ROBUST LOCALIZATION OF THE BEST ERROR WITH FINITE ELEMENTS IN THE REACTION-DIFFUSION NORM

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Abstract. We consider the approximation in the reaction-diffusion norm with continuous finite elements and prove that the best error is equivalent to a sum of the local best errors on pairs of elements. The equivalence constants do not depend on the ratio of diffusion to reaction. We illustrate the usefulness of this result with two applications. First, we discuss robustness and locking properties of continuous finite elements with respect to the reaction-diffusion norm. Second, we derive local error functionals that ensure robust performance of adaptive tree approximation in the reaction-diffusion norm.

1. Introduction

Finite element methods are well-established for the numerical solution of elliptic and parabolic problems. An important aspect in their mathematical understanding and foundation are the approximation properties of finite elements spaces. In view of adaptive mesh refinement, the local features of the latter under minimal regularity assumptions are of interest.

The most basic finite element approach to the homogeneous Dirichlet problem for Poisson's equation leads to the following approximation problem: Approximate a function $u \in H^1_0(\Omega)$ in the H^1-seminorm with functions from a space S consisting of continuous piecewise polynomials of degree $\leq \ell$ associated with a given simplicial mesh $\mathcal{T}$. In this context one of the authors [23] proved that

$$\inf_{v\in S} \|\nabla(u-v)\|_\Omega \approx \left(\sum_{K\in\mathcal{T}} \inf_{P\in\mathbb{P}_\ell(K)} \|\nabla(u-P)\|_K^2 \right)^{1/2}, \tag{1.1}$$

i.e. the global best error is equivalent to the ℓ_2-norm of the local best errors on elements. Notice that the right-hand side does not involve any coupling between elements and that no additional regularity of u is invoked. Thus, the (broken) best error with continuous and possibly discontinuous piecewise polynomial functions is about the same. If u disposes of additional piecewise regularity, this result and the Bramble-Hilbert Lemma readily imply error bounds. Moreover, it shows that adaptive tree approximation [8] by P. Binev and R. DeVore with the local best errors as error functionals yields near best meshes for the global best error on the left-hand side.

In view of problems with extreme parameters, it is important that approximation properties are robust. An important and basic example for such a problem is given by reaction-dominated diffusion, whose stationary variant is also of interest in the discretization of the heat equation. In this context the H^1-seminorm in (1.1) is replaced by the so-called reaction-diffusion norm

$$|||\cdot|||^2 := \|\cdot\|^2 + \varepsilon \|\nabla\cdot\|^2, \qquad \text{with } \varepsilon > 0, \tag{1.2}$$

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and one is interested in a variant of (1.1) where the hidden constants are independent of the parameter ε. For $\varepsilon \searrow 0$, the reaction-diffusion norm (1.2) becomes the L^2-norm, while for $\varepsilon \nearrow \infty$, it formally corresponds to the H^1-seminorm appearing in (1.1). The exact counterpart of (1.1) for the reaction-diffusion norm cannot be robust; this arises from the fact that, for $\varepsilon = 0$, a discontinuous piecewise constant function yields 0 for the sum of the local best errors, but not in general for the global best error. In other words, the global best error with discontinuous piecewise polynomial functions may be much smaller than the one with continuous ones.

In this article we establish robust alternatives for (1.1) with the reaction-diffusion norm. Our departure point is to replace the local spaces $\mathbb{P}_\ell(K) = S|_K$ by the spaces $S|_{\omega_\mathcal{T}(E)}$ of continuous piecewise polynomials on pairs $\omega_\mathcal{T}(E)$ of elements sharing an internal face $E \in \mathcal{E}_\Omega$. The derived results imply the robust relationships

$$\begin{aligned} \inf_{v\in S} |||u-v|||_\Omega &\approx \left(\sum_{E\in\mathcal{E}_\Omega} \inf_{V\in S|_{\omega_\mathcal{T}(E)}} |||u-V|||^2_{\omega_\mathcal{T}(E)} \right)^{1/2} \\ &\lesssim \left(\sum_{K\in\mathcal{T}} \inf_{P\in\mathbb{P}_\ell(K)} \left[|||u-P|||^2_K + \frac{|K|}{|\partial K|} \|u-P\|^2_{\partial K} \right] \right)^{1/2}, \end{aligned} \tag{1.3}$$

as well as the non-obvious parts of the following comments:

- The first sum provides a robust localization in terms of best errors that, in contrast to (1.1), take the coupling between elements into account.
- The second sum involves best errors that are localized to single elements as in (1.1) but augmented by a trace term. For $\varepsilon = 0$, it is a robust localization of the norm that I. Babuška et al. [1, 2] use to show a close-to-L^2 quasi-optimality of the corresponding Galerkin method.
- The non-robustness of the continuous elements in (1.1) with the reaction-diffusion norm is closely related to the trace augmentation. The augmentation cannot be a robust lower bound of the global best error in the reaction-diffusion norm, because it requires an additional 1/2-derivative for $\varepsilon = 0$. Its overestimation and the non-robustness are limited by and may actually reach $\varepsilon^{-1/4}$, but they are restricted to low regularity.

As the equivalence (1.1), the relationships in (1.3) have several applications. In what follows, we address in particular:

- *Error bounds with piecewise regularity.* Estimating the second sum by means of the Bramble-Hilbert lemma on single elements leads to error bounds that do not invoke regularity across interelement faces. The latter has the advantage that the bounds are also sharp if the target function is close to a discrete function.
- *L^2-error of DG methods.* We determine the critical penalization of the interelement jumps in DG methods that annihilates the aforementioned advantage of possibly discontinuous discrete functions.
- *Adaptive tree approximation.* Adopting the idea of minimal rings in P. Binev et al. [7] to pairs, we modify the first sum so that a regrouping of its terms defines local error functionals which are still relatively easy to implement and weakly subadditive on conforming leaves. Consequently, tree approximation as in P. Binev [6] can be used to construct robust near best approximations in the reaction-diffusion norm. This can be applied to create non-asymptotic benchmarks for the corresponding adaptive Galerkin method and for coarsening. The latter may be useful in the solution of parabolic problems by implicit Euler-Galerkin methods or in the solution of elliptic semilinear problems by means of adaptively discretized infinite-dimensional Newton iterations, similar to those in A. Cohen et al. [11].

The article is organized as follows. In §2 we show that the hidden constant of the nontrivial inequality of (1.1) for the reaction-diffusion norm blows up for $\varepsilon \searrow 0$. In §3 we fix notations, while in §4 we show that localization results like the first part of (1.3) follow from a suitable property of a quasi-interpolation operator. This is exploited, in §5 and §8 respectively, to prove the first part of (1.3) and its counterpart for minimal pairs. Section §6 analyzes the non-robustness of §2 more precisely, thereby deriving alternative ways to compute the local best errors in the first part of (1.3) and verifying its second part. Finally, we extend in §7 our results to conforming approximation of functions with vanishing boundary values.

Throughout the article we indicate the L^2-norm with respect to the d-dimensional Lebesgue measure restricted to a measurable set $\Omega \subset \mathbb{R}^d$ simply with $\|\cdot\|_\Omega$. With a slight abuse of notation, the same symbol is used also for L^2-norms with respect to restrictions of the $(d-1)$-dimensional Hausdorff measure. Moreover, we shall write $|\cdot|_{k;\Omega}$ for the highest order seminorm of the Sobolev space $H^k(\Omega)$ over an open set $\Omega \subset \mathbb{R}^d$.

2. Decoupling of elements is not robust

The purpose of this section is to show that, for the reaction-diffusion norm $|||\cdot|||$, the '$\lesssim$'-part in (1.1) cannot hold with a constant independent of ε. The counterexample provides functions $u_\varepsilon \in H^1(\Omega)$ converging to a discontinuous function $u_0 \notin H^1(\Omega)$ such that the global best error is bounded from below independently of ε, while the local best errors decrease with $\sqrt[4]{\varepsilon}$.

We consider the one-dimensional domain $\Omega = (-2,2) \subset \mathbb{R}$ and assume that $\mathcal{T}$ is a partition of Ω into intervals such that 0 is a breakpoint. Let S be the space of continuous functions that are piecewise polynomial of degree at most ℓ with respect to $\mathcal{T}$. We denote by $\mathcal{P}_\mathcal{T}$ the L^2-projection onto S, by $\mathcal{R}^\varepsilon_\mathcal{T}$ the Ritz projection onto S with respect to the reaction-diffusion norm and by $\mathcal{R}^\varepsilon_K$ the local counterpart of $\mathcal{R}^\varepsilon_\mathcal{T}$. The functions $u_\varepsilon \in H^1(\Omega)$ and $u_0 \in L^2(\Omega)$ are given by

$$u_\varepsilon(x) = \begin{cases} -1 & \text{for } x < -\sqrt{\varepsilon}, \\ \frac{x}{\sqrt{\varepsilon}} & \text{for } -\sqrt{\varepsilon} < x < \sqrt{\varepsilon}, \\ 1 & \text{for } \sqrt{\varepsilon} < x, \end{cases} \qquad u_0(x) = \begin{cases} -1 & \text{for } x < 0, \\ 1 & \text{for } 0 < x. \end{cases}$$

On one hand, for the local best errors, the fact that $u_{0|K} \in \mathbb{P}_\ell(K)$ for every $K \in \mathcal{T}$ implies

$$\sum_{K\in\mathcal{T}} |||u_\varepsilon - \mathcal{R}^\varepsilon_K u_\varepsilon|||^2_K \le \sum_{K\in\mathcal{T}} |||u_\varepsilon - u_0|||^2_K = \int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} \varepsilon\,|u_\varepsilon'|^2 + |u_\varepsilon - u_0|^2 = \frac{8}{3}\sqrt{\varepsilon}.$$

On the other hand, for the global best error, it holds

$$\begin{aligned} |||u_\varepsilon - \mathcal{R}^\varepsilon_\mathcal{T} u_\varepsilon|||_\Omega &\ge \|u_\varepsilon - \mathcal{R}^\varepsilon_\mathcal{T} u_\varepsilon\|_\Omega \\ &\ge \|u_\varepsilon - \mathcal{P}_\mathcal{T} u_\varepsilon\|_\Omega \to \|u_0 - \mathcal{P}_\mathcal{T} u_0\|_\Omega > 0 \qquad \text{for } \varepsilon \searrow 0, \end{aligned}$$

since u_ε converges to u_0 in the L^2-norm, the L^2-projection onto S is continuous and $u_0 \notin S$. Consequently, the constant in the nontrivial inequality of (1.1) for the reaction-diffusion norm has to grow at least with $\varepsilon^{-1/4}$. In §6 below, we shall see that this rate is the worst possible one.

This simple example reflects a more general situation. Consider in fact any conforming simplicial triangulation in $\mathbb{R}^d$ with at least one $(d-1)$-dimensional face belonging to two simplices and fix an element $K \in \mathcal{T}$ containing such an interelement face. Taking $u_\varepsilon(x) = \min\{1, \varepsilon^{-1/2}\mathrm{dist}(x, \mathbb{R}^d \setminus K)\}$, $x \in \Omega$, and $u_0 = \chi_K$ and reasoning as above shows that the aforementioned constant blows up again with $\varepsilon^{-1/4}$ as $\varepsilon \searrow 0$.

These observations suggest to modify (1.1) for the reaction-diffusion norm by invoking local best errors that take into account the continuity constraint across faces.

3. Meshes, coverings, and basis functions

We denote by $\mathcal{T}$ a conforming simplicial mesh of a polyhedral domain $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, by $\mathcal{E}(\mathcal{T})$ the set of its $(d-1)$-dimensional faces, and by $\mathcal{E}_\Omega(\mathcal{T})$ the subset of $\mathcal{E}$ of those faces which are not contained in the boundary $\partial\Omega$. Except for §8, the mesh $\mathcal{T}$ will be fixed and then we write simply $\mathcal{E}$ or $\mathcal{E}_\Omega$. If $K \in \mathcal{T}$ is an element and $E \in \mathcal{E}$ is a face, we write $|K|$ and $|E|$ for its d-dimensional Lebesgue and $(d-1)$-dimensional Hausdorff measure, respectively. For every face $E \in \mathcal{E}$, the set

$$\omega_{\mathcal{T}}(E) := \bigcup \{K \in \mathcal{T} : \partial K \supseteq E\}$$

is the union of elements sharing the face E. It consists of two elements if $E \in \mathcal{E}_\Omega$ and of one element otherwise. We stress that E belongs to various meshes and that $\omega_{\mathcal{T}}(E)$ actually depends on $\mathcal{T}$ too. Moreover, we denote the set of nodes of S by

$$\mathcal{N} := \bigcup_{K\in\mathcal{T}} \mathcal{N}_K \quad \text{with} \quad \mathcal{N}_K := \left\{ \sum_{i=0}^d \frac{\alpha_i}{\ell} a_i : \alpha \in \mathbb{N}_0^{d+1}, \sum_{i=0}^d \alpha_i = \ell \right\}$$

whenever $a_0, \dots, a_d$ are the vertices of a simplex K. In accordance with the definitions of $\mathcal{N}_K$ and $\mathcal{E}_\Omega$, a subscript E, Ω, etc. to $\mathcal{N}$, $\mathcal{E}$ indicates that only those nodes, faces which are contained in the index set are considered.

A family $\mathcal{W}$ of subdomains of Ω is a *β-finite covering* of $\mathcal{T}$ if for every element $K \in \mathcal{T}$

- there exists $\omega \in \mathcal{W}$ with $\omega \supseteq K$ and
- it holds $\sum_{\omega\in\mathcal{W}} \chi_\omega \le \beta$ on $\overset{\circ}{K}$,

where χ_ω stands for the characteristic function of ω. The families $\{K\}_{K\in\mathcal{T}}$ and $\mathcal{W}_{\mathcal{T}} := \{\omega_{\mathcal{T}}(E)\}_{E\in\mathcal{E}_\Omega}$ in (1.1) and (1.3) are 1-finite and $(d+1)$-finite coverings, respectively. Notice that β arises in (1.1) and (1.3) as multiplicative constant in the straight-forward inequality. Another β-finite covering appears in §8. We shall also need 'local' coverings associated with elements. Let $\mathcal{W}$ be a β-finite covering of $\mathcal{T}$. A *family of local coverings* from $\mathcal{W}$ for $\mathcal{T}$ is given by a map that associates a subset $\mathcal{A}_K$ of $\mathcal{W}$ with each element $K \in \mathcal{T}$ such that

- for every $K \in \mathcal{T}$, there is a subdomain $\omega \in \mathcal{A}_K$ such that $\omega \supseteq K$,
- the cardinality of the subsets $\mathcal{A}_K$ is bounded independently of $K \in \mathcal{T}$,
- the number of elements with $\mathcal{A}_K \ni \omega$ is bounded independently of $\omega \in \mathcal{W}$.

