On the Algebraic Structure of Conditional Events

Tommaso Flaminio[1], Lluis Godo[2(✉)], and Hykel Hosni[3(✉)]

[1] Dipartimento di Scienze Teoriche e Applicate, Università dell'Insubria, Via Mazzini 5, 21100 Varese, Italy
tommaso.flaminio@uninsubria.it

[2] Artificial Intelligence Research Institute (IIIA - CSIC), Campus de la Univ. Autònoma de Barcelona s/n, 08193 Bellaterra, Spain
godo@iiia.csic.es

[3] London School of Economics, Houghton Street, London WC2A 2AE, UK
h.hosni@lse.ac.uk

Abstract. This paper initiates an investigation of conditional measures as simple measures on conditional events. As a first step towards this end we investigate the construction of *conditional algebras* which allow us to distinguish between the logical properties of conditional events and those of the conditional measures which we can be attached to them. This distinction, we argue, helps us clarifying both concepts.

Keywords: Conditionals events · Uncertain reasoning · Conditional algebra

1 Introduction and Motivation

This paper offers a logico-algebraic perspective on conditionals which is motivated by a number of pressing problems in field of logic-based uncertain reasoning. Indeed, conditionals play a fundamental role both in qualitative and in quantitative uncertain reasoning. The former is a consequence of the very fruitful interaction between philosophical logic and artificial intelligence, which linked the semantic approaches to conditionals of the 1970s, mainly Stalnaker's and D. Lewis's to the proof-theoretic and model-theoretic development of non monotonic consequence relations in the 1990s (see [14]). But it is in quantitative uncertain reasoning that conditionals play their most delicate role leading to the key concept of conditional probability. Despite the apparent simplicity of the "ratio definition", on which more below, the notion of conditional probability is far from being uncontroversial. Makinson, for instance, points out in [15] how some rather undesirable behaviour can arise when conditioning on what he refers to the "critical zone". Things get inevitably more complicated if we move to non-classical probability logic, i.e. probability defined on non-classical logics, a rapidly expanding research field. Yet the problem with conditional probability arises in much simpler contexts that those just mentioned.

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Logician Ernest Adams is well-known for putting forward

Adam's Thesis: Conditional probability is the probability that the conditional is true.

The thesis is quite plausible if one reasons as follows. Let θ be a sentence in some propositional language, and let its probability be denoted by $\mu(\theta)$. Then it is very natural to interpret $\mu(\theta)$ as the *probability that* θ *is true*. This is certainly compatible with the Bayesian operational definition of subjective probability as the price a rational agent would be willing to bet on event θ [7]. Now, if θ is of the form $p \to q$, then $\mu(\theta)$ appears to be naturally intepreted as the probability that q *is true given that* p *is also true*, i.e. $\mu(p \mid q)$. But this gives rise to the

Lewis's Triviality: Adams' thesis implies that $\mu(\theta \mid \phi) = \mu(\theta)$.

So, either $\to$ is not truth-functional or Adams' thesis is wrong, and the two alternatives are exclusive.[1]

This paper initiates a research project which aims at tackling the foundational difficulties related to conditional probability by radically changing the perspective. In a nutshell our overall goal is to investigate conditional algebras in such a way as to see *conditional measures as simple measures (possibly with further properties) on conditional events*, i.e. the elements inhabiting conditional algebras. Hence we adopt a two-fold perspective on conditionals. First, we characterize conditional events as elements of an algebra which we term *conditional algebra*. Within such structures conditionals are *simple* objects, a terminology whose meaning will be apparent in a short while. Second, since we are interested in modelling various epistemic attitudes that agents may exhibit in connection to conditional events – and in particular rational belief under uncertainty – we are ultimately interested investigating appropriate *measures* to be attached to conditionals. This paper prepares the stage by focusing on the first objective.

