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Threshold resummation at N^{3}LL accuracy and soft-virtual cross sections at N^{3}LO

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Abstract

We consider QCD radiative corrections to the production of colorless high-mass systems in hadron collisions. We show that the recent computation of the soft-virtual corrections to Higgs boson production at N^{3}LO [1] together with the universality structure of soft-gluon emission can be exploited to extract the general expression of the hard-virtual coefficient that contributes to threshold resummation at N^{3}LL accuracy. The hard-virtual coefficient is directly related to the process-dependent virtual amplitude through a universal (process-independent) factorization formula that we explicitly evaluate up to three-loop order. As an application, we present the explicit expression of the soft-virtual N^{3}LO corrections for the production of an arbitrary colorless system. In the case of the Drell–Yan process, we confirm the recent result of Ref. [2].

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The authors of Ref. [1] have recently presented the result of the calculation of the cross section for the threshold production of the Higgs boson at hadron colliders at the next-to-next-to-next-to-leading order (N^3LO) in perturbative QCD. This result has prompted the observation [2] that the Higgs boson calculation contains information on soft-gluon radiation that can be implemented to explicitly determine the N^3LO threshold cross section for the Drell–Yan (DY) process. In the present contribution, we exploit the universality (process-independent) structure [3] of soft-gluon contributions near partonic threshold and the specific calculation of Ref. [1]. We show how the results of Refs. [1] and [3] can be straightforwardly combined and used to extract the general expression of the hard-virtual coefficient that contributes to threshold resummation at next-to-next-to-next-to-leading-logarithmic (N^3LL) accuracy for the cross section of a generic (and arbitrary) colorless high-mass system produced in hadron collisions. The threshold resummation formula for the production cross section can also be perturbatively expanded up to N^3LO, and for the specific case of the DY process we recover the result of Ref. [2].

The N^3LO Higgs boson results of Ref. [1] complete a cross section calculation that requires the evaluation of several independent ingredients related to collinear-counterterm factors [4,5] and to real- [6,7] and virtual-radiation [8–10] contributions. One of these ingredients is the three-loop virtual amplitude [9,10] $gg \to H$ for Higgs boson production through gluon fusion (the three-loop results of Refs. [9,10] use the large-m_{top} approximation). As discussed in Ref. [3], all-order soft-gluon resummation [11–13] for the hadroproduction cross section of a generic colorless high-mass system can be expressed in a process-independent form, whose sole process-dependent information is encoded in the virtual amplitude of the specific process. Therefore, using the cross section of Ref. [1] and the virtual amplitude of Refs. [9,10] for the specific case of Higgs boson production, we can apply the formulation of Ref. [3] and we can explicitly determine the entire process-independent information that contributes to soft-gluon resummation for a generic production process up to the three-loop level. In the following we recall the formalism of soft-gluon resummation (by mainly following the notation of Section 5 in Ref. [3]) and we present and illustrate our three-loop results.

We consider the inclusive hard-scattering reaction

$$h_1(p_1) + h_2(p_2) \to F\big(\{q_i\}\big) + X, \tag{1}$$

where the collision of the two hadrons h_1 and h_2 with momenta p_1 and p_2 produces the triggered final state F, and X denotes the accompanying final-state radiation. The observed final state F is a *generic* system of one or more *colorless* particles (with momenta q_i), such as lepton pairs (produced by the DY mechanism), photon pairs, vector bosons, Higgs boson(s), and so forth. We focus on the *total* cross section[2] for the process in Eq. (1) at fixed value M of the invariant mass of the triggered final state F (i.e., we integrate the differential cross section over the momenta q_i with the constraint $(\sum_i q_i)^2 = M^2$). In the simplest case, the final-state system F consists of a single ('on-shell') particle of mass M (for example, F can be a vector boson or a Higgs boson). The total cross section $\sigma_F(p_1, p_2; M^2)$ for the production of the system F is computable in QCD perturbation theory according to the following factorization formula:

[2] The formalism of soft-gluon resummation can be further elaborated and extended to include the dependence on final-state kinematical variables such as, for instance, the rapidity of the final state F (see, e.g., Refs. [13–16]).

$$\sigma_F\big(p_1, p_2; M^2\big) = \sum_{a_1,a_2} \int_0^1 dz_1 \int_0^1 dz_2\, \hat{\sigma}^F_{a_1a_2}\big(\hat{s} = z_1 z_2 s; M^2; \alpha_S\big(M^2\big)\big) f_{a_1/h_1}\big(z_1, M^2\big) f_{a_2/h_2}\big(z_2, M^2\big), \quad (2)$$

where $s = (p_1 + p_2)^2 \simeq 2p_1 \cdot p_2$, $\hat{\sigma}^F_{a_1a_2}$ is the total partonic cross section for the inclusive partonic process $a_1a_2 \to F + X$ and, for simplicity, the parton densities $f_{a_i/h_i}(z_i, M^2)$ $(i = 1, 2)$ are evaluated at the scale M^2 (the inclusion of an arbitrary factorization scale μ_F in the parton densities and in the partonic cross sections can be implemented in a straightforward way by using the Altarelli–Parisi evolution equations of $f_{a/h}(z, \mu_F^2)$). The partonic cross section $\hat{\sigma}^F_{a_1a_2}(\hat{s}; M^2; \alpha_S(M^2))$ depends on the mass M of the system F, on the centre-of-mass energy $\sqrt{\hat{s}}$ of the colliding partons a_1 and a_2, and it is a renormalization-group invariant quantity that can be perturbatively computed as series expansion in powers of $\alpha_S(M^2)$. Considering, for instance, the inclusive partonic channel $c\bar{c} \to F + X$, we can write

$$\hat{\sigma}^F_{c\bar{c}}\big(\hat{s}; M^2; \alpha_S\big(M^2\big)\big) = \sigma^{(0)}_{c\bar{c}\to F}\big(M^2; \alpha_S\big(M^2\big)\big) \sum_{n=0}^{\infty} \left(\frac{\alpha_S(M^2)}{\pi}\right)^n z g^{F(n)}_{c\bar{c}}(z) \quad (3)$$

where $z = M^2/\hat{s}$,

$$g^{F(0)}_{c\bar{c}}(z) = \delta(1 - z), \quad (4)$$

and $\sigma^{(0)}_{c\bar{c}\to F}$ is the lowest-order cross section for the partonic process $c\bar{c} \to F$. Since the system F is colorless, the lowest-order cross section is determined by the partonic processes of quark–antiquark annihilation ($c = q, \bar{q}$) and/or gluon fusion ($c = g$) (in the case of $q\bar{q}$-annihilation the quark and antiquark can have different flavors, such as, for instance, if $F = W^{\pm}$). Perturbative expressions that are analogous to Eq. (3) can be written for the partonic cross sections $\hat{\sigma}^F_{a_1a_2}$ of all the other partonic channels. Using the renormalization-group evolution of the QCD running coupling $\alpha_S(q^2)$, we can equivalently expand $\hat{\sigma}^F_{a_1a_2}$ in powers of $\alpha_S(\mu_R^2)$, with corresponding perturbative coefficients $g^{F(n)}_{a_1a_2}$ that explicitly depend on M^2/μ_R^2, where μ_R is an arbitrary renormalization scale. Throughout the paper we use parton densities as defined in the $\overline{\text{MS}}$ factorization scheme, and $\alpha_S(q^2)$ is the QCD running coupling in the $\overline{\text{MS}}$ renormalization scheme.

The kinematical variable $z = M^2/\hat{s}$ in Eq. (3) parametrizes the distance from the partonic threshold. The limit $z \to 1$ specifies the kinematical region that is close to the partonic threshold. In this region the partonic cross section $\hat{\sigma}^F_{a_1a_2}$ receives large QCD radiative corrections that are proportional to the singular functions

$$\mathcal{D}_m(z) \equiv \left[\frac{1}{1-z} \ln^m(1 - z)\right]_+ \quad (m = 0, 1, \ldots), \quad (5)$$

where the subscript '+' denotes the customary 'plus-distribution'. The all-order perturbative resummation of these logarithmic contributions (including all the singular contributions that are proportional to $\delta(1 - z)$) can be systematically performed by working in Mellin (N-moment) space [11,12]. The Mellin transform $\hat{\sigma}_N(M^2)$ of the partonic cross section $\hat{\sigma}(\hat{s}; M^2)$ is defined as

$$\hat{\sigma}^F_{a_1a_2, N}\big(M^2; \alpha_S\big(M^2\big)\big) \equiv \int_0^1 dz\, z^{N-1} \hat{\sigma}^F_{a_1a_2}\big(\hat{s} = M^2/z; M^2; \alpha_S\big(M^2\big)\big). \quad (6)$$

In Mellin space, the threshold region $z \to 1$ corresponds to the limit $N \to \infty$, and the plus-distributions of Eq. (5) become powers of $\ln N$ ($(\frac{1}{1-z}\ln^m(1-z))_+ \to \ln^{m+1} N +$ 'subleading logs'). These logarithmic contributions are evaluated to all perturbative orders by using threshold resummation [11,12]. Neglecting terms that are relatively suppressed by powers of $1/N$ in the limit $N \to \infty$, we write

$$\hat{\sigma}^{F}_{c\bar{c},\,N}\big(M^2;\alpha_S\big(M^2\big)\big) = \hat{\sigma}^{F\,(\mathrm{res})}_{c\bar{c},\,N}\big(M^2;\alpha_S\big(M^2\big)\big)\big[1+\mathcal{O}(1/N)\big]. \tag{7}$$

Note that we are considering only the partonic channel $c\bar{c} \to F + X$, with $c\bar{c} = q\bar{q}$ and/or $c\bar{c} = gg$, since the other partonic channels give contributions that are of $\mathcal{O}(1/N)$. In this paper, we use the Mellin-space formalism of threshold resummation [11,12] that we have just introduced. Related formulations of threshold resummation for hadron–hadron collisions can be found, for instance, in Ref. [17] (which is exploited to derive the results of Ref. [2]) and in Refs. [18–20].

