposition 3.1.3, that says that the geometric Picard number of X_η is at most 19.

Proof of Proposition 3.1.3. The family $\mathfrak{X}$ is parametrised by the affine line $\mathbb{A}^1$, which has dimension 1. Lemma 3.3.21 shows that $\mathfrak{X}$ is not isotrivial and so, by Theorem 1.2.50, the geometric Picard number of X_η can be at most 19.

On the other hand, the Picard lattice $\operatorname{Pic}\overline{X_\eta}$ contains the lattice Λ', that has rank 19, therefore $\operatorname{Pic}\overline{X_\eta}$ has rank at least 19.

The statement follows. □

Remark 3.3.22. It is possible to prove that $\rho(\overline{X_\eta}) \geq 19$ also without using the explicit divisors listed above. In fact we will show that there is a dominant rational map from X_η to a surface having geometric Picard number at least 19. From this, using Proposition 1.2.36.(4), it follows that also the geometric Picard number of X_η is at least 19.

Consider the dominant rational morphism $\phi\colon \mathbb{P} \dashrightarrow \mathbb{P}$ defined by

$$(x : y : z : w) \mapsto ((yz)^2 : (xz)^2 : (xy)^2 : (xyz)^3 w).$$

It is easy to see that ϕ maps the surface X to

$$X'\colon w^2 = (yz)^3 + (xz)^3 + (xy)^3 + t(xyz)^2.$$

The surface $X' \subset \mathbb{P}$ is not smooth: it has three D_4-singularities at $(0 : 0 : 1 : 0), (0 : 1 : 0 : 0), (1 : 0 : 0 : 0)$. Blowing up X' at

$Q = (0:0:1:0)$ we obtain the surface $X'' = \mathrm{Bl}_Q X' \subset \mathbb{P} \times \mathbb{P}^1(s_1, s_2)$ defined by

$$\begin{cases} s_1 y &= s_2 x \\ w^2 &= (yz)^3 + (xz)^3 + (xy)^3 + t(xyz)^2. \end{cases}$$

Consider the affine patch $X'' \cap \{x \neq 0, s_1 \neq 0\}$, defined by the equation

$$W^2 = s^3 Z^3 + Z^3 + s^3 + ts^2 Z^2,$$

where $W = w/x^3, Z = z/x$ and $s = s_2/s_1$. Eventually, after the change of variables

$$\begin{cases} W' &= (s^3+1)W \\ Z' &= (s^3+1)Z \end{cases}$$

one can see that $X'' \cap \{x \neq 0, s_1 \neq 0\}$ is birational to the surface given by

$$W'^2 = Z'^3 + ts^2 Z'^2 + s^3(s^3+1)^2,$$

that is a cubic base-change of the surface

$$E\colon W^2 = Z^3 + ts^2 Z^2 + s^5(s+1)^2,$$

obtained by sending s to s^3.

One can easily notice that the natural projection of $E \subseteq \mathbb{A}^2 \times \mathbb{A}^1$ onto $\mathbb{A}^1$, defined by

$$\pi_s\colon ((Z,W),s) \mapsto s,$$

induces an elliptic fibration $E \to \mathbb{A}_1$. The generic fiber E_ϵ of E has j-invariant

$$j(E_\epsilon) = -2^8 \frac{t^6 s^2}{(s+1)^2(27s^2 + (4t^3+54)s + 27)}$$

and discriminant

$$\Delta(E_\epsilon) = -2^4 s^{10}(s+1)^2(27s^2 + (4t^3+54)s + 27).$$

It follows that the fibers above the points $s = 0, -1, \gamma_1, \gamma_2$, with γ_i such that $\gamma_i^2 + (4t^3/27 + 2)\gamma_i + 1 = 0$, for $i = 1, 2$, are singular. Doing analogous computations on the affine patch $X'' \cap \{x \neq 0, s_2 \neq 0\}$ one can see that also the fiber above the point $\infty = (0:1)$ is singular. So E has five singular fibers, namely the fibers above the points

$$(0:1), (1:0), (1:-1), (1:\gamma_1), (1:\gamma_2).$$

Using the characterisation of singular fibers (for example, see [Sil94, Table IV.9.4.1]), the fibers above these points are of type II^*, II^*, I_2, I_1, and I_1, respectively (here we use Kodaira's notation for singular fibers, see [Kod64]) and hence they have $9, 9, 2, 1$, and 1 irreducible components, respectively. From Tate-Shioda formula (see [Shi90, Theorem 1.3 and Corollary 5.3]), it follows that $\rho(\overline{X''}) \geq 19$.

$$\begin{array}{ccccc} X & \overset{\phi}{\dashrightarrow} & X' & \overset{\mathrm{bl}_Q}{\dashrightarrow} & X'' \\ & & & & \downarrow{\scriptstyle \pi_s} \\ & & & & \mathbb{P}^1 \end{array}$$

Since the composition $\mathrm{bl}_Q \circ \phi$ is a dominant rational map, $\rho(\overline{X}) = \rho(\overline{X''})$ (cf. Proposition 1.2.36.(4)), and so $\rho(\overline{X}) \geq 19$.

Corollary 3.3.23. *The lattice* Λ' *is a proper finite-index sublattice of* $\operatorname{Pic} \overline{X_\eta}$.

Proof. By construction, the lattice Λ' is contained in $\operatorname{Pic} \overline{X_\eta}$, and they both have rank 19. Hence Λ' is a finite-index sublattice of $\operatorname{Pic} \overline{X_\eta}$.

Then all we need to show is that Λ is not equal to $\operatorname{Pic} \overline{X_\eta}$. Assume, by contradiction, that $\Lambda' = \operatorname{Pic} \overline{X_\eta}$. Then the transcendental lattice $T(\overline{X_\eta}) = (\operatorname{Pic} \overline{X_\eta})^\perp \subset H^2(X_\eta, \mathbb{Z})$ has rank $3 = 22 - 19$ and the discriminant lattice A_T is isomorphic to the discriminant lattice A_P of $\operatorname{Pic} \overline{X_\eta}$ (see Proposition 1.1.12). By Proposition 3.3.14 the discriminant group A_P is isomorphic to $C_2^{16} \times C_6 \times C_{12}^2$, implying that it cannot be generated by fewer than 19 elements, i.e., $\ell(A_P) = 19$. Since A_P and A_T are isomorphic, $\ell(A_T) = 19$. From Lemma 1.1.11 it then follows that $19 = \ell(A_T) \leq \operatorname{rk}(T(\overline{X_\eta})) = 3$, getting a contradiction. $\square$

Corollary 3.3.24. *The automorphism group* $\operatorname{Aut}(X_\eta)$ *embeds into the group of isometries of* $\operatorname{Pic} \overline{X_\eta}$.

Proof. By Proposition 1.2.47, the statement is true if $\operatorname{Pic} \overline{X_\eta}$ is a lattice of odd rank with discriminant not a power of 2. From Corollary 3.3.23

we know that $\operatorname{Pic}\overline{X_\eta}$ has rank 19 and that Λ' is a finite-index sublattice of $\operatorname{Pic}\overline{X_\eta}$. The lattice Λ' has discriminant $2^{21}\,3^3$ (cf. Proposition 3.3.14) and therefore, using Lemma 1.1.5, the discriminant $\operatorname{Pic}\overline{X_\eta}$ is congruent to 6 up to square factors. Hence, it cannot be a power of 2. □

3.4 The field of definition of $\operatorname{Pic}\overline{X_\eta}$

Even though we know that Λ' cannot be the full geometric Picard lattice of X_η (see Corollary 3.3.23), the fact that Λ' has finite index inside $\operatorname{Pic}\overline{X_\eta}$ allows us to say something about the field of definition of $\operatorname{Pic}\overline{X_\eta}$ (cf. Proposition 3.4.8).

Let us recall the notation introduced in Subsection 3.3.3. K is the field $\mathbb{Q}(t)$, we fixed an algebraic closure $\overline{K}$ of K such that $\overline{\mathbb{Q}} \subset \overline{K}$. The element $\zeta_n \in \overline{\mathbb{Q}}$ is a n-th root of unity, for $n \in \{3, 4, 6, 12\}$, such that $\zeta_n = \zeta_{12}^{12/n}$. For $i = 0, 1, 2$, the element $\beta_i \in \overline{K}$ is such that $\beta_i^2 = t + 3\zeta_3^i$. The field K_1 is the field $K = K(\zeta_{12}, \beta_0, \beta_1, \beta_2)$.

Remark 3.4.1. The elements $t + 3, t + 3\zeta_3$, and $t + 3\zeta_3^2$ generate a subgroup B of order 8 inside $K(\zeta_{12})^\times/(K(\zeta_{12})^\times)^2$. By Kummer theory (cf. Theorem [Mil15, Theorem 5.28]), the subgroup B corresponds to the extension of $K(\zeta_{12})$ obtained by adjoining the square roots of all the elements of B. The field we obtain adjoining these square roots is $K_1 = K(\zeta_{12}, \beta_0, \beta_1, \beta_2)$. Then, by Theorem [Mil15, Theorem 5.28], the extension $K_1/K(\zeta_{12})$ has degree 8 and exponent 2.

We defined $c_i \in \overline{K}$, for $i = 0, 1, 2$, to be the roots of the polynomial $h(v) := v^3 + tv^2 + 4 \in K[v]$. The polynomial h has discriminant $\Delta = -16(t^3 + 27) = (4\zeta_4\beta_0\beta_1\beta_2)^2$, and δ denotes the element $4\zeta_4\beta_0\beta_1\beta_2 \in K_1$, a square root of Δ. The field K_2 is the field obtained by adjoining c_0 to K_1, that is, $K_2 := K(\zeta_{12}, \beta_0, \beta_1, \beta_2, c_0)$.

Lemma 3.4.2. *The polynomial h splits completely over K_2, that is, c_0, c_1, c_2 are elements of K_2.*

Proof. By definition of K_2 we have that c_0 is in there. The discriminant of h is $\Delta = -16(D^3 + 27) = (4\zeta_4\beta_0\beta_1\beta_2)^2$, that is, Δ is a square inside K_1 and $\delta = 4\zeta_4\beta_0\beta_1\beta_2$ is one of its square roots. This means that adjoining a root of h to K_1 we get a splitting field for h. □

Remark 3.4.3. It is possible to explicitly write the roots c_1, c_2 in terms of the generators of K_2, the elements $\zeta_{12}, \beta_0, \beta_1, \beta_2, c_0$. Namely, one can then see that the other two roots of h are

$$\frac{-t - c_0 \pm \epsilon}{2},$$

where $\epsilon = \frac{\delta}{c_0(3c_0+2t)}$ and $\delta = 4\zeta_4\beta_0\beta_1\beta_2$, the square root of the discriminant Δ of h.