Whilst we are unaware of other proposals which separate the logico-algebraic properties of conditionals from those of conditional measures, the notion of conditional algebra has been investigated in the context of the so-called *Goodman-Nguyen-van Fraassen algebras*. Since we will be in an ideal position to compare this approach with ours after having introduced some formal details, suffice it to mention now that the notion of Conditional Event Algebra (CEA) introduced in [10] differs quite substantially from our notion of conditional algebra.

Let θ, ϕ be sentences in a classical logic propositional language. We denote the conditional assertion "ϕ given θ" by $\phi \mid \theta$. It will sometimes be convenient to refer to ϕ as the *consequent* of the conditional and to θ as its *antecedent*. When presented with a conditional of this form, there are three *distinct* questions that we may ask:

[1] Among many other references, the reader may get an idea of the arguments in support of Adam's thesis which sees the probability of a conditional as conditional probability from [2,10,11,19], and from the arguments which reject it as ill-founded from [13,16].

(1) what are the syntactic properties of $\phi \mid \theta$?
(2) what are the semantic properties of $\phi \mid \theta$?
(3) what properties should be satisfied by a (rational) measure of belief on $\phi \mid \theta$?

This paper focusses on (1) and provides an algebraic interpretation for (2), leaving the investigation of question (3) as future research. The answers put forward in this paper can be informally illustrated as follows:

1. Though it makes perfectly good sense to distinguish, in the conditional $\phi \mid \theta$, the antecedent from the consequent, we will assume that conditional events are *simple* objects which live in a conditional structure. The fundamental consequence of this approach is that the "global" properties, so to speak, of conditionals are defined for the underlying algebraic structure and not at the object level of the conditional formula.

2. The semantic properties of conditionals are also given at the level of the conditional algebra. For instance, by suitably constraining the ideals of a particular freely generated Boolean algebra we will be in a position to characterize the semantic properties we want conditional events to satisfy. As will become apparent, all the results of this paper fail for *counterfactuals*. The reason for this lies in the adoption of a principle which we refer to as the *rejection constraint* according to which a conditional $\phi \mid \theta$ is (semanticaly) meaningless if the antecedent fails to be true (under a suitably defined valuation). This property, as we shall shortly see, is motivated by reflections on conditional events.

2 The Logic of Conditionals

The most general feature on conditionals is that they express some form of hypothetical assertion: the assertion of the consequent based *on the supposition* that the antecedent is satisfied (with respect, of course, to a suitably defined semantics). As Quine put it some four decades ago:

> [An] affirmation of the form 'if p then q' is commonly felt less as an affirmation of a conditional than as a conditional affirmation of the consequent. If, after we have made such an affirmation, the antecedent turns out true, then we consider ourselves committed to the consequent, and are ready to acknowledge error if it proves false. *If, on the other hand, the antecedent turns out to have been false, our conditional affirmation is as if it had never been made* ([20] Added emphasis)

The idea here is that the semantic evaluation of a conditional (in this interpretation) amounts to a two-step procedure. We first check the antecedent. If this is rejected, the conditional ceases to mean anything at all. Otherwise we move on to evaluating the consequent. Note that is in full consonance with de Finetti's semantics for conditional events, an interpretation which lies at the foundation of his *betting interpretation* of subjective probability [7] and which can be extended to more general measures of belief [8]. In particular, with respect to a fixed possible world v,

$$\text{a bet on } \theta \mid \phi \text{ is } \begin{cases} \text{won if} & v(\phi) = v(\theta) = 1; \\ \text{lost if} & v(\phi) = 1 \text{ and } v(\theta) = 0; \\ \text{called off if} & v(\phi) = 0. \end{cases}$$

The final clause is of course the most interesing one, for it states that under the valuation which assigns 0 to the conditioning event, a conditional bet must be called off (all paid monies are returned). This property is what we will henceforth name *Rejection Constraint* stating that in the process of realization of a conditional bet into a fixed world v, we must agree to invalidate bets made on conditionals whose antecedents are evaluated to 0.[2] An immediate consequence of this is that any expression of the form $\theta \mid \perp$ cannot be considered a conditional event. Indeed, in this interpretation, it does not make sense to bet on a conditional whose antecedent is false independently on the possible world v in which the conditional is realized, because it would be always rejected.