The expression $\hat{\sigma}^{F\,(\mathrm{res})}_{c\bar{c},\,N}$ in the right-hand side of Eq. (7) embodies all the perturbative terms that are logarithmically enhanced or *constant* in the limit $N \to \infty$. The partonic cross section $\hat{\sigma}^{F\,(\mathrm{res})}_{c\bar{c},\,N}$ has a universal (process-independent) all-order structure that is given by the following threshold-resummation formula [11–13,21–23]:

$$\hat{\sigma}^{F\,(\mathrm{res})}_{c\bar{c},\,N}\big(M^2;\alpha_S\big(M^2\big)\big) = \sigma^{(0)}_{c\bar{c}\to F}\big(M^2;\alpha_S\big(M^2\big)\big)C^{\mathrm{th}}_{c\bar{c}\to F}\big(\alpha_S\big(M^2\big)\big)\Delta_{c,\,N}\big(M^2\big). \tag{8}$$

The factor $\sigma^{(0)}_{c\bar{c}\to F}$ obviously depends on the produced final-state system F, and it is simply proportional to the square of the lowest-order scattering amplitude $\mathcal{M}^{(0)}_{c\bar{c}\to F}$ (see Eq. (22)) of the partonic process $c\bar{c} \to F$. The factor $C^{\mathrm{th}}_{c\bar{c}\to F}$ also depends on the produced final-state system F and, therefore, it includes a process-dependent component. The factor $\Delta_{c,\,N}$ is process-independent: it does not depend on the final-state system F, and it only depends on the type ($c = q$ or $c = g$) of colliding partons.

The factor $\Delta_{c,\,N}$ is entirely due to soft-parton radiation [11,12]. This radiative factor resums all the perturbative contributions $\alpha_S^n \ln^m N$ (including some constant terms, i.e. terms with $m = 0$), and it has the following all-order form:

$$\begin{aligned}&\Delta_{c,N}\big(M^2\big)\\&\quad= \exp\left\{\int_0^1 dz\,\frac{z^{N-1}-1}{1-z}\left[2\int_{M^2}^{(1-z)^2M^2}\frac{dq^2}{q^2}A_c\big(\alpha_S\big(q^2\big)\big) + D_c\big(\alpha_S\big((1-z)^2M^2\big)\big)\right]\right\},\end{aligned} \tag{9}$$

where $A_c(\alpha_S)$ and $D_c(\alpha_S)$ are perturbative series in α_S,

$$A_c(\alpha_S) = \left(\frac{\alpha_S}{\pi}\right)A_c^{(1)} + \left(\frac{\alpha_S}{\pi}\right)^2 A_c^{(2)} + \left(\frac{\alpha_S}{\pi}\right)^3 A_c^{(3)} + \left(\frac{\alpha_S}{\pi}\right)^4 A_c^{(4)} + \mathcal{O}\big(\alpha_S^5\big), \tag{10}$$

$$D_c(\alpha_S) = \left(\frac{\alpha_S}{\pi}\right)^2 D_c^{(2)} + \left(\frac{\alpha_S}{\pi}\right)^3 D_c^{(3)} + \mathcal{O}\big(\alpha_S^4\big). \tag{11}$$

The function $A_c(\alpha_S)$ is produced by radiation that is soft and collinear to the direction of the colliding partons c and $\bar{c}$. The effect of soft non-collinear radiation is embodied in the function $D_c(\alpha_S)$. The perturbative coefficients $A_c^{(1)}$, $A_c^{(2)}$ [12,24,25] and $A_c^{(3)}$ [4,23] are explicitly known. They read

$$
\begin{aligned}
A_c^{(1)} &= C_c, \\
A_c^{(2)} &= \frac{1}{2} K\, C_c, \qquad K = C_A\left(\frac{67}{18} - \frac{\pi^2}{6}\right) - \frac{5}{9} n_F, \\
A_c^{(3)} &= C_c\Bigg(\left(\frac{245}{96} - \frac{67}{216}\pi^2 + \frac{11}{720}\pi^4 + \frac{11}{24}\zeta_3\right) C_A^2 + \left(-\frac{209}{432} + \frac{5}{108}\pi^2 - \frac{7}{12}\zeta_3\right) C_A\, n_F \\
&\quad + \left(-\frac{55}{96} + \frac{1}{2}\zeta_3\right) C_F\, n_F - \frac{1}{108} n_F^2\Bigg),
\end{aligned}
\tag{12}
$$

where n_F is the number of quark flavors, N_c is the number of colors, and the color factors are $C_F = (N_c^2 - 1)/(2N_c)$ and $C_A = N_c$ in $SU(N_c)$ QCD. The color coefficient C_c depends on the type c of colliding partons, and we have $C_c = C_F$ if $c = q$ and $C_c = C_A$ if $c = g$. The perturbative expansion of $D_c(\alpha_S)$ starts at $\mathcal{O}(\alpha_S^2)$ (i.e., $D_c^{(1)} = 0$), and the perturbative coefficients $D_c^{(2)}$ [21,26] and $D_c^{(3)}$ [27,28] are explicitly known. They read

$$
\begin{aligned}
D_c^{(2)} &= C_c\left(C_A\left(-\frac{101}{27} + \frac{11}{18}\pi^2 + \frac{7}{2}\zeta_3\right) + n_F\left(\frac{14}{27} - \frac{1}{9}\pi^2\right)\right), \\
D_c^{(3)} &= C_c\Bigg(C_A^2\left(-\frac{297\,029}{23\,328} + \frac{6139}{1944}\pi^2 - \frac{187}{2160}\pi^4 + \frac{2509}{108}\zeta_3 - \frac{11}{36}\pi^2\zeta_3 - 6\zeta_5\right) \\
&\quad + C_A\, n_F\left(\frac{31\,313}{11\,664} - \frac{1837}{1944}\pi^2 + \frac{23}{1080}\pi^4 - \frac{155}{36}\zeta_3\right) \\
&\quad + C_F\, n_F\left(\frac{1711}{864} - \frac{1}{12}\pi^2 - \frac{1}{180}\pi^4 - \frac{19}{18}\zeta_3\right) \\
&\quad + n_F^2\left(-\frac{58}{729} + \frac{5}{81}\pi^2 + \frac{5}{27}\zeta_3\right)\Bigg).
\end{aligned}
\tag{13}
$$

Using Eq. (9), the coefficients $A_c^{(1)}$, $A_c^{(2)}$, $A_c^{(3)}$, $D_c^{(2)}$, $D_c^{(3)}$ in Eqs. (12)–(13) and the coefficient $A_c^{(4)}$ in Eq. (10) explicitly determine soft-gluon resummation up to N^{3}LL accuracy. The fourth-order coefficient $A_c^{(4)}$ is still unknown. Numerical approximations of $A_c^{(4)}$ [23] indicate that this coefficient can have a small quantitative effect in practical applications of threshold resummation. By direct inspection of Eqs. (12) and (13), we note that the dependence on c (the type of colliding parton) of the perturbative functions $A_c(\alpha_S)$ and $D_c(\alpha_S)$ is entirely specified up to $\mathcal{O}(\alpha_S^3)$ by the overall color factor C_c. To highlight this overall dependence, we introduce the notation

$$
A_c(\alpha_S) = C_c\left(\frac{\alpha_S}{\pi}\right)\left(1 + \left(\frac{\alpha_S}{\pi}\right)\gamma_{\rm cusp}^{(1)} + \left(\frac{\alpha_S}{\pi}\right)^2 \gamma_{\rm cusp}^{(2)}\right) + \left(\frac{\alpha_S}{\pi}\right)^4 A_c^{(4)} + \mathcal{O}\big(\alpha_S^5\big),
\tag{14}
$$

so that $\gamma_{\rm cusp}^{(1)} \equiv A_c^{(2)}/C_c = K/2$ and $\gamma_{\rm cusp}^{(2)} \equiv A_c^{(3)}/C_c$ (see Eq. (12)) are universal QCD coefficients (namely, they do not depend on the type c of colliding parton). This overall dependence on C_c, which is customarily named as Casimir scaling relation, follows from the soft-parton origin of both $A_c(\alpha_S)$ and $D_c(\alpha_S)$, and it is eventually a consequence of non-abelian exponentiation [29] for soft-gluon radiation. The validity of the Casimir scaling relation (14) beyond $\mathcal{O}(\alpha_S^3)$ is a subject of current theoretical investigations (see Ref. [30] and references therein). More detailed comments on the structure of soft-gluon radiation are postponed below Eq. (42).

