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A Model-free analysis of discrete time Financial Markets

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Contents

[1]Chapter 1 is based on the paper: Universal Arbitrage Aggregator in Discrete Time Markets under Uncertainty, joint work with M. Frittelli and M. Maggis, forthcoming on *Finance & Stochastics*

[2]Chapter 2 is based on the preprint: Model-free Superhedging duality, joint work with M. Frittelli and M. Maggis, arXiv 1506.06608.

[3]Chapter 3 is based on the working paper: Arbitrage and Hedging in Model Independent Markets with frictions.

Introduction

The *State Preference Model* or *Asset Pricing Model* is the base of any mathematical description of Financial Markets. It postulates that the price of d financial assets is known at a certain initial time $t_0 = 0$ (today), while the price at future times $t > 0$ (tomorrow) is unknown and therefore it is given by a certain random outcome. The natural framework for the formalization of this model is that of Stochastic Analysis. We essentially need to fix a set of events Ω, where any $\omega \in \Omega$ represents a possible state of the world, and for any future time $t \in I$, we need a random vector $S_t : \Omega \to R^d$ which provides the price of the d assets $S_t(\omega)$ if the state of the world ω occurs. Typical examples for the set of future times I are $I = \{0, \ldots, T\}$ (discrete time) and $I = [0, T]$ (continuous time) for a certain fixed $T > 0$ named time horizon. The financial market will be also enriched by the specification of a σ-algebra $\mathcal{F}$ and a filtration $\mathbb{F} := \{\mathcal{F}_t\}_{t\in I} \subseteq \mathcal{F}$ with the requirement that the process $S := (S_t)_{t\in I}$ is $\mathbb{F}$-adapted. The interpretation for this reasonable assumption is that it is not possible to forecast the possible future values of the price process S with the current information available in the market (which is specified by the filtration $\mathbb{F}$). We can therefore assert that a Financial Market model is given by the quadruple $(\Omega, \mathcal{F}, \mathbb{F}, S)$ and we note that, so far, no probability measure is introduced, neither seems required for the specification of the model.
In modern Financial Markets a great variety of securities are traded every day. Most of them are contracts on some underlying assets (e.g. derivative) and the prices for exchanging such contracts are not directly given by the *law of supply and demand*. Mathematical models are therefore developed in order to answer two key questions:

Pricing.: What is a fair price for a traded security according to well established economical principles?

Hedging.: Every trade is connected to some risks arising from unfavourable future events. How does an agent *cover* such possible risks?

These two questions represent the quintessence of Mathematical Finance and the first, fundamental, answers are contained in the so-called *Fundamental Theorem of Asset Pricing* (FTAP) and *Super-hedging duality*.

1. On Fundamental Theorem of Asset Pricing

In a nutshell the Fundamental Theorem of Asset Pricing asserts that any reasonable pricing system must be an expectation under a certain (risk-neutral) probability measure and viceversa. A pricing system is reasonable if it does not admit *arbitrage opportunities* i.e. it is not possible to create a portfolio of financial securities in such a way that the initial investment is zero (or even negative), while the final outcome is always non negative (and in some cases strictly positive). If this was allowed it would be possible to make an arbitrary large profit with no risk. A first intuition for this equivalence can be accredited to De Finetti for his work on *coherence* and *previsions* (see [**deF70**]), while the first systematic approach for understanding the deep relation among no arbitrage pricing and risk-neutral pricing can be found in the work of Ross on Arbitrage Pricing Theory (see e.g. [**Ross76, Ross77**]) and further developed in [**Hu82**]. Later on in the case of Ω being a finite set of events a version of FTAP has been proven by Harrison and Pliska [**HP81**] (see also [**HK79, K81**]) using geometric arguments and separation in finite dimensional spaces.

THEOREM. *Let $\Omega = \{\omega_1, \dots, \omega_n\}$ and let $s = (s^1, \dots, s^d)$ the initial prices of d assets with random outcome $S(\omega) = (S^1(\omega), \dots, S^d(\omega))$ for any $\omega \in \Omega$.*

$$\begin{array}{c} \nexists H \in \mathbb{R}^d \text{ such that } H \cdot s \le 0 \\ \text{and } H \cdot S(\omega) \ge 0 \text{ with } > \text{ for some } \omega \in \Omega \end{array} \iff \begin{array}{c} \exists Q \in \mathfrak{P} \text{ such that } E_Q[S^i] = s^i \\ \text{and } Q(\omega_i) > 0 \ \forall i = 1, \dots, n \end{array} \tag{1}$$

where $\mathfrak{P}$ is the class of probability measures on Ω.

It is immediately clear that in the finite setting no probability measure is needed for the specification of the model since impossible events are automatically excluded from the construction of the state space Ω. On the other hand, linear pricing rules consistent with the observed prices $s^1, \dots s^d$, and not violating the No Arbitrage condition, turn out to be (risk-neutral) probabilities with full support i.e. they assign positive measure to any state of the world. By introducing a *reference probability measure* P with full support and defining an arbitrage as a portfolio with $H \cdot s \le 0$, $P(H \cdot S(\omega) \ge 0) = 1$ and $P(H \cdot S(\omega) > 0) > 0$, the thesis in Theorem 1 can be restated as

$$\text{There is No Arbitrage} \iff \exists Q \sim P \text{ such that } E_Q[S^i] = s^i \quad \forall i = 1, \dots d \tag{2}$$

This identification allows non-trivial extensions of the Fundamental Theorem of Asset Pricing to the case of a general, infinite dimensional Ω. Since it is well known that, on such a space, it is not possible to find a single measure Q with the property that $Q(\{\omega\}) > 0$ for any $\omega \in \Omega$ Theorem 1 cannot hold with Ω being an infinite dimensional space. The extension suggested by (2) is instead possible and it has been proven by Dalang-Morton-Willinger in the celebrated work [**DMW90**], by use of measurable selection arguments. Nevertheless, this apparently innocuous passage, carries out what is, at the matter of facts a model assumption i.e. the choice of a reference probability measure P. This aspect has been recently criticized especially after the recent financial crises: while it is certainly possible to estimate the probability distribution of a certain asset S from historical data, this estimation might be not accurate or, even worse, it might be no longer representative of the stochastic evolution of S due to the prominent dynamic nature of real world markets. For these situations, the unreliability of the measure P opened new and interesting challenges in several branches of Mathematical Finance under the name of Knightian Uncertainty. In particular it has renewed the attention on foundational issues such as option pricing rules and arbitrage conditions which is the main topic of Chapter 1 of this thesis. Along the lines of the previous discussion we can distinguish two extreme cases:

(1) We are completely sure about the reference probability measure P. In this case, the classical notion of No Arbitrage or No Free Lunch with Vanishing Risk can be successfully applied. In discrete markets several different proofs of the Fundamental Theorem of Asset Pricing have been provided after the seminal paper [**DMW90**]. Schachermayer in

[**S92**] proposed a simplified approach based on orthogonality in Hilbert spaces; the key result in the paper is the closure of a certain cone of superreplicable contingent claims with respect to convergence in probability, in the spirit of [**St90**]. A different point of view has been considered by Rogers in [**R94**] who exploited the solution of a utility maximization problem to construct the density of an equivalent martingale measure. Several alternative techniques have been implemented in order to simplify the original proof by avoiding measurable selection arguments such as in [**KK94, KS01b, JS98**]. In continuous time the problem is much more involved and requires a deeper analysis on No Arbitrage conditions as well as the use of sophisticated tools from the general theory of semi-martingales (see e.g. [**DS94, DS98**]).

(2) We face complete uncertainty about any probabilistic model and therefore we must describe our model independently of any probability. In this case we might adopt a model independent (weak) notion of No Arbitrage. A pioneering contribution was given by Hobson in the paper [**Ho98**] where the problem of pricing exotic options is tackled under model mis-specification. In his approach the key assumption is the existence of a martingale measure for the market, consistent with the prices of some observed vanilla options (see also [**BHR01, CO11, DOR14**] for further developments). In [**DH07**], Davis and Hobson relate the previous problem to the absence of Model Independent Arbitrages, by the mean of semi-static strategies. A step forward towards a model-free version of the First Fundamental Theorem of Asset Pricing in discrete time was formerly achieved by Riedel [**Ri15**] in a one period market and by Acciaio at al. [**AB13**] in a more general setup.

Between cases 1. and 2., there is the possibility to accept that the model could be described in a probabilistic setting, but we cannot assume the knowledge of a specific reference probability measure but at most of a set of priors, which leads to the new theory of Quasi-sure Stochastic Analysis as in [**BK12, DHP11, DM06, Pe10, STZ11, STZ11a**]. The idea is that the classical probability theory can be reformulated as far as the single reference probability P is replaced by a class of (possibly non-dominated) probability measures $\mathcal{P}'$. This is the case, for example, of uncertain volatility (e.g. [**STZ11a**]) where, in a general continuous time market model, the volatility is only known to lie in a certain interval $[\sigma_m, \sigma_M]$.
In the theory of arbitrage for non-dominated sets of priors, important results were provided by Bouchard and Nutz [**BN15**] in discrete time. A suitable notion of arbitrage opportunity with respect to a class $\mathcal{P}'$, named $NA(\mathcal{P}')$, was introduced and it was shown that the no arbitrage condition is equivalent to the existence of a family $\mathcal{Q}'$ of martingale measure having the same polar sets of $\mathcal{P}'$. In continuous time markets, a similar topic has been recently investigated also by Biagini et al. [**BBKN14**].

Bouchard and Nutz [**BN15**] answer the following question: which is a good notion of arbitrage opportunity for all **admissible** probabilistic models $P \in \mathcal{P}'$ (i.e. one single H that works as an arbitrage for all admissible models) ? To pose this question one has to know **a priori** which are the admissible models, i.e. we have to exhibit a subset of probabilities $\mathcal{P}'$. On the contrary we want to investigate arbitrage conditions and robustness properties of markets that are described independently of any reference probability or set of priors.

To this aim we introduce a flexible notion of arbitrage that we denominate *Arbitrage de la classe* $\mathcal{S}$ (see Definition 1.1). Since, loosely speaking, an arbitrage opportunity is a a riskless portfolio which yields a positive profit in *some* state of the world (denoted by $\mathcal{V}_H^+ \subseteq \Omega$), in order to formally describe this economical principle we need to specify the meaning of a "riskless portfolio"and that of a "true gain". While, in a model-free setup, the former can be naturally considered as a strategy whose returns are non-negative in any state of the world (i.e. $\forall \omega \in \Omega$), it is less intuitively and more debatable the concept of a true gain. This is exactly the role attributed to the class $\mathcal{S}$. We say that a riskless portfolio is an arbitrage if it yields a strictly positive return on a sufficiently *significant* set of events belonging to $\mathcal{S}$ (i.e. $\mathcal{V}_H^+ \supseteq A$ with $A \in \mathcal{S}$). Several definitions of arbitrage considered in the literature can be seen as a particular case of this general postulate. The strongest notion is obtained with $\mathcal{S} = \{\Omega\}$, meaning that we have a true gain if we can make a profit in any state of the world; the weakest notion is instead given by $\mathcal{S} = \{\text{any non-empty measurable set}\}$, meaning that we can consider a true gain whenever it is achieved for at least one state of the world. In Chapter 1 we provide a model independent version of the Fundamental Theorem of Asset pricing for a generic class $\mathcal{S}$ linking the choice of the class of significant set to the richness of the set of martingale measures. Note that, for a particular choice of the class $\mathcal{S}$, the No Arbitrage assumption does not preclude the existence of riskless portfolios with strictly positive gain on a non-significant set. This situation does not arise in the classical case and it considerably complicates the analysis of the relations between No Arbitrage conditions and existence of martingale measures. A first insight into the problem was formerly given in [**DH07**]: suppose you have two call options C_1, C_2 with the same initial price c_0 but with different strikes $K_2 > K_1$. Anyone would agree that in this market there is an arbitrage opportunity. Observe however that the strategy $C_1 - C_2$ yields a positive gain if the price of the underlying asset ends above K_1 at maturity. On the contrary if an agent is convinced that the price of the underlying will remain below K_1 she would implement a different strategy, namely a short position in one of the two options (since they will never be exercized). In Chapter 1 we formally describe situations where there might be a disagreement on the effective arbitrage strategy and we mathematically treat them by means of a measurable multifunction, that we called *Universal Arbitrage Aggregator* and whose task is precisely to capture all the inefficiencies of the market. This technical tool is the key ingredient that allows us to show a general model-free version of the Fundamental Theorem of Asset Pricing.

2. On Super-hedging duality

As in the classic theory the Fundamental Theorem of Asset Pricing represents the groundwork for a formal option pricing theory based on martingale measures. The existence of this peculiar type of measures and the justification for their use as pricing rules rely on a strong economical basis such as the absence of arbitrage opportunities. When there is exactly one martingale measure there is no doubt on the choice of the pricing rule and we therefore univocally assign a single price to any contingent claim. Markets that exhibit such a behaviour are called *complete* but, in real world situations, this is typically not the case. The reason is two-fold. On one hand agents usually evaluate the same security differently one from another, as it is evident from the existence of spreads between bid and ask prices. On the other hand complete markets are not attractive for investors since when there is agreement on the value of a certain claim there is less room for making profits. In general we therefore have a whole class of prices for the same contingent claim g, corresponding to different possible choice of the pricing martingale measure in $\mathcal{M}$. The Super-hedging duality Theorem relates the supremum of these prices to the cheapest portfolio that gives a payoff at least as good as g (called super-hedging strategy). It is well known that in the classical case the convexity of the set of equivalent martingale measures $\mathcal{M}_e(P)$ guarantees that the set of admissible prices for g is an interval, and it is given by $(\inf_{Q\in\mathcal{M}_e(P)} E_Q[g], \sup_{Q\in\mathcal{M}_e(P)} E_Q[g])$. Outside this interval P-Classical arbitrage opportunities can be obtained. More importantly we have that

$$\sup_{Q\in\mathcal{M}_e(P)} E_Q[g] = \inf\{x \in \mathbb{R} \mid \exists H \in \mathcal{H} \text{ such that } x + (H\cdot S)_T \geq g \ P\text{-a.s.}\}$$

It is therefore natural to pose the question: Can we show an analogous Super-hedging duality Theorem when no reference probability is considered? The relevance of the problem is revealed by the increasing amount of literature on this topic in the last decades. Breeden and Litzenberger in [**BL78**] observed that the prices of some European call options, with the same maturity T, reveal information on the marginal distribution μ of the underlying price process at time T under the risk-neutral measure. This key observation has two consequences. Firstly it enables to price other vanilla derivatives with the same maturity; since these options depend only on the value of the underlying at time T, the knowledge of the distribution μ is sufficient. Secondly it also permit to provide robust bounds for exotic path-dependent options.

The first work in this direction is due to Hobson [**Ho98**] who exploited Skorokhod Embedding Problem techniques in order to find robust bounds for the price of a lookback option. The idea is the following: for a certain path-dependent option the first step is to write a pathwise lower/upper bound by means of a *semi-static* strategy i.e. a linear combination of payoffs of some vanilla options (static part) and dynamic trading in the underlying (dynamic part). Since any martingale measure, compatible with the estimated marginal μ, will assign the same price to these portfolios, the obtained bounds can be legitimate considered model-independent. The second step is to show that they are also tight, meaning that there exist a model for the price process S which is compatible with μ and which attains the boundaries. This is exactly where the Skorokhod Embedding Problem comes into play (see the survey [**Ho11**] or [**Ob10**] and the reference therein for a full account).

Another important stream of research started with the reformulation of the superhedging duality in the framework of the Monge-Kantorovich optimal mass transport. Given two probability space (X_1, μ_1), (X_2, μ_2) the problem amounts to find a "cheap" way of transporting μ_1 to μ_2. Any transport plan is given by a probability measure in the product space $X_1 \times X_2$ with marginals μ_1 and μ_2 while the cost function is specified by a map g. If we now recall that the marginal distribution of the price process S_T can be estimated from market data and that, obviously, the initial price S_0 is observable we can easily identify $\mu_1 := \delta_{S_0}$ and $\mu_2 := \mu$ where δ_x is the Dirac measure centered in x. By defining the cost function as the payoff of a certain path-dependent option g we have that the primal problem corresponds to minimize the expectations of g over the set of probability measures compatible with the estimated marginal. Note that differently from the original Monge-Kantorovich problem it is necessary to impose an additional constraint, namely, that the transport plans need to be martingale measures which complicates the analysis. Nevertheless in many interesting cases the dual problem can be successfully rewritten in terms of a sub-hedging problem, again by means of semi-static strategies. By exploiting the optimal mass transport duality, versions of the superhedging theorem can be obtained both in discrete and continuous time as in [**BHLP13, DS13, DS14b, GHLT14, HL0ST15, HO15, TT13**].

In any of these papers the underlying process S is the canonical process and given a set of vanilla options $\{\Phi_j\}_{j\in J}$ (with no loss of generality with zero-initial price) a semi-static trading strategy is said to be a superhedge for the claim g if its terminal payoff dominates g in *any* state of the world, i.e. the following version of the superhedging problem has been studied

$$\inf\left\{x\in\mathbb{R}\mid \exists (H,h)\in\mathcal{H} \text{ such that } x+(H\cdot S)_T(\omega)+\sum_{j=1}^{k}h_j\Phi_j(\omega)\geq g(\omega)\ \forall\omega\in\Omega\right\} \tag{3}$$

While this requirement appears to be very reasonable from a model-independent point of view, in some cases it turns out to be too restrictive in order to recover a perfect duality respect to the set of No Arbitrage prices given by martingale measures. In Chapter 2 we provide an example of a market where a duality gap appears unless some artificial assumptions are imposed on the payoff of the claim g. Once again this is essentially a consequences of the fact that No Arbitrage conditions in the model-free setup are, in general, compatible with the existence of riskless portfolios with strictly positive gain on certain non-significant sets. While it is certainly true that the set of events where this is possible is negligible it is likewise true that the corresponding set of trajectories for the underlying price process is inefficient so that an agent should not be interested in hedging such a risk. For this reason we propose to weaken the requirement of a pathwise dominating inequality as in (3) with the validity of the same inequality on an efficient set $\Omega_\Phi\subseteq\Omega$. A full description and characterization of this set is given in Chapter 2 where we also show some measurability properties of Ω_Φ. This modification of the problem turns out to be crucial to fill the duality gap and to obtain the validity of a model-free version of the superhedging theorem in a general setting (see Theorem 2.2). A restriction of the set of paths considered for super-replication can also be found in [**HO15**] but it is different in spirit. Differently from our approach this set is not endogenously determined by the market but it is, on the contrary, determined by the modeller whenever she have additional information that allows her to narrow the set of possible scenarios.

3. Models with transaction costs

The last part of the thesis is devoted to the extension of the previous results to the case of a discrete time Model Independent framework when proportional transaction costs are taken into account. The mathematical tools that we employed in the frictionless case are based on measurable selection combined with a geometric point of view. This methodology applies very well to the case of markets with proportional transaction cost which consequently appears to be a natural prosecution of our analysis.

The setting proposed by Kabanov et al. (see e.g. [**KS01a, KRS02**]) based on *solvency cones*, has already a geometric nature and allows the extension of the aforementioned classical result on the Fundamental Theorem of Asset Pricing with Ω finite (see [**HP81**]) to the case of proportional

transaction costs as in [**KS01a**]. The result connects the absence of arbitrage to the existence of a price process with values in the bid-ask spread which is a martingale under a certain risk-neutral probability. This kind of process have been subsequently denominated "Consistent Price Systems" (CPS), by Schachermayer [**S04**] where the equivalence between absence of Arbitrage and existence of CPS has been proven on a general space $(\Omega, \mathcal{F}, P)$. In this paper Schachermayer pointed out that in order to establish this duality the right concept to use is what he called Robust No Arbitrage. This concept formalize the idea that if a market is arbitrage free then, for a sufficiently small reduction of the transaction costs, the market should maintain the arbitrage free property. The same condition is used in the Chapter 3 dropping the reference probability. The possibility of shrinking the bid-ask spread, even for an arbitrary small amount, is crucial in order to avoid undesirable complications. We have already discussed that in the frictionless case different agents might disagree on the effective strategy which realize an arbitrage opportunity. This controversy relies on a delicate linear dependence among the price processes of different assets. When these prices can be perturbarted, by the presence of frictions in the market, this dependence diseappear and the agents not only recognize an arbitrage opportunity but they also agree on the strategy which they should undertake to take advantage of that.
In Chapter 3 we consider the model-free version of the notion of arbitrage introduced in [**S04**] and we provide a Fundamental Theorem of Asset Pricing in this framework. To the best of our knowledge version of this Theorem in this context has not yet been studied. Only a very short literature is indeed available for the robust case, when a class of (possibly non-dominated) set of priors is considered, recent results in this direction are given by [**BZ13, BN14**].
In the second part of Chapter 3 we investigate the superhedging Theorem in the presence of proportional transaction costs. In the classical framework of a fixed probability measure there is a huge literature on this topic (for a non exhaustive list see [**BT00, CK96, CPT99, LS97, K99, SSC95, S14**]. Likewise the case of the Fundamental Theorem of Asset Pricing there are very few results in the model-free case. A first important paper on this topic is given by Dolinsky and Soner [**DS14**] where the case of a discrete time single-asset market is considered with constant proportional transaction costs. By defining a Monge-Kanotorovich optimization problem and exploiting optimal transport techniques the authors succeeded to show that the superhedging price of a path-dependent European option g coincides with the supremum of the expectation of g in the set of proability measure called *approximate martingale measures*. Roughly speaking a probability measure belongs to this set if for any $u \geq t$, the conditional expectation of S_u at time t is contained in the interval $((1-k)S_t, (1+k)S_t)$ where k models the proportional transaction costs. A more recent paper from the same authors [**DS15**] study the case of a continuous time market with one risky asset and with semi-static strategies in some vanilla options allowed. Two hedging problems are considered: in the first one it is required that the super-replication needs to hold for any path in Ω; in the second just in the P-a.s. for a suitable P with conditional full support. Using convex duality techniques they show that the two optimization problems have the same value.
In Chapter 3 we consider the hedging problem in a d-dimensional discrete time market with (random) proportional transaction costs.

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CHAPTER 1

Arbitrage and Martingales[1]

We consider a financial market described by a discrete time adapted stochastic process $S := (S_t)_{t \in I}$, $I = \{0, \dots, T\}$, defined on $(\Omega, \mathcal{F}, \mathbb{F})$, $\mathbb{F} := (\mathcal{F}_t)_{t \in I}$, with $T < \infty$ and taking values in $\mathbb{R}^d$ (see Section 1). Note we are not imposing any restriction on S so that it may describe generic financial securities (for examples, stocks and/or options). Differently from previous approaches in literature, in our setting the measurable space $(\Omega, \mathcal{F})$ and the price process S defined on it are given, and we investigate the properties of martingale measures for S induced by no arbitrage conditions. The class $\mathcal{H}$ of admissible trading strategies is formed by all $\mathbb{F}$-predictable d-dimensional stochastic processes and we denote with $\mathcal{M}$ the set of all probability measures under which S is an $\mathbb{F}$-martingale and with $\mathcal{P}$ the set of all probability measures on $(\Omega, \mathcal{F})$. We introduce a flexible definition of Arbitrage which allows us to characterize the richness of the set $\mathcal{M}$ in a unified framework.

Arbitrage *de la classe* $\mathcal{S}$. Let:

$$\mathcal{V}_H^+ = \{\omega \in \Omega \mid \ V_T(H)(\omega) > 0\},$$

where $V_T(H) = \sum_{t=1}^T H_t \cdot (S_t - S_{t-1})$ is the final value of the strategy H. It is natural to introduce several notion of Arbitrage accordingly to the properties of the set $\mathcal{V}_H^+$.

Definition 1.1. *Let $\mathcal{S}$ be a class of measurable subsets of Ω such that $\varnothing \notin \mathcal{S}$. A trading strategy $H \in \mathcal{H}$ is an Arbitrage de la classe $\mathcal{S}$ if*

- $V_0(H) = 0$, $V_T(H)(\omega) \geq 0 \ \forall \omega \in \Omega$ *and* $\mathcal{V}_H^+$ *contains a set in* $\mathcal{S}$.

The class $\mathcal{S}$ has the role to translate mathematically the meaning of a "true gain". When a probability P is given (the "reference probability") then we agree on representing a true gain as $P(V_T(H) > 0) > 0$ and therefore the classical no arbitrage condition can be expressed: no losses $P(V_T(H) < 0) = 0$ implies no true gain $P(V_T(H) > 0) = 0$. In a similar fashion, when a subset $\mathcal{P}'$ of probability measures is given, one may replace the P-a.s. conditions above with $\mathcal{P}$-q.s conditions, as in [**BN15**]. However, if we can not or do not want to rely on a set of probability measures a priori assigned, we may well use another concept: there is a true gain if $\mathcal{V}_H^+$ contains a set considered *significant*. This is exactly the role attributed to the class $\mathcal{S}$ which is the core of Section 2. Families of sets, not determined by some probability measures, have been already used in the context of the first and second Fundamental Theorem of Asset Pricing respectively by Battig Jarrow [**BJ99**] and Cassese [**C08**] (see Section 3.1 for a more specific comparison).

[1]Chapter 1 is based on the paper: Universal Arbitrage Aggregator in Discrete Time Markets under Uncertainty, joint work with M. Frittelli and M. Maggis, forthcoming on *Finance & Stochastics*

In order to investigate the properties of the martingale measures induced by No Arbitrage conditions of this kind we first study (see Section 3) the structural properties of the market adopting a geometrical approach in the spirit of [**HP81**] but with Ω being a general Polish space, instead of a finite sample space. In particular, we characterize the class $\mathcal{N}$ of the $\mathcal{M}$-polar sets i.e. those $N \subset \Omega$ such that there is no martingale measure that can assign a positive measure to N. In the model independent framework the set $\mathcal{N}$ is induced by the market since the set of martingale measure has not to withstand to any additional condition (such as being equivalent to a certain P). Once these polar sets are identified we explicitly build in Section 3.6 a process $H^{\bullet}$ which depends only on the price process S and satisfies:

- $V_T(H^{\bullet})(\omega) \geq 0 \ \forall \omega \in \Omega$
- $N \subseteq \mathcal{V}^+_{H^{\bullet}}$ for every $N \in \mathcal{N}$.

This strategy is a measurable selection of a set valued process $\mathbb{H}$, that we baptize **Universal Arbitrage Aggregator** since for any P, which is not absolutely continuous with respect to $\mathcal{M}$, an arbitrage opportunity H^P (in the classical sense) can be found among the values of $\mathbb{H}$. All the inefficiencies of the market are captured by the process $H^{\bullet}$ but, in general, it fails to be $\mathbb{F}$-predictable. To recover predictability we need to enlarge the natural filtration of the process S by considering a suitable technical filtration $\widetilde{\mathbb{F}} := \{\widetilde{\mathcal{F}}_t\}_{t \in I}$ which does not affect the set of martingale measures, i.e. any martingale measure $Q \in \mathcal{M}$ can be uniquely extended to a martingale measure $\widetilde{Q}$ on the enlarged filtration.

This allows us to prove, in Section 3.6, the main result of the Chapter:

THEOREM 1.2. *Let $(\Omega, \widetilde{\mathcal{F}}_T, \widetilde{\mathbb{F}})$ be the enlarged filtered space as in Section 3.5 and let $\widetilde{\mathcal{H}}$ be the set of d-dimensional discrete time $\widetilde{\mathbb{F}}$-predictable stochastic process. Then*

$$\textit{No Arbitrage de la classe } \mathcal{S} \textit{ in } \widetilde{\mathcal{H}} \Leftrightarrow \mathcal{M} \neq \varnothing \textit{ and } \mathcal{N} \textit{ does not contain sets of } \mathcal{S}$$

In other words, properties of the family $\mathcal{S}$ have a dual counterpart in terms of polar sets of the pricing functional.

In Section 3.6 we further provide our version of the Fundamental Theorem of Asset Pricing: the equivalence between absence of Arbitrage de la classe $\mathcal{S}$ in $\widetilde{\mathcal{H}}$ and the existence of martingale measures $Q \in \mathcal{M}$ with the property that $Q(C) > 0$ for all $C \in \mathcal{S}$.

Model Independent Arbitrage. When $\mathcal{S} := \{\Omega\}$ then the Arbitrage de la classe $\mathcal{S}$ corresponds to the notion of a Model Independent Arbitrage. As Ω never belongs to the class of polar sets $\mathcal{N}$, from Theorem 1.2 we directly obtain the following result.

THEOREM 1.3.

$$\textit{No Model Independent Arbitrage in } \widetilde{\mathcal{H}} \Longleftrightarrow \mathcal{M} \neq \varnothing.$$

An analogous result has been obtained in [**AB13**] when considering a single risky asset S as the canonical process on the path space $\Omega = \mathbb{R}^T_+$, a possibly uncountable collection of options $(\varphi_\alpha)_{\alpha \in A}$ whose prices are known at time 0, and when trading is possible through semi-static strategies (see also [**Ho11**] for a detailed discussion). Assuming the existence of an option φ_0 with a specific payoff, equivalence in Theorem 1.3 is achieved in the original measurable space $(\Omega, \mathcal{F}, \mathbb{F}, \mathcal{H})$. In our setup, although we are free to choose a $(d+k)$-dimensional process S for modeling a finite number of options (k) on possibly different underlying (d), the class $\widetilde{\mathcal{H}}$ of admissible strategies are dynamic in every S^i for $i = 1, \dots d + k$. In order to incorporate the case of semi-static strategies we would need to consider restrictions on $\widetilde{\mathcal{H}}$ and for this reason the two results are not directly comparable.

Arbitrage with respect to open sets. In the topological context, in order to obtain full support martingale measures, the suitable choice for $\mathcal{S}$ is the class of open sets. This selection determines the notion of Arbitrage with respect to open sets, which we shorten as "Open Arbitrage":

- **Open Arbitrage** is an admissible trading strategy H such that $V_0(H) = 0$, $V_T(H)(\omega) \geq 0\ \forall \omega \in \Omega$ and $\mathcal{V}_H^+$ contains an open set.

This concept admits the following dual reformulation (see Section 5, Proposition 1.64). An Open Arbitrage consists in a trading strategy $H \in \mathcal{H}$ and a non empty weakly open set $\mathcal{U} \subseteq \mathcal{P}$ such that

$$\text{for all } P \in \mathcal{U},\ V_T(H) \geq 0\ P\text{-a.s. and } P(\mathcal{V}_H^+) > 0. \tag{4}$$

The robust feature of an open arbitrage is therefore evident from this dual formulation, as a certain strategy H satisfies (4) if it represents an arbitrage in the classical sense for a whole open set of probabilities. In addition, if H is such strategy and we disregard any finite subset of probabilities then H remains an Open Arbitrage. Moreover every weakly open subset of $\mathcal{U}$ contains a full support probability P (see Lemma 1.57) under which H is a P-Arbitrage in the classical sense. Full support martingale measures can be efficiently used whenever we face model mis-specification, since they have a well spread support that captures the features of the sample space of events without neglecting significantly large parts. In Dolinski and Soner [**DS14**] the equivalence of a local version of NA and the existence of full support martingale measures has been proven (see Section 2.5, [**DS14**]) in a continuous time market determined by one risky asset with proportional transaction costs.

Feasibility and approximating measures. In Section 4 we answer the question: which are the markets that are feasible in the sense that the properties of the market are nice for "**most**" probabilistic models? Clearly this problem depends on the choice of the feasibility criterion, but to this aim we do not need to exhibit a priori a subset of probabilities. On the opposite, given a market (described without reference probability), the induced set of No Arbitrage models (probabilities) for that market will determine if the market itself is feasible or not. What is needed here is a good notion of "**most**" probabilistic models.
More precisely given the price process S defined on $(\Omega, \mathcal{F})$, we introduce the set $\mathcal{P}_0$ of probability measures that exhibit No Arbitrage in the classical sense:

$$\mathcal{P}_0 = \{P \in \mathcal{P} \mid \text{No Arbitrage with respect to } P\}. \tag{5}$$

When

$$\overline{\mathcal{P}_0}^{\tau} = \mathcal{P}$$

with respect to some topology τ the market is feasible in the sense that any "bad" reference probability can be approximated by No Arbitrage probability models. We show in Proposition 1.58 that this property is equivalent to the existence of a full support martingale measure if we choose τ as the weak* topology.

One other contribution of this thesis, proven in Section 4, is the following characterization of feasible markets and absence of Open Arbitrage in terms of existence of full support martingale measures. We denote with $\mathcal{P}_+ \subset \mathcal{P}$ the set of full support probability measures.

THEOREM 1.4. *The following are equivalent:*

(1) The market is feasible, i.e $\overline{\mathcal{P}_0}^{\sigma(\mathcal{P},C_b)} = \mathcal{P}$*;*
(2) There exists $P \in \mathcal{P}_+$ *s.t. No Arbitrage w.r.to* P *(in the classical sense) holds true;*
(3) $\mathcal{M} \cap \mathcal{P}_+ \neq \varnothing$*;*
(4) No Open Arbitrage holds with respect to admissible strategies $\widetilde{\mathcal{H}}$*.*

Riedel [**Ri15**] already pointed out the relevance of the concept of full support martingale measures in a probability-free set up. Indeed in a one period market model and under the assumption that the price process is continuous with respect to the state variable, he showed that the absence of a one point arbitrage (non-negative payoff, with strict positivity in at least one point) is equivalent to the existence of a full support martingale measure. As shown in Section 5.1, this equivalence is no longer true in a multiperiod model (or in a single period model with non trivial initial σ-algebra), even for price processes continuous in ω. In this Chapter we consider a multi-assets multi-period model without ω-continuity assumptions on the price processes and we develop the concept of open arbitrage, as well as its dual reformulation, that allows for the equivalence stated in the above theorem.
Finally, we present a number of simple examples that point out: the differences between single period and multi-period models (examples 1.13, 1.66, 1.67); the geometric approach to absence of arbitrage and existence of martingale measures (Section 3.1); the need in the multi-period setting of the disintegration of the atoms (example 1.26); the need of the one period anticipation of some polar sets (example 1.32).

1. Financial Markets

We will assume that (Ω, d) is a Polish space and $\mathcal{F} = \mathcal{B}(\Omega)$ is the Borel σ-algebra induced by the metric d. The requirement that Ω is Polish is used in Section 3.3 to guarantee the existence of a proper regular conditional probability, see Theorem 1.28. We fix a finite time horizon $T \geq 1$, a finite set of time indices $I := \{0, \dots, T\}$ and we set: $I_1 := \{1, \dots, T\}$. Let $\mathbb{F} := \{\mathcal{F}_t\}_{t \in I}$ be a filtration with $\mathcal{F}_0 = \{\varnothing, \Omega\}$ and $\mathcal{F}_T \subseteq \mathcal{F}$. We denote with $\mathcal{L}(\Omega, \mathcal{F}_t; \mathbb{R}^d)$ the set of $\mathcal{F}_t$-measurable random variables $X : \Omega \to \mathbb{R}^d$ and with $\mathcal{L}(\Omega, \mathbb{F}; \mathbb{R}^d)$ the set of adapted processes $X = (X_t)_{t \in I}$ with $X_t \in \mathcal{L}(\Omega, \mathcal{F}_t; \mathbb{R}^d)$.
The market consists of one non-risky asset $S^0_t = 1$ for all $t \in I$, constantly equal to 1, and $d \geq 1$ risky assets $S^j = (S^j_t)_{t \in I}$, $j = 1, \dots, d$, that are real-valued adapted stochastic processes. Let $S = [S^1, \dots, S^d] \in \mathcal{L}(\Omega, \mathbb{F}; \mathbb{R}^d)$ be the d-dimensional vector of the (discounted) price processes.
In this Chapter we focus on arbitrage conditions, and therefore without loss of generality we will restrict our attention to self-financing trading strategies of zero initial cost. Therefore, we may assume that a trading strategy $H = (H_t)_{t \in I_1}$ is an $\mathbb{R}^d$-valued predictable stochastic process: $H = [H^1, \dots, H^d]$, with $H_t \in \mathcal{L}(\Omega, \mathcal{F}_{t-1}; \mathbb{R}^d)$, and we denote with $\mathcal{H}$ the class of all trading

strategies. The (discounted) value process $V(H) = (V_t(H))_{t\in I}$ is defined by:

$$V_0(H) := 0, \quad V_t(H) := \sum_{i=1}^{t} H_i \cdot (S_i - S_{i-1}), \quad t \geq 1.$$

A (discrete time) financial market is therefore assigned, without any reference probability measure, by the quadruple $[(\Omega, d); (\mathcal{B}(\Omega), \mathbb{F}); S; \mathcal{H}]$ satisfying the previous conditions.

NOTATION 1.5. *For $\mathcal{F}$-measurable random variables X and Y, we write $X > Y$ (resp. $X \geq Y$, $X = Y$) if $X(\omega) > Y(\omega)$ for all $\omega \in \Omega$ (resp. $X(\omega) \geq Y(\omega)$, $X(\omega) = Y(\omega)$ for all $\omega \in \Omega$).*

1.1. Probability and martingale measures. Let $\mathcal{P} := \mathcal{P}(\Omega)$ be the set of all probabilities on $(\Omega, \mathcal{F})$ and $C_b := C_b(\Omega)$ the space of continuous and bounded functions on Ω. Except when explicitly stated, we endow $\mathcal{P}$ with the weak* topology $\sigma(\mathcal{P}, C_b)$, so that $(\mathcal{P}, \sigma(\mathcal{P}, C_b))$ is a Polish space (see [**AB06**] Chapter 15 for further details). The convergence of P_n to P in the topology $\sigma(\mathcal{P}, C_b)$ will be denoted by $P_n \overset{w}{\to} P$ and the $\sigma(\mathcal{P}, C_b)$ closure of a set $\mathcal{Q} \subseteq \mathcal{P}$ will be denoted with $\overline{\mathcal{Q}}$.

We define the *support* of an element $P \in \mathcal{P}$ as

$$supp(P) = \bigcap \{C \in \mathcal{C} \mid P(C) = 1\}$$

where $\mathcal{C}$ are the closed sets in (Ω, d). Under our assumptions the support is given by

$$supp(P) = \{\omega \in \Omega \mid P(B_\varepsilon(\omega)) > 0 \text{ for all } \varepsilon > 0\},$$

where $B_\varepsilon(\omega)$ is the open ball with radius ε centered in ω.

DEFINITION 1.6. *We say that $P \in \mathcal{P}$ has full support if $supp(P) = \Omega$ and we denote with*

$$\mathcal{P}_+ := \{P \in \mathcal{P} \mid supp(P) = \Omega\}$$

the set of all probability measures having full support.

Observe that $P \in \mathcal{P}_+$ if and only if $P(A) > 0$ for every open set A. Full support measures are therefore important, from a topological point of view, since they assign positive probability to all open sets.

DEFINITION 1.7. *The set of $\mathbb{F}$-martingale measures is*

$$\mathcal{M}(\mathbb{F}) = \{Q \in \mathcal{P} \mid S \text{ is a } (Q, \mathbb{F})\text{-martingale}\}. \tag{6}$$

and we set: $\mathcal{M} := \mathcal{M}(\mathbb{F})$, when the filtration is not ambiguous, and

$$\mathcal{M}_+ = \mathcal{M} \cap \mathcal{P}_+.$$

DEFINITION 1.8. *Let $P \in \mathcal{P}$ and $\mathcal{G} \subseteq \mathcal{F}$ be a sub σ-algebra of $\mathcal{F}$. The generalized conditional expectation of a non negative $X \in \mathcal{L}(\Omega, \mathcal{F}, \mathbb{R})$ is defined by:*

$$E_P[X \mid \mathcal{G}] := \lim_{n\to+\infty} E_P[X \wedge n \mid \mathcal{G}],$$

*and for $X \in \mathcal{L}(\Omega, \mathcal{F}, \mathbb{R})$ we set $E_P[X \mid \mathcal{G}] := E_P[X^+ \mid \mathcal{G}] - E_P[X^- \mid \mathcal{G}]$, where we adopt the convention $\infty - \infty = -\infty$. All basic properties of the conditional expectation still hold true (see for example [**FKV09**]). In particular if $Q \in \mathcal{M}$ and $H \in \mathcal{H}$ then $E_Q[H_t \cdot (S_t - S_{t-1}) \mid \mathcal{F}_{t-1}] = H_t \cdot E_Q[(S_t - S_{t-1}) \mid \mathcal{F}_{t-1}] = 0$ Q-a.s., so that $E_Q[V_T(H)] = 0$ Q-a.s.*

2. Arbitrage de la classe $\mathcal{S}$

Let $H \in \mathcal{H}$ and recall that $\mathcal{V}_H^+ := \{\omega \in \Omega \mid V_T(H)(\omega) > 0\}$ and that $V_0(H) = 0$.

DEFINITION 1.9. *Let $P \in \mathcal{P}$. A P-**Classical Arbitrage** is a trading strategy $H \in \mathcal{H}$ such that:*

- $V_T(H) \geq 0$ *$P-$a.s., and* $P(\mathcal{V}_H^+) > 0$

We denote with $NA(P)$ the absence of P-Classical Arbitrage.

Recall the definition of Arbitrage de la classe $\mathcal{S}$ stated in the Introduction.

DEFINITION 1.10. *Some examples of Arbitrage de la classe $\mathcal{S}$:*

(1) H is a $1p$-Arbitrage when $\mathcal{S} = \{C \in \mathcal{F} \mid C \neq \varnothing\}$. This is the weakest notion of arbitrage since $\mathcal{V}_H^+$ might reduce to a single point. The $1p$-Arbitrage corresponds to the definition given in [**Ri15**]*. This can be easily generalized to the following notion of n point Arbitrage: H is an np-Arbitrage when*

$$\mathcal{S} = \{C \in \mathcal{F} \mid C \text{ has at least } n \text{ elements}\},$$

and might be significant for Ω (at most) countable.

(2) H is an Open Arbitrage when $\mathcal{S} = \{C \in \mathcal{B}(\Omega) \mid C \text{ open non-empty}\}$.