The latter observation, leads us to impose a second constraint to our analysis, namely we will require the algebra of conditional events to be Boolean. This property of conditionals is what we will call *Boolean Constraint* and it is essentially motivated to provide conditional events with an algebraic structure which is a suitable domain of uncertainty measures. Indeed, as recalled in Sect. 1, in our future work we will investigate simple (i.e. unconditional) uncertainty measures on conditional algebras. Moreover, Sect. 4 presents an algebraic construction that defines conditional Boolean algebras in a modular way.

3 Algebraic Preliminaries

For every Boolean algebra A we denote by $\delta : A \times A \to A$ the well known *symmetric difference* operator. In other words δ stands for the following abbreviation: for every $x, y \in A$,

$$\delta(x, y) = (x \vee y) \wedge \neg(x \wedge y) = \neg(x \leftrightarrow y). \tag{1}$$

In any Boolean algebra A, the following equations hold:

(i) $\delta(x, y) = \delta(y, x)$
(ii) $\delta(x, \delta(y, z)) = \delta(\delta(x, y), z)$
(iii) $\delta(\delta(x, y), \delta(y, z)) = \delta(x, z)$
(iv) $\delta(x, \perp) = x$
(v) $\delta(x, x) = \perp$

Therefore, in particular δ is (i) commutative; (ii) associative; and (iv) has $\perp$ as neutral element.

The following proposition collects further properties of δ. Owing to space limitations we are forced omit proofs.

Proposition 1. *The following hold in any Boolean algebra A:*

(a) $\delta(x, y) = \perp$ *iff* $x = y$
(b) $\delta(x, z) \leq \delta(x, y) \vee \delta(y, z)$
(c) $\delta(x, y) = \delta(\neg x, \neg y)$
(d) $\delta(x \vee y, z \vee k) = \delta(x, z) \vee \delta(y, k)$.

[2] Note that the Rejection Constraint forces us to exclude counterfactual conditionals from our analysis.

A non-empty subset $\mathfrak{i}$ of a Boolean algebra A is said to be an *ideal* of A if: (1) $\bot \in \mathfrak{i}$; (2) for any $x, y \in \mathfrak{i}$, $x \vee y \in \mathfrak{i}$; (3) if $x \in \mathfrak{i}$, and $y \leq x$, then $y \in \mathfrak{i}$. If $X \subseteq A$, denote by $\mathfrak{I}(X)$, the ideal generated by X, i.e. the least ideal (w.r.t. inclusion) containing X. For every $x \in A$, we denote by $\downarrow x$ the *principal ideal* of A generated by x, i.e. $\downarrow x = \{y \in A : y \leq x\} = \mathfrak{I}(\{x\})$.

Proposition 2. *Let A be a Boolean algebra, and let $\mathfrak{i}$ be an ideal of A. Then for every $x, y \in A$, the equation $x = y$ is valid in the quotient algebra $A/\mathfrak{i}$ iff $\delta(x, y) \in \mathfrak{i}$.*

Remark 1. The above Proposition 2 immediately implies that, whenever $\mathfrak{i}$ is a proper ideal, and $\neg\delta(x, y) \in \mathfrak{i}$, then the quotient algebra $A/\mathfrak{i}$ makes valid $\neg(x = y)$. In fact if $\neg\delta(x, y) \in \mathfrak{i}$, then $\delta(x, y) \notin \mathfrak{i}$ (otherwise $\delta(x, y) \vee \neg\delta(x, y) = \top \in \mathfrak{i}$, and hence $\mathfrak{i}$ would not be proper) iff in $A/\mathfrak{i}$, $\neg(x = y)$ holds true i.e. $x \neq y$.

3.1 On the Conjunction of Conditionals

Let A be a Boolean algebra, and denote by $A \mid A$ the set $\{a \mid b : a, b \in A\}$. The problem of defining operations between the objects in $A \mid A$ has been discussed extensively in the context of measure-free conditionals [6].