The numbers

$$\alpha_1 := \max_{K\in\mathcal{T}} \#\mathcal{A}_K \quad \text{and} \quad \alpha_2 := \max_{\omega\in\mathcal{W}} \#\{K \in \mathcal{T} : \mathcal{A}_K \ni \omega\}$$

will be called, respectively, the maximal cardinality and the overlapping index of $\{\mathcal{A}_K\}_{K\in\mathcal{T}}$. If $\mathcal{W} = \{K\}_{K\in\mathcal{T}}$, then, for each element $K \in \mathcal{T}$, the subdomains $\mathcal{A}_K = \{K' \in \mathcal{T} : K \cap K' \neq \emptyset\}$ cover the patch around K. This family appears in the proof of (1.1).

The space

$$S := S^{\ell,0}(\mathcal{T}) = \{v \in C^0(\Omega) : \ \forall\, K \in \mathcal{T},\, v \in \mathbb{P}_\ell(K)\}$$

consists of all continuous functions that are piecewise polynomial over $\mathcal{T}$. Given a set $\omega \subset \Omega$, we indicate its restriction with

$$S|_\omega := \{v \in C^0(\omega) : \exists \tilde{v} \in S,\ \tilde{v}|_\omega = v\}.$$

In particular, for any element $K \in \mathcal{T}$, it holds $S|_K = \mathbb{P}_\ell(K)$ and, for any $E \in \mathcal{E}_\Omega$, we have $S|_{\omega_{\mathcal{T}}(E)} = \{v \in C^0(\omega_{\mathcal{T}}(E)) : \forall K \in \mathcal{T} \text{ with } K \subseteq \omega_{\mathcal{T}}(E),\, v \in \mathbb{P}_\ell(K)\}$.

We denote by $\{\phi_z\}_{z\in\mathcal{N}}$ the nodal basis, for which each ϕ_z is uniquely determined by

$$\phi_z \in S \qquad \text{and} \qquad \forall y \in \mathcal{N} \quad \phi_z(y) = \delta_{yz}.$$

Given an element $K \in \mathcal{T}$, the $L^2(K)$-dual basis functions $\{\psi_z^K\}_{z\in\mathcal{N}_K}$ are such that, for every $z \in \mathcal{N}_K$, it holds

$$\psi_z^K \in \mathbb{P}_\ell(K) \qquad \text{and} \qquad \forall y \in \mathcal{N}_K \quad \int_K \psi_z^K \phi_y = \delta_{zy}.$$

We thus have, for every $p \in \mathbb{P}_\ell(K)$ and for every $z \in \mathcal{N}_K$,

$$p(z) = \int_K p\psi_z^K. \tag{3.1}$$

We also recall some basic scaling properties of different norms of ϕ_z and ψ_z^K. We denote by $\hat{K} := \mathrm{conv}\{0, e_1, \dots, e_d\}$ the reference d-simplex, by $\hat{h} := \mathrm{diam}(\hat{K})$ the diameter of $\hat{K}$, by $\{\hat{\phi}_{\hat{z}}\}$, $\{\hat{\psi}_{\hat{z}}\}$ respectively the basis and dual basis functions on $\hat{K}$. For every element $K \in \mathcal{T}$, there exists an affine transformation $F : \mathbb{R}^d \to \mathbb{R}^d$ with $F(\hat{K}) = K$. Given also $z \in \mathcal{N}_K$, there are different choices for $\hat{z} = F^{-1}(z)$. While $\|\hat{\phi}_{\hat{z}}\|_{\hat{K}}$ is independent of this choice, $\|\nabla\hat{\phi}_{\hat{z}}\|_{\hat{K}}$ does depend on it. In order to have a unique value for the latter, we take F such that $\hat{z}$ has a minimal sum of its coordinates. Since $\hat{\psi}_{\hat{z}} = (\det B)\psi_z^K \circ F$, where B is the non-singular matrix associated to F, the transformation rule and the proof of [9, Theorems 15.1 and 15.2] imply

$$\|\phi_z\|_K = \frac{|K|^{1/2}}{|\hat{K}|^{1/2}}\|\hat{\phi}_{\hat{z}}\|_{\hat{K}}, \qquad \left\|\psi_z^K\right\|_K = \frac{|\hat{K}|^{1/2}}{|K|^{1/2}}\|\hat{\psi}_{\hat{z}}\|_{\hat{K}}, \tag{3.2}$$

$$\|\nabla\phi_z\|_K \le \frac{\hat{h}|K|^{1/2}}{\rho_K|\hat{K}|^{1/2}}\|\nabla\hat{\phi}_{\hat{z}}\|_{\hat{K}}, \tag{3.3}$$

where ρ_K denotes the maximum diameter of a ball inscribed in K. These inequalities are related to the following quantities, which describe properties of the mesh entering our results. Denoting by $\widetilde{\omega}_{\mathcal{T}}(K)$ the set of the elements $\widetilde{K} \in \mathcal{T}$ touching an element $K \in \mathcal{T}$, we define

$$\mu_{\mathcal{T}} := \max_{K\in\mathcal{T}} \max_{\widetilde{K}\in\widetilde{\omega}_{\mathcal{T}}(K)} \frac{|K|^{1/2}}{|\widetilde{K}|^{1/2}} \quad \text{and} \quad \sigma_{\mathcal{T}} := \max_{K\in\mathcal{T}} \max_{\widetilde{K}\in\widetilde{\omega}_{\mathcal{T}}(K)} \frac{h_{\widetilde{K}}}{\rho_K}. \tag{3.4}$$

The quantity $\mu_{\mathcal{T}}$ measures the local quasi-uniformity of $\mathcal{T}$ in terms of element volumes, while $\sigma_{\mathcal{T}}$ is related to the element shapes and the local quasi-uniformity in terms of the element diameters: in fact, it is bounded by $\max_{K\in\mathcal{T}} h_K/\rho_K$, which is small for isotropic meshes, and a counterpart of $\mu_{\mathcal{T}}$ for element diameters.

4. Localization and interpolation

In this section we reduce the problem of localizing the global best error to the problem of defining a global quasi-interpolation operator that is locally near best. Roughly speaking, the latter means that the difference between the interpolant and a local best approximation is bounded, up to a constant, by a finite sum of local best errors. The results of this section are used in §5 and §8.

Let $|||\cdot|||_\Omega$ be a norm on $H^1(\Omega)$ such that its square is set-additive. Then it holds $|||\cdot|||_\Omega^2 = \sum_{K\in\mathcal{T}} |||\cdot|||_K^2$. Moreover let $\mathcal{W}$ be a β-finite covering such that $|||\cdot|||_\omega$ is well-defined for all $\omega \in \mathcal{W}$. For every subdomain $\omega \in \mathcal{W}$, let $\mathcal{Q}_\omega : H^1(\omega) \to S|_\omega$ be the local operator which maps a function to its corresponding best approximation in $S|_\omega$ with respect to $|||\cdot|||_\omega$. We thus have, for all $\omega \in \mathcal{W}$ and every $u \in H^1(\omega)$,

$$|||u - \mathcal{Q}_\omega u|||_\omega = \inf_{V\in S|_\omega} |||u - V|||_\omega. \tag{4.1}$$

Definition 4.1 (Local near best interpolation)**.** An interpolation operator Π is near best with respect to $\mathcal{W} \ni \omega \mapsto |||\cdot|||_\omega$, if there exist a family $\{\mathcal{A}_K\}_{K\in\mathcal{T}}$ of local coverings from $\mathcal{W}$ and a constant $C_{\rm int} \geq 0$ such that, for every $K \in \mathcal{T}$ and $z \in \mathcal{N}_K$,

$$\sum_{z\in\mathcal{N}_K} |\mathcal{Q}_\omega u(z) - \Pi u(z)|\, |||\phi_z|||_K \leq C_{\rm int} \sum_{\omega'\in\mathcal{A}_K} |||u - \mathcal{Q}_{\omega'} u|||_{\omega'}, \tag{4.2}$$

with some $\omega \in \mathcal{A}_K$ such that $\omega \supseteq K$.

Local near best interpolation entails global near best interpolation. This reveals the proof of the following proposition.

Proposition 4.2 (Localization by interpolation)**.** *Let $\mathcal{W}$ be a β-finite covering of $\mathcal{T}$. If there exists an interpolation operator into S that is near best with respect to $\mathcal{W} \ni \omega \mapsto |||\cdot|||_\omega$, then, for every $u \in H^1(\Omega)$,*

$$\inf_{v\in S} |||u - v|||_\Omega \leq C_{\rm loc} \left(\sum_{\omega\in\mathcal{W}} \inf_{V\in S|_\omega} |||u - V|||_\omega^2 \right)^{1/2} \leq \beta C_{\rm loc} \inf_{v\in S} |||u - v|||_\Omega \tag{4.3}$$

where the constant $C_{\rm loc} = \sqrt{\alpha_1\alpha_2}(1 + C_{\rm int})$ of localization is determined by the quantities appearing in Definition 4.1; α_1 and α_2 are, respectively, the maximal cardinality and the overlapping index of the family of local coverings therein.

Proof. Let $\Pi : H^1(\Omega) \to S$ be an interpolation operator that is locally near best with respect to $\mathcal{W} \ni \omega \mapsto |||\cdot|||_\omega$. Bounding the infimum on the left-hand side of (4.3) by $|||u - \Pi u|||_\Omega$ and writing the norm as a sum over elements results in

$$\inf_{v\in S} |||u - v|||_\Omega \leq |||u - \Pi u|||_\Omega = \left(\sum_{K\in\mathcal{T}} |||u - \Pi u|||_K^2 \right)^{1/2}.$$

Fix an element $K \in \mathcal{T}$ and choose a subdomain $\omega \supseteq K$ as in Definition 4.1. The triangle inequality then yields

$$|||u - \Pi u|||_K \leq |||u - \mathcal{Q}_\omega u|||_K + |||\Pi u - \mathcal{Q}_\omega u|||_K.$$

Since both $\Pi u|_K$ and $\mathcal{Q}_\omega u|_K$ are in $\mathbb{P}_\ell(K)$, we can represent them in terms of the local nodal basis $\{\phi_z\}_{z\in\mathcal{N}_K}$ and obtain

$$|||\Pi u - \mathcal{Q}_\omega u|||_K \leq \sum_{z\in\mathcal{N}_K} |\mathcal{Q}_\omega u(z) - \Pi u(z)|\, |||\phi_z|||_K.$$

Using (4.2) and inserting back up to the first inequality, we get

$$\inf_{v\in S} |||u - v|||_\Omega \leq \left(\sum_{K\in\mathcal{T}} \left[(1 + C_{\rm int}) \sum_{\omega'\in\mathcal{A}_K} |||u - \mathcal{Q}_{\omega'} u|||_{\omega'} \right]^2 \right)^{1/2}.$$

As $\#\mathcal{A}_K \leq \alpha_1$ we can use $\left(\sum_{i=1}^n a_i\right)^2 \leq n \sum_{i=1}^n a_i^2$ with $n \leq \alpha_1$. Moreover every $\omega \in \mathcal{W}$ belongs to at most α_2 of the sets A_K and so, upon rearranging terms, we arrive at

$$\inf_{v\in S} |||u - v|||_\Omega \leq \sqrt{\alpha_1\alpha_2}(1 + C_{\rm int}) \left(\sum_{\omega\in\mathcal{W}} |||u - \mathcal{Q}_\omega u|||_\omega^2 \right)^{1/2}.$$

In view of (4.1), this proves the first inequality in (4.3). To verify the second one, take $v \in S$, observe $|||u - \mathcal{Q}_\omega u|||_\omega \leq |||u - v|||_\omega$ for any subdomain $\omega \in \mathcal{W}$ and recall that every element $K \in \mathcal{T}$ appears in at most β subdomains $\omega \in \mathcal{W}$. □

Our task is now reduced to find an operator Π that is locally near best. Since point values are in general not defined for the norms of our interest, the definition of Π below invokes local average processes. More precisely, for nodes belonging to several elements, we shall use dual basis functions $\psi_z^{\widetilde{K}}$, where $\widetilde{K}$ is fixed for each such node z. This entails that $\Pi u|_K$ on some element $K \in \mathcal{T}$ may depend also on $u|_{\widetilde{K}}$ on certain other elements $\widetilde{K}$. To deal with this dislocation in a locally near best manner, we invoke suitable paths of overlapping subdomains from $\mathcal{W}$, which will lead to the definitions of the sets $\mathcal{A}_K$. The following proposition applies also to the covering in §8, whose subdomains are in general not unions of elements of the mesh $\mathcal{T}$.

Proposition 4.3 (Dislocation control)**.** *Let $\mathcal{W}$ be a β-finite covering of $\mathcal{T}$ and $K, \widetilde{K} \in \mathcal{T}$ two elements sharing a node $z \in \mathcal{N}$. If there exist a finite sequence $\{\omega_j\}_{j=1}^n \subset \mathcal{W}$ and $\nu \in (0,1]$ such that*

- *$\omega_1 \supseteq K$ and $\omega_n \supseteq \widetilde{K}$,*
- *any intersection $\omega_j \cap \omega_{j+1}$ is a simplex T_j containing z and there is an element $K_j \in \mathcal{T}$ which again contains z and satisfies $|T_j| \geq \nu |K_j|$,*

then

$$\left| \mathcal{Q}_{\omega_1} u(z) - \int_{\widetilde{K}} u \psi_z^{\widetilde{K}} \right| \leq 2\nu^{-1/2} \|\hat{\psi}_{\hat{z}}\|_{\hat{K}} \sum_{j=1}^{n} \frac{|\hat{K}|^{1/2}}{|K_j|^{1/2}} \left\| \mathcal{Q}_{\omega_j} u - u \right\|_{L^2(\omega_j)} .$$

with $K_n = \widetilde{K}$.

Comparing with Proposition 4.2, we notice that the bound involves not best errors but L^2-errors of best approximations.