Let $E := K(\delta, c_0) \subset K_2$ be the field obtained by adjoining the elements $\delta, c_0 \in K_2$ to K.

Let $F := K(\beta_0) \subset K_2$ be the field obtained by adjoining $\beta_0 \in K_2$ to K.

Let $L := K(\beta_1, \beta_2) \subset K_2$ be the field given by adjoining $\beta_1, \beta_2 \in K_2$ to K.

Lemma 3.4.4. *The following statements hold.*

1. *The extension E/K is a Galois extension of degree* 6 *with Galois group* $\mathrm{Gal}(E/K) \cong S_3$.

2. *The extension F/K is a Galois extension of degree* 2 *with Galois group* $\mathrm{Gal}(F/K) \cong C_2$.

3. *The extension L/K is a Galois extension of degree* 8 *with Galois group* $\mathrm{Gal}(L/K) \cong D_4$.

4. *The fields E, F, and L intersects pairwise trivially, that is, the intersection of any two of them equals K.*

5. *The compositum field $E \cdot F \cdot L$ equals K_2.*

Proof. 1. By construction, the field E is the splitting field of the cubic polynomial $h = v^3 + tv^2 + 4$, that is irreducible over K and whose discriminant is not a square in K. The statement follows.

2. The field F is the splitting field of the second degree polynomial $v^2 - (3 + t)$. The statement trivially follows.

3. The field L is the splitting field of the polynomial

$$l = v^4 + (-2t + 3)v^2 + t^2 - 3t + 9,$$

and so L/K is a Galois extension. The roots of l are $\pm\beta_1, \pm\beta_2$, therefore the Galois group $\mathrm{Gal}(L/K)$ is generated by $\gamma_1, \gamma_2, \gamma$, where γ_1 changes the sign of β_1, γ_2 changes the sign of β_2, and γ switches β_1 and β_2. Since L/K is Galois, we have the following chain of equalities: $\#\,\mathrm{Gal}(L/K) = [L : K] = 8$. One can easily check that $\gamma\gamma_1 \neq \gamma_1\gamma$, and that these two are the only elements of order 4 of $\mathrm{Gal}(L/K)$. Summarising, $\mathrm{Gal}(L/K)$ is a non-abelian group of order 8 with exactly two elements of order 4. Then $\mathrm{Gal}(L/K)$ must be isomorphic to D_4.

4. By explicit computations.

5. The compositum field $E \cdot F \cdot L$ is by construction contained in K_2, since E, F, and L are all defined as subsets of K_2. Then, we only need to show that other inclusion. The field K_2 is obtained by adjoining $\zeta_{12}, \beta_0, \beta_1, \beta_2, c_0$ to K. From the definition of E, F, and L, the elements $\beta_0, \beta_1, \beta_2, c_0$ are inside $E \cdot F \cdot L$, as well as $4\zeta_4\beta_0\beta_1\beta_2 = \delta \in E$ (see Remark 3.4.3) and $\frac{1}{3}(\beta_1^2 - t) = \zeta_3 \in L$. Then the element
$$\alpha = \frac{4\zeta_3\beta_0\beta_1\beta_2}{4\zeta_4\beta_0\beta_1\beta_2} = \frac{\zeta_3}{\zeta_4}$$
is also inside $E \cdot F \cdot L$. Recall that $\zeta_3 = \zeta_{12}^4$ and $\zeta_4 = \zeta_{12}^3$, then
$$\alpha = \frac{\zeta_3}{\zeta_4} = \frac{\zeta_{12}^4}{\zeta_{12}^3} = \zeta_{12}.$$
This proves that $K_2 \subseteq E \cdot F \cdot L$ and, therefore, $K_2 = E \cdot F \cdot L$.

□

Theorem 3.4.5. *The field extension* K_2/K *is a Galois extension of degree* $2^5 \cdot 3$. *The Galois group* $\mathrm{Gal}(K_2/K)$ *is isomorphic to the group*
$$S_3 \times C_2 \times D_4.$$

Proof. By Lemma 3.4.4.(5) and (4) we have that $K_2 = E \cdot F \cdot L$ and that E, F, and L intersect pairwise trivially. It follows that
$$\mathrm{Gal}(K_2/K) \cong \mathrm{Gal}(E/K) \times \mathrm{Gal}(F/K) \times \mathrm{Gal}(L/K).$$

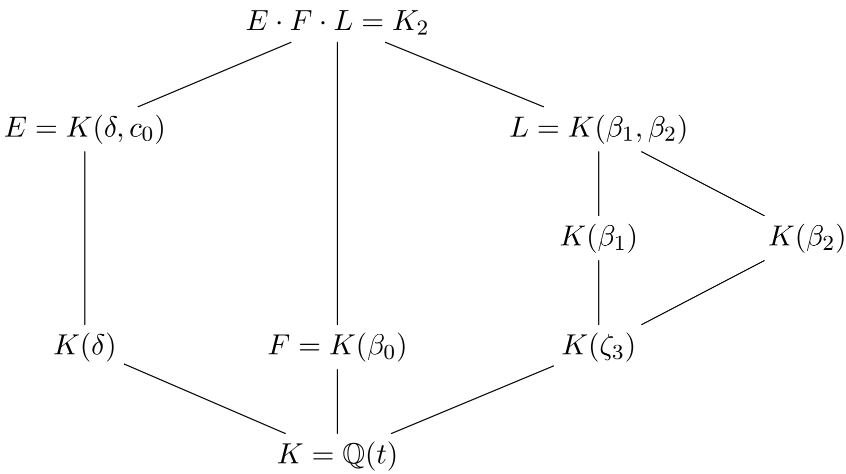


Figure 3.2: An alternative description of K_2.

From Lemma 3.4.4.(1)–(3) we know that

$$\begin{aligned} \mathrm{Gal}(E/K) &\cong S_3, \\ \mathrm{Gal}(F/K) &\cong C_2, \\ \mathrm{Gal}(L/K) &\cong D_4. \end{aligned}$$

The statement follows. □

Remark 3.4.6. Let t_0 be an element of $\overline{\mathbb{Q}}$, and let K_{t_0} and K_{2,t_0} the specializations to t_0 of K and K_2, respectively. Trivially, we have that $[K_{2,t_0} : K_{t_0}] \leq [K_2 : K]$; sometimes, this inequality can be strict. This is the case, for example, if $t_0 = \zeta_{12}$: in this case, in fact, we have that $[K_{2,t_0} : K_{t_0}] = 2^3\, 3$.

One might then ask whether there exists a t_0 such that K_{2,t_0} is exactly K_{t_0}. The answer to this question is positive: with the help of Maarten Derickx, we have been able to find an element $\alpha \in \overline{\mathbb{Q}}$, living in a number field of degree 64, such that $[K_{2,\alpha} : K_\alpha] = 1$. See [Fes16] for the explicit computations.

Remark 3.4.7. For future reference it can be useful to explicitly describe an isomorphism between $\mathrm{Gal}(K_2/K)$ and $S_3 \times C_2 \times D_4$. In order to do so, we will present five automorphisms $\tau_i \in \mathrm{Gal}(K_2/K)$, with

$i = 1, 2, 3, 4, 5$, such that:

$$\begin{aligned} \operatorname{Gal}(E/K) &= \langle \tau_1, \tau_2 \rangle \cong S_3; \\ \operatorname{Gal}(F/K) &= \langle \tau_3 \rangle \cong C_2; \\ \operatorname{Gal}(L/K) &= \langle \tau_4, \tau_5 \rangle \cong D_4. \end{aligned}$$

The field K_2 is generated by $c_0, \zeta_{12}, \beta_0, \beta_1, \beta_2$ over K, so to describe an element $\tau \in \operatorname{Gal}(K_2/K)$ it is enough to describe its action on those elements. The action of τ_i on those generators of K_2 over K is listed in the table below. For the convenience of the reader, the table also lists the action of τ_i, for $i = 1, ..., 5$, on other interesting elements of K_2.

	c_0	c_1	c_2	δ	ζ_{12}	ζ_4	ζ_3	β_0	β_1	β_2
τ_1	c_0	c_2	c_1	$-\delta$	ζ_{12}^7	$-\zeta_4$	ζ_3	β_0	β_1	β_2
τ_2	c_1	c_2	c_0	δ	ζ_{12}	ζ_4	ζ_3	β_0	β_1	β_2
τ_3	c_0	c_1	c_2	δ	ζ_{12}^7	$-\zeta_4$	ζ_3	$-\beta_0$	β_1	β_2
τ_4	c_0	c_1	c_2	δ	ζ_{12}^{11}	$-\zeta_4$	ζ_3^2	β_0	$-\beta_2$	β_1
τ_5	c_0	c_1	c_2	δ	ζ_{12}^7	$-\zeta_4$	ζ_3	β_0	β_1	$-\beta_2$

Recalling the notation introduced in Section 1.2, we say that K_2 is the field of definition of a class $D \in \operatorname{Pic} X_\eta$ if $\operatorname{Gal}(\overline{K}/K_2)$ is the stabilizer of D inside $G_K := \operatorname{Gal}(\overline{K}/K)$; we say that $D \in \operatorname{Pic} X_\eta$ can be defined over K_2 if $\operatorname{Gal}(\overline{K}/K_2)$ is contained in the stabilizer of D inside G_K.

We say that K_2 is *the field of definition of* $\operatorname{Pic}\overline{X_\eta}$ if $\operatorname{Gal}(K_2/K)$ acts freely on $\operatorname{Pic}\overline{X_\eta}$; we say that $\operatorname{Pic}\overline{X_\eta}$ *can be defined over* K_2 if all the elements of $\operatorname{Pic}\overline{X_\eta}$ can be defined over K_2.

Proposition 3.4.8. *The lattice* $\operatorname{Pic}\overline{X_\eta}$ *can be defined over* K_2.

Proof. First we claim that the lattice Λ' can be defined over K_2. In order to see this just notice that all the divisors in $\mathcal{D}'$ can be defined over K_2, as well as the automorphisms in G. Therefore we can conclude that all the divisors in $G\mathcal{D}'$ are fixed by $\operatorname{Gal}(\overline{K}/K_2)$ and hence, the lattice Λ' can be defined over K_2.