Whilst widespread consensus exists about defining the negation of a conditional as $\neg(a \mid b) = \neg a \mid b$, there are at least three major proposals competing for the definition of conjunction:

(Schay, Calabrese) $(a \mid b)\ \&_1\ (c \mid d) = [(b \to a) \wedge (d \to c) \mid (b \vee d)]$ (cf. [5,21] and see also [1] where this conjunction between conditionals is called *quasi-conjunction*).

(Goodman and Nguyen) $(a \mid b)\ \&_2\ (c \mid d) = (a \wedge c) \mid [(\neg a \wedge b) \vee (\neg c \wedge d) \vee (b \vee d)]$ (cf. [11])

(Schay) $(a \mid b)\ \&_3\ (c \mid d) = (a \wedge c) \mid (b \wedge d)$ (cf. [21])

Disjunctions $\oplus_1, \oplus_2$ and $\oplus_3$ among conditionals, are defined by De Morgan's laws from $\&_1, \&_2$ and $\&_3$ above. Schay [21], and Calabrese [5] show that $\&_1$, and $\oplus_1$ are not distributive with respect to each other, and hence the class $A \mid A$ of conditionals, endowed with $\&_1$ and $\oplus_1$ is no longer a Boolean algebra. Therefore $\&_1$ does not satisfy the Boolean constraint mentioned in the introductory Section. For this reason we reject $\&_1$ as a suitable definition of conjunction.

Similarly, the Boolean constraint leads us to reject also $\&_3$, and $\oplus_3$ as candidates for defining conjunction and disjunction betwen conditionals. Indeed, if we defined the usual order relations by

1. $(a_1 \mid b_1) \leq_1 (a_2 \mid b_2)$ iff $(a_1 \mid b_1)\ \&_3\ (a_2 \mid b_2) = (a_1 \wedge a_2 \mid b_1 \wedge b_2) = (a_1 \mid b_1)$,
2. $(a_1 \mid b_1) \leq_2 (a_2 \mid b_2)$ iff $(a_1 \mid b_1) \oplus_3 (a_2 \mid b_2) = (a_1 \vee a_2 \mid b_1 \wedge b_2) = (a_2 \mid b_2)$,

then $\leq_1 \neq \leq_2$. To see this, let a be a fixed element in A. Then $(a \mid \top)\&_3(a \mid a) = (a \mid a)$ and hence $(a \mid a) \leq_1 (a \mid \top)$. On the other hand $(a \mid \top) \oplus_3 (a \mid a) = (a \mid a)$ as well, and therefore $(a \mid \top) \leq_2 (a \mid a)$ for every $a \in A$, and in particular for a such that $a \mid a \neq a \mid \top$. Conversely, it is easy to see that, if we restrict to the class of those conditionals $a_i \mid b$ with a fixed antecedent b, then $\leq_1=\leq_2$. Therefore $\&_3$ is suitable as a definition of conjunction only for those conditionals $a_1 \mid b_1$ and $a_2 \mid b_2$, such that $b_1 = b_2$. Interestingly enough, when restricted to this class of conditionals, $\&_2$ and $\&_3$ do coincide.

It is worth noticing that the above conjunctions are defined in order to make the class $A \mid A$ of conditional objects closed under $\&_i$, and hence an algebra. Therefore for every $a_1, b_1, a_2, b_2 \in A$, and for every $i = 1, 2, 3$, there exists $c, d \in A$ such that, $(a_1 \mid b_1) \&_i (a_2 \mid b_2) = (c \mid d)$. This leads us to introduce a further constraint:

Context Constraint (CC): Let $a_1 \mid b_1$, $a_2 \mid b_2$ be conditionals in $A \mid A$. If $b_1 = b_2$, then the conjunction $(a_1 \mid b_1)\ AND\ (a_2 \mid b_2)$ is a conditional in the form $c \mid d$, and in that case $d = b_1 = b_2$.