Proof. For every $j = 2, \dots, n$, we add and subtract $\mathcal{Q}_{\omega_j} u(z)$, which is well-defined thanks to $z \in \omega_j$, and use the triangle inequality to get

$$\left| \mathcal{Q}_{\omega_1} u(z) - \int_{\widetilde{K}} u \psi_z^{\widetilde{K}} \right| \leq \left| \mathcal{Q}_{\omega_n} u(z) - \int_{\widetilde{K}} u \psi_z^{\widetilde{K}} \right| + \sum_{j=1}^{n-1} \left| \mathcal{Q}_{\omega_j} u(z) - \mathcal{Q}_{\omega_{j+1}} u(z) \right| .$$

We bound the terms on the right-hand side separately. Exploiting property (3.1) and the Cauchy-Schwarz inequality, we obtain

$$\left| \mathcal{Q}_{\omega_n} u(z) - \int_{\widetilde{K}} u \psi_z^{\widetilde{K}} \right| = \left| \int_{\widetilde{K}} (\mathcal{Q}_{\omega_n} u - u) \psi_z^{\widetilde{K}} \right| \leq \left\| \mathcal{Q}_{\omega_n} u - u \right\|_{L^2(\widetilde{K})} \left\| \psi_z^{\widetilde{K}} \right\|_{L^2(\widetilde{K})} ,$$

and similarly, for every $j = 1, \dots, n-1$,

$$\begin{aligned} \left| \mathcal{Q}_{\omega_j} u(z) - \mathcal{Q}_{\omega_{j+1}} u(z) \right| &= \left| \int_{T_j} (\mathcal{Q}_{\omega_j} u - \mathcal{Q}_{\omega_{j+1}} u) \psi_z^{T_j} \right| \\ &\leq \left\| \mathcal{Q}_{\omega_j} u - \mathcal{Q}_{\omega_{j+1}} u \right\|_{L^2(T_j)} \left\| \psi_z^{T_j} \right\|_{L^2(T_j)} \\ &\leq \left(\left\| \mathcal{Q}_{\omega_j} u - u \right\|_{L^2(T_j)} + \left\| \mathcal{Q}_{\omega_{j+1}} u - u \right\|_{L^2(T_j)} \right) \left\| \psi_z^{T_j} \right\|_{L^2(T_j)} . \end{aligned}$$

When summing the last inequality over j, the L^2-norm of $\mathcal{Q}_{\omega_j} u - u$ appears on both T_j and T_{j-1}. We bound both contributions by the L^2-norm on ω_j and combine this with the scaling property (3.2) of ψ_z and $|T_j| \geq \nu |K_j|$. Finally, for simplification, we incorporate the term with $\widetilde{K}$ into the sum and obtain the claimed inequality. □

For the covering $\mathcal{W}_\mathcal{T} = \{\omega_\mathcal{T}(E)\}_{E \in \mathcal{E}_\Omega}$ appearing in (1.3), the existence of the path in Proposition 4.3 follows from the following property of the mesh $\mathcal{T}$.

Definition 4.4 (Face-connectedness)**.** A conforming simplicial mesh $\mathcal{T}$ is face-connected, if for every element pair K, $\widetilde{K} \in \mathcal{T}$ sharing a node $z \in \mathcal{N}$, there exists a pairwise disjoint finite sequence $\{K_j\}_{j=1}^n \subset \mathcal{T}$ such that

- $K_1 = K$ and $K_n = \widetilde{K}$,
- each intersection $K_j \cap K_{j+1} \in \mathcal{E}_\Omega$ is an interelement face containing z.

The length n of the path is bounded in terms of

$$\bar{n} := \max_{z \in \mathcal{N}} \#\{K \in \mathcal{T} : K \ni z\}. \tag{4.4}$$

Notice that, for a fixed node $z \in \mathcal{N}$, there exists a finite sequence as in Definition 4.4 whenever the interior of the support of ϕ_z is connected. The latter is verified if z is interior to Ω or if z lies on a Lipschitz boundary. Non-Lipschitz boundaries are more subtle: it holds, e.g., on the boundary of the slit domain $\Omega = \{x \in \mathbb{R}^2 : |x| \leq 1\} \setminus ([0,1] \times \{0\})$, but not in the origin in case of $\Omega = \{x \in \mathbb{R}^2 : |x_1| < |x_2|\}$; see also [23] for further discussion.

Essentially, Propositions 4.2 and 4.3 cover both (1.1) and (1.3); in fact, [23] uses a variant of Proposition 4.3, where the intersection of subdomains are faces. The following definition specifies the property of the covering $\mathcal{W}$ that is crucial for the robustness in the first part of (1.3).

Definition 4.5 (Internal face covering)**.** A β-finite covering $\mathcal{W}$ covers interelement faces internally if for every interelement face $E \in \mathcal{E}_\Omega$, there exists $\omega \in \mathcal{W}$ such that its interior almost contains E, i.e. $E \subset \omega$ and $|E \cap \mathring{\omega}| = |E|$.

While $\mathcal{W}_\mathcal{T}$ covers interelement faces internally, $\mathcal{T}$ does not. As a consequence, the local best errors associated with $\mathcal{W}_\mathcal{T}$ take into account the continuity constraint across interelement faces, a feature that is crucial for robustness by the observations in §2.

5. Robust localization to pairs of elements

The purpose of this section is to prove the first part of (1.3). The reaction-diffusion norm (1.2) has the L^2-norm and the H^1-seminorm as limiting cases for $\varepsilon \searrow 0$ and $\varepsilon \nearrow \infty$, respectively. We consider these cases first in a unified manner by applying Proposition 4.2 with the same covering; the associated interpolation operators differ only in the involved local best approximations.

Throughout this section the mesh $\mathcal{T}$ is face-connected and, in view of §2, we choose the covering $\mathcal{W}_\mathcal{T} = \{\omega_\mathcal{T}(E)\}_{E \in \mathcal{E}_\Omega}$ and start with the L^2-norm, which appears the more critical limiting case.

5.1. Pure reaction norm.

We first introduce an interpolation operator and then show that it is locally near best with respect to $|||\cdot|||_\omega = \|\cdot\|_{L^2(\omega)}$, $\omega \in \mathcal{W}_\mathcal{T}$. For simplicity, we write $\mathcal{P}_E u$ for the best approximation to u in $S|_{\omega_\mathcal{T}(E)}$ with respect to $\|\cdot\|_{\omega_\mathcal{T}(E)} = \|\cdot\|_{L^2(\omega_\mathcal{T}(E))}$.

The definition of the interpolation operator relies on a classification of the nodes. For nodes that are interior to an element, we define the corresponding nodal value of Π with the help of a best approximation. For the other nodes that belong to several elements or are on the boundary $\partial\Omega$, we use the averaging technique of L. R. Scott and S. Zhang [21]. More precisely,

- for every element $K \in \mathcal{T}$, we fix a face $E = E_K \in \mathcal{E}_\Omega$ such that $E_K \subset \partial K$ and
- for every $z \in \mathcal{N} \cap \Sigma$ with $\Sigma := \cup_{K \in \mathcal{T}} \partial K$, we fix an element $K_z \in \mathcal{T}$ such that $z \in \partial K_z$.

Given $u \in H^1(\Omega)$, we then set

$$\Pi^0 u := \sum_{z\in\mathcal{N}} u_z \phi_z \quad \text{with} \quad u_z = \begin{cases} \mathcal{P}_{E_K} u(z) & \text{if } z \in \mathcal{N}_K^\circ \text{ with } K \in \mathcal{T}, \\ \displaystyle\int_{K_z} u \psi_z^{K_z} & \text{if } z \in \mathcal{N} \cap \Sigma. \end{cases} \tag{5.1}$$

Notice that in general $\Pi^0 u|_K$ depends not only on $u|_K$ but also on $u|_{\widetilde{K}}$ for neighboring elements $\widetilde{K}$.

In order to verify that Π^0 is locally near best, we fix an element $K \in \mathcal{T}$, write $E := E_K$ for short and choose $\omega = \omega_{\mathcal{T}}(E)$ in (4.2). This is an admissible choice for ω since $\omega_{\mathcal{T}}(E) \supseteq K$. It is also a natural choice because in this way, for every $z \in \mathcal{N}_K^\circ$, we have

$$|\mathcal{P}_E u(z) - \Pi^0 u(z)| = 0. \tag{5.2}$$

Otherwise, if $z \in \mathcal{N}_K \cap \Sigma$, we exploit Proposition 4.3. Since $\mathcal{T}$ is face-connected, there exists a finite sequence of faces $\{E_j\}_{j=1}^n$ such that the corresponding sequence $\{\omega_j\}_{j=1}^n := \{\omega_{\mathcal{T}}(E_j)\}_{j=1}^n \subset \mathcal{W}_{\mathcal{T}}$ satisfies

- $\omega_1 = \omega_{\mathcal{T}}(E) \supseteq K$ and $\omega_n \supseteq K_z$,
- each intersection $K_j := \omega_j \cap \omega_{j+1}$ is an element of $\mathcal{T}$ containing z.

We can therefore apply Proposition 4.3 with $\nu = 1$ and get

$$\left|\mathcal{P}_E u(z) - \Pi^0 u(z)\right| \le 2\|\hat{\psi}_{\hat{z}}\|_{\hat{K}} \sum_{j=1}^n \frac{|\hat{K}|^{1/2}}{|K_j|^{1/2}} \left\|\mathcal{P}_{E_j} u - u\right\|_{\omega_j} \tag{5.3}$$

with $K_n := K_z$. Combining (5.2) and (5.3) with the scaling property (3.2), we obtain

$$\begin{aligned} \sum_{z\in\mathcal{N}_K} \left|\mathcal{P}_E u(z) - \Pi^0 u(z)\right| \|\phi_z\|_K &\le 2 \sum_{z\in\mathcal{N}_{\partial K}} \|\hat{\psi}_{\hat{z}}\|_{\hat{K}} \|\hat{\phi}_{\hat{z}}\|_{\hat{K}} \sum_{j=1}^n \frac{|K|^{1/2}}{|K_j|^{1/2}} \left\|\mathcal{P}_{E_j} u - u\right\|_{\omega_j} \\ &\le M_0 \max_{\widetilde{K}\in\widetilde{\omega}_{\mathcal{T}}(K)} \frac{|K|^{1/2}}{|\widetilde{K}|^{1/2}} \sum_{\widetilde{E}\in\gamma_{\mathcal{T}}(K)} \left\|\mathcal{P}_{\widetilde{E}} u - u\right\|_{\omega_{\mathcal{T}}(\widetilde{E})}, \end{aligned}$$

where

$$M_0 = M_0(\ell, d) := 2 \textstyle\sum_{\hat{z}\in\partial\hat{K}} \|\hat{\psi}_{\hat{z}}\|_{\hat{K}} \|\hat{\phi}_{\hat{z}}\|_{\hat{K}},$$

the symbol $\widetilde{\omega}_{\mathcal{T}}(K)$ stands for the set of the elements $\widetilde{K} \in \mathcal{T}$ touching K, and $\gamma_{\mathcal{T}}(K) := \{E \in \mathcal{E}_\Omega : E \cap \partial K \neq \emptyset\}$ are the faces in the skeleton of $\widetilde{\omega}_{\mathcal{T}}(K)$. In order to achieve a bound that is independent of K, we recall $\mu_{\mathcal{T}}$ from (3.4) and arrive at

$$\sum_{z\in\mathcal{N}_K} |\mathcal{P}_E u(z) - \Pi^0 u(z)| \|\phi_z\|_K \le \mu_{\mathcal{T}} M_0 \sum_{\widetilde{E}\in\gamma_{\mathcal{T}}(K)} \left\|\mathcal{P}_{\widetilde{E}} u - u\right\|_{\omega_{\mathcal{T}}(\widetilde{E})}. \tag{5.4}$$

We observe that $\#\gamma_{\mathcal{T}}(K)$ is bounded in terms of d and $\bar{n}$ from (4.4). Given $\widetilde{E} \in \mathcal{E}_\Omega$, the same holds for $\#\{K \in \mathcal{T} : \gamma_{\mathcal{T}}(K) \ni \widetilde{E}\}$. We can therefore apply Proposition 4.2 and get the following theorem.

Theorem 5.1 (Localization of best L^2-error)**.** *For every $u \in L^2(\Omega)$ it holds*

$$\inf_{v\in S} \|u - v\|_\Omega \le C_{\mathrm{loc}} \left(\sum_{E\in\mathcal{E}_\Omega} \inf_{V\in S|_{\omega_{\mathcal{T}}(E)}} \|u - V\|^2_{\omega_{\mathcal{T}}(E)} \right)^{\frac{1}{2}} \le (d+1) C_{\mathrm{loc}} \inf_{v\in S} \|u - v\|_\Omega$$

where the localization constant C_{loc} depends on the dimension d, the polynomial degree ℓ, $\mu_{\mathcal{T}}$ from (3.4)*, and $\bar{n}$ from* (4.4)*.*

Notice that there is no explicit dependence on the shape regularity of $\mathcal{T}$ but a dependence on the local quasi-uniformity of $\mathcal{T}$ through $\bar{n}$ and $\mu_{\mathcal{T}}$.

5.2. **Pure diffusion seminorm.** The counterpart of Theorem 5.1 for the H^1-seminorm follows from (1.1). In this subsection we present an alternative approach relying on an interpolation operator that is very close to the one in §5.1. The obtained inequalities will turn out useful for dealing with the reaction-diffusion norm.

Since $\|\nabla\cdot\|_\omega = \|\nabla\cdot\|_{L^2(\omega)}$ is only a seminorm, best approximations in $S|_\omega$ are only unique up to a constant. This freedom allows to bound the L^2-errors appearing in Proposition 4.3 by local best errors in the H^1-seminorm with the help of the Poincaré inequality. We are thus led to the following local best approximation operators: given $u \in H^1(\Omega)$ and any $E \in \mathcal{E}_\Omega$, let $\mathcal{R}_E$ be the best approximation to u in $S|_{\omega_\mathcal{T}(E)}$ with respect to $\|\nabla\cdot\|_{\omega_\mathcal{T}(E)}$ such that

$$\int_{\omega_\mathcal{T}(E)} \mathcal{R}_E u = \int_{\omega_\mathcal{T}(E)} u. \tag{5.5}$$

The interpolation operator Π^∞ is then given by (5.1) where $\mathcal{P}_{E_K}$ is replaced by $\mathcal{R}_{E_K}$. Consequently, Π^0 and Π^∞ differ only in the involved local best approximations. Before embarking on the proof that Π^∞ is locally near best, we provide the following tailor-made Poincaré inequality.

Lemma 5.2 (Poincaré inequality for element pairs)**.** *Let ω be the union of two adjacent elements K_1, K_2 sharing a face $E = K_1 \cap K_2$. For every $v \in H^1(\omega)$ it holds*

$$\left\| v - \frac{1}{|\omega|}\int_\omega v \right\|_\omega \le C_P h_\omega \|\nabla v\|_\omega ,$$

where $C_P \le \left(\frac{1}{\pi^2} + \frac{1}{d^2}\right)^{1/2}$ and $h_\omega := \max\{\operatorname{diam}(K_1), \operatorname{diam}(K_2)\}$.