In this paper we focus on the threshold-resummation factor $C_{c\bar{c}\to F}^{\rm th}$. The factor $C_{c\bar{c}\to F}^{\rm th}$ embodies all the remaining N-independent contributions (i.e., terms that are constant in the limit

$N \to \infty$) to the partonic cross section in Eq. (8). This factor is definitely process dependent, and it has the general perturbative expansion

$$C^{\mathrm{th}}_{c\bar{c}\to F}(\alpha_S) = 1 + \sum_{n=1}^{\infty} \left(\frac{\alpha_S}{\pi}\right)^n C^{\mathrm{th}\,(n)}_{c\bar{c}\to F}. \tag{15}$$

Despite its process dependence, in Ref. [3] we have discussed and shown that the all-order factor $C^{\mathrm{th}}_{c\bar{c}\to F}(\alpha_S)$ involves a minimal amount of process-dependent information. This information is entirely due to the renormalized all-loop scattering amplitude $\mathcal{M}_{c\bar{c}\to F}$ of the (elastic-production) partonic process $c\bar{c} \to F$. Having $\mathcal{M}_{c\bar{c}\to F}$, we can introduce the corresponding hard-virtual amplitude $\widetilde{\mathcal{M}}^{\mathrm{th}}_{c\bar{c}\to F}$ for threshold resummation by using a process-independent (universal) factorization formula that has the following all-order expression [3]:

$$\widetilde{\mathcal{M}}^{\mathrm{th}}_{c\bar{c}\to F} = \left[1 - \tilde{I}^{\mathrm{th}}_c\left(\epsilon, M^2\right)\right]\mathcal{M}_{c\bar{c}\to F}. \tag{16}$$

The subtraction operator $\tilde{I}^{\mathrm{th}}_c(\epsilon, M^2)$ in Eq. (16) is a renormalization-group invariant quantity that does not depend on the specific final-state system F: it only depends on the type ($c = q$ or $c = g$) of colliding partons and on a scale that is set by the invariant mass M of the system F. The factor $C^{\mathrm{th}}_{c\bar{c}\to F}(\alpha_S)$ is then directly related to $\widetilde{\mathcal{M}}^{\mathrm{th}}_{c\bar{c}\to F}$. In the simple case where the system F consists of a single particle of mass M, the direct relation is [3]

$$\alpha_S^{2k}\left(M^2\right)C^{\mathrm{th}}_{c\bar{c}\to F}\left(\alpha_S\left(M^2\right)\right) = \frac{|\widetilde{\mathcal{M}}^{\mathrm{th}}_{c\bar{c}\to F}|^2}{|\mathcal{M}^{(0)}_{c\bar{c}\to F}|^2} \quad (F\text{: single particle}), \tag{17}$$

where the value k of the power of $\alpha_S(M^2)$ and the lowest-order amplitude $\mathcal{M}^{(0)}_{c\bar{c}\to F}$ are precisely defined in Eq. (22). The relation in Eq. (17) can be straightforwardly generalized to the more general case where the system F is formed by two or more particles with momenta q_i (see Eq. (1)). The generalization simply follows from the fact that we are considering the cross section integrated over the final-state momenta q_i and, therefore, we have

$$\begin{aligned} &\sigma^{(0)}_{c\bar{c}\to F}\left(M^2; \alpha_S\left(M^2\right)\right)C^{\mathrm{th}}_{c\bar{c}\to F}\left(\alpha_S\left(M^2\right)\right) \\ &\quad = \int\limits_{PS(\{q_i\};M)} \left|\widetilde{\mathcal{M}}^{\mathrm{th}}_{c\bar{c}\to F}\left(\{q_i\}\right)\right|^2 \quad (F\text{: multiparticle system}). \end{aligned} \tag{18}$$

Here we have introduced a shorthand (symbolic) notation: the symbol $\int_{PS(\{q_i\};M)}$ denotes the properly normalized (see Eq. (23)) phase space integration over the final-state momenta $\{q_i\}$ at fixed value of the their total invariant mass M. The extension from Eq. (17) to Eq. (18) derives from the simple key observation that the operator $\tilde{I}^{\mathrm{th}}_c(\epsilon, M^2)$ in Eq. (16) is completely independent of the final-state momenta q_i and, therefore, the q_i-dependence of $\widetilde{\mathcal{M}}^{\mathrm{th}}_{c\bar{c}\to F}(\{q_i\})$ is entirely and directly given by the q_i-dependence of the scattering amplitude $\mathcal{M}_{c\bar{c}\to F}(\{q_i\})$. In Ref. [3] we obtained the explicit expression of the subtraction operator $\tilde{I}^{\mathrm{th}}_c$ up to the second order in the QCD coupling α_S. In this paper we extend those results and compute $\tilde{I}^{\mathrm{th}}_c$ to the third order in α_S.

Before presenting our results, we give more details on the notation that is used in Eqs. (16)–(18). The all-loop scattering amplitude $\mathcal{M}_{c\bar{c}\to F}$ of the partonic process $c\bar{c} \to F$ contains ultraviolet (UV) and infrared (IR) singularities, which are regularized in $d = 4 - 2\epsilon$ space-time dimensions. To be definite we use the customary scheme of conventional dimensional regularization (CDR). Before performing renormalization, the multiloop QCD amplitude has a perturbative dependence on powers of $\alpha_S^u \mu_0^{2\epsilon}$, where α_S^u is the bare coupling and μ_0 is the dimensional-regularization scale. In the following we work with the renormalized on-shell scattering amplitude that is obtained from the corresponding unrenormalized amplitude by just expressing the bare coupling α_S^u in terms of the running coupling $\alpha_S(\mu_R^2)$ according to the $\overline{\rm MS}$ scheme relation

$$\alpha_S^u \mu_0^{2\epsilon} S_\epsilon = \alpha_S\big(\mu_R^2\big)\,\mu_R^{2\epsilon} Z\big(\alpha_S\big(\mu_R^2\big),\epsilon\big), \qquad S_\epsilon = (4\pi)^\epsilon e^{-\epsilon\gamma_E}, \tag{19}$$

$$Z(\alpha_S,\epsilon) = 1 - \alpha_S\frac{\beta_0}{\epsilon} + \alpha_S^2\left(\frac{\beta_0^2}{\epsilon^2} - \frac{\beta_1}{2\epsilon}\right) - \alpha_S^3\left(\frac{\beta_0^3}{\epsilon^3} - \frac{7}{6}\frac{\beta_0\beta_1}{\epsilon^2} + \frac{\beta_2}{3\epsilon}\right) + \mathcal{O}\big(\alpha_S^4\big), \tag{20}$$

where γ_E is the Euler number, μ_R is the renormalization scale and β_0, β_1 and β_2 are the first three coefficients of the QCD β-function [8]:

$$\begin{aligned}
&12\pi\beta_0 = 11C_A - 2n_F, \qquad 24\pi^2\beta_1 = 17C_A^2 - 5C_A n_F - 3C_F n_F,\\
&64\pi^3\beta_2 = \frac{2857}{54}C_A^3 - \frac{1415}{54}C_A^2 n_F - \frac{205}{18}C_A C_F n_F + C_F^2 n_F\\
&\qquad\qquad + \frac{79}{54}C_A n_F^2 + \frac{11}{9}C_F n_F^2.
\end{aligned} \tag{21}$$