The Context constraint is better understood by pointing out that, whenever the object $(a_1 \mid b_1)\ AND\ (a_2 \mid b_2)$ cannot be reduced to a conditional $c \mid d$, then necessarily $b_1 \neq b_2$.

Note that each of the $\&_i$'s above satisfy the stronger requirement, denoted by (CC)', that for every $a_1 \mid b_1$, and $a_2 \mid b_2$, $(a_1 \mid b_1)\ AND\ (a_2 \mid b_2)$ is a conditional in the form $c \mid d$ (but in general $d \neq b_1$, and $d \neq b_2$). This stronger condition ensures in fact that $A \mid A$ is closed under $\&_i$, and hence makes $\&_i$ a total operator on $A \mid A$. On the other hand, as we are going to show in the next section, our construction of conditional algebra, defines a structure whose domain strictly contains all the elements $a \mid b$ for a in A, and b belonging to a particular subset of A guaranteeing the satisfaction of our Rejection constraint. This allows us to relax this condition of closure as stated above. Indeed, for every pair of conditionals of the form $a_1 \mid b_1$ and $a_2 \mid b_2$ belonging to the conditional algebra, their conjunction will always be an element of the algebra (i.e. the conjunction is a total, and not a partial, operation), but in general it will be not in the form $c \mid d$. Therefore we will provide a definition for conjunction between conditionals that satisfies (CC), but not, in general, (CC)'. Moreover our definition of conjunction behaves as $\&_2$, and $\&_3$ whenever restricted to those conditionals $(a_1 \mid b_1), (a_2 \mid b_2)$ with $b_1 = b_2$.

4 Conditional Boolean Algebras

We now show how a conditional Boolean algebra can be built up from any Boolean algebra A and a non-empty $\{\bot\}$-free subset of A, which we will call a *bunch* of A, and denote by A'.

Let A be any Boolean algebra and let $A \times A'$ be the cartesian product of A and A' (as sets). We denote by

$$\mathcal{F}(A \times A') = (\mathcal{F}(A \times A'), \wedge^{\mathcal{F}}, \vee^{\mathcal{F}}, \neg^{\mathcal{F}}, \bot^{\mathcal{F}}, \top^{\mathcal{F}})$$

the Boolean algebra freely generated by the pairs $(a, b) \in A \times A'$ (cf. [4][II §10]). Consider the following elements in $\mathcal{F}(A \times A')$: for every $x, z \in A$, $y, k \in A'$,

$x_1 \in A$ and $z_1 \in A'$ with $x_1 \not\geq z_1$, and $x_2 \in A$ and $y_2, z_2 \in A'$ such that $x_2 \to y_2 = y_2 \to z_2 = \top$

(**t**1) $\delta((y,y), \top^{\mathcal{F}})$
(**t**2) $\delta((x,y) \wedge^{\mathcal{F}} (z,y), (x \wedge z, y))$
(**t**3) $\delta(\neg^{\mathcal{F}}(x,y), (\neg x, y))$
(**t**4) $\delta((x \wedge y, y), (x,y))$
(**t**5) $\neg\delta((x_1, z_1), (z_1, z_1))$
(**t**6) $\delta((x_2, z_2), (x_2, y_2) \wedge^{\mathcal{F}} (y_2, z_2))$.

Consider the proper ideal $\mathfrak{C}$ of $\mathcal{F}(A \times A')$ that is generated by the set of all the instances of the above introduced terms (**t**1)-(**t**6).

Definition 1. *For every Boolean algebra A and every bunch A' of A, we say that the quotient algebra $\mathcal{C}(A, A') = \mathcal{F}(A \times A')/\mathfrak{C}$ is the* conditional algebra *of A and A'.*

Thus, every conditional algebra $\mathcal{C}(A, A')$ is a quotient of a free Boolean algebra, whence is Boolean. So our Boolean constraint is satisfied.