Proof. Since the mean value on ω is the best approximating constant with respect to the L^2-norm, we can substitute it with the mean value on the common face E:

$$\left\| v - \frac{1}{|\omega|}\int_\omega v \right\|_\omega^2 \le \left\| v - \frac{1}{|E|}\int_E v \right\|_\omega^2 = \sum_{i=1}^2 \left\| v - \frac{1}{|E|}\int_E v \right\|_{K_i}^2 .$$

Thanks to the trace identity from [24, Proposition 4.2], we may write

$$\frac{1}{|E|}\int_E v = \frac{1}{|K_i|}\int_{K_i} v + \frac{1}{d|K_i|}\int_{K_i} \mathbf{q}_{i,E}\cdot\nabla v, \tag{5.6}$$

where $\mathbf{q}_{i,E}(x) := x - z_i$ and z_i is the vertex of K_i opposite to E. Moreover the classical Poincaré inequality on convex domains, see [4, 19], implies

$$\begin{aligned}
&\left\| v - \frac{1}{|K_i|}\int_{K_i} v - \frac{1}{d|K_i|}\int_{K_i} \mathbf{q}_{i,E}\cdot\nabla v \right\|_{K_i}^2 \\
&\quad = \left\| v - \frac{1}{|K_i|}\int_{K_i} v \right\|_{K_i}^2 + \frac{1}{d^2|K_i|}\left(\int_{K_i} \mathbf{q}_{i,E}\cdot\nabla v\right)^2 \\
&\quad \le \operatorname{diam}(K_i)^2 \left(\frac{1}{\pi^2} + \frac{1}{d^2}\right) \|\nabla v\|_{K_i}^2 .
\end{aligned}$$

Combining (5.6) and the two inequalities yields the claim. □

In order to show that Π^∞ is locally near best with respect to $\|\nabla\cdot\|_{\omega_\mathcal{T}(E)}$, $E \in \mathcal{E}_\Omega$, we fix an element $K \in \mathcal{T}$, write $E := E_K$ for short and, as in §5.1, we choose $\omega = \omega_\mathcal{T}(E)$ in (4.2). If $z \in \mathcal{N}_K^\circ$, we have again

$$|\mathcal{R}_E u(z) - \Pi^\infty u(z)| = 0. \tag{5.7}$$

Otherwise, if $z \in \mathcal{N}_K \cap \Sigma$, we apply also Proposition 4.3 with $\nu = 1$ and obtain

$$|\mathcal{R}_E u(z) - \Pi^\infty u(z)| \leq 2\|\hat{\psi}_{\hat{z}}\|_{\hat{K}} \sum_{j=1}^{n} \frac{|\hat{K}|^{1/2}}{|K_j|^{1/2}} \left\|\mathcal{R}_{E_j} u - u\right\|_{\omega_j}$$

with $K_n = K_z$. Here we invoke the Poincaré inequality Lemma 5.2, which yields

$$|\mathcal{R}_E u(z) - \Pi^\infty u(z)| \leq 2C_P\|\hat{\psi}_{\hat{z}}\|_{\hat{K}} \sum_{j=1}^{n} h_{\omega_\mathcal{T}(E_j)} \frac{|\hat{K}|^{1/2}}{|K_j|^{1/2}} \left\|\nabla(\mathcal{R}_{E_j} u - u)\right\|_{\omega_j} \tag{5.8}$$

with $h_{\omega_\mathcal{T}(E)} = \max_{K\in\mathcal{T}, K\subseteq\omega_\mathcal{T}(E)} \operatorname{diam}(K)$. Combining (5.7) and (5.8) with (3.3) leads to

$$\begin{aligned}
&\sum_{z\in\mathcal{N}_K} |\mathcal{R}_E u(z) - \Pi^\infty u(z)| \, \|\nabla\phi_z\|_K \\
&\quad \leq 2\hat{h} \sum_{z\in\mathcal{N}_{\partial K}} \|\hat{\psi}_{\hat{z}}\|_{\hat{K}} \|\nabla\hat{\phi}_{\hat{z}}\|_{\hat{K}} C_P \sum_{j=1}^{n} \frac{h_{\omega_\mathcal{T}(E_j)}|K|^{1/2}}{\rho_K |K_j|^{1/2}} \left\|\nabla(\mathcal{R}_{E_j} u - u)\right\|_{\omega_\mathcal{T}(E_j)} \\
&\quad \leq M_\infty \max_{\widetilde{K}\in\widetilde{\omega}_\mathcal{T}(K)} \frac{|K|^{1/2}}{|\widetilde{K}|^{1/2}} \max_{\widetilde{K}\in\widetilde{\omega}_\mathcal{T}(K)} \frac{h_{\widetilde{K}}}{\rho_K} \sum_{\widetilde{E}\in\gamma_\mathcal{T}(K)} \left\|\nabla(\mathcal{R}_{\widetilde{E}} u - u)\right\|_{\omega_\mathcal{T}(\widetilde{E})},
\end{aligned}$$

where $h_{\widetilde{K}} := \operatorname{diam}(\widetilde{K})$ and

$$M_\infty = M_\infty(\ell, d) := 2\sqrt{2} C_P \textstyle\sum_{\hat{z}\in\partial\hat{K}} \|\hat{\psi}_{\hat{z}}\|_{\hat{K}} \|\nabla\hat{\phi}_{\hat{z}}\|_{\hat{K}}.$$

We thus see that the use of the Poincaré inequality compensates the scaling of $\|\nabla\phi_z\|_K$, if the shape parameter $\sigma_\mathcal{T}$ from (3.4) is moderate. We therefore have

$$\begin{aligned}
&\sum_{z\in\mathcal{N}_K} |\mathcal{R}_E u(z) - \Pi^\infty u(z)| \, \|\nabla\phi_z\|_K \\
&\qquad \leq \sigma_\mathcal{T}\mu_\mathcal{T} M_\infty \sum_{\widetilde{E}\in\gamma_\mathcal{T}(K)} \left\|\nabla(\mathcal{R}_{\widetilde{E}} u - u)\right\|_{\omega_\mathcal{T}(\widetilde{E})}
\end{aligned} \tag{5.9}$$

and can apply Proposition 4.2. Apart from the dependencies listed in Theorem 5.1, the involved constant depends in addition on $\sigma_\mathcal{T}$ in (3.4).

5.3. **Reaction-diffusion norm.** We now turn to the main result of this section: the robust localization of the best error in the reaction-diffusion norm. To this end, we follow the lines of §5.1 and §5.2, combining their results. For simplicity, we write $|||\cdot|||_\omega$ for $(\|\cdot\|_\omega^2 + \varepsilon\,\|\nabla\cdot\|_\omega^2)^{1/2}$.

Here we use the following local best approximation operators: given $u \in H^1(\Omega)$ and $E \in \mathcal{E}_\Omega$, let $\mathcal{R}_E^\varepsilon u$ be the best approximation in $S|_{\omega_\mathcal{T}(E)}$ with respect to the norm $|||\cdot|||_{\omega_\mathcal{T}(E)}$. Then, for every $v \in S|_{\omega_\mathcal{T}(E)}$, it holds

$$\varepsilon \int_{\omega_\mathcal{T}(E)} \nabla u \cdot \nabla v + \int_{\omega_\mathcal{T}(E)} uv = \varepsilon \int_{\omega_\mathcal{T}(E)} \nabla\mathcal{R}_E^\varepsilon u \cdot \nabla v + \int_{\omega_\mathcal{T}(E)} \mathcal{R}_E^\varepsilon u \, v.$$

Testing with $v = 1$ yields

$$\int_{\omega_\mathcal{T}(E)} \mathcal{R}_E^\varepsilon u = \int_{\omega_\mathcal{T}(E)} u, \tag{5.10}$$

which shows that (5.5) is a natural choice. The interpolation operator Π^ε is then given by (5.1) where $\mathcal{P}_{E_K}$ is replaced by $\mathcal{R}_{E_K}^\varepsilon$. Thus, Π^ε differs from Π^0 and Π^∞ only in the choice of the local best approximations. This and (5.10) entail that the counterparts of (5.4) and (5.9) for Π^ε and $\mathcal{R}_{E_K}^\varepsilon$ hold.

In order to show that Π^ε is locally near best in a robust manner, we again fix $K \in \mathcal{T}$, write $E := E_K$ for short and choose $\omega = \omega_\mathcal{T}(E)$ in (4.2). Due to the equivalence

of norms on a finite dimensional space, there exist constants $c_N = c_N(\ell, d)$ and $C_N = C_N(\ell, d)$ such that, for every reference node $\hat{z}$,

$$c_N \|\hat{\phi}_{\hat{z}}\|_{\hat{K}} \leq \|\nabla \hat{\phi}_{\hat{z}}\|_{\hat{K}} \leq C_N \|\hat{\phi}_{\hat{z}}\|_{\hat{K}}.$$

Using (3.2) and (3.3), we derive from this

$$|||\phi_z|||_K \leq \left(1 + \varepsilon \frac{C_N^2 \hat{h}^2}{\rho_K^2}\right)^{1/2} \|\phi_z\|_K \tag{5.11}$$

on one hand and

$$|||\phi_z|||_K \leq \left(1 + \frac{1}{\varepsilon} \frac{h_K^2}{c_N^2 \hat{\rho}^2}\right)^{1/2} \varepsilon^{1/2} \|\nabla \phi_z\|_K \tag{5.12}$$

on the other. Inequality (5.11) and the counterpart of (5.4) imply

$$\sum_{z \in \mathcal{N}_K} |\mathcal{R}_E^\varepsilon u(z) - \Pi^\varepsilon u(z)| \, |||\phi_z|||_K \leq \mu_{\mathcal{T}} M_0 \sqrt{1 + \varepsilon \frac{C_N^2 \hat{h}^2}{\rho_K^2}} \sum_{\widetilde{E} \in \gamma_{\mathcal{T}}(K)} \left\| \mathcal{R}_{\widetilde{E}}^\varepsilon u - u \right\|_{\omega_{\mathcal{T}}(\widetilde{E})},$$

while (5.12) and the counterpart of (5.9) give

$$\sum_{z \in \mathcal{N}_K} |\mathcal{R}_E^\varepsilon u(z) - \Pi^\varepsilon u(z)| \, |||\phi_z|||_K \leq \mu_{\mathcal{T}} \sigma_{\mathcal{T}} M_\infty \sqrt{1 + \frac{1}{\varepsilon} \frac{h_K^2}{c_N^2 \hat{\rho}^2}} \sum_{\widetilde{E} \in \gamma_{\mathcal{T}}(K)} \varepsilon^{1/2} \left\| \nabla (\mathcal{R}_{\widetilde{E}}^\varepsilon u - u) \right\|_{\omega_{\mathcal{T}}(\widetilde{E})}.$$

Combining the last two inequalities, we arrive at

$$\sum_{z \in \mathcal{N}_K} |\mathcal{R}_E^\varepsilon u(z) - \Pi^\varepsilon u(z)| \, |||\phi_z|||_K \leq \mu_{\mathcal{T}} M_\varepsilon \sum_{\widetilde{E} \in \gamma_{\mathcal{T}}(K)} |||\mathcal{R}_{\widetilde{E}}^\varepsilon u - u|||_{\omega_{\mathcal{T}}(\widetilde{E})} \tag{5.13}$$

where

$$M_\varepsilon := \min \left\{ M_0 \sqrt{1 + \varepsilon \frac{C_N^2 \hat{h}^2}{\rho_K^2}}, M_\infty \sigma_{\mathcal{T}} \sqrt{1 + \frac{1}{\varepsilon} \frac{h_K^2}{c_N^2 \hat{\rho}^2}} \right\}$$

satisfies $\lim_{\varepsilon \searrow 0} M_\varepsilon = M_0$ and $\lim_{\varepsilon \nearrow \infty} M_\varepsilon = M_\infty \sigma_{\mathcal{T}}$ as well as

$$M_\varepsilon \leq \max\{M_0, M_\infty \sigma_{\mathcal{T}}\} \sqrt{1 + \frac{C_N}{c_N} \frac{\hat{h}}{\hat{\rho}} \frac{h_K}{\rho_K}}.$$

Using this in Proposition 4.2 provides the main result of this section:

Theorem 5.3 (Robust localization for reaction-diffusion norm)**.** *For every $\varepsilon > 0$ and $u \in H^1(\Omega)$, it holds*

$$\inf_{v \in S} |||u - v|||_\Omega \leq C_{\mathrm{loc}} \left(\sum_{E \in \mathcal{E}_\Omega} \inf_{V \in S|_{\omega_{\mathcal{T}}(E)}} |||u - V|||_{\omega_{\mathcal{T}}(E)}^2 \right)^{\frac{1}{2}} \leq (d+1) C_{\mathrm{loc}} \inf_{v \in S} |||u - v|||_\Omega$$

where $|||\cdot|||$ stands for $\left(\|\cdot\|^2 + \varepsilon \|\nabla \cdot\|^2 \right)^{1/2}$ and the localization constant C_{loc} depends on the dimension d, the polynomial degree ℓ, $\bar{n}$ from (4.4), $\mu_{\mathcal{T}}$ and $\sigma_{\mathcal{T}}$ from (3.4), but not on ε.

If ε is small with respect to $\min_{K \in \mathcal{T}} \rho_K^2$, the localization constant C_{loc} can be chosen independently of the shape parameter $\sigma_{\mathcal{T}}$.

6. Local best errors: single elements versus pairs

According to §2 and §5.3, the best errors in the reaction-diffusion norm (1.2) on single elements do not provide a robust localization, while those on pairs of elements do. This section analyzes their difference. As side-products, we derive the remaining part of (1.3), alternative ways to compute best errors on element pairs, and give regularity assumptions ensuring that the global best error in the reaction-diffusion norm is free from locking under uniform refinement.

Throughout this section ω is the union of two simplices K_1 and K_2 with common face $E = K_1 \cap K_2$. For example, $\omega = \omega_{\mathcal{T}}(E)$ with $E \in \mathcal{E}_\Omega$. We write $h_i := h_{K_i}$ and $\rho_i := \rho_{K_i}$ for short, $i = 1, 2$, and set

$$\sigma_E := \frac{\max\{h_1, h_2\}}{\min\{\rho_1, \rho_2\}} \quad \text{and} \quad h_E := \frac{\min\{|K_1|, |K_2|\}}{|E|}.$$

Moreover denote by V_E the best approximation in $S|_\omega := S^{\ell,0}(\{K_1, K_2\})$ with respect to the reaction-diffusion norm $|||\cdot|||_\omega$, whence

$$|||u - V_E|||_\omega = \inf_{V \in S|_\omega} |||u - V|||_\omega. \tag{6.1}$$

6.1. **Jump augmentation and interpolation.** If one wants to bound (6.1), one may be tempted to replace the space $S|_\omega$ by the smaller one $\mathbb{P}_\ell(\omega)$ and then to apply the Bramble-Hilbert lemma. The resulting bound however requires regularity across the interelement face E, like e.g. $(|u|^2_{1;\omega} + \varepsilon |u|^2_{2;\omega})^{1/2}$. The latter comes with overestimation: in particular, if $u_{|\omega} \in S|_\omega$, then the bound is (formally) ∞, while (6.1) vanishes. This observation suggests to apply the Bramble-Hilbert lemma on single elements. On the other hand, the best errors on single elements alone do not provide a robust upper bound. We therefore first identify a quantity that fills the gap between (6.1) and the best errors on K_1 and K_2.