The renormalized all-loop amplitude $\mathcal{M}_{c\bar{c}\to F}$ has the perturbative (loop) expansion

$$\mathcal{M}_{c\bar{c}\to F} = \big(\alpha_S\big(M^2\big)M^{2\epsilon}\big)^k\left[\mathcal{M}^{(0)}_{c\bar{c}\to F} + \sum_{n=1}^{\infty}\left(\frac{\alpha_S(M^2)}{2\pi}\right)^n \mathcal{M}^{(n)}_{c\bar{c}\to F}\right], \tag{22}$$

where the value k of the overall power of α_S depends on the specific process (for instance, $k=0$ in the case of the vector boson production process $q\bar{q}\to V$, and $k=1$ in the case of the Higgs boson production process $gg\to H$ through a heavy-quark loop). Note that the lowest-order term $\mathcal{M}^{(0)}_{c\bar{c}\to F}$ is not necessarily a tree-level amplitude (for instance, it involves a quark loop in the cases $gg\to H$ and $gg\to\gamma\gamma$). If F is a multiparticle system, using the shorthand notation of Eq. (18), we can write the lowest-order cross section as

$$\sigma^{(0)}_{c\bar{c}\to F}\big(M^2;\alpha_S\big(M^2\big)\big) = \alpha_S^{2k}\big(M^2\big)\int\limits_{PS(\{q_i\};M)} \big|\mathcal{M}^{(0)}_{c\bar{c}\to F}\big(\{q_i\}\big)\big|^2, \tag{23}$$

which (implicitly) fixes the overall normalization of the phase space integration. The perturbative terms $\mathcal{M}^{(l)}_{c\bar{c}\to F}$ $(l=1,2,3,\ldots)$ are UV finite, but they still depend on ϵ: in particular, they contain ϵ-pole contributions and, therefore, they are IR divergent as $\epsilon\to 0$. The IR divergent contributions to the scattering amplitude $\mathcal{M}_{c\bar{c}\to F}$ have a universal (process-independent) structure [31–34] that is explicitly known up to the three-loop $(l=3)$ level [35]. The subtraction operator $\tilde{I}_c^{\rm th}(\epsilon, M^2)$ in Eq. (16) has the perturbative expansion

$$\tilde{I}_c^{\rm th}\big(\epsilon, M^2\big) = \sum_{n=1}^{\infty}\left(\frac{\alpha_S(M^2)}{2\pi}\right)^n \tilde{I}_c^{{\rm th}(n)}(\epsilon), \tag{24}$$

and the perturbative terms $\tilde{I}_c^{{\rm th}(n)}(\epsilon)$ contain IR divergent contributions (ϵ-poles) and a definite amount of IR finite contributions. The IR divergent contributions to $\tilde{I}_c^{\rm th}(\epsilon, M^2)$ are exactly those that are necessary to cancel the IR divergences of the renormalized all-loop amplitude $\mathcal{M}_{c\bar{c}\to F}$. Therefore, the hard-virtual amplitude $\widetilde{\mathcal{M}}^{\rm th}_{c\bar{c}\to F}$ in Eq. (16) is IR finite order-by-order in perturbation theory, and it can be evaluated in the limit $\epsilon\to 0$. The threshold resummation coefficient $C^{\rm th}_{c\bar{c}\to F}(\alpha_S(M^2))$ can be directly computed in the four-dimensional limit $\epsilon\to 0$ (though, this limit is not explicitly denoted in the right-hand side of Eqs. (17) and (18)). The perturbative expansion of $\widetilde{\mathcal{M}}^{\rm th}_{c\bar{c}\to F}$ is completely analogous to that of $\mathcal{M}_{c\bar{c}\to F}$ (see Eq. (22)) with the replacement $\mathcal{M}^{(n)}_{c\bar{c}\to F} \to \widetilde{\mathcal{M}}^{{\rm th}(n)}_{c\bar{c}\to F}$. Note that $\widetilde{\mathcal{M}}^{{\rm th}(0)}_{c\bar{c}\to F} = \mathcal{M}^{(0)}_{c\bar{c}\to F}$, and the higher-order contributions $\widetilde{\mathcal{M}}^{{\rm th}(n)}_{c\bar{c}\to F}$ $(n \geq 1)$ are obtained from Eq. (16) in terms of $\mathcal{M}^{(l)}_{c\bar{c}\to F}$ and $\tilde{I}_c^{{\rm th}(l)}(\epsilon)$ at equal or lower orders, i.e. with $l \leq n$ (see, e.g., Eqs. (48) and (49) in Ref. [3]). For simplicity, the perturbative expansions on the right-hand side of Eqs. (22) and (24) are expressed in powers of $\alpha_{\rm S}(M^2)$. Note, however, that $\mathcal{M}_{c\bar{c}\to F}$ and $\tilde{I}_c^{\rm th}(\epsilon, M^2)$ are separately renormalization-group invariant quantities. Therefore, they can be equivalently expanded as powers series in $\alpha_{\rm S}(\mu_R^2)$, with corresponding perturbative terms that depend on M^2/μ_R^2 (see, e.g., Eqs. (50)–(57) in Ref. [3]). The equivalent expansions are simply obtained by using Eq. (19) to directly express $\alpha_{\rm S}(M^2)$ in terms of $\alpha_{\rm S}(\mu_R^2)$ and integer powers of $(M^2/\mu_R^2)^{-\epsilon}$.

In Ref. [3] we derived the explicit expression of the first-order and second-order subtraction operators $\tilde{I}_c^{{\rm th}(1)}(\epsilon)$ and $\tilde{I}_c^{{\rm th}(2)}(\epsilon)$. To extend the results to the third order, we introduce a more compact (though completely equivalent) all-order representation. The operator $\tilde{I}_c^{\rm th}(\epsilon, M^2)$ can be written as

$$1 - \tilde{I}_c^{\rm th}(\epsilon, M^2) = \exp\bigl\{R_c\bigl(\epsilon, \alpha_{\rm S}\bigl(M^2\bigr)\bigr) - i\Phi_c\bigl(\epsilon, \alpha_{\rm S}\bigl(M^2\bigr)\bigr)\bigr\}, \tag{25}$$

where R_c and Φ_c are *real* functions. The function $\Phi_c(\epsilon, M^2)$ is the IR divergent Coulomb phase that originates from the virtual contributions to the all-loop amplitude $\mathcal{M}_{c\bar{c}\to F}$. Its explicit expression up to $\mathcal{O}(\alpha_{\rm S}^3)$ [35] reads

$$\begin{aligned}-i\Phi_c(\epsilon, \alpha_{\rm S}) = \frac{i\pi\, C_c}{2\epsilon}\biggl\{\biggl(\frac{\alpha_{\rm S}}{\pi}\biggr) + \biggl(\frac{\alpha_{\rm S}}{\pi}\biggr)^2 \frac{1}{2}\biggl(\gamma^{(1)}_{\rm cusp} - \frac{\beta_0\pi}{\epsilon}\biggr) \\ + \biggl(\frac{\alpha_{\rm S}}{\pi}\biggr)^3 \frac{1}{3}\biggl(\gamma^{(2)}_{\rm cusp} - \frac{1}{\epsilon}\gamma^{(1)}_{\rm cusp}\,\beta_0\pi + \frac{1}{\epsilon}\pi^2\biggl(\frac{\beta_0^2}{\epsilon} - \beta_1\biggr)\biggr)\biggr\} + \mathcal{O}\bigl(\alpha_{\rm S}^4\bigr).\end{aligned} \tag{26}$$

The function $R_c(\epsilon, \alpha_{\rm S})$ contains IR finite terms and all the remaining IR divergent terms (in the limit $\epsilon \to 0$) in the exponent of Eq. (25). This perturbative function can be decomposed as follows:

$$R_c(\epsilon, \alpha_{\rm S}) = R_c^{\rm soft}(\epsilon, \alpha_{\rm S}) + R_c^{\rm coll}(\epsilon, \alpha_{\rm S}), \tag{27}$$

where

$$\begin{aligned}R_c^{\rm soft}(\epsilon, \alpha_{\rm S}) = C_c\biggl(\frac{\alpha_{\rm S}}{\pi} R^{{\rm soft}(1)}(\epsilon) + \biggl(\frac{\alpha_{\rm S}}{\pi}\biggr)^2 R^{{\rm soft}(2)}(\epsilon) + \biggl(\frac{\alpha_{\rm S}}{\pi}\biggr)^3 R^{{\rm soft}(3)}(\epsilon)\biggr) \\ + \mathcal{O}\bigl(\alpha_{\rm S}^4\bigr),\end{aligned} \tag{28}$$

$$R_c^{\rm coll}(\epsilon, \alpha_{\rm S}) = \frac{\alpha_{\rm S}}{\pi} R_c^{{\rm coll}(1)}(\epsilon) + \biggl(\frac{\alpha_{\rm S}}{\pi}\biggr)^2 R_c^{{\rm coll}(2)}(\epsilon) + \biggl(\frac{\alpha_{\rm S}}{\pi}\biggr)^3 R_c^{{\rm coll}(3)}(\epsilon) + \mathcal{O}\bigl(\alpha_{\rm S}^4\bigr). \tag{29}$$