By Corollary 3.3.23, we know that Λ' is a finite-index sublattice in $\operatorname{Pic}\overline{X_\eta}$. Let $m \geq 1$ be the index $[\operatorname{Pic}\overline{X_\eta} : \Lambda']$. Now let N be an element of $\operatorname{Pic}\overline{X_\eta}$. Then mN is in Λ', that is, it can be written as linear combination of elements of $G\mathcal{D}'$. It follows that mN can be defined over K_2 and so $\operatorname{Gal}(\overline{K}/K_2)$ fixes mN. Since the Galois action is linear and $\operatorname{Pic}\overline{X_\eta}$ is torsion-free, it follows that $\operatorname{Gal}(\overline{K}/K_2)$ fixes N

too, i.e., N can be defined over K_2. The statement follows from the generality of N. □

Remark 3.4.9. The natural action of the absolute Galois group G_K on the geometric Picard group induces a map from G_K to the group of isometries of $\operatorname{Pic}\overline{X_\eta}$. Let H_K be the kernel of this map.

$$0 \to H_K \to G_K \to \mathcal{O}(\operatorname{Pic}\overline{X_\eta})$$

Then Proposition 3.4.8 can be rephrased by saying that $\operatorname{Gal}(\overline{K}/K_2)$ is contained in H_K. Later we will see that in fact $H_K = \operatorname{Gal}(\overline{K}/K_2)$, that is, K_2 is the field of definition of $\operatorname{Pic}\overline{X_\eta}$. (cf. Remark 3.7.4).

3.5 More divisors

By Corollary 3.3.23 we know that Λ' is not the full geometric Picard lattice of X_η. In order to generate $\operatorname{Pic}\overline{X_\eta}$ we then need more divisors on $\overline{X_\eta}$. Combining different techniques (cf. Remark 3.5.1) we managed to find more plane conics with splitting pull-back on $\overline{X_\eta}$.

Remark 3.5.1. If we add all the divisors coming from the del Pezzo surfaces of degree 1 of which X_η is a double cover (cf. Subsection 3.3.2) to the ones in the set $\mathcal{D}'$, and we take the union of their G-orbits, then this set generates a sublattice of $\operatorname{Pic}\overline{X_\eta}$ that is bigger than Λ', but that can still be proven not to be the full geometric Picard lattice, using an argument as in Corollary 3.3.23.

Failing in finding other six-tangent conics with particular symmetric equations, we decided to go for an extensive search. A generic conic inside $\mathbb{P}^2_K$ is given by a linear combination of the six monomials of degree 2 in x, y, z. The field K is a function field over an infinite field, and performing computations on K or over some algebraic extension of K, like K_2, requires a lot of computational power. Therefore an extensive search for six-tangent plane conics, running through all the possible 6-tuples of coefficients, looks infeasible over K_2 or K. This is why, once again, we reduced our computations to a finite field.

Fix an integer $t_0 \in \mathbb{Z}$, and an integral model for X_{t_0}. Let p be a prime of good reduction for the fixed integral model of X_{t_0} and recall the notation introduced after stating Proposition 3.3.14. Let K_{2,t_0} be the number field obtained by specializing K_2 to t_0, and let $\mathfrak{p}$ be a prime of $\mathcal{O}_{t_0}$ above p. Let $G\mathcal{D}'_{t_0,\mathfrak{p}} := \{(D)_{t_0,\mathfrak{p}} \;:\; D \in G\mathcal{D}'\}$ be the set given by first specializing to t_0 and then reducing modulo $\mathfrak{p}$ the divisors in $\mathcal{D}'$. Let Λ_0 be the sublattice generated by the divisors of $G\mathcal{D}'_{t_0,\mathfrak{p}}$. Notice that using the specialization and the reduction maps, we get an isometry between Λ' and Λ_0. From Proposition 3.4.8, it follows that Λ_0 can be defined over $\kappa(\mathfrak{p})$.

Let m be the positive integer for which $\kappa(\mathfrak{p}) = \mathbb{F}_{p^m}$. Then we run through all the 6-tuples $a = (a_0, ..., a_5) \in (\mathbb{F}_{p^m})^6$ such that the conic

$$C_a\colon a_0x^2 + a_1y^2 + a_2z^2 + a_3xy + a_4xz + a_5yz = 0$$

intersects $B_{t_0,p}$ with even multiplicity everywhere. Notice that in this case we have a finite number of 6-tuples to run through, 'only' p^{6m} (or $6\,p^{5m}$, if we assume at least one coefficient to always be non-zero), and that for a computer performing computations over a finite field is much easier than performing computations over (an algebraic extension of) a function field.

For each conic C_a found in this way, we compute the lattice Λ_a inside $\operatorname{Pic}\overline{X_{t_0,p}}$, generated by the irreducible components of the pull-back of C_a on $X_{t_0,p}$ and the divisors of $G\mathcal{D}'_{t_0,\mathfrak{p}}$.

Then the rank of Λ_a is greater than or equal to the rank of Λ_0 and, if equality holds, then $\det(\Lambda_a) \leq \det(\Lambda_0)$. Since the specialization map and the reduction map are both injective and both have torsion-free cokernel (cf. Propositions 1.2.51 and 1.2.52), if $\operatorname{rk}\Lambda_a = \operatorname{rk}(\Lambda_0)$ and $\det(\Lambda_a) < \det(\Lambda_0)$ then we know that C_a lifts to a curve $C \subset \mathbb{P}^2_{\overline{K}}$ such that: the pull-back D of C on $\overline{X_\eta}$ splits into two irreducible components (by Proposition 1.2.27); the classes inside $\operatorname{Pic}\overline{X_\eta}$ of the two irreducible components of D are not in Λ'. Then the components D, together with the divisors on $\overline{X_\eta}$ we already have, generate a bigger sublattice of $\operatorname{Pic}\overline{X_\eta}$.

Lifting the conic $C_a \subset \mathbb{P}^2_{\mathbb{F}_{p^m}}$ to a conic $C \subset \mathbb{P}^2_{K_2}$ was hard, since the coefficients of the equation defining C are roots of polynomials over K to be computed inside K_2. We divided this process into two steps:

1. first we lift C_a to a conic C_{t_0} defined over some number field;

2. then we lift C_{t_0} to a conic C defined over K_2.

The first step is the hardest one, and we accomplished it by looking at the equation of C_a, looking for symmetries and vanishing coefficients, hoping that these phenomena would reflect a symmetry or a vanishing coefficient also in characteristic 0 (this was not always the case). Such assumption would make the computations over K_2 easier, possibly easy enough for a computer to be handled.

The second step was accomplished by considering the coefficients of C_{t_0} for different values of t_0, and then interpolating these values in terms of t.

Let $\mathcal{D} = \{D_1, ..., D_5\}$ be the set of divisors on $\overline{X_\eta}$ given by

$$D_1 = D'_1, D_2 = D'_2, D_3 = D'_3,$$

and

$$D_4\colon \begin{cases} 2xy - c_1 z^2 & = 0 \\ x^3 - y^3 - w & = 0 \end{cases}$$

$$D_5\colon \begin{cases} a_5 x^2 + c_5(y^2 + z^2) + yz & = 0 \\ r_5 x^3 + v_5 xyz - w & = 0 \end{cases}$$

where

$$a_5 = \frac{\zeta_{12}(-\zeta_6 + 2)}{9}(\beta_0\beta_1 + \beta_0\beta_2 + \beta_1\beta_2 + t),$$
$$c_5 = \frac{\zeta_{12}(\zeta_6 - 2)}{3},$$
$$r_5 = \frac{\zeta_{12}(\zeta_6 - 2)}{9}(2\beta_0\beta_1\beta_2 + (2t - 3)\beta_0 + (2t - 3\zeta_3)\beta_1 + (2t + 3\zeta_6)\beta_2),$$
$$v_5 = -\beta_0 - \beta_1 - \beta_2.$$

Remark 3.5.2. The divisor D_4 was obtained by considering the generators of the geometric Picard lattice of a del Pezzo surface of which X_η is a double cover (see Subsection 3.3.2).

The divisor D_5 was found using the technique described in Remark 3.5.1.

For every $i = 1, ..., 5$, the divisor $D_i \subset \overline{X_\eta}$ is defined by two equations, namely $f_i = w - g_i = 0$, where f_i and g_i are two homogeneous polynomial in x, y, z of degree 2 and 3 respectively. Since the polynomial f_i has no w-term, we denote by C_i the conic of $\mathbb{P}^2_K$ it defines.

Proposition 3.5.3. *For every $i \in \{1, ..., 5\}$, the following statements hold:*

1. *the conic $C_i \subset \mathbb{P}^2_K$ intersects the branch locus B_η of π with even multiplicity everywhere;*
2. *the divisor D_i on $\overline{X_\eta}$ is an irreducible component of the pull-back of C_i via π;*
3. *the curve $D_i \subset \overline{X_\eta}$ is isomorphic to the conic C_i.*

Proof. Analogous to Proposition 3.3.13. □

Let $G\mathcal{D} := \{\, sD_i \ : \ s \in G,\ i \in \{1, ..., 5\} \,\}$ denote the set obtained by letting the automorphisms of G act on the elements of $\mathcal{D}$ and let Λ be the sublattice of $\operatorname{Pic}\overline{X_\eta}$ generated by the elements of $G\mathcal{D}$.

Proposition 3.5.4. *The lattice Λ is an even lattice of rank 19, signature $(1, 19)$, discriminant $2^5 3^3$ and discriminant group isomorphic to $C_6 \times C_{12}^2$.*

Proof. The proof goes as the proof of Proposition 3.3.14. See [Fes16]. □

Remark 3.5.5. The set $\mathcal{D}$ is minimal in order to obtain a lattice of rank 19 and discriminant $2^5\, 3^3$. For any $j \in \{1, ..., 5\}$, the set

$$G(\mathcal{D} - \{D_j\}) = \{sD_i : s \in G,\ i \in \{1, ..., 5\} - \{j\}\}$$

generates either a lattice of rank less than 19 or a lattice with rank 19 and discriminant at least $2^5\, 3^5$. See [Fes16] for the explicit computations in `MAGMA`.