Lemma 6.1 (Jump augmentation)**.** *Let $C_P > 0$ be a constant and $u \in H^1(\omega)$. If $P_i \in \mathbb{P}_\ell(K_i)$ satisfies the Poincaré inequality*

$$\|u - P_i\|_{K_i} \le C_P h_i \|\nabla(u - P_i)\|_{K_i}, \qquad i = 1, 2, \tag{6.2}$$

then it holds

$$|||u - V_E|||_\omega \le C \left(h_E^{1/2} \|P_1 - P_2\|_E + \sum_{i=1}^{2} |||u - P_i|||_{K_i} \right),$$

where C depends on C_P, ℓ, d, σ_E, but is independent of ε.

Proof. We may assume $|K_2| \ge |K_1|$ without loss of generality. Define $\widetilde{V} \in S|_\omega$ by

$$\widetilde{V}(z) := \begin{cases} P_i(z) & z \in \mathcal{N}_{K_i \setminus E} \text{ with } i \in \{1, 2\}, \\ P_2(z) & z \in \mathcal{N}_E. \end{cases}$$

Thanks to (6.1) we have $|||u - V_E|||_\omega \le |||u - \widetilde{V}|||_\omega$. Observing

$$|||u - \widetilde{V}|||_{K_2} = |||u - P_2|||_{K_2} \quad \text{and} \quad |||u - \widetilde{V}|||_{K_1} \le |||u - P_1|||_{K_1} + |||P_1 - \widetilde{V}|||_{K_1},$$

we are left with establishing a suitable bound for $|||P_1 - \widetilde{V}|||_{K_1}$. To this end, we proceed similarly to §5.3. We expand $P_1 - \widetilde{V}$ with respect to the nodal basis on K_1 and obtain

$$|||P_1 - \widetilde{V}|||_{K_1} \le \textstyle\sum_{z \in \mathcal{N}_E} |P_1(z) - P_2(z)| \, |||\phi_z|||_{K_1} \tag{6.3}$$

because $|P_1(z) - \widetilde{V}(z)| = 0$ for every $z \in \mathcal{N}_{K_1 \setminus E}$. For the shared nodes $z \in \mathcal{N}_E$, we have

$$|P_1(z) - P_2(z)| \leq \int_E |(P_1 - P_2)\psi_z^E| \leq \frac{|\hat{E}|^{1/2}}{|E|^{1/2}} \|P_1 - P_2\|_E \, \|\hat{\psi}_{\hat{z}}\|_{\hat{E}} \tag{6.4}$$

by (3.2). Taking also (5.11) into account, the last two inequalities imply

$$\begin{aligned} |||P_1 - \widetilde{V}|||_{K_1} &\leq \sum_{z \in \mathcal{N}_E} |P_1(z) - P_2(z)| \sqrt{1 + \varepsilon \frac{C_N^2 \hat{h}^2}{\rho_1^2}} \, \|\phi_z\|_{K_1} \\ &\leq m_0 \sqrt{1 + \varepsilon \frac{C_N^2 \hat{h}^2}{\rho_1^2}} h_E^{1/2} \, \|P_1 - P_2\|_E \end{aligned} \tag{6.5}$$

with

$$m_0 = \frac{|\hat{E}|^{1/2}}{|\hat{K}|^{1/2}} \sum_{\hat{z} \in \hat{E}} \|\hat{\psi}_{\hat{z}}\|_{\hat{E}} \|\hat{\phi}_{\hat{z}}\|_{\hat{K}}.$$

To derive an alternative bound when ε is big, we first observe that the trace identity (5.6) and the Poincaré inequalities (6.2) yield

$$\|u - P_i\|_E^2 \leq C_P \left(C_P + \frac{2}{d} \right) \frac{h_i^2 |E|}{|K_i|} \, \|\nabla(u - P_i)\|_{K_i}^2 , \quad i = 1, 2. \tag{6.6}$$

Using this in (6.4) gives

$$|P_1(z) - P_2(z)| \leq \sqrt{C_P \left(C_P + \frac{2}{d} \right) |\hat{E}|} \, \|\hat{\psi}_{\hat{z}}\|_{\hat{E}} \sum_{j=1}^{2} \frac{h_j}{|K_j|^{1/2}} \, \|\nabla(u - P_j)\|_{K_j} ,$$

which together with (6.3), (5.12) and (3.3) implies

$$\begin{aligned} |||P_1 - \widetilde{V}|||_{K_1} &\leq \sum_{z \in \mathcal{N}_E} |P_1(z) - P_2(z)| \sqrt{1 + \frac{1}{\varepsilon} \frac{h_1^2}{c_N^2 \hat{\rho}^2}} \, \varepsilon^{1/2} \|\nabla \phi_z\|_{K_1} \\ &\leq m_\infty \sqrt{1 + \frac{1}{\varepsilon} \frac{h_1^2}{c_N^2 \hat{\rho}^2}} \sum_{j=1}^{2} \frac{h_j}{\rho_1} \varepsilon^{1/2} \, \|\nabla(u - P_j)\|_{K_j} , \end{aligned} \tag{6.7}$$

where

$$m_\infty = m_\infty(C_P, \ell, d) := \sqrt{C_P \left(C_P + \frac{2}{d} \right) \frac{|\hat{E}|}{|\hat{K}|}} \, \hat{h} \sum_{\hat{z} \in \hat{E}} \|\hat{\psi}_{\hat{z}}\|_{\hat{E}} \|\nabla \hat{\phi}_{\hat{z}}\|_{\hat{K}}.$$

Combining (6.5) and (6.7) yields

$$|||P_1 - \widetilde{P}|||_{K_1} \leq C \left(h_E^{1/2} \, \|P_1 - P_2\|_E + \sum_{i=1}^{2} |||u - P_i|||_{K_i} \right)$$

with

$$C = \max\{m_0, m_\infty \sigma_E\} \sqrt{1 + \frac{C_N}{c_N} \frac{\hat{h}}{\hat{\rho}} \frac{h_1}{\rho_1}}$$

and thus establishes the claimed inequality. □

Next, we check that the jump term in the upper bound in Lemma 6.1 does not overestimate, if we choose $P_i \in \mathbb{P}_\ell(K_i)$ as the best approximations to u with respect to the reaction-diffusion norm $|||\cdot|||$, that is

$$|||u - P_i|||_{K_i} = \inf_{P \in \mathbb{P}_\ell(K_i)} |||u - P|||_{K_i} \quad i = 1, 2. \tag{6.8}$$

This leads to the following characterization of the best error on a pair by the best approximation on its single elements. Notice also that this characterization is less costly to compute, although still some communication between elements is necessary.

Theorem 6.2 (Sharp jump augmentation)**.** *For any $u \in H^1(\omega)$, it holds*

$$|||u - V_E|||_\omega \approx \left(h_E \|P_1 - P_2\|_E^2 + \sum_{i=1}^{2} |||u - P_i|||_{K_i}^2 \right)^{\frac{1}{2}}$$

whenever P_i are given by (6.8)*. The hidden constants depend on d, ℓ, σ_E, but are independent of ε.*

Proof. We start by bounding $|||u - V_E|||_\omega$ from above. The choice (6.8) implies $\int_{K_i} P_i = \int_{K_i} u$; see (5.10). Therefore the classical Poincaré inequality on convex domains, see [4, 19], ensures that (6.2) holds with $C_P \leq \frac{1}{\pi}$ and Lemma 6.1 yields the desired bound.

In order to bound $|||u - V_E|||_\omega$ from below, we first observe

$$\sum_{i=1}^{2} |||u - P_i|||_{K_i}^2 \leq \sum_{i=1}^{2} |||u - V_E|||_{K_i}^2 = |||u - V_E|||_\omega^2 \tag{6.9}$$

using (6.8). Therefore the critical term is the jump term. To bound it, we first add and subtract V_E and use the triangle inequality:

$$\|P_1 - P_2\|_E \leq \|P_1 - V_E\|_E + \|P_2 - V_E\|_E \,. \tag{6.10}$$

Since P_i and V_E are both polynomials on E, we write their expansion with respect to the nodal basis on E. Every $z \in \mathcal{N}_E$ is also a node of K_i and so we can use (3.1) on K_i. Using also the Cauchy-Schwarz inequality and the scaling properties (3.2), we derive the following explicit inverse inequality:

$$\begin{aligned} h_E^{1/2} \|P_i - V_E\|_E &\leq h_E^{1/2} \sum_{z \in \mathcal{N}_E} \int_{K_i} |(P_i - V_E)\psi_z^{K_i}| \, \|\phi_z\|_E \\ &\leq \frac{|K_1|^{1/2}}{|E|^{1/2}} \sum_{z \in \mathcal{N}_E} \|P_i - V_E\|_{K_i} \|\hat{\psi}_{\hat{z}}\|_{\hat{K}} \|\hat{\phi}_{\hat{z}}\|_{\hat{E}} \frac{|\hat{K}|^{1/2}}{|\hat{E}|^{1/2}} \frac{|E|^{1/2}}{|K_i|^{1/2}} \\ &\leq \widetilde{m}_0 \|P_i - V_E\|_{K_i} \,, \end{aligned}$$

where we have assumed $|K_2| \geq |K_1|$ without loss of generality and

$$\widetilde{m}_0 = \widetilde{m}_0(\ell, d) := \frac{|\hat{K}|^{1/2}}{|\hat{E}|^{1/2}} \sum_{\hat{z} \in \hat{E}} \|\hat{\psi}_{\hat{z}}\|_{\hat{K}} \|\hat{\phi}_{\hat{z}}\|_{\hat{E}}.$$

Inserting u and recalling (6.8), we get

$$h_E^{1/2} \|P_i - V_E\|_E \leq 2\widetilde{m}_0 |||u - V_E|||_{K_i}. \tag{6.11}$$

Combining (6.9), (6.10) and (6.11), we conclude

$$h_E \|P_1 - P_2\|_E^2 + \sum_{i=1}^{2} |||u - P_i|||_{K_i}^2 \leq \left(1 + 8\widetilde{m}_0^2\right) |||u - V_E|||_\omega^2,$$

where the constant depends only on ℓ and d. □

Theorem 6.2 shows that best approximations on elements augmented with interelement jumps provide also a robust localization. As a consequence, the non-robustness in §2 is caused by the absence of these jump terms. This fact is illustrated by the following corollary for the limiting case $\varepsilon = 0$.

Corollary 6.3 (Sharp jump augmentation for L^2)**.** *Let $u \in L^2(\omega)$ and denote by V_E, P_i the best approximations to u in $S|_\omega$ and $\mathbb{P}_\ell(K_i)$, respectively with respect to the L^2-norm. Then*

$$\|u - V_E\|_\omega \approx \left(h_E \|P_1 - P_2\|_E^2 + \sum_{i=1}^{2} \|u - P_i\|_{K_i}^2 \right)^{\frac{1}{2}},$$

where the hidden constants depend on ℓ and d, but are independent of σ_E.

Proof. Consider only the L^2-part of the reaction-diffusion norm in the proof of Theorem 6.2. □

Corollary 6.3 is of interest in the analysis of the L^2-error of discontinuous Galerkin (DG) methods.

Remark 6.4 (Jump penalization in DG methods)**.** In the light of §2, the space

$$S^{\ell,-1}(\mathcal{T}) := \{v \in L^\infty(\Omega) : \forall K \in \mathcal{T},\, v \in \mathbb{P}_\ell(K)\}$$

of possibly discontinuous piecewise polynomial functions offers advantages over $S^{\ell,0}(\mathcal{T})$ in particular when approximating in the L^2-norm. This suggests to use DG methods based upon subspaces of $S^{\ell,-1}(\mathcal{T})$ in such cases. Yet, DG methods have to be stabilized by penalizing jumps over interelement faces. More precisely, the stability estimate usually involves terms of the form $\delta_E \|[v]\|_E^2$, where $[v]$ denotes the jump of $v \in S^{\ell,-1}(\mathcal{T})$ across $E \in \mathcal{E}_\Omega$ and δ_E is a suitable scaling factor. Theorem 5.1 and Corollary 6.3 reveal that such DG method does not exploit the aforementioned advantages of $S^{\ell,-1}(\mathcal{T})$ over $S^{\ell,0}(\mathcal{T})$ whenever $\delta_E \gtrsim h_E$.

The function $\widetilde{V}$ in the proof of Lemma 6.1 with the polynomials (6.8) is an example of a robust near best approximation in $S|_\omega$ with respect to $|||\cdot|||_\omega$. The best approximating polynomials (6.8) require in general the solution of two linear systems. Using suitable Scott-Zhang interpolation, we can obtain such robust near best approximations that are explicit and therefore easier to compute. To this end, for any node $z \in \mathcal{N}_\omega$, we fix a simplex K_z^E such that $z \in K_z^E \subset \omega$ holds. With these choices, we define $\Pi_E u \in S|_\omega$ by

$$\forall z \in \mathcal{N}_\omega \qquad \Pi_E u(z) := \int_{K_z^E} u \psi_z^E, \tag{6.12}$$

where ψ_z^E stand for the dual basis functions of the simplex K_z^E corresponding to z.

Lemma 6.5 (Local robust near best interpolation)**.** *The local interpolation operator Π_E given by* (6.12) *is a robust near best approximation in $S|_\omega$ with respect to the reaction-diffusion norm $|||\cdot|||_\omega$: for any $u \in H^1(\omega)$, it holds*

$$|||u - \Pi_E u|||_\omega \le C |||u - V_E|||_\omega$$

where C depends on d, ℓ, σ_E, but is independent of ε.

Proof. The claim follows essentially from the arguments in the proof of Theorem 5.3, because the definition of (6.12) is quite close to the definitions of Π^ε in §5. In fact, considering $\{K_1, K_2\}$ as a mesh for ω, both (5.1) and (6.12) use the Scott-Zhang approach for the nodes on the skeleton of ω. For the remaining nodes, Π^ε invokes local best approximations, while (6.12) uses again the Scott-Zhang approach, where here we have even $z \in \mathring{K}_z^E$. Consequently, property (5.2) is not given but can be replaced by arguments that are similar to the ones for skeleton nodes, yet are simpler because there is no dislocation. □

It is worth observing that Π^ε itself does not depend on ε and therefore $\Pi^\varepsilon u$ is a simultaneous near best approximation with respect to $|||\cdot|||_\omega$ for all $\varepsilon > 0$. Similarly, the appropriately defined Scott-Zhang interpolation operator is a simultaneous near best approximation with respect to $|||\cdot|||_\Omega$ for all $\varepsilon > 0$. This however comes at the price of a relationship with the global best error $\inf_{v\in S}|||u - v|||_\Omega$ that is less close than the one in Theorem 5.3: indeed, since (5.2) does not hold, the constant in the upper bound is expected to be larger, and the constant in the lower bound does not depend only on d.