We will denote atomic elements of $A \times A'$ by $a \mid b$ instead of (a, b). In a conditional algebra $\mathcal{C}(A, A')$ we therefore have *atomic conditionals* in the form $a \mid b$ for $a \in A$, and $b \in A'$, and also *compound conditionals* being those elements in $\mathcal{C}(A, A')$ that are the algebraic terms definable in the language of Boolean algebras, modulo the identification induced by $\mathfrak{C}$. The operations on $\mathcal{C}(A, A')$ are denoted using the following notation, which is to be interpreted in the obvious way:

$$\mathcal{C}(A, A') = (\mathcal{C}(A, A'), \cap_{\mathfrak{C}}, \cup_{\mathfrak{C}}, \neg_{\mathfrak{C}}, \bot_{\mathfrak{C}}, \top_{\mathfrak{C}}).$$

The construction of $\mathcal{C}(A, A')$, and in particular the role of the ideal $\mathfrak{C}$, is best illustrated by means of an example.

Example 1. Let A be the four elements Boolean algebra $\{\top, a, \neg a, \bot\}$, and consider the bunch $A' = A \setminus \{\bot\}$. Then $A \times A' = \{(\top, \top), (\top, a), (\top, \neg a), (a, \top), (a, a), (a, \neg a), (\neg a, \top), (\neg a, a), (\neg a, \neg a), (\bot, \top), (\bot, a), (\bot, \neg a)\}$. The cartesian product $A \times A'$ has cardinality 12, whence $\mathcal{F}(A \times A')$ is the free Boolean algebra of cardinality $2^{2^{12}}$, i.e. the finite Boolean algebra of 2^{12} atoms. The conditional algebra $\mathcal{C}(A, A')$ is then obtained as the quotient of $\mathcal{F}(A \times A')$ by the ideal $\mathfrak{C}$ generated by (**t**1)-(**t**6). Having in mind Proposition 2, we can easily see that the ideal $\mathfrak{C}$ of $\mathcal{F}(A \times A')$ specifically *forces* the free algebra $\mathcal{F}(A \times A')$ about which elements are equal as conditionals. For instance, following Proposition 3 (see below), in $\mathcal{C}(A, A')$ the following equations hold: $\top \mid \top = a \mid a = (\neg a) \mid (\neg a)$; $(\top \mid \top) \cap_{\mathfrak{C}} (a \mid \top) = (\top \wedge a) \mid \top = (a \mid \top)$; $\neg_{\mathfrak{C}}(\top \mid \top) = \bot \mid \top$, $\neg_{\mathfrak{C}}(a \mid \neg a) = (\neg a) \mid (\neg a) = \top \mid \top$.

Notice that the conditional algebra $\mathcal{C}(A, A')$ can be defined as a quotient of the free Boolean algebra $\mathcal{F}(X)$ by $\mathfrak{C}$, where X is the subset of $A \times A'$ whose pairs are not redundant under $\mathfrak{C}$, i.e. $X = \{(x_i, y_i) \in A \times A' : \forall i \neq j,\ \delta((x_i, y_i), (x_j, y_j)) \notin \mathcal{C}\} = \{(\top, \top), (a, \top), (\neg a, \top), (\bot, \top)\}$. Therefore $\mathcal{F}(X)$ is the free Boolean algebra with 2^4 atoms.

Proposition 3. *Every conditional algebra $\mathcal{C}(A, A')$ satisfies the following equations:*

(**e**1) *For all $y \in A'$, $y \mid y = \top_{\mathfrak{C}}$*
(**e**2) *For all $x, z \in A$ and $y \in A'$, $(x \mid y) \cap_{\mathfrak{C}} (z \mid y) = (x \wedge z) \mid y$*
(**e**3) *For all $x \in A$ and $y \in A'$, $\neg_{\mathfrak{C}}(x \mid y) = (\neg x \mid y)$*
(**e**4) *For all $x \in A$, for all $y \in A'$, $(x \wedge y \mid y) = (x \mid y)$*
(**e**5) *For all $x, y \in A$, if $(x \mid \top) = (y \mid \top)$, then $x = y$*
(**e**6) *For all $y \in A'$, $\neg y \mid y = \bot_{\mathfrak{C}}$*
(**e**7) *For all $x, z \in A$, and $y \in A'$, $(x \mid y) \cup_{\mathfrak{C}} (z \mid y) = (x \vee z \mid y)$*
(**e**8) *For all $x \in A$ and $y, z \in A'$ such that $x \to y = y \to z = \top$, $(x \mid z) = (x \mid y) \cap_{\mathfrak{C}} (y \mid z)$*