Corollary 3.5.6. *The lattice Λ is isometric to the lattice* (3.4) *in Theorem 3.1.4.*

Proof. Let Σ be the lattice given in (3.4). Notice that both Λ and Σ are indefinite even lattices. From Proposition 3.5.4 we know that $\ell(A_\Lambda) = 3 < 17 = \operatorname{rk}(\Lambda) - 2$. Then, by Proposition 1.1.15, we have that Λ and Σ are isometric if and only if they have the same rank, signature, and discriminant group. One can easily see that these invariants of Σ are the same as the invariants of Λ given in Proposition 3.5.4. □

Corollary 3.5.7. *The lattice* Λ *is a finite-index sublattice of* $\operatorname{Pic} \overline{X_\eta}$*. The index* $[\operatorname{Pic} \overline{X_\eta} : \Lambda]$ *divides* 12*.*

Proof. In order to show that Λ is a finite-index sublattice it is enough to recall that the rank of $\operatorname{Pic} \overline{X_\eta}$ is 19 (cf. Corollary 3.3.23) and that Λ has indeed rank 19.

The second statement follows from Lemma 1.1.5, recalling that $\det \Lambda = 2^5\, 3^3$. □

3.6 The proof of the main theorem

In this section we show the proof of Theorem 3.1.4. The strategy of the proof is the same used by Michael Stoll and Damiano Testa in proving [ST10, Theorem 7].

Using the same notation as before, let $\Lambda \subseteq \operatorname{Pic} \overline{X_\eta}$ be the sublattice of $\operatorname{Pic} \overline{X_\eta}$ generated by the divisors inside $G\mathcal{D} := \{sD \ : \ s \in G,\ D \in \mathcal{D}\}$. In what follows, for the sake of easy notation, we will denote the geometric Picard lattice of X_η by simply P.

In the first part of the section we restate some results proved in Section 1.1, keeping in mind that $P = \operatorname{Pic} \overline{X_\eta}$ is an even lattice and Λ is a finite-index sublattice of P (cf. Corollary 3.5.7).

Let $p \in \mathbb{Z}$ be a prime and consider the quotient groups $\Lambda/p\Lambda$ and P/pP. They also have the structure of $\mathbb{F}_p$-vector spaces. As in Section 1.1, if x is an element of Λ, we denote by $[x]_\Lambda$ and $[x]_P$ its class inside $\Lambda/p\Lambda$ and P/pP respectively.

The inclusion map $\Lambda \hookrightarrow P$ induces the group homomorphism

$$\iota_p \colon \Lambda/p\Lambda \to P/pP$$

defined by $[x]_\Lambda \to [x]_P$.

Let Λ_p denote the kernel of ι_p.

Lemma 3.6.1. *The following equality holds:*

$$\Lambda_p = \frac{\Lambda \cap pP}{p\Lambda}.$$

Proof. Lemma 1.1.17. □

Let $[x]_P$ be an element of P/pP, and define the homomorphism

$$[x]^* \colon \Lambda/p\Lambda \to \mathbb{Z}/p\mathbb{Z}$$

by sending $[y]_\Lambda \in \Lambda/p\Lambda$ to $b_{P,p}([x]_P, [y]_P)$, where $b_{P,p}$ is defined as in Section 1.1. By lemmas 1.1.18 and 1.1.19 we can define the following morphism:

$$\phi_{P,p} \colon P/pP \to \operatorname{Hom}(\Lambda/p\Lambda, \mathbb{Z}/p\mathbb{Z}),$$

defined by sending $[x]_P$ to $[x]^*$. In the same way, we define the morphism

$$\phi_{\Lambda,p} \colon \Lambda/p\Lambda \to \operatorname{Hom}(\Lambda/p\Lambda, \mathbb{Z}/p\mathbb{Z}).$$

Let k_p denote the kernel of $\phi_{\Lambda,p}$.

Lemma 3.6.2. *Let p be any prime. The following diagram is commutative.*

$$\begin{array}{ccccccc}
0 & \longrightarrow & \Lambda_p & \hookrightarrow & \Lambda/p\Lambda & \xrightarrow{\iota_p} & P/pP \\
 & & \big\downarrow & & \big\| & & \big\downarrow \phi_{P,p} \\
0 & \longrightarrow & k_p & \hookrightarrow & \Lambda/p\Lambda & \xrightarrow[\phi_{\Lambda,p}]{} & \operatorname{Hom}(\Lambda, \mathbb{Z}/p\mathbb{Z})
\end{array}$$

Proof. See Lemma 1.1.21. □

Since Λ is an even lattice, we can define k'_p to be the subset of k_p given by

$$\{[\lambda]_\Lambda \in k_p \mid \lambda^2 \equiv 0 \bmod 2p^2\}.$$

Lemma 3.6.3. *Then k'_p contains Λ_p and it is fixed by all the isometries of Λ.*

Proof. See Lemma 1.1.23. □

Remark 3.6.4. Notice that while Λ_p also depends on $\operatorname{Pic} X_\eta$, the sets k_p and k'_p depend only on Λ.

Lemma 3.6.5. *The sublattice $\Lambda \subseteq \operatorname{Pic} \overline{X_\eta}$ is equal to $\operatorname{Pic} \overline{X_\eta}$ if and only if the map ι_p is injective for every prime p whose square divides $\det \Lambda$.*

Proof. See Proposition 1.1.24 and Remark 1.1.29. □

In Section 3.2 we provided the subgroup G of automorphisms of X_η and we showed that it embeds into the isometry group of $\operatorname{Pic}\overline{X_\eta}$. In Section 3.4 we computed the Galois group $\operatorname{Gal}(K_2/K)$, where K_2 is the splitting field of $\operatorname{Pic}\overline{X_\eta}$. The group $\operatorname{Gal}(K_2/K)$ acts on the classes of $\operatorname{Pic}\overline{X_\eta}$, by acting on the coefficients of a representative of a class inside $\operatorname{Pic}\overline{X_\eta}$. This action induces a group homomorphism

$$\operatorname{Gal}(K_2/K) \to \mathcal{O}(\operatorname{Pic}\overline{X_\eta}).$$

Let $\tilde{G}$ be the group generated by G and the image of $\operatorname{Gal}(K_2/K)$ inside $\mathcal{O}(\operatorname{Pic}\overline{X_\eta})$, and let $\tilde{\mathcal{D}}$ be the set of divisors obtained by letting $\tilde{G}$ act on $\mathcal{D}$, namely

$$\tilde{\mathcal{D}} := \{\, sD_i \ : \ s \in \tilde{G},\ i \in \{1, ..., 5\}\,\}.$$

Let $\tilde{\Lambda}$ be the sublattice of $\operatorname{Pic}\overline{X_\eta}$ generated by the divisors inside $\tilde{\mathcal{D}}$.

Proposition 3.6.6. *The following equality holds:*

$$\Lambda = \tilde{\Lambda}.$$

Proof. Since the group G embeds inside $\tilde{G}$, the lattice Λ is a sublattice of $\tilde{\Lambda}$. As in Proposition 3.3.14, one can see that $\tilde{\Lambda}$ has same rank and determinant as Λ. See [Fes16] for the explicit computations. The statement follows. □

Corollary 3.6.7. *The action of $\tilde{G}$ on $\operatorname{Pic}\overline{X_\eta}$ induces an action on Λ.*

Proof. The lattice Λ is the lattice generated by the divisors inside $\tilde{\mathcal{D}}$, and $\tilde{\mathcal{D}}$ is stable under the action of $\tilde{G}$. □

We now have all the ingredients to prove Theorem 3.1.4. The proof consists of two main steps. Let p be either 2 or 3. First we explicitly compute k'_p, a subset of $\Lambda/p\Lambda$ that contains Λ_p but that is independent of $\operatorname{Pic}\overline{X_\eta}$; then we show that every non-zero element inside k'_p is not an element of Λ_p.

Proof of Theorem 3.1.4. By Corollary 3.5.7 we know that the lattice Λ is of finite index inside $\operatorname{Pic}\overline{X_\eta}$. We want to show that Λ is the full geometric Picard lattice.

By Lemma 3.6.5 we have that to prove the statement it is enough to prove that the map ι_p is injective for every prime p whose square divides $\det \Lambda$. By Proposition 3.5.4 we have that Λ has discriminant $2^5\, 3^3$ and, therefore, to prove the statement it suffices to prove the injectivity of ι_p for $p = 2, 3$.

Let p be equal to 2 or 3. From Corollary 3.6.7 we know that $\tilde{G}$ acts on Λ. Since the $\mathbb{F}_p$-vector space Λ_p is the kernel of a $\tilde{G}$-equivariant homomorphism, it is $\tilde{G}$-invariant. So if an element $[x]$ is in Λ_p, its whole $\tilde{G}$-orbit $\tilde{G}[x]$ is contained in Λ_p. Since the discriminant of Λ is $2^5\, 3^3$, by Proposition 1.1.28, the $\mathbb{F}_p$-vector space Λ_p can have dimension at most 2 and 1, for $p = 2, 3$ respectively. Since Λ_p is stable under the action of $\tilde{G}$, it follows that the $\tilde{G}$-orbit of every element in Λ_p spans an $\mathbb{F}_p$-vector space of dimension at most 2 or 1, for $p = 2, 3$ respectively.

Analogous statements hold if we consider the action of just G, instead of the whole $\tilde{G}$.

Let $p = 2$. In [Fes16] we computed the subset k'_p. Inside k'_p we found only one non-trivial G-orbit spanning a vector subspace of dimension at most 2. Let W denote this subspace. The subspace W has dimension 2, and it admits a basis $\{w_1, w_2\}$ such that

$$w_1 = [E_1]_\Lambda$$
$$w_2 = [E_2]_\Lambda,$$

where $E_1 := \psi_{0,3,0} D_4 - D_4$ and $E_2 := \tau_2^2 \bar{\sigma}_{(x,y)}(\psi_{0,3,0} D_3 - D_3)$. Using the same technique used in Proposition 3.3.14 one is able to check that $E_1^2 = E_2^2 = -8$. Assume w_1 is an element of Λ_p, then E_1 is an element of Λ that is 2-divisible in P, say $E_1 = 2C$, for some $C \in P$. Since $E_1^2 = -8$, the class C is a -2-class, and then either C' or $-C'$ is effective (cf. Lemma 1.2.35). By construction $E_1 = E_{1,1} - E_{1,2}$, where $E_{1,1} = \psi_{0,3,0} D_4$ and $E_{1,2} = D_4$. Note that both $E_{1,1}$ and $E_{1,2}$ are elements of $\tilde{G}\mathcal{D}$, so $E_{1,1}^2 = E_{1,2}^2 = -2$. Let H be the hyperplane class in $\operatorname{Pic} X_\eta$, and notice that it is ample (in fact $3H$ is very ample). Since $E_1 = 2C$ with C a -2-class and H is ample, we have that the intersection number $H \cdot E_1 = 2H \cdot C$ is either positive or negative (according to whether C or $-C$ is effective); on the other hand, $E_1 = E_{1,1} - E_{1,2}$, and so $H \cdot E_1 = H \cdot E_{1,1} - H \cdot E_{1,2} = 2 - 2 = 0$, yielding a contradiction. Therefore E_1 cannot be 2-divisible. The same argument holds for E_2 as well as for any other element of W, since the orbit of every element of W spans the whole W, as in k'_p there are no 1-dimensional subspaces generated by $\tilde{G}$-orbits. So we have shown that ι_2 is injective.