6.2. **Trace augmentation.** In order to bound the right-hand side of Theorem 6.2, one may add and subtract the trace of u in the jump term and apply the triangle inequality. This simple idea leads to the following proposition, which establishes the second part of (1.3), uses only best errors on single elements and avoids any communication between the elements.

Proposition 6.6 (Trace augmentation)**.** *For any $u \in H^1(\omega)$, it holds*

$$|||u - V_E|||_\omega \leq C \sum_{i=1}^{2} \inf_{P\in\mathbb{P}_\ell(K_i)} \left(|||u - P|||_{K_i}^2 + \frac{|K_i|}{|\partial K_i|} \|u - P\|_{\partial K_i}^2 \right)^{1/2}.$$

where the constant C depends on ℓ, d, σ_E, but is independent of ε.

Proof. Let P_i, $i = 1, 2$, be the best approximations associated with the two infima in the claimed bound. It suffices to verify that they satisfy (6.2). In fact, inserting u in the jump term of Lemma 6.1 yields the claim. Fix $i \in \{1, 2\}$ and write $K := K_i$ and $v := u - P_i$ for short. Similarly as (5.10), we obtain

$$\int_K v + \frac{|K|}{|\partial K|} \int_{\partial K} v = 0$$

from the optimality of P_i. Hence we can write

$$\|v\|_K^2 = \left\| v - \frac{1}{|K|} \int_K v \right\|_K^2 + \frac{1}{4|K|} \left(\frac{|K|}{|\partial K|} \int_{\partial K} v - \int_K v \right)^2.$$

Adding the trace identity, cf. (5.6), for every face of K in a suitable weighted manner, we obtain a vector field q_K such that

$$\left| \frac{|K|}{|\partial K|} \int_{\partial K} v - \int_K v \right| = \left| \int_K q_K \cdot \nabla v \right| \leq \frac{h_K}{d} |K|^{1/2} \|\nabla v\|_K .$$

Consequently the classical Poincaré inequality on convex domains, see [4, 19], shows that (6.2) holds with $C_P \leq \left(\frac{1}{\pi} + \frac{1}{4d^2}\right)^{1/2}$. □

The key difference between the right-hand sides in Theorem 6.2 and Proposition 6.6 is that the latter is only defined in the limiting case $\varepsilon = 0$, if u has a trace on ∂K, that is, if u has at least 1/2-derivative in that u is an element of the Besov space $B_1^{1/2}\big(L^2(K)\big)$; see, e.g., L. R. Scott [20]. As a consequence, the opposite inequality of Proposition 6.6 cannot be robust. To see this, consider a family $\{u_\varepsilon\}_{\varepsilon>0} \subset H^1(K)$ of functions such that $\|u_\varepsilon\|_K^2 + \varepsilon \|\nabla u_\varepsilon\|_K^2 = 1$ and $\|u_\varepsilon - u_0\|_K \to 0$ as $\varepsilon \searrow 0$, where $u_0 \in L^2(K) \setminus B_1^{1/2}\big(L^2(K)\big)$. Then, as $\varepsilon \searrow 0$, the right-hand side in Proposition 6.6 becomes ∞, while its left-hand side remains bounded. If $K = [0, 1]$ the functions u_ε in §2 are an example for such a sequence.

In spite of this non-robustness for low regularity, the inequality in Proposition 6.6 is interesting in its own right. Let us illustrate this by a connection to Galerkin approximations in $S^{\ell,0}(\mathcal{T})$ and three applications.

The bound in Proposition 6.6 is closely related to quasi-optimality properties of the Galerkin approximation U in $S_0^{\ell,0}(\mathcal{T}) := S^{\ell,0}(\mathcal{T}) \cap H_0^1(\Omega)$ to the solution of

$-\Delta u = f$ in Ω, $u = 0$ on $\partial\Omega$. In the one-dimensional case, I. Babuška and J. Osborn [1, (6.5)] have shown that, while the Galerkin approximation U is not quasi-optimal with respect to the plain $L^2(\Omega)$-norm, it is so for the augmented $L^2(\Omega)$-norm

$$\|v\|_{L^2(\mathcal{T})} := \left(\|v\|_\Omega^2 + \sum_{E \in \mathcal{E}_\Omega} h_E \, \|v\|_E^2 \right)^{1/2}. \tag{6.13}$$

Multidimensional generalizations follow from I. Babuska et al. [2] and K. Eriksson and C. Johnson [13, Lemma 8.1], which however assume (weak) quasi-uniformity of the mesh $\mathcal{T}$ and convexity of the domain Ω. Remarkably, the global best error in this norm is equivalent to the sum of the bounding local best errors in Proposition 6.6 in the case of $\varepsilon = 0$:

$$\inf_{v \in S^{\ell,0}(\mathcal{T})} \|u - v\|_{L^2(\mathcal{T})}^2 \approx \sum_{K \in \mathcal{T}} \inf_{P \in \mathbb{P}_\ell(K)} \left(\|u - P\|_K^2 + \frac{|K|}{|\partial K|} \|u - P\|_{\partial K}^2 \right).$$

In fact, the '$\lesssim$'-direction is straight-forward, while the other one can be shown with the help of Theorem 5.1 and Proposition 6.6. Thus, the aforementioned difference between Theorem 6.2 and Proposition 6.6 describes, in a local manner, what the Galerkin approximation U lacks to be quasi-optimal in $L^2(\Omega)$.

Next, we use Proposition 6.6 to show that the scaling $\varepsilon^{-1/4}$ of non-robustness in §2 is the worst possible one. To this end, we replace v by $|v|^2$ in (5.6) and apply the Cauchy-Schwarz inequality to obtain

$$|||v|||_K^2 + \frac{|K|}{|\partial K|} \|v\|_{\partial K}^2 \le 2|||v|||_K^2 + \frac{2}{d} h_K \, \|v\|_K \, \|\nabla v\|_K . \tag{6.14}$$

Combining this with

$$2 \, \|v\|_K \, \|\nabla v\|_K \le \frac{1}{\sqrt{\varepsilon}} \, \|v\|_K^2 + \sqrt{\varepsilon} \, \|\nabla v\|_K^2 \le \varepsilon^{-1/2} |||v|||_K^2,$$

we get

$$\inf_{P \in \mathbb{P}_\ell(K)} \left(|||u - P|||_K^2 + \frac{|K|}{|\partial K|} \|u - P\|_{\partial K}^2 \right) \le \left(2 + \frac{1}{d} \frac{h_K}{\sqrt{\varepsilon}} \right) \inf_{P \in \mathbb{P}_\ell(K)} |||u - P|||_K^2,$$

which proves the third part of (1.3). Observe that the powers of ε here and in the example of §2 coincide. Consequently, the power of ε here cannot be improved and the example in §2 is a worst case.

Next, we illustrate how Proposition 6.6 can be combined with the Bramble-Hilbert Lemma on single elements to derive an error bound in terms of piecewise regularity. To this end, for any element $K \in \mathcal{T}$, denote by $P_K \in \mathbb{P}_1$ the polynomial given by

$$\forall |\alpha| \le 1 \quad \int_K \partial^\alpha P_K = \int_K \partial^\alpha u.$$

Thanks to (6.14), the Poincaré inequality [4, 19], and the fact that the mean value is the best constant with respect to the $L^2(K)$-norm, this polynomial verifies

$$\begin{aligned} |||u - P_K|||_K^2 + \frac{|K|}{|\partial K|} \|u - P_K\|_{\partial K}^2 & \\ &\le 2|||u - P_K|||_K^2 + \frac{2h_K}{d} \, \|u - P_K\|_K \, \|\nabla(u - P_K)\|_K \\ &\le \frac{2}{\pi} \left(\frac{1}{\pi} + \frac{1}{d} \right) \left(|u|_{1;K}^2 + \varepsilon |u|_{2;K}^2 \right) h_K^2. \end{aligned}$$

Hence Theorem 5.3 and Proposition 6.6 yield the error bound

$$\inf_{v\in S}|||u-v|||_\Omega \le C\left[\sum_{K\in\mathcal{T}} h_K^2\left(|u|_{1;K}^2+\varepsilon|u|_{2;K}^2\right)\right]^{1/2} \tag{6.15}$$

in terms of weighted broken Sobolev norms. Observe that the right-hand side of (6.15) vanishes whenever its left-hand sides vanishes. It is also worth mentioning that (6.15) does not follow by means of known error bounds for Lagrange, Clément [10] or Scott-Zhang [21] interpolation.

Finally, we discuss with the help of (6.15) robustness properties of $S^{\ell,0}$. In particular we shall show that, under suitable regularity assumptions with respect to ε, uniform refinement yields locking-free global best errors in $S^{\ell,0}$ in the sense of [3, Definition 2.1]. Motivated by (6.15) as well as regularity theorems for reaction-diffusion boundary value problems, we consider a family of functions $\{u_\varepsilon\}_{\varepsilon>0}\subset H^2(\Omega)$ such that

$$|u_\varepsilon|_{1;\Omega}^2+\varepsilon|u_\varepsilon|_{2;\Omega}^2\le 1. \tag{6.16}$$

Applying (6.15) then yields

$$\inf_{v\in S}|||u_\varepsilon-v|||_\Omega \le C(\#\mathcal{T})^{-1/d}$$

with C independent of ε for any mesh of a quasi-uniform and shape-regular family. Combining this uniform error bound as in [3] with n-width results for $\varepsilon=1$, we get that $\inf_{v\in S}|||u_\varepsilon-v|||_\Omega$ is free from locking under (6.16). We may replace the latter, which requires one additional derivative in a ε-uniform manner, by a variant assuming $s\in[1/2,\ell+1]$ additional derivatives, where in the borderline case $s=1/2$ one uses the seminorm of the Besov space $B_1^{1/2}(L^2(\Omega))$ instead of the fractional Sobolev seminorm. Then a similar, but more technical argument with a sharp trace theorem and Taylor polynomials averaged on maximal balls of elements as in [12] ensures a robust error bound with $(\#\mathcal{T})^{-s/d}$. On the other hand, §2 implies that the global best errors of $S^{\ell,0}$ under uniform refinement show locking whenever $s<1/2$.

7. Robust localization with Dirichlet boundary conditions

In this section we briefly discuss the modifications of our results if the boundary values of the target function are imposed on the approximants. This is of interest, for example, when conforming finite element methods are applied to the homogeneous Dirichlet problem of the reaction-diffusion equation. For simplicity, we consider target functions in $H_0^1(\Omega)$ approximated by elements from $S_0:=S_0^{\ell,0}(\mathcal{T}):=S^{\ell,0}(\mathcal{T})\cap H_0^1(\Omega)$.

Considering $u_\varepsilon(x)=\min\{1,\varepsilon^{-1/2}\mathrm{dist}(x,\partial\Omega)\}$, $x\in\Omega$, as in §2 reveals the following: if any local best error on a subdomain ω with positive $(d-1)$-dimensional Hausdorff measure $|\partial\omega\cap\partial\Omega|>0$ does not take the boundary condition into account, robustness cannot hold. This suggests the following modification of the setting for, e.g., Theorem 5.3. We associate to every $E\in\mathcal{E}_\Omega$ the local space

$$S_E:=\begin{cases} S_0|_{\omega_\mathcal{T}(E)} & \text{if } E\in\mathcal{E}_\partial,\\ S|_{\omega_\mathcal{T}(E)} & \text{otherwise,}\end{cases} \tag{7.1}$$

where $\mathcal{E}_\partial:=\{E'\in\mathcal{E}:\omega_\mathcal{T}(E')\text{ has a face on }\partial\Omega\}$. Notice that we have $S_E\neq S_0|_{\omega_\mathcal{T}(E)}$ if and only if $\omega_\mathcal{T}(E)\cap\partial\Omega$ is not empty but does not contain a $(d-1)$-dimensional face. This however, at least, does not create a problem for the second inequality in Theorem 5.3 since $S_E\supseteq S_0|_{\omega_\mathcal{T}(E)}$.

The interpolation operator Π_0^ε now has to vanish on the domain boundary $\partial\Omega$. To this end, the elegant approach of averaging on boundary faces in [21], which has been adopted in [23], cannot be applied because the use of traces does not allow for (5.3) and so for robustness. We therefore use the original approach of suppressing the boundary nodes in P. Clément [10] and set

$$\Pi_0^\varepsilon u := \sum_{z\in\mathcal{N}_\Omega} u_z \phi_z$$

with u_z as is (5.1), where $\mathcal{P}_{E_K} u$ is replaced by the best approximation $\mathcal{R}^\varepsilon_{0,E_K} u$ in $S_{\omega_\mathcal{T}(E_K)}$ to u with respect to the reaction-diffusion norm. Consequently, Π^ε and Π_0^ε differ only at boundary nodes and at nodes invoking a face in $\mathcal{E}_\partial$.

Nevertheless the counterpart of (5.4) holds. To see this, consider $K \in \mathcal{T}$ such that $K \cap \partial\Omega$ is non-empty and notice that only the boundary nodes $z \in \mathcal{N}_K \cap \partial\Omega$ are critical. If K has a face on the boundary $\partial\Omega$, then the same holds for $\omega_\mathcal{T}(E_K)$ and so we have $u_z = 0 = \mathcal{R}^\varepsilon_{0,E_K} u(z)$ for all boundary nodes $z \in \mathcal{N}_K \cap \partial\Omega$. If the intersection $K\cap\partial\Omega$ is only a k-face with $k < d-1$, we can find a path $\{\omega_\mathcal{T}(E_j)\}_{j=1}^n$ of pairs such that $\omega_\mathcal{T}(E_1) \supseteq K$ and $\omega_\mathcal{T}(E_n)$ has a face on $\partial\Omega$. Hence we can bound $|\mathcal{R}^\varepsilon_{0,E_K} u(z)| = |\mathcal{R}^\varepsilon_{0,E_K} u(z) - \mathcal{R}^\varepsilon_{0,E_n} u(z)|$ as in the proof of Proposition 4.3 for all boundary nodes $z \in \mathcal{N}_K \cap \partial\Omega$.