Remark 2. (1) As we have already stated, for all $a_1 \mid b_1, a_2 \mid b_2 \in A \times A'$, their conjunction is the element $(a_1 \mid b_1) \cap_{\mathfrak{C}} (a_2 \mid b_2)$ that belongs to the conditional algebra by definition. Notice that $(a_1 \mid b_1) \cap_{\mathfrak{C}} (a_2 \mid b_2) = (c \mid d)$ iff, from Proposition 2, $\delta((a_1 \mid b_1) \cap_{\mathfrak{C}} (a_2 \mid b_2), (c \mid d)) \in \mathfrak{C}$. Therefore (**t2**) ensures that, if $b_1 = b_2 = d$, then $(a_1 \mid d) \cap_{\mathfrak{C}} (a_2 \mid d) = (c \mid d)$ (see Proposition 3 (**e**2)). Therefore our Context constraint (CC) is satisfied. Also notice that (CC)' is not satisfied in general by the conjunction we have defined in $\mathcal{C}(A, A')$. In fact, when $b_1 \neq b_2$, we cannot ensure in general $(a_1 \mid b_1) \cap_{\mathfrak{C}} (a_2 \mid b_2)$ to be atomic, and hence in the form $(c \mid d)$. In any case $\mathcal{C}(A, A')$ is closed under $\cap_{\mathfrak{C}}$.

(2) The Rejection constraint introduced in Sect. 2, forces our construction to drop $\bot$ from the algebra intended to contain the antecedents of conditionals. For this reason we defined the *bunch* as a bottom-free subset of A. Notice that if we allowed the conditional algebra to represent counterfactual conditionals (i.e. had we not imposed the Rejection constraint), the resulting algebraic structure would have not be Boolean as shown in [19,22]. In this sense, the Rejection constraint can be seen as being closely connected to the Boolean one.

In a conditional algebra $\mathcal{C}(A, A')$, as in any Boolean algebra, one can define the order relation $\leq$ by the letting

$$(x \mid y) \leq (z \mid k) \text{ iff } (x \mid y) \cap_{\mathfrak{C}} (z \mid k) = (x \mid y). \tag{2}$$

Proposition 4. *In every conditional algebra $\mathcal{C}(A, A')$ the following hold:*

(**o**1) *For every $x, y \in A$, and for every $z \in A'$, $(x \mid y) \leq (z \mid z)$; moreover $(x \mid z) \geq (z \mid z)$, implies $x \geq z$*
(**o**2) *For every $x, y \in A$ and $z \in A'$, if $x \leq y$, then $(x \mid z) \leq (y \mid z)$ (where clearly $x \leq y$ means with respect to A). In particular $x \leq y$ iff $(x \mid \top) \leq (y \mid \top)$*
(**o**3) *For every $x \in A$ and $y \in A'$, if $x \not\geq y$, then $(x \mid y) \neq (y \mid y)$, and in particular $(x \mid y) < (y \mid y)$*
(**o**4) *For every $x, y \in A$ and $z \in A'$, if $(x \mid z) \neq (y \mid z)$, then $x \neq y$. In particular $x \neq y$ iff $(x \mid \top) \neq (y \mid \top)$*
(**o**5) *For every $x \in A'$, $(\top \mid x) = (x \mid x) = \top_{\mathfrak{C}}$, and $(\bot \mid x) = (\neg x \mid x) = \bot_{\mathfrak{C}}$*
(**o**6) *For every $x, y \in A$ and $z, k \in A'$, $(x \mid k) \cap_{\mathfrak{C}} (y \mid z) = (x \mid k)$ iff $(x \mid k) \cup_{\mathfrak{C}} (y \mid z) = (y \mid z)$*