Let $p = 3$. We computed the subset k'_p. Among the vectors in k'_p, we looked for those whose orbit under $\tilde{G}$ spans a 1-dimensional $\mathbb{F}_3$-vector space. There are no such vectors. See [Fes16] for the explicit computations.

In this way we proved that ι_3 is also injective, and therefore the injective morphism

$$\iota\colon \Lambda \to \operatorname{Pic} \overline{X_\eta}$$

is an isomorphism. □

3.7 Some consequences

Theorem 3.1.4 can be useful for gaining additional information about the geometric Picard lattice of every fiber of $\overline{\mathfrak{X}}$. Also, using the computations done to generate the lattice $\tilde{\Lambda}$, it is possible to compute the Galois module structure of $\operatorname{Pic} \overline{X_\eta}$.

Corollary 3.7.1. *Let $t_0 \in \overline{\mathbb{Q}}$ be an algebraic number. Then the surface X_{t_0} has either geometric Picard lattice isomorphic to* (3.4) *or geometric Picard number 20.*

Proof. By Lemma 1.2.51 we know that the specialization map

$$\operatorname{sp}_{t_0}\colon \operatorname{Pic} \overline{X_\eta} \to \operatorname{Pic} \overline{X_{t_0}}$$

is injective and has torsion free cokernel.

This implies that the rank of $\operatorname{Pic} \overline{X_{t_0}}$ is greater than or equal to 19 and that

$$\operatorname{Pic} \overline{X_{t_0}} / \operatorname{Pic} \overline{X_\eta} = \operatorname{coker}(\operatorname{sp}_{t_0}) \cong \mathbb{Z}^{\rho(\overline{X_{t_0}}) - \rho(\overline{X_\eta})}.$$

Since X_{t_0} is a K3 surface, the rank of $\operatorname{Pic} \overline{X_{t_0}}$ can be at most 20. So $\rho(\overline{X_{t_0}}) \in \{19, 20\}$.

Assume that $\rho(\overline{X_{t_0}}) = 19$. Then

$$\operatorname{coker}(\operatorname{sp}_{t_0}) = \operatorname{Pic} \overline{X_{t_0}} / \operatorname{Pic} \overline{X_\eta} \cong \mathbb{Z}^{19-19} = \{1\}$$

and therefore $\operatorname{Pic} \overline{X_\eta} \cong \operatorname{Pic} \overline{X_{t_0}}$. □

This result gives us some information about the rational points on each smooth fiber of $\mathfrak{X}$. Recall that if X is a $K3$ surface defined over a number field K, we say that X *has potentially dense rational points* if there is a finite field extension K'/K such that the set $X(K')$ of K'-rational points is Zariski dense inside $X(\mathbb{C})$.

Corollary 3.7.2. *Let $t_0 \in \overline{\mathbb{Q}}$ be an algebraic number such that X_{t_0} is smooth. Then the K3 surface X_{t_0} defined over the number field $\mathbb{Q}(t_0)$ admits an elliptic fibration. Also, X_{t_0} has potentially dense rational points.*

Proof. By Corollary 3.7.1 we have that $\rho(\overline{X_{t_0}}) \geq 19 > 5$. Then, by [Huy15, Proposition 11.1.3.(ii)], X_{t_0} admits an elliptic fibration. The second statement immediately follows from [BT00, Theorem 1.1] or [BT00, Theorem 1.4 and Remark 1.5]. □

Proposition 3.7.3. *The Galois group* $\operatorname{Gal}(K_2/K)$ *acts faithfully on* $\operatorname{Pic}\overline{X_\eta}$*, that is, K_2 is the field of definition of* $\operatorname{Pic}\overline{X_\eta}$.

Proof. By definition, the list $\tilde{\mathcal{D}}$ of divisors on $\overline{X_\eta}$ is stable under the action of $\tilde{G}$, and so it is stable under the action of $\operatorname{Gal}(K_2/K)$. Combining Proposition 3.6.6 and Theorem 3.1.4, the lattice generated by $\tilde{\Lambda}$ is $\operatorname{Pic}\overline{X_\eta}$. Using the action of $\operatorname{Gal}(K_2/K)$ on $\tilde{\Lambda}$, in [Fes16] we explicitly computed the 19×19 matrices representing the action of $\operatorname{Gal}(K_2/K)$ on $\operatorname{Pic}\overline{X_\eta}$. One can then check that none of these matrices is the identity matrix. □

Remark 3.7.4. Let G_K denote the absolute Galois group of K, and let H_K be the kernel of the map from G_K to $\mathcal{O}(\overline{X_\eta})$ induced by the action of G_K on $\operatorname{Pic}\overline{X_\eta}$.

$$0 \to H_K \to G_K \to \mathcal{O}(\operatorname{Pic}\overline{X_\eta})$$

In Remark 3.4.9 we have seen that $\operatorname{Gal}(\overline{K}/K_2)$ is contained in H_K. Then from Proposition 3.7.3 it follows that $\operatorname{Gal}(\overline{K}/K_2) = H_K$. From this we also get that $\operatorname{Gal}(K_2/K)$ embeds into $\mathcal{O}(\operatorname{Pic}\overline{X_\eta})$, in fact

$$\operatorname{Gal}(K_2/K) \cong \operatorname{Gal}(\overline{K}/K)/\operatorname{Gal}(\overline{K}/K_2) \cong G_K/H_K \hookrightarrow \mathcal{O}(\operatorname{Pic}\overline{X_\eta}).$$

Theorem 3.7.5. *Considering the action of* $\operatorname{Gal}(K_2/K)$ *on* $\operatorname{Pic}\overline{X_\eta}$*, the following statements hold.*

1. *$H^0(\mathrm{Gal}(K_2/K), \operatorname{Pic} \overline{X_\eta})$ is isomorphic to $\mathbb{Z}$ and it is generated by the class of the hyperplane section of X_η;*
2. *$H^1(\mathrm{Gal}(K_2/K), \operatorname{Pic} \overline{X_\eta})$ is isomorphic to C_2^3;*
3. *for every non-trivial subgroup $H \subseteq \mathrm{Gal}(K_2/K)$, we have*
$$H^1(H, \operatorname{Pic} \overline{X_\eta}) \cong C_2^i,$$
with $i \in \{0, 1, 2, 3, 4, 5, 6, 8, 10, 12\}$;
4. *there are 49 normal subgroups N of $\mathrm{Gal}(K_2/K)$ for which the group $H^1(N, \operatorname{Pic} \overline{X_\eta})$ is trivial;*
5. *there are 47 normal subgroups N of $\mathrm{Gal}(K_2/K)$ for which the group $H^1(N, \operatorname{Pic} \overline{X_\eta})$ is non-trivial.*

Proof. By explicit computations. See [Fes16]. □

Using Theorem 3.1.4, it is also possible to deduce information about the transcendental lattice.

Proposition 3.7.6. *The transcendental lattice $T(\overline{X_\eta})$ is isometric to a sublattice of $U(3) \oplus A_2(4)$ of rank 3, signature $(2, 1)$, determinant $2^5\,3^3$, and discriminant group isomorphic to $\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z}$.*

Proof. The lattice $T(\overline{X_\eta})$ is the orthogonal complement of $\operatorname{Pic} \overline{X_\eta}$ inside $H^2(X, \mathbb{Z})$. Then from Theorem 3.1.4 and Proposition 1.2.36 it immediately follows that $T(\overline{X_\eta})$ has rank 3 and signature $(2, 1)$.

From Theorem 3.1.4 and Proposition 1.1.12 it follows that $T(\overline{X_\eta})$ has determinant $2^5\,3^3$, and discriminant group $\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z}$.

We only need to show that $T(\overline{X_\eta})$ embeds into $U(3) \oplus A_2(4)$. In order to see this, recall that at the end of Section 3.2 we have seen that there is a subgroup G_s acting symplectically (and faithfully) on $H^2(\overline{X_\eta}, \mathbb{Z})$. Then, by Lemma 1.2.48, we have that $T(\overline{X_\eta})$ is contained in $H^2(\overline{X_\eta}, \mathbb{Z})^{G_s}$, the sublattice of $H^2(\overline{X_\eta}, \mathbb{Z})$ invariant under G_s. In [Has12], Hashimoto gives a complete list of abstract groups acting symplectically on the second cohomology group of a K3 surface. For each such group, he also computes the sublattice fixed by the group, that depends only on the abstract group and not on the surface. Keeping in mind that G_s is isomorphic to the group $\mathfrak{A}_{4,3}$ (cf. Remark 3.2.12), the statement follows from the tables in [Has12, Subsections 10.2 and 10.3]. □

Chapter 4

A determinantal quartic K3 surface with prescribed Picard lattice

In this chapter we present a determinantal quartic K3 surface whose Picard lattice is isomorphic to a particular lattice of rank 2. This construction is made interesting by Oguiso in [Ogu15], where he showed that K3 surfaces with such a Picard lattice admit a fixed point free automorphism of positive entropy and can be embedded into $\mathbb{P}^3$ as quartic surfaces. In [FGvGvL13], it is shown that in fact such surfaces can be embedded as *determinantal* quartic surfaces, and an explicit example of such a surface is provided, giving also an explicit description of the automorphism predicted by Oguiso. Here the contribution of the author of this thesis to that paper is presented, except for Proposition 4.2.2 and Remark 4.2.3, due to Bert van Geemen and Alice Garbagnati. All the material presented here is part of a joint work with Alice Garbagnati, Bert van Geemen, and Ronald van Luijk, and it can be found in [FGvGvL13].