Inequality (5.9) hinges on a Poincaré-type inequality on pairs. If $E' \in \mathcal{E}_\partial$, then (5.10) may not be correct and the counterpart of (5.9) is built on the following Friedrichs' inequality, which is a tailor-made variant of Lemma 5.1 in [24].

Lemma 7.1 (Friedrichs inequality on element pairs)**.** *Let ω be the union of two adjacent elements K_1, K_2 sharing a face $E = K_1 \cap K_2$. Moreover let E_0 be a face of K_1. For every $v \in H^1(\omega)$ with $\int_{E_0} v = 0$, it holds*

$$\|v\|_\omega \leq C_F h_\omega \|\nabla v\|_\omega ,$$

where $h_\omega := \max\{\operatorname{diam}(K_1), \operatorname{diam}(K_2)\}$ and C_F depends on d and the shape parameter of $\{K_1, K_2\}$.

Proof. Adding and subtracting the mean value over the common face E, we obtain

$$\|v\|_\omega \leq \left\| v - \frac{1}{|E|}\int_E v \right\|_\omega + |\omega|^{1/2} \left| \frac{1}{|E|}\int_E v \right| .$$

We treat the first term on the right-hand side as in the proof of Lemma 5.2. For the second term, we use twice the trace identity (5.6) to get

$$\frac{1}{|E|}\int_E v = \frac{1}{d|K_1|}\int_{K_1} (z_{E_0} - z_E)\cdot\nabla v,$$

where z_{E_0} and z_E are the vertices opposite to E_0 and E, respectively. Hence

$$\begin{aligned} |\omega|^{1/2} \left| \frac{1}{|E|}\int_E v \right| &\leq \left(|K_1|^{1/2} + |K_2|^{1/2} \right) \frac{|z_{E_0} - z_E| \, \|\nabla v\|_{K_1}}{d|K_1|^{1/2}} \\ &\leq \left(\frac{\operatorname{diam}(K_1)}{d} + \frac{|K_2|^{1/2}|z_{E_0} - z_E|}{d|K_1|^{1/2}} \right) \|\nabla v\|_{K_1} . \end{aligned}$$

Since

$$\frac{|K_2|^{1/2}|z_{E_0} - z_E|}{|K_1|^{1/2}} \leq \frac{|E|^{1/2}\operatorname{diam}(K_2)^{1/2}|z_{E_0} - z_E|}{|E|^{1/2}\operatorname{dist}(z_E, E)^{1/2}} \leq \frac{|z_{E_0} - z_E|^{1/2}}{\operatorname{dist}(z_E, E)^{1/2}} h_\omega,$$

we conclude

$$\|v\|_\omega \leq \left(C_P + \frac{1}{d} + \frac{|z_{E_0} - z_E|^{1/2}}{d \operatorname{dist}(z_E, E)^{1/2}} \right) h_\omega \|\nabla v\|_\omega . \qquad \square$$

We thus obtain the following variant of Theorem 5.3.

Theorem 7.2 (Robust localization with boundary condition)**.** *Using* (7.1)*, it holds*

$$\inf_{v\in S_0(\mathcal{T})}|||u-v|||_\Omega \approx \left(\sum_{E\in\mathcal{E}_\Omega}\inf_{V\in S_E}|||u-V|||^2_{\omega_\mathcal{T}(E)}\right)^{\frac12}$$

for every $u\in H^1_0(\Omega)$. The hidden constants have the same dependencies as the corresponding constants in Theorem 5.3.

8. Robust localization for tree approximation

Adaptive tree approximation is a form of nonlinear approximation due to P. Binev and R. DeVore [8]. In this section we give a variant of the localization in Theorem 5.3 that is useful for tree approximation in the reaction-diffusion norm. The robustness of the localization ensures robust performance of tree approximation.

The need and manner of modifying Theorem 5.3 arises from assumptions in tree approximation and properties of the employed adaptive refinement technique. We therefore start by recalling them, in a form suitable for our purposes. In what follows, we consider only $d\geq 2$, because the case $d=1$ is of different nature.

8.1. **Bisection and tree approximation.** We adopt bisection of simplices to produce adaptively refined meshes. This technique was introduced by W. Mitchell [16], I. Kossaczký [14], and J. M. Maubach [15]. Roughly speaking, it subdivides a simplex into two of equal volume by bisecting an assigned edge, the refinement edge. This leads to a tree structure and, by suitably assigning the refinement edge of successive bisections, one can produce conforming meshes with bounded shape parameter. This is important for our purposes, because both conformity and boundedness of the shape parameter are exploited for the localization in Theorem 5.3.

Bisection for $d=2$ is less technical than for $d\geq 3$. We therefore recall these two cases separately so that a reader who is interested only in $d=2$ can skip the parts for $d\geq 3$. For proofs, we refer to the articles by Binev et al. [7] and by Stevenson [22] or the accounts [17, 18].

In the case $d=2$, the refinement edge can be assigned via a labeling of the edges. Given a triangle with edge labeling $(l,l+1,l+1)$, its bisection connects the midpoint of the edge labeled l with the opposite vertex. This produces two new triangles and the three new edges containing the midpoint obtain the label $l+2$. The refinement edge of a triangle is thus the one with the lowest label. The procedure is started from a conforming initial mesh $\mathcal{T}_0$ by assigning to every triangle an edge labeling $(0,1,1)$ such that

(8.1) the labels of an edge shared by two triangles coincide.

Such a labeling always exists. In conforming refinements of $\mathcal{T}_0$, the labels of an edge shared by two triangles continue to coincide.

In the case $d\geq 3$, the refinement edge is assigned with the help of an ordering of the vertices and an additional type. If $K=(a_0,\dots,a_d)_t$ denotes a tagged simplex with ordered vertices $a_0,\dots,a_d\in\mathbb{R}^d$ and type $t\in\{0,\dots,d-1\}$, its bisection generates a new vertex $\bar a:=\frac12(a_0+a_d)$ and the two children

$$(a_0,\bar a,a_1,\dots,a_{d-1})_{(t+1)\bmod d},\quad (a_d,\bar a,a_1,\dots a_t,a_{d-1},\dots,a_{t+1})_{(t+1)\bmod d}. \tag{8.2}$$

The refinement edge of K is thus $[a_0,a_d]$. Notice that K and its so-called reflection $K_R=(a_d,a_1,\dots,a_t,a_{d-1},\dots,a_{t+1},a_0)_t$ have the same children. Putting $d=2$ in (8.2), the type gets superfluous and we get the aforementioned subdivision rule for $d=2$. Assumption (8.1) generalizes to the so-called matching condition. To formulate it, we need the following three notions about neighboring simplices:

- Two simplices are neighbors whenever their intersection is a common face.

- Two tagged simplices K and K' are called reflected neighbors whenever they are neighbors and the ordered sequence of vertices of K or its reflection K_R coincides with that of K' for all but one position.
- Two neighbors K and K' match if the following holds: if one of the refinement edges is included in the common face, then K and K' are reflected neighbors, otherwise the pair of their neighboring children are reflected neighbors.

The matching condition for the conforming initial mesh $\mathcal{T}_0$ then reads as follows:

(8.3) all simplices of $\mathcal{T}_0$ have the same type and all its neighbors are matching.

This can be always met by replacing $\mathcal{T}_0$ with a certain refinement of itself.

For general $d \geq 2$, assume that $\mathcal{T}_0$ is a conforming mesh of Ω verifying (8.1) or (8.3). Then bisection generates a so-called master tree B_∞. This is the forest of infinite binary trees, the roots of which correspond to elements of the initial mesh and each element is connected with its children, which in turn are connected to their children etc. An element $K' \in B_\infty$ is a descendant of $K \in B_\infty$ whenever $K' \subseteq K$. The simplices in B_∞ form a shape-regular family.

We say that B is a finite subtree of B_∞ and write $B \in \mathbb{B}$ whenever $\#B < \infty$, $B \subset B_\infty$ and, for every element in B that has a descendant in B, both children are in B. The symbol $\mathcal{L}(B)$ stands for the set of leaves, i.e. the elements without children, of a subtree B. The leaves $\mathcal{L}(B)$ form a mesh of some subset of Ω, which in general is not conforming. We set $\mathbb{B}_0 := \{B \in \mathbb{B} \mid B \supseteq \mathcal{T}_0\}$ and let $\mathbb{B}^c$ denote the subfamily of the trees in $\mathbb{B}$ whose leaves form a conforming mesh. Then

$$\mathbb{T} := \{\mathcal{L}(B) : B \in \mathbb{B}_0\} \quad \text{and} \quad \mathbb{T}^c := \{\mathcal{L}(B) : B \in \mathbb{B}_0 \cap \mathbb{B}^c\}$$

collect all (conforming, for $\mathbb{T}^c$) meshes of Ω that can be produced with bisection from the initial mesh $\mathcal{T}_0$.

Given a subtree $B \in \mathbb{B}$, we denote by complete(B) the smallest subtree in $\mathbb{B}^c$ containing B. Noteworthy, a single bisection applied to a tree in $\mathbb{B}^c$ may require an arbitrary number of additional bisections to re-establish conformity. If the roots of B form a conforming mesh $\mathcal{R}(B)$ satisfying (8.1) or (8.3) in place of $\mathcal{T}_0$, then complete(B) can be constructed by means of a recursive procedure and it holds

$$N\big(\text{complete}(B)\big) \leq C_{\text{conf}} N(B), \tag{8.4}$$

where $N(B) := \#(B \setminus \mathcal{L}(B))$ stands for the number of bisections and C_{conf} depends only on d and $\mathcal{R}(B)$. If B is single rooted, i.e. $\#\mathcal{R}(B) = 1$, then (8.1) or (8.3) is verified and C_{conf} depends only on d.

In adaptive tree approximation, one assumes that the error associated with a conforming mesh $\mathcal{T} \in \mathbb{T}^c$ is given in terms of so-called local error functionals, i.e.

$$E(\mathcal{T}) := \sum_{K \in \mathcal{T}} e(K), \tag{8.5}$$

where $e(K) \geq 0$ is a positive real number for any element $K \in B_\infty$. Roughly speaking, the goal is to construct conforming meshes with an almost optimal balance of error E and cost. Since the cost is bounded from below by the number $\#\mathcal{T}$ of mesh elements, the best global errors

$$E_n := \min\{E(\mathcal{T}) : \mathcal{T} \in \mathbb{T}^c,\ \#\mathcal{T} \leq n\} \tag{8.6}$$

set benchmarks. An algorithm constructs near best conforming meshes if there exist constants $C_1 \geq 1$ and $c_2 \in (0, 1]$ such that any output mesh $\mathcal{T}_*$ satisfies

$$E(\mathcal{T}_*) \leq C_1 E_n \quad \text{whenever} \quad n \leq c_2 \#\mathcal{T}_*. \tag{8.7}$$

The number of competing meshes in (8.6) grows exponentially with the budget n of mesh elements. Nevertheless, one can construct near best meshes at essentially linear cost, if one assumes a property of the local error functionals in (8.5) that is

called subadditivity. Here we shall need the following variant: the error functional e is weakly subadditive on conforming leaves if there exists a constant $C_0 \geq 1$ such that

$$\sum_{K\in\mathcal{L}(B)} e(K) \leq C_0 e(K^\star) \quad \text{whenever } B \in \mathbb{B}^c \text{ has the single root } K^\star. \tag{8.8}$$

In other words: a possible growth of the error under conforming refinement is limited irrespective of the refinement depth.

In light of the piecewise structure in (8.5), one may try to construct near best meshes by applying a greedy strategy on the local error functionals. Counterexamples however show that this can fail. A remedy is to apply the greedy strategy not to the local error functionals but to modified ones that take the refinement history into account. The following recursive definition is due to P. Binev [5]:

$$\eta(K) := \begin{cases} e(K) & \text{if } K \in \mathcal{T}_0, \\ \dfrac{e(K)\eta(K^\star)}{e(K)+\eta(K^\star)} & \text{if } e(K)+\eta(K^\star) > 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $K^\star$ stands for the parent of a non-root element $K \notin \mathcal{T}_0$. Furthermore, for notational convenience, we define $E(B)$ also for all trees $B \in \mathbb{B}_0$, using $\mathcal{L}(B)$ in place of $\mathcal{T}$ on the left hand side of (8.5). The variant of tree approximation that is suitable for our purposes then reads as follows:

TreeApprox

Require: a tolerance $\mathrm{tol} > 0$
Ensure: a conforming mesh $\mathcal{T}_* \in \mathbb{T}^c$ with $E(\mathcal{T}) \leq \mathrm{tol}$
1: $B_0 := \mathcal{T}_0$, $i := 0$
2: **while** $E(B_i) > \mathrm{tol}/C_0$ **do**
3: choose some leaf $K_i \in \mathcal{L}(B_i)$ that maximizes η in $\mathcal{L}(B_i)$
4: produce a new tree $B_{i+1} \in \mathbb{B}_0$ by bisecting K_i
5: increment i
6: **end while**
7: $\mathcal{T}_* := \mathcal{L}(\mathrm{complete}(B_i))$

The number of operations and evaluations of e is $O(\#\mathcal{T}\log\#\mathcal{T})$; for a modification of the algorithm with $O(\#\mathcal{T})$, see [8, Remark 5.3]. In any case, this is asymptotically much smaller than the number of competing subtrees in (8.6).