The two components $R_c^{\rm soft}$ and $R_c^{\rm coll}$ of Eq. (27) have a soft and collinear origin, respectively. The ϵ-dependent perturbative coefficients on the right-hand side of Eqs. (28) and (29) read

$$R^{{\rm soft}(1)}(\epsilon) = \frac{1}{2\epsilon^2} + R^{{\rm fin}(1)}, \tag{30}$$

$$R_c^{{\rm coll}(1)}(\epsilon) = \frac{\gamma_c}{2\epsilon}, \tag{31}$$

$$R^{\mathrm{soft}(2)}(\epsilon) = -\frac{3}{8}\frac{\beta_0\pi}{\epsilon^3} + \frac{1}{8\epsilon^2}\gamma_{\mathrm{cusp}}^{(1)} - \frac{1}{16\epsilon}d_{(1)} + R^{\mathrm{fin}(2)}, \tag{32}$$

$$R_c^{\mathrm{coll}(2)}(\epsilon) = -\frac{\beta_0\pi}{4\epsilon^2}\gamma_c + \frac{1}{8\epsilon}\gamma_c^{(1)}, \tag{33}$$

$$R^{\mathrm{soft}(3)}(\epsilon) = \frac{11\beta_0^2 - 8\beta_1\epsilon}{36\epsilon^4}\pi^2 - \frac{5}{36\epsilon^3}\beta_0\pi\gamma_{\mathrm{cusp}}^{(1)} + \frac{1}{18\epsilon^2}\gamma_{\mathrm{cusp}}^{(2)} + \frac{1}{24\epsilon^2}\beta_0\pi d_{(1)} - \frac{1}{48\epsilon}d_{(2)} + R^{\mathrm{fin}(3)}, \tag{34}$$

$$R_c^{\mathrm{coll}(3)}(\epsilon) = \frac{\gamma_c}{6\epsilon^2}\left(\frac{(\beta_0\pi)^2}{\epsilon} - \beta_1\pi^2\right) - \beta_0\pi\frac{\gamma_c^{(1)}}{12\epsilon^2} + \frac{1}{24\epsilon}\gamma_c^{(2)}. \tag{35}$$

The coefficients γ_c, $\gamma_c^{(1)}$ and $\gamma_c^{(2)}$ in Eqs. (31), (33) and (35) depend on the parton flavor $c = q, g$ and they have a collinear origin. They are equal to the coefficients of the term proportional to $\delta(1-z)$ (i.e., to the virtual contribution) in the leading order (LO), next-to-leading order (NLO) and next-to-next-to-leading order (NNLO) collinear splitting functions [4], and their explicit values[3] are

$$\begin{aligned}
\gamma_q &= \frac{3}{2}C_F,\\
\gamma_q^{(1)} &= \left(\frac{3}{8} - \frac{1}{2}\pi^2 + 6\zeta_3\right)C_F^2 + \left(\frac{17}{24} + \frac{11}{18}\pi^2 - 3\zeta_3\right)C_F C_A + \left(-\frac{1}{12} - \frac{1}{9}\pi^2\right)C_F n_F,\\
\gamma_q^{(2)} &= C_F^3\left(\frac{29}{16} + \frac{3}{8}\pi^2 + \frac{\pi^4}{5} + \frac{17}{2}\zeta_3 - \frac{2}{3}\pi^2\zeta_3 - 30\zeta_5\right)\\
&\quad + C_F^2 C_A\left(\frac{151}{32} - \frac{205}{72}\pi^2 - \frac{247}{1080}\pi^4 + \frac{211}{6}\zeta_3 + \frac{1}{3}\pi^2\zeta_3 + 15\zeta_5\right)\\
&\quad + C_A^2 C_F\left(-\frac{1657}{288} + \frac{281}{81}\pi^2 - \frac{\pi^4}{144} - \frac{194}{9}\zeta_3 + 5\zeta_5\right)\\
&\quad + C_F^2 n_F\left(-\frac{23}{8} + \frac{5}{36}\pi^2 + \frac{29}{540}\pi^4 - \frac{17}{3}\zeta_3\right) + C_F n_F^2\left(-\frac{17}{72} + \frac{5}{81}\pi^2 - \frac{2}{9}\zeta_3\right)\\
&\quad + C_F C_A n_F\left(\frac{5}{2} - \frac{167}{162}\pi^2 + \frac{\pi^4}{360} + \frac{25}{9}\zeta_3\right),
\end{aligned} \tag{36}$$

$$\begin{aligned}
\gamma_g &= \frac{11}{6}C_A - \frac{1}{3}n_F,\\
\gamma_g^{(1)} &= \left(\frac{8}{3} + 3\zeta_3\right)C_A^2 - \frac{2}{3}C_A n_F - \frac{1}{2}C_F n_F,\\
\gamma_g^{(2)} &= C_A^3\left(\frac{79}{16} + \frac{\pi^2}{18} + \frac{11}{432}\pi^4 + \frac{67}{3}\zeta_3 - \frac{1}{3}\pi^2\zeta_3 - 10\zeta_5\right)\\
&\quad + C_A^2 n_F\left(-\frac{233}{144} - \frac{\pi^2}{18} - \frac{\pi^4}{216} - \frac{10}{3}\zeta_3\right)\\
&\quad + \frac{1}{8}C_F^2 n_F - \frac{241}{144}C_A C_F n_F + \frac{29}{144}C_A n_F^2 + \frac{11}{72}C_F n_F^2.
\end{aligned} \tag{37}$$

[3] In Ref. [3] we used a slightly different notation, and the coefficient $\gamma_{c(1)}$ therein is related to $\gamma_c^{(1)}$ as $\gamma_c^{(1)} = -\gamma_{c(1)}/8$.

The coefficients $d_{(1)}$ and $d_{(2)}$ in Eqs. (32) and (34) have a soft origin, and their values read

$$d_{(1)} = \left(\frac{28}{27} - \frac{1}{18}\pi^2\right) n_F + \left(-\frac{202}{27} + \frac{11}{36}\pi^2 + 7\zeta_3\right) C_A, \tag{38}$$

$$\begin{aligned} d_{(2)} = {} & C_A^2\left(-\frac{136\,781}{5832} + \frac{6325}{1944}\pi^2 - \frac{11}{45}\pi^4 + \frac{329}{6}\zeta_3 - \frac{11}{9}\pi^2\zeta_3 - 24\zeta_5\right) \\ & + C_A n_F\left(\frac{5921}{2916} - \frac{707}{972}\pi^2 + \frac{\pi^4}{15} - \frac{91}{27}\zeta_3\right) \\ & + C_F n_F\left(\frac{1711}{216} - \frac{\pi^2}{12} - \frac{\pi^4}{45} - \frac{38}{9}\zeta_3\right) \\ & + n_F^2\left(\frac{260}{729} + \frac{5}{162}\pi^2 - \frac{14}{27}\zeta_3\right). \end{aligned} \tag{39}$$

The coefficients $R^{\mathrm{fin}(1)}$ and $R^{\mathrm{fin}(2)}$ determine the IR finite part on the right-hand side of Eqs. (30) and (32): their explicit values are known [3] and read[4]

$$R^{\mathrm{fin}(1)} = -\frac{\pi^2}{8}, \tag{40}$$

$$R^{\mathrm{fin}(2)} = C_A\left(\frac{607}{648} - \frac{469}{1728}\pi^2 + \frac{\pi^4}{288} - \frac{187}{144}\zeta_3\right) + n_F\left(-\frac{41}{324} + \frac{35}{864}\pi^2 + \frac{17}{72}\zeta_3\right). \tag{41}$$

The first-order and second-order results in Eqs. (30)–(33) were obtained in Ref. [3]. The three-loop expressions in Eqs. (34) and (35) and, especially, the value of the IR finite part $R^{\mathrm{fin}(3)}$ in Eq. (34) are the main new results of the present paper. The explicit value of the third-order coefficient $R^{\mathrm{fin}(3)}$ is

$$\begin{aligned} R^{\mathrm{fin}(3)} = {} & \left(\frac{5\,211\,949}{1\,679\,616} - \frac{578\,479}{559\,872}\pi^2 + \frac{9457}{311\,040}\pi^4 + \frac{19}{326\,592}\pi^6 \right. \\ & \left. - \frac{64\,483}{7776}\zeta_3 + \frac{121}{192}\pi^2\zeta_3 + \frac{67}{72}\zeta_3^2 - \frac{121}{144}\zeta_5\right) C_A^2 \\ & + \left(-\frac{412\,765}{839\,808} + \frac{75\,155}{279\,936}\pi^2 - \frac{79}{9720}\pi^4 + \frac{154}{81}\zeta_3 - \frac{11}{288}\pi^2\zeta_3 - \frac{1}{24}\zeta_5\right) C_A n_F \\ & + \left(-\frac{42\,727}{62\,208} + \frac{605}{6912}\pi^2 + \frac{19}{12\,960}\pi^4 + \frac{571}{1296}\zeta_3 - \frac{11}{144}\pi^2\zeta_3 + \frac{7}{36}\zeta_5\right) C_F n_F \\ & + \left(-\frac{2}{6561} - \frac{101}{7776}\pi^2 + \frac{37}{77\,760}\pi^4 - \frac{185}{1944}\zeta_3\right) n_F^2. \end{aligned} \tag{42}$$

We note that the phase factor $e^{-i\Phi_c}$ in Eq. (25) is physically (and practically) harmless to the purpose of computing the threshold resummation coefficient $C^{\mathrm{th}}_{c\bar{c}\to F}$ in Eqs. (17) and (18). Indeed, $e^{-i\Phi_c}$ produces a corresponding overall phase factor contribution to $\widetilde{\mathcal{M}}^{\mathrm{th}}_{c\bar{c}\to F}$ in Eq. (16) and, therefore, $e^{-i\Phi_c}$ gives a vanishing contribution to $|\widetilde{\mathcal{M}}^{\mathrm{th}}_{c\bar{c}\to F}|^2$ and, hence, to $C^{\mathrm{th}}_{c\bar{c}\to F}$. We recall [3] that this phase factor has been introduced in $\tilde{I}_c^{\mathrm{th}}$ to the sole practical (aesthetical) purpose of canceling the IR divergent Coulomb phase of the virtual amplitude $\mathcal{M}_{c\bar{c}\to F}$, so that

[4] In Ref. [3], the IR finite part of $\tilde{I}_c^{\mathrm{th}(1)}$ and $\tilde{I}_c^{\mathrm{th}(2)}$ is specified by using a different notation in terms of the coefficients $\delta^{\mathrm{th}}_{(1)}$ and $\delta^{\mathrm{th}}_{(2)}$ therein.