4.1 The main result

Let k be any field, and let x_0, x_1, x_2, x_3 denote the coordinates of $\mathbb{P}^3_k$. Let $X \subset \mathbb{P}^3$ be a surface. We say that X is *determinantal* if it is defined by an equation of the form

$$X\colon \det M = 0,$$

where M is a square matrix whose entries are linear homogeneous polynomials in x_0, x_1, x_2, x_3.

Let $L = L_{(4,2,-4)}$ be the rank 2 lattice with Gram matrix

$$\begin{pmatrix} 4 & 2 \\ 2 & -4 \end{pmatrix}. \tag{4.1}$$

The following is the main result of this chapter.

Theorem 4.1.1. *Let $R = \mathbb{Z}[x_0, x_1, x_2, x_3]$ and let $M \in M_4(R)$ be any 4×4 matrix whose entries are homogeneous polynomials of degree 1 and such that M is congruent modulo 2 to the matrix*

$$M_0 = \begin{pmatrix} x_0 & x_2 & x_1 + x_2 & x_2 + x_3 \\ x_1 & x_2 + x_3 & x_0 + x_1 + x_2 + x_3 & x_0 + x_3 \\ x_0 + x_2 & x_0 + x_1 + x_2 + x_3 & x_0 + x_1 & x_2 \\ x_0 + x_1 + x_3 & x_0 + x_2 & x_3 & x_2 \end{pmatrix}. \tag{4.2}$$

Denote by X the complex surface in $\mathbb{P}^3$ given by $\det M = 0$. Then X is a K3 surface and its Picard lattice is isometric to L.

Remark 4.1.2. Let $\varphi \in \mathbb{R}$ be the real number given by

$$\varphi := \frac{1 + \sqrt{5}}{2},$$

and let $K := \mathbb{Q}(\varphi)$ be the number field obtained by adjoining φ to $\mathbb{Q}$; Notice that $K = \mathbb{Q}(\sqrt{5})$. Let $\mathcal{O}_K$ be the ring of integers of K. Then $\mathcal{O}_K = \mathbb{Z}[\varphi]$. The ring $\mathcal{O}_K$ has the structure of a $\mathbb{Z}$-module of rank 2, and $(1, \varphi)$ is a basis. If $x = a + b\varphi$ is an element of $\mathcal{O}_K$, denote by $\overline{x}$ the Galois conjugate of x. So, if $x = r + s\sqrt{5}$, then $\overline{x} = r - s\sqrt{5}$; it follows that $\overline{\varphi} = 1 - \varphi$, and hence

$$\overline{a + b\varphi} = a + b - b\varphi.$$

We define the bilinear form $b\colon \mathcal{O}_K \times \mathcal{O}_K \to \mathbb{Z}$ by

$$(x, y) \mapsto 2(x\overline{y} + \overline{x}y).$$

It is easy to see that b is a symmetric, non-degenerate bilinear form of $\mathcal{O}_K$. Then $(\mathcal{O}_K, b)$ is an integral lattice of rank 2. If we consider the basis $(1, \varphi)$, we immediately see that $(\mathcal{O}_K, b)$ is isometric to the lattice L defined in 4.1.

4.2 Proof of the main result

In this section we give a proof of Theorem 4.1.1. Let L be the lattice defined in 4.1, and let $R = \mathbb{Z}[x_0, x_1, x_2, x_3]$ and let $M \in M_4(R)$ be any 4×4 matrix whose entries are homogeneous polynomials of degree 1 and such that M is congruent modulo 2 to the matrix M_0 given in (4.2). From now until the end of the section, let X be the complex surface defined by $\det M = 0$.

We will first show that X is a K3 surface with a Picard lattice admitting L as sublattice. Then we will show that X has Picard number at most 2, and finally we will prove that L is the whole Picard lattice of X, hence proving Theorem 4.1.1.

Lemma 4.2.1. *Let X be defined as before; then X is smooth.*

Proof. Let X_2 be the surface over $\mathbb{F}_2$ defined by $\det M_0 = 0 \mod 2$. Using a computer, one can check that X_2 is smooth. Notice that X equals the reduction of X modulo 2. Then it follows that X is smooth. □

Proposition 4.2.2. *Let X be defined as before. Then X is a complex K3 surface and L can be embedded into* $\operatorname{Pic} X$*.*

Proof. It immediately follows from [FGvGvL13, Proposition 2.2]. □

Remark 4.2.3. As shown in the proof of [FGvGvL13, Proposition 2.2], it is easy to find two divisors of X generating inside $\operatorname{Pic} X$ a sublattice isometric to L. By [Bea00, Proposition 6.2], the surface X admits a projective normal curve of degree 6 and genus 3; let $C \in \operatorname{Pic} X$ be the class of that curve and let $H \in \operatorname{Pic} X$ be the hyperplane class. Then the sublattice $\langle H, C\rangle \subseteq \operatorname{Pic} X$ has Gram matrix

$$\begin{pmatrix} 4 & 6 \\ 6 & 4 \end{pmatrix}$$

and it is isometric to L.

The previous proposition implies that 2 is a lower bound for $\rho(X)$. To show that 2 is also an upper bound for $\rho(X)$, we follow [FGvGvL13, Section 5] and we use a method described in [vL07]. For the definition of the étale cohomology groups $H^i_{\text{ét}}(S, \mathbb{Q}_\ell)$ and $H^i_{\text{ét}}(S, \mathbb{Q}_\ell(1))$ for a scheme S, with values in $\mathbb{Q}_\ell$ or its Tate twist $\mathbb{Q}_\ell(1)$, we refer to [Tat65] and [Mil80, p. 163–165] .

Recall the definition of the étale cohomology groups given in 1.2.37. The following results show how to give an upper bound for the geometric Picard number of S.

Proposition 4.2.4. *Let K be a number field with ring of integers $\mathcal{O}$, let $\mathfrak{p}$ be a prime of $\mathcal{O}_K$ with residue field k, and let $\mathcal{O}_\mathfrak{p}$ be the localization of $\mathcal{O}$ at $\mathfrak{p}$. Let $\mathfrak{S}$ be a smooth projective surface over $\mathcal{O}_\mathfrak{p}$ and set $\overline{S} = \mathfrak{S} \times_{\mathcal{O}_\mathfrak{p}} \overline{K}$ and $S_{\overline{k}} = \mathfrak{S} \times_{\mathcal{O}_\mathfrak{p}} \overline{k}$. Let ℓ be a prime not dividing $q = \#k$. Let F_q^* denote the automorphism of $H^2_{\acute{e}t}(S_{\overline{k}}, \mathbb{Q}_\ell(1))$ induced by the q-th power Frobenius $F_q \in \mathrm{Gal}(\overline{k}/k)$.*

The rank of $\mathrm{Pic}\,\overline{S}$ is at most the number of eigenvalues of F_q^* that are roots of unity, counted with multiplicity.

Proof. Combining Lemma 1.2.52 and Remark 1.2.37 we get a chain of primitive embeddings of lattices

$$\mathrm{Pic}\,\overline{S} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \hookrightarrow \mathrm{Pic}\,S_{\overline{k}} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \hookrightarrow H^2_{\text{ét}}(S_{\overline{k}}, \mathbb{Q}_\ell(1)) \,,$$

and hence an upper bound for the rank of $\mathrm{Pic}\,S_{\overline{k}} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell$ is an upper bound for the rank of $\mathrm{Pic}\,\overline{S}$ too.

Let c be an element of $\mathrm{Pic}\,X_{\overline{k}}$; then c is represented by a divisor of $X_{\overline{k}}$, say $c = [C]$, for some $C \in \mathrm{Div}\,X_{\overline{k}}$. Since $\mathrm{Pic}\,S_{\overline{k}} \cong \mathrm{NS}\,S_{\overline{k}}$ is finitely generated (cf. Theorem 1.2.7), it follows that some power of Frobenius acts as the identity on $\mathrm{Pic}\,\overline{k}$. This means that the rank of $\mathrm{Pic}\,S_{\overline{k}}$ is at most the number of eigenvalues of F_q^* that are roots of unity, counted with multiplicity, and therefore the rank of $\mathrm{Pic}\,\overline{S}$ is so too.

See also [FGvGvL13, Proposition 5.2] and/or [vL07, Proposition 6.2 and Corollary 6.4]. □

Proposition 4.2.5. *Let S be a K3 surface over a finite field $k \cong \mathbb{F}_q$. As in Proposition 4.2.4, let F_q^* denote the automorphism of $H^2_{\acute{e}t}(S_{\overline{k}}, \mathbb{Q}_\ell(1))$*

induced by the q-th power Frobenius $F_q \in \mathrm{Gal}(\overline{k}/k)$, and for any n, let $\mathrm{Tr}((F_q^*)^n)$ denote the trace of $(F_q^*)^n$. Then we have

$$\mathrm{Tr}\left((F_q^*)^n\right) = \frac{\#S(\mathbb{F}_{q^n}) - 1 - q^{2n}}{q^n}.$$

Furthermore, the characteristic polynomial $f(t) = \det(t - F_q^*) \in \mathbb{Q}[t]$ of F_q^* has degree 22 and satisfies the functional equation

$$t^{22} f(t^{-1}) = \pm f(t).$$

Proof. Let F_S be the q-th power absolute Frobenius of S, which acts as the identity on the k-rational points of S and by raising to the q-th power on the coordinate rings of affine open subsets of X. The geometric Frobenius $\varphi = F_S \times 1$ on $S \times_k \overline{k} = \overline{S}$ is an endomorphism of $\overline{S}$ over $\overline{k}$ (cf. [Mil80, proof of V.2.6 and pages 290–291]). The set of fixed points of φ^n is $S(\mathbb{F}_{q^n})$. The Weil conjectures (see [Mil80, Section VI.12], recall that these were proven by Deligne) state that the eigenvalues of φ^* acting on $H^i_{\text{ét}}(\overline{S}, \mathbb{Q}_\ell)$ have absolute value $q^{i/2}$. Since S is a K3 surface, we have $\dim H^i_{\text{ét}}(\overline{S}, \mathbb{Q}_\ell) = 1, 0, 22, 0, 1$ for $i = 0, 1, 2, 3, 4$, respectively (see 1.2.37), so the Lefschetz trace formula for φ^n (see [Mil80, Theorems VI.12.3 and VI.12.4]) yields

$$\begin{aligned}\#S(\mathbb{F}_{q^n}) = \sum_{i=0}^{4} (-1)^i \,\mathrm{Tr}\left((\varphi^*)^n | H^i_{\text{ét}}(\overline{S}, \mathbb{Q}_\ell)\right) = \\ = 1 + \mathrm{Tr}\left((\varphi^*)^n | H^2_{\text{ét}}(\overline{S}, \mathbb{Q}_\ell)\right) + q^{2n}.\end{aligned} \tag{4.3}$$