Theorem 8.1 (Near best conforming meshes)**.** *If the error functional e is weakly subadditive on conforming leaves, then the algorithm* TreeApprox *produces near best conforming meshes. The involved constants depend only on the dimension d, the initial mesh $\mathcal{T}_0$, and C_0 in* (8.8)*.*

Proof. If (8.8) were without the restriction to conforming leaves, then the claim would follow from the arguments establishing [6, Theorem 2.1]. Using an idea of P. Binev exploiting (8.4), one can allow for the restriction to conforming leaves at the cost of bigger constants. Details, which have been checked by the authors, will be found in an up-coming version of [6]. □

8.2. **Minimal pairs and local error functionals.** In order to exploit the algorithm TreeApprox and Theorem 8.1 for approximation in the reaction-diffusion norm, we need to find an error functional e such that

- for any conforming refinement $\mathcal{T} \in \mathbb{T}^c$ of $\mathcal{T}_0$, the best reaction-diffusion norm error in $S^{\ell,0}(\mathcal{T})$ is equivalent to the global error $E(\mathcal{T})$ from (8.5),

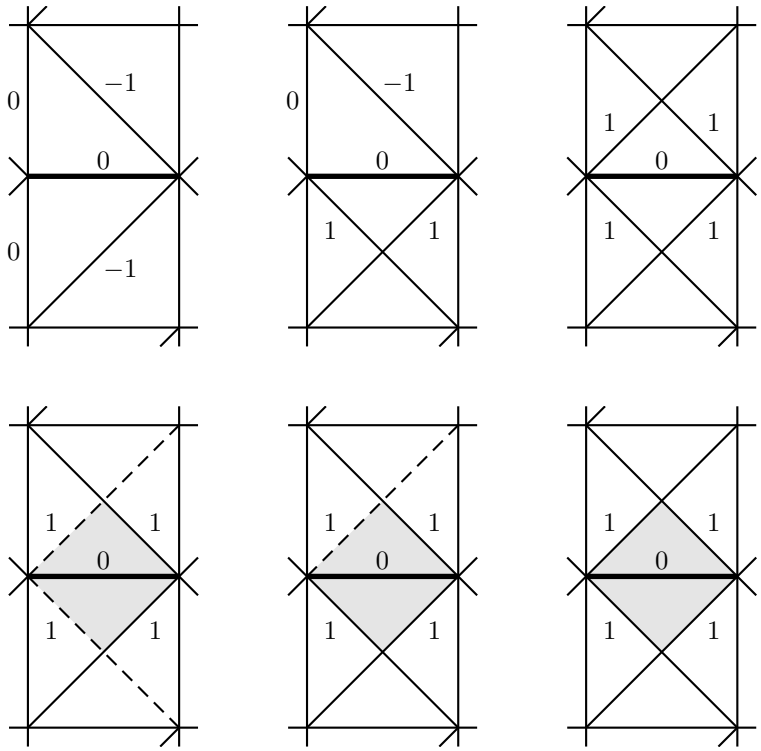

FIGURE 1. (Two-dimensional minimal pairs) Top row: different element pairs of an edge E (bold line); the labeling of the edges is such that E has label 0 and the refinement edge of a triangle has the lowest label. Bottom row: corresponding minimal pairs (gray) with virtual refinements (dashed lines).

- e is weakly subadditive, at least on conforming leaves; compare with (8.8).

In view of Theorem 5.3, one may be tempted to define a local error functional $e(K)$ by simply collecting the local best errors on the pairs associated with the faces of K. An element $K \in B_\infty$ however typically belongs to various triangulations and therefore $e(K)$ would not be well-defined. We therefore mimic the idea of 'minimal ring' in P. Binev et al. [7] and introduce the following variant of $\omega_{\mathcal{T}}(E)$: given a face $E \in \mathcal{E}(K)$ of any simplex $K \in B_\infty$, we define

$$\omega_\star(E) := \bigcap_{\mathcal{T} \in \mathbb{T}^c : E \in \mathcal{E}(\mathcal{T})} \omega_{\mathcal{T}}(E).$$

If E is an interelement face of some mesh $\mathcal{T} \in \mathbb{T}^c$, then $\omega_\star(E)$ is the union of two elements K_1' and K_2' that belong to some virtual refinement of $\mathcal{T}$ and are such that $K_1' \cap K_2' = E$. Figure 1 illustrates the 'generic' minimal pairs in the case $d = 2$. The following observations ensure in particular that the minimal pairs still cover interelement faces internally.

Remark 8.2 (Properties of $\omega_\star$)**.** For any conforming mesh $\mathcal{T} \in \mathbb{T}^c$, there hold:

(i) Let $K \in \mathcal{T}$ and $E \in \mathcal{E}(K)$. Then $K \subseteq \omega_\star(E)$ if and only if the refinement edge of K is contained in the face E.
(ii) Let $E \in \mathcal{E}(\mathcal{T})$. For every $K \in \mathcal{T}$ with $K \subseteq \omega_{\mathcal{T}}(E)$, the set $\omega_\star(E) \cap K$ is an element of $\mathcal{T}$ or of some virtual refinement of $\mathcal{T}$ so that $|\omega_\star(E) \cap K| \geq |K|/2$.

(iii) For any $K \in \mathcal{T}$, there exists $E \in \mathcal{E}(\mathcal{T})$ such that $\omega_\star(E) \supseteq K$. Moreover it is possible to take $E \in \mathcal{E}_\Omega(\mathcal{T})$ if and only if there exists a face E of K such that $E \nsubseteq \partial\Omega$ and E contains the refinement edge of K.
(iv) The family $\mathcal{W}_\star := \{\omega_\star(E)\}_{E\in\mathcal{E}}$ is a d-finite covering of $\mathcal{T}$ and covers interelement faces internally (see Definition 4.5).

Proof. We start with (i). Since $E \subseteq \partial K$, we have $K \subseteq \omega_{\mathcal{T}}(E)$. Denoting by e_R the refinement edge of K, we consider the two cases $e_R \subseteq E$ and $e_R \nsubseteq E$ separately. In the first case, if K is bisected, E is also bisected and therefore E cannot be a face of any conforming refinement $\mathcal{T}' \in \mathbb{T}^{\mathrm{c}}$ of $\mathcal{T}$. Hence $K \subseteq \omega_\star(E)$. Next, consider the second case $e_R \nsubseteq E$. If K is bisected, one of its children K' still contains E and its refinement edge is contained in E. Consequently $K' \subseteq \omega_\star(E)$ and $K' \subsetneq K$ entails $K \nsubseteq \omega_\star(E)$.

In order to verify (ii), we note from the proof of (i) that either $\omega_\star(E) \cap K = K$ or $\omega_\star(E) \cap K = K'$, where K' is a child of K. Since bisection yields $|K'| = |K|/2$, we have also $|\omega_\star(E) \cap K| \geq |K|/2$.

Finally, taking a face which contains the refinement edge of K and applying (i) shows (iii), which then together with (i) implies (iv). □

Using minimal pairs, we define the following local error functional in connection with the reaction-diffusion norm:

$$e(K) := \sum_{E\in\mathcal{E}(K)} \inf_{V\in S|_{\omega_\star(E)}} |||u - V|||^2_{\omega_\star(E)}, \qquad K \in B_\infty. \tag{8.9}$$

This indeed depends only on K because each $\omega_\star(E)$ depends only on E and the tagged simplices of the initial mesh $\mathcal{T}_0$. Let us first verify that these error functionals have the desired subadditivity (8.8).

Proposition 8.3 (Weak subadditivity on conforming leaves)**.** *Let $B \in \mathbb{B}$ be a finite subtree with single root K and assume that its leaves form a conforming mesh of K. Then the functional e in* (8.9) *satisfies*

$$\sum_{K'\in\mathcal{L}(B)} e(K') \leq 2d\, e(K).$$

Proof. For every face E of K, denote by V_E the best approximation in $S|_{\omega_\star(E)}$ to u with respect to the $|||\cdot|||_{\omega_\star(E)}$-norm. For every face E' of some leave $K' \in \mathcal{L}(B)$, there exists $E \subset K$ such that $\omega_\star(E') \subset \omega_\star(E)$ and so

$$|||u - V_{E'}|||_{\omega_\star(E')} \leq |||u - V_E|||_{\omega_\star(E')}.$$

Notice that a given point in $\omega_\star(E)$ is contained in at most d pairs $\omega_\star(E')$ with $E' \in \mathcal{E}(\mathcal{L}(B))$. Hence we obtain

$$\begin{aligned}\sum_{K'\in\mathcal{L}(B)} e(K') &\leq 2 \sum_{E'\in\mathcal{E}(\mathcal{L}(B))} |||u - V_E|||^2_{\omega_\star(E')} \leq 2d \sum_{E\in\mathcal{E}(K)} |||u - V_E|||^2_{\omega_\star(E)} \\ &= 2d\, e(K).\end{aligned}$$

□

It is worth mentioning that, in view of the overlapping of the local error functionals, C_0 cannot be 1 as for the error functionals arising from (1.1). Moreover, for trees with non-conforming leaves, C_0 cannot be bounded independently of the refinement depth.

The next theorem is the announced variant of Theorem 5.3 and establishes the equivalence of the error functionals (8.9) and the best reaction-diffusion norm error.

Theorem 8.4 (Localization with minimal pairs)**.** *For any $u \in H^1(\Omega)$ and any conforming mesh $\mathcal{T} \in \mathbb{T}^c$, it holds*

$$\inf_{v \in S^{\ell,0}(\mathcal{T})} |||u - v|||_\Omega \approx \left[\sum_{K \in \mathcal{T}} e(K) \right]^{1/2},$$

where the hidden constants depend only on d, ℓ, $\mathcal{T}_0$ but not on ε.

Proof. Writing $S := S^{\ell,0}(\mathcal{T})$ for short, the result follows from

$$\inf_{v \in S} |||u - v|||_\Omega \approx \left(\sum_{E \in \mathcal{E}(\mathcal{T})} \inf_{V \in S|_{\omega_\star(E)}} |||u - V|||^2_{\omega_\star(E)} \right)^{1/2} \tag{8.10}$$

where the hidden constants depend on d, ℓ, and the shape parameter $\sigma_{\mathcal{T}}$. In fact, $E(B)$ regroups only the terms of the sum inside the square root of (8.10) and the shape parameter $\sigma_{\mathcal{T}}$ is bounded in terms of the $\sigma_{\mathcal{T}_0}$; see, e.g., [17, Corollary 4.1].

The proof of (8.10) resembles the one of Theorem 5.3 and we restrict ourselves to emphasize the differences.

In order to define the interpolation operator, we fix K_z for every $z \in \mathcal{N} \cap \Sigma$ as before but, for every $K \in \mathcal{T}$, we fix E_K such that $\omega_\star(E_K) \supseteq K$ using Remark 8.2 (iii). The latter allows to choose $\omega = \omega_\star(E_K)$ in the verification of (4.2) but requires to incorporate the faces on the domain boundary $\partial\Omega$ in the localization (8.10); see the second part of Remark 8.2 (iii). The interpolation operator $\Pi^\varepsilon_\star$ is then given by (5.1) where $\mathcal{P}_{E_K}$ is replaced by $\mathcal{R}^\varepsilon_{\star,E_K}$, the best approximation operator associated with $S|_{\omega_\star(E_K)}$ and the reaction-diffusion norm.

We show that $\Pi^\varepsilon_\star$ is locally near best with respect to the covering $\mathcal{W}_\star$ in Remark 8.2 (iv). To this end, we fix $K \in \mathcal{T}$, write $E := E_K$ and choose $\omega = \omega_\star(E)$ in (4.2). Again, we have $|\mathcal{R}^\varepsilon_{\star,E} u(z) - \Pi^\varepsilon_\star u(z)| = 0$ for $z \in \mathcal{N}^\circ_K$ and exploit Proposition 4.3 for $z \in \mathcal{N}_K \cap \Sigma$. For the covering $\mathcal{W}_\star$ however, the construction of the path of subdomains is more involved.

Since $\mathcal{T}$ is face-connected, there exists a sequence $\{K_i\}_{i=1}^r$ of elements of $\mathcal{T}$ such that $K_1 = K$, $K_r = K_z$, and each intersection $K_i \cap K_{i+1} \in \mathcal{E}_\Omega$ is an interelement face containing z. We write $\widetilde{E}_i := K_i \cap K_{i+1}$ for the intersections and, for every element K_i, we choose a face E_{K_i} such that $\omega_\star(E_{K_i}) \supseteq K_i$. We then construct the path $\{\omega_j\}_{j=1}^n := \{\omega_\star(E_j)\}_{j=1}^n$ of subdomains by means of the following algorithm:

$E_1 := E$, $j := 1$
if $E_1 \neq \widetilde{E}_1$ **then**
  $E_2 := \widetilde{E}_1$, $j := j + 1$
endif
for $i = 1, \dots, r-2$ **do**
  if $|\omega_\star(\widetilde{E}_i) \cap \omega_\star(\widetilde{E}_{i+1})| \geq |K_{i+1}|/2$ **then**
    $E_{j+1} := \widetilde{E}_{i+1}$, $j := j + 1$
  else
    $E_{j+1} := E_{K_{i+1}}$, $E_{j+2} := \widetilde{E}_{i+1}$, $j := j + 2$
  endif
endfor
$E_{j+1} := E_{K_r}$

In view of Remark 8.2, this path is admissible for Proposition 4.3 with $\nu = 1/2$. We therefore can follow the lines in the proof of Theorem 5.3, replacing pairs by minimal pairs. Taking into account $\nu = 1/2$, $h_{\omega_\star(E')} \leq h_{\omega_{\mathcal{T}}(E')}$ and the fact that more faces are involved, we derive

$$\sum_{z\in\mathcal{N}_K} |\mathcal{R}^\varepsilon_{\star,E}u(z) - \Pi^\varepsilon_\star u(z)|\|\phi_z\|\!|_K \le \sqrt{2}\mu_{\mathcal{T}} M_\varepsilon \sum_{\widetilde{E}\in\bar\gamma_{\mathcal{T}}(K)} \|\!|\mathcal{R}^\varepsilon_{\star,\widetilde{E}}u - u|\!\|_{\omega_\star(\widetilde{E})}. \tag{8.11}$$

with $\bar\gamma_{\mathcal{T}}(K) := \gamma_{\mathcal{T}}(K) \cup \{E \in \mathcal{E} : E \subset \partial\widetilde{\omega}_{\mathcal{T}}(K)\}$. Since $\#\bar\gamma_{\mathcal{T}}(K)$ and $\#\{K \in \mathcal{T} : E \in \bar\gamma_{\mathcal{T}}(K)\}$ are still bounded in terms of $\bar n$ and d, Proposition 4.2 ensures (8.10). □

The combination of Proposition 8.3, Theorems 8.1 and 8.4 leads to following result involving the best reaction-diffusion norm errors

$$E(u,\mathcal{T})_\varepsilon := \inf_{v\in S^{\ell,0}(\mathcal{T})} \|\!|u-v|\!\|_\Omega, \qquad \mathcal{T}\in\mathbb{T}^c.$$

Theorem 8.5 (Robust near best approximation)**.** *For any $u \in H^1(\Omega)$, the algorithm* TreeApprox *with the error functionals* (8.9) *outputs a conforming mesh $\mathcal{T}_* \in \mathbb{T}^c$ such that*

$$E(u,\mathcal{T}_*)_\varepsilon \le C_1 \inf_{\mathcal{T}\in\mathbb{T}^c:\#\mathcal{T}\le n} E(u,\mathcal{T})_\varepsilon \quad \text{whenever} \quad n \le c_2\#\mathcal{T}_*$$

where the constants C_1 and c_2 depend on d, ℓ, the initial mesh $\mathcal{T}_0$ but not on ε.

Remark 8.6 (Boundary conditions)**.** If $u \in H^1_0(\Omega)$, we may replace the discrete space $S^{\ell,0}(\mathcal{T})$ by the smaller one $S^{\ell,0}_0(\mathcal{T})$ and use the counterparts of (7.1) for the minimal pairs in the definition of (8.9). Proposition 8.3, Theorem 8.4 and therefore Theorem 8.5 continue to hold for these slightly modified error functionals.

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