$\widetilde{\mathcal{M}}^{\rm th}_{c\bar{c}\to F}$ itself (and not only $|\widetilde{\mathcal{M}}^{\rm th}_{c\bar{c}\to F}|^2$) is IR finite in the limit $\epsilon \to 0$. We note that $\widetilde{\mathcal{M}}^{\rm th}_{c\bar{c}\to F}$ can also be redefined by including equally harmless contributions that are purely real (rather than phase factors). We can consider a *multiplicative* redefinition $\widetilde{\mathcal{M}}^{\rm th}_{c\bar{c}\to F} \to F(\alpha_S,\epsilon)\widetilde{\mathcal{M}}^{\rm th}_{c\bar{c}\to F}$, where F is an arbitrary perturbative function (i.e., $F = 1 + \mathcal{O}(\alpha_S)$) such that it is equal to unity in the limit $\epsilon \to 0$ (i.e., $F = 1 + \mathcal{O}(\epsilon^m)$ with $m = 1, 2, \ldots$). Since $\widetilde{\mathcal{M}}^{\rm th}_{c\bar{c}\to F}$ is IR finite, this multiplicative redefinition gives a vanishing contribution to $\widetilde{\mathcal{M}}^{\rm th}_{c\bar{c}\to F}$ in the four-dimensional limit $\epsilon \to 0$. Such harmless multiplicative redefinition corresponds to the replacement $(1 - \tilde{I}_c^{\rm th}) \to F(\alpha_S,\epsilon)\,(1 - \tilde{I}_c^{\rm th})$ or, equivalently, to the replacement $R_c(\epsilon,\alpha_S) \to R_c(\epsilon,\alpha_S) + \ln F(\alpha_S,\epsilon) = R_c(\epsilon,\alpha_S) + \mathcal{O}(\epsilon^m)$, with $m = 1, 2, \ldots$, in Eq. (25) (we have used $\ln F(\alpha_S,\epsilon) = \mathcal{O}(\epsilon^m)$). Therefore, we see that terms of $\mathcal{O}(\epsilon^m)$, with $m = 1, 2, \ldots$, in $R_c(\epsilon,\alpha_S)$ are harmless. In our explicit expressions (see Eqs. (27)–(35)) of $R_c(\epsilon,\alpha_S)$ we have not included any of these terms, whereas the explicit expressions of $\tilde{I}_c^{\rm th(1)}(\epsilon)$ and $\tilde{I}_c^{\rm th(2)}(\epsilon)$ that are presented in Ref. [3] include contributions that are due to this type of harmless terms.

The derivation of the factorization formula (16), its origin and the general structure of the subtraction operator $\tilde{I}_c^{\rm th}(\epsilon, M^2)$ in Eq. (25) were discussed in Ref. [3]. Here we limit ourselves to presenting the main conclusions of our reasoning [3] in a very concise form (we refer to Sections 4.1 and 5 of Ref. [3] for an extended discussion). We have already recalled the origin of the phase factor $e^{-i\Phi_c}$ in Eq. (25). We then recall [3] that the remaining contributions to $\tilde{I}_c^{\rm th}$ (i.e., the factor e^{R_c} in Eq. (25)) have a soft and collinear origin, as specified by the decomposition in Eq. (27). The collinear contributions are embodied in the factor $e^{R_c^{\rm coll}}$, and they are entirely due to the *virtual* part of the collinear-counterterm factor that is introduced in the (bare) partonic cross sections to factorize the $\overline{\rm MS}$ parton densities (see Eq. (2)). Since we are considering parton densities in the $\overline{\rm MS}$ factorization scheme, this collinear-counterterm factor is completely and explicitly specified up to $\mathcal{O}(\alpha_S^3)$ [4] and, in particular, the perturbative function $R_c^{\rm coll}(\epsilon,\alpha_S)$ in Eq. (29) includes only ϵ-pole contributions (see Eqs. (31), (33) and (35)) with no additional IR finite terms. The soft contributions to $\tilde{I}_c^{\rm th}$ are embodied in the factor $e^{R_c^{\rm soft}}$. They are due to the soft part of the $\overline{\rm MS}$ collinear counterterm [4] and to the inelastic processes $c\bar{c} \to F + X$, where the radiated final-state system X includes only soft partons. The soft-parton contribution of the inelastic processes can be determined by using universal (process-independent) soft factorization formulae [36–40] of the corresponding scattering amplitudes. In Ref. [41], the soft-parton contribution to the total cross section was explicitly computed up to NNLO in a process-independent form by using soft factorization formulae up to $\mathcal{O}(\alpha_S^2)$ [37–39]. A corresponding process-independent calculation at N^{3}LO can be performed by using soft factorization formulae at $\mathcal{O}(\alpha_S^3)$ [7,42]. As discussed in Ref. [42], soft-factorization results from Refs. [7,38,39,42] and the soft limit of the results in Ref. [6] can be combined and used to reproduce [42] the results of the N^{3}LO cross sections for Higgs boson [1] and DY production [2]. However, as discussed and pointed out in Ref. [3], much information on the soft contribution to $\tilde{I}_c^{\rm th}$ can be obtained independently of detailed computations. Indeed, due to non-abelian eikonal exponentiation [29], the intensity of soft radiation from the parton c is simply proportional to the Casimir coefficient C_c of that parton (this conclusion is certainly valid up to $\mathcal{O}(\alpha_S^3)$ [29]). Therefore, $R_c^{\rm soft}(\epsilon,\alpha_S)$ can be expressed by factorizing the overall coefficient C_c as in Eq. (28). This Casimir scaling behavior is completely analogous to that of the functions $A_c(\alpha_S)$ (see Eq. (14)), $D_c(\alpha_S)$ (see Eqs. (11) and (13)) and $\Phi_c(\epsilon,\alpha_S)$ (see Eq. (26)), since all these functions are entirely due to soft-parton contributions [3]. The perturbative coefficients $R^{{\rm soft}(n)}(\epsilon)$, with $n = 1, 2, 3$, in Eq. (28) are completely process independent and they can be determined by considering a single specific process. In particular, $R^{{\rm soft}(n)}(\epsilon)$ contains IR divergent contributions (ϵ-pole terms) and IR finite contributions. These IR divergent terms of soft-parton origin are due to real emission contributions, but they are constrained (because of the real–virtual cancellation mechanism of IR divergences) to be exactly equal to the corresponding IR divergent terms due to virtual radiation. Therefore, the ϵ-pole terms in Eqs. (30), (32) and (34) are completely specified by the explicit calculation of either the quark or gluon form factors [35] (as recalled below, the process independence of these terms is consistent with the universality structure of the IR divergent contributions to the QCD scattering amplitudes [31,33,34]). It follows that the IR finite coefficients $R^{\mathrm{fin}(n)}$ $(n = 1, 2, 3)$ are the only terms that are not explicitly determined by using our general reasoning [3]. Owing to their universality, the explicit computation of a single process is sufficient to extract the values of these IR finite coefficients. As illustrated below, we use the N^{3}LO Higgs boson results of Ref. [1] to obtain the value of $R^{\mathrm{fin}(3)}$ in Eq. (42).