For the remainder of this proof we restrict our attention to the middle cohomology, so $H^i_{\text{ét}}$ with $i = 2$. By the (proven) Weil conjectures, the characteristic polynomial $f_\varphi(t) = \det(t - \varphi^* | H^2_{\text{ét}}(\overline{S}, \mathbb{Q}_\ell))$ is a polynomial in $\mathbb{Z}[t]$ satisfying the functional equation $t^{22} f_\varphi(q^2/t) = \pm q^{22} f_\varphi(t)$ (note that the polynomial $P_2(X, t) = \det(1 - \varphi^* t | H^2_{\text{ét}}(\overline{S}, \mathbb{Q}_\ell))$ of [Mil80, Section VI.12], is the reverse of f_φ). Let $\varphi^*(1)$ denote the action on $H^2_{\text{ét}}(\overline{S}, \mathbb{Q}_\ell(1))$ (with a Tate twist) induced by φ. Note that the fact that $\varphi^*(1)$ acts on the middle cohomology is not reflected in the notation. The eigenvalues of $\varphi^*(1)$ differ from those of φ^* on $H^2_{\text{ét}}(\overline{S}, \mathbb{Q}_\ell)$ by a factor q (see [Tat65]), so we have

$$\mathrm{Tr}\left((\varphi^*)^n | H^2_{\text{ét}}(\overline{S}, \mathbb{Q}_\ell)\right) = q \cdot \mathrm{Tr}\left(\varphi^*(1)^n\right), \tag{4.4}$$

and the characteristic polynomial $f_\varphi^{(1)} \in \mathbb{Q}[t]$ of $\varphi^*(1)$ satisfies the functional equation $q^{22} f_\varphi^{(1)}(t) = f_\varphi(qt)$, and thus also the equation $t^{22} f_\varphi^{(1)}(1/t) = \pm f_\varphi^{(1)}(t)$. It follows that the eigenvalues, and hence the characteristic polynomials, of $\varphi^*(1)$ and $\varphi^*(1)^{-1}$ coincide. Finally, the product of the geometric Frobenius $\varphi = F_S \times 1$ and the Galois automorphism $1 \times F_q$ on $S \times_k \overline{k} = \overline{S}$ is the absolute Frobenius $F_{\overline{S}}$, which acts as the identity on the cohomology groups, so the maps $\varphi^*(1)$ and F_q^* act as inverses of each other (see [Mil80, Lemma VI.13.2 and Remark VI.13.5,] and [Tat65, Chapter 3]). We conclude $f = f_\varphi^{(1)}$ and $\mathrm{Tr}\left((F_q^*)^n\right) = \mathrm{Tr}(\varphi^*(1)^{-n}) = \mathrm{Tr}(\varphi^*(1)^n)$, which, together with (4.3) and (4.4), implies the proposition. □

Proposition 4.2.6. *Let X be defined as at the beginning of the section Then $\rho(X) \leq 2$.*

Proof. Let $\mathfrak{S}$ denote the surface over the localization $\mathbb{Z}_{(2)}$ of $\mathbb{Z}$ at the prime 2 given by $\det M = 0$, and write S' and $\overline{S'}$ for the reductions $\mathfrak{S}_{\mathbb{F}_2}$ and $\mathfrak{S}_{\overline{\mathbb{F}}_2}$, respectively. One checks that S' is smooth and $\mathfrak{S}$ is reduced, for instance with `MAGMA` [BCP97]. Since $\operatorname{Spec}\mathbb{Z}_{(2)}$ is integral and regular of dimension 1, the scheme $\mathfrak{S}$ is integral, and the map $\mathfrak{S} \to \operatorname{Spec}\mathbb{Z}_{(2)}$ is dominant, it follows from [Har77, Proposition III.9.7], that $\mathfrak{S}$ is flat over $\operatorname{Spec}\mathbb{Z}_{(2)}$. Since the fiber over the closed point is smooth, it follows from [Liu02, Definition 4.3.35], that $\mathfrak{S}$ is smooth over $\operatorname{Spec}\mathbb{Z}_{(2)}$. Therefore, $S = \mathfrak{S}_\mathbb{C}$ is smooth as well, so S and S' are K3 surfaces. Let F_2^* denote the automorphism of $H^2_{\text{ét}}(\overline{S'}, \mathbb{Q}_\ell(1))$ induced by Frobenius $F_2 \in \mathrm{Gal}(\overline{\mathbb{F}}_2/\mathbb{F}_2)$.

The divisor classes in $H^2_{\text{ét}}(\overline{S'}, \mathbb{Q}_\ell(1))$ defined by the hyperplane class and the curve C as in Remark 4.2.3 span a two-dimensional subspace V on which F_2^* acts as the identity. We denote the linear map induced by F_2^* on the quotient $W := H^2_{\text{ét}}(\overline{S}_2, \mathbb{Q}_\ell(1))/V$ by $F_2^*|_W$, so that $\mathrm{Tr}(F_2^*)^n = \mathrm{Tr}(F_2^*|_V)^n + \mathrm{Tr}(F_2^*|_W)^n = 2 + \mathrm{Tr}(F_2^*|_W)^n$ for every integer n. From Proposition 4.2.4, we obtain

$$\mathrm{Tr}(F_2^*|_W)^n = -2 + \frac{\#S'(\mathbb{F}_{2^n}) - 1 - 2^{2n}}{2^n} .$$

We counted the number of points in $S'(\mathbb{F}_{2^n})$ for $n = 1, \ldots, 10$ with `MAGMA`. The results are in the table below.

n	1	2	3	4	5	6	7	8	9	10
$\#S'(\mathbb{F}_{2^n})$	6	26	90	258	1146	4178	17002	64962	260442	1044786
$\mathrm{Tr}(F_2^*\|_W)^n$	$-\frac{3}{2}$	$\frac{1}{4}$	$\frac{9}{8}$	$-\frac{31}{16}$	$\frac{57}{32}$	$-\frac{47}{64}$	$\frac{361}{128}$	$-\frac{1087}{256}$	$-\frac{2727}{512}$	$-\frac{5839}{1024}$

If $\lambda_1, \ldots, \lambda_{20}$ denote the eigenvalues of $F_2^*|_W$, then the trace of $(F_2^*|_W)^n$ equals

$$\mathrm{Tr}(F_2^*|_W)^n = \lambda_1^n + \ldots + \lambda_{20}^n ,$$

i.e., it is the n-th power sum symmetric polynomial in the eigenvalues of $F_2^*|_W$. Let e_n denote the elementary symmetric polynomial of degree n in the eigenvalues of $F_2^*|_W$ for $n \geq 0$. Using Newton's identities

$$ne_n = \sum_{i=1}^{n} (-1)^{i-1} e_{n-i} \cdot \mathrm{Tr}(F_2^*|_W)^i$$

and $e_0 = 1$, we compute the values of e_n for $n = 1, \ldots, 10$. They are listed in the following table.

n	1	2	3	4	5	6	7	8	9	10
e_n	$-\frac{3}{2}$	1	0	0	0	0	$\frac{1}{2}$	0	-1	2

We denote the characteristic polynomial of a linear operator T by f_T, so that

$$f_{F_2^*} = f_{F_2^*|_V} \cdot f_{F_2^*|_W} = (t-1)^2 f_{F_2^*|_W} .$$

Because $f_{F_2^*}$ satisfies the functional equation of Proposition 4.2.5, the polynomial $f_{F_2^*|_W}$ satisfies $t^{20} f_{F_2^*|_W}(t^{-1}) = \pm f_{F_2^*|_W}(t)$. Since the middle coefficient $e_{10} = 2$ of t^{10} in $f_{F_2^*|_W}$ is nonzero, the sign in this functional equation is $+1$, so $f_{F_2^*|_W}$ is palindromic and we get

$$\begin{aligned} f_{F_2^*|_W} &= t^{20} - e_1 t^{19} + e_2 t^{18} - \cdots + e_{10} t^{10} - e_9 t^9 + \cdots - e_1 t + 1 \\ &= t^{20} + \tfrac{3}{2} t^{19} + t^{18} - \tfrac{1}{2} t^{13} + t^{11} + 2t^{10} + t^9 - \tfrac{1}{2} t^7 + t^2 + \tfrac{3}{2} t + 1. \end{aligned}$$

With `MAGMA`, one checks that this polynomial is irreducible over $\mathbb{Q}$, and as it is not integral, its roots are not algebraic integers, so none of its roots is a root of unity. Hence, the polynomial $f_{F_2^*} = (t-1)^2 f_{F_2^*|_W}$ has exactly two roots that are a root of unity. This implies that F_2^* has only two eigenvalues (counted with multiplicity) that are roots of unity, and so, by Proposition 4.2.4, it follows that the rank of the Picard group $\operatorname{Pic} S \cong \operatorname{Pic} \mathfrak{S}_{\overline{\mathbb{Q}}})$ is bounded by two from above. □

We have now all the elements to prove Theorem 4.1.1.

Proof of Theorem 4.1.1. By Proposition 4.2.2 we have $\rho(X) \geq 2$ and L can be embedded into $\operatorname{Pic} X$.

By Proposition 4.2.6 we have that $\rho(X) \leq 2$. It follows that $\rho(X) = 2$ and L is a finite index sublattice of $\operatorname{Pic} X$.