Remark 3. As we have already observed, every conditional algebra $\mathcal{C}(A, A')$ is finite whenever A is finite. So, if A is finite, $\mathcal{C}(A, A')$ is atomic. Moreover, since the canonical homomorphism $h_{\mathfrak{C}} : \mathcal{F}(A \times A') \to \mathcal{C}(A, A')$ is onto, we have:

$$2^{2^{|A \times A'|}} = |\mathcal{F}(A \times A')| \geq |\mathcal{C}(A, A')|.$$

Finally, recall that a conditional probability on a Boolean algebra A is a map $\mu : A \times A' \to [0,1]$, where A' is a bunch of A, such that:

(μ1) For all $x \in A'$, $\mu(x \mid x) = 1$,
(μ2) If $x_1, x_2 \in A$, $x_1 \wedge x_2 = 0$ and $y \in A'$, $\mu(x_1 \vee x_2 \mid y) = \mu(x_1 \mid y) + \mu(x_2 \mid y)$,
(μ3) If $x \in A$ and $y \in A'$, $\mu(x \mid y) = \mu(x \wedge y \mid y)$,
(μ4) If $x \in A$ and $y, z \in A'$ such that $x \to y = y \to z = \top$, then $\mu(x \mid z) = \mu(x \mid y) \cdot \mu(y \mid z)$.

Theorem 1. *Let A be a Boolean algebra, A' a bunch of A and let $\mu : \mathcal{C}(A, A') \to$ [0,1] be a simple (i.e. unconditional) probability further satisfying: for all $x \in A$ and $y, z \in A'$ such that $x \to y = y \to z = \top$*

$$\mu((x \mid y) \cap_{\mathfrak{C}} (y \mid z)) = \mu(x \mid y) \cdot \mu(y \mid z). \tag{3}$$

Then, μ satisfies all the axioms of a conditional probability on A.

Proof. The properties (μ1) and (μ3) respectively follow from Proposition 3 (**e**1), (**e**4) together, with the normalization property for probability measures: $\mu(\top) = 1$. In order to show (μ2), notice that whenever $x_1 \wedge x_2 = \bot$, then from Proposition 3 (**e**2), for every $y \in A'$, $(x_1 \mid y) \cap_{\mathfrak{C}} (x_2 \mid y) = (x_1 \wedge x_2 \mid y) = (\bot \mid y) = \bot_{\mathfrak{C}}$. Therefore, since μ is additive, $\mu((x_1 \mid y) \cup_{\mathfrak{C}} (x_2 \mid y)) = \mu(x_1 \mid y) + \mu(x_2 \mid y)$. Therefore ($\mu$2) also holds because by Proposition 3 (**e**7), $(x_1 \mid y) \cup_{\mathfrak{C}} (x_2 \mid y) = (x_1 \vee x_2) \mid y$. Finally, by Proposition 3 (**e**8) together with (3), if $x \in A$ and $y, z \in A'$ are such that $x \to y = y \to z = \top$, $\mu(x \mid z) = \mu((x \mid y) \wedge_{\mathfrak{C}} (y \mid z)) = \mu(x \mid y) \cdot \mu(y \mid z)$.

5 Conclusions and Further Work

The results reported in this paper constitute a first step towards providing a rather flexible framework for conditionals which builds on the distinction between the properties of a conditional event and those of a conditional measure. Our next step will involve relaxing the Boolean constraint, a relaxation which implies a substantial generalization of the Rejection constraint as well and that may have a significant impact on our understanding of conditional *many-valued* probability, a topic to which considerable research effort has been devoted in the past decade, see e.g. [9,12,17,18]). Another interesting prospective (pointed out by one of the referees) is to look at the conditional as a partial operation on a Boolean algebra and apply techniques of theory of partial algebras [3].

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