Before considering the evaluation of $R^{\mathrm{fin}(3)}$, we present some additional comments on the structure of Eqs. (25)–(39) and on the connection between real- and virtual-emission contributions. As we have discussed, the subtraction operator $(1 - \tilde{I}_c^{\mathrm{th}})$ in Eqs. (16) and (25) includes the Coulomb phase factor $e^{-i\Phi_c}$ and an additional factor of soft and collinear origin. In Eq. (25) we express this additional factor by using the exponentiated form e^{R_c}. The exponentiated form, which is completely equivalent to its direct expansion in powers of α_S, is more compact in view of the factorization and exponentiation properties of both soft and collinear contributions. Owing to factorization we can write $e^{R_c} = e^{R_c^{\mathrm{coll}}} e^{R_c^{\mathrm{soft}}}$, i.e. we can introduce the decomposition in Eq. (27). The collinear factor $e^{R_c^{\mathrm{coll}}}$ is entirely due to the virtual part of the collinear counterterm of the $\overline{\mathrm{MS}}$ parton densities, and its exponentiated structure is eventually a consequence of the customary solution of the Altarelli–Parisi evolution equations in terms of an exponentiated evolution operator. Indeed (as stated below Eq. (35)) the exponent R_c^{coll} is directly determined by the coefficients $\gamma_c, \gamma_c^{(1)}$ and $\gamma_c^{(2)}$ of the virtual part of the Altarelli–Parisi splitting functions. The factor $e^{R_c^{\mathrm{soft}}}$ is due to real emission of soft partons: it fulfills non-abelian eikonal exponentiation and, therefore, we can express the exponent R_c^{soft} through the Casimir scaling relation (28). The soft/collinear structure of $(1 - \tilde{I}_c^{\mathrm{th}}) \propto e^{R_c^{\mathrm{coll}}} e^{R_c^{\mathrm{soft}}}$ does not originate from virtual contributions to the scattering amplitude $\mathcal{M}_{c\bar{c}\to F}$, but the IR divergent terms in Eqs. (28)–(35) exactly match the analogous universal structure of the IR divergent virtual contributions to $\mathcal{M}_{c\bar{c}\to F}$. The IR divergent virtual contributions [31–35] include dominant and subdominant ϵ-poles. The dominant poles have a soft-collinear origin and are controlled by the perturbative function $A_c(\alpha_S)$ in Eq. (10) or, equivalently, the function $\gamma_{\mathrm{cusp}}(\alpha_S)$ in Eq. (14). The subdominant poles originate from either collinear (and non-soft) or soft (and non-collinear) contributions and they are controlled by the collinear coefficients in Eqs. (36)–(37) and the soft coefficients in Eqs. (38)–(39). We also note that the real emission contribution to the partonic cross section of Eq. (8) is separated in two different factors: the N-independent factor $e^{R_c^{\mathrm{soft}}}$ (which contributes to $(1 - \tilde{I}_c^{\mathrm{th}})$ and, hence, to $C^{\mathrm{th}}_{c\bar{c}\to F}$) and the $\ln N$-dependent radiative factor $\Delta_{c,N}$ of Eq. (9). These two factors have a soft origin and they are not fully independent. In particular, the coefficients of the dominant IR poles of $R_c^{\mathrm{soft}}(\epsilon, \alpha_S)$ are directly related to the dominant $\ln N$-dependence of $\Delta_{c,N}$ (as given by the perturbative function $A_c(\alpha_S)$). The subdominant $\ln N$-dependence of $\Delta_{c,N}$ is due to the soft-parton function $D_c(\alpha_S)$, whose perturbative coefficients $D_c^{(n)}$ are related to the soft-parton coefficients $C_c\, d_{(n-1)}$ and $C_c\, R^{\mathrm{fin}(n-1)}$ of $R_c^{\mathrm{soft}}(\epsilon, \alpha_S)$: this relation between $\ln N$ terms, ϵ-poles and IR finite terms is discussed and worked out in Refs. [27,28]. We note that using the general analysis of Refs. [27,28] and our result for $R^{\mathrm{fin}(3)}$ in Eq. (42), the fourth-order coefficient $D_c^{(4)}$ of $D_c(\alpha_S)$ can be determined in terms of the ϵ-poles at $\mathcal{O}(\alpha_S^4)$ (once they become available).

To evaluate the third-order coefficient $R^{\text{fin}(3)}$, we consider the perturbative expansion of the resummation formula in Eq. (8), which contains all the terms which are not suppressed in the large-N limit, namely, the logarithmically-enhanced terms and the constant terms as $N \to \infty$. We consider the N^{3}LO contribution (see, e.g., the Appendix E in Ref. [22]) and we transform it back to z space to obtain the general expression of the N^{3}LO term $g_{c\bar{c}}^{F(3)}(z)$ of Eq. (3) in the threshold limit $z \to 1$. We find

$$
\begin{aligned}
g_{c\bar{c}}^{F(3)}(z) = {} & 8\big(A_c^{(1)}\big)^3 \mathcal{D}_5 - \frac{40}{3}\beta_0\pi\big(A_c^{(1)}\big)^2 \mathcal{D}_4 \\
& + \left(-\frac{32}{3}\pi^2\big(A_c^{(1)}\big)^3 + 8C^{\text{th}(1)}_{c\bar{c}\to F}\big(A_c^{(1)}\big)^2 + 16A_c^{(1)}A_c^{(2)} + \frac{16}{3}(\beta_0\pi)^2 A_c^{(1)}\right)\mathcal{D}_3 \\
& + \big(160\zeta_3\big(A_c^{(1)}\big)^3 - 4\beta_0\pi\, A_c^{(1)}C^{\text{th}(1)}_{c\bar{c}\to F} + 8\beta_0\pi^3\big(A_c^{(1)}\big)^2 \\
& - 8\beta_0\pi A_c^{(2)} + 6A_c^{(1)}D_c^{(2)} - 4A_c^{(1)}\beta_1\pi^2\big)\mathcal{D}_2 \\
& + \bigg(4\big(A_c^{(3)} + A_c^{(2)}C^{\text{th}(1)}_{c\bar{c}\to F} + A_c^{(1)}C^{\text{th}(2)}_{c\bar{c}\to F}\big) - \frac{16}{3}A_c^{(1)}A_c^{(2)}\pi^2 \\
& - \frac{8}{3}\big(A_c^{(1)}\big)^2 C^{\text{th}(1)}_{c\bar{c}\to F}\pi^2 - \frac{4}{9}\pi^4\big(A_c^{(1)}\big)^3 - 4\beta_0\pi\big(D_c^{(2)} + 24\big(A_c^{(1)}\big)^2\zeta_3\big)\bigg)\mathcal{D}_1 \\
& + \bigg(\left(192\zeta_5 - \frac{64}{3}\pi^2\zeta_3\right)\big(A_c^{(1)}\big)^3 + 16A_c^{(1)}\zeta_3\big(2A_c^{(2)} + A_c^{(1)}C^{\text{th}(1)}_{c\bar{c}\to F}\big) \\
& + \frac{4}{9}\big(A_c^{(1)}\big)^2\beta_0\pi^5 + C^{\text{th}(1)}_{c\bar{c}\to F}D_c^{(2)} + D_c^{(3)} - \frac{2}{3}A_c^{(1)}D_c^{(2)}\pi^2\bigg)\mathcal{D}_0 \\
& + \bigg(C^{\text{th}(3)}_{c\bar{c}\to F} - \frac{2}{45}A_c^{(1)}A_c^{(2)}\pi^4 - \frac{1}{45}\big(A_c^{(1)}\big)^2 C^{\text{th}(1)}_{c\bar{c}\to F}\pi^4 \\
& + \left(\frac{160}{3}\zeta_3^2 - \frac{116}{2835}\pi^6\right)\big(A_c^{(1)}\big)^3 + 4A_c^{(1)}D_c^{(2)}\zeta_3 \\
& + \frac{16}{3}\big(A_c^{(1)}\big)^2\beta_0\pi\big(\pi^2\zeta_3 - 12\zeta_5\big)\bigg)\delta(1-z) + \ldots, \qquad (43)
\end{aligned}
$$

where $\mathcal{D}_m = \mathcal{D}_m(z)$ are the plus-distributions defined in Eq. (5), and the dots in the right-hand side of Eq. (43) denote additional terms that are less singular in the limit $z \to 1$ (i.e., terms that are relatively suppressed by some powers of $(1-z)$). The terms that are explicitly denoted in the right-hand side of Eq. (43) define the soft-virtual (SV) approximation of the N^{3}LO contribution $g_{c\bar{c}}^{F(3)}(z)$ to the partonic cross section. These terms depend on the universal perturbative coefficients $A_c^{(n)}$, $D_c^{(n)}$ (see Eqs. (12) and (13)) and on the process-dependent coefficients $C^{\text{th}(n)}_{c\bar{c}\to F}$ with $n \le 3$.