Since $\det L = -20$, from Lemma 1.1.5 it follows that the index $[\operatorname{Pic} X : L]$ can only be 1 or 2. Assume $[\operatorname{Pic} X : L] = 2$, and let D be an element of $\operatorname{Pic} X$ that is not in L. Let H, C be defined as in Remark 4.2.3, (namely the hyerplane section class and the class of a curve of degree 6 and degree 3), then (H, C) is a basis of L and $D = \frac{aH+bC}{2}$ It follows that $D^2 = a^2 + 3ab + b^2$. Since L is an even lattice, D^2 is even and so a and b are both even. Then $D = \frac{a}{2}H + \frac{b}{2}C$ is inside L, getting a contradiction. The contradiction comes from the assumption that $[\operatorname{Pic} X : L] = 2$. So $[\operatorname{Pic} X : L] = 1$ and this concludes the proof. □

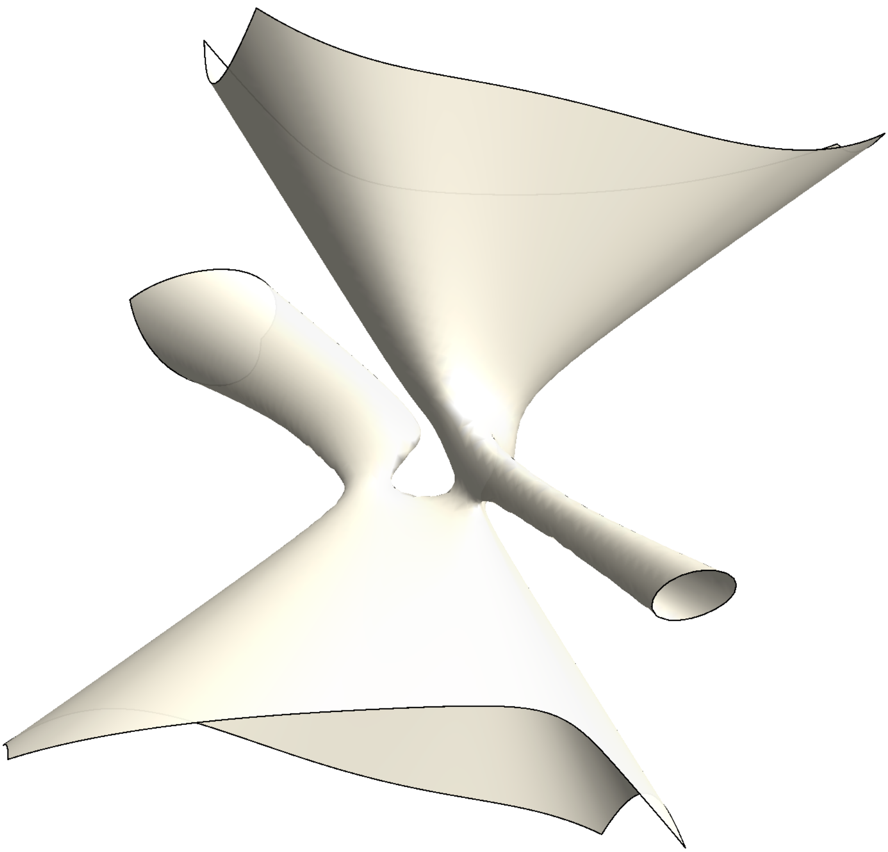


Figure 4.1: A visual rendition of the real points of an affine patch of the complex K3 surface given by $\det M_0 = 0$.

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Summary

In the thesis titled "Topics in the arithmetic of del Pezzo and K3 surfaces", the author presents the results he achieved during his PhD. Some of these results have already been published, or are anyway available online, see [FGvGvL13], [FvL15] and [FvL16].

The thesis consists of four chapters: the first one is meant to introduce the notation and the basic results that are needed in order to address the problems treated in the rest of the thesis; each of the other three chapters deals with a different problem.

In [STVA14], C. Salgado, D. Testa, and A. Várilly-Alvarado show that all del Pezzo surfaces of degree 2 over a finite field are unirational, except possibly for three isomorphism classes of surfaces. In 2015, Ronald van Luijk and the author of the thesis show that the statement holds also for the remaining three cases. This result is presented in the second chapter.

During the Arizona Winter School 2015, A. Varilly-Alvarado, one of the lecturer of the winter school, asked to compute the Galois module structure of the Picard lattice of the K3 surfaces in a given 1-dimensional family. Chapter 3 provides an answer to that question. The whole chapter is joint work with F. Bouyer, E. Costa, C. Nicholls, and M. West.

In [Ogu15], K. Oguiso proved that if a K3 surface S has Picard lattice isometric to a particular rank 2 lattice, then S admits a fixed point free automorphism of positive entropy and can be embedded into $\mathbb{P}^3$ as a quartic surface. In the same paper, Oguiso remarks that "it seems extremely hard but highly interesting to write down explicitly the equation of S and the action of g in terms of the global homogeneous coordinates of $\mathbb{P}^3$, for at least one of such pairs" (cf. [Ogu15, Remark

4.2]). In [FGvGvL13], A. Garbagnati, B. van Geemen, R. van Luijk, and the author of the thesis provide an explicit example of such S and g, described using the global homogeneous coordinates of $\mathbb{P}^3$. The contribution of the author of the thesis to the paper is presented in the fourth and last chapter.

Samenvatting

In dit proefschrift, getiteld “Topics in the arithmetic of del Pezzo and K3 surfaces”, beschrijft de auteur zijn resultaten behaald tijdens zijn doctoraalonderzoek. Een gedeelte van deze resultaten is reeds gepubliceerd, of is online beschikbaar, zie [FGvGvL13], [FvL15] and [FvL16].

Het proefschrift is onderverdeeld in vier hoofdstukken: het eerste hoofdstuk biedt een introductie tot de notatie en de resultaten die ten grondslag liggen aan de latere hoofdstukken; ieder van de drie volgende hoofdstukken behandelt een op zichzelf staand probleem.

In [STVA14] laten C. Salgado, D. Testa en A. Várilly-Alvarado zien dat alle del Pezzo oppervlakken van graad 2 over een eindig lichaam unirationeel zijn, afgezien van mogelijk drie isomorfieklassen van zulke oppervlakken. In 2015, hebben Ronald van Luijk en de auteur van dit proefschrift laten zien dat het resultaat ook waar is voor deze overige drie isomorfieklassen. Dit resultaat is het hoofdresultaat van hoofdstuk twee.

Tijdens de Arizona Winter School 2015 werd door een van de sprekers, A. Várilly-Alvarado, de vraag gesteld hoe de Galois moduulstructuur van het Picard rooster van de K3-oppervlakken in een gegeven 1-dimensionale familie te bepalen. Het antwoord op deze vraag is een gevolg van samenwerking met F. Bouyer, E. Costa, C. Nicholls en M. West en is te vinden in hoofdstuk drie.

In [Ogu15] bewijst K. Oguiso dat als het Picard rooster van een K3-oppervlak S isometrisch is aan een specifiek rooster van rang 2, dat er dan een automorfisme van S van positieve entropie bestaat dat geen dekpunten heeft. Bovendien kan S in dit geval worden ingebed in $\mathbb{P}^3$ als een vierdegraads oppervlak. In ditzelfde artikel, merkt Oguiso op dat het zeer interessant en heel moeilijk lijkt om de vergelijking van S

en de actie van g in termen van globale homogene coordinaten op $\mathbb{P}^3$ expliciet op te schrijven, voor tenminste een van zulke paren (cf. [Ogu15, Remark 4.2]). In [FGvGvL13] geven A. Garbagnati, B. van Geemen, R. van Luijk en de auteur van dit proefschrift een expliciet voorbeeld hiervan. De bijdrage van de auteur van dit proefschrift aan dat artikel staat in hoofdstuk vier.

Sommario

Nella tesi intitolata “Topics in the arithmetic of del Pezzo and K3 surfaces”, l’autore espone i risultati da lui raggiunti durante gli anni del dottorato. Alcuni di questi risultati sono già stati pubblicati, o sono comunque disponibili online, si veda [FGvGvL13], [FvL15] and [FvL16].

La tesi consiste di quattro capitoli: il primo è dedicato all’introduzione di nozioni e risultati basilari, necessari alla trattazione dei problemi considerati nel resto della tesi; ognuno dei successivi capitoli è dunque dedicato a un diverso problema.

In [STVA14], C. Salgado, D. Testa, e A. Várilly-Alvarado dimostrano che tutte le superfici di del Pezzo di grado 2 su un campo finito sono unirazionali, con l’eventuale eccezione di tre superfici, a meno di isomorfismi. Nel 2015, Ronald van Luijk e l’autore della tesi hanno dimostrato l’unirazionalità anche di questi tre casi rimanenti. Questo risultato è esposto nel secondo capitolo.

Durante l’Arizona Winter School 2015, A. Várilly-Alvarado, uno dei lecturer della scuola invernale, chiese di calcolare la struttura di modulo di Galois del reticolo di Picard delle superfici K3 appartenenti a una particolare famiglia unidimensionale. Nel terzo capitolo viene fornita una risposta a tale domanda. L’intero capitolo è un lavoro congiunto con F. Bouyer, E. Costa, C. Nicholls, e M. West.

In [Ogu15], K. Oguiso dimostra che se S è una superficie K3 con reticolo di Picard isometrico a un particolare reticolo di rango 2, allora S ammette un automorfismo g con entropia positiva e senza punti fissi e può essere immersa in $\mathbb{P}^3$ come superficie quartica. Nello stesso articolo, Oguiso commenta che “sembra estremamente difficile ma altamente interessante descrivere esplicitamente l’equazione di S e l’azione di g usando le coordinate di $\mathbb{P}^3$” (cf. [Ogu15, Remark 4.2], tradotto dall'inglese).

In [FGvGvL13], A. Garbagnati, B. van Geemen, R. van Luijk, e l'autore della tesi forniscono un esempio esplicito di tali S e g, descritto usando le coordinate di $\mathbb{P}^3$. Il contributo dell'autore della tesi all'articolo è presentato nel quarto e ultimo capitolo.

Curriculum vitae

Dino Festi was born on 23$^{\text{rd}}$ August 1988, in Salerno (Italy). There he attended the Liceo scientifico "Giovanni da Procida", where he graduated in 2007.

In 2010 he obtained a bachelor degree in mathematics, at the university of Salerno, under the supervision of prof. Patrizia Longobardi.

In the same year, he was awarded an ALGANT Master scholarship, allowing him to study in Padova and in Leiden. In 2012, under the supervision of dr. Ronald van Luijk, he obtained a joint master degree in mathematics, at the universities of Padova, Leiden, and Bordeaux.

After the master, he started a joint PhD program at the universities of Leiden and Milan, under the supervision of dr. Ronald van Luijk and prof. Lambertus van Geemen, graduating in July 2016.

As of October 2016, he will hold a three-year Post-Doc position at the Gutenberg University of Mainz.

Dino enjoys practicing yoga and football, and playing the electric bass guitar.