(3) H is a $\mathcal{P}'$-q.s. Arbitrage when $\mathcal{S} = \{C \in \mathcal{F} \mid P(C) > 0 \text{ for some } P \in \mathcal{P}'\}$, for a fixed family $\mathcal{P}' \subseteq \mathcal{P}$. Notice that $\mathcal{S} = (\mathcal{N}(\mathcal{P}'))^c$, the complements of the polar sets of $\mathcal{P}'$. Then there are No $\mathcal{P}'$-q.s. Arbitrage if:

$$H \in \mathcal{H} \text{ such that } V_T(H)(\omega) \geq 0 \ \forall \omega \in \Omega \Rightarrow V_T(H) = 0 \ \mathcal{P}'\text{-q.s.}$$

This definition is similar to the No Arbitrage condition in [**BN15**]*, the only difference being that here we require $V_T(H)(\omega) \geq 0 \ \forall \omega \in \Omega$, while in the cited reference it is only required $V_T(H) \geq 0$ $\mathcal{P}'$-q.s.. Hence No $\mathcal{P}'$-q.s. Arbitrage is a condition weaker than No Arbitrage in* [**BN15**].

(4) H is a P-a.s. Arbitrage when $\mathcal{S} = \{C \in \mathcal{F} \mid P(C) > 0\}$ for a fixed $P \in \mathcal{P}$. As in the previous example the No P-a.s. Arbitrage is a weaker condition than the No P-Classical Arbitrage condition, the only difference being that here we require $V_T(H)(\omega) \geq 0 \ \forall \omega \in \Omega$, while in the classical definition it is only required $V_T(H) \geq 0$ P-a.s.

(5) H is a Model Independent Arbitrage when $\mathcal{S} = \{\Omega\}$, in the spirit of [**AB13, DH07, CO11**].

(6) H is an ε-Arbitrage when $\mathcal{S} = \{C_\varepsilon(\omega) \mid \omega \in \Omega\}$, where $\varepsilon > 0$ is fixed and $C_\varepsilon(\omega)$ is the closed ball in (Ω, d) of radius ε and centered in ω.

Obviously, for any class $\mathcal{S}$,

$$\text{No } 1p\text{-Arb.} \Rightarrow \text{No Arb. de la classe } \mathcal{S} \Rightarrow \text{No Model Ind. Arb.} \tag{7}$$

and these notions depend only on the properties of the financial market and are not necessarily related to any probabilistic models.

REMARK 1.11. *The No Arbitrage concepts defined above, as well as the possible generalization of No Free Lunch de la classe $\mathcal{S}$, can be considered also in more general, continuous time, financial* *market models. We choose to present our theory in the discrete time framework, as the subsequent results in the next sections will rely crucially on the discrete time setting.*

EXAMPLE 1.12. *The flexibility of our approach relies on the arbitrary choice of the class $\mathcal{S}$. Consider $\Omega = C^0([0,T];\mathbb{R})$ which is a Polish space once endowed with the supremum norm $\|\cdot\|_\infty$. We may consider two classes*

$$\mathcal{S}^\infty = \{\text{ open balls in } \|\cdot\|_\infty\} \quad \text{and} \quad \mathcal{S}^1 = \{\text{ open balls in } \|\cdot\|_1\}$$

where $\|\omega\|_1 = \int_0^T |\omega(t)|dt$. Notice that since the integral operator $\int_0^T |\cdot|dt : C^0([0,T];\mathbb{R}) \to \mathbb{R}$ is $\|\cdot\|_\infty$-continuous every open ball in $\|\cdot\|_1$ is also open in $\|\cdot\|_\infty$. Hence every Arbitrage de la classe $\mathcal{S}^1$ is also an Arbitrage de la classe $\mathcal{S}^\infty$ but not the converse.
For instance consider a market described by an underlying process S^1 and a digital option S^2, where trading is allowed only in a set of finite times $\{0,1,\dots,T-1\}$. Define $S_0^1(\omega) = s_0$ for every $\omega \in \Omega$ and $S_t^1(\omega) = \omega(t)$ for the underlying and $S_t^2(\omega) = \mathbf{1}_B(\omega)\mathbf{1}_T(t)$ for the option where the set B is given by $B := \{\omega \mid S_t^1(\omega) \in (s_0 - \varepsilon, s_0 + \varepsilon)\ \forall t \in [0,T]\}$. A long position in the option at time $T-1$ is an arbitrage de la classe $\mathcal{S}^\infty$ even though there does not exist any arbitrage de la classe $\mathcal{S}^1$.

2.1. Defragmentation. When the reference probability $P \in \mathcal{P}$ is fixed, the market admits a P-Classical Arbitrage if and only if there exists $t \in \{1,\dots,T\}$ and a random vector $\eta \in L^0(\Omega,\mathcal{F}_{t-1},P;\mathbb{R}^d)$ such that $\eta \cdot (S_t - S_{t-1}) \geq 0$ P-a.s. and $P(\eta \cdot (S_t - S_{t-1}) > 0) > 0$ (see [**DMW90**] or [**FS04**], Proposition 5.11). In our context the existence of an Arbitrage de la classe $\mathcal{S}$, over a certain time interval $[0,T]$, does not necessarily imply the existence of a single time step where the arbitrage is realized on a set in $\mathcal{S}$. It might happen, instead, that the agent needs to implement a strategy over multiple time steps to achieve an arbitrage de la classe $\mathcal{S}$. The following example shows exactly a simple case in which this phenomenon occurs. Recall that $\mathcal{L}(\Omega,\mathcal{F};\mathbb{R}^d)$ is the set of $\mathbb{R}^d$-valued $\mathcal{F}$-measurable random variables on Ω.

EXAMPLE 1.13. *Consider a 2 periods market model composed by two risky assets S^1, S^2 on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$ which are described by the following trajectories*

$$S^1 : \begin{array}{cccccc} & & 3 \rightarrow & 3 & \omega \in A_1 \\ & \nearrow & & 5 & \omega \in A_2 \\ & & & \nearrow & & \\ 2 \rightarrow & & 2 & & & \\ & & & \searrow & & \\ & \searrow & & 1 & \omega \in A_3 \\ & & 1 \rightarrow & 1 & \omega \in A_4 \end{array} \qquad S^2 : \begin{array}{cccccc} & & 7 \rightarrow & 7 & \omega \in A_1 \\ & \nearrow & & 3 & \omega \in A_2 \\ & & & \nearrow & & \\ 2 \rightarrow & & 2 & & & \\ & & & \searrow & & \\ & \searrow & & 1 & \omega \in A_3 \\ & & 1 \rightarrow & 1 & \omega \in A_4 \end{array}$$

Consider $H_1 = (-1,+1)$ and $H_2 = (\mathbf{1}_{A_2 \cup A_3}, -\mathbf{1}_{A_2 \cup A_3})$.
Then $H_1 \cdot (S_1 - S_0) = 4\mathbf{1}_{A_1}$ and $H_2 \cdot (S_2 - S_1) = 2\mathbf{1}_{A_2}$. Choosing $A_1 = \mathbb{Q} \cap (0,1)$, $A_2 = (\mathbb{R} \setminus \mathbb{Q}) \cap (0,1)$ and $A_3 = [1,+\infty)$, $A_4 = (-\infty,0]$ we observe that an Open Arbitrage can be obtained only by a two step strategy, while in each step we have only 1*p-Arbitrages.*
In general the multi step strategy realizes the Arbitrage de la classe $\mathcal{S}$ at time T even though it does not yield necessarily a positive gain at each time: i.e. there might exist a $t < T$ such that $\{V_t(H) < 0\} \neq \varnothing$. This is the case of Example 1.32.

In the remaining of this section $\Delta S_t = [S_t^1 - S_{t-1}^1, \dots, S_t^d - S_{t-1}^d]$.

LEMMA 1.14. *The strategy $H \in \mathcal{H}$ is a 1p-Arbitrage if and only if there exists a time $t \in I_1$, an $\alpha \in \mathcal{L}(\Omega, \mathcal{F}_{t-1}; \mathbb{R}^d)$ and a non empty $A \in \mathcal{F}_t$ such that*

$$\begin{aligned} \alpha(\omega) \cdot \Delta S_t(\omega) &\geq 0 \qquad \forall\, \omega \in \Omega \\ \alpha(\omega) \cdot \Delta S_t(\omega) &> 0 \qquad \text{on } A. \end{aligned} \tag{8}$$

PROOF. ($\Rightarrow$) Let $H \in \mathcal{H}$ be a $1p$-Arbitrage. Set

$$\bar{t} = \min\{t \in \{1, \dots, T\} \mid V_t(H) \geq 0 \text{ with } V_t(H)(\omega) > 0 \text{ for some } \omega \in \Omega\}.$$

If $\bar{t} = 1$, $\alpha = H_1$ satisfies the requirements. If $\bar{t} > 1$, $\{V_{\bar{t}-1}(H) < 0\} \neq \varnothing$ or $\{V_{\bar{t}-1}(H) = 0\} = \Omega$. In the first case, for $\alpha = H_{\bar{t}} \mathbf{1}_{\{V_{\bar{t}-1}(H) < 0\}}$ we have $\alpha \cdot \Delta S_{\bar{t}} \geq 0$ with strict inequality on $\{V_{\bar{t}-1}(H) < 0\}$. In the latter case $\alpha = H_{\bar{t}}$ satisfies the requirements.
($\Leftarrow$) Take $\alpha \in \mathcal{L}(\Omega, \mathcal{F}_{t-1}; \mathbb{R}^d)$ as by assumption and define $H \in \mathcal{H}$ by $H_s = 0$ for every $s \neq t$ and $H_t = \alpha$. Hence $V_T(H) = V_t(H)$ so that $V_T(H) \geq 0$. Note that $\{\omega \in \Omega \mid V_T(H)(\omega) > 0\} = \{\omega \in \Omega \mid \alpha \cdot \Delta S_t(\omega) > 0\}$ and the proof is complete. ☐

REMARK 1.15. *Notice that only the implication ($\Leftarrow$) of the previous Lemma holds true for Open Arbitrage. This means that there exists an Open Arbitrage if we can find a time $t \in I_1$, an $\alpha \in \mathcal{L}(\Omega, \mathcal{F}_{t-1}; \mathbb{R}^d)$ and a set $A \in \mathcal{F}_t$ containing an open set such that (8) holds true. Similarly for Arbitrage de la classe $\mathcal{S}$. On the other hand the converse is false in general as shown by Example 1.13.*

The following Lemma provides a full characterization of Arbitrages de la classe $\mathcal{S}$ by the mean of a multi-step decomposition of the strategy.

LEMMA 1.16 (Defragmentation). *The strategy $H \in \mathcal{H}$ is an Arbitrage de la classe $\mathcal{S}$ if and only if there exists:*

- *a finite family $\{U_t\}_{t \in I}$ with $U_t \in \mathcal{F}_t$, $U_t \cap U_s = \varnothing$ for every $t \neq s$ and $\bigcup_{t \in I} U_t$ contains a set in $\mathcal{S}$;*
- *a strategy $\widehat{H} \in \mathcal{H}$ such that $V_T(\widehat{H}) \geq 0$ on Ω, and $\widehat{H}_t \cdot \Delta S_t > 0$ on U_t for any $U_t \neq \varnothing$.*

PROOF. ($\Rightarrow$) Let $H \in \mathcal{H}$ be an Arbitrage de la classe $\mathcal{S}$. Define $B_t = \{V_t(H) > 0\}$ and

$$\begin{aligned} U_1 = B_1 \quad &\Rightarrow \quad H_1 \cdot \Delta S_1(\omega) > 0 \quad \forall\, \omega \in U_1 \\ U_2 = B_1^c \cap B_2 \quad &\Rightarrow \quad H_2 \cdot \Delta S_2(\omega) > 0 \quad \forall\, \omega \in U_2 \\ U_{T-1} = B_1^c \cap \ldots \cap B_{T-2}^c \cap B_{T-1} \quad &\Rightarrow \quad H_{T-1} \cdot \Delta S_{T-1}(\omega) > 0 \quad \forall\, \omega \in U_{T-1} \\ U_T = B_1^c \cap \ldots \cap B_{T-2}^c \cap B_{T-1}^c \cap \mathcal{V}_H^+ \quad &\Rightarrow \quad H_T \cdot \Delta S_T(\omega) > 0 \quad \forall\, \omega \in U_T \end{aligned}$$

From the definition of $\{U_1, U_2, \dots, U_T\}$ we have that $\mathcal{V}_H^+ \subseteq \bigcup_{i=1}^T U_i$. Set $\widehat{H}_1 = H_1$ and consider the strategy for every $2 \leq t \leq T$ given by

$$\widehat{H}_t(\omega) = H_t(\omega) \mathbf{1}_{D_{t-1}}(\omega) \quad \text{where } D_{t-1} = \left(\bigcup_{s=1}^{t-1} U_s \right)^c .$$

By construction $\widehat{H} \in \mathcal{H}$ and $\widehat{H}_t \cdot \Delta S_t(\omega) > 0$ for every $\omega \in U_t$.
($\Leftarrow$) The converse implication is trivial. ☐

3. Arbitrage de la classe $\mathcal{S}$ and Martingale Measures

Before addressing this topic in its full generality we provide some insights into the problem and we introduce some examples that will help to develop the intuition on the approach that we adopt. The required technical tools will then be stated in Sections 3.2 and 3.3.

Consider the family of polar sets of $\mathcal{M}$

$$\mathcal{N} := \{A \subseteq A' \in \mathcal{F} \mid Q(A') = 0 \; \forall \; Q \in \mathcal{M}\}.$$

In Nutz and Bouchard [**BN15**] the notion of $NA(\mathcal{P}')$ for any fixed family $\mathcal{P}' \subseteq \mathcal{P}$ is defined by:

$$V_T(H) \geq 0 \; \mathcal{P}' - q.s. \Rightarrow V_T(H) = 0 \; \mathcal{P}' - q.s.$$

where H is a predictable process which is measurable with respect to the universal completion of $\mathbb{F}$. One of the main results in [**BN15**] asserts that, under $NA(\mathcal{P}')$, there exists a class $\mathcal{Q}'$ of martingale measures which shares the same polar sets of $\mathcal{P}'$. If we take $\mathcal{P}' = \mathcal{P}$ then $NA(\mathcal{P})$ is equivalent to No (universally measurable) $1p$-Arbitrage, since $\mathcal{P}$ contains all Dirac measures. In addition, the class of polar sets of $\mathcal{P}$ is empty. In Section 3.4 we will show that this same result is true also in our setting as a consequence of Proposition 1.34. The existence of a class of martingale measures with no polar sets implies that $\forall \omega \in \Omega$ there exists $Q \in \mathcal{M}$ such that $Q(\{\omega\}) > 0$ and since Ω is a separable space we can find a dense set $D := \{\omega_n\}_{n=1}^{\infty}$, with associated $Q^n \in \mathcal{M}$, such that $\sum_{n=1}^{\infty} \frac{1}{2^n} Q^n$ is a full support martingale measure (see Lemma 1.76).

Proposition 1.17. *We have the following implications*

(1) *No $1p$-Arbitrage $\Longrightarrow \mathcal{M}_+ \neq \emptyset$.*
(2) *$\mathcal{M}_+ \neq \emptyset \Longrightarrow$ No Open Arbitrage.*

Proof. The proof of 1. is postponed to Section 3.4.
We prove 2. by observing that for any open set O and $Q \in \mathcal{M}_+$ we have $Q(O) > 0$. Since for any $H \in \mathcal{H}$ such that $V_T(H) \geq 0$ we have $Q(\mathcal{V}_H^+) = 0$, then $\mathcal{V}_H^+$ does not contain any open set. □

Example 1.18. *Note however that the existence of a full support martingale measure is compatible with $1p$-Arbitrage so that the converse implication of 1. in Proposition 1.17 does not hold. Let $(\Omega, \mathcal{F}) = (\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$. Consider the market with one risky asset: $S_0 = 2$ and*

$$S_1 = \begin{cases} 3 & \omega \in \mathbb{R}^+ \setminus \mathbb{Q} \\ 2 & \omega \in \mathbb{Q}^+ \end{cases} \tag{9}$$

Then obviously there exists a $1p$-Arbitrage even though there exist full support martingale measures (those probabilities assigning positive mass only to each rational).

As soon as we weaken No $1p$-Arbitrage, by adopting any other no arbitrage conditions in Definition 1.10, there is no guarantee of the existence of martingale measures, as shown in Section 3.1. In order to obtain the equivalence between $\mathcal{M} \neq \varnothing$ and No Model Independent Arbitrage (the weakest among the No Arbitrage conditions de la classe $\mathcal{S}$) we will enlarge the filtration, as explained in Section 3.5.

3.1. Examples. This section provides a variety of counterexamples to many possible conjectures on the formulation of the Fundamental Theorem of Asset Pricing in the model-free framework. A financially meaningful example is the one of two call options with the same spot price $p_1 = p_2$ but with strike prices $K_1 > K_2$, formulated in [**DH07**], which already highlights that the equivalence between absence of model independent arbitrage and existence of martingale measures is not possible.

We consider a one period market (i.e. $T = 1$) with $(\Omega, \mathcal{F}) = (\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$ and with $d = 2$ risky assets $S = [S^1, S^2]$, in addition to the riskless asset $S^0 = 1$. Admissible trading strategies are represented by vectors $H = (\alpha, \beta) \in \mathbb{R}^2$ so that

$$V_T(H) = \alpha \Delta S^1 + \beta \Delta S^2,$$

where $\Delta S^i = S_1^i - S_0^i$ for $i = 1, 2$. Let $S_0 = [S_0^1, S_0^2] = [2, 2]$,

$$S_1^1 = \begin{cases} 3 & \omega \in \mathbb{R}^+ \setminus \mathbb{Q} \\ 2 & \omega \in \mathbb{Q}^+ \end{cases} \quad ; \quad S_1^2 = \begin{cases} 1 + \exp(\omega) & \omega \in \mathbb{R}^+ \setminus \mathbb{Q} \\ 1 & \omega = 0 \\ 1 + \exp(-\omega) & \omega \in \mathbb{Q}^+ \setminus \{0\} \end{cases} \tag{10}$$

and $\mathcal{F} = \mathcal{F}^S$. We notice the following simple facts.

(1) There are no martingale measures:

$$\mathcal{M} = \varnothing.$$

Indeed, if we denote by $\mathcal{M}_i$ the set of martingale measures for the i^{th} asset we have $\mathcal{M}_1 = \{Q \in \mathcal{P} \mid Q(\mathbb{R}^+ \setminus \mathbb{Q}) = 0\}$ and $\forall Q \in \mathcal{M}_2,\ Q(\mathbb{R}^+ \setminus \mathbb{Q}) > 0$.

(2) The final value of the strategy $H = (\alpha, \beta) \in \mathbb{R}^2$ is

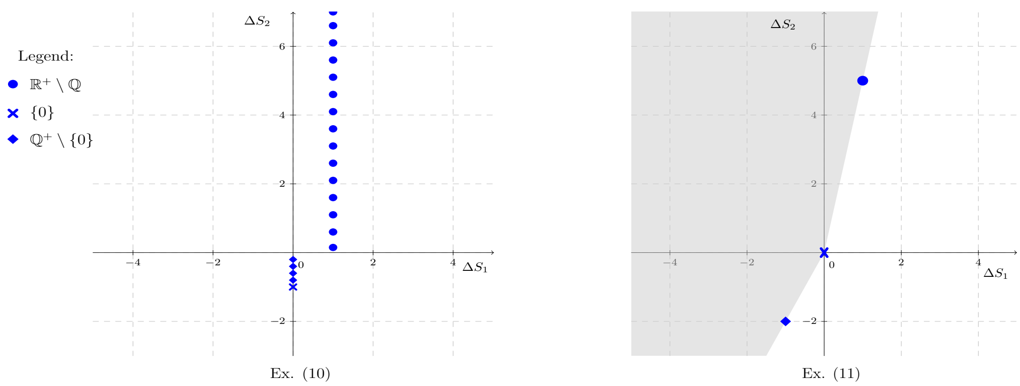
$$V_T(H) = \begin{cases} \alpha + \beta(\exp(\omega) - 1) & \omega \in \mathbb{R}^+ \setminus \mathbb{Q} \\ -\beta & \omega = 0 \\ \beta(\exp(-\omega) - 1) & \omega \in \mathbb{Q}^+ \setminus \{0\} \end{cases}.$$

Only the strategies $H \in \mathbb{R}^2$ having $\beta = 0$ and $\alpha \geq 0$ satisfy $V_T(H)(\omega) \geq 0$ for all $\omega \in \Omega$. For $\beta = 0$ and $\alpha > 0$, $\mathcal{V}_H^+ = \mathbb{R}^+ \setminus \mathbb{Q}$ and therefore there are **No Open Arbitrage** and **No Model Independent Arbitrage** (but $\mathcal{M} = \varnothing$). This fact persists even if we impose boundedness restrictions on the process S or on the admissible strategies H, as the following modification of the example shows: let $S_0 = [2, 2]$ and take $S_1^1 = [2 + \exp(-\omega)]\mathbf{1}_{\mathbb{R}^+ \setminus \mathbb{Q}} + 2\mathbf{1}_{\mathbb{Q}^+}$ and $S_1^2 = [1 + \exp(\omega) \wedge 4]\mathbf{1}_{\mathbb{R}^+ \setminus \mathbb{Q}} + \mathbf{1}_{\{0\}} + [1 + \exp(-\omega)]\mathbf{1}_{\mathbb{Q}^+ \setminus \{0\}}$.

(3) Set $\mathcal{H}^+ := \{H \in \mathcal{H} \mid V_T(H) \geq 0 \text{ and } V_0(H) = 0\}$ so that we have $\bigcup_{H \in \mathcal{H}^+} \mathcal{V}_H^+ = \mathbb{R}^+ \setminus \mathbb{Q} \subsetneqq \Omega$. This shows that the condition $\mathcal{M} = \varnothing$ is not equivalent to $\bigcup_{H \in \mathcal{H}^+} \mathcal{V}_H^+ = \Omega$ i.e. it is not true that the set of martingale measures is empty iff for every ω there exists a strategy H that gives positive wealth on ω and $V_0(H) = 0$. In order to recover the equivalence between these two concepts (as in Proposition 1.43) we need to enlarge the filtration in the way explained in Section 3.5.

(4) By fixing any probability P there exists a P-Classical Arbitrage, since the (probabilistic) Fundamental Theorem of Asset Pricing holds true and $\mathcal{M} = \varnothing$. Indeed:

FIGURE 1. In examples (10) and (11), 0 does not belong to the relative interior of the convex set generated by the points $\{[\Delta S^1(\omega), \Delta S^2(\omega)]\}_{\omega\in\Omega}$ and hence there exists an hyperplane which separates the points.



(a) If $P(\mathbb{R}^+ \setminus \mathbb{Q}) = 0$, then $\beta = -1$ ($\alpha = 0$) yield a P-Classical arbitrage, since $\mathcal{V}_H^+ = \mathbb{Q}^+$ and $P(\mathcal{V}_H^+) = 1$

(b) If $P(\mathbb{R}^+ \setminus \mathbb{Q}) > 0$ then $\beta = 0$ and $\alpha = 1$ yield a P-Classical arbitrage, since $\mathcal{V}_H^+ = \mathbb{R}^+ \setminus \mathbb{Q}$ and $P(\mathcal{V}_H^+) > 0$.

(5) Instead, by adopting the definition of a P-a.s. Arbitrage ($V_T(H)(\omega) \geq 0$ for all $\omega \in \Omega$ and $P(\mathcal{V}_H^+) > 0$), there are two possibilities:

(a) If $P(\mathbb{R}^+ \setminus \mathbb{Q}) = 0$, **No P-a.s. Arbitrage** holds, since only the strategies $H \in \mathbb{R}^2$ having $\beta = 0$ and $\alpha \geq 0$ satisfies $V_T(H)(\omega) \geq 0$ for all $\omega \in \Omega$ and $\mathcal{V}_H^+ = \mathbb{R}^+ \setminus \mathbb{Q}$.

(b) If $P(\mathbb{R}^+ \setminus \mathbb{Q}) > 0$, then $\beta = 0$ and $\alpha = 1$ **yield a P-a.s. arbitrage**, since $\mathcal{V}_H^+ = \mathbb{R}^+ \setminus \mathbb{Q}$ and $P(\mathcal{V}_H^+) > 0$.

(6) Geometric approach: If we plot the vector $[\Delta S^1, \Delta S^2]$ on the real plane (see Figure 1) we see that there exists a unique separating hyperplane given by the vertical axis. As a consequence 1p-Arbitrage can arise only by investment in the first asset ($\beta = 0$). For a separating hyperplane we mean an hyperplane in $\mathbb{R}^d$ passing by the origin and such that one of the associated half-space contains (not necessarily strictly contains) all the image points of the random vector $[\Delta S^1, \Delta S^2]$. Let us now consider this other example on $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$. Let $S_0 = [2, 2]$, and

$$S_1^1 = \begin{cases} 3 & \omega \in \mathbb{R}^+ \setminus \mathbb{Q} \\ 2 & \omega = 0 \\ 1 & \omega \in \mathbb{Q}^+ \setminus \{0\} \end{cases} \qquad S_1^2 = \begin{cases} 7 & \omega \in \mathbb{R}^+ \setminus \mathbb{Q} \\ 2 & \omega = 0 \\ 0 & \omega \in \mathbb{Q}^+ \setminus \{0\} \end{cases} \tag{11}$$

In both examples (10) and (11) there exist separating hyperplanes i.e. a $1p$-Arbitrage can be obtained (see Figure 1). In example (10) $\mathcal{M}$ is empty and we find a unique separating hyperplane: this hyperplane cannot give a strict separation of the set $[\Delta S^1(\omega), \Delta S^2(\omega)]_{\omega\in\mathbb{Q}^+}$ even though $\mathbb{Q}^+$ does not support any martingale measure. In example (11) $\mathcal{M} = \{\delta_{\omega=0}\}$, only the event $\{\omega = 0\}$ supports a martingale measure and there exists an infinite number of hyperplanes which strictly separates the image of both polar sets $\mathbb{R}^+ \setminus \mathbb{Q}$ and $\mathbb{Q}^+ \setminus \{0\}$, namely, those separating the convex grey region in Figure 1.

In conclusion the previous examples show that in a model-free environment the existence of a martingale measure can not be implied by arbitrage conditions - *at least of the type considered so far*. This is an important difference between the model-free and quasi-sure analysis approach (see for example [**BN15**]):

- Model free approach: we deduce the 'richness' of the set $\mathcal{M}$ of martingale measures starting directly from the underlying market structure $(\Omega, \mathcal{F}, S)$ and we analyze the class of polar sets with respect to $\mathcal{M}$.
- Quasi sure approach: the class of priors $\mathcal{P}' \subseteq \mathcal{P}$ and its polar sets are given and one formulates a No-Arbitrage type condition to guarantee the existence of a class of martingale measures which has the same polar sets as the set of priors.

An alternative definition of Arbitrage. The notion of No P-Classical Arbitrage, $P(V_T(H) < 0) = 0$ implies $P(V_T(H) > 0) = 0$, can be rephrased as: $V_0(H) = 0$ and

$$\{V_T(H) < 0\} \text{ is negligible } \Rightarrow \{V_T(H) > 0\} \text{ is negligible} \tag{12}$$

or in our setting

$$\mathcal{V}_H^- \text{ does not contain sets in } \mathcal{S} \Rightarrow \mathcal{V}_H^+ \text{ does not contain sets in } \mathcal{S}. \tag{13}$$

where $\mathcal{V}_H^- := \{\omega \in \Omega \mid V_T(H)(\omega) < 0\}$. In the definition (13) we are giving up the requirement $V_T(H) \geq 0$, and so the differences with respect to the existence of arbitrage opportunities showed in Item 5 of the example in this section disappear. However, this alternative definition of arbitrage does not work well, as shown by the following example. Consider $(\Omega, \mathcal{F}) = (\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$, a one period market with one risky asset: $S_0 = 2$,

$$S_1 = \begin{cases} 3 & \omega \in [1, \infty) \\ 2 & \omega = [0,1) \setminus \mathbb{Q} \\ 1 & \omega \in [0,1) \cap \mathbb{Q} \end{cases} \tag{14}$$

Consider the strategy of buying the risky asset: $H = 1$. Then $\mathcal{V}_H^- = [0,1) \cap \mathbb{Q}$ does not contain an open set, $\mathcal{V}_H^+ = [1, \infty)$ contains open sets. Therefore, there is an Open Arbitrage (in the modified definition obtained from (13)) but there are full support martingale measures, for example $Q([0,1) \cap \mathbb{Q}) = Q([1, \infty)) = \frac{1}{2}$. Notice also that by enlarging the filtration the Open Arbitrage would persist.
A concept of no arbitrage similar to (12) was introduced by Cassese [**C08**], by adopting an ideal $\mathcal{N}$ of "negligible" sets - not necessarily derived from probability measures. In a continuous time setting, he proves that the absence of such an arbitrage is equivalent to the existence of a finitely additive "martingale measure". Our results are not comparable with those by [**C08**] since the markets are clearly different, we do not require any structure on the family $\mathcal{S}$ and [**C08**] works with finitely additive measures. In addition, the example (14) just discussed shows the limitation in our setting of the definition (12) for finding martingale probability measures with the appropriate properties.

3.2. Technical Lemmata. Recall that $S = (S_t)_{t \in I}$ is an $\mathbb{R}^d$-valued stochastic process defined on a Polish space Ω endowed with its Borel σ-algebra $\mathcal{F} = \mathcal{B}(\Omega)$ and $I_1 := \{1, \dots, T\}$.

Through the rest of the Chapter we will make use of the natural filtration $\mathcal{F}^S = \{\mathcal{F}_t^S\}_{t\in I}$ of the process S and for ease of notation we will not specify S, but simply write $\mathcal{F}_t$ for $\mathcal{F}_t^S$.

For the sake of simplicity we indicate by $\mathbf{Z} := Mat(d\times(T+1);\mathbb{R})$ the space of $d\times(T+1)$ matrices with real entries representing the space of all the possible trajectories of the price process. Namely for every $\omega \in \Omega$ we have $(S_0(\omega), S_1(\omega), \ldots, S_T(\omega)) = (z_0, z_1, \ldots, z_T) =: z \in \mathbf{Z}$. Fix $s \leq t$: for any $z \in \mathbf{Z}$ we indicate , the components from s to t by $z_{s:t} = (z_s, \ldots, z_t)$ and $z_{t:t} = z_t$. Similarly $S_{s:t} = (S_s, S_{s+1}, \ldots, S_t)$ represents the process from time s to t.
We denote with $ri(K)$ the relative interior of a set $K \subseteq \mathbb{R}^d$. In this section we will make extensive use of the geometric properties of the image in $\mathbb{R}^d$ of the increments of the price process $\Delta S_t := S_t - S_{t-1}$ relative to a set $\Gamma \subseteq \Omega$. The typical sets that we will consider are the level sets $\Gamma = \Sigma_{t-1}^z$, where:

$$\Sigma_{t-1}^z := \{\omega \in \Omega \mid S_{0:t-1}(\omega) = z_{0:t-1}\} \in \mathcal{F}_{t-1},\ z \in \mathbf{Z},\, t \in I_1 \tag{15}$$

and $\Gamma = A_{t-1}^z$, the intersection of the level set Σ_{t-1}^z with a set $A \in \mathcal{F}_{t-1}$:

$$A_{t-1}^z := \{\omega \in A \mid S_{0:t-1}(\omega) = z_{0:t-1}\} \in \mathcal{F}_{t-1}. \tag{16}$$

For any $\Gamma \subseteq \Omega$ define the convex cone:

$$(\Delta S_t(\Gamma))^{cc} := co\left(conv\big(\Delta S_t(\Gamma)\big)\right) \cup \{0\} \subseteq \mathbb{R}^d. \tag{17}$$

If $0 \in ri(\Delta S_t(\Gamma))^{cc}$ we cannot apply the hyperplane separating theorem to the convex sets $\{0\}$ and $ri(\Delta S_t(\Gamma))^{cc}$, namely, there is no $H \in \mathbb{R}^d$ that satisfies $H \cdot \Delta S_t(\omega) \geq 0$ for all $\omega \in \Gamma$ with strict inequality for some of them. As intuitively evident, and shown in Corollary 1.21 below, $0 \in ri(\Delta S_t(\Gamma))^{cc}$ if and only if No $1p$-Arbitrage are possible on the set Γ, since a trading strategy on Γ with a non-zero payoff always yields both positive and negative outcomes.
In this situation, for $\Gamma = \Sigma_{t-1}^z$, the level set is not suitable for the construction of a $1p$-Arbitrage opportunity and sets with this property are naturally important for the construction of a martingale measure. We wish then to identify, for $\Gamma = \Sigma_{t-1}^z$ satisfying $0 \notin ri(\Delta S_t(\Gamma))^{cc}$, those subset of Σ_{t-1}^z that retain this property. This result is contained in the following key Lemma 1.20.

Observe first that for a convex cone $K \subseteq \mathbb{R}^d$ such that $0 \notin ri(K)$ we can consider the family $V = \{v \in \mathbb{R}^d \mid \|v\| = 1 \text{ and } v \cdot y \geq 0\ \forall\, y \in K\}$ so that

$$\overline{K} = \bigcap_{v\in V} \{y \in \mathbb{R}^d \mid v \cdot y \geq 0\} = \bigcap_{n\in\mathbb{N}} \{y \in \mathbb{R}^d \mid v_n \cdot y \geq 0\},$$

where $\{v_n\} = (\mathbb{Q}^d \cap V) \setminus \{0\}$.

DEFINITION 1.19. *Adopting the above notations, we will call $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} v_n \in V$ the standard separator.*

LEMMA 1.20. *Fix $t \in I_1$ and $\Gamma \neq \varnothing$. If $0 \notin ri(\Delta S_t(\Gamma))^{cc}$ then there exist $\beta \in \{1, \ldots, d\}$, $H^1, \ldots, H^\beta$, $B^1, \ldots, B^\beta, B^*$ with $H^i \in \mathbb{R}^d$, $B^i \subseteq \Gamma$ and $B^* := \Gamma \setminus \left(\cup_{j=1}^{\beta} B^j\right)$ such that:*

(1) $B^i \neq \varnothing$ for all $i = 1, \ldots \beta$, and $\{\omega \in \Gamma \mid \Delta S_t(\omega) = 0\} \subseteq B^*$ which may be empty;
(2) $B^i \cap B^j = \varnothing$ if $i \neq j$;
(3) $\forall i \leq \beta,\ H^i \cdot \Delta S_t(\omega) > 0$ for all $\omega \in B^i$ and $H^i \cdot \Delta S_t(\omega) \geq 0$ for all $\omega \in \cup_{j=i}^{\beta} B^j \cup B^*$.
(4) $\forall H \in \mathbb{R}^d$ s.t. $H \cdot \Delta S_t \geq 0$ on B^* we have $H \cdot \Delta S_t = 0$ on B^*.

Moreover, for $z \in \mathbf{Z}$, $A \in \mathcal{F}_{t-1}$ and $\Gamma = A^z_{t-1}$ (or $\Gamma = \Sigma^z_{t-1}$) we have $B^i, B^* \in \mathcal{F}_t$ and

$$H(\omega) := \sum_{i=1}^{\beta} H^i \mathbf{1}_{B^i}(\omega) \tag{18}$$

is an $\mathcal{F}_t$-measurable random variable that is uniquely determined when we adopt for each H^i the standard separator.
Clearly in these cases, β, H^i, H, B^i and B^* will depend on t and z and whenever necessary they will be denoted by $\beta_{t,z}$, $H^i_{t,z}$, $H_{t,z}$, $B^i_{t,z}$ and $B^*_{t,z}$.

Proof. Set $A^0 := \Gamma$ and $K^0 = (\Delta S_t(\Gamma))^{cc} \subseteq \mathbb{R}^d$ and the possibly empty set $\Delta_0 := \{\omega \in A^0 \mid \Delta S_t(\omega) = 0\}$.

Step 1:: The set $K^0 \subseteq \mathbb{R}^d$ is non-empty and convex and so $ri(K^0) \neq \varnothing$. From $0 \notin ri(K^0)$ there exists a standard separator $H^1 \in \mathbb{R}^d$ such that we have: (i) $H^1 \cdot \Delta S_t(\omega) \geq 0$ for all $\omega \in A^0$; (ii) $B^1 := \{\omega \in A^0 \mid H^1 \cdot \Delta S_t(\omega) > 0\}$ is non-empty.
Set $A^1 := (A^0 \setminus B^1) = \{\omega \in A^0 \mid H^1 \cdot \Delta S_t(\omega) = 0\}$ and let $K^1 := (\Delta S_t(A^1))^{cc}$, which is a non-empty convex set with $dim(K^1) \leq d-1$.
If $0 \in ri(K^1)$ (this includes the case $K^1 = \{0\}$) the procedure is complete: one cannot separate $\{0\}$ from the relative interior of K^1. The conclusion is that $\beta = 1$, $B^* = A^1 = A^0 \setminus B^1$ which might be empty, and $\Delta_0 \subseteq B^*$. Notice that if $K^1 = \{0\}$ then $B^* = \Delta_0$ which might be empty. Otherwise:

Step 2:: If $0 \notin ri(K^1)$ we find the standard separator $H^2 \in \mathbb{R}^d$ such that $H^2 \cdot \Delta S_t(\omega) \geq 0$, for all $\omega \in A^1$, and $B^2 := \{\omega \in A^1 \mid H^2 \cdot \Delta S_t(\omega) > 0\}$ is non-empty. Denote $A^2 := (A^1 \setminus B^2)$ and let $K^2 = (\Delta S_t(A^2))^{cc}$ with $dim(K^2) \leq d-2$.
If $0 \in ri(K^2)$ (this includes $K^2 = \{0\}$) the procedure is complete and we have the conclusions with $\beta = 2$ and $B^* = A^1 \setminus B^2 = A^0 \setminus (B^1 \cup B^2)$, and $\Delta_0 \subseteq B^*$. Notice that if $K^1 = \{0\}$ then $B^* = \Delta_0$. Otherwise:

...:

Step d-1: If $0 \notin ri(K^{d-2})$ ... Set $B^{d-1} \neq \varnothing$, $A^{d-1} = (A^{d-2} \setminus B^{d-1})$, $K^{d-1} = (\Delta S_t(A^{d-1}))^{cc}$ with $dim(K^{d-1}) \leq 1$. If $0 \in ri(K^{d-1})$ the procedure is complete. Otherwise:

Step d:: We necessarily have $0 \notin ri(K^{d-1})$, so that $dim(K^{d-1}) = 1$, and the convex cone K^{d-1} necessarily coincides with a half-line with origin in 0. We find a separator $H^d \in \mathbb{R}^d$ with $B^d := \{\omega \in A^{d-1} \mid H^d \cdot \Delta S_t(\omega) > 0\} \neq \varnothing$ and the set

$$B^* := \{\omega \in A^{d-1} \mid \Delta S_t(\omega) = 0\} = \{\omega \in A^0 \mid \Delta S_t(\omega) = 0\} = \Delta_0$$

satisfies: $B^* = A^{d-1} \setminus B^d$. Set $A^d := A^{d-1} \setminus B^d = B^* = \Delta_0$ and $K^d := (\Delta S_t(A^d))^{cc}$. Then $K^d = \{0\}$.

Since $dim(\Delta S_t(\Gamma))^{cc} \leq d$ we have at most d steps. In case $\beta = d$ we have $\Gamma = A^0 = \bigcup_{i=1}^d B^i \cup \Delta_0$. To prove the last assertion we note that for any fixed t and z, B^i are $\mathcal{F}_t$-measurable since $B^i = A^z_{t-1} \cap (f \circ S_t)^{-1}((0,\infty))$ where $f : \mathbb{R}^d \mapsto \mathbb{R}$ is the continuous function $f(x) = H^i \cdot (x - z_{t-1})$ with $H^i \in \mathbb{R}^d$ fixed. $\square$

COROLLARY 1.21. *Let $t \in I_1$, $z \in \mathbf{Z}$, $A \in \mathcal{F}_{t-1}$, $\Gamma = A^z_{t-1}$. Then $0 \in ri(\Delta S_t(\Gamma))^{cc}$ if and only if there are No 1p-Arbitrage on Γ, i.e.:*

$$\text{for all } H \in \mathbb{R}^d \text{ s.t. } H(S_t - z_{t-1}) \geq 0 \text{ on } \Gamma \text{ we have } H(S_t - z_{t-1}) = 0 \text{ on } \Gamma. \tag{19}$$

PROOF. Let $0 \notin ri(\Delta S_t(\Gamma))^{cc}$. Then from Lemma 1.20-3) with $i = 1$ we obtain a $1p$-Arbitrage H^1 on $\Gamma = \cup_{j=1}^{\beta} B^j \cup B^*$, since $B^1 \neq \varnothing$. Viceversa, if $0 \in ri(\Delta S_t(\Gamma))^{cc}$ we obtain (19) from the argument following equation (17). $\square$

DEFINITION 1.22. *For $A \in \mathcal{F}_{t-1}$ and $\Gamma = A^z_{t-1}$ we naturally extend the definition of $\beta_{t,z}$ in Lemma 1.20 to the case of $0 \in ri(\Delta S_t(\Gamma))^{cc}$ using*

$$\beta_{t,z} = 0 \dot{\Leftrightarrow} 0 \in ri(\Delta S_t(\Gamma))^{cc}$$

with $B^0_{t,z} = \varnothing$ and $B^*_{t,z} = A^z_{t-1} \in \mathcal{F}_{t-1}$. In this case, we also extend the definition of the random variable in (18) as $H_{t,z}(\omega) \equiv 0$.

COROLLARY 1.23. *Let $t \in I_1$, $z \in \mathbf{Z}$, $A \in \mathcal{F}_{t-1}$ and $\Gamma = A^z_{t-1}$ with $0 \notin ri(\Delta S_t(\Gamma))^{cc}$. For any $P \in \mathcal{P}$ s.t. $P(\Gamma) > 0$ let*

$$j := \inf\{1 \leq i \leq \beta \mid P(B^i_{t,z}) > 0\}.$$

If $j < \infty$ the trading strategy $H(s, \omega) := H^j \mathbf{1}_\Gamma(\omega) \mathbf{1}_{\{t\}}(s)$ is a P-Classical Arbitrage.

PROOF. From Lemma 1.20 we obtain: $H^j \Delta S_t(\omega) > 0$ on $B^j_{t,z}$ with $P(B^j_{t,z}) > 0$; $H^j \Delta S_t(\omega) \geq 0$ on $\bigcup_{i=j}^{\beta_{t,z}} B^i_{t,z} \cup B^*_{t,z}$ and $P(B^k_{t,z}) = 0$ for $1 \leq k < j$. $\square$

REMARK 1.24. *Let $D \subseteq \mathbb{R}^d$ and $C := (D)^{cc} \subseteq \mathbb{R}^d$ be the convex cone generated by D. If $0 \in ri(C)$ then for any $x \in D$ there exist a finite number of elements $x_j \in D$ such that 0 is a convex combination of $\{x, x_1, \ldots, x_m\}$ with a strictly positive coefficient of x. Indeed, fix $x \in D$ and recall that for any convex set $C \subseteq \mathbb{R}^d$ we have*

$$ri(C) := \{z \in C \mid \forall x \in C \ \exists \varepsilon > 0 \text{ s.t. } z - \varepsilon(x - z) \in C\}.$$

As $0 \in ri(C)$ and $x \in D \subseteq C$ we obtain $-\varepsilon x \in C$, for some $\varepsilon > 0$, and thus: $\frac{\varepsilon}{1+\varepsilon} x + \frac{1}{1+\varepsilon}(-\varepsilon x) = 0$. Since $-\varepsilon x \in C$ then it is a linear combination with non negative coefficients of elements of D and we obtain: $\frac{\varepsilon}{1+\varepsilon} x + \frac{1}{1+\varepsilon} \sum_{j=1}^m \alpha_j x_j = 0$, which can be rewritten as: $\lambda x + \sum_{j=1}^m \lambda_j x_j = 0$, with $x_j \in D$, $\lambda + \sum_{j=1}^m \lambda_j = 1$, $\lambda > 0$ and $\lambda_j \geq 0$. When the set $D \subseteq \mathbb{R}^d$ is the set of the image points of the increment of the price process $[\Delta S_t(\omega)]_{\omega \in \Gamma}$, for a fixed time t, this observation shows that, however we choose $\omega \in \Gamma$ we can construct a conditional martingale measure, relatively to the period $[t-1, t]$, which assigns a strictly positive weight to ω and has finite support. The measure is determined by the coefficients $\{\lambda, \lambda_1, \ldots, \lambda_m\}$ in the equation: $0 = \lambda \Delta S_t(\omega) + \sum_{j=1}^m \lambda_j \Delta S_t(\omega_j)$. This heuristic argument is made precise in the following Corollary and it will be used also in the proof of Proposition 1.34.

COROLLARY 1.25. *Let z, t, $\Gamma = A^z_{t-1}$ and $B^*_{t,z}$ as in Lemma 1.20.*

$$\text{For all } U \subseteq B^*_{t,z},\ U \in \mathcal{F} \quad \text{there exists } Q \in \mathcal{M}(B^*_{t,z}) \text{ s.t. } Q(U) > 0$$

where $\mathcal{M}(B) = \{Q \in \mathcal{P} \mid Q(B) = 1$ and $E_Q[S_t \mid \mathcal{F}_{t-1}] = S_{t-1}$ Q-a.s.$\}$, for $B \in \mathcal{F}$.

PROOF. From Lemma 1.20-4) there are no $1p$-Arbitrage restricted to $\Gamma = B^*_{t,z}$. Applying Corollary 1.21 this implies that $0 \in ri(\Delta S_t(B^*_{t,z}))^{cc}$. Take any $\omega \in U \subseteq B^*_{t,z}$. Applying Remark 1.24 to the set $D := \Delta S_t(B^*_{t,z})$ and to $x := \Delta S_t(\omega) \in D$, we deduce the existence of $\{\omega_1, \dots, \omega_m\} \subseteq B^*_{t,z}$ and non negative coefficients $\{\lambda_t(\omega_1), \dots, \lambda_t(\omega_m)\}$ and $\lambda_t(\omega) > 0$ such that:

$$\lambda_t(\omega) + \sum_{j=1}^{m} \lambda_t(\omega_j) = 1 \ \text{ and } \ 0 = \lambda_t(\omega)\Delta S_t(\omega) + \sum_{j=1}^{m} \lambda_t(\omega_j)\Delta S_t(\omega_j).$$

Since $\{\omega_1, \dots, \omega_m\} \subseteq B^*_{t,z}$ and $\omega \in B^*_{t,z}$ we have $S_{t-1}(\omega_j) = z_{t-1}$ and $S_{t-1}(\omega) = z_{t-1}$. Therefore:

$$0 = \lambda_t(\omega)(S_t(\omega) - z_{t-1}) + \sum_{j=1}^{m} \lambda_t(\omega_j)(S_t(\omega_j) - z_{t-1}), \tag{20}$$

so that $Q(\{\omega\}) = \lambda_t(\omega)$ and $Q(\{\omega_j\}) = \lambda_t(\omega_j)$, for all j, give the desired probability. $\square$

EXAMPLE 1.26. *Let* $(\Omega, \mathcal{F}) = (\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$ *and consider a single period market with* $d = 3$ *risky asset* $S_t = [S^1_t, S^2_t, S^3_t]$ *with* $t = 0, 1$ *and* $S_0 = [2, 2, 2]$. *Let*

$$S^1_1(\omega) = \begin{cases} 1 & \omega \in \mathbb{R}^+ \setminus \mathbb{Q} \\ 2 & \omega \in \mathbb{Q} \cap [1/2, +\infty) \\ 3 & \omega \in \mathbb{Q} \cap [0, 1/2) \end{cases} \quad S^2_1(\omega) = \begin{cases} 2 & \omega \in \mathbb{R}^+ \setminus \mathbb{Q} \\ 1 + \omega^2 & \omega \in \mathbb{Q} \cap [1/2, +\infty) \\ 1 + \omega^2 & \omega \in \mathbb{Q} \cap [0, 1/2) \end{cases}$$

$$S^3_1(\omega) = \begin{cases} 2 + \omega^2 & \omega \in \mathbb{R}^+ \setminus \mathbb{Q} \\ 2 & \omega \in \mathbb{Q} \cap [1/2, +\infty) \\ 2 & \omega \in \mathbb{Q} \cap [0, 1/2) \end{cases}$$

Fix $t = 1$ and $z \in \mathbf{Z}$ with $z_0 = S_0$. It is easy to check that in this case $\beta_{t,z} = 2$ with $B^1_{t,z} = \mathbb{R}^+ \setminus \mathbb{Q}$, $B^2_{t,z} = \mathbb{Q} \cap [0, 1/2)$, $B^*_{t,z} = \mathbb{Q} \cap [1/2, +\infty)$. The corresponding strategies $H = [h_1, h_2, h_3]$ (standard in the sense of Lemma 1.20) are given by $H^1_{t,z} = [0, 0, 1]$ and $H^2_{t,z} = [1, 0, 0]$. Note that $H^1_{t,z}$ is a 1p-arbitrage with $\mathcal{V}^+_{H^1_{t,z}} = B^1_{t,z}$. We have therefore that $B^1_{t,z}$ is a null set with respect to any martingale measure. The strategy $H^2_{t,z}$ satisfies $V_T(H^2_{t,z}) \geq 0$ on $(B^1_{t,z})^c$ with $\mathcal{V}^+_{H^2_{t,z}} = B^2_{t,z}$ hence, $B^2_{t,z}$ is also an $\mathcal{M}$-polar set. This example shows the need of a multiple separation argument, as it is not possible to find a *single* separating hyperplane $H \in \mathbb{R}^d$ such that the image points of $B^1_{t,z} \cup B^2_{t,z}$ (which is $\mathcal{M}$-polar), through the random vector ΔS, are strictly contained in one of the associated half-spaces. We have indeed that $B^2_{t,z}$ is a subset of $\{\omega \in \Omega \mid H^1_{t,z}(S_1 - S_0) = 0\}$ where $H^1_{t,z}$ is the only 1p-arbitrage in this market.
The corollaries 1.23 and 1.25 show the difference between the sets B^i and B^*. Restricted to the time interval $[t-1, t]$, a probability measure whose mass is concentrated on B^* admits an equivalent martingale measure while for those probabilities that assign positive mass to at least one B^i an arbitrage opportunity can be constructed. We can summarize the possible situations as follows.

COROLLARY 1.27. *For* $\Gamma = A^z_{t-1}$, *with* $A \in \mathcal{F}_{t-1}$, *and* $\mathcal{M}(B)$ *defined in Corollary 1.25 we have:*

(1) $B^*_{t,z} = A^z_{t-1} \Leftrightarrow$ *No 1p-Arbitrage on* $A^z_{t-1} \overset{\cdot}{\Leftrightarrow} 0 \in ri(\Delta S_t(A^z_{t-1}))^{cc}$.
(2) $B^*_{t,z} = \varnothing \Leftrightarrow 0 \notin conv\big(\Delta S_t(A^z_{t-1})\big)$

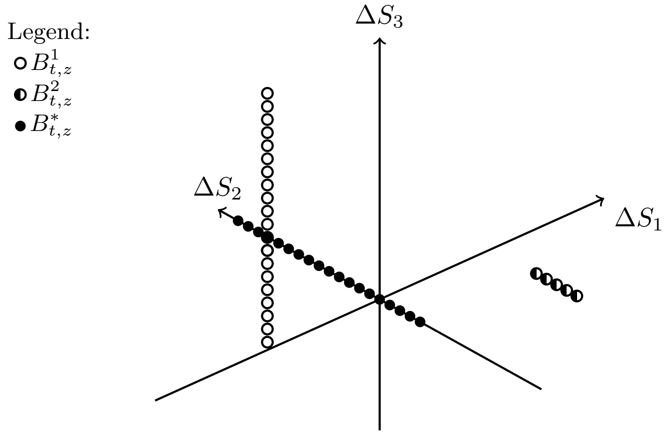


FIGURE 2. Decomposition of Ω in Example 1.26

(3) $\beta_{t,z} = 1$ *and* $B^*_{t,z} \neq \varnothing \implies$ *there exists* $H \in \mathbb{R}^d$, $H \neq 0$ *such that* $B^*_{t,z} = \{\omega \in A^z_{t-1} \mid H(S_t(\omega) - z_{t-1}) = 0\}$ *is "martingalizable" i.e.* $\forall U \subset B^*_{t,z}$, $U \in \mathcal{F}$ *there exists* $Q \in \mathcal{M}(B^*_{t,z})$ *s.t.* $Q(U) > 0$.

PROOF. Equivalence 1. immediately follows from Corollary 1.21 and Definition 1.22. To show 2. we use the sets K^i for $i = 1, \dots, \beta_{t,z}$ and the other notations from the proof of Lemma 1.20. Suppose first that $0 \notin conv\big(\Delta S_t(\Gamma)\big)$ which implies $0 \notin ri(\Delta S_t(\Gamma))^{cc}$ and $\Delta_0 = \varnothing$. From the assumption we have $0 \notin conv\big(\Delta S_t(C)\big)$ for any subset $C \subseteq \Gamma$ so, in particular, $0 \notin ri(K^i)$ unless $K^i = \{0\}$. This implies $B^*_{t,z} = \Delta_0 = \varnothing$.
Suppose now $0 \in conv\big(\Delta S_t(\Gamma)\big)$. If $0 \in ri(\Delta S_t(\Gamma))^{cc}$, by Definition 1.22 we have $B^*_{t,z} = \Gamma$ which is non empty. Suppose then $0 \notin ri(\Delta S_t(\Gamma))^{cc}$. As $0 \in conv\big(\Delta S_t(\Gamma)\big)$ there exists $n \geq 1$ such that: $0 = \sum_{j=1}^n \lambda_j (S_t(\omega_j) - z_{t-1})$, with $\sum_{j=1}^n \lambda_j = 1$, $\lambda_j > 0$ and $\omega_j \in \Gamma$ for all j. If 0 is an extremal point then $n = 1$, $S_t(\omega_1) - z_{t-1} = 0$ and $\{\omega_1\} \in \Delta_0 \subseteq B^*_{t,z}$. If $n \geq 2$ we have $0 \in conv(\Delta S_t\{\omega_1, \dots, \omega_n\})$ so that for any $H \in \mathbb{R}^d$ that satisfies $H \cdot \Delta S_t(\omega_i) \geq 0$ for any $i = 1, \dots, n$ we have $H \cdot \Delta S_t(\omega_i) = 0$. Hence $\{\omega_1, \dots, \omega_n\} \subseteq B^*_{t,z}$ by definition of $B^*_{t,z}$. We conclude by showing 3. From Lemma 1.20 items 3 and 4, if we select $H = H^1$ then $\{\omega \in \Gamma \mid H^1(S_t(\omega) - z_{t-1}) = 0\} = \Gamma \setminus B^1_{t,z} = B^*_{t,z} \neq \varnothing$ and on $B^*_{t,z}$ we may apply Corollary 1.25. $\square$

3.3. On $\mathcal{M}$-polar sets. We consider for any $t \in I$ the σ-algebra $\mathfrak{F}_t := \bigcap_{Q \in \mathcal{M}} \mathcal{F}^Q_t$, where $\mathcal{F}^Q_t$ is the Q-completion of $\mathcal{F}_t$. $\mathfrak{F}_t$ is the universal completion of $\mathcal{F}_t$ with respect to $\mathcal{M} = \mathcal{M}(\mathbb{F})$. Notice that the introduction of this enlarged filtration needs the knowledge *a priori* of the whole class $\mathcal{M}$ of martingale measures. Recall that any measure $Q \in \mathcal{M}$ can be uniquely extended to a measure $\overline{Q}$ on the enlarged σ-algebra $\mathfrak{F}_T$ so that we can write with slight abuse of notation $\mathcal{M}(\mathbb{F}) = \mathcal{M}(\mathfrak{F})$ where $\mathfrak{F} := \{\mathfrak{F}_t\}_{t \in I}$.

We wish to show now that under any martingale measure the sets $B^i_{t,z}$ (and their arbitrary unions) introduced in Lemma 1.20 must be null-sets. To this purpose we need to recall some properties of a proper regular conditional probability (see Theorems 1.1.6, 1.1.7 and 1.1.8 in Stroock-Varadhan [**SV06**]).

THEOREM 1.28. *Let $(\Omega, \mathcal{F}, Q)$ be a probability space, where Ω is a Polish space, $\mathcal{F}$ is the Borel σ-algebra, $Q \in \mathcal{P}$. Let $\mathcal{A} \subseteq \mathcal{F}$ be a countably generated sub σ-algebra of $\mathcal{F}$. Then there exists a proper regular conditional probability, i.e. a function $Q_{\mathcal{A}}(\cdot,\cdot) : (\Omega, \mathcal{F}) \mapsto [0,1]$ such that:*

a) for all $\omega \in \Omega$, $Q_{\mathcal{A}}(\omega, \cdot)$ is a probability measure on $\mathcal{F}$;
b) for each $B \in \mathcal{F}$, the function $Q_{\mathcal{A}}(\cdot, B)$ is a version of $Q(B \mid \mathcal{A})(\cdot)$;
c) $\exists N \in \mathcal{A}$ with $Q(N) = 0$ such that $Q_{\mathcal{A}}(\omega, B) = 1_B(\omega)$ for $\omega \in \Omega \setminus N$ and $B \in \mathcal{A}$
d) In addition, if $X \in L^1(\Omega, \mathcal{F}, Q)$ then $E_Q[X \mid \mathcal{A}](\omega) = \int_\Omega X(\tilde{\omega}) Q_{\mathcal{A}}(\omega, d\tilde{\omega})$ $Q - a.s.$

Recall that $\mathcal{F}_t = \mathcal{F}_t^S$, $t \in I$, is countably generated.

LEMMA 1.29. *Fix $t \in I_1 = \{1, \dots, T\}$, $A \in \mathcal{F}_{t-1}$, $Q \in \mathcal{M}$ and for $z \in \mathbf{Z}$ consider the set $A_{t-1}^z := \{\omega \in A \mid S_{0:t-1}(\omega) = z_{0:t-1}\}$. Then*

$$\bigcup_{z \in \mathbf{Z}} \{\omega \in A_{t-1}^z \ s.t. \ Q_{\mathcal{F}_{t-1}}(\omega, \cup_{i=1}^{\beta_{t,z}} B_{t,z}^i) > 0\}$$

is a subset of an $\mathcal{F}_{t-1}$-measurable Q-null set.

PROOF. If $Q(A) = 0$ there is nothing to show. Suppose now $Q(A) > 0$. In this proof we set for the sake of simplicity $X := S_t$, $Y := E_Q[X \mid \mathcal{F}_{t-1}] = S_{t-1}$ Q-a.s. $\beta := \beta_{t,z}$ and $\mathcal{A} := \mathcal{F}_{t-1} = \mathcal{F}_{t-1}^S$. Set

$$D_t^z := \left\{\omega \in A_{t-1}^z \text{ such that } Q_{\mathcal{A}}(\omega, \cup_{i=1}^{\beta} B_{t,z}^i) > 0\right\}.$$

If $z \in \mathbf{Z}$ is such that $0 \in ri(\Delta S_t(A_{t-1}^z))^{cc}$ then $\cup_{i=1}^{\beta} B_{t,z}^i = \varnothing$ and $D_t^z = \varnothing$. So we can consider only those $z \in \mathbf{Z}$ such that $0 \notin ri(\Delta S_t(A_{t-1}^z))^{cc}$. Fix such z.
Since $\mathcal{A} = \mathcal{F}_{t-1}^S$ is countably generated, Q admits a proper regular conditional probability $Q_{\mathcal{A}}$. From Theorem 1.28 d) we obtain:

$$Y(\omega) = \int_\Omega X(\tilde{\omega}) Q_{\mathcal{A}}(\omega, d\tilde{\omega}) \quad Q - a.s.$$

As $A_{t-1}^z \in \mathcal{A}$, by Theorem 1.28 c) there exists a set $N \in \mathcal{A}$ with $Q(N) = 0$ so that $Q_{\mathcal{A}}(\omega, A_{t-1}^z) = 1$ on $A_{t-1}^z \setminus N$ and therefore we have

$$\int_\Omega X(\tilde{\omega}) Q_{\mathcal{A}}(\omega, d\tilde{\omega}) = \int_{A_{t-1}^z} X(\tilde{\omega}) Q_{\mathcal{A}}(\omega, d\tilde{\omega}) \quad \forall \omega \in A_{t-1}^z \setminus N. \tag{21}$$

Since $0 \notin ri(\Delta S_t(\Gamma))^{cc}$ we may apply Lemma 1.20: for any $i = 1, \dots, \beta$, there exists $H^i \in \mathbb{R}^d$ such that $H^i \cdot (X(\tilde{\omega}) - z_{t-1}) \geq 0$ for all $\tilde{\omega} \in \cup_{l=i}^{\beta} B_{t,z}^l \cup B_{t,z}^*$ and $H^i \cdot (X(\tilde{\omega}) - z_{t-1}) > 0$ for every $\tilde{\omega} \in B_{t,z}^i$. Now we fix $\omega \in D_t^z \setminus N \subseteq A_{t-1}^z \setminus N$. Then the index $j := \min\{1 \leq i \leq \beta \mid Q_{\mathcal{A}}(\omega, B_{t,z}^i) > 0\}$ is well defined and: i) $H^j \cdot (X(\tilde{\omega}) - z_{t-1}) > 0$ on $B_{t,z}^j$, (ii) $Q_{\mathcal{A}}(\omega, B_{t,z}^j) > 0$ iii) $H^j \cdot (X(\tilde{\omega}) - z_{t-1}) \geq 0$ on $\cup_{l=j}^{\beta} B_{t,z}^l \cup B_{t,z}^*$; iv) $Q_{\mathcal{A}}(\omega, B_{t,z}^i) = 0$ for $i < j$. From i) and ii) we obtain

$$Q_{\mathcal{A}}(\omega, A_{t-1}^z \cap \{H^j \cdot (X - z_{t-1}) > 0\}) \geq Q_{\mathcal{A}}(\omega, B_{t,z}^j) > 0.$$

From iii) and iv) we obtain:

$$\begin{aligned} Q_{\mathcal{A}}(\omega, \{H^j \cdot (X - z_{t-1}) \geq 0\}) &\geq Q_{\mathcal{A}}(\omega, \cup_{l=j}^{\beta} B_{t,z}^l \cup B_{t,z}^*) \\ &\geq Q_{\mathcal{A}}(\omega, A_{t-1}^z) - Q_{\mathcal{A}}(\omega, \cup_{i<j} B_{t,z}^i) = 1. \end{aligned}$$

Hence

$$H^j \cdot \left(\int_{A^z_{t-1}} X(\tilde{\omega}) Q_{\mathcal{A}}(\omega, d\tilde{\omega}) - z_{t-1} \right) = \int_{A^z_{t-1}} H^j \cdot (X(\tilde{\omega}) - z_{t-1}) Q_{\mathcal{A}}(\omega, d\tilde{\omega}) > 0$$

and therefore, from equation (21) and from $z_{t-1} = Y(\omega)$, we have:

$$H^j \cdot \left(\int_{\Omega} X(\tilde{\omega}) Q_{\mathcal{A}}(\omega, d\tilde{\omega}) - Y(\omega) \right) > 0.$$

As this holds for any $\omega \in D^z_t \setminus N$ we obtain:

$$D^z_t \setminus N \subseteq \{\omega \in \Omega \mid Y(\omega) \neq \int_{\Omega} X(\tilde{\omega}) Q_{\mathcal{A}}(\omega, d\tilde{\omega})\} =: N^* \in \mathcal{F}_{t-1}$$

with $Q(N^*) = 0$. Hence, $D^z_t \subseteq N \cup N^* := N_0$ with $Q(N_0) = 0$ and N_0 not dependent on z. As this holds for every $z \in \mathbf{Z}$ we conclude that $\bigcup_{z \in \mathbf{Z}} D^z_t \subseteq N_0$. □

Corollary 1.30. *Fix $t \in I_1$ and $Q \in \mathcal{M}$. If*

$$\mathfrak{B}_t := \bigcup_{z \in \mathbf{Z}} \bigcup_{i=1}^{\beta_{t,z}} B^i_{t,z}$$

for $B^i_{t,z}$ given in Lemma 1.20 with $\Gamma = \Sigma^z_{t-1}$ or $\Gamma = A^z_{t-1}$ (defined in equations (15) *and* (16)*), then $\mathfrak{B}_t$ is a subset of an $\mathcal{F}_t$-measurable Q null set.*

Proof. First we consider the case $\Gamma = \Sigma^z_{t-1}$ and $B^i_{t,z}$ given in Lemma 1.20 with $\Gamma = \Sigma^z_{t-1}$. As in the previous proof, we denote the σ-algebra $\mathcal{F}_{t-1}$ with $\mathcal{A} := \mathcal{F}_{t-1}$. Notice that if $z \in \mathbf{Z}$ is such that $0 \in ri(\Delta S_t(\Gamma))^{cc}$ then $\cup_{i=1}^{\beta_{t,z}} B^i_{t,z} = \varnothing$, hence we may assume that $0 \notin ri(\Delta S_t(\Gamma))^{cc}$. From the proof of Lemma 1.29

$$\bigcup_{z \in \mathbf{Z}} D^z_t \subseteq N_0 = N \cup N^*$$

with $Q(N_0) = 0$. Notice that if $\omega \in \Omega \setminus N_0$ then, for all $z \in \mathbf{Z}$, either $\omega \notin \Sigma^z_{t-1}$ or $Q_{\mathcal{A}}(\omega, \cup_{i=1}^{\beta_{t,z}} B^i_{t,z}) = 0$. Hence $\omega \in \Sigma^z_{t-1} \setminus N_0$ implies $Q_{\mathcal{A}}(\omega, \cup_{i=1}^{\beta_{t,z}} B^i_{t,z}) = 0$. By Theorem 1.28 c) we have $Q_{\mathcal{A}}(\omega, (\Sigma^z_{t-1})^c) = 0$ for all $\omega \in \Sigma^z_{t-1} \setminus N_0$.

Fix now $\omega \in \Sigma^z_{t-1} \setminus N_0$ and consider the completion $\mathcal{F}_t^{Q_{\mathcal{A}}(\omega,\cdot)}$ of $\mathcal{F}_t$ and the unique extension on $\mathcal{F}_t^{Q_{\mathcal{A}}(\omega,\cdot)}$ of $Q_{\mathcal{A}}(\omega,\cdot)$, which we denote with $\widehat{Q}_{\mathcal{A}}(\omega,\cdot) : \mathcal{F}_t^{Q_{\mathcal{A}}(\omega,\cdot)} \to [0,1]$.

From $Q_{\mathcal{A}}(\omega, (\Sigma^z_{t-1})^c) = 0$ we deduce that $\mathfrak{B}_t \cap (\Sigma^z_{t-1})^c \in \mathcal{F}_t^{Q_{\mathcal{A}}(\omega,\cdot)}$ and $\widehat{Q}_{\mathcal{A}}(\omega, \mathfrak{B}_t \cap (\Sigma^z_{t-1})^c) = 0$. From $\mathfrak{B}_t \cap \Sigma^z_{t-1} = \cup_{i=1}^{\beta_{t,z}} B^i_{t,z}$ and $Q_{\mathcal{A}}(\omega, \cup_{i=1}^{\beta_{t,z}} B^i_{t,z}) = 0$ we deduce: $\mathfrak{B}_t \cap \Sigma^z_{t-1} \in \mathcal{F}_t^{Q_{\mathcal{A}}(\omega,\cdot)}$ and $\widehat{Q}_{\mathcal{A}}(\omega, \mathfrak{B}_t \cap \Sigma^z_{t-1}) = 0$. Then $\mathfrak{B}_t = (\mathfrak{B}_t \cap \Sigma^z_{t-1}) \cup (\mathfrak{B}_t \cap (\Sigma^z_{t-1})^c) \in \mathcal{F}_t^{Q_{\mathcal{A}}(\omega,\cdot)}$ and $\widehat{Q}_{\mathcal{A}}(\omega, \mathfrak{B}_t) = 0$. Since $\omega \in \Sigma^z_{t-1} \setminus N_0$ was arbitrary, we showed that $\widehat{Q}_{\mathcal{A}}(\omega, \mathfrak{B}_t) = 0$ for all $\omega \in \Sigma^z_{t-1} \setminus N_0$ and all $z \in \mathbf{Z}$. Since $\bigcup_{z \in \mathbf{Z}} (\Sigma^z_{t-1} \setminus N_0) = \Omega \setminus N_0$ we have:

$$\mathfrak{B}_t \in \mathcal{F}_t^{Q_{\mathcal{A}}(\omega,\cdot)} \text{ and } \widehat{Q}_{\mathcal{A}}(\omega, \mathfrak{B}_t) = 0 \text{ for all } \omega \in \Omega \setminus N_0 \text{ with } \ Q(N_0) = 0. \tag{22}$$

Now consider the σ-algebra

$$\widehat{\mathcal{F}}_t = \bigcap_{\omega \in \Omega \setminus N_0} \mathcal{F}_t^{Q_{\mathcal{A}}(\omega,\cdot)}$$

and observe that $\mathfrak{B}_t \in \widehat{\mathcal{F}}_t$. Notice that if a subset $B \subseteq \Omega$ satisfies: $B \subseteq C$ for some $C \in \mathcal{F}_t$ with $Q_{\mathcal{A}}(\omega, C) = 0$ for all $\omega \in \Omega \setminus N_0$, then

$$Q(C) = \int_{\Omega} Q_{\mathcal{A}}(\omega, C) Q(d\omega) = \int_{\Omega \setminus N_0} Q_{\mathcal{A}}(\omega, C) Q(d\omega) = 0,$$

so that $B \in \mathcal{F}_t^Q$. This shows that $\mathcal{F}_t \subseteq \widehat{\mathcal{F}}_t \subseteq \mathcal{F}_t^Q$. Hence $\mathfrak{B}_t \in \mathcal{F}_t^Q$. Let $\widehat{Q} : \widehat{\mathcal{F}}_t \to [0,1]$ be defined by $\widehat{Q}(\cdot) := \int_\Omega \widehat{Q}_\mathcal{A}(\omega,\cdot)Q(d\omega)$. Then $\widehat{Q}$ is a probability which satisfies $\widehat{Q}(B) = Q(B)$ for every $B \in \mathcal{F}_t$ and therefore is an extension on $\widehat{\mathcal{F}}_t$ of Q. Since $\overline{Q} : \mathcal{F}_t^Q \to [0,1]$ is the unique extension on $\mathcal{F}_t^Q$ of Q and $\mathcal{F}_t \subseteq \widehat{\mathcal{F}}_t \subseteq \mathcal{F}_t^Q$ then $\widehat{Q}$ is the restriction of $\overline{Q}$ on $\widehat{\mathcal{F}}_t$ and

$$\overline{Q}(\mathfrak{B}_t) = \widehat{Q}(\mathfrak{B}_t) = \int_\Omega \widehat{Q}_\mathcal{A}(\omega,\mathfrak{B}_t)Q(d\omega) = \int_{\Omega\setminus N_0} \widehat{Q}_\mathcal{A}(\omega,\mathfrak{B}_t)Q(d\omega) = 0.$$

Suppose now $A \in \mathcal{F}_{t-1}$, $\Gamma = A^z_{t-1}$ and set $\mathfrak{C}_t := \bigcup_{z\in\mathbf{Z}}\{\cup_{i=1}^{\beta_{t,z}} B^i_{t,z}\}$ where $B^i_{t,z}$ is given in Lemma 1.20 with $\Gamma = A^z_{t-1}$. Fix any $\omega \in A$. Then $\Sigma_t^{S_{0:T}(\omega)} \subseteq A$ since $A \in \mathcal{F}_{t-1}$. As a consequence $\mathfrak{C}_t \subseteq \mathfrak{B}_t$. $\square$

COROLLARY 1.31. *Fix $t \in I_1 = \{1,\dots,T\}$ and for $A \in \mathcal{F}_{t-1}$ consider $A^z_{t-1} = \{\omega \in A \mid S_{0:t-1}(\omega) = z_{0:t-1}\} \neq \varnothing$.*
Then for any $Q \in \mathcal{M}$ the set $\bigcup\{A^z_{t-1} \mid 0 \notin conv\big(\Delta S_t(A^z_{t-1})\big)\}$ is a subset of an $\mathcal{F}_{t-1}$-measurable Q-null set and as a consequence is an $\mathcal{M}$-polar set.

PROOF. From Corollary 1.27, the condition $0 \notin conv\big(\Delta S_t(A^z_{t-1})\big)$ implies that $\cup_{i=1}^{\beta_{t,z}} B^i_{t,z} = A^z_{t-1}$. From Theorem 1.28 we have $Q_\mathcal{A}(\omega, A^z_{t-1}) = 1$ on $A^z_{t-1}\setminus N$, $D^z_t = \{\omega \in A^z_{t-1} | Q_{\mathcal{F}_{t-1}}(\omega, A^z_{t-1}) > 0\} \supseteq A^z_{t-1} \setminus N$ and

$$\Big(\bigcup\{A^z_{t-1} \mid 0 \notin conv\big(\Delta S_t(A^z_{t-1})\big)\} \setminus N\Big) \subseteq \bigcup_{z\in\mathbf{Z}} D^z_t \subseteq N_0 \in \mathcal{F}_{t-1}.$$

$\square$

3.3.1. *Backward effect in the multiperiod case.* The following example shows that additional care is required in the multi-period setting:

EXAMPLE 1.32. *Let $\Omega = \{\omega_1,\omega_2,\omega_3,\omega_4\}$ and consider a single risky asset S_t with $t = 0,1,2$.*

$$S_0 = 7 \qquad S_1(\omega) = \begin{cases} 8 & \omega \in \{\omega_1,\omega_2\} \\ 3 & \omega \in \{\omega_3,\omega_4\} \end{cases} \qquad S_2(\omega) = \begin{cases} 9 & \omega = \omega_1 \\ 6 & \omega = \omega_2 \\ 5 & \omega = \omega_3 \\ 4 & \omega = \omega_4 \end{cases}$$

Fix $z \in \mathbf{Z}$ with the first two components (z_0, z_1) equal to $(7,3)$.
First period: *$\Sigma^z_0 = \Omega$ and $0 \in ri(conv(\Delta S_1(\Sigma^z_0))) = (-4,1)$ and there exists Q_1 such that $Q_1(\omega_i) > 0$ for $i = 1,2,3,4$ and $S_0 = E_{Q_1}[S_1]$. If we restrict the problem to the first period only, there exists a full support martingale measure for (S_0,S_1) and there are no $\mathcal{M}$-polar sets.*
Second period: *$\Sigma^z_1 = \{\omega_3,\omega_4\}$, $0 \notin conv(\Delta S_2(\Sigma^z_1)) = [1,2]$ and hence Σ^z_1 is not supported by any martingale measure for S, i.e. if $Q \in \mathcal{M}$ then $Q(\{\omega_3,\omega_4\}) = 0$.*
Backward*: As $\{\omega_3,\omega_4\}$ is a Q null set for any martingale measure $Q \in \mathcal{M}$, then $Q(\{\omega_1,\omega_2\}) = 1$. This reflects into the first period by means of $0 \notin conv(\Delta S_1(\{\omega_1,\omega_2\})) = \{1\}$ and we deduce that also $\{\omega_1,\omega_2\}$ is not supported by any martingale measure, implying $\mathcal{M} = \varnothing$.*
This example thus shows that new $\mathcal{M}$-polar sets (as $\{\omega_3,\omega_4\}$) can arise at later times creating a backward effect on the existence martingale measures. In order to detect these situations at time t, we shall need to anticipate certain polar sets at posterior times.

More formally we need to consider the following iterative procedure. Let

$$\begin{aligned}\Omega_T :=& \quad \Omega \\ \Omega_{t-1} :=& \quad \Omega_t \setminus \bigcup_{z\in\mathbf{Z}} \{\Sigma^z_{t-1} \mid 0 \notin conv\big(\Delta S_t(\widetilde{\Sigma}^z_{t-1})\big)\}, \quad t \in I_1,\end{aligned}$$

where

$$\widetilde{\Sigma}^z_{t-1} := \{\omega \in \Omega_t \mid S_{0:t-1} = z_{0:t-1}\}, \quad t \in I_1.$$

We show that the set $B^i_{t,z}$ obtained from Lemma 1.20 with $\Gamma = \widetilde{\Sigma}^z_t$ belong to the family of polar set of $\mathcal{M}(\mathbb{F})$:

$$\mathcal{N} := \{A \subseteq A' \in \mathcal{F} \;\mid\; Q(A') = 0 \;\forall\, Q \in \mathcal{M}(\mathbb{F})\}$$

More precisely,

LEMMA 1.33. *For all $t \in I_1$ and $z \in \mathbf{Z}$ consider the sets $B^i_{t,z}$ from Lemma 1.20 with $\Gamma = \widetilde{\Sigma}^z_{t-1}$. Let*

$$\widetilde{\mathfrak{B}}_t := \bigcup_{z\in\mathbf{Z}} \left\{\cup_{i=1}^{\beta_{t,z}} B^i_{t,z}\right\} \quad \mathfrak{D}_{t-1} := \bigcup_{z\in\mathbf{Z}} \left\{\Sigma^z_{t-1} \mid 0 \notin conv(\Delta S_t(\widetilde{\Sigma}^z_{t-1}))\right\}$$

For any $Q \in \mathcal{M}$, $\widetilde{\mathfrak{B}}_t$ is a subset of a $\mathcal{F}_t$-measurable Q-null set and $\mathfrak{D}_{t-1}$ is a subset of an $\mathcal{F}_{t-1}$-measurable Q-null set.

PROOF. We prove this by backward induction. For $t = T$ the assertion is true from Corollary 1.30 and Corollary 1.31. Suppose now the claim holds true for any $k+1 \le t \le T$. From the inductive hypothesis there exists $N^Q_k \in \mathcal{F}_k$ such that $\mathfrak{D}_k \subseteq N^Q_k$ with $Q(N^Q_k) = 0$. Introduce the auxiliary $\mathcal{F}_k$-measurable random variable

$$X^Q_k := S_{k-1}\mathbf{1}_{N^Q_k} + S_k\mathbf{1}_{(N^Q_k)^c} \tag{23}$$

and notice that $E_Q[X^Q_k \mid \mathcal{F}_{k-1}] = S_{k-1}$ Q-a.s. From $\Delta X^Q_k := X^Q_k - S_{k-1} = 0$ on N^Q_k and $\Omega \setminus N^Q_k \subseteq \Omega \setminus \mathfrak{D}_k$, we can deduce that

$$0 \notin ri(\Delta S_k(\widetilde{\Sigma}^z_{k-1}))^{cc} \Longrightarrow 0 \notin ri(\Delta X^Q_k(\Sigma^z_{k-1}))^{cc} \tag{24}$$

which implies $\widetilde{\mathfrak{B}}_k \subseteq \mathfrak{B}_k(X^Q_k) \cup N^Q_k$ where we denote $\mathfrak{B}_k(X^Q_k)$ the set obtained from Corollary 1.30 with $\Gamma = \Sigma^z_{k-1}$ and X^Q_k which replaces S_k. According to Corollary 1.30 we find $M^Q_k \in \mathcal{F}_k$ with $Q(M^Q_k) = 0$ so that $\widetilde{\mathfrak{B}}_k \subseteq \mathfrak{B}_k(X^Q_k) \cup N^Q_k \subseteq M^Q_k \cup N^Q_k$. Since Q is arbitrary we have the thesis.
We now show the second assertion.
For every $Q \in \mathcal{M}$ and $\underline{\varepsilon} = (\varepsilon, \dots, \varepsilon) \in \mathbb{R}^d$ with $\varepsilon > 0$ we can define

$$S^Q_k = (S_{k-1} + \underline{\varepsilon})\mathbf{1}_{N^Q_k \cup M^Q_k} + S_k\mathbf{1}_{(N^Q_k \cup M^Q_k)^c} \tag{25}$$

and $E_Q[S^Q_k \mid \mathcal{F}_{k-1}] = S_{k-1}$. With $\Delta S^Q_k := S^Q_k - S_{k-1}$ we claim

$$\mathfrak{D}_{k-1} \subseteq \bigcup_{z\in\mathbf{Z}} \{\Sigma^z_{k-1} \mid 0 \notin conv(\Delta S^Q_k(\Sigma^z_{k-1}))\}. \tag{26}$$

Indeed let $z \in \mathbf{Z}$ such that $\Sigma^z_{k-1} \subseteq \mathfrak{D}_{k-1}$ and observe that

$$0 \notin conv(\Delta S_k(\widetilde{\Sigma}^z_{k-1})) \Leftrightarrow 0 \notin conv(\Delta S_k(\Sigma^z_{k-1} \setminus \mathfrak{D}_k)). \tag{27}$$

Since $\Sigma^z_{k-1} \setminus N^Q_k \subseteq \Sigma^z_{k-1} \setminus \mathfrak{D}_k \subseteq \widetilde{\mathfrak{B}}_k \subseteq N^Q_k \cup M^Q_k$, then

$$\begin{aligned}\Sigma^z_{k-1} &= (\Sigma^z_{k-1} \cap N^Q_k) \cup (\Sigma^z_{k-1} \setminus N^Q_k) \subseteq N^Q_k \cup M^Q_k \\ &\subseteq \bigcup_{z \in \mathbf{Z}} \{\Sigma^z_{k-1} \mid 0 \notin conv(\Delta S^Q_k(\Sigma^z_{k-1}))\}\end{aligned}$$

for any $\Sigma^z_{k-1} \subseteq \mathfrak{D}_{k-1}$. Hence the claim since $\bigcup_z \{\Sigma^z_{k-1} \mid 0 \notin conv(\Delta S^Q_k(\Sigma^z_{k-1}))\}$ is a subset of an $\mathcal{F}_{k-1}$-measurable Q-null set. □

3.4. On the maximal $\mathcal{M}$-polar set and the support of martingale measures. The sets introduced in Sections 3.2 and 3.3.1 provide a geometric decomposition of Ω in two parts, $\Omega = \Omega_* \cup \Omega^c_*$ specified in Proposition 1.34 below. The set Ω_* contains those events ω supported by martingale measures, namely, for any of those events it is possible to construct a martingale measure (even with finite support) that assign positive probability to ω. Observe that such a decompostion is induced by S and it is determined prior to arbitrage considerations.

PROPOSITION 1.34. *Let $\{\Omega_t\}_{t \in I}$ as defined in Section 3.5 and, for any $z \in \mathbf{Z}$, let $\beta_{t,z}$ and $B^*_{t,z}$ be the index β and the set B^* from Lemma 1.20 with $\Gamma = \widetilde{\Sigma}^z_{t-1}$. Define*

$$\Omega_* := \bigcap_{t=1}^{T} \left(\bigcup_{z \in \mathbf{Z}} B^*_{t,z} \right).$$

We have the following

$$\mathcal{M} \neq \varnothing \iff \Omega_* \neq \varnothing \iff \mathcal{M} \cap \mathcal{P}_f \neq \varnothing,$$

where

$$\mathcal{P}_f := \{P \in \mathcal{P} \mid supp(P) \text{ is finite}\}$$

is the set of probability measures whose support is a finite number of $\omega \in \Omega$.
If $\mathcal{M} \neq \varnothing$ then for any $\omega_* \in \Omega_*$ there exists $Q \in \mathcal{M}$ such that $Q(\{\omega_*\}) > 0$, so that Ω^c_* is the maximal $\mathcal{M}$-polar set, i.e. Ω^c_* is an $\mathcal{M}$-polar set and

$$\forall N \in \mathcal{N} \text{ we have } N \subseteq \Omega^c_*. \tag{28}$$

PROOF. Observe first that:

$$\Omega^c_* = \bigcup_{t=1}^{T} \widetilde{\mathfrak{B}}_t.$$

From Lemma 1.33, $\widetilde{\mathfrak{B}}_t$ is an $\mathcal{M}$-polar set for any $t \in I_1$, which implies Ω^c_* is an $\mathcal{M}$-polar set. Suppose now that $\Omega_* = \varnothing$ so that $\Omega = \bigcup_{t=1}^T \widetilde{\mathfrak{B}}_t$ is a polar set. We can conclude that $\mathcal{M} = \varnothing$.
Suppose now that $\Omega_* \neq \varnothing$. We show that for every $\omega_* \in \Omega_*$ there exists a $Q \in \mathcal{M}$ such that $Q(\{\omega_*\}) > 0$. Observe now that for any $t \in I_1$ and for any $\omega \in \Omega_*$, $0 \in ri(\Delta S_t(B^*_{t,z}))^{cc}$ with $z = S_{0:T}(\omega)$. As we did in Corollary 1.25, we apply Remark 1.24 and conclude that there exists a finite number of elements of $B^*_{t,z}$, named $C_t(\omega) := \{\omega, \omega_1, \ldots, \omega_m\} \subseteq B^*_{t,z}$, such that

$$S_{t-1}(\omega) = \lambda_t(\omega) S_t(\omega) + \sum_{j=1}^{m} \lambda_t(\omega_j) S_t(\omega_j) \tag{29}$$

where $\lambda_t(\omega) > 0$ and $\lambda_t(\omega) + \sum_{j=1}^m \lambda_t(\omega_j) = 1$.
Fix now $\omega_* \in \Omega_*$. We iteratively build a set Ω^T_f which is suitable for being the finite support of a discrete martingale measure (and contains ω_*).

Start with $\Omega_f^1 = C_1(\omega_*)$ which satisfies (29) for $t=1$. For any $t>1$, given Ω_f^{t-1}, define $\Omega_f^t := \{C_t(\omega) \mid \omega \in \Omega_f^{t-1}\}$. Once Ω_f^T is settled, it is easy to construct a martingale measure via (29):

$$Q(\{\omega\}) = \prod_{t=1}^{T} \lambda_t(\omega) \quad \forall \omega \in \Omega_f^T$$

Since, by construction, $\lambda_t(\omega_*) > 0$ for any $t \in I_1$, we have $Q(\{\omega_*\}) > 0$ and $Q \in \mathcal{M} \cap \mathcal{P}_f$.

To show (28) just observe from the previous line that Ω_* is not $\mathcal{M}$-polar, while $\Omega_*^c = \bigcup_{t=1}^T \widetilde{\mathfrak{B}}_t$ is $\mathcal{M}$-polar thanks to Lemma 1.33. □

PROOF OF PROPOSITION 1.17. The absence of $1p$-Arbitrages readly implies that $\Omega_* = \Omega$ (see Corollary 1.27). Take a dense subset $\{\omega_n\}_{n\in\mathbb{N}}$ of Ω: from Proposition 1.34 for any ω_n there exists a martingale measure $Q^n \in \mathcal{M}$ such that $Q^n(\{\omega_n\}) > 0$. From Lemma 1.76 in the Appendix $Q := \sum_{i=1}^{\infty} \frac{1}{2^i} Q^i \in \mathcal{M}$, moreover $Q(\{\omega_n\}) > 0$ $\forall n \in \mathbb{N}$. Since $\{\omega_n\}_{n\in\mathbb{N}}$ is dense, Q is a full support martingale measure. □

3.5. Enlarged Filtration and Universal Arbitrage Aggregator. In Sections 3.2 and 3.3 we solve the problem of characterizing the $\mathcal{M}$-polar sets of a certain market model on a fixed time interval $[t-1,t]$ for $t \in I_1 = \{1,\dots,T\}$. In particular, if we look at the level sets Σ_{t-1}^z of the price process $(S_t)_{t\in I}$, we can identify the component of these sets that must be polar (Corollary 1.30) which coincides with the whole level set when $0 \notin conv(\Delta S_t(\Sigma_{t-1}^z))$ (Corollary 1.31). Further care is required in the multiperiod case due to the backward effects (see Section 3.3.1), but nevertheless a full characterization of $\mathcal{M}$-polar sets is obtained in Section 3.4 .

In this section we build a predictable strategy that embrace all the inefficiencies of the market. Unfortunately, even on a single time-step, the polar set given by Corollary 1.30 belongs, in general, to $\mathfrak{F}_t$ (the universal $\mathcal{M}$-completion), hence the trading strategies suggested by equation (18) in Lemma 1.20 fail to be predictable. This reflects into the necessity of enlargement of the original filtration by anticipating some one step-head information. Under this filtration enlargement, which depends only on the underlying structure of the market, the set of martingale measures will not change (see Lemma 1.41).

DEFINITION 1.35. *We call Universal Arbitrage Aggregator the strategy*

$$H_t^\bullet(\omega)\mathbf{1}_{\Sigma_{t-1}^z} := \begin{cases} H_{t,z}(\omega) & on \quad \bigcup_{i=1}^{\beta_{t,z}} B_{t,z}^i \\ 0 & on \quad \Sigma_{t-1}^z \setminus \bigcup_{i=1}^{\beta_{t,z}} B_{t,z}^i \end{cases} \tag{30}$$

for $t \in I_1 = \{1,\dots,T\}$, where $z \in \mathbf{Z}$ satisfies $z_{0:t-1} = S_{0:t-1}(\omega)$ and $H_{t,z}$, $B_{t,z}^i, B_{t,z}^*$ comes from (18) *and Lemma 1.20 with $\Gamma = \widetilde{\Sigma}_{t-1}^z$.*
This strategy is predictable with respect to the enlarged filtration $\widetilde{\mathbb{F}} = \{\widetilde{\mathcal{F}}_t\}_{t\in I}$ given by

$$\widetilde{\mathcal{F}}_t \;\;:\;\; = \mathcal{F}_t \vee \sigma(H_1^\bullet,\dots,H_{t+1}^\bullet),\ t \in \{0,\dots,T-1\} \tag{31}$$

$$\widetilde{\mathcal{F}}_T \;\;:\;\; = \mathcal{F}_T \vee \sigma(H_1^\bullet,\dots,H_T^\bullet). \tag{32}$$

REMARK 1.36. *The strategy $H^\bullet$ in equation* (30) *satisfies $V_T(H^\bullet) \geq 0$ and*

$$\mathcal{V}_{H^\bullet}^+ = \bigcup_{t=1}^{T} \widetilde{\mathfrak{B}}_t. \tag{33}$$

Indeed, from Lemma 1.20 $H_{t,z}\cdot\Delta S_t > 0$ *on* $\bigcup_{i=1}^{\beta_{t,z}} B^i_{t,z}$, *so that* $\bigcup_{t=1}^{T}\widetilde{\mathfrak{B}}_t \subseteq \mathcal{V}^+_{H^\bullet}$. *On the other hand,* $\mathcal{V}^+_{H^\bullet} \subseteq \{H^\bullet_t \neq 0$ *for some* $t\} \subseteq \bigcup_{t=1}^{T}\widetilde{\mathfrak{B}}_t$.
For $t<T$ *we therefore conclude that* $\widetilde{\mathcal{F}}_t \subseteq \mathcal{F}_t \vee \bigcup_{s=1}^{t+1}\mathcal{N}_s \subseteq \mathfrak{F}_t$, *where*

$$\mathcal{N}_t := \left\{ A = \bigcup_{z\in\mathbf{V}}\bigcup_{i\in J(z)} B^i_{t,z} \mid \textit{for some} \quad \begin{array}{c}\mathbf{V}\subseteq\mathbf{Z}\\ J(z)\subseteq\{1,\dots,\beta_{t,z}\}\end{array}\right\} \cup \mathfrak{D}_t,$$

while for $t=T$, $\widetilde{\mathcal{F}}_T \subseteq \mathcal{F}_T \vee \bigcup_{s=1}^{T}\mathcal{N}_s \subseteq \mathfrak{F}_T$.
For any $Q\in\mathcal{M}$ *and* $t\in I$, *any element of* $\mathcal{N}_t$ *is a subset of a* $\mathcal{F}_t$*-measurable* Q*-null set* .

From now on we will assume that the class of admissible trading strategies $\widetilde{\mathcal{H}}$ is given by all $\widetilde{\mathbb{F}}$ predictable processes. We can rewrite the definition of Arbitrage de la classe $\mathcal{S}$ using strategies adapted to $\widetilde{\mathbb{F}}$. Namely, an Arbitrage de la classe $\mathcal{S}$ with respect to $\widetilde{\mathcal{H}}$ is an $\widetilde{\mathbb{F}}$-predictable processes $H=[H^1,\dots,H^d]$ such that $V_T(H)\geq 0$ and $\{V_T(H)>0\}$ contains a set in $\mathcal{S}$.

Remark 1.37. *No Arbitrage de la classe* $\mathcal{S}$ *with respect to* $\widetilde{\mathcal{H}}$ *implies No Arbitrage de la classe* $\mathcal{S}$ *with respect to* $\mathcal{H}$.

Remark 1.38. **(Financial interpretation of the filtration enlargement)** *Fix* $t\in I_1$, $z\in\mathbf{Z}$, *the event* $\Sigma^z_{t-1}=\{S_{0:t-1}=z_{0:t-1}\}$ *and suppose the market presents the opportunity given by* $0\notin ri(\Delta S_t(\Sigma^z_{t-1}))^{cc}$. *Consider two probabilities* $P_k\in\mathcal{P}$, $k=1,2$, *for which* $P_k(\Sigma^z_{t-1})>0$. *Following Lemma 1.20, if* $j_k:=\inf\{i=1,\dots,\beta \mid P_k(B^i_{t,z})>0\}<\infty$, *then the rational choice for the strategy is* H^{j_k}, *as shown in Corollary 1.23. Thus it is possible that* $j_k<\infty$ *holds for both probabilities, so that the two agents represented by* P_1 *and* P_2 *agree that* Σ^z_{t-1} *is a non-efficient level set of the market, although it is possible that* $j_1\neq j_2$ *so that they might not agree on the trading strategy* H^{j_k} *that establish the* P_k*-Classical Arbitrage on* Σ^z_{t-1}. *In such case, these two arbitrages are realized on different subsets of* Σ^z_{t-1} *and generate different payoffs. Nevertheless note that any of these agents is able to find an arbitrage opportunity among the finite number of trading strategies* $\{H^i_{t,z}\}_{i=1}^{\beta_{t,z}}$ *given by Lemma 1.20 (recall* $\beta_{t,z}\leq d$*). The filtration enlargement allows to embrace them all. This can be referred to the analogous discussion in* [**DH07**]*: "A weak arbitrage opportunity is a situation where we know there must be an arbitrage but we cannot tell, without further information, what strategy will realize it".*

We expand on this argument more formally. Recall that Lemma 1.20 provides a partition of any level set $\widetilde{\Sigma}^z_{t-1}$ with the following property: for any $\omega\in\Omega^c_*$ there exists a unique set $B^i_{t,z}$, identified by $i=i(\omega)$, such that $\omega\in B^i_{t,z}$ with $z=S_{0:T}(\omega)$. Define therefore, for any $t\in I_1$ the multifunction

$$\mathbb{H}_t(\omega):=\left\{H\in\mathbb{R}^d \mid H\cdot\Delta S_t(\widehat{\omega})\geq 0 \text{ for any } \widehat{\omega}\in\cup_{j=i(\omega)}^{\beta_{t,z}}B^j_{t,z}\cup B^*_{t,z}\right\} \tag{34}$$

if $\omega\in\Omega^c_*$ and $\mathbb{H}_t(\omega)=\{0\}$ otherwise.
Observe that for any $t\in I_1$, if ω_1,ω_2 satisfy $S_{0:t-1}(\omega_1)=S_{0:t-1}(\omega_2)$ and $i(\omega_1)=i(\omega_2)$ they belong to the same $B^i_{t,z}$ and $\mathbb{H}_t(\omega_1)=\mathbb{H}_t(\omega_2)$. In other words $\mathbb{H}_t$ is constant on any $B^i_{t,z}$ and therefore for any open set $V\subseteq\mathbb{R}^d$ we have

$$\{\omega\in\Omega \mid \mathbb{H}_t(\omega)\cap V\neq\varnothing\}=\bigcup_{z\in\mathbf{Z}}\bigcup_{i=1}^{\beta_{t,z}}\{B^i_{t,z} \mid \mathbb{H}_t(B^i_{t,z})\cap V\neq\varnothing\}$$

from which $\mathbb{H}_t$ is measurable with respect to $\mathcal{F}_{t-1} \vee \bigcup_{s=1}^{t} \mathcal{N}_s$. Note that since $H_t^\bullet(\omega) \in \mathbb{H}_t(\omega)$ for any $\omega \in \Omega$, we have that $H_t^\bullet$ is a selection of $\mathbb{H}_t$ with the same measurability. We now show how the process $\mathbb{H} := (\mathbb{H}_t)_{t\in I_1}$ provides P-Classical Arbitrage as soon as we choose a probabilistic model $P \in \mathcal{P}$ which is not absolutely continuous with respect to the capacity $\nu(A) := \sup_{Q\in\mathcal{M}} Q(A)$, $A \in \mathcal{F}$ (see Lemma 1.68 for more details on the properties of ν). The case of $P \ll \nu$ is discussed in Remark 1.40.

THEOREM 1.39. *Let $\mathbb{H}$ be defined in* (34)*. If $P \in \mathcal{P}$ is not absolutely continuous with respect to ν then there exists an $\mathbb{F}^P$-predictable trading strategy H^P which is a P-Classical Arbitrage and*

$$H^P(\omega) \in \mathbb{H}(\omega) \qquad P\text{-}a.s.$$

where $\mathcal{F}_t^P$ denote the P-completion of $\mathcal{F}_t$ and $\mathbb{F}^P := \{\mathcal{F}_t^P\}_{t\in I}$.

PROOF. See Appendix 6.1. □

From Lemma 1.68 if $P \in \mathcal{P}$ fulfills the hypothesis of Theorem 1.39 there exists an $\mathcal{F}$-measurable set $F \subseteq (\Omega_*)^c$ with $P(F) > 0$. Note that from Remark 1.69 such a P always exists if $\Omega_*^c \neq \varnothing$. Theorem 1.39 asserts therefore that for any probabilistic models which supports Ω_*^c an $\mathbb{F}^P$-*predictable* arbitrage opportunity can be found among the values of the set-valued process $\mathbb{H}$. This property suggested us to baptize $\mathbb{H}$ as the Universal Arbitrage Aggregator and thus $H^\bullet$ as a (standard) selection of the Universal Arbitrage Aggregator. Note that we could have considered a different selection of $\mathbb{H}$ satisfying the essential requirement (33). Since this choice does not affect any of our results we simply take $H^\bullet$.

REMARK 1.40. *Recall from (5) that any $P \in (\mathcal{P}_0)^c$ admits a P-Classical Arbitrage opportunity. We can distinguish between two different classes in $(\mathcal{P}_0)^c$.*
The first one is: $\mathcal{P}_{\mathcal{M}} := \{P \in (\mathcal{P}_0)^c \mid P << \nu\}$ or, in other words, an element $P \in (\mathcal{P}_0)^c$ belong to $\mathcal{P}_{\mathcal{M}}$ iff any subset of Ω_*^c is P-null. Then for each probability P in this class, there exists a probability P' with larger support that annihilates any P-Classical Arbitrage opportunity. Recall Example 1.26 where $\Omega_* = \mathbb{Q}\cap[1/2,+\infty)$. By choosing $P = \delta_{\{\frac{1}{2}\}} \in \mathcal{P}_{\mathcal{M}}$ we clearly have P-Classical Arbitrages. Nevertheless by simply taking $P' = \lambda\delta_{\{\frac{1}{2}\}} + (1-\lambda)\delta_{\{2\}}$ for some $0 < \lambda < 1$ this market is arbitrage free. From a model-independent point of view these situations must not considered as market inefficiencies since they vanish as soon as more trajectories are considered. This feature is captured by the Universal Arbitrage Aggregator by means of the property: $H^\bullet = 0$ on Ω_*.
On the other hand when $P \in (\mathcal{P}_0)^c \setminus \mathcal{P}_{\mathcal{M}}$ then P assigns a positive measure to some $\mathcal{M}$-polar $\mathcal{F}$-measurable set $F \in \mathcal{N}$. Therefore, any other $P' \in \mathcal{P}$ with larger support will satisfy $P'(F) > 0$ and the probabilistic model $(\Omega, \mathcal{F}, \mathbb{F}, S, P')$ will also exhibit P'-Classical Arbitrages. In the case of Example 1.26 $\Omega_*^c = B^1 \cup B^2$ where $B^1 = \mathbb{R}^+ \setminus \mathbb{Q}$ and $B^2 = \mathbb{Q} \cap [0, 1/2)$. If $P(\Omega_*^c) > 0$ the market exhibits a P-Classical Arbitrage, but this is still valid for any probabilistic model given by P' with $P << P'$. In particular if $P'(B^1) > 0$ then $H^1 := [0, 0, 1]$ is a P'-Classical Arbitrage, while if $P'(B^1) = 0$ and $P'(B^2) > 0$ then $H^2 := [1, 0, 0]$ is the desired strategy. In this example, $H_1^\bullet = H^1 1_{\{\mathbb{R}^+\setminus\mathbb{Q}\}} + H^2 1_{\{\mathbb{Q}\cap[0,1/2)\}}$.

LEMMA 1.41. *$\mathcal{M}(\mathbb{F}) \leftrightarrows \mathcal{M}(\widetilde{\mathbb{F}})$ with the following meaning*

- *the restriction of any* $\widetilde{Q} \in \mathcal{M}(\widetilde{\mathbb{F}})$ *to* $\mathcal{F}_T$ *belongs to* $\mathcal{M}(\mathbb{F})$*;*
- *any* $Q \in \mathcal{M}(\mathbb{F})$ *can be uniquely extended to an element of* $\mathcal{M}(\widetilde{\mathbb{F}})$

Proof. Let $\widetilde{Q} \in \mathcal{M}(\widetilde{\mathbb{F}})$ and $Q \in \mathcal{P}(\Omega)$ be the restriction to $\mathcal{F}_T$. For any $t \in I_1$ and $A \in \mathcal{F}_{t-1}$ we have $E_Q[(S_t - S_{t-1})\mathbf{1}_A] = E_{\widetilde{Q}}[(S_t - S_{t-1})\mathbf{1}_A] = 0$. Let now $Q \in \mathcal{M}(\mathbb{F})$. There exists a unique extension to $\widetilde{\mathcal{F}}_T$ of Q that we call $\widetilde{Q}$. For any $\widetilde{A} \in \widetilde{\mathcal{F}}_{t-1}$ with $t \in I_1$ there exists $A \in \mathcal{F}_{t-1}$ such that $\widetilde{Q}(\widetilde{A}) = \widetilde{Q}(A) = Q(A)$. Hence $E_{\widetilde{Q}}[(S_t - S_{t-1})\mathbf{1}_{\widetilde{A}}] = E_{\widetilde{Q}}[(S_t - S_{t-1})\mathbf{1}_A] = E_Q[(S_t - S_{t-1})\mathbf{1}_A] = 0$, where the first equality follows from $\widetilde{Q}(\widetilde{A} \backslash A) = 0$ and the second one from the $\mathcal{F}_T$-measurability of $(S_t - S_{t-1})\mathbf{1}_A$. We conclude that $E_{\widetilde{Q}}[S_t \mid \widetilde{\mathcal{F}}_{t-1}] = S_{t-1}$, hence $\widetilde{Q} \in \mathcal{M}(\widetilde{\mathbb{F}})$. □

Remark 1.42. *The filtration enlargement* $\widetilde{\mathbb{F}}$ *has been introduced to guarantee the aggregation of* 1*p-Arbitrages on the sets* $B^i_{t,z}$ *obtained from Lemma 1.20 with* $\Gamma = \widetilde{\Sigma}^z_{t-1}$*. If indeed we follow* [**C12**] *we can consider any collection of probability measures* $\Theta_t := \{P^i_{t,z}\}$ *on* $(\Omega, \mathcal{F})$ *such that* $P^i_{t,z}(B^i_{t,z}) = 1$*. Observe first that*

$$\mathcal{F}_t^{\Theta_t} \supseteq \sigma\left(\bigcup\{B^i_{t,z} \mid z \in \mathbf{V}, i \in J(z)\}\right)$$

with $\mathbf{V}$ *and* $J(z)$ *arbitrary. For any* $P^i_{t,z}$ *we have indeed that* $\mathcal{F}_t^{P^i_{t,z}}$ *contains any subset of* $(B^i_{t,z})^c$*. Therefore if* $A = \bigcup\{B^i_{t,z} \mid z \in \mathbf{V},\, i \in J(z)\}$ *we have*

- *if* $z \notin \mathbf{V}$ *or* $i \notin J(z)$ *then* $A \in \mathcal{F}_t^{P^i_{t,z}}$ *trivially because* $A \subset (B^i_{t,z})^c$
- *if* $z \in \mathbf{V}$ *and* $i \in J(z)$ *then* $A \in \mathcal{F}_t^{P^i_{t,z}}$ *because* $A = B^i_{t,z} \cup \bar{A}$ *with* $\bar{A} \subseteq (B^i_{t,z})^c$

It is easy to check that Θ_t *has the Hahn property on* $\mathcal{F}_t$ *as defined in Definition 3.2,* [**C12**]*, with* $\Phi_t := \Theta_t \mid_{\mathcal{F}_t}$*. We can therefore apply Theorem 3.16 in* [**C12**] *to find an* $\mathcal{F}_t^{\Theta_t}$*- measurable function* H_t *such that* $H_t = H^i_{t,z}$ $P^i_{t,z}$*-a.s. which means that* $H_t(\omega) = H^i_{t,z}$ *for every* $\omega \in B^i_{t,z}$*.*

3.6. Main Results. Our aim now is to show how the absence of arbitrage de la classe $\mathcal{S}$ provides a pricing functional *via* the existence of a martingale measure with nice properties. Clearly the "No 1p-Arbitrage" condition is the strongest that one can assume in this model independent framework and we have shown in Proposition 1.17 that it automatically implies the existence of a full support martingale measure. On the other hand we are interested in characterizing those markets which can exhibit 1p-Arbitrages but nevertheless admits a rational system of pricing rules.

The set Ω_* introduced in Section 3.4 has a clear financial interpretation as it represents the set of events for which No 1p-Arbitrage can be found. This is the content of the following Proposition. Let $(\Omega, \widetilde{\mathcal{F}}_T, \widetilde{\mathbb{F}})$, $\widetilde{\mathcal{H}}$ as in Section 3.5 and define

$$\widetilde{\mathcal{H}}^+ := \left\{H \in \widetilde{\mathcal{H}} \mid \ V_T(H)(\omega) \geq 0 \,\forall\, \omega \in \Omega \text{ and } V_0(H) = 0\right\}.$$

Proposition 1.43. (1) $\mathcal{V}^+_{H^\bullet} = \bigcup_{H \in \widetilde{\mathcal{H}}^+} \mathcal{V}^+_H = \Omega^c_*$

(2) $\mathcal{M} \neq \varnothing$ *if and only if* $\bigcup_{H \in \widetilde{\mathcal{H}}^+} \mathcal{V}^+_H$ *is strictly contained in* Ω*.*

Proof. (2) follows from (1) and Proposition 1.34. Indeed: $\mathcal{M} \neq \varnothing$ iff $\Omega_* \neq \varnothing$ iff $\Omega^c_* \subsetneq \Omega$ iff $\bigcup_{H \in \widetilde{\mathcal{H}}^+} \mathcal{V}^+_H \subsetneq \Omega$. Now we prove (1). Given (33), we only need to show the inclusion $\bigcup_{H \in \widetilde{\mathcal{H}}^+} \mathcal{V}^+_H \subseteq \Omega^c_*$. Let $\overline{\omega} \in \bigcup_{H \in \widetilde{\mathcal{H}}^+} \mathcal{V}^+_H$, then there exists $\overline{H} \in \widetilde{\mathcal{H}}^+$ and $t \in I_1$ such that $\overline{H}_t(\omega) \cdot \Delta S_t(\omega) \geq 0$ $\forall \omega \in \Omega$ and $\overline{H}_t(\overline{\omega}) \cdot \Delta S_t(\overline{\omega}) > 0$. Let $z = S_{0:T}(\overline{\omega})$. From Lemma 1.20 there exists $i \in \{1, \ldots, \beta_{t,z}\}$ such that $\overline{\omega} \in B^i_{t,z}$ hence we conclude that $\overline{\omega} \in \widetilde{\mathfrak{B}}_t$ and therefore $\overline{\omega} \in \Omega^c_*$. □

PROOF OF THEOREM 1.2. We prove that

$$\exists \text{ an Arbitrage de la classe } \mathcal{S} \text{ in } \widetilde{\mathcal{H}} \iff \mathcal{M} = \varnothing \text{ or } \mathcal{N} \text{ contains sets of } \mathcal{S}.$$

Notice that if $H \in \widetilde{\mathcal{H}}$ satisfies $V_T(H)(\omega) \geq 0$ $\forall \omega \in \Omega$ then, if $\mathcal{M} \neq \varnothing$, $\mathcal{V}_H^+ \in \mathcal{N}$, otherwise $0 < \mathbb{E}_Q[V_T(H)] = V_0(H) = 0$ for $Q \in \mathcal{M}$. If there exists an $\widetilde{\mathcal{H}}$-Arbitrage de la classe $\mathcal{S}$ then $\mathcal{V}_H^+$ contains a set in $\mathcal{S}$ and therefore $\mathcal{N}$ contains a set in $\mathcal{S}$. If instead $\mathcal{M} = \varnothing$ we already have the thesis. For the opposite implication, we exploit the Universal Arbitrage $H^\bullet \in \widetilde{\mathcal{H}}$ as defined in equation (30) satisfying $V_T(H^\bullet)(\omega) \geq 0$ $\forall \omega \in \Omega$ and $\mathcal{V}_{H^\bullet}^+ = \bigcup_{t=1}^T \widetilde{\mathfrak{B}}_t = \Omega_*^c$. If $\mathcal{M} = \varnothing$ then, by Proposition 1.34, $\Omega_*^c = \Omega$ and $H^\bullet$ is an $\widetilde{\mathcal{H}}$-Model Independent Arbitrage and hence (from (7)) $H^\bullet$ is also an Arbitrage de la classe $\mathcal{S}$. If $\mathcal{M} \neq \varnothing$ and $\mathcal{N}$ contains a set C in $\mathcal{S}$ then $C \subseteq \Omega_*^c = \mathcal{V}_{H^\bullet}^+$ from (28) and Proposition 1.43, item 1. Therefore $H^\bullet$ is an $\widetilde{\mathcal{H}}$-Arbitrage de la classe $\mathcal{S}$. □

DEFINITION 1.44. *Define the following convex subset of* $\mathcal{P}$*:*

$$\mathcal{R}_\mathcal{S} := \{Q \in \mathcal{P} \mid Q(C) > 0 \text{ for all } C \in \mathcal{S}\}. \tag{35}$$

The martingale measures having the property of the class $\mathcal{R}_\mathcal{S}$ will be associated to the Arbitrage de la classe $\mathcal{S}$.

EXAMPLE 1.45. *We consider the examples introduced in Definition 1.10. Suppose there are no Model Independent Arbitrage in* $\widetilde{\mathcal{H}}$*. From Theorem 1.2 we obtain:*

(1) $1p$*-Arbitrage:* $\mathcal{S} = \{C \in \mathcal{F} \mid C \neq \varnothing\}$.
- *No* $1p$*-Arbitrage in* $\widetilde{\mathcal{H}}$ *iff* $\mathcal{N} = \varnothing$;
- $\mathcal{R}_\mathcal{S} = \mathcal{P}_+$*, if* Ω *finite or countable; otherwise* $\mathcal{R}_\mathcal{S} = \varnothing$.
- *In the case of np-Arbitrage we have:*

$$\mathcal{R}_\mathcal{S} = \{Q \in \mathcal{P} \mid Q(A) > 0 \text{ for all } A \subseteq \Omega \text{ having at least } n \text{ elements}\}$$

No np-Arbitrage in $\widetilde{\mathcal{H}}$ *iff* $\mathcal{N}$ *does not contain elements having more than* $n-1$ *elements.*

(2) Open Arbitrage: $\mathcal{S} = \{C \in \mathcal{B}(\Omega) \mid C \text{ open non-empty}\}$.
- *No Open Arbitrage in* $\widetilde{\mathcal{H}}$ *iff* $\mathcal{N}$ *does not contain non-empty open sets;*
- $\mathcal{R}_\mathcal{S} = \mathcal{P}_+$.

(3) $\mathcal{P}'$*-q.s. Arbitrage:* $\mathcal{S} = \{C \in \mathcal{F} \mid P(C) > 0 \text{ for some } P \in \mathcal{P}'\}$, $\mathcal{P}' \subseteq \mathcal{P}$.
- *No* $\mathcal{P}'$*-q.s. Arbitrage in* $\widetilde{\mathcal{H}}$ *iff* $\mathcal{N}$ *may contain only* $\mathcal{P}'$*-polar sets;*
- $\mathcal{R}_\mathcal{S} = \{Q \in \mathcal{P} \mid P' \ll Q \text{ for all } P' \in \mathcal{P}'\}$.

(4) P*-a.s. Arbitrage:* $\mathcal{S} = \{C \in \mathcal{F} \mid P(C) > 0\}$, $P \in \mathcal{P}$.
- *No* P*-a.s. Arbitrage in* $\widetilde{\mathcal{H}}$ *iff* $\mathcal{N}$ *may contain only* P*-null sets;*
- $\mathcal{R}_\mathcal{S} = \{Q \in \mathcal{P} \mid P \ll Q\}$.

(5) Model Independent Arbitrage: $\mathcal{S} = \{\Omega\}$.
- $\mathcal{R}_\mathcal{S} = \mathcal{P}$.

(6) ε*-Arbitrage:* $\mathcal{S} = \{C_\varepsilon(\omega) \mid \omega \in \Omega\}$*, where* $\varepsilon > 0$ *is fixed and* $C_\varepsilon(\omega)$ *is the closed ball in* (Ω, d) *of radius* ε *and centered in* ω.
- *No* ε*-Arbitrage in* $\widetilde{\mathcal{H}}$ *iff* $\mathcal{N}$ *does not contain closed balls of radius* ε*;*
- $\mathcal{R}_\mathcal{S} = \{Q \in \mathcal{P} \mid Q(C_\varepsilon(\omega)) > 0 \text{ for all } \omega \in \Omega\}$.

Corollary 1.46. *Suppose that the class $\mathcal{S}$ has the property:*

$$\exists \{C_n\}_{n\in\mathbb{N}} \subseteq \mathcal{S} \text{ s.t. } \forall C \in \mathcal{S}\ \exists \overline{n} \text{ s.t. } C_{\overline{n}} \subseteq C. \tag{36}$$

Then:

$$\text{No Arb. de la classe } \mathcal{S} \text{ in } \widetilde{\mathcal{H}} \Longleftrightarrow \mathcal{M} \cap \mathcal{R}_{\mathcal{S}} \neq \varnothing. \tag{37}$$

Proof. Suppose $Q \in \mathcal{M} \cap \mathcal{R}_{\mathcal{S}} \neq \varnothing$. Then any polar set $N \in \mathcal{N}$ does not contain sets in $\mathcal{S}$ (otherwise, if $C \in \mathcal{S}$ and $C \subseteq N$ then $Q(C) > 0$ and $Q(N) = 0$, a contradiction). Then, from Theorem 1.2, No Arbitrage de la classe $\mathcal{S}$ holds true. Conversely, suppose that No Arbitrage de la classe $\mathcal{S}$ holds true so that $\mathcal{M} \neq \varnothing$ and let $\{C_n\}_{n\in C} \subseteq \mathcal{S}$ be the collection of sets in the assumption. From Theorem 1.2, we obtain that $N \in \mathcal{N}$ does not contain any set in $\mathcal{S}$, and so each set C_n is not a polar set, hence for each n there exists $Q_n \in \mathcal{M}$ such that $Q_n(C_n) > 0$. Set $Q := \sum_{n=1}^{\infty} \frac{1}{2^n} Q_n \in \mathcal{M}$ (see Lemma 1.76). Take any $C \in \mathcal{S}$ and let $C_{\overline{n}} \subseteq C$. Then

$$Q(C) \geq \frac{1}{2^{\overline{n}}} Q_{\overline{n}}(C) \geq \frac{1}{2^{\overline{n}}} Q_{\overline{n}}(C_{\overline{n}}) > 0$$

and $Q \in \mathcal{M} \cap \mathcal{R}_{\mathcal{S}}$. □

Corollary 1.47. *Let $\mathcal{S}$ be the class of non empty open sets. Then the condition (36) is satisfied and therefore*

$$\text{No Open Arbitrage in } \widetilde{\mathcal{H}} \Longleftrightarrow \mathcal{M}_+ \neq \varnothing. \tag{38}$$

Proof. Consider a dense countable subset $\{\omega_n\}_{n\in\mathbb{N}}$ of Ω, as Ω is Polish. Consider the open balls:

$$B^m(\omega_n) := \left\{ \omega \in \Omega \mid d(\omega, \omega_n) < \frac{1}{m} \right\},\ m \in \mathbb{N},$$

The density of $\{\omega_n\}_{n\in\mathbb{N}}$ implies that $\Omega = \bigcup_{n\in\mathbb{N}} B^m(\omega_n)$ for any $m \in \mathbb{N}$. Take any open set $C \subseteq \Omega$. Then there exists some $\overline{n}$ such that $\omega_{\overline{n}} \in C$. Take $\overline{m} \in \mathbb{N}$ sufficiently big so that $B^{\overline{m}}(\omega_{\overline{n}}) \subseteq C$. □

Corollary 1.48. *Suppose that Ω is finite or countable. Then the condition (36) is fulfilled and therefore:*

$$\text{No Arb. de la classe } \mathcal{S} \text{ in } \widetilde{\mathcal{H}} \Longleftrightarrow \mathcal{M} \cap \mathcal{R}_{\mathcal{S}} \neq \varnothing. \tag{39}$$

In particular:

$$\text{No 1}p\text{-Arbitrage in } \widetilde{\mathcal{H}} \quad \Longleftrightarrow \quad \mathcal{M}_+ \neq \emptyset. \tag{40}$$

$$\text{No } P\text{-a.s. Arbitrage in } \widetilde{\mathcal{H}} \quad \Longleftrightarrow \quad \exists\, Q \in \mathcal{M} \text{ s.t. } P \ll Q. \tag{41}$$

$$\text{No } \mathcal{P}'\text{-q.s. Arbitrage in } \widetilde{\mathcal{H}} \quad \Longleftrightarrow \quad \exists\, Q \in \mathcal{M} \text{ s.t. } P' \ll Q\ \forall P' \in \mathcal{P}'. \tag{42}$$

Proof. Define $\mathcal{S}_0 := \{\{\omega\} \mid \omega \in \Omega \text{ such that there exists } C \in \mathcal{S} \text{ with } \omega \in C\}$. Then $\mathcal{S}_0$ is at most a countable set and satisfies condition (36). □

Remark 1.49. *While (40) holds also for 1p-Arbitrage in $\mathcal{H}$ (see Proposition 1.65)), (41) and (42) can not be improved. Indeed, by replacing in the example (10) $\mathbb{R}^+$ with $\mathbb{Q}^+$ and $\mathbb{Q}^+$ with $\mathbb{N}$, Ω is countable, we still have $\mathcal{M} = \varnothing$ but there are No P-a.s. Arbitrage in $\mathcal{H}$ if $P(\mathbb{Q}^+ \setminus \mathbb{N}) = 0$ (see Section 3.1, item 5 (a)).*

Remark 1.50. *There are other families of sets satisfying condition (36). For example, in a topological setting, nowhere dense subset of Ω (those having closure with empty interior) are often considered "negligible" sets. Then the class of sets which are the complement of nowhere dense sets, satisfies condition (36).*

Remark 1.51. *Condition (36) is not necessary to obtain the desired equivalence (37). Consider for example the class $\mathcal{S}$ defining ε-Arbitrage in Example 1.45 item 6. In such a case condition (36) fails, as soon as Ω is uncountable. However, we now prove that (37) holds true, when $\Omega = \mathbb{R}$. We already know by the previous proof that $\mathcal{M} \cap \mathcal{R}_{\mathcal{S}} \neq \varnothing$ implies No Arbitrage de la classe $\mathcal{S}$ in $\widetilde{\mathcal{H}}$. For the converse, from No Arbitrage de la classe $\mathcal{S}$ in $\widetilde{\mathcal{H}}$ we know that each element in $\mathcal{S} := \{[r-\varepsilon, r+\varepsilon] \mid r \in \mathbb{R}\}$ is not a polar set. Consider the countable class*

$$G := \{[q-\varepsilon, q+\varepsilon] \mid q \in \mathbb{Q}\} \subseteq \mathcal{S}.$$

Each set $G_n \in G$ is not a polar set, hence for each n there exists $Q_n \in \mathcal{M}$ such that $Q_n(G_n) > 0$. Set $\overline{Q} := \sum_{n=1}^{\infty} \frac{1}{2^n} Q_n \in \mathcal{M}$ (see Lemma 1.76). The set

$$D := \{r \in \mathbb{R} \mid \overline{Q}([r-\varepsilon, r+\varepsilon]) = 0\}$$

is at most countable. Indeed, any two distinct intervals $J := [r-\varepsilon, r+\varepsilon]$ and $J' := [r'-\varepsilon, r'+\varepsilon]$, with $r, r' \in D$, must be disjoint, otherwise for a rational q between r and r' we would have: $[q-\varepsilon, q+\varepsilon] \subseteq J \cup J'$ and thus $\overline{Q}([q-\varepsilon, q+\varepsilon]) = 0$, which is impossible by construction of $\overline{Q}$. For each $r_n \in D$ the set $[r_n-\varepsilon, r_n+\varepsilon] \in \mathcal{S}$ is not a polar set, hence for each n there exists $\widehat{Q}_n \in \mathcal{M}$ such that $\widehat{Q}_n([r_n-\varepsilon, r_n+\varepsilon]) > 0$. Set $\widehat{Q} := \sum_{n=1}^{\infty} \frac{1}{2^n} \widehat{Q}_n \in \mathcal{M}$. Thus $Q := \frac{1}{2}\overline{Q} + \frac{1}{2}\widehat{Q} \in \mathcal{M} \cap \mathcal{R}_{\mathcal{S}}$ is the desired measure.

4. Feasible Markets

We extend the classical notion of arbitrage with respect to a single probability measure $P \in \mathcal{P}$ to a class of probabilities $\mathcal{R} \subseteq \mathcal{P}$ as follows:

Definition 1.52. *The market admits **$\mathcal{R}$-Arbitrage** if*

- *for all $P \in \mathcal{R}$ there exists a P-Classical Arbitrage.*

We denote with No $\mathcal{R}$-Arbitrage the property: for some $P \in \mathcal{R}$, $NA(P)$ holds true.

Remark 1.53 (Financial interpretation of $\mathcal{R}$-Arbitrage.). *If a model admits an $\mathcal{R}$-Arbitrage then the agent will not be able to find a fair pricing rule, whatever model $P \in \mathcal{R}$ he will choose. However, the presence of an $\mathcal{R}$-Arbitrage only implies that for each P there exists a trading strategy H^P which is a P-Classical Arbitrage and this is a different concept respect to the existence of one single trading strategy H that realizes an arbitrage for all $P \in \mathcal{R}$. In the particular case of $\mathcal{R} = \mathcal{P}$ this notion was firstly introduced in* [**DH07**] *as "Weak Arbitrage opportunity" and further studied in* [**CO11, DOR14**] *and the reference therein. The No $\mathcal{R}$-Arbitrage property above should not be confused with the condition $NA(\mathcal{R})$ introduced by Bouchard and Nutz* [**BN15**] *and recalled in Section 3 as well as in Definition 1.10, item 3.*

We set:

$$\mathcal{P}_e(P) = \{P' \in \mathcal{P} \mid P' \sim P\}, \qquad \mathcal{M}_e(P) = \{Q \in \mathcal{M} \mid Q \sim P\}$$

In discrete time financial markets the Dalang-Morton-Willinger Theorem applies, so that $NA(P)$ iff $\mathcal{M}_e(P) \neq \varnothing$.

Proposition 1.54. *Suppose that $\mathcal{R} \subseteq \mathcal{P}$ has the property: $P \in \mathcal{R}$ implies $\mathcal{P}_e(P) \subseteq \mathcal{R}$. Then*

$$\text{No } \mathcal{R}\text{-Arbitrage iff } \mathcal{M} \cap \mathcal{R} \neq \varnothing.$$

In particular

$$\begin{aligned} \text{No } \mathcal{R}_{\mathcal{S}}\text{-Arbitrage iff } \mathcal{M} \cap \mathcal{R}_{\mathcal{S}} &\neq \varnothing, \\ \text{No } \mathcal{P}_{+}\text{-Arbitrage iff } \mathcal{M}_{+} &\neq \varnothing, \\ \text{No } \mathcal{P}\text{-Arbitrage iff } \mathcal{M} &\neq \varnothing. \end{aligned}$$

where $\mathcal{R}_{\mathcal{S}}$ is defined in (35) *and all arbitrage conditions here are with respect to $\mathcal{H}$.*

Proof. Suppose $Q \in \mathcal{M} \cap \mathcal{R} \neq \varnothing$. Since $Q \in \mathcal{R}$ and $NA(Q)$ holds true we have No $\mathcal{R}$-Arbitrage. Viceversa, suppose No $\mathcal{R}$-Arbitrage holds true. Then there exists $P \in \mathcal{R}$ for which $NA(P)$ holds true and therefore there exists $Q \in \mathcal{M}_e(P)$. The assumption $\mathcal{P}_e(P) \subseteq \mathcal{R}$ implies $Q \in \mathcal{M}_e(P) := \mathcal{M} \cap \mathcal{P}_e(P) \subseteq \mathcal{M} \cap \mathcal{R}$. The particular cases follows from the fact that $\mathcal{R}_{\mathcal{S}}$ has the property: $P \in \mathcal{R}_{\mathcal{S}}$ implies $\mathcal{P}_e(P) \subseteq \mathcal{R}_{\mathcal{S}}$. □

Remark 1.55. *As a result of the previous proposition, whenever (37), (38), (39) hold true each (equivalent) condition in (37), (38), (39) is also equivalent to: No $\mathcal{R}_{\mathcal{S}}$-Arbitrage in $\mathcal{H}$ (with $\mathcal{S} :=$ {open sets} for (38)).*

Given the measurable space $(\Omega, \mathcal{F})$ and the price process S defined on it, in this section we investigate the properties of the set of arbitrage free (for S) probabilities on $(\Omega, \mathcal{F})$. A minimal reasonable requirement on the financial market is the existence of at least one probability $P \in \mathcal{P}$ that does not allow any P-Classical Arbitrage. Recall from the Introduction the definition of the set

$$\mathcal{P}_0 = \{P \in \mathcal{P} \mid \mathcal{M}_e(P) \neq \varnothing\}.$$

By Proposition 1.54 and the definition of $\mathcal{P}_0$ it is clear that:

$$\text{No } \mathcal{P}\text{-Arbitrage} \Leftrightarrow \mathcal{M} \neq \varnothing \Leftrightarrow \mathcal{P}_0 \neq \varnothing,$$

and each one of these conditions is equivalent to No Model Independent Arbitrage with respect to $\widetilde{\mathcal{H}}$ (Theorem 1.3). When $\mathcal{P}_0 \neq \varnothing$, it is possible that only very few models (i.e. a "small" set of probability measures - the extreme case being $|\mathcal{P}_0| = 1$) are arbitrage free. On the other hand, the financial market could be very "well posed", so that for "most" models no arbitrage is assured - the extreme case being $\mathcal{P}_0 = \mathcal{P}$.
To distinguish these two possible occurrences we analyze the conditions under which the set $\mathcal{P}_0$ is dense in $\mathcal{P}$: in this case even if there could be some particular models allowing arbitrage opportunities, the financial market is well posed for most models.

Definition 1.56. *The market is feasible if $\overline{\mathcal{P}_0} = \mathcal{P}$*

Recall that we are here considering the $\sigma(\mathcal{P}, C_b)$- closure.
In Proposition 1.58 we characterize feasibility with the existence of a full support martingale measure, a condition independent of any a priori fixed probability.

Lemma 1.57. *For all $P \in \mathcal{P}_+$*

$$\overline{\mathcal{P}_e(P)} = \mathcal{P} \text{ and } \mathcal{P}_+ \text{ is } \sigma(\mathcal{P}, C_b)\text{-dense in } \mathcal{P}.$$

Proof. It is well know that under the assumption that (Ω, d) is separable, $\mathcal{P}_+ \neq \varnothing$. Let us first show that $\forall a \in \Omega$ we have that $\delta_a \in \overline{\mathcal{P}_e(P)}$ where $P \in \mathcal{P}_+$ and δ_a is the point mass probability measure in a. Let

$$A_n := \left\{\omega \in \Omega \ : \quad d(a,\omega) < \frac{1}{n}\right\}.$$

This set is open in the topology induced by d and, since P has full support, $0 < P(A_n) < 1$. Define the conditional probability measure $P_n := P(\cdot \mid A_n)$. For all $0 < \lambda < 1$, the measure $P_\lambda := \lambda P(\cdot \mid A_n^c) + (1-\lambda)P(\cdot \mid A_n)$ has full support, is equivalent to P and P_λ converges weakly to $P(\cdot \mid A_n)$ as $\lambda \downarrow 0$. Hence: $P_n \in \overline{\mathcal{P}_e(P)}$. In order to show that $P_n \overset{w}{\to} \delta_a$ we prove that $\forall\, G$ open $\liminf P_n(G) \geq \delta_a(G)$. If $a \in G$ then $\delta_a(G) = 1$ and $P(G \cap A_n) = P(A_n)$ eventually so we have that $\liminf P_n(G) = 1 = \delta_a(G)$. Otherwise if $a \notin G$ then $\delta_a(G) = 0$ and the inequality is obvious. Since $\forall a \in \Omega$ we have that $\delta_a \in \overline{\mathcal{P}_e(P)}$ then $co(\{\delta_a \ : \ a \in \Omega\}) \subseteq \overline{\mathcal{P}_e(P)}$ and from the density of the probability measures with finite support in $\mathcal{P}$ (respect to the weak topology) it follows that $\overline{\mathcal{P}_e(P)} = \mathcal{P}$. The last assertion is obvious since $\mathcal{P}_e(P) \subseteq \mathcal{P}_+$ for each $P \in \mathcal{P}_+$. □

Proposition 1.58. *The following assertions are equivalent:*

- *(1)* $\mathcal{M}_+ \neq \emptyset$;
- *(2) No* $\mathcal{P}_+$*-Arbitrage;*
- *(3)* $\mathcal{P}_0 \cap \mathcal{P}_+ \neq \emptyset$;
- *(4)* $\overline{\mathcal{P}_0 \cap \mathcal{P}_+} = \mathcal{P}$;
- *(5)* $\overline{\mathcal{P}_0} = \mathcal{P}$.

Proof. Since $\mathcal{M}_+ \neq \emptyset \Leftrightarrow$ No $\mathcal{P}_+$-Arbitrage by Proposition 1.54 and No $\mathcal{P}_+$-Arbitrage $\Leftrightarrow$ $\mathcal{P}_0 \cap \mathcal{P}_+ \neq \emptyset$ by definition, $1), 2), 3)$ are clearly equivalent.
Let us show that 3) $\Rightarrow$ 4): Assume that $\mathcal{P}_0 \cap \mathcal{P}_+ \neq \emptyset$ and observe that if $P \in \mathcal{P}_0 \cap \mathcal{P}_+$ then $\mathcal{P}_e(P) \subseteq \mathcal{P}_0 \cap \mathcal{P}_+$, which implies that $\overline{\mathcal{P}_e(P)} \subseteq \overline{\mathcal{P}_0 \cap \mathcal{P}_+} \subseteq \mathcal{P}$. From Lemma 1.57 we conclude that 4) holds.
Observe now that the implication 4) $\Rightarrow$ 5) holds trivially, so we just need to show that 5) $\Rightarrow$ 3). Let $P \in \mathcal{P}_+$. If P satisfies NA(P) there is nothing to show, otherwise by 5) there exist a sequence of probabilities $P_n \in \mathcal{P}_0$ such that $P_n \overset{w}{\to} P$ and the condition NA(P_n) holds $\forall n \in \mathbb{N}$. Define $P^* := \sum_{n=1}^{+\infty} \frac{1}{2^n} P_n$ and note that for this probability the condition NA(P^*) holds true, so we just need to show that P^* has full support. Assume by contradiction that $supp(P^*) \subset \Omega$. Then there exist an open set O such that $P^*(O) = 0$ and $P(O) > 0$ since P has full support. From $P_n(O) = 0$ for all n, and $P_n \overset{w}{\to} P$ we obtain $0 = \liminf P_n(O) \geq P(O) > 0$, a contradiction. □

Remark 1.59. *From the previous proof we observe that if the market is feasible then* $\overline{\bigcup_{P \in \mathcal{P}_0} supp(P)} = \Omega$ *and no "significantly large parts" of* Ω *are neglected by no arbitrage models* $P \in \mathcal{P}_0$.

Proof of Theorem 1.4. Proposition 1.58 guarantees: 1. $\Leftrightarrow$ 2. $\Leftrightarrow$ 3. and Corollary 1.47 assures: 3. $\Leftrightarrow$ 4. □

The case of a countable space Ω. When $\Omega = \{\omega_n \mid n \in \mathbb{N}\}$ is countable it is possible to provide another characterization of feasibility using the norm topology instead of the weak topology on $\mathcal{P}$. More precisely, we consider the topology induced by the total variation norm. A sequence of probabilities P_n *converges in variation* to P if $\|P_n - P\| \to 0$, where the variation norm of a signed measure R is defined by:

$$\|R\| = \sup_{(A_i,\dots,A_n)\in\mathcal{F}} \sum_{i=1}^{n} |R(A_i)|, \tag{43}$$

and $(A_i,\dots,A_n)$ is a finite partition of Ω.

Lemma 1.60. *Let Ω be a countable space. Then $\forall P \in \mathcal{P}_+$*

$$\overline{\mathcal{P}_e(P)}^{\|\cdot\|} = \overline{\mathcal{P}_+}^{\|\cdot\|} = \mathcal{P}.$$

Proof. Since Ω is countable we have that

$$\mathcal{P} = \{P := \{p_n\}_1^\infty \in \ell^1 \;\mid\; p_n \geq 0 \;\forall n \in \mathbb{N},\; \|P\|_1 = 1\},$$

$$\mathcal{P}_+ = \{P \in \mathcal{P} \;\mid\; p_n > 0 \;\forall n \in \mathbb{N}\},$$

with $\|\cdot\|_1$ the ℓ^1 norm. Observe that in the countable case $\mathcal{P}_e(P) = \mathcal{P}_+$ for every $P \in \mathcal{P}_+$. So we only need to show that for any $P \in \mathcal{P}$ and any $\varepsilon > 0$ there exists $P' \in \mathcal{P}_+$ s.t. $\|P - P'\|_1 \leq \varepsilon$.
Let $P \in \mathcal{P} \setminus \mathcal{P}_+$. Then $P = \{p_n\}_1^\infty \in \ell^1$ and there exists at least one index n for which $p_n = 0$. Let $\alpha > 0$ be the constant satisfying

$$\sum_{n\in\mathbb{N} \text{ s.t. } p_n=0} \frac{\alpha}{2^n} = 1.$$

There also exists one index n, say n_1, for which $1 \geq p_{n_1} > 0$. Let $p := p_{n_1} > 0$.
For any positive $\varepsilon < p$, define $P' = \{p'_n\}$ by: $p'_{n_1} = p - \frac{\varepsilon}{2}$, $p'_n = p_n$ for all $n \neq n_1$ s.t. $p_n > 0$, $p'_n = \frac{\alpha}{2^n}\frac{\varepsilon}{2}$ for all n s.t. $p_n = 0$. Then $p'_n > 0$ for all n and $\sum_{n=1}^\infty p'_n = \sum_{n \text{ s.t.} p_n > 0} p_n = 1$, so that $P' \in \mathcal{P}_+$ and $\|P - P'\|_1 = \varepsilon$. □

Remark 1.61. *In the general case, when Ω is uncountable, while it is still true that $\overline{\mathcal{P}_+}^{\|\cdot\|} = \mathcal{P}$, it is no longer true that $\overline{\mathcal{P}_e(P)}^{\|\cdot\|} = \mathcal{P}$ for any $P \in \mathcal{P}_+$.*
Take $\Omega = [0,1]$ and $\mathcal{P}_e(\lambda)$ the set of probability measures equivalent to Lebesgue. It is easy to see that $\delta_0 \notin \overline{\mathcal{P}_e(\lambda)}^{\|\cdot\|}$ since $\|P - \delta_0\| \geq P((0,1]) = 1$.

Proposition 1.62. *If Ω is countable, the following conditions are equivalent:*

(1) $\mathcal{M}_+ \neq \emptyset$;
(2) No $\mathcal{P}_+$-Arbitrage;
(3) $\mathcal{P}_0 \cap \mathcal{P}_+ \neq \emptyset$;
(4) $\overline{\mathcal{P}_0}^{\|\cdot\|} = \mathcal{P}$,

where $\|\cdot\|$ is the total variation norm on $\mathcal{P}$

Proof. Using Lemma 1.60 the proof is straightforward using the same techniques as in Proposition 1.58. □

5. On Open Arbitrage

In the introduction we already illustrated the interpretation and robust features of the dual formulation of Open Arbitrage. In order to prove the equivalence between Open Arbitrage and (4) consider the following definition and recall that $\mathcal{V}_H^+ := \{\omega \in \Omega \mid V_T(H)(\omega) > 0\}$.

DEFINITION 1.63. *Let τ be a topology on $\mathcal{P}$ and $\mathfrak{H}$ be a class of trading strategies. Set*

$$W(\tau, \mathfrak{H}) = \left\{ H \in \mathfrak{H} \mid \begin{array}{c} \text{there exists a non empty } \tau - \text{open set } \mathcal{U} \subseteq \mathcal{P} \text{ such that} \\ \forall P \in \mathcal{U} \quad V_T(H) \geq 0 \ P\text{-a.s.} \quad \text{and} \quad P(\mathcal{V}_H^+) > 0 \end{array} \right\}$$

Clearly, $W(\tau, \mathfrak{H})$ consists of the trading strategies satisfying condition (4) with respect to the appropriate topology and the measurability requirement. The first item in the next proposition is the announced equivalence. The second item shows that the analogue equivalence is true also with respect to the class $\widetilde{\mathcal{H}}$. Therefore, in Theorem 1.4 we could add to the four equivalent conditions also the dual formulation of Open Arbitrage with respect to $\widetilde{\mathcal{H}}$.

PROPOSITION 1.64. *(1) Let $\sigma := \sigma(\mathcal{P}, C_b)$ and $\|\cdot\|$ the variation norm defined in (43). Then:*

$$\begin{array}{ccl} H \in W(\|\cdot\|, \mathcal{H}) & \Longleftrightarrow & H \in \mathcal{H} \text{ is a } 1p\text{-Arbitrage} \\ \Uparrow & & \\ H \in W(\sigma, \mathcal{H}) & \Longleftrightarrow & H \in \mathcal{H} \text{ is an Open Arbitrage} \end{array}$$

In addition, if $H \in W(\sigma, \mathcal{H})$ then $V_T(H)(\omega) \geq 0$ for all $\omega \in \Omega$.
(2) Let $\mathcal{F} = \mathcal{B}(\Omega)$ be the Borel σ-algebra and let $\widetilde{\mathcal{F}}$ be a σ-algebra such that $\mathcal{F} \subseteq \widetilde{\mathcal{F}}$. Define the set

$$\widetilde{\mathcal{P}} := \{\widetilde{P} : \widetilde{\mathcal{F}} \to [0,1] \mid \widetilde{P} \text{ is a probability}\},$$

and endow $\widetilde{\mathcal{P}}$ with the topology $\widetilde{\sigma} := \sigma(\widetilde{\mathcal{P}}, C_b)$. The class of admissible trading strategies $\widetilde{\mathcal{H}}$ is given by all $\widetilde{\mathbb{F}}$- predictable processes. Then

$$H \in W(\widetilde{\sigma}, \widetilde{\mathcal{H}}) \iff H \in \widetilde{\mathcal{H}} \text{ is an Open Arbitrage in } \widetilde{\mathcal{H}}$$

In addition, if $H \in W(\widetilde{\sigma}, \widetilde{\mathcal{H}})$ then $V_T(H)(\omega) \geq 0$ for all $\omega \in \Omega$.

PROOF. We prove (1) and we postpone the proof of (2) to the Appendix.
(a) H is a $1p$-Arbitrage $\Rightarrow H \in W(\|\cdot\|, \mathcal{H})$. Let $H \in \mathcal{H}$ be a $1p$-Arbitrage. Then $V_T(H)(\omega) \geq 0 \ \forall \omega \in \Omega$ and there exists a probability P such that $P(\mathcal{V}_H^+) > \varepsilon > 0$. From the implication $\|P - Q\| < \varepsilon \Rightarrow |P(C) - Q(C)| < \varepsilon$ for every $C \in \mathcal{F}$, we obtain: $\bar{P}(\mathcal{V}_H^+) > 0 \ \forall \bar{P} \in B_\varepsilon(P)$, where $B_\varepsilon(P)$ is the ball of radius ε centered in P. Hence $H \in W(\|\cdot\|, \mathcal{H})$.
(b) $H \in W(\|\cdot\|, \mathcal{H}) \Rightarrow H$ is a $1p$-Arbitrage. If $H \in W(\|\cdot\|, \mathcal{H})$ then $V_T(H) \geq 0$ P-a.s. for all P in the open set $\mathcal{U}$. We need only to show that $B := \{\omega \in \Omega \mid V_T(H)(\omega) < 0\}$ is empty. By contradiction, let $\omega \in B$, take any $P \in \mathcal{U}$ and define the probability $P_\lambda := \lambda\delta_\omega + (1-\lambda)P$. Since $V_T(H) \geq 0$ P-a.s. we must have $P(\omega) = 0$, otherwise $P(B) > 0$. However, $P_\lambda(B) \geq P_\lambda(\omega) = \lambda > 0$ for all positive λ and P_λ will eventually belongs to $\mathcal{U}$, as $\lambda \downarrow 0$, which contradicts $V_T(H) \geq 0$ P-a.s. for any $P \in \mathcal{U}$.
(c) $H \in W(\sigma, \mathcal{H}) \Rightarrow H \in W(\|\cdot\|, \mathcal{H})$. This claim is trivial because every weakly open set is also open in the norm topology.
(d) If $H \in W(\sigma, \mathcal{H})$ then $V_T(H)(\omega) \geq 0$ for all $\omega \in \Omega$. This follows from (c) and (b).

(e) $H \in W(\sigma, \mathcal{H}) \Rightarrow H$ is an Open Arbitrage. Suppose $H \in W(\sigma, \mathcal{H})$, so that $V_T(H)(\omega) \geq 0 \ \forall \omega \in \Omega$. We claim that $(\mathcal{V}_H^+)^c = \{\omega \in \Omega \mid V_T(H) = 0\}$ is not dense in Ω. This will imply the thesis as $int(\mathcal{V}_H^+)$ will then be a non empty open set on which $V_T(H) > 0$. Suppose by contradiction that $\overline{(\mathcal{V}_H^+)^c} = \Omega$. We know by Lemma 1.77 in the Appendix that the set $\mathcal{Q}$ of embedded probabilities $co(\{\delta_\omega\} \mid \omega \in (\mathcal{V}_H^+)^c)$ is weakly dense in $\mathcal{P}$ and hence it intersects, in particular, the weakly open set $\mathcal{U}$ in the definition of $W(\sigma, \mathcal{H})$. However, for every $P \in \mathcal{Q}$ we have $V_T(H) = 0$ P-a.s. and so H is not in $W(\sigma, \mathcal{H})$.
(f) H is an Open Arbitrage $\Rightarrow H \in W(\sigma, \mathcal{H})$. Note first that if F is a closed subset of Ω, then $\mathcal{P}(F) := \{P \in \mathcal{P} \mid \ supp(P) \subset F\}$ is a $\sigma(\mathcal{P}, C_b)$ closed face of $\mathcal{P}$ from Th. 15.19 in [**AB06**]. If H is an Open Arbitrage then $\mathcal{V}_H^+$ contains an open set and in particular $G := \overline{(\mathcal{V}_H^+)^c}$ is a closed set strictly contained in Ω. Observe then that $\mathcal{U} := (\mathcal{P}(G))^c$ is a non empty open set of probabilities that fulfills the properties in the definition of $W(\sigma, \mathcal{H})$. $\square$

The following proposition is an improvement of (40), as the $1p$-Arbitrage is defined with respect to $\mathcal{H}$.

Proposition 1.65. *For Ω countable: No $1p$-Arbitrage in $\mathcal{H} \iff \mathcal{M}_+ \neq \emptyset$.*

Proof. As a consequence of Propositions 1.17 and 1.64 we only need to prove $\mathcal{M}_+ \neq \emptyset \Longrightarrow W(\|\cdot\|, \mathcal{H}) = \varnothing$. From Proposition 1.62 item 4) we have $\mathcal{M}_+ \neq \emptyset \Longrightarrow \overline{\mathcal{P}_0}^{\|\cdot\|} = \mathcal{P}$ and so for every (norm) open set $\mathcal{U} \subseteq \mathcal{P}$ there exists $P \in \mathcal{P}_0 \cap \mathcal{U}$ for which $NA(P)$ holds, which implies $W(\|\cdot\|, \mathcal{H}) = \varnothing$. $\square$

5.1. On the continuity of S with respect to ω. Consider first a one period market $I = \{0,1\}$ with $S_0 = s_0 \in \mathbb{R}^d$ and S_1 a random outcome *continuous* in ω. Then every $1p$-Arbitrage generates an Open Arbitrage (this was shown by [**Ri15**] and is intuitively clear). From Proposition 1.17, No $1p$-Arbitrage implies $\mathcal{M}_+ \neq \varnothing$ and therefore No Open Arbitrage. We then conclude that, in this particular case, the three conditions are all equivalent and Theorem 1.4 holds without the enlargement of the natural filtration so that we recover in particular the result stated in [**Ri15**].

Differently from the one period case, in the multi-period setting it is no longer true that No Open Arbitrage and No $1p$-Arbitrage (with respect to admissible strategies $\mathcal{H}$) are equivalent, as shown by the following examples. Moreover, even with S continuous in ω, No Open Arbitrage is not equivalent to $\mathcal{M}_+ \neq \varnothing$ as long as we do not enlarge the filtration as in Section 3.5.

Example 1.66. *Consider $\Omega = [0,1] \times [0,1]$, $\mathcal{F} = \mathcal{B}_{[0,1]} \otimes \mathcal{B}_{[0,1]}$ and the canonical process given by $S_1(\omega) = \omega_1$ and $S_2(\omega) = \omega_2$. Clearly for any $\omega = (\omega_1, \omega_2)$ such that $\omega_1 \in (0,1)$ we have that $0 \in ri(\Delta S_2(\Sigma_1^\omega))^{cc}$. On the other hands for $\overline{\omega} = (1, \omega_2)$ or $\widehat{\omega} = (0, \omega_2)$ we have $1p$-Arbitrages since $S_2(\overline{\omega}) \leq S_1(\overline{\omega})$ with $<$ for any $\omega_2 \neq 1$ and $S_2(\widehat{\omega}) \geq S_1(\widehat{\omega})$ with $>$ for any $\omega_2 \neq 0$. Denote by $\Sigma^1 = \{S_1 = 1\}$ and $\Sigma^0 = \{S_1 = 0\}$ then $\alpha(\omega) = -\mathbf{1}_{\Sigma^1} + \mathbf{1}_{\Sigma^0}$ is a $1p$-Arbitrage which does not admit any open arbitrage since neither Σ^1 nor Σ^0 are open sets, and any strategy which is not zero on $(\Sigma^1 \cup \Sigma^0)^c$ gives both positive and negative payoffs.*

Example 1.67. *We show an example of a market with S continuous in ω, with no Open Arbitrage in $\mathcal{H}$ and $\mathcal{M}_+ = \varnothing$. Let us first introduce the following continuous functions on $\Omega = [0, +\infty)$*

$$\varphi^m_{a,b}(\omega) := \begin{cases} m(\omega-a) & \omega\in[a,\frac{a+b}{2}] \\ -m(\omega-b) & \omega\in[\frac{a+b}{2},b] \\ 0 & otherwise \end{cases} \quad \phi^m_{a,b}(\omega) := \begin{cases} m(\omega-a) & \omega\in[a,a+1] \\ m & \omega\in[a+1,b-1] \\ -m(\omega-b) & \omega\in[b-1,b] \\ 0 & otherwise \end{cases}$$

with $a,b,m\in\mathbb{R}$. Define the continuous (in ω) stochastic process $(S_t)_{t=0,1,2,3}$

$$\begin{aligned} S_0(\omega) &:= \frac{1}{2} \\ S_1(\omega) &:= \phi^1_{[0,3]}(\omega)+\phi^1_{[3,6]}(\omega)+\sum_{k=3}^{\infty}\varphi^1_{[2k,2k+2]}(\omega) \\ S_2(\omega) &:= \phi^{\frac{1}{2}}_{[0,3]}(\omega)+\phi^{\frac{1}{2}}_{[3,6]}(\omega)+\sum_{k=3}^{\infty}\varphi^2_{[2k,2k+2]}(\omega) \\ S_3(\omega) &:= \varphi^2_{[0,3]}(\omega)+\varphi^{\frac{1}{4}}_{[3,6]}(\omega)+\varphi^4_{[6,8]}(\omega)+\sum_{k=4}^{\infty}\varphi^{4k}_{[2k+1-\frac{1}{k},2k+1+\frac{1}{k}]}(\omega) \end{aligned}$$

It is easy to check that given $z\in\mathbf{Z}$ such that $z_{0:2}=[\frac{1}{2},1,2]$, we have $\Sigma^z_2=\{2k+1\}_{k\geq 3}$ and $H:=\mathbf{1}_{\Sigma^z_2}$ is the only 1p-Arbitrage opportunity in the market. One can also check that $\mathcal{V}^+_H=\Sigma^z_2$, as a consequence, H is not an Open Arbitrage and

$$Q(\{2k+1\}_{k\geq 3})=0 \text{ for any } Q\in\mathcal{M} \tag{44}$$

Consider now $\hat{z}\in\mathbf{Z}$ with $\hat{z}_{0:2}=[\frac{1}{2},1,\frac{1}{2}]$ and the corresponding level set $\Sigma^{\hat{z}}_2$. It is easy to check that

$$\Sigma^{\hat{z}}_2=[1,2]\cup[4,5] \quad \text{and} \quad \Delta S_2<0 \text{ on } \Sigma^{\hat{z}}_2 \tag{45}$$

Observe now that $z_{0:1}=\hat{z}_{0:1}$ and that $\Sigma^z_1=[1,2]\cup[4,5]\cup\{2k+1\}_{k\geq 3}$. We therefore have

$$S_2(\omega)=\begin{cases} 2 & \omega\in\{2k+1\}_{k\geq 3} \\ \frac{1}{2} & \omega\in[1,2]\cup[4,5] \end{cases} \quad \text{for } \omega\in\Sigma^z_1$$

From $S_1(\omega)=1$ on Σ^z_1, (44) and (45), any martingale measure must satisfy $Q([1,2]\cup[4,5])=0$. In other words there exist polar sets with non-empty interior which implies $\mathcal{M}_+=\varnothing$.

6. Appendix

6.1. proof of Theorem 1.39.

LEMMA 1.68 (Lebesgue decomposition of P). *Let $\nu:=\sup_{Q\in\mathcal{M}}Q$. For any $P\in\mathcal{P}$ there exists a set $F\in\mathcal{F}$ such that $F\subseteq\Omega^c_*$, and the measures $P_c(\cdot):=P(\cdot\setminus F)$ and $P_s(\cdot):=P(\cdot\cap F)$ satisfy*

$$P_c\ll\nu,\ P_s\perp\nu \qquad \text{and} \qquad P=P_c+P_s \tag{46}$$

PROOF. We wish to apply Theorem 4.1 in [**LYL07**] to $\mu=P\in\mathcal{P}$ and $\nu=\sup_{Q\in\mathcal{M}}Q$. It is easy to check that: 1) μ and ν are monotone $[0,1]$-valued set functions on $\mathcal{F}$ satisfying $\mu(\varnothing)=0$ and $\nu(\varnothing)=0$; 2) P is *exhaustive*, i.e. if $\{A_n\}_{n\in\mathbb{N}}$ is a disjoint sequence then $P(A_n)\to 0$ (indeed, $1\geq P(\cup_n A_n)=\sum_n P(A_n)\geq 0\Rightarrow P(A_n)\to 0$; 3) ν *is weakly null additive*: if $A,B\in\mathcal{F}$ with $\nu(A)=\nu(B)=0$ then $\nu(A\cup B)=0$ (indeed, if $\nu(A)=\nu(B)=0$ then for any $Q\in\mathcal{M}$,

$Q(A) = Q(B) = 0$ which implies $Q(A \cup B) = 0$ and $\nu(A \cup B) = 0$); 4) ν is *continuous from below.* Indeed if $A_n \nearrow A$ then $Q(A_n) \uparrow Q(A)$, $Q(A) = \sup_n Q(A_n)$ and

$$\lim_{n\to\infty} \nu(A_n) = \sup_n \nu(A_n) = \sup_n \sup_{Q\in\mathcal{M}} Q(A_n) = \sup_{Q\in\mathcal{M}} \sup_n Q(A_n) = \nu(A).$$

Hence μ and ν satisfy all the assumptions of Theorem 4.1 in [**LYL07**] and hence we obtain the existence of $F \in \mathcal{F}$ such that $\nu(F) = 0$ and the decomposition in (46) holds true. From Proposition 1.34, $\forall A \in \mathcal{F}$ such that $A \subseteq \Omega_*$ we have $\nu(A) > 0$. Therefore, $F \subseteq \Omega_*^c$ and this concludes the proof. $\square$

REMARK 1.69. *Observe that if $\Omega_*^c \neq \varnothing$ the set of probability measures with non trivial singular part P_s is non-empty. Simply take, for instance, any convex combination of $\{\delta_\omega \mid \omega \in \Omega_*^c\}$.*

Preliminary considerations. We want to consider now the probabilistic model $(\Omega, \{\mathcal{F}_t^P\}_{t\in I}, S, P)$ and we need therefore to pass from ω-wise considerations to P-a.s considerations. For this reason we first need to construct an auxiliary process S_t^P with the property $S_t^P = S_t$ P-a.s for any $t \in I$ in the same spirit of Lemma 1.33.
Let $P_{\Delta S_T}(\cdot,\cdot) : \Omega \times \mathcal{B}(\mathbb{R}^d) \mapsto [0,1]$ be the conditional distribution of ΔS_T and denote $\Upsilon_{\Delta S_T}$ its random support. Define as in Rokhlin [**Ro08**] the set $A_{\Delta S_T} := \{0 \notin \mathrm{ri}(\mathrm{conv}\Upsilon_{\Delta S_T})\}$. It may happen that $P(A_{\Delta S_T}) = 0$. In this case $\mathfrak{B}_T$ and $\mathfrak{D}_{T-1}$ as in Lemma 1.33 are subset of P-null sets (respectively in $\mathcal{F}_T$ and $\mathcal{F}_{T-1}$). Construct iteratively X_t^P and S_t^P as in (23) and (25). Denote $\Delta X_t^P := X_t^P - S_{t-1}$ and let

$$\tau := \min\left\{t \in I_1 \mid P\left(A_{\Delta X_t^P}\right) > 0\right\}. \tag{47}$$

Observe that τ is well defined since, from Lemma 1.33, if $P(A_{\Delta X_t^P}) = 0$ for any $t \geq 1$ we have that $\bigcup_{t\in I_1} \widetilde{\mathfrak{B}}_t = \Omega_*^c$ is a subset of a P-null set (cfr (24)). This is a contradiction since P is not absolutely continuous with respect to ν, henceforth the set F from Lemma 46 satisfies $F \subseteq \Omega_*^c$ and $P(F) > 0$. From now on we still denote by $\{S_t\}_{t\in I}$ the P-a.s. version of the process given by $\{S_t \mathbf{1}_{t<\tau} + X_t^P \mathbf{1}_{t\geq\tau}\}_{t\in I}$.

REMARK 1.70. *For any $t \in I_1$ denote $P_{t-1}(\cdot,\cdot) : (\Omega, \mathcal{F}) \mapsto [0,1]$ the conditional probability of P on $\mathcal{F}_{t-1}$. Recall from Theorem 1.28 c] that there exists $N_1 \in \mathcal{F}_{t-1}$ with $P(N_1) = 0$ such that for any $\omega \in \Omega \setminus N_1$ we have $P_{t-1}(\omega, \Sigma_{t-1}^{z(\omega)}) = 1$ where $z(\omega) = S_{0:T}(\omega)$.*

Construction of a P-arbitrage from $\mathbb{H}$. Recall that τ is defined in 47 and denote $A_\tau := A_{\Delta S_\tau}$. For any $\omega \in \Omega$ the level set $\Sigma_{\tau-1}^z$ can be decomposed as $\Sigma_{\tau-1}^z = \cup_{i=1}^{\beta_{\tau,z}} B_{\tau,z}^i \cup B_{\tau,z}^*$. Define for any $z \in \mathbf{Z}$

$$j_z := \inf\left\{j \in \{1,\dots,\beta_{\tau,z}\} \mid P(\omega, B_{\tau,z}^j) > 0\ \forall\omega \in \Sigma_{\tau-1}^z\right\}$$

and recall that $P(\cdot, B_{\tau,z}^j)$ is constant on $\Sigma_{\tau-1}^z$ (Theorem 1.28 b]). Define $N_2 := \bigcup_{z\in\mathbf{Z}_f} \cup_{i=1}^{j_z-1} B_{\tau,z}^i$ where $\mathbf{Z}_f := \{z \in \mathbf{Z} \mid j_z < \infty\}$. N_2 is a $\bar{P}$-null set since for any $\omega \in N_1^c$ we have $\bar{P}(\omega, N_2) = \bar{P}(\omega, \cup_{i=1}^{j_z-1} B_{\tau,z}^i) = 0$ hence $\bar{P}(N_2) = \bar{P}(N_1 \cap N_2) + \bar{P}(N_1^c \cap N_2) = 0$ (see also Lemma 1.73 below). Recall that $\bar{P}(\cdot)$ and $\bar{P}(\omega,\cdot)$ denote the completion of $P(\cdot)$ and $P(\omega,\cdot)$ respectively.

Denote $N := N_1 \cup N_2$. We are now able to define the following multifunction $\Psi : \Omega \mapsto 2^{\mathbb{R}^d}$ with values in the power set of $\mathbb{R}^d$.

$$\Psi(\omega) := \begin{cases} \Delta S_\tau \left(\Sigma_{\tau-1}^{z(\omega)} \cap N^c \right) & \omega \in N^c \\ \varnothing & \text{otherwise} \end{cases} \tag{48}$$

In Lemma 1.71 we show that Ψ is $\mathcal{F}_{\tau-1}^P$-measurable. We apply now an argument similar to [**Ro08**]. Denote $\mathbb{S}_1^d$ the unitary closed ball in $\mathbb{R}^d$, $\text{lin}(\chi)$ the linear space generated by χ and χ° the polar cone of χ. By preservation of measurability (see Proposition 1.75) the (closed-valued) multifunction

$$\omega \mapsto G_0(\omega) := \text{lin}(\Psi(\omega)) \cap (-\text{cone } \Psi(\omega))^\circ \cap \mathbb{S}_1^d$$

is also $\mathcal{F}_{\tau-1}^P$-measurable and $G_0(\omega) \neq \varnothing$ iff $\omega \in A_\tau \cap N^c$, hence $A_\tau = \{0 \notin \text{ri}(\text{conv}\Upsilon_{\Delta S_\tau^P})\}$ is $\mathcal{F}_{\tau-1}^P$-measurable. Note that we already have that $G_0(\omega) \subseteq \mathbb{H}_\tau(\omega)$ for P-a.e. $\omega \in \Omega$. Indeed fix $\omega \notin N$ and consider the level set $\Sigma_{\tau-1}^{z(\omega)}$ and its decomposition as in Lemma 1.20. By construction of G_0 we have that any $g \in G_0(\omega) \neq \varnothing$ satisfies $g \cdot \Delta S_\tau(\omega) \geq 0$ for any $\omega \in \cup_{i=j_z}^{\beta_{\tau,z}} B_{\tau,z}^i \cup B_{\tau,z}^*$ and thus $g \in \mathbb{H}(\omega)$.

Nevertheless, the random set $G_0(\omega)$ contains those $g \in \mathbb{S}_1^d$ such that $g \cdot \Delta S_\tau(\omega) = 0$. Thus, we will not extract a measurable selection from G_0 but we will rather consider for any $n \in \mathbb{N}$ the following closed-valued multifunction

$$\omega \mapsto G_n(\omega) := \text{lin}(\Psi(\omega)) \cap \left\{ v \in \mathbb{R}^d \mid \langle v, s \rangle \geq \frac{1}{n} \quad \forall s \in \Psi(\omega) \setminus \{0\} \right\} \cap \mathbb{S}_1^d, \quad n \geq 1$$

and seek for a measurable selection of $G := \cup_{n=0}^\infty G_n$. From Lemma 1.74 all the random sets G_n are $\mathcal{F}_{\tau-1}^P$-measurable and therefore the same is true for G. Now, for any $n \geq 0$, let $\tilde{H}_n$ a measurable selection of G_n on $\{G_n \neq \varnothing\}$ which always exists for a (measurable) closed-valued multifunction with $\tilde{H}_n(\omega) = 0$ if $G_n(\omega) = \varnothing$. Define therefore

$$H_k := \sum_{n=0}^{k} \tilde{H}_n \qquad \text{and} \quad B_k := \mathcal{V}_{H_k}^+ \tag{49}$$

By construction B_k is an increasing sequence of sets converging to $\cup_z B_{\tau,z}^{j_z}$ which is therefore measurable and it satisfies

$$P(\cup_z B_{\tau,z}^{j_z}) = \int_{\Omega \setminus N} P(\omega, B_{\tau,z}^{j_z}) dP(\omega) \geq \int_{A_\tau \setminus N} P(\omega, B_{\tau,z}^{j_z}) dP(\omega) > 0$$

which follows from the definition of conditional probability, $P(A_\tau) > 0$ and $P(\omega, B_{\tau,z}^{j_z}) > 0$ for every $\omega \in A_\tau \setminus N$. We can therefore conclude that there exists $m \geq 0$ such that $P(B_m) > 0$ and since obviously $H_m \Delta S_\tau \geq 0$ we have that H_m is a P-arbitrage. The normalized random variable $H_\tau^P := H_m(\omega)/\|H_m(\omega)\|$ is a measurable selector of the multifunction G_0 since it satisfies $H_\tau^P(\omega) \in \cup_{n=1}^m G_n(\omega) \subseteq G(\omega) \subseteq \mathbb{H}_\tau(\omega)$ P-a.s. and thus the desired strategy is given by $H_s^P = H_\tau^P \mathbf{1}_\tau(s)$.

LEMMA 1.71. *The multifunction Ψ defined in* (48) *is $\mathcal{F}_{\tau-1}^P$-measurable.*

PROOF. Recall that by definition the multifunction Ψ is measurable iff for any open set $V \subseteq \mathbb{R}^d$ we have $\{\omega \mid \Psi(\omega) \cap V \neq \varnothing\}$ is a measurable set. Observe that

$$\Psi^{-1}(V) := \{\omega \mid \Psi(\omega) \cap V \neq \varnothing\} = S_{\tau-1}^{-1} \left[S_{\tau-1} \left(\Delta S_\tau^{-1}(V) \cap N^c \right) \right] \cap N^c$$

Let us show that the complement of this set is $\mathcal{F}^P_{\tau-1}$-measurable from which the thesis will follow. Observe that for any function f and for any set A we have $(f^{-1}(A))^c = f^{-1}(A^c)$ so that

$$\begin{aligned} (\Psi^{-1}(V))^c &= S^{-1}_{\tau-1}\left[S_{\tau-1}\left(\Delta S^{-1}_{\bar{t}}(V)\cap N^c\right)^c\right]\cup N \\ &= S^{-1}_{\tau-1}\left[S_{\tau-1}\left((\Delta S^{-1}_{\tau}(V))^c\cup N\right)\right]\cup N \\ &= S^{-1}_{\tau-1}\left[S_{\tau-1}\left(\Delta S^{-1}_{\tau}(V^c)\cup N\right)\right]\cup N \end{aligned}$$

Note now that $A_1 := \Delta S^{-1}_\tau(V^c)\cup N$ is an analytic set since it is union of a Borel set and a $\bar{P}$-null set. The set $B_1 := S_{\tau-1}(A_1)$ is an analytic subset of $\mathbb{R}^d$ since S is a Borel function and image of an analytic set through a Borel measurable function is analytic. Finally $A_2 := S^{-1}_{\tau-1}(B_1)$ is an analytic subset of Ω since pre-image of an analytic set through a Borel measurable function is analytic. Since P-completion of $\mathcal{F}$ contains any analytic set, $A_2\cup N$ is also analytic and belongs to $\mathcal{F}^P_{\tau-1}$.

Remark 1.72. *For sure $A_2\cup N$ is analytic and belongs to $\mathcal{F}^P$. The heuristic for $A_2\cup N$ belonging to $\mathcal{F}^P_{\tau-1}$ should be that this set is union of atoms of $\mathcal{F}^P_{\tau-1}$. More formally, since B_1 is analytic in $\mathbb{R}^d$ for any measure μ there exists F, G such that $B_1 = F\cup G$ with F a Borel set and G a subset of μ-null measure (because analytic sets are in the completion of $\mathcal{B}$ respect to any measure μ). Taking μ as the distribution of $S_{\tau-1}$ under P we have $A_2 = S^{-1}_{\tau-1}(F)\cup S^{-1}_{\tau-1}(G)$. Since $S^{-1}_{\tau-1}(F)\in\mathcal{F}_{\tau-1}$ and $S^{-1}_{\tau-1}(G)$ is a subset of a $\mathcal{F}_{\tau-1}$-measurable P-null set, we have $A_2\in\mathcal{F}^P_{\tau-1}$ and hence also $A_2\cup N$.*

□

Lemma 1.73. *Let $(\Omega,\mathcal{F},P)$ a probability space and $\mathcal{G}$ a sub σ-algebra of $\mathcal{F}$. Let $P_{\mathcal{G}}(\omega,\cdot)$ the conditional probability of P on $\mathcal{G}$. Then*

$$\bar{P}(A) = \int_{\Omega\setminus N(A)} \bar{P}_{\mathcal{G}}(\omega,A)dP(\omega) \qquad A\in\mathcal{F}^P \tag{50}$$

where $\bar{P}_{\mathcal{G}}(\omega,\cdot)$ is the completion of $P_{\mathcal{G}}(\omega,\cdot)$ and $N(A)\in\mathcal{G}$ is a P-null set which depends on A.

Proof. It is easy to see that every set in $\mathcal{F}^P$ is union of a set $F\in\mathcal{F}$ and a subset of a P-null set. For any $F\in\mathcal{F}$, $\bar{P}(F) = P(F)$ and $P_{\mathcal{G}}(\omega,F) = \bar{P}_{\mathcal{G}}(\omega,F)$ so equality (50) is obvious from the definition of conditional probability (with $N(F)=\varnothing$). Let A be a subset of a P-null set A_1. $0 = P(A_1) = \int_\Omega P_{\mathcal{G}}(\omega,A_1)dP(\omega)$ which means that $P_{\mathcal{G}}(\omega,A_1)=0$ P-a.s. Thus, we also have $\bar{P}_{\mathcal{G}}(\omega,A)=0$ P-a.s. and by taking $N(A)=\{\omega\in\Omega : P_{\mathcal{G}}(\omega,A_1)>0\}\in\mathcal{G}$ equality (50) follows. □

Measurable selection results.

Lemma 1.74. *Let $(\Omega,\mathcal{A})$ a measurable space and $\Psi:\Omega\mapsto 2^{\mathbb{R}^d}$ an $\mathcal{A}$-measurable multifunction. Let $\varepsilon>0$ then*

$$\Psi^\varepsilon : \omega\mapsto \left\{v\in\mathbb{R}^d \mid \langle v,s\rangle\geq\varepsilon \quad \forall s\in\Psi(\omega)\setminus\{0\}\right\}$$

is an $\mathcal{A}$-measurable multifunction.

Proof. Observe first that for $v\in\mathbb{R}^d$

$$\begin{aligned} \langle v,s\rangle\geq\varepsilon \quad \forall s\in\Psi(\omega)\setminus\{0\} &\Leftrightarrow \langle v,s\rangle\geq\varepsilon \quad \forall s\in\overline{\Psi}(\omega)\setminus\{0\} \\ &\Leftrightarrow \langle v,s\rangle\geq\varepsilon \quad \forall s\in D(\omega)\setminus\{0\} \end{aligned} \tag{51}$$

where $D(\omega)$ is a dense subset of $\Psi(\omega)$. This is obvious by continuity of the scalar product. With no loss of generality we can then consider Ψ closed valued and we denote by ψ_n its Castaing representation (see Theorem 14.5 in [**RW98**] for details). For any $n \in \mathbb{N}$ consider the following closed-valued multifunction:

$$\Lambda_n(\omega) = \begin{cases} \{v \in \mathbb{R}^d \mid \langle v, \psi_n(\omega)\rangle \geq \varepsilon\} & \text{if } \omega \in \text{dom } \Psi,\ \psi_n(\omega) \neq 0 \\ \mathbb{R}^d & \text{if } \omega \in \text{dom } \Psi,\ \psi_n(\omega) = 0 \\ \varnothing & \text{otherwise} \end{cases}$$

We claim that Λ_n is measurable $\forall n \in \mathbb{N}$ from which the map $\omega \mapsto \bigcap_{n\in\mathbb{N}} \Lambda_n(\omega)$ is also measurable (cfr Proposition 1.75). From (51) we thus conclude that Ψ^ε is measurable.
We are only left to show the claim. To this end observe that $\Lambda_n(\omega)$ has non-empty interior on $\{\Lambda_n \neq \varnothing\}$. Therefore for any open set $V \subseteq \mathbb{R}^d$ we have

$$\{\omega \in \Omega \mid \Lambda_n(\omega) \cap V \neq \varnothing\} = \{\omega \in \Omega \mid \text{int}(\Lambda_n(\omega)) \cap V \neq \varnothing\}.$$

Note now that

$$\{\omega \in \Omega \mid \text{int}(\Lambda_n(\omega)) \cap V \neq \varnothing\} = \psi_n^{-1}\left(\Pi_y\left(\Pi_x^{-1}(V) \cap \langle\cdot,\cdot\rangle^{-1}(\varepsilon,\infty)\right)\right) \cup \psi_n^{-1}(0)$$

which is measurable (when ψ_n is measurable) from the continuity of $\langle\cdot,\cdot\rangle$ and from the open mapping property of the projections $\Pi_x, \Pi_y : \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{R}^d$. □

PROPOSITION 1.75. *[Proposition 14.2-11-12 [**RW98**]] Consider a certain family of $\mathcal{A}$-measurable set-valued functions. The following operations preserve $\mathcal{A}$-measurability: countable unions, countable intersections (if the functions are closed-valued), finite linear combination, convex/linear/affine hull, generated cone, polar set, closure.*

6.2. Complementary results. Recall that we are assuming that Ω is a Polish space.

LEMMA 1.76. *Let $Q_i \in \mathcal{M}$ for any $i \in \mathbb{N}$. Then*

$$Q := \sum_{i\in\mathbb{N}} \frac{1}{2^i} Q_i \in \mathcal{M}$$

PROOF. We first observe that $Q \in \mathcal{P}$ hence we just need to show that is a martingale measure. Consider the measures $Q_k := \sum_{i=1}^k \frac{1}{2^i} Q_i$, which are not probabilities, and note that for each k we have: $\int_\Omega 1_B \Delta S_t dQ_k = 0$ if $B \in \mathcal{F}_{t-1}$. We observe that $\|Q_k - Q\| \to 0$ for $k \to \infty$, where $\|\cdot\|$ is the total variation norm. We have indeed that

$$\sup_{A\in\mathcal{F}} |Q_k(A) - Q(A)| = \sup_{A\in\mathcal{F}} \sum_{i=k+1}^{\infty} \frac{1}{2^i} Q_i(A) = \sum_{i=k+1}^{\infty} \frac{1}{2^i} \to 0 \text{ as } k \to \infty.$$

In particular we have $Q_k(A) \uparrow Q(A)$ for any $A \in \mathcal{F}$. Representing any simple function f as $\sum_{j=1}^{n(f)} a_j(f) \mathbf{1}_{A_j}$, we obtain for a non negative random variable X

$$\lim_{k\to\infty} \int_\Omega X dQ_k = \lim_{k\to\infty} \sup_{f\in\mathfrak{S}} \sum_{j=1}^{n(f)} a_j(f) Q_k(A_j) = \sup_k \sup_{f\in\mathfrak{S}} \sum_{j=1}^{n(f)} a_j(f) Q_k(A_j)$$

$$= \sup_{f\in\mathfrak{S}} \sup_k \sum_{j=1}^{n(f)} a_j(f) Q_k(A_j) = \sup_{f\in\mathfrak{S}} \sum_{j=1}^{n(f)} a_j(f) Q(A_j) = \int_\Omega X dQ$$

where $\mathfrak{S}$ are the simple function less or equal than X. For any $B \in \mathcal{F}_{t-1}$ we then have:

$$E_Q[1_B \Delta S_t] = \int_\Omega (1_B \Delta S_t)^+ dQ - \int_\Omega (1_B \Delta S_t)^- dQ$$
$$= \lim_{k\to\infty} \int_\Omega (1_B \Delta S_t)^+ dQ_k - \lim_{k\to\infty} \int_\Omega (1_B \Delta S_t)^- dQ_k = \lim_{k\to\infty} \int_\Omega 1_B \Delta S_t dQ_k = 0.$$

□

LEMMA 1.77. *For any dense set $D \subseteq \Omega$, the set of probabilities $co(\{\delta_\omega\}_{\omega\in D})$ is $\sigma(\mathcal{P}, C_b)$ dense in $\mathcal{P}$.*

PROOF. Take $\omega^* \notin D$ and let $\omega_n \to \omega^*$. Note that for every open set G we have $\liminf \delta_{\omega_n}(G) \geq \delta_{\omega^*}(G)$ and this is equivalent to the weak convergence $\delta_{\omega_n} \xrightarrow{w} \delta_{\omega^*}$. Observe that for every set X we have

$$\overline{co(X)} = \overline{co}(X) := \bigcap \{C \mid C \text{ convex closed containing } X\} = \overline{co}(\overline{X}).$$

Hence, by taking $X = \{\delta_\omega\}_{\omega\in D}$ and by $\sigma(\mathcal{P}, C_b)$ density of the set of measures with finite support in $\mathcal{P}$, we obtain the thesis. □

LEMMA 1.78. *Let $\mathcal{F} = \mathcal{B}(\Omega)$ be the Borel σ-algebra and let $\widetilde{\mathcal{F}}$ be a σ-algebra such that $\mathcal{F} \subseteq \widetilde{\mathcal{F}}$. The set $\widetilde{\mathcal{P}} := \{\widetilde{P} : \widetilde{\mathcal{F}} \to [0,1] \mid \widetilde{P}$ is a probability$\}$ is endowed with the topology $\sigma(\widetilde{\mathcal{P}}, C_b)$. Then*

(1) If $A \subseteq \Omega$ is dense in Ω, then $co(\{\delta_\omega\}_{\omega\in A})$ is $\sigma(\widetilde{\mathcal{P}}, C_b)$ dense in $\widetilde{\mathcal{P}}$. Notice that any element $Q \in co(\{\delta_\omega\}_{\omega\in A})$ can be extended to $\widetilde{\mathcal{F}}$.

(2) If $D \subseteq \Omega$ is closed then

$$\widetilde{\mathcal{P}}(D) := \{\widetilde{P} \in \widetilde{\mathcal{P}} \mid supp(\widetilde{P}) \subseteq D\}$$

is $\sigma(\widetilde{\mathcal{P}}, C_b)$ closed, where the support is well-defined by

$$supp(\widetilde{P}) := \bigcap \{C \in \mathcal{C} \mid \widetilde{P}(C) = 1\}$$

and $\mathcal{C}$ are the closed sets in (Ω, d).

PROOF. By construction for any $\widetilde{P} \in \widetilde{\mathcal{P}}$ we have $\int f d\widetilde{P} = \int f dP$ for any $f \in C_b$ where $P \in \mathcal{P}$ is the restriction of $\widetilde{P}$ to $\mathcal{F}$.
To show the first claim we choose any $\widetilde{P} \in \widetilde{\mathcal{P}}$. Consider $P \in \mathcal{P}$ the restriction of $\widetilde{P}$ to $\mathcal{F}$. Then from Lemma 1.77 there exists a sequence $Q_n \in co(\{\delta_\omega\}_{\omega\in A})$ such that $\int f dQ_n \to \int f dP$ for every $f \in C_b$. As a consequence $\int f dQ_n \to \int f d\widetilde{P}$, for every $f \in C_b$.
To show the second claim consider any net $\{\widetilde{P}_\alpha\}_\alpha \subset \widetilde{\mathcal{P}}(D)$ such that $\widetilde{P}_\alpha \xrightarrow{w} \widetilde{P}$. We want to show that $\widetilde{P} \in \widetilde{\mathcal{P}}(D)$. Consider P_α, P the restriction to $\mathcal{F}$ of $\widetilde{P}_\alpha, \widetilde{P}$ respectively. Then $P_\alpha \xrightarrow{w} P$. Notice that by definition $supp(P_\alpha) = supp(\widetilde{P}_\alpha) \subseteq D$ and $supp(P) = supp(\widetilde{P})$. Moreover $\mathcal{P}(D) = \{P \in \mathcal{P} \mid supp(P) \subseteq D\}$ is $\sigma(\mathcal{P}, C_b)$ closed (Theorem 15.19 in [**AB06**]) so that $D \supseteq supp(P) = supp(\widetilde{P})$. □

PROOF OF PROPOSITION 1.64, ITEM (2). Recall that an Open Arbitrage in $\widetilde{\mathcal{H}}$ is a $\widetilde{\mathbb{F}}$-predictable processes $H = [H^1, \ldots, H^d]$ such that $V_T(H) \geq 0$ and $\mathcal{V}_H^+ = \{V_T(H) > 0\}$ contains an open set. First we show that $H \in W(\widetilde{\sigma}, \widetilde{\mathcal{H}})$ implies $V_T(H)(\omega) \geq 0$ for all $\omega \in \Omega$. We need only to show that the set $B := \{\omega \in \Omega \mid V_T(H)(\omega) < 0\}$ is empty. By contradiction, let $\omega \in B$, take any $P \in \mathcal{U}$ and define the probability $P_\lambda := \lambda\delta_\omega + (1-\lambda)P$. Since $V_T(H) \geq 0$ P-a.s. we must have $P(\omega) = 0$, otherwise $P(B) > 0$. However, $P_\lambda(B) \geq P_\lambda(\omega) = \lambda > 0$ for all positive λ and P_λ will belongs to

$\mathcal{U}$, as $\lambda \downarrow 0$, which contradicts $V_T(H) \geq 0$ P-a.s. for any $P \in \mathcal{U}$. To prove the equivalence, assume first that $H \in W(\widetilde{\sigma}, \widetilde{\mathcal{H}})$. We claim that $(\mathcal{V}_H^+)^c = \{\omega \in \Omega \mid V_T(H) = 0\}$ is not dense in Ω. This will imply the thesis as the open set $int(\mathcal{V}_H^+)$ will then be a not empty on which $V_T(H) > 0$. Suppose by contradiction that $\overline{(\mathcal{V}_H^+)^c} = \Omega$. We know by Lemma 1.78 that the corresponding set $\mathcal{Q}$ of embedded probabilities $co(\{\delta_\omega\}_{\omega \in (\mathcal{V}_H^+)^c})$ is weakly dense in $\widetilde{\mathcal{P}}$ and hence it intersects, in particular, the weakly open set $\mathcal{U}$. However, for every $P \in \mathcal{Q}$ we have $V_T(H) = 0$ P-a.s. and so this contradicts the assumption. Suppose now that $H \in \widetilde{H}$ is an Open Arbitrage. Note that from Lemma 1.78 if F is a closed subset of Ω, then $\widetilde{\mathcal{P}}(F) := \{P \in \widetilde{\mathcal{P}} \mid \ supp(P) \subset F\}$ is $\sigma(\widetilde{\mathcal{P}}, C_b)$ closed. Since H is an Open Arbitrage then $\mathcal{V}_H^+$ contains an open set and in particular $G := \overline{(\mathcal{V}_H^+)^c}$ is a closed set strictly contained in Ω. Observe then that $(\widetilde{\mathcal{P}}(G))^c$ is a non empty $\sigma(\widetilde{\mathcal{P}}, C_b)$ open set of probabilities such that for all $P \in \mathcal{U}$ we have $V_T(H) \geq 0$, P-a.s. and $P(\mathcal{V}_H^+) > 0$. $\square$

CHAPTER 2

Model-free Superhedging duality[1]

We adopt the following setting and notations: let Ω be a Polish space and $\mathcal{F} = \mathcal{B}(\Omega)$ be the Borel sigma-algebra; $T \in \mathbb{N}$, $I := \{0, ..., T\}$, $S = (S_t)_{t\in I}$ be an $\mathbb{R}^d$-valued stochastic process on $(\Omega, \mathcal{F})$ representing the price process of $d \in \mathbb{N}$ assets; $\mathcal{P}$ be the set of all probability measures on $(\Omega, \mathcal{F})$; $\mathbb{F}^S := \{\mathcal{F}_t^S\}_{t\in I}$ be the natural filtration and $\mathbb{F} := \{\mathcal{F}_t\}_{t\in I}$ be the Universal Filtration, namely

$$\mathcal{F}_t := \bigcap_{P\in\mathcal{P}} \mathcal{F}_t^S \vee \mathcal{N}_t^P, \text{ where } \mathcal{N}_t^P = \{N \subseteq A \in \mathcal{F}_t^S \mid P(A) = 0\};$$

$\mathcal{H}$ be the class of $\mathbb{F}$-predictable stochastic processes, with values in $\mathbb{R}^d$, representing the family of admissible trading strategies; $(H \cdot S)_T := \sum_{t=1}^T \sum_{j=1}^d H_t^j (S_t^j - S_{t-1}^j) = \sum_{t=1}^T H_t \cdot \Delta S_t$ be the gain up to time T from investing in S adopting the strategy H. We denote

$$\begin{aligned} \mathcal{M} :=& \quad \{Q \in \mathcal{P} \mid S \text{ is an } \mathbb{F}\text{-martingale under } Q\}, \\ \mathcal{P}_f :=& \quad \{Q \in \mathcal{P} \mid \text{supp}(Q) \text{ is finite}\}, \\ \mathcal{M}_f :=& \quad \mathcal{M} \cap \mathcal{P}_f, \end{aligned}$$

where the support of $P \in \mathcal{P}$ is defined by $\text{supp}(P) = \bigcap\{C \in \mathcal{F} \mid C \text{ closed, } P(C) = 1\}$. The family of $\mathcal{M}$-polar sets is given by $\mathcal{N} := \{N \subseteq A \in \mathcal{F} \mid Q(A) = 0\ \forall Q \in \mathcal{M}\}$ and a property is said to hold *quasi surely* (q.s.) if it holds outside a polar set. We adopt the convention $\infty - \infty = -\infty$ for those random variables g whose positive and negative part is not integrable. We are also assuming the existence of a numeraire asset $S_t^0 = 1$ for all $t \in I$.

The aim of this Chapter is the proof of the following discrete time, model independent version of the superhedging theorem.

Theorem 2.1 (Superhedging). *Let $g : \Omega \mapsto \mathbb{R}$ be an $\mathcal{F}$-measurable random variable. Then*

$$\begin{aligned} & \inf\{x \in \mathbb{R} \mid \exists H \in \mathcal{H} \text{ such that } x + (H \cdot S)_T \geq g\ \mathcal{M}\text{-q.s.}\} \\ =\ & \inf\{x \in \mathbb{R} \mid \exists H \in \mathcal{H} \text{ such that } x + (H \cdot S)_T(\omega) \geq g(\omega)\ \forall \omega \in \Omega_*\} \\ =\ & \sup_{Q\in\mathcal{M}_f} E_Q[g] = \sup_{Q\in\mathcal{M}} E_Q[g], \end{aligned}$$

where

$$\Omega_* := \{\omega \in \Omega \mid \exists Q \in \mathcal{M} \text{ s.t. } Q(\omega) > 0\}. \tag{52}$$

[1] Chapter 2 is based on the preprint: Model-free Superhedging duality, joint work with M. Frittelli and M. Maggis, arXiv 1506.06608.

Probability free set up. In the statement of the superhedging theorem there is no reference to any a priori assigned probability measure and the notions of $\mathcal{M}$, $\mathcal{H}$ and Ω_* only depend on the measurable space $(\Omega, \mathcal{F})$ and the price process S. In general the class $\mathcal{M}$ is not dominated. In case $\mathcal{M} = \varnothing$ then $\Omega_* = \varnothing$ and the theorem is trivial, as each term in the equalities of Theorem 2.1 is equal to $-\infty$, provided we convene that any $\mathcal{M}$-q.s. inequalities hold true when $\mathcal{M} = \varnothing$. For this reason we will assume without loss of generality $\mathcal{M} \neq \varnothing$, and recall that this condition can be reformulated in terms of absence of Model Independent $\widetilde{\mathcal{H}}$-Arbitrages (see Chapter 1).
We are not imposing any restriction on S so that it may describe generic financial securities (for examples, stocks and/or options). However, in the framework of Theorem 2.1 the class $\mathcal{H}$ of admissible trading strategies requires dynamic trading in all assets. In Theorem 2.2 below we easily extend this setup to the case of semi-static trading on a finite number of options.
As illustrated in Section 3, we explicitly show that the initial cost of the cheapest portfolio that dominates a contingent claim g on *every possible path*

$$\inf \{x \in \mathbb{R} \mid \exists H \in \mathcal{H} \text{ such that } x + (H \cdot S)_T(\omega) \geq g(\omega)\ \forall \omega \in \Omega\} \tag{53}$$

can be strictly greater than $\sup_{Q \in \mathcal{M}} E_Q[g]$, unless some artificial assumptions are imposed on g or on the market. In order to avoid these restrictions on the class of derivatives, it is crucial to select the correct set of paths (i.e. Ω_*) where the superhedging strategy can be efficiently employed.

On the set Ω_*. In Theorem 2.1, the pathwise model independent inequality in (53), is replaced with an inequality involving only those $\omega \in \Omega$ which are weighted by at least one martingale measure $Q \in \mathcal{M}$. In Chapter 1 (see also Proposition 2.9) it is shown the existence of the maximal $\mathcal{M}$-polar set N_*, namely a set $N_* \in \mathcal{N}$ containing any other set $N \in \mathcal{N}$. Moreover

$$\Omega_* = (N_*)^C. \tag{54}$$

The inequality $x + (H \cdot S)_T \geq g$ $\mathcal{M}$-q.s. holds by definition outside any $\mathcal{M}$-polar set and therefore it is equivalent, thanks to (54), to the inequality $x + (H \cdot S)_T(\omega) \geq g(\omega)\ \forall \omega \in \Omega_*$, which justifies the first equality in Theorem 2.1. The set Ω_* can be equivalently determined (see Proposition 2.9) via the set $\mathcal{M}_f$ of martingale measures with finite support, a property that turns out to be crucial in several proofs.
We stress that we do not make any ad hoc assumptions on the discrete time financial model and notice that Ω_* is determined only by S: indeed the set $\mathcal{M}$ can be written also as $\mathcal{M} = \{Q \in \mathcal{P} \mid S \text{ is an } \mathbb{F}^S\text{-martingale under } Q\}$. One of the main technical results of this Chapter is the proof that the set Ω_* is an *analytic set* (Proposition 2.17) and so our findings show that the natural setup for studying this problem is $(\Omega, S, \mathbb{F}, \mathcal{H})$. We also point out that we could replace any sigma-algebra $\mathcal{F}_t$ with the sub sigma-algebra generated by the analytic sets of $\mathcal{F}_t^S$.

Superhedging with semi-static strategies on options and stocks. We now allow for the possibility of static trading in a finite number of options. Let us add to the previous market k options $\Phi = (\phi^1, ..., \phi^k)$ which expires at time T and assume without loss of generality that they have zero initial cost. We assume that each ϕ^j is an $\mathcal{F}$-measurable random variable. Define $h\Phi := \sum_{j=1}^k h^j \phi^j$, $h \in \mathbb{R}^k$, and

$$\mathcal{M}_\Phi := \{Q \in \mathcal{M}_f \mid E_Q[\phi^j] = 0\ \forall j = 1, ..., k\} = \{Q \in \mathcal{M}_f \mid E_Q[h\Phi] = 0\ \forall h \in \mathbb{R}^k\}, \tag{55}$$

which are the options-adjusted martingale measures, and

$$\Omega_\Phi := \{\omega \in \Omega \mid \exists Q \in \mathcal{M}_\Phi \text{ s.t. } Q(\omega) > 0\} \subseteq \Omega_*. \tag{56}$$

We have by definition that for every $Q \in \mathcal{M}_\Phi$ the support satisfies $supp(Q) \subseteq \Omega_\Phi$. We define the superhedging price when semi-static strategies are allowed by

$$\pi_\Phi(g) := \inf \left\{ x \in \mathbb{R} \mid \exists (H,h) \in \mathcal{H} \times \mathbb{R}^k \text{ such that } x + (H \cdot S)_T(\omega) + h\Phi(\omega) \geq g(\omega) \ \forall \omega \in \Omega_\Phi \right\}. \tag{57}$$

With the same methodology used in the proof of Theorem 2.1 we will obtain in Section 4.3 the superhedging duality with semi-static strategies:

THEOREM 2.2 (Super-hedging with options). *Let $g : \Omega \mapsto \mathbb{R}$ and $\phi^j : \Omega \mapsto \mathbb{R}$, $j = 1, ..., k$, be $\mathcal{F}$-measurable random variables. Then*

$$\pi_\Phi(g) = \sup_{Q \in \mathcal{M}_\Phi} E_Q[g].$$

Comparison with the related literature. In the classical case when a reference probability is fixed, this subject was originally studied by El Karoui and Quenez [**KQ95**]; see also [**Ka97**] and [**DS94**] and the references cited therein.

In [**BN15**] a superhedging theorem is proven in the case of a non-dominated class of priors $\mathcal{P}' \subseteq \mathcal{P}$. The result strongly relies on two technical hypothesis: (i) The state space Ω has a product structure, $\Omega = \Omega_1^T$, where Ω_1 is a certain fixed Polish space and Ω_1^t is the t-fold product space; (ii) The set of priors $\mathcal{P}'$ is also obtained as a collection of product measures $P := P_0 \otimes \ldots \otimes P_T$ where every P_t is a measurable selector of a certain random class $\mathcal{P}'_t \subseteq \mathcal{P}(\Omega_1)$. $\mathcal{P}'_t(\omega)$ represents the set of possible models for the t-th period, given state ω at time t. An essential requirement on $\mathcal{P}'_t$ is that the graph($\mathcal{P}'_t$) must be an analytic subset of $\Omega_1^t \times \mathcal{P}(\Omega_1)$. These assumptions are crucial in order to apply the measurable selection and stochastic control arguments which lead to the proof of the superhedging theorem. In our setting we do not impose restrictions on the state space Ω so the result cannot be deduced from [**BN15**] for $\mathcal{P}' = \mathcal{M}$. Moreover, even in the case of $\Omega = \Omega_1^T$, the class of martingale probability measures $\mathcal{M}$ is endogenously determined by the market and we do not require that it satisfies any additional restrictions. Furthermore, the techniques employed to deduce our version of the superhedging duality theorem are completely different, as they rely on the results of [**BFM14**].

Different approaches are taken in [**AB13, Ri15**]. In [**Ri15**] the continuity assumptions on the assets allow to embed the problem in the linear programming framework and to obtain the desired equality in a one period market. In [**AB13**] from a model independent version of the Fundamental Theorem of Asset Pricing they deduce the following superhedging duality (Theorem 1.4)

$$\inf \left\{ x \in \mathbb{R} \mid \exists (H,h) \in \mathcal{H} \times \mathbb{R}^k \text{ s.t. } x + (H \cdot S)_T(\omega) + h\Phi(\omega) \geq g(\omega) \ \forall \omega \in \Omega \right\} = \sup_{Q \in \mathcal{M}_\Phi} E_Q[g]. \tag{58}$$

They assume a discrete time market, with one dimensional canonical process S on the path space $\Omega = [0, \infty)^T$ and an arbitrary (but non empty) set of options on S available for static trading. Theorem 1.4 in [**AB13**] relies on two additional technical assumptions: (i) The existence of an option with super-linearly growing and convex payoff; (ii) The upper semi-continuity of the claim g.

The example in Section 3 shows that without the upper semi-continuity of the claim g the duality in (58) fails and it also points out that the reason for this is the insistence of superhedging over the whole space Ω, instead of over the relevant set of paths Ω_*. Our result holds for a d-dimensional (not necessarily canonical) process S and does not necessitate of any specific technical assumptions, nor of the existence of any options.

1. Aggregation results

In this section we investigate when certain conditions (like superhedging or hedging) which hold Q-a.s. for all $Q \in \mathcal{M}$, ensure the validity of the correspondent pathwise conditions on Ω_*.
For $\mathcal{G}$-measurable random variables X and Y, we write $X > Y$ if $X(\omega) > Y(\omega)$ for all $\omega \in \Omega$. When we specify $X > Y$ on a measurable set $A \subset \Omega$ it means that $X(\omega) > Y(\omega)$ holds for all $\omega \in A$. Similarly for $X \geq Y$ and $X = Y$. We recall that absence of classical arbitrage opportunities, with respect to a probability $P \in \mathcal{P}$, is denoted by $NA(P)$. For an arbitrary sigma-algebra $\mathcal{G}$ we set

$$\begin{aligned}\mathcal{L}(\Omega,\mathcal{G}) :=&\quad \{f : \quad \Omega \to \mathbb{R} \mid \mathcal{G}\text{-measurable }\},\\ \mathcal{L}(\Omega,\mathcal{G})_+ :=&\quad \{f \in \quad \mathcal{L}(\Omega,\mathcal{G}) \mid f \geq 0\}.\end{aligned}$$

The linear space of attainable random payoffs with zero initial cost is given by

$$\mathcal{K} := \{(H \cdot S)_T \in \mathcal{L}(\Omega,\mathcal{F}) \mid H \in \mathcal{H}\}.$$

Recall that the set of events supporting martingale measures Ω_* is defined in (52) and observe that the convex cones

$$\mathcal{C} := \quad \{f \in \quad \mathcal{L}(\Omega,\mathcal{F}) \mid f \leq k \text{ on } \Omega_* \text{ for some } k \in \mathcal{K}\}, \tag{59}$$

$$\mathcal{C}(Q) := \quad \{f \in \quad \mathcal{L}(\Omega,\mathcal{F}) \mid f \leq k \ Q\text{-a.s. for some } k \in \mathcal{K}\}. \tag{60}$$

are related by $\mathcal{C} \subseteq \mathcal{C}(Q)$, if $Q \in \mathcal{M}$.
The main Theorem 2.1 relies on the following cornerstone proposition that will be proven in Section 4, as its proof requires several technical arguments.

PROPOSITION 2.3. *Let $g \in \mathcal{L}(\Omega,\mathcal{F})$ and define*

$$\pi_*(g) \quad : \quad = \inf\{x \in \mathbb{R} \mid \ \exists H \in \mathcal{H} \ s.t. \ x + (H \cdot S)_T \geq g \quad on \ \Omega_*\} \tag{61}$$

$$\pi_Q(g) \quad : \quad = \inf\{x \in \mathbb{R} \mid \ \exists H \in \mathcal{H} \ s.t. \ x + (H \cdot S)_T \geq g \quad Q\text{- } a.s. \ \}. \tag{62}$$

Then

$$\pi_*(g) \quad = \quad \sup_{Q \in \mathcal{M}_f} \pi_Q(g) \tag{63}$$

$$\mathcal{C} \quad = \quad \bigcap_{Q \in \mathcal{M}_f} \mathcal{C}(Q). \tag{64}$$

In particular, if $\pi_*(g) < +\infty$ the infimum is a minimum.

COROLLARY 2.4. *Let $g \in \mathcal{L}(\Omega,\mathcal{F})$ and $x \in \mathbb{R}$. If for every $Q \in \mathcal{M}_f$ there exists $H^Q \in \mathcal{H}$ such that $x + (H^Q \cdot S)_T \geq g$ Q-a.s. then there exists $H \in \mathcal{H}$ such that $x + (H \cdot S)_T(\omega) \geq g(\omega)$ for every $\omega \in \Omega_*$.*

PROOF. By assumption, $g - x \in \mathcal{C}(Q)$ for every $Q \in \mathcal{M}_f$. From $\mathcal{C} = \bigcap_{Q\in\mathcal{M}_f} \mathcal{C}(Q)$ we obtain $g - x \in \mathcal{C}$. $\square$

COROLLARY 2.5 (Perfect hedge). *Let $g \in \mathcal{L}(\Omega, \mathcal{F})$. If for every $Q \in \mathcal{M}_f$ there exists $H^Q \in \mathcal{H}, x^Q \in \mathbb{R}$ such that $x^Q + (H^Q \cdot S)_T = g$ Q-a.s. then there exists $H \in \mathcal{H}, x \in \mathbb{R}$ such that $x + (H \cdot S)_T(\omega) = g(\omega)$ for every $\omega \in \Omega_*$, and $x^Q = x$ for every $Q \in \mathcal{M}_f$.*

PROOF. Note first that, from the hypothesis, for every $Q \in \mathcal{M}_f$ there exists $H^Q \in \mathcal{H}$, $x^Q \in \mathbb{R}$ such that $x^Q + (H^Q \cdot S)_T(\omega) = g(\omega)$ for every $\omega \in supp(Q)$. We first show that x^Q does not depend on Q. Assume there exist $Q_1, Q_2 \in \mathcal{M}_f$ such that $x^{Q_1} < x^{Q_2}$. For every $\lambda \in (0, 1)$ set $Q_\lambda := \lambda Q_1 + (1-\lambda)Q_2 \in \mathcal{M}_f$. Then there exist $H^{Q_\lambda} \in \mathcal{H}$ and $x^{Q_\lambda} \in \mathbb{R}$ such that $x^{Q_\lambda} + (H^{Q_\lambda} \cdot S)_T(\omega) = g(\omega)$ for every $\omega \in supp(Q_\lambda) = supp(Q_1) \cup supp(Q_2)$. Therefore $x^{Q_\lambda} + (H^{Q_\lambda} \cdot S)_T(\omega) = g(\omega)$ for every $\omega \in supp(Q_i)$, for any $i = 1, 2$, and from $NA(Q_i)$ we necessarily have that $x^{Q_\lambda} = x^i$.
Since $x + (H^Q \cdot S)_T(\omega) = g(\omega)$ for every $\omega \in supp(Q)$ we can apply Corollary 2.4 which implies the existence of $H \in \mathcal{H}$ such that $x + (H \cdot S)_T(\omega) \geq g(\omega)$ on Ω_*. Moreover $x - x + ((H - H^Q) \cdot S)_T(\omega) \geq g(\omega) - g(\omega)$ for every $\omega \in supp(Q)$ implies $((H - H^Q) \cdot S)_T(\omega) \geq 0$ for every $\omega \in supp(Q)$. Since $NA(Q)$ holds, we conclude $((H - H^Q) \cdot S)_T(\omega) = 0$ for every $\omega \in supp(Q)$. Thus for every $Q \in \mathcal{M}_f$ we have $x + (H \cdot S)_T(\omega) = g(\omega)$ on $supp(Q)$ and hence the thesis follows from Proposition 4.18 [**BFM14**] (or Proposition 2.9). $\square$

COROLLARY 2.6 (Bipolar representation). *Let $\mathcal{C}$ be defined in (59). Then*

$$\mathcal{C} = \{g \in \mathcal{L}(\Omega, \mathcal{F}) \mid E_Q[g] \leq 0 \; \forall \, Q \in \mathcal{M}_f\} \tag{65}$$

PROOF. Clearly $\mathcal{C} \subseteq \{g \in \mathcal{L}(\Omega, \mathcal{F}) \mid E_R[g] \leq 0 \; \forall \, R \in \mathcal{M}_f\} =: \widetilde{\mathcal{C}}$. Fix $Q \in \mathcal{M}_f$ and observe that $L^0(\Omega, \mathcal{F}, Q) \equiv L^1(\Omega, \mathcal{F}, Q) \equiv L^\infty(\Omega, \mathcal{F}, Q)$. For $g \in \mathcal{L}(\Omega, \mathcal{F})$ we denote with the capital letter G the corresponding equivalence class $G \in L^0(\Omega, \mathcal{F}, Q)$. The quotient of $\mathcal{K}$ and $\mathcal{C}(Q)$ with respect to the Q-a.s. identification $\sim_Q$ are denoted respectively by

$$\begin{aligned}
\mathcal{K}_Q \quad &: \; = \{K \in L^0(\Omega, \mathcal{F}, Q) \mid K = (H \cdot S)_T \; Q - \text{a.s.}, \; H \in \mathcal{H}\}, \\
\mathcal{C}_Q \quad &: \; = \{G \in L^0(\Omega, \mathcal{F}, Q) \mid \exists K \in \mathcal{K}_Q \text{ such that } G \leq K \; Q - \text{a.s.}\} = \mathcal{K}_Q - L^0_+(\Omega, \mathcal{F}, Q).
\end{aligned}$$

Now we may follow the classical arguments: the convex cone $\mathcal{C}_Q$ is closed in probability with respect to Q (see e.g. [**KS01a**] Theorem 1). As $Q \in \mathcal{M}_f$, $\mathcal{C}_Q$ is also closed in $L^1(\Omega, \mathcal{F}, Q)$ and therefore:

$$(\mathcal{C}_Q)^0 = \{Z \in L^\infty(\Omega, \mathcal{F}, Q) \mid E[ZG] \leq 0 \; \forall G \in \mathcal{C}_Q\} \subseteq L^\infty_+(\Omega, \mathcal{F}, Q).$$

Notice that $R \ll Q$ and $R \in \mathcal{M}_f$ if and only if $R \ll Q$ and $\frac{dR}{dQ} \in (\mathcal{C}_Q)^0$. Hence:

$$\begin{aligned}
(\mathcal{C}_Q)^{00} \quad &= \quad \left\{G \in L^1(\Omega, \mathcal{F}, Q) \mid E[ZG] \leq 0 \; \forall Z \in (\mathcal{C}_Q)^0\right\} \\
&= \quad \left\{G \in L^1(\Omega, \mathcal{F}, Q) \mid E_R[G] \leq 0 \; \forall R \ll Q \text{ s.t. } \frac{dR}{dQ} \in (\mathcal{C}_Q)^0\right\} \\
&= \quad \left\{G \in L^1(\Omega, \mathcal{F}, Q) \mid E_R[G] \leq 0 \; \forall R \ll Q \text{ s.t. } R \in \mathcal{M}_f\right\}
\end{aligned} \tag{66}$$

Let $g \in \widetilde{\mathcal{C}}$. By the characterization in (66) the corresponding G belongs to $(\mathcal{C}_Q)^{00}$. By the bipolar theorem $\mathcal{C}_Q = (\mathcal{C}_Q)^{00}$ and therefore $G \in \mathcal{C}_Q$ and $g \in \mathcal{C}(Q)$ (as defined in (60)). Since this holds for any $Q \in \mathcal{M}_f$, from $\mathcal{C} = \bigcap_{Q\in\mathcal{M}_f} \mathcal{C}(Q)$ (Proposition 2.3) we conclude that $g \in \mathcal{C}$. $\square$

REMARK 2.7. *One may ask whether the bipolar duality* (65) *implies that* $\mathcal{C}$ *is closed with respect to some topology. To answer this question let us introduce on* $\mathcal{L}(\Omega,\mathcal{F})$ *the following equivalence relation: for any* $X,Y\in\mathcal{L}(\Omega,\mathcal{F})$

$$X\sim Y \text{ if and only if } X(\omega)-Y(\omega)=k(\omega) \text{ for some } k\in\mathcal{K} \text{ and for every } \omega\in\Omega_*.$$

Consider the quotient space $\mathbf{L}(\Omega,\mathcal{F})=\mathcal{L}(\Omega,\mathcal{F})/\sim$ *and the vector space* V_f *generated by* $\mathcal{M}_f$. *We first claim that the couple* $(\mathbf{L}(\Omega,\mathcal{F}),V_f)$ *is a separated dual pair under the bilinear form* $\langle\cdot,\cdot\rangle:\mathbf{L}(\Omega,\mathcal{F})\times V_f\to\mathbb{R}$ *defined by:* $\langle[X],\mu\rangle\mapsto E_\mu[X]$, *for any* $X\in[X]$. *Notice that the form* $\langle[X],\mu\rangle\mapsto E_\mu[X]$ *is well posed as* $E_\mu[k]=0$ *for all* $k\in\mathcal{K}$ *and the pairing is obviously bilinear. Clearly if* $\mu\neq 0$ *then there exists* $\omega\in\Omega_*$ *such that* $\mu(\{\omega\})\neq 0$ *and* $E_\mu[\mathbf{1}_\omega]\neq 0$. *Thus we have showed that* $\langle[X],\mu\rangle=0$, *for every* $[X]$, *implies* $\mu=0$.
We now prove that $\langle[X],\mu\rangle=0$ *for every* μ *implies* $[X]=[0]$. *By contradiction assume* $[X]\neq[0]$. *By assumption,* X *can not be replicable at a non zero cost. Observe that if* $X\in[X]$ *is replicable at zero cost in any market* $(\Omega,\mathcal{F},\mathbb{F},S;Q)$ *for any possible choice* $Q\in\mathcal{M}_f$ *then by Corollary 2.5* X *is pathwise replicable for every* $\omega\in\Omega_*$, *or in other words:* $[X]=[0]$.
Hence our assumption $[X]\neq[0]$ *implies that there exists a* $Q\in\mathcal{M}_f$ *such that the market* $(\Omega,\mathcal{F},\mathbb{F},S;Q)$ *is not complete, so that* $\mathcal{M}_e(Q):=\{Q^*\sim Q\mid Q^*\in\mathcal{M}\}\}\neq\{Q\}$, *and* $X\in[X]$ *is not replicable in such market. Then*

$$\inf_{Q^*\in\mathcal{M}_e(Q)}E_{Q^*}[X]<\sup_{Q^*\in\mathcal{M}_e(Q)}E_{Q^*}[X].$$

As $Q\in\mathcal{M}_f$ *has finite support,* $\mathcal{M}_e(Q)\subset\mathcal{M}_f$ *and there exists a* $\mu\in\mathcal{M}_e(Q)\subset V_f$ *such that* $E_\mu[X]\neq 0$, *which is a contradiction.*
Now we conclude that the cone $\mathcal{C}/_\sim$ *is closed with respect to the weak topology* $\sigma(\mathbf{L}(\Omega,\mathcal{F}),V_f)$. *Indeed, from* (65) *we obtain that*

$$\mathcal{C}/_\sim=\{[g]\in\mathbf{L}(\Omega,\mathcal{F})\mid E_Q[g]\leq 0\;\forall Q\in\mathcal{M}_f\}=\bigcap_{Q\in\mathcal{M}_f}\{[g]\in\mathbf{L}(\Omega,\mathcal{F})\mid E_Q[g]\leq 0\}$$

is the intersection of $\sigma(\mathbf{L}(\Omega,\mathcal{F}),V_f)$*-closed sets.*

2. Proof of Theorem 2.1

As shown by the following result from [**BF04**], the abstract version of the superhedging theorem is a simple consequence that the cone $\mathcal{C}$ and its bipolar cone coincide.

THEOREM 2.8 (Theorem 10, [**BF04**]). *Let* L,L' *be two vector spaces and let* $<\cdot,\cdot>: L\times L'\to\mathbb{R}$ *be a bilinear form. Let* $G\subseteq L$ *be a convex cone satisfying* $G^{00}=G$, *where* $G^0{:=}\{z\in L'\mid <g,z>\leq 0\;\forall g\in G\}$, $G^{00}{:=}\{g\in L\mid <g,z>\leq 0\;\forall z\in G^0\}$, *and assume the existence of an element* $\mathbf{1}\in L$ *such that* $-\mathbf{1}\in G$. *If the set* $N_1\triangleq\{z\in G^0\mid <\mathbf{1},z>=1\}$ *is not empty then for all* $h\in L$ *we have:*

$$\inf\{x\in\mathbb{R}\mid h-x\mathbf{1}\in G\}=\sup\{<h,z>\mid z\in N_1\}. \tag{67}$$

We also recall from [**BFM14**] the relevant properties of the set Ω_* that will be needed several times in the proofs.

Proposition 2.9 (Proposition 4.18, [**BFM14**]). *In the setting described in the Introduction of this Chapter we have*

$$\begin{aligned}\mathcal{M} &\neq \varnothing \Longleftrightarrow \Omega_* \neq \varnothing \Longleftrightarrow \mathcal{M}_f \neq \varnothing \\ \Omega_* &= \{\omega \in \Omega \mid \exists Q \in \mathcal{M}_f \ s.t. \ Q(\omega) > 0\}. \end{aligned} \tag{68}$$

The complement of Ω_* is the maximal $\mathcal{M}$-polar set.

Proof of Theorem 2.1. As already stated in the introduction, we may assume w.l.o.g. that $\mathcal{M} \neq \varnothing$, or equivalently $\mathcal{M}_f \neq \varnothing$. The first equality of the theorem holds because of the definition of $\mathcal{M}$-q.s. inequality and the fact that Ω_* is the maximal $\mathcal{M}$-polar set.
Step 1: Here we show that

$$\inf\{x \in \mathbb{R} \mid \exists H \in \mathcal{H} \text{ such that } x + (H \cdot S)_T(\omega) \geq g(\omega)\ \forall \omega \in \Omega_*\} = \sup_{Q \in \mathcal{M}_f} E_Q[g].$$

Consider V_f the vector space generated by $\mathcal{M}_f$. The couple $(\mathcal{L}(\Omega, \mathcal{F}), V_f)$ form a (not separated) dual pair under the bilinear form

$$\langle \cdot, \cdot \rangle : \mathcal{L}(\Omega, \mathcal{F}) \times V_f \to \mathbb{R} \quad \langle X, \mu \rangle \mapsto E_\mu[X].$$

Set $G := \mathcal{C}$. Adopting for G^0, G^{00} and N_1 the notations of Theorem 2.8, we observe that $G^0 = (V_f)_+$, $N_1 = \{\mu \in (V_f)_+ \mid E_\mu[\mathbf{1}_\Omega] = 1\} = \mathcal{M}_f \neq \varnothing$.
In addition, by Corollary 2.6 we obtain $G = G^{00}$ and from (67) we then conclude:

$$\inf\{x \in \mathbb{R} \mid g - x \in \mathcal{C}\} = \sup\{E_Q[g] \mid Q \in \mathcal{M}_f\}.$$

Step 2: We end the proof by showing that for any $g \in \mathcal{L}(\Omega, \mathcal{F})$

$$\sup_{Q \in \mathcal{M}} E_Q[g] = \sup_{Q \in \mathcal{M}_f} E_Q[g], \tag{69}$$

where we adopt the convention $\infty - \infty = -\infty$ for those random variables g whose positive and negative part is not integrable. Set:

$$m := \sup_{Q \in \mathcal{M}} E_Q[g], \quad l := \sup_{Q \in \mathcal{M}_f} E_Q[g].$$

We obviously have that $l \leq m$ so that we only have to prove the converse inequality. If $l = \infty$ there is nothing to prove. Suppose then $l < \infty$. We first show that

$$\text{if } Q \in \mathcal{M} \text{ satisfy } E_Q[g] > l \Rightarrow E_Q[g] = \infty \tag{70}$$

Suppose indeed by contradiction that there exists $Q \in \mathcal{M} \setminus \mathcal{M}_f$ such that $l < E_Q[g] < \infty$. Consider now an arbitrary version of the process $g_t := E_Q[g \mid \mathcal{F}_t]$ and extend the original market with the asset $S_t^{d+1} := g_t$ for $t \in I$. We obviously have that Q is a martingale measure for the extended market and from Proposition 2.9 this implies the existence of a finite support martingale measure Q_f which, by construction, belongs to $\mathcal{M}_f$. Since $E_{Q_f}[g] = g_0 > l$, which is the supremum of the expectations of g over $\mathcal{M}_f$, we have a contradiction.
From (70) we readily infer that if $m < \infty$ then $l = m$. We are only left to study the case of $m = \infty$ and we show that this is not possible under the hypothesis $l < \infty$. Consider first the class of martingale measures $\mathcal{Q}(g) \subset \mathcal{M}$ such that $E_Q[g^-] = \infty$. We obviously have that $\mathcal{Q}(g) \cap \mathcal{M}_f = \varnothing$, moreover, since $l < m = \infty$ from (70) and from $\infty - \infty = -\infty$, there exists $\widetilde{Q} \in \mathcal{M} \setminus \mathcal{Q}(g)$ such

that $E_{\widetilde{Q}}[g]=\infty$ and $E_{\widetilde{Q}}[g^-]<\infty$. Consider now the sequence of claims $g_n := g\wedge n$ for any $n\in\mathbb{N}$. From $E_{\widetilde{Q}}[g^-]<\infty$ and Monotone Convergence Theorem we have $E_{\widetilde{Q}}[g\wedge n]\uparrow E_{\widetilde{Q}}[g]=\infty$, hence, there exists $\overline{n}\in\mathbb{N}$ such that $\overline{n}\geq E_{\widetilde{Q}}[g\wedge\overline{n}]>l$. Note now that

$$\sup_{Q\in\mathcal{M}_f} E_Q[g\wedge\overline{n}]\leq \sup_{Q\in\mathcal{M}_f} E_Q[g]=l<E_{\widetilde{Q}}[g\wedge\overline{n}] \tag{71}$$

Applying (70) to $g\wedge\overline{n}$ we get $E_{\widetilde{Q}}[g\wedge\overline{n}]=+\infty$, which is a contradiction since the contingent claim $g\wedge\overline{n}$ is bounded.

3. Example: forget about superhedging everywhere!

Let $(\Omega,\mathcal{F})=(\mathbb{R}^+,\mathcal{B}(\mathbb{R}^+))$. Consider a one period market ($T=1$) defined by a non-risky asset $S^0_t\equiv 1$ for $t=0,1$ (interest rate is zero) and a single risky asset $S^1_T(\omega)=\omega$ with initial price $S^1_0:=s_0>0$. In this market we also have two options $\Phi=(\phi^0,\phi^1)$, where $\phi^0:=f^0(S_T)$ is a butterfly spread option and $\phi^1:=f^1(S_T)$ is a power option, i.e.

$$f^0(x):=(x-K_0)^+-2(x-(K_0+1))^+ +(x-(K_0+2))^+$$
$$f^1(x):=(x^2-K_1)^+.$$

Assume $K_0>s_0$, $K_1>(K_0+2)^2$ and that these options are traded at prices $c_0=0$ and $c_1>0$ respectively. Set $c=(c_0,c_1)$. The payoffs of these financial instruments are shown in Figure 1 for $K_0=2$, $K_1=25$:

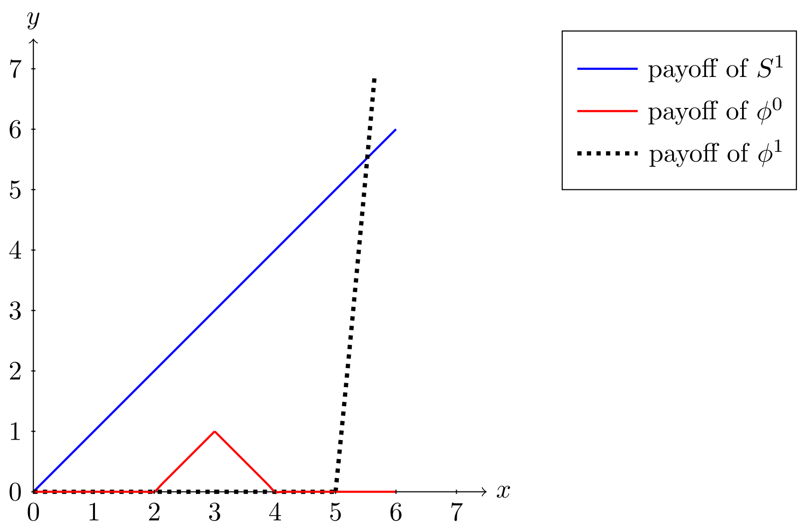


FIGURE 1. Payoffs.

DEFINITION 2.10. *(1) There exists a* model independent arbitrage *(in the sense of Acciaio et al.* [**AB13**]*) if* $\exists(H,h)\in\mathcal{H}\times\mathbb{R}^k$ *such that* $(H\cdot S)_T(\omega)+h(\Phi(\omega)-c)>0\ \forall\omega\in\Omega$.
(2) There exists a one point arbitrage *(in the sense of* [**BFM14**]*) if* $\exists(H,h)\in\mathcal{H}\times\mathbb{R}^k$ *such that* $(H\cdot S)_T(\omega)+h(\Phi(\omega)-c)\geq 0\ \forall\omega\in\Omega$ *and* $(H\cdot S)_T(\omega)+h(\Phi(\omega)-c)>0$ *for some* $\omega\in\Omega$.

It is clear that any long position in the option ϕ^0 is a one point arbitrage but it is not a model independent arbitrage. We have indeed that there are No Model Independent Arbitrage as:

$$\mathcal{M}_\Phi\neq\varnothing.$$

More precisely, any $Q \in \mathcal{M}_\Phi$ must satisfy $Q\left((K_0, K_0+2)\right) = 0$, so that (K_0, K_0+2) is an $\mathcal{M}_\Phi$-polar set, nevertheless,

$$\Omega_\Phi = \mathbb{R}^+ \setminus (K_0, K_0+2).$$

One possible way to see this is to observe that on $\Gamma := \mathbb{R}^+ \setminus (K_0, K_0+2)$ the option ϕ^0 has zero payoff and zero initial cost so that any probability P, with $\mathrm{supp}(P) \subseteq \Gamma$, that is a martingale measure for S^1, ϕ^1, is also a martingale measure for S^0, S^1, ϕ^0, ϕ^1. Take now $\omega_1 = 0$, $\omega_2 \in (K_0+2, \sqrt{K_1})$, $\omega_3 > \sqrt{K_1 + c_1}$ and observe that the corresponding points $x_1 := (-s_0, -c_1)$, $x_2 := (\omega_2 - s_0, -c_1)$ and $x_3 := (\omega_3 - s_0, \phi^1(\omega_3) - c_1))$ clearly belong to $conv(\Delta X(\omega) \mid \omega \in \Gamma)$ where ΔX is the random vector $[S_1^1 - s_0; \phi^1 - c_1]$. Consider now $\varepsilon := \frac{1}{2}\min\{c_1, s_0, |\omega_2 - s_0|\}$ so that for ω_3 sufficiently large we have

$$B_\varepsilon(0) \subseteq conv(\Delta X(\omega) \mid \omega \in \{\omega_1, \omega_2, \omega_3\}) \subseteq conv(\Delta X(\omega) \mid \omega \in \Gamma).$$

We have therefore that 0 is in the interior of $conv(\Delta X(\omega) \mid \omega \in \Gamma)$ and from Corollary 4.11 item 1) in [**BFM14**], $\Omega_\Phi = \Gamma = \mathbb{R}^+ \setminus (K_0, K_0+2)$. Note, moreover, that this is true for any value of the price $c_1 > 0$.

Consider now the digital options $g_i = F_i(S_T)$, $i = 1, 2$, with

$$\begin{aligned} F_1(x) &= \mathbf{1}_{(K_0, K_0+2)}(x), \\ F_2(x) &= \mathbf{1}_{[K_0, K_0+2]}(x) \end{aligned}$$

which differ only at the extreme points of the interval (K_0, K_0+2) and observe that F_2 is upper semi-continuous while F_1 is not. From the previous remark g_1 has price zero under any martingale measure $Q \in \mathcal{M}_\Phi$, so that

$$\sup_{Q \in \mathcal{M}_\Phi} E_Q[g_1] = 0. \tag{72}$$

Define:

$$\pi_\Omega(g) := \inf\left\{x \in \mathbb{R} \mid \exists (H, h) \in \mathcal{H} \times \mathbb{R}^k \text{ such that } x + (H \cdot S)_T(\omega) + h\Phi(\omega) \geq g(\omega)\ \forall \omega \in \Omega\right\}$$

and recall that

$$\pi_\Phi(g) := \inf\left\{x \in \mathbb{R} \mid \exists (H, h) \in \mathcal{H} \times \mathbb{R}^k \text{ such that } x + (H \cdot S)_T(\omega) + h\Phi(\omega) \geq g(\omega)\ \forall \omega \in \Omega_\Phi\right\}$$

Claim 2.11. *In this market:*

(1) $\pi_\Phi(g_1) = \sup_{Q \in \mathcal{M}_\Phi} E_Q[g_1] = 0$ *and* $\pi_\Phi(g_2) = \sup_{Q \in \mathcal{M}_\Phi} E_Q[g_2]$;
(2) $\pi_\Omega(g_1) = \min\left\{\frac{s_0}{K_0}, 1\right\} > \sup_{Q \in \mathcal{M}_\Phi} E_Q[g_1] = 0$;
(3) $\pi_\Omega(g_2) = \sup_{Q \in \mathcal{M}_\Phi} E_Q[g_2]$.

Remark 2.12. *(i) Item (1) is in agreement with the conclusion of Theorem 2.2.*
(ii) Item (2) shows instead that the superhedging duality with respect to the whole Ω *does not hold for the claim* g_1 *(which is even bounded). Note that in this example all the hypothesis of Theorem 1.4 in* [**AB13**] *are satisfied except for the upper semi-continuity of* g_1.

As the comparison between g_1 and g_2 in items (2) and (3) shows, the assumption of upper semi-continuity of the claim seems artificial from the financial point of view, even though necessary for the validity of Theorem 1.4 in [**AB13**].

Our results demonstrates that it is possible to obtain a superhedging duality on the relevant set Ω_Φ (or Ω_* when there are no options) for *any measurable claim*, regardless of the continuity assumptions (as well as without the existence of an option with super-linear payoff).

PROOF OF THE CLAIM 2.11. Item (1) holds thanks to Theorem 2.2. Notice also that the equalities $\pi_\Phi(g_1) = 0 = \sup_{Q\in\mathcal{M}_\Phi} E_Q[g_1]$ are consequences of (72) and the fact that $(H,h) = (0,0)$ is a superhedging strategy for g_1 on Ω_Φ. As g_2 is upper semi-continuous, the superhedging duality in item (3) holds thanks to Theorem 1.4 in [**AB13**], see (58). In the remaining of this section we conclude the proof by showing $\pi_\Omega(g_1) = \min\left\{\frac{s_0}{K_0}, 1\right\} = \frac{s_0}{K_0}$ (by the assumption $K_0 > s_0$) and hence item (2).

Let us consider the model independent superhedging strategies i.e. the set of $(H,h) \in \mathbb{R}^2 \times \mathbb{R}^2$ such that $x + (H\cdot S)_T(\omega) + h\Phi(\omega) \geq g_1(\omega)$ for any $\omega \in \Omega$. Any admissible trading strategy is given by $(H,h) := [H^0, H^1, h^0, h^1] \in \mathbb{R}^4$ which correspond to positions in the securities $[S^0, S^1, \phi^0, \phi^1]$ so that

$$\begin{array}{rll} \text{price:} & V_0(H,h) := & H^0 + H^1 s_0 + h^1 c_1 \\ \text{payoff:} & V_T(H,h) := & H^0 + H^1\omega + h^0\phi^0(\omega) + h^1\phi^1(\omega) \end{array} \tag{73}$$

Trivial super-hedges There are two immediate strategies whose terminal payoff is a super-hedge for g_1.

(1) S^0 (i.e. $H^0 = 1$ in (73) and $H^1 = h^0 = h^1 = 0$) with initial cost 1.
(2) $\frac{1}{K_0}S^1$ (i.e. $H^1 = \frac{1}{K_0}$ in (73) and $H^0 = h^0 = h^1 = 0$) with initial cost $\frac{s_0}{K_0}$.

Consider now a generic superhedging strategy (H,h) for the option g_1 and suppose first that $H^1 \geq 0$.

Observe that for every $\omega \in [0, K_0]$ we have: $V_T(H,h)(\omega) = H^0 + H^1\omega$ and $g_1(\omega) = 0$. If $H^0 < 0$ there exists $\widetilde{\omega} \in [0, K_0]$ such that $H^0 + H^1\widetilde{\omega} < 0 = g_1(\widetilde{\omega})$ so that the strategy does not dominate the payoff of g_1. Necessarily $H^0 \geq 0$.

$h^1 \neq 0$ **is not optimal for super-hedging** g_1**:** If $h^1 \neq 0$ we necessarily have $h^1 \geq 0$, otherwise $V_T(H,h)(\omega) < 0$ for ω large enough (because of the super-linearity of f^1) and (H,h) is not a super-hedge for g_1. Since $f^1(x) = 0$ on (K_0, K_0+2) and $c_1 > 0$, the most convenient super-hedge is with $h^1 = 0$ (cfr Figure 2).

: From now on with no loss of generality $h^1 = 0$.

$h^0 \neq 0$ **is not optimal for super-hedging** g_1**:** Since ϕ^0 has a positive payoff, if $h^0 \neq 0$ we might take $h^0 \geq 0$ otherwise we have a better super-hedge (at the same cost) by replacing $h^0\phi^0$ with the zero portfolio. Suppose now $h^0 > 0$. By recalling that $H^0, H^1 \geq 0$ we note that $V_T(H,h)$ as in (73) satisfies

$$\inf_{\omega\in(K_0,K_0+2)} H^0 + H^1\omega + h^0\phi^0(\omega) = H^0 + H^1K_0$$

so that the same super-hedge is achieved by trading only in S^0 and S^1. In other words with no loss of generality $h^0 = 0$ (cfr Figure 3)

We finally discuss the case $H^1 < 0$.

This is, in general, a more expensive choice for the strategy (H,h). Indeed we have, for instance, that for $\widetilde{\omega} = K_0 + 1$, $H^1S^1(\widetilde{\omega}) = H^1(K_0+1) < 0$ while $g_1(\widetilde{\omega}) = 1$. Since for any strategy $(H,h) \in \mathbb{R}^4$, $V_T(H,h)(\widetilde{\omega}) = H^0 + H^1\widetilde{\omega}$ we need $H^0 \geq 1 - H^1(K_0+1)$, hence, the initial price

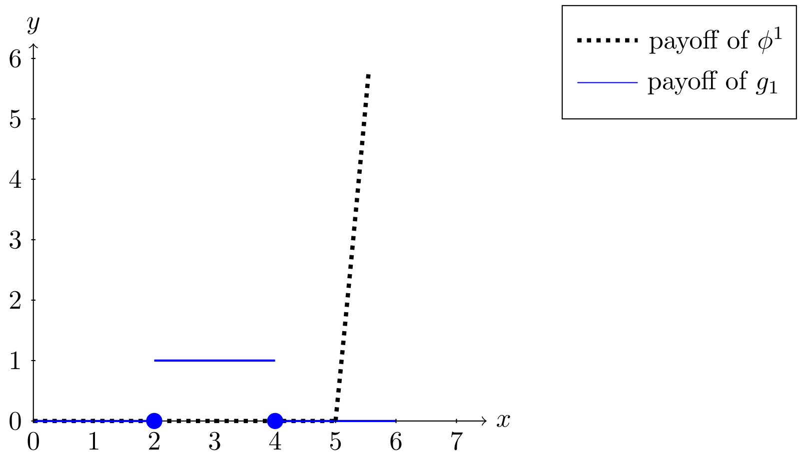


FIGURE 2. ϕ^1 has no positive wealth on $(K_0, K_0 + 2)$.

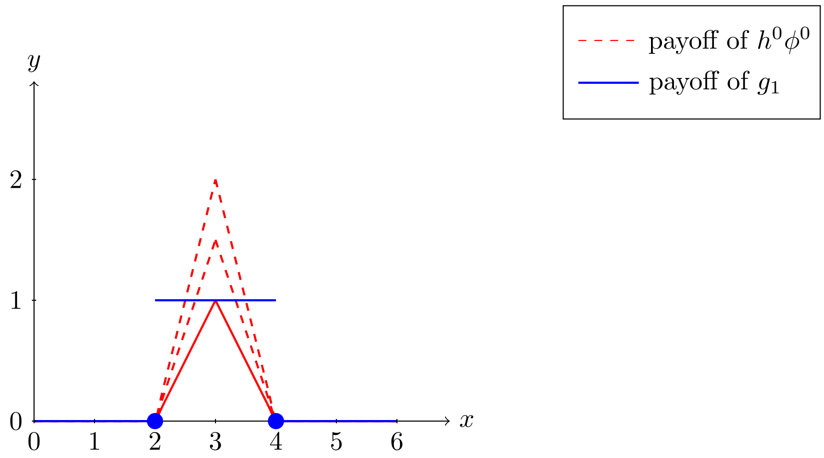


FIGURE 3. $h^0\phi^0$ does not dominate g_1 on $(K_0, K_0 + \varepsilon)$ for any h^0 with $\varepsilon = \varepsilon(h^0)$

$V_0(H, h) \geq 1 - H^1(K_0 + 1 - s_0)$. By choosing the parameters s_0, K_0 such that $K_0 + 1 - s_0 < 0$ any superhedging strategy with $H^1 < 0$ is more expensive than the trivial super-hedge given by $H^0 = 1, H^1 = h^0 = h^0 = 0$. Note moreover that in order to cover the losses in H^1S^1 for large value of ω we would need to take a long position in the option ϕ^1 (whose payoff dominates S^1) for an additional cost of $h^1c_1 > 0$ with $h^1 > -H^1 > 0$.

We can conclude that the cheapest super-replicating strategies are, in general, given by $H^0S^0 + H^1S^1$ with $H^0, H^1 \geq 0$ and it is easy to see that

$$\pi_\Omega(g_1) = \min\left\{\frac{s_0}{K_0}, 1\right\} = \frac{s_0}{K_0} > 0.$$

$\square$

4. Technical results and proofs

Recall that $\{\mathcal{F}_t\}_{t\in I}$ is the universal filtration which satisfies in particular that $\mathcal{F}_t$ contains the family of analytic sets of $(\Omega,\mathcal{F}_t^S)$ for any $t\in I$.
We indicate by $Mat(d\times(T+1);\mathbb{R})$ the space of $d\times(T+1)$ matrices with real entries representing the set of all the possible trajectories of the price process: for every $\omega\in\Omega$ we have $(S_0(\omega),S_1(\omega),...,S_T(\omega))\in Mat(d\times(T+1);\mathbb{R})$. Fix $t\le T$: we indicate $S_{0:t}=(S_0,S_1,...,S_t)$ and recall that $S_{0:t}^{-1}(A)=\{\omega\in\Omega\mid S_{0:t}(\omega)\in A\}$ for $A\subset Mat(d\times(t+1);\mathbb{R})$. We set $\Delta S_t:=S_t-S_{t-1}$, $t=1,...,T$.

4.1. Ω_* and Ω_Φ are analytic sets.

Lemma 2.13. *The set $\mathcal{P}_f=\{P\in\mathcal{P}\mid P$ has finite support$\}$ is an analytic subset of $\mathcal{P}$ endowed with the sigma-algebra generated by the $\sigma(\mathcal{P},C_b)$ topology.*

Proof. Set $E=\{\delta_\omega\mid\omega\in\Omega\}$ which is $\sigma(\mathcal{P},C_b)$ closed (Th. 15.8 [**AB06**]) and observe that $\mathcal{P}_f$ is the convex hull of E. Consider for any $n\in\mathbb{N}$ the simplex $\Delta_n\subset\mathbb{R}^n$ and the map

$$\gamma_n:E^n\times\Delta_n\longrightarrow\mathcal{P}_f$$

defined by $\gamma_n(\delta_{\omega_1},\dots,\delta_{\omega_n},\lambda_1,\dots,\lambda_n)=\sum_{i=1}^n\lambda_i\delta_{\omega_i}$ which is a continuous function in the product topology. Since $E^n\times\Delta_n$ is closed in the product topology of the Borel Space $\mathcal{P}^n\times\mathbb{R}^n$, then the image $\gamma_n(E^n\times\Delta_n)$ is analytic (Proposition 7.40 [**BS78**]). Finally we notice that $\mathcal{P}_f=\bigcup_n\gamma_n(E^n\times\Delta_n)$ which is therefore analytic, being countable union of analytic sets. □

Definition 2.14. *Let $\mathcal{L}^\infty(\Omega,\mathcal{F}):=\{f\in\mathcal{L}(\Omega,\mathcal{F})\mid f$ is bounded$\}$. A subset $\mathcal{U}\subset\mathcal{P}_f$ is countably determined if there exists a countable set $L\subseteq\mathcal{L}^\infty(\Omega,\mathcal{F})$ such that*

$$\mathcal{U}:=\{\mu\in\mathcal{P}_f\mid E_\mu[f]\le 0,\forall f\in L\}$$

Lemma 2.15. *If $\mathcal{U}\subseteq\mathcal{P}_f$ is countably determined then it is analytic.*

Proof. For each $f_n\in L$ define

$$F_n:\mathcal{P}\to\mathbb{R}\text{ such that }F_n(\mu)=\int_\Omega f_n d\mu.$$

From Theorem 15.13 in [**AB06**], F_n is Borel measurable so that

$$\mathcal{U}:=\{\mu\in\mathcal{P}_f\mid E_\mu[f_n]\le 0\text{ for all }n\in\mathbb{N}\}=\bigcap_{n\in\mathbb{N}}(F_n)^{-1}(-\infty,0]\cap\mathcal{P}_f$$

is analytic, being countable intersection of analytic sets. □

Lemma 2.16. *Let $Z_1(\omega):=\max_{i=1,\dots,d}\max_{u=0,\dots,T}|S_u^i(\omega)|$, $Z_2(\omega):=\max_{j=1,\dots,k}|\phi^j(\omega)|$ and $Z=\max(Z_1,Z_2)$ then*

$$\begin{aligned}\mathcal{P}_Z&=\left\{\mu\in\mathcal{P}_f\mid\exists Q\in\mathcal{M}_f\text{ such that }\frac{dQ}{d\mu}=\frac{c(\mu)}{1+Z}\right\}\\ \mathcal{P}_{Z,\Phi}&=\left\{\mu\in\mathcal{P}_f\mid\exists Q\in\mathcal{M}_\Phi\text{ such that }\frac{dQ}{d\mu}=\frac{c(\mu)}{1+Z}\right\}\end{aligned}$$

are analytic subsets of $\mathcal{P}$ where $c(\mu)=E_\mu\left[(1+Z)^{-1}\right]^{-1}$.

Proof. Assume $\mathcal{P}_Z \neq \varnothing$ (resp. $\mathcal{P}_{Z,\Phi} \neq \varnothing$) otherwise there is nothing to prove. Fix any $t \in \{1,...,T\}$. Let $Mat(d\times t;\mathbb{Q})$ be the countable set of $d\times t$ matrices with rational entries and denote its elements by q_n, $n\in\mathbb{N}$. For $q_n \in Mat(d\times t;\mathbb{Q})$, consider the set $\{A_n\}$ with $A_n = \{\omega\in\Omega \mid S_{0:t-1}\le q_n\}\in\mathcal{F}_{t-1}$. With a slight abuse of notation $S_{0:t-1}\le q_n$ stands for $S^i_u \le [q_n]_{i,u}$ for every $i=1,\dots,d$ and $u=0,1,\dots,t-1$. Define

$$\begin{aligned} f^i_n := &\quad \left(\frac{S^i_t - S^i_{t-1}}{1+Z}\right)\mathbf{1}_{A_n}\in\mathcal{L}^\infty(\Omega,\mathcal{F}),\\ g^j := &\quad \left(\frac{\phi^j}{1+Z}\right)\in\mathcal{L}^\infty(\Omega,\mathcal{F}). \end{aligned} \tag{74}$$

The following sets

$$\begin{aligned} \mathcal{U} := &\quad \left\{\mu\in\mathcal{P}_f \mid E_\mu[f^i_n]=0\ \forall i,n\right\}\\ \mathcal{U}_\Phi := &\quad \left\{\mu\in\mathcal{P}_f \mid E_\mu[f^i_n]=0 \text{ and } E_\mu[g^j]=0\ \forall i,n,j\right\} \end{aligned}$$

are analytic since they are countably determined. We now show that $\mathcal{U}=\mathcal{P}_Z$ and $\mathcal{U}_\Phi=\mathcal{P}_{Z,\Phi}$ and this will complete the proof.

For any fixed $\mu\in\mathcal{U}$ we have by construction:

$$\int_\Omega \frac{S^i_t}{1+Z}\mathbf{1}_{A_n}d\mu = \int_\Omega \frac{S^i_{t-1}}{1+Z}\mathbf{1}_{A_n}d\mu \quad \text{for every } A_n. \tag{75}$$

Consider the finite set of matrices $\{s_j\}_{j=1}^m := \{S_{0:t-1}(\omega)\in Mat(d\times t;\mathbb{R}) \mid \omega\in supp(\mu)\}$ where $m=m(\mu)$ depends on μ. Since μ has finite support, for any $j=1,\dots,m$ we may find A_{n_i} with $i=0,\dots,d\times t$ such that

$$\mu(B_j)=\mu\left(A_{n_0}\setminus \cup_{i=1}^{d(t-1)}A_{n_i}\right)$$

where $B_j := \{S_{0:t-1}=s_j\}$. We conclude

$$\int_\Omega \frac{S^i_t}{1+Z}\mathbf{1}_{B_j}d\mu = \int_\Omega \frac{S^i_{t-1}}{1+Z}\mathbf{1}_{B_j}d\mu \quad \text{for every } j=1,\dots,m$$

and $E_\mu\left(\frac{S^i_t}{1+Z}\mid\mathcal{F}_{t-1}\right)=E_\mu\left(\frac{S^i_{t-1}}{1+Z}\mid\mathcal{F}_{t-1}\right)$. Define Q by $\frac{dQ}{d\mu}:=\frac{c}{1+Z}$ where $c:=c(\mu)>0$ is the normalization constant. Then , $Q\sim\mu$, $Q\in\mathcal{P}_f$ and:

$$E_\mu\left(\frac{S^i_t}{1+Z}\mid\mathcal{F}_{t-1}\right)=E_\mu\left(\frac{S^i_{t-1}}{1+Z}\mid\mathcal{F}_{t-1}\right) \text{ if and only if } E_Q\left(S^i_t\mid\mathcal{F}_{t-1}\right)=S^i_{t-1}. \tag{76}$$

Thus we can conclude $Q\in\mathcal{M}_f$ and $\mathcal{U}\subseteq\mathcal{P}_Z$. Take now $\mu\in\mathcal{P}_Z$ then there exists Q such that $E_Q\left(S^i_t\mid\mathcal{F}_{t-1}\right)=S^i_{t-1}$ and $\frac{dQ}{d\mu}=\frac{c}{1+Z}$. From Equation (76) we have that condition (75) holds and hence $\mu\in\mathcal{U}$.

Recall that $\mathcal{M}_\Phi$ is defined in (55) and consider now $\mu\in\mathcal{U}_\Phi\subseteq\mathcal{U}$. Then there exists $Q\in\mathcal{M}_f$ such that $\frac{dQ}{d\mu}=\frac{c(\mu)}{1+Z}$. Moreover $E_\mu[g^j]=0$ for every $j=1,\dots,k$ so that, by (74), $E_Q[\phi^j]=0$. In this way $\mathcal{U}_\Phi\subseteq\mathcal{P}_{Z,\Phi}$. Take now $\mu\in\mathcal{P}_{Z,\Phi}$ then $\mu\in\mathcal{P}_Z$ from the previous part of the proof. Moreover there exists $Q\in\mathcal{M}_\Phi$ such that $E_Q\left(\Phi^j\right)=0$ and $\frac{dQ}{d\mu}=\frac{c}{1+Z}$. Again by (74) we have $E_\mu[g^j]=0$ for every $j=1,\dots,k$ and hence $\mu\in\mathcal{U}_\Phi$. □

Proposition 2.17. *Ω_* and Ω_Φ are analytic subsets of $(\Omega,\mathcal{F})$.*

Proof. Consider the Baire space $\mathbb{N}^{\mathbb{N}}$ of all sequences of natural numbers. In this proof we denote by $B_\varepsilon(\omega)$ the closed ball of radius ε, centered in ω in (Ω, d).
Consider a dense subset $\{\omega_i\}_{i=1}^\infty$ of Ω. For any $\mathbf{n} = (n_1, ..., n_k, ...) \in \mathbb{N}^{\mathbb{N}}$ we denote by $\mathbf{n}(1), \ldots, \mathbf{n}(k)$ the first k terms (i.e. $n_1, ..., n_k$). Define

$$A_{\mathbf{n}(1)} := B_1(\omega_{\mathbf{n}(1)}).$$

Let now $\{\omega_{\mathbf{n}(1),i}\}_{i=1}^\infty$ a dense subset of $A_{\mathbf{n}(1)}$ we define

$$A_{\mathbf{n}(1),\mathbf{n}(2)} := B_{\frac{1}{2}}(\omega_{\mathbf{n}(1),\mathbf{n}(2)}) \cap A_{\mathbf{n}(1)}.$$

At the k^{th} step we shall have $\{\omega_{\mathbf{n}(1),\ldots,\mathbf{n}(k-1),i}\}_{i=1}^\infty$ a dense subset of $A_{\mathbf{n}(1),\ldots,\mathbf{n}(k-1)}$ and we define the closed set

$$A_{\mathbf{n}(1),\ldots,\mathbf{n}(k)} := B_{\frac{1}{k}}(\omega_{\mathbf{n}(1),\ldots,\mathbf{n}(k)}) \cap A_{\mathbf{n}(1),\ldots,\mathbf{n}(k-1)}.$$

Notice that for any $\omega \in \Omega$ there will exists an $\mathbf{n} \in \mathbb{N}^{\mathbb{N}}$ such that

$$\bigcap_{k\in\mathbb{N}} A_{\mathbf{n}(1),\ldots,\mathbf{n}(k)} = \{\omega\}. \tag{77}$$

We consider the *nucleus* of the Souslin scheme given by

$$\bigcup_{\mathbf{n}\in\mathbb{N}^{\mathbb{N}}} \bigcap_{k\in\mathbb{N}} A_{\mathbf{n}(1),\ldots,\mathbf{n}(k)} \times \{Q \in \mathcal{P}_Z \mid Q(A_{\mathbf{n}(1),\ldots,\mathbf{n}(k)}) > 0\}. \tag{78}$$

Observe that $A_{\mathbf{n}(1),\ldots,\mathbf{n}(k)}$ closed in Ω implies $\{Q \in \mathcal{P} \mid Q(A_{\mathbf{n}(1),\ldots,\mathbf{n}(k)}) \geq \frac{1}{m}\}$ is $\sigma(\mathcal{P}, C_b)$-closed from Corollary 15.6 in [**AB06**]. Therefore

$$\{Q \in \mathcal{P} \mid Q(A_{\mathbf{n}(1),\ldots,\mathbf{n}(k)}) > 0\} = \bigcup_m \{Q \in \mathcal{P} \mid Q(A_{\mathbf{n}(1),\ldots,\mathbf{n}(k)}) \geq \frac{1}{m}\}$$

is Borel measurable in $(\mathcal{P}, \sigma(\mathcal{P}, C_b))$. By Lemma 2.16 we have that $\{Q \in \mathcal{P}_Z \mid Q(A_{\mathbf{n}(1),\ldots,\mathbf{n}(k)}) > 0\}$ is analytic. We can conclude that $A_{\mathbf{n}(1),\ldots,\mathbf{n}(k)} \times \{Q \in \mathcal{P}_Z \mid Q(A_{\mathbf{n}(1),\ldots,\mathbf{n}(k)}) > 0\}$ is an analytic subset of $\Omega \times \mathcal{P}$ (which is a Polish space).
From Lemma 2.16 we observe that any $\mu \in \mathcal{P}_Z$ admits an equivalent martingale measure with finite support. From $\Omega_* = \{\omega \in \Omega \mid \exists Q \in \mathcal{M}_f \text{ s.t. } Q(\omega) > 0\}$, if $\omega \notin \Omega_*$ then $\omega \notin supp(\mu)$ for any $\mu \in \mathcal{P}_Z$. Taking (77) into account, if $\omega \notin \Omega_*$ we can find a large enough $\bar{k}$ such that $A_{\mathbf{n}(1),\ldots,\mathbf{n}(\bar{k})} \cap supp(\mu) = \varnothing$. We then have

$$\bigcap_{k\in\mathbb{N}} A_{\mathbf{n}(1),\ldots,\mathbf{n}(k)} \times \{Q \in \mathcal{P}_Z \mid Q(A_{\mathbf{n}(1),\ldots,\mathbf{n}(k)}) > 0\} = \begin{cases} \{\omega\} \times \mathcal{P}_\omega & \text{if } \omega \in \Omega_* \\ \varnothing & \text{if } \omega \notin \Omega_* \end{cases},$$

where $\mathcal{P}_\omega = \{Q \in \mathcal{P}_Z \mid Q(\{\omega\}) > 0\}$.
From Proposition 7.35 and Proposition 7.41 in [**BS78**] any kernel of a Souslin scheme of analytic sets is again an analytic set. Then

$$\bigcup_{\mathbf{n}\in\mathbb{N}^{\mathbb{N}}} \bigcap_{k\in\mathbb{N}} A_{\mathbf{n}(1),\ldots,\mathbf{n}(k)} \times \{Q \in \mathcal{P}_Z \mid Q(A_{\mathbf{n}(1),\ldots,\mathbf{n}(k)}) > 0\} = \Omega_* \times \bigcup_{\omega\in\Omega_*} \mathcal{P}_\omega$$

is analytic in $\Omega \times \mathcal{P}$. Since the projection $\Pi : \Omega \times \mathcal{P} \to \Omega$ is continuous we finally deduce that Ω_* is analytic.
For Ω_Φ repeat the same proof replacing $\mathcal{P}_Z$ with $\mathcal{P}_{Z,\Phi}$. $\square$

Remark 2.18. *In one-period markets ($T = 1$), Ω_* is a Borel measurable set. To see this observe that if there are no one point arbitrages then $\Omega_* = \Omega \in \mathcal{B}(\Omega)$ by Proposition 4.11 in* [**BFM14**]. *When this condition is violated, there exists a strategy $H^1 \in \mathbb{R}^d$ such that $H^1 \cdot (S_1 - S_0) \geq 0$ and $B^1 := \{\omega \in \Omega \mid H^1 \cdot (S_1(\omega) - S_0) > 0\}$ is non-empty and Borel measurable. Indeed $B^1 = (f \circ S_1)^{-1}(0, \infty)$ with $f(x) := H^1 \cdot (x - S_0)$ continuous and S_1 Borel measurable. Observe now that, restricted to the set $\Omega \setminus B^1$, one asset is redundant (say S^d) so that the market can be described by $(S^0, \dots, S^{d-1})$. If there is no one point arbitrage we have $\Omega_* = \Omega \setminus B^1 \in \mathcal{B}(\Omega)$. Otherwise we can iteratively repeat the same argument to construct $B^i := \{\omega \in \Omega \setminus \cup_{j=1}^{i-1} B^j \mid H^i \cdot (S_1(\omega) - S_0) > 0\} \in \mathcal{B}(\Omega)$ and dropping iteratively one additional asset. Since the number of assets is finite the procedure takes $\beta \leq d$ steps. On the resulting set there are no one point arbitrages so that $\Omega_* = (\cup_{i=1}^{\beta} B^j)^C \in \mathcal{B}(\Omega)$.*

4.2. On the key Proposition 2.3.

Remark 2.19. *We point out at this stage that Ω_* is not only analytic but also it belongs to $\mathcal{F}_T$ where $\mathcal{F}_T$ is the universal completion of $\sigma(S_t \mid t \leq T)$. Indeed $\Omega_* \subseteq S_{0:T}^{-1}(S_{0:T}(\Omega_*))$. Moreover for any $\omega_1 \in S_{0:T}^{-1}(S_{0:T}(\Omega_*))$ there exists $\omega_2 \in \Omega_*$ such that $S_{0:T}(\omega_1) = S_{0:T}(\omega_2)$. Therefore for any $Q \in \mathcal{M}_f$ such that $Q(\{\omega_2\}) > 0$ and $Q(\{\omega_1\}) = 0$, the measure $\tilde{Q}$ such that $\tilde{Q}(\{\omega_1\}) := Q(\{\omega_2\})$, $\tilde{Q}(\{\omega_2\}) := 0$ and $\tilde{Q} = Q$ elsewhere is a martingale measure. Necessarily $\omega_1 \in \Omega_*$.*

In the proof of Proposition 2.3 we will make use of the following simple fact: set $\Omega_*^T := \Omega_* \in \mathcal{F}_T$ then by backward recursion we have

$$\Omega_*^t := S_{0:t}^{-1}(S_{0:t}(\Omega_*^{t+1})) \in \mathcal{F}_t, \quad \Omega_*^{t+1} \subseteq \Omega_*^t \text{ for any } t = 0, \dots, T-1, \text{ and } \quad \Omega_* = \bigcap_{t=1}^{T} \Omega_*^t.$$

Notice that Ω_*^t can be interpreted as the $\mathcal{F}_t$-measurable projection of Ω_* since $\Omega_*^t = S_{0:t}^{-1}(S_{0:t}(\Omega_*))$.

We also recall that the condition No one point arbitrage holds true on Ω_*. If indeed there exists $H \in \mathcal{H}$ such that $(H \cdot S)_T \geq 0$ with $(H \cdot S)_T(\omega) > 0$ for some $\omega \in \Omega_*$, then any measure P such that $P(\omega) > 0$ cannot be a martingale measure, which contradicts (52).

4.2.1. *Proof of Proposition 2.3.* We show, in several steps, that $\pi_*(g) = \sup_{Q \in \mathcal{M}_f} \pi_Q(g)$ where π_* and π_Q are defined in (61) and (62) and $g \in \mathcal{L}(\Omega, \mathcal{F})$.

Step 1: The first step is to construct, for any $1 \leq t \leq T$, an $\mathcal{F}_{t-1}$-measurable random set $R_{t,X,D} \subseteq \mathbb{R}^{d+1}$ whose interpretation is the following: if ω occurs, any $H^1, \dots H^d, H^{d+1} \in R_{t,X,D}(\omega)$ represents a strategy at time $t-1$ that allows to super-hedge the random variable X at time t, for any trajectory in $D \subseteq \Omega$. Here H^{d+1} represents the investment in the non-risky asset. Note that we need to incorporate the additional feature given by the choice of the set D since we want to super-hedge the random variable g only on $\Omega_* \subseteq \Omega$.

Recall $\Delta S_t = S_t - S_{t-1}$. Consider, for an arbitrary $1 \leq t \leq T$, $D \in \mathcal{F}_t$ and $X \in \mathcal{L}(\Omega, \mathcal{F})$ the multifunction

$$\psi_{t,X,D} : \omega \mapsto \left\{ [\Delta S_t(\widetilde{\omega}); 1; X(\widetilde{\omega})] \, \mathbf{1}_D \mid \widetilde{\omega} \in \Sigma_{t-1}^{\omega} \right\} \subseteq \mathbb{R}^{d+2}$$

where $[\Delta S_t; 1; X] \, \mathbf{1}_D = \left[\Delta S_t^1 \mathbf{1}_D, \dots, \Delta S_t^d \mathbf{1}_D, \mathbf{1}_D, X \mathbf{1}_D \right]$ and Σ_{t-1}^{ω} is the level set of the trajectory ω up to time $t-1$ i.e. $\Sigma_{t-1}^{\omega} = \{\widetilde{\omega} \in \Omega \mid S_{0:t-1}(\widetilde{\omega}) = S_{0:t-1}(\omega)\}$. We show that $\psi_{t,X,D}$ is an

$\mathcal{F}_{t-1}$-measurable multifunction. For any open set $O \subseteq \mathbb{R}^d \times \mathbb{R}^2$

$$\{\omega \in \Omega \mid \psi_{t,X,D}(\omega) \cap O \neq \varnothing\} = S_{0:t-1}^{-1}\left(S_{0:t-1}(B)\right) \text{ where } B = ([\Delta S_t; 1; X]\, \mathbf{1}_D)^{-1}(O).$$

First $[\Delta S_t, 1, X]\, \mathbf{1}_D$ is an $\mathcal{F}$-measurable random vector then $B \in \mathcal{F}$. Second S_u is a Borel measurable function for any $0 \leq u \leq t-1$ so that we have, as a consequence of Theorem III.18 in [**DM82**], that $S_{0:t-1}(B)$ belongs to the sigma-algebra generated by the analytic sets in $Mat(d \times t; \mathbb{R})$ endowed with its Borel sigma-algebra. Applying now Theorem III.11 in [**DM82**] we deduce that $S_{0:t-1}^{-1}(S_{0:t-1}(B)) \in \mathcal{F}_{t-1}$ and hence the desired measurability for $\psi_{t,X,D}$.
By preservation of measurability (see [**RW98**] for instance) the multifunction

$$\psi^*_{t,X,D}(\omega) := \left\{H \in \mathbb{R}^{d+2} \mid H \cdot y \leq 0 \quad \forall y \in \psi_{t,X,D}(\omega)\right\}$$

is also $\mathcal{F}_{t-1}$-measurable and thus, the same holds true for $-\psi^*_{t,X,D} \cap \mathbb{R}^{d+1} \times \{-1\}$. The projection on the first $d+1$ components, $R_{t,X,D} := \Pi_{x_1,\dots,x_{d+1}}(-\psi^*_{t,X,D} \cap \mathbb{R}^{d+1} \times \{-1\})$, provides the building blocks for the super-replicating strategy for g. By the previous construction we have indeed that

$$R_{t,X,D}(\omega) = \left\{H \in \mathbb{R}^{d+1} \mid H^{d+1}\mathbf{1}_D + \sum_{i=1}^{d} H^i \Delta S_t^i(\widetilde{\omega})\mathbf{1}_D \geq X(\widetilde{\omega})\mathbf{1}_D \quad \forall \widetilde{\omega} \in \Sigma_{t-1}^{\omega}\right\} \tag{79}$$

Notice that if $D \cap \Sigma_{t-1}^{\omega} = \varnothing$ then $R_{t,X,D}(\omega) = \mathbb{R}^{d+1}$. Note also that $R_{t,X,D}$ is, by construction, a closed set.
Denote by $\Pi_{x_{d+1}}(R_{t,X,D})$ the projection on the $(d+1)$-th component, which is a random interval in $\mathbb{R}$ with possible values $\{\varnothing\}, \{\mathbb{R}\}$. Observe now that the projection is continuous and that the infimum of a real-valued random set A preserve the measurability since

$$\{\omega \in \Omega \mid \inf\{a \mid a \in A(\omega)\} < y\} = \{\omega \in \Omega \mid A(\omega) \cap (-\infty, y) \neq \varnothing\}$$

Conclude, therefore, that $X_{t-1} := \inf \Pi_{x_{d+1}}(R_{t,X,D})$ is an $\mathcal{F}_{t-1}$-measurable function with values in $\mathbb{R} \cup \{\pm\infty\}$.
Step 2. We prove that for every $\omega \in \{|X_{t-1}| < \infty\}$ the infimum in X_{t-1} is actually a minimum. To this aim fix $\omega \in \{|X_{t-1}| < \infty\}$ and notice that there might exist $L \in \mathbb{R}^d \setminus \{0\}$ such that $L \cdot \Delta S_t = 0$ on $\Sigma_{t-1}^{\omega} \cap \Omega_*^t$, meaning that some assets are redundant on this level set. We can reduce the number of assets by selecting $i_1, \dots, i_k \in (1, ..., d)$ such that $l_1 \Delta S_t^{i_1} + \ldots + l_k \Delta S_t^{i_k} = 0$ implies $l_j = 0$ for every $j = 1, \dots, k$. Consider the closed set

$$\widetilde{R}(\omega) = \left\{H \in R_{t,X,D}(\omega) \mid H^{i_j} = 0 \text{ for every } j = 1, \dots, k\right\}$$

and observe that

$$\begin{aligned} X_{t-1}(\omega) &= \inf \Pi_{x_{d+1}}\left(R_{t,X,D}(\omega)\right) = \inf \Pi_{x_{d+1}}(\widetilde{R}(\omega)) \\ &= \inf \Pi_{x_{d+1}}\left(\widetilde{R}(\omega) \cap \left\{\mathbb{R}^d \times [X_{t-1}(\omega), X_{t-1}(\omega)+1]\right\}\right). \end{aligned}$$

The set $Ko(\omega) := \widetilde{R}(\omega) \cap \left\{\mathbb{R}^d \times [X_{t-1}(\omega), X_{t-1}(\omega)+1]\right\}$ is closed being the intersection of closed sets. We claim that $Ko(\omega)$ is bounded. By contradiction, suppose it is unbounded. Let $\hat{H}_n = (H_n, H_n^{d+1}) \in Ko(\omega) \subset \mathbb{R}^d \times \mathbb{R}$, such that $\|H_n\| \to +\infty$. By definition $H_n^{i_j} = 0$ for every $j = 1, \dots, k$ and H_n^{d+1} is bounded by $X_{t-1}(\omega) + 1$. For any $\widetilde{\omega} \in D \cap \Sigma_{t-1}^{\omega}$ and any n we have

$$\frac{X_{t-1}(\omega)+1}{\|H_n\|} + \frac{H_n}{\|H_n\|} \cdot \Delta S_t(\widetilde{\omega}) \geq \frac{X_t(\widetilde{\omega})}{\|H_n\|}.$$

Since $\frac{H_n}{\|H_n\|}$ lies on the unit sphere of $\mathbb{R}^d$, we can extract a subsequence converging to H^* with $\|H^*\| = 1$. Therefore passing to the limit over this subsequence we have $H^* \cdot \Delta S_t(\widetilde{\omega}) \geq 0$ for every $\widetilde{\omega} \in D \cap \Sigma^{\omega}_{t-1}$. From No one point arbitrage condition we deduce $H^* \cdot \Delta S_t = 0$ on $D \cap \Sigma^{\omega}_{t-1}$. Since $H_n \in Ko(\omega)$ then $(H^*)^{i_j} = 0$ on the redundant assets and thus $H^* = 0$ which is a contradiction. The set $Ko(\omega)$ is closed and bounded in $\mathbb{R}^{d+1}$, hence compact. From the continuity of the projection $\Pi_{x_{d+1}}(Ko(\omega))$ is compact, so that the infimum is attained.

Step 3: We now provide a backward procedure which yields the super-replication price and the corresponding optimal strategy. By classical arguments, when we fix a reference probability $Q \in \mathcal{M}_f$ this procedure yields two processes $X_t(Q)$ and $H_t(Q)$ such that

$$g \leq \sum_{u=t+1}^{T} H_u(Q) \cdot \Delta S_u + X_t(Q) = \sum_{t=1}^{T} H_t(Q) \cdot \Delta S_t + X_0(Q) \quad Q-\text{a.s.} \tag{80}$$

where $X_t(Q)$ represents the minimum amount of cash that we need at time t in order to super-hedge g in the Q-a.s. sense. Recall that from $NA(Q)$ we necessarily have $X_t(Q) > -\infty$ on $\text{supp}(Q)$. With no loss of generality set $X_t(Q)(\omega) = -\infty$ for any $\omega \notin \text{supp}(Q)$. Now we prove the pathwise counterpart of (80):

Set $X_T := g$ and $D_T := \Omega_*$ which belongs to $\mathcal{F}_T$ by Remark 2.19 and consider first the random set R_{T,X_T,D_T}. The random variable $X_{T-1} := \inf \Pi_{x_{d+1}}(R_{T,X_T,D_T})$ represents the minimum amount of cash that we need at time $T-1$ in order to super-hedge g on Ω_*. X_{T-1} is therefore the $\mathcal{F}_{T-1}$-measurable random variable that needs to be super-replicated at time $T-2$.
For $t = T-1, \dots, 0$ we indeed iterate the procedure by taking $X_t := \inf \Pi_{x_{d+1}}(R_{t+1,X_{t+1},D_{t+1}})$, $D_t = S^{-1}_{0:t}(S_{0:t}(D_{t+1})) \in \mathcal{F}_t$ and the random set $R_{t+1,X_{t+1},D_{t+1}}$ as defined before. We again have that X_t is an $\mathcal{F}_t$-measurable function with values in $\mathbb{R} \cup \{\pm\infty\}$.

This backward procedure yields the super-hedging price X_0 on Ω_* but also provide the corresponding cheapest portfolio as follows: note first that for every $\omega \in \Omega_*$, $X_t(\omega) > -\infty$. If this is not the case there exists a sequence $(H_n, x_n)_{n\in\mathbb{N}} \in \mathbb{R}^d \times \mathbb{R}$ such that $x_n \downarrow -\infty$, $x_n + H_n \Delta S_{t+1}(\widetilde{\omega}) \geq X_{t+1}(\widetilde{\omega})$ for every $\widetilde{\omega} \in D_{t+1} \cap \Sigma^{\omega}_t$ and hence Q-a.s. for every $Q \in \mathcal{M}_f$ such that $Q(\Sigma^{\omega}_t) > 0$. This would lead to a contradiction with $X_t(Q) > -\infty$. From now on we therefore assume that $X_t(\omega) > -\infty$. In the case $X_t(\omega) < \infty$ for every $t = 0, \dots, T-1$, Step 2 provides that X_t is actually a minimum. The $\mathcal{F}_t$-measurable multifunction given by $\Pi_{x_1,\dots,x_d}(R_{t+1,X_{t+1},D_{t+1}} \cap \{\mathbb{R}^d \times X_t\})$ is therefore non-empty for every $t = 0, \dots, T-1$ and thus admits a measurable selector H_{t+1}. The strategy $H_1, \dots, H_T$ satisfy the inequalities

$$\begin{aligned} g &\leq H_T \cdot \Delta S_T + X_{T-1} && \text{on } D_T \\ X_{T-1} &\leq H_{T-1} \cdot \Delta S_{T-1} + X_{T-2} && \text{on } D_{T-1} \\ & \cdots \\ X_1 &\leq H_1 \cdot \Delta S_1 + X_0 && \text{on } D_1 \end{aligned}$$

and it represents a super-hedge on $\Omega_* = \bigcap_{t=1}^T D_t$ as

$$g \le H_T \cdot \Delta S_T + X_{T-1} \le \sum_{t=T-1}^{T} H_t \cdot \Delta S_t + X_{T-2} \le \ldots \le \sum_{t=1}^{T} H_t \cdot \Delta S_t + X_0 \tag{81}$$

holds true for any $\omega \in \Omega_*$. When instead $X_t(\omega) = \infty$ for some $\omega \in \Omega_*$ and for some $t \ge 0$ then by simply taking $X_u \equiv \infty$ and H_u arbitrary for every $u \le t$, the inequality (81) is trivially satisfied.

Step 4: In order to prove (63) we recursively show that $X_t(\omega) = \sup_{Q\in\mathcal{M}_f} X_t(Q)(\omega)$ for any $\omega \in \Omega_*$ which, in particular, implies $X_0 = \sup_{Q\in\mathcal{M}_f} X_0(Q)$. Obviously $X_t(\omega) \ge X_t(Q)(\omega)$ for any $\omega \in \Omega_*$ so that $X_t \ge \sup_{Q\in\mathcal{M}_f} X_t(Q)$. Thus, we need only to prove the reverse inequality.

For $t = T$ the claim is obvious: $X_T = g$. By backward recursion suppose now it holds true for any u with $t+1 \le u \le T$ i.e. $X_u(\omega) = \sup_{Q\in\mathcal{M}_f} X_u(Q)(\omega)$ for any $\omega \in \Omega_*$.
From the recursive hypothesis in order to find a super-replication strategy with the same price for any $Q \in \mathcal{M}_f$ we need to super-replicate X_{t+1}. We fix a level set Σ_t^ω and recall that X_t is $\mathcal{F}_t$-measurable, hence it is constant on Σ_t^ω. We first treat two trivial cases:

- If $X_{t+1}(\omega) = \infty$ for some $\omega \in \Omega_*$ then the claim is not super-replicable at a finite cost hence the thesis follows with $X_0 = \sup_{Q\in\mathcal{M}_f} X_0(Q) = \infty$.
- If $\Sigma_t^\omega \cap \Omega_*^{t+1} = \varnothing$ we have two consequences: Σ_t^ω is an $\mathcal{M}_f$-polar set, hence by assumption, $X_t(Q) = -\infty$ on Σ_t^ω, for any $Q \in \mathcal{M}_f$. Moreover, as explained after equation (79), $\Pi_{x_{d+1}}(R_{t+1,X_{t+1},D_{t+1}}) = \mathbb{R}$ so that $X_t(\omega) = -\infty$ and the desired equality follows.

From now on we therefore assume $X_{t+1} < \infty$ and $\Sigma_t^\omega \cap \Omega_*^{t+1} \neq \varnothing$. Define, for any $y \in \mathbb{R}$, the set

$$\Gamma_y := co\left(conv\left\{[\Delta S_{t+1}(\widetilde{\omega}); y - X_{t+1}(\widetilde{\omega})] \mid \widetilde{\omega} \in \Sigma_t^\omega \cap \Omega_*^{t+1}\right\}\right)$$

We claim that

$$0 \in int(\Gamma_y) \Longrightarrow X_t > y \tag{82}$$

Indeed from $0 \in int(\Gamma_y)$ there is no non zero $(H,h) \in \mathbb{R}^d \times \mathbb{R}$, such that either $h(y - X_{t+1}) + H \cdot \Delta S_{t+1} \ge 0$ or $h(y - X_{t+1}) + H \cdot \Delta S_{t+1} \le 0$ on $\Sigma_t^\omega \cap \Omega_*^{t+1}$. In particular there is no $H \in \mathbb{R}^d$ such that $y + H \cdot \Delta S_{t+1} \ge X_{t+1}$. Since, as in Step 3, X_t is actually a minimum with a corresponding optimal super-hedging strategy, (82) follows.

Premise: As in Step 1, we may suppose, without loss of generality, that if for some $H \in \mathbb{R}^d$, $H \cdot \Delta S_{t+1} = 0$ on $\Sigma_t^\omega \cap \Omega_*^{t+1}$ then $H = 0$. In fact if this is not the case we can reduce, with an analogous procedure, the number of assets needed for super-replication on the level set .

We now distinguish two cases.

Case 1: Suppose there exist $(H,h,\alpha) \in \mathbb{R}^{d+2}$ with $(H,h,\alpha) \neq 0$ such that $h(y - X_{t+1}) + H \cdot \Delta S_{t+1} = \alpha$ on $\Sigma_t^\omega \cap \Omega_*^{t+1}$. We claim that $h \neq 0$. Indeed, if $h = 0$ then $\alpha \neq 0$, since $H \cdot \Delta S_{t+1} = 0$ implies $(H,h,\alpha) = 0$. However, $\alpha \neq 0$ implies $H \cdot \Delta S_{t+1} = \alpha$ on $\Sigma_t^\omega \cap \Omega_*^{t+1}$ which would yield a trivial one point arbitrage on Ω_*, hence a contradiction.
Since $h \neq 0$ we have $y - \frac{\alpha}{h} + \frac{H}{h} \cdot \Delta S_{t+1} = X_{t+1}$ on $\Sigma_t^\omega \cap \Omega_*^{t+1}$: this means that X_{t+1} is replicable implementing the strategy $\bar{H} := \frac{H}{h}$ in the risky assets and $X_t = y - \frac{\alpha}{h}$ in the non-risky asset. If now for some $Q \in \mathcal{M}_f$ such that $Q(\Sigma_t^\omega) > 0$, we have the existence of $x \le X_t$ and $H_x \in \mathbb{R}^d$ such that $x + H_x \cdot \Delta S_{t+1} \ge X_{t+1}$ Q-a.s. then

$x - X_t + (H_x - \bar{H})\Delta S_{t+1} \geq 0$ Q-a.s. hence, since $NA(Q)$ holds true, $x \geq X_t$. Therefore $X_t = X_t(Q)$ on Σ_{t-1}^{ω}.

Case 2: If a triplet $(H, h, \alpha) \in \mathbb{R}^{d+2}$ such as in Case 1 does not exist then we define

$$\bar{y} = \sup\left\{y \in \mathbb{R} \mid \exists\, H \in \mathbb{R}^d : \; y + H \cdot \Delta S_{t+1} \leq X_{t+1} \text{ on } \Sigma_t^{\omega} \cap \Omega_*^{t+1}\right\}.$$

Obviously $\bar{y} < X_t$ otherwise we are back to Case 1. For every $0 < \varepsilon < X_t - \bar{y}$ and for every $H \in \mathbb{R}^d$ neither $X_t - \varepsilon + H\Delta S_{t+1} \geq X_{t+1}$ nor $X_t - \varepsilon + H\Delta S_{t+1} \leq X_{t+1}$ holds true on $\Sigma_t^{\omega} \cap \Omega_*^{t+1}$. Moreover if there exists $h \in \mathbb{R}$ such that $h(X_t - \varepsilon - X_{t+1}) + H\Delta S_{t+1} \geq 0$ (or $h(X_t - \varepsilon - X_{t+1}) + H\Delta S_{t+1} \leq 0$) on $\Sigma_t^{\omega} \cap \Omega_*^{t+1}$ necessarily h would be 0 (otherwise simply divide by h). In such a case $H\Delta S_{t+1} \geq 0$ (or $H\Delta S_{t+1} \leq 0$) on $\Sigma_t^{\omega} \cap \Omega_*^{t+1}$ and by absence of one point arbitrage we get $H\Delta S_{t+1} = 0$ and hence $H = 0$. For this reason neither $h(X_t - \varepsilon - X_{t+1}) + H\Delta S_{t+1} \geq 0$ nor $h(X_t - \varepsilon - X_{t+1}) + H\Delta S_{t+1} \leq 0$ for any $(H, h) \in \mathbb{R}^{d+1} \setminus \{0\}$ so that $0 \in int\Gamma_{X_t - \varepsilon}$.

Take $\{\omega_i\}_{i=1}^k \subset \Sigma_t^{\omega} \cap \Omega_*$ (with $k \leq d$) such that $\{[\Delta S_{t+1}(\omega_i); X_t - \varepsilon - X_{t+1}(\omega_i)] \mid i = 1, \ldots, k\}$ are linearly independent and generates the same linear space in $\mathbb{R}^{d+1}$ as $\Gamma_{X_t - \varepsilon}$. By Proposition 2.9, and the convexity of the set of martingale measures, there exists $Q \in \mathcal{M}_f$ such that $Q(\{\omega_i\}) > 0$ for any $i = 1, \ldots, k$. For such a Q we get

$$\Gamma_{X_t - \varepsilon} = co\,(conv\{[\Delta S_{t+1}(\widetilde{\omega}); X_t - \varepsilon - X_{t+1}(\widetilde{\omega})] \mid \widetilde{\omega} \in supp(Q) \cap \Sigma_t^{\omega}\})$$

so that, from $0 \in int\Gamma_{X_t - \varepsilon}$, there exists no $H(Q) \in \mathbb{R}^d$ such that $X_t - \varepsilon + H(Q) \cdot \Delta S_{t+1} \geq X_{t+1}$ Q-a.s. We can conclude that $X_t \geq \sup_{Q \in \mathcal{M}_f} X_t(Q) \geq X_t - \varepsilon$. Letting $\varepsilon \downarrow 0$ we get $\sup_{Q \in \mathcal{M}_f} X_t(Q) = X_t$ as desired.

Step 5: finally we prove (64). Notice that $\mathcal{C} \subseteq \bigcap_{Q \in \mathcal{M}_f} \mathcal{C}(Q)$. Moreover if $g \in \bigcap_{Q \in \mathcal{M}_f} \mathcal{C}(Q)$ then (80) holds with $X_0(Q) \leq 0$ for every $Q \in \mathcal{M}_f$. Therefore also in Equation (81) we have $X_0 = \sup_{Q \in \mathcal{M}_f} X_0(Q) \leq 0$ and $g \leq \sum_{t=1}^T H_t \cdot \Delta S_t$ on Ω_* i.e. $g \in \mathcal{C}$.

4.3. Proof of Theorem 2.2. Recall that π_Φ is defined in (57) and $\mathcal{M}_\Phi$ in (55). Set

$$\widetilde{\pi}_\Phi(g) := \inf\left\{x \in \mathbb{R} \mid \exists H \in \mathcal{H} \text{ such that } x + (H \cdot S)_T(\omega) \geq g(\omega)\ \forall \omega \in \Omega_\Phi\right\}.$$

LEMMA 2.20. *Let $g : \Omega \mapsto \mathbb{R}$ and $\phi^j : \Omega \mapsto \mathbb{R}$, $j = 1, ..., k$, be $\mathcal{F}$-measurable random variables. Then*

$$\pi_\Phi(g) = \inf_{h \in \mathbb{R}^k} \widetilde{\pi}_\Phi(g - h\Phi).$$

PROOF. For every $h \in \mathbb{R}^k$ we have $\pi_\Phi(g) \leq \widetilde{\pi}_\Phi(g - h\Phi)$ so that $\pi_\Phi(g) \leq \inf_{h \in \mathbb{R}^k} \widetilde{\pi}_\Phi(g - h\Phi)$. By contradiction assume $\pi_\Phi(g) < \inf_{h \in \mathbb{R}^k} \widetilde{\pi}_\Phi(g - h\Phi)$, then there exist $(\bar{x}, \bar{h}, \bar{H}) \in (\mathbb{R}, \mathbb{R}^k, \mathcal{H})$ such that

$$\bar{x} < \inf_{h \in \mathbb{R}^k} \widetilde{\pi}_\Phi(g - h\Phi) \quad \text{and}$$
$$\bar{x} + (\bar{H} \cdot S)_T(\omega) + \bar{h}\Phi(\omega) \geq g(\omega) \text{ for all } \omega \in \Omega_\Phi$$

Clearly we have a contradiction since

$$\bar{x} < \widetilde{\pi}_\Phi(g - \bar{h}\Phi) = \inf\left\{x \in \mathbb{R} \mid \exists H \in \mathcal{H} \text{ s. t. } x + (H \cdot S)_T(\omega) \geq g(\omega) - \bar{h}\Phi(\omega)\ \forall \omega \in \Omega_\Phi\right\} \leq \bar{x}.$$

□

Proof of Theorem 2.2. Since also Ω_Φ is analytic (Proposition 2.17), by comparing the definition of Ω_Φ in (56) with (68), we may repeat step by step the same arguments used in the proof of Theorem 2.1 and Proposition 2.3 replacing Ω_* with Ω_Φ. We then conclude that $\widetilde{\pi}_\Phi(g) = \sup_{Q\in\mathcal{M}_\Phi} E_Q[g]$ for any $\mathcal{F}$-measurable random variable g. Since $E_Q[h\Phi] = 0$ for all $Q \in \mathcal{M}_\Phi$ and $h \in \mathbb{R}^k$, for the $\mathcal{F}$-measurable random variable $g - h\Phi$ we have

$$\widetilde{\pi}_\Phi(g - h\Phi) = \sup_{Q\in\mathcal{M}_\Phi} E_Q[g - h\Phi] = \sup_{Q\in\mathcal{M}_\Phi} E_Q[g],\ \forall h \in \mathbb{R}^k.$$

The Lemma 2.20 then implies: $\pi_\Phi(g) = \inf_{h\in\mathbb{R}^k} \widetilde{\pi}_\Phi(g - h\Phi) = \sup_{Q\in\mathcal{M}_\Phi} E_Q[g]$. $\square$

CHAPTER 3

Models with proportional transaction costs[1]

Arbitrage and Consistent Price Systems. We consider here a model independent version of the Robust No Arbitrage condition introduced in [**S04**]. Whenever this condition holds true the broker still have room for proposing a discount on the bid-ask spread without creating with this operation arbitrage opportunities. In this sense the terminology "robustness" of the No Arbitrage condition should be interpreted rather than the probability-free setup. Differently from the approach of [**S04**] we are not defining arbitrage in terms of physical units of assets, while we are choosing a numeràire and we are evaluating a sure gain in terms of the value process of a certain strategy. Nevertheless we show the same equivalence under the name of FTAP:

There are No Robust Model Independent Arbitrage iff there exists a CPS (83)

A related paper in this direction is the recent work of Bayraktar and Zhang (see [**BZ13**]). In this paper the authors replaced the single reference probability with a (possibly non-dominated) set of priors $\mathcal{P}$ and considered the case of a multi-period market with a single risky asset. By using a strong continuity assumption and the tools of Quasi-Sure Analysis they were able to show the analogous equivalence (83). We point out that even by choosing the extreme class of priors $\mathcal{P}$ as the set of all possible probabilities $\mathfrak{P}$ the Model Independent case is not covered and hence the desired equivalence is not automatically achieved. In particular, in this Chapter we study a multi-period, multi-asset model and we show that when no reference probabilities are fixed, we do not need any continuity assumption in order to show (83). To this aim we will make use of the general theory of random sets and measurable selection which have already been considered by Rokhlin in [**Ro08**] for the probabilistic case. Nevertheless in [**Ro08**] the author provided an equivalent condition to the existence of CPSs based on random sets. This condition turns out to be also equivalent to No Robust Arbitrage due to the equivalence (83) which was already known from [**S04**]. Since in this Chapter we do not have (83) while, on the contrary, it is exactly what we want to show, the extension to the model-free setup of some results of [**Ro08**] is only partially useful.

Super-hedging Theorem. The second part of the Chapter is devoted to the proof of the Super-hedging Theorem in the presence of proportional transaction costs. Denote $\mathcal{Q}$ the class of probability measure $Q \in \mathfrak{P}$ such that there exists a price process S with values in the bid-ask spread for which the couple (Q, S) is a consistent price systems. Recall that from (83) this class is non-empty if No Model Independent Arbitrage holds true. Denote also by $\mathcal{S}$, the family of processes $\widetilde{S}$ for which $\mathcal{Q}_{\widetilde{S}} \neq \varnothing$. For a given claim g, in Section 3, we formally prove the following

[1]Chapter 3 is based on the working paper: Arbitrage and Hedging in Model Independent Markets with frictions.

equality

$$\sup_{Q\in\mathcal{Q}} \mathbb{E}_Q[g] = \inf\{x \in \mathbb{R} \mid \exists H \in \mathcal{H} \text{ s.t. } x + V_T(H) \geq g \quad \forall \omega \in \Omega_*\} =: \overline{p}(g) \tag{84}$$

where $\mathcal{Q}$ is the set of probability measures Q for which there exist a process $\widetilde{S} \in \mathcal{S}$. The set $\Omega_* \subseteq \Omega$ for which we require the superhedging inequality is given by

$$\Omega_* := \{\omega \in \Omega \mid \exists Q \in \mathcal{Q} \text{ such that } Q(\{\omega\}) > 0\}$$

and we denominate it the support of the consistent price system CPS (See Definition (3.14)). As a consequence of (83) this equality is meaningful when the condition No Robust Model Independent Arbitrage holds true but nevertheless, by assuming the convention that the superhedging inequality is always satisfied when $\Omega_* = \varnothing$, then (84) is true in general.
The idea of the proof is very simple. By simply writing explicitly the value process $V_T(H)$ of a certain strategy H we obtain that

$$V_T(H) = \sum_{t=0}^{T}\sum_{j=1}^{d}\left(H_t^j - H_{t+1}^j\right)\left(\overline{S}_t^j \mathbf{1}_{\{H_t^j \leq H_{t+1}^j\}} + \underline{S}_t^j \mathbf{1}_{\{H_{t+1}^j \leq H_t^j\}}\right)$$

where $\underline{S}_t^j$ and $\overline{S}_t^j$ are, respectively the cost of selling and buying a share of asset j at time t. We observe that for any H, the value process $V_T(H)$ is simply the (discrete time) stochastic integral of a certain process S^H laying at the boundaries of the bid-ask spread. Since any process $\widetilde{S} \in \mathcal{S}$ defines a frictionless market it is possible to compute the superhedging price for g that we denote $\overline{p}_{\widetilde{S}}$. From Chapter 2 the equality

$$\sup_{Q\in\mathcal{Q}_{\widetilde{S}}} \mathbb{E}_Q[g] = \overline{p}_{\widetilde{S}}(g)$$

holds true and when $\overline{p}_{\widetilde{S}}(g)$ is finite it is actually a minimum with a corresponding set of cheapest strategies $\mathcal{H}^{\widetilde{S}}$. If there exist now a strategy $H^{\widetilde{S}} \in \mathcal{H}^{\widetilde{S}}$ such that the stochastic integral $(H^{\widetilde{S}} \circ \widetilde{S})_T$ and the value process $V_T(H^{\widetilde{S}})$ require the same initial capital to superreplicate g we then have

$$\overline{p}_{\widetilde{S}}(g) + (H^{\widetilde{S}} \circ \widetilde{S})_T \geq g \qquad \text{and} \qquad \overline{p}_{\widetilde{S}}(g) + V_T(H^{\widetilde{S}}) \geq g$$

From which $\sup_{Q\in\mathcal{Q}} \mathbb{E}_Q[g] \geq \overline{p}(g)$ holds true which is the difficult part in showing (84).

In order to show the existence of such a process we construct an auxiliary set-valued superhedging problem (see Definition 3.16) by considering at a certain time t the whole set of random vectors which are convex combinations of random vectors at time $t+1$. Note that for such processes an obvious conditional martingale with finite support exists and it is given by the convex combination. We will show in Section 3 that by solving the set-valued superhedging problem we will obtain the desired $\widetilde{S} \in \mathcal{S}$.

1. Setting and notations

Fix $(\Omega, \mathcal{B}(\Omega))$ a measurable space, where Ω is Polish, and $\mathcal{F} := \mathcal{B}(\Omega)$ is the Borel sigma-algebra. Let $\mathfrak{P} = \mathfrak{P}(\Omega)$ be the set of probability measures on $(\Omega, \mathcal{F})$. We consider a discrete time interval $I = \{0, \dots, T\}$ on a finite time horizon $T \in \mathbb{N}$ and we introduce a $(d+1)$-dimensional stochastic process $(S_t)_{t\in I}$ which is Borel-measurable and which represents the discrete time evolution of the price process of $d+1$ assets where the first one serves as a numeràire. With no loss of generality we may therefore assume $S_t^0 \equiv 1$ for any $t \in I$. The setup of Kabanov et al. (for example [**KS01a, KRS02**]) can be defined also when a reference probability is absent. For any $t \in I$ a Borel-measurable stochastic matrix $\Lambda_t = [\lambda_t^{ij}]_{i,j=0,\dots,d}$ is given, where any λ_t^{ij} models the transaction cost for exchanging one unit of the asset i for the corresponding value in units of the asset j, at time t. Following the notation of Kabanov and Stricker [**KS01a**] and Schachermayer [**S04**], one can also define the matrix $\Pi_t = [\pi_t^{ij}]_{i,j=0,\dots,d}$ given by

$$\pi_t^{ij} := \frac{S^j}{S^i}(1 + \lambda_t^{ij})$$

where any π_t^{ij} represents the physical unit of asset i that an agent need to exchange, at time t, for having one unit of asset j. Clearly $\lambda_t^{ii} = 0$ and consequently $\pi_t^{ii} = 1$ for any $t \in I$. A standard assumption is that agents are smart enough to take advantage of favourable exchange between assets so that, for any $t \in I$, for any $\omega \in \Omega$, one may assume

$$\pi_t^{ij} \le \pi_t^{ik_1}\pi_t^{k_1k_2}\cdots\pi_t^{k_nj}$$

for any combination of asset $k_1, \dots, k_n$.

Differently from the frictionless case when an agent wants to implement a trading strategy she needs to consider the cost of rebalancing the portfolio after each trade date. The definition of self-financing strategies, goes as follows:

Definition 3.1. *Denote by e_i with $i = 0, \dots d$ the vector of the canonical base of $\mathbb{R}^{d+1}$ and define*

$$K_t := co\left(conv\left\{e_i, \pi_t^{ij}e_i - e_j \mid i,j = 0,\dots,d\right\}\right)$$

*the so-called **solvency cone**. Any portfolio in K_t can be indeed reduced to the 0 portfolio up to suitable exchanges of assets and up to "throwing away" some money if necessary. The **cone of portfolio available at cost 0** at time t, is simply given by $-K_t$ and $F_t := K_t \cap -K_t$ is the set of portfolio which are exchangeable with the zero portfolio.*
*A **self-financing** trading strategy $H := (H_t)_{1\le t\le T}$ is a predictable process with*

$$H_t - H_{t-1} \in -K_{t-1} \quad \textit{for any } t = 1,\dots,T$$

meaning that rebalancing the portfolio is obtained at zero cost.

In this paper the asset S^0 serves as a numeràire and the value of any portfolio is evaluated in terms of S^0. This amounts to the choice of $\pi_t^{ij} = \pi_t^{i0}\pi_t^{0j}$ in the above setting for any $t \in I$. We have therefore that the stochastic interval $[\frac{1}{\pi^{j0}}, \pi^{0j}]$ represents the bid-ask spread of the asset $j \in \{1,\dots,d\}$.

Notation 3.2. *In the following, the bid-ask spread $\left[\frac{1}{\pi_t^{j0}}, \pi_t^{0j}\right]$ will be shortly denoted as $[\underline{S}_t^j, \overline{S}_t^j]$ for $t = 0,\dots,T$ and $j = 1,\dots,d$.*

For any $t \in I$, for any $\omega \in \Omega$, define

$$C_t(\omega) := \left[\underline{S}_t^1(\omega), \overline{S}_t^1(\omega)\right] \times, \dots, \left[\underline{S}_t^d(\omega), \overline{S}_t^d(\omega)\right] \subseteq \mathbb{R}^d \tag{85}$$

Assumption 3.3. *We assume that $int(C_t) \neq \varnothing$, known as the* efficient friction *hypothesis, modelling non-trivial transaction costs, and we assume that, for every ω fixed, $C_t(\omega)$ is bounded.*

We finally set $\mathbb{F}^S := \{\mathcal{F}_t^S\}_{t\in I}$, where $\mathcal{F}_t^S := \sigma\{\underline{S}_u, \overline{S}_u \mid 0 \le u \le t\}$ denotes the natural filtration of the processes $\underline{S}$ and $\overline{S}$, and we consider the Universal Filtration $\mathbb{F} := \{\mathcal{F}_t\}_{t\in I}$, namely,

$$\mathcal{F}_t := \bigcap_{P\in\mathfrak{P}} \mathcal{F}_t^S \vee \mathcal{N}_t^P, \text{ where } \mathcal{N}_t^P = \{N \subseteq A \in \mathcal{F}_t^S \mid P(A) = 0\};$$

For any $0 \le t \le T$, we denote by $\mathcal{L}^0(\mathcal{F}_t; V)$ the set of $\mathcal{F}_t$-measurable functions with values in $V \subseteq \mathbb{R}^d$. For technical purposes we will also adopt the following notation:

NOTATION 3.4. *For a random set Ψ in $\mathbb{R}^d$ we denote by Ψ^* the (positive) dual of Ψ and for $\varepsilon > 0$ we introduce the ε-dual of Ψ as*

$$\begin{aligned} \Psi^*(\omega) &:= \{v \in \mathbb{R}^d \mid v \cdot x \ge 0 \quad \forall x \in \Psi(\omega)\} \\ \Psi^\varepsilon(\omega) &:= \left\{v \in \mathbb{R}^d \mid v \cdot x \ge \varepsilon \quad \forall x \in \Psi(\omega) \setminus \{0\}\right\} \end{aligned}$$

which they both preserve the same measurability as Ψ as discussed in the Appendix (see Lemma 3.25 and Proposition 3.27).

2. Arbitrage and Consistent Price Systems

In this Section we consider the class of strategies on $\{0, \dots, T+1\}$ of the form $(H_0, \dots, H_{T+1})$ where $H_0 = H_{T+1} = 0$ and $H_1, \dots, H_T$ is a self-financing strategy as in Definition 3.1. Denote by $\mathcal{H}$ the class of admissible strategies. Since $H_{T+1} = H_0 + \sum_{t=0}^T \xi_t$ with $\xi_t \in -K_t$ any admissible strategy has no initial endowment ($H_0 = 0$), it is implemented by subsequently rebalancing the portfolio at zero cost and at time T any open position must be closed ($H_{T+1} = 0$).
We consider the value process $V_t(H)$ of a certain admissible strategy $H \in \mathcal{H}$ as the position in the numeràire S^0 at time t after rebalancing. The terminal value is given by

$$V_T(H) = \sum_{t=0}^T \sum_{j=1}^d \left(H_t^j - H_{t+1}^j\right) \left(\overline{S}_t^j \mathbf{1}_{\{H_t^j \le H_{t+1}^j\}} + \underline{S}_t^j \mathbf{1}_{\{H_{t+1}^j \le H_t^j\}}\right) \tag{86}$$

One can easily verify the above formula. If, for instance, at time t the agent switch from a long position to a short one in asset j then she needs to liquidate H_t^j obtaining $H_t^j \underline{S}_t^j$ and then selling H_{t+1}^j shares of the asset at the same price, yielding $(H_t^j - H_{t+1}^j)\underline{S}_t^j$ which coincides with the second term in (86) since obviously $H_{t+1}^j \le H_t^j$. If instead she wants only to diminish the amount of shares in the long position, then $H_{t+1}^j \le H_t^j$ and she needs to liquidate the amount $H_t^j - H_{t+1}^j$ obtaining in return $(H_t^j - H_{t+1}^j)\underline{S}_t^j$. The remaining cases follow similarly.

Using a similar argument as in Schachermayer [**S04**] we may introduce, and motivate, the following definition of Arbitrage,

DEFINITION 3.5. *We say that a bid-ask process $\widetilde{\Pi}$ has smaller transaction costs than Π if and only if for any $\omega \in \Omega$, for any $t \in I$*

$$\left[\frac{1}{\widetilde{\pi}_t^{j0}}, \widetilde{\pi}_t^{0j}\right] \subsetneq \left[\frac{1}{\pi_t^{j0}}, \pi_t^{0j}\right] \qquad \textit{for any } j = 1, \dots, d$$

Observe that clearly $V_T(H)$ depends also on Π and, in particular, $V_T(H)(\widetilde{\Pi}) > V_T(H)(\Pi)$ if $\widetilde{\Pi}$ has smaller transaction costs than Π. We will omit this dependence when it is clear from the context.

DEFINITION 3.6. *Consider a market with bid-ask spread* Π. *We say that a trading strategy* $H \in \mathcal{H}$ *is an Enhanceable Model Independent Arbitrage if for any arbitrary small reduction of the transaction costs* $\widetilde{\Pi}$ *we have* $V_T(H)(\widetilde{\Pi}) > 0$ *for any* $\omega \in \Omega$.

This definition is the model-free version of the Robust Arbitrage condition introduced (in negation form NA^r) in [**S04**] but in order to avoid misleading terminology in the context of model uncertainty we decide to stick to the introduced definition. If the condition No Enhanceable Arbitrage holds true the broker still have room for proposing a discount on the transaction costs without creating arbitrage opportunity. On the contrary if this condition is not satisfied it is sufficient to have an infinitely small discount to get an arbitrage opportunity on a certain set of events. Since transaction costs are often subject of negotiation it looks quite natural to consider markets that exclude these possibilities.

Before stating our version of the Fundamental Theorem of Asset pricing we lastly need to formulate the definition of the so-called consistent price systems, in this model-free context.

DEFINITION 3.7. *We say that a couple* $(Q, \tilde{\mathbb{S}})$ *is a consistent price system on* $[0,T]$ *if* $\tilde{\mathbb{S}} := (\tilde{S}_t)_{t\in I}$ *is a* $(d+1)$*-dimensional,* $\mathbb{F}$*-adapted stochastic process with* $\tilde{S}^0_t \equiv 1$*, for any* $t \in I$ *and which is a martingale under the measure* $Q \in \mathfrak{P}(\Omega)$*. In addition* $\tilde{S}^j_t$ *takes values in the interior of the bid ask-spread defined by* Π *i.e.*

$$\tilde{S}^j_t \in \left(\frac{1}{\pi^{j0}_t} \ , \pi^{0j}_t \right)$$

for any $\omega \in \Omega$ *and for any* $j = 1, \ldots, d$.
Denote $\mathcal{M}_\Pi$ *the class of price systems consistent with* Π.

2.1. Model free FTAP. We are now ready to introduce one of our main results. The arguments used in [**KS01a**] and [**S04**] to show the probabilistic versions of this Theorem are based on properties of the dual cones K^*_t. The formulation of the problem illustrated in Section 2, allows for making use of a geometric approach similar in spirit as the one used in [**BFM14**] for showing the Fundamental Theorem in the frictionless case, which is essentially based on separating hyperplane theorems in finite dimensional spaces and measurable selection arguments. We will also use an iterative modification of the bid-ask spread in order to capture the arbitrage opportunities. This idea is similar in spirit as in [**BZ13**] but different in its implementation. In particular no additional hypothesis on S such as continuity is required. Differently from previous approaches we also stress that we do not solve first the problem for the one period case and then expanding to the multi-period case but we directly tackle the dynamic case. This appear to be very natural in the context of transaction cost since, arbitrage strategies might involve different times of execution. The simple example in the Introduction of [**BZ13**] clarify this intuition: consider a single asset with deterministic bid-ask spread $[1,3]$ at time 0 and $[2,4]$ $[3.5,5]$ at time 1 and 2 respectively. There is an arbitrage opportunity given by the strategy: buy at time 0 and selling at time 2.

THEOREM 3.8. *Let* $\mathcal{M}_\Pi$ *the set of consistent price systems as in Definition 3.7. We have* $\mathcal{M}_\Pi = \varnothing$ *iff there exists an Enhanceable Model Independent Arbitrage*

Before giving the proof we need some preliminary results.

For any $t \in I$, for any $\omega \in \Omega$, define iteratively, the following random sets

$$\begin{aligned} &\Theta_{T+1}(\omega) := \mathbb{R}^d \\ &\Theta_{t-1}(\omega) := C_{t-1}(\omega) \cap \overline{conv}\left(\Theta_t(\Sigma^\omega_{t-1})\right) \quad \text{for } t = T+1 \ldots, 1 \end{aligned} \tag{87}$$

where Σ^ω_{t-1} denotes the level set of ω i.e. $\Sigma^\omega_{t-1} = \{\widetilde{\omega} \in \Omega \mid S_{0:t-1}(\widetilde{\omega}) = S_{0:t-1}(\omega)\}$. Here $S_{0:t-1}(\omega)$ is a shorthand for the trajectory of the process S up to time $t-1$. Since $\mathbb{F}^S \subseteq \mathbb{