In the case of Higgs boson production ($gg \to H$) by gluon fusion, the SV N^{3}LO expression in Eq. (43) exactly corresponds to the result of the explicit computation performed in Ref. [1]. The first-order and second-order coefficients $C^{\text{th}(1)}_{gg\to F}$ and $C^{\text{th}(2)}_{gg\to F}$ are known (they can be determined by our process-independent resummation formalism up to $\mathcal{O}(\alpha_S^2)$ or, equivalently, they can be extracted from the SV NNLO results of Refs. [26,43]). Therefore, comparing Eq. (43) with the result in Eq. (10) of Ref. [1], we can extract the coefficient $C^{\text{th}(3)}_{gg\to F}$ and we find

$$
\begin{aligned}
C^{\mathrm{th}(3)}_{gg\to H} = {} & C_A^3\biggl(\frac{215\,131}{5184}+\frac{16\,151}{7776}\pi^2-\frac{1765}{15\,552}\pi^4+\frac{1}{2160}\pi^6 \\
& -\frac{15\,649}{432}\zeta_3-\frac{77}{144}\pi^2\zeta_3+\frac{3}{2}\zeta_3^2+\frac{869}{144}\zeta_5\biggr) \\
& +C_A^2 n_F\biggl(-\frac{98\,059}{5184}-\frac{35}{243}\pi^2+\frac{2149}{38\,880}\pi^4+\frac{29}{8}\zeta_3-\frac{29}{72}\pi^2\zeta_3+\frac{101}{72}\zeta_5\biggr) \\
& +C_A C_F n_F\biggl(-\frac{63\,991}{5184}-\frac{71}{216}\pi^2+\frac{11}{6480}\pi^4+\frac{13}{2}\zeta_3+\frac{1}{2}\pi^2\zeta_3+\frac{5}{2}\zeta_5\biggr) \\
& +C_F^2 n_F\biggl(\frac{19}{18}+\frac{37}{12}\zeta_3-5\zeta_5\biggr) \\
& +C_A n_F^2\biggl(\frac{2515}{1728}-\frac{133}{1944}\pi^2-\frac{19}{3240}\pi^4+\frac{43}{108}\zeta_3\biggr) \\
& +C_F n_F^2\biggl(\frac{4481}{2592}-\frac{23}{432}\pi^2-\frac{1}{3240}\pi^4-\frac{7}{6}\zeta_3\biggr).
\end{aligned}
\tag{44}
$$

To be precise, the coefficient $C^{\mathrm{th}(3)}_{gg\to H}$ in Eq. (44) corresponds to the perturbative expansion that is defined by Eq. (3) after having rescaled the partonic cross section with the Wilson coefficient of the effective point-like coupling ggH [9] (this definition exactly corresponds to that used in Eq. (4) of Ref. [1]). Having the information in Eq. (44) and using Eqs. (16) and (17), we apply the operator $(1-\tilde{I}_c^{\mathrm{th}})$ of Eq. (25) to the three-loop gluon form factor [10] and we can extract the coefficient $R^{\mathrm{fin}(3)}$ in Eq. (34). We find the explicit value that is presented in Eq. (42).

The coefficient $R^{\mathrm{fin}(3)}$ completely determines the explicit expression of the process-independent subtraction operator $\tilde{I}_c^{\mathrm{th}}$ up to $\mathcal{O}(\alpha_S^3)$. Using this expression and Eqs. (16)–(18), the threshold resummation coefficient $C^{\mathrm{th}}_{c\bar{c}\to F}(\alpha_S)$ for an arbitrary process $c\bar{c}\to F$ is straightforwardly and explicitly computable up to the three-loop order once the corresponding three-loop scattering amplitude $\mathcal{M}_{c\bar{c}\to F}$ for that process is known.

As an application of our general formalism and results, we can consider the production of a vector boson V ($V=Z,W^\pm$) by the DY process $q\bar{q}\to V$. Using the subtraction operator $(1-\tilde{I}_c^{\mathrm{th}})$ and the results for the quark form factor up to three-loop order [10], we can compute the coefficients $C^{\mathrm{th}(n)}_{q\bar{q}\to V}$ with $n=1,2,3$. We find

$$
C^{\mathrm{th}(1)}_{q\bar{q}\to V} = C_F\biggl(-4+\frac{\pi^2}{3}\biggr),
\tag{45}
$$

$$
\begin{aligned}
C^{\mathrm{th}(2)}_{q\bar{q}\to V} = {} & C_F^2\biggl(\frac{511}{64}-\frac{35}{48}\pi^2+\frac{\pi^4}{40}-\frac{15}{4}\zeta_3\biggr)+C_F C_A\biggl(-\frac{1535}{192}+\frac{37}{54}\pi^2-\frac{\pi^4}{240}+\frac{7}{4}\zeta_3\biggr) \\
& +C_F n_F\biggl(\frac{127}{96}-\frac{7}{54}\pi^2+\frac{1}{2}\zeta_3\biggr),
\end{aligned}
\tag{46}
$$

$$
\begin{aligned}
C^{\mathrm{th}(3)}_{q\bar{q}\to V} = {} & C_F^3\biggl(-\frac{5599}{384}-\frac{65}{576}\pi^2-\frac{17}{320}\pi^4+\frac{803}{136\,080}\pi^6 \\
& -\frac{115}{16}\zeta_3+\frac{5}{24}\pi^2\zeta_3+\frac{1}{2}\zeta_3^2+\frac{83}{4}\zeta_5\biggr) \\
& +C_F^2 C_A\biggl(\frac{74\,321}{2304}-\frac{6593}{5184}\pi^2+\frac{94}{1215}\pi^4-\frac{2309}{272\,160}\pi^6
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{8653}{432}\zeta_3 + \frac{53}{54}\pi^2\zeta_3 + \frac{37}{12}\zeta_3^2 - \frac{689}{72}\zeta_5\Big) \\
&+ C_A^2 C_F\Big(-\frac{1\,505\,881}{62\,208} + \frac{281}{128}\pi^2 + \frac{14\,611}{311\,040}\pi^4 + \frac{829}{272\,160}\pi^6 \\
&+ \frac{82\,385}{5184}\zeta_3 - \frac{221}{288}\pi^2\zeta_3 - \frac{25}{12}\zeta_3^2 - \frac{51}{16}\zeta_5\Big) \\
&+ C_A C_F n_F\Big(\frac{110\,651}{15\,552} - \frac{7033}{7776}\pi^2 - \frac{1439}{77\,760}\pi^4 - \frac{94}{81}\zeta_3 + \frac{13}{72}\pi^2\zeta_3 - \frac{\zeta_5}{8}\Big) \\
&+ C_F^2 n_F\Big(-\frac{421}{192} + \frac{329}{1296}\pi^2 - \frac{223}{19\,440}\pi^4 + \frac{869}{216}\zeta_3 - \frac{7}{27}\pi^2\zeta_3 - \frac{19}{18}\zeta_5\Big) \\
&+ C_F n_F^2\Big(-\frac{7081}{15\,552} + \frac{151}{1944}\pi^2 + \frac{\pi^4}{486} - \frac{79}{324}\zeta_3\Big) \\
&+ C_F N_{F,V}\Big(\frac{N_c^2-4}{N_c}\Big)\Big(\frac{1}{8} + \frac{5}{96}\pi^2 - \frac{\pi^4}{2880} + \frac{7}{48}\zeta_3 - \frac{5}{6}\zeta_5\Big),
\end{aligned}
\tag{47}
$$

where $N_{F,V}$ is a factor originating by diagrams where the virtual gauge boson does not couple directly to the initial state quarks [10], and it is proportional to the charge weighted sum of the quark flavors. The explicit expressions of the coefficients $A_c^{(n)}$ and $D_c^{(n)}$ up to $\mathcal{O}(\alpha_S^3)$ and the expressions of $C^{\mathrm{th}(n)}_{q\bar{q}\to V}$ in Eqs. (45)–(47) can be inserted in Eq. (43) to obtain the explicit expression of the SV N^{3}LO cross section for the DY process. The ensuing result is in agreement with the result in Ref. [2].

In this paper we have considered the processes in which an arbitrary colorless system F with high mass is produced in hadronic collisions. We have focused on the structure of the perturbative QCD contributions near partonic threshold. Such contributions are controlled by universal resummation factors plus a process dependent hard-virtual function. As discussed in Ref. [3], the hard-virtual function is directly related to the process-dependent virtual amplitude through a universal factorization formula that depends on a process-independent subtraction operator. The results that were documented in Ref. [3] determine the structure of the subtraction operator (and, thus, of the hard-virtual function) up to a universal perturbative function with purely numerical perturbative coefficients that were explicitly computed up to the second-order in α_S. In this paper we have pointed out that the recent computation of the soft-virtual corrections to Higgs boson production at N^{3}LO [1] is sufficient to extend those results to the third-order in α_S, and we have explicitly computed the corresponding perturbative coefficient. The results presented in this paper can be used to perform soft-gluon resummation up to N^{3}LL accuracy[5] for the production of an arbitrary colorless system F in hadron collisions. Equivalently, they allow us to determine the explicit form of the N^{3}LO corrections to the production cross section near partonic threshold, once the corresponding three-loop scattering amplitude $\mathcal{M}_{c\bar{c}\to F}$ is available. We have applied our results to the DY process and we have presented the explicit expression of the hard-virtual function up to N^{3}LO, confirming the result of Ref. [2] for the DY cross section at N^{3}LO.

[5] A quantitative study of Higgs boson production at N^{3}LL accuracy, with the inclusion of the soft-virtual contribution at N^{3}LO, is presented in a very recent paper [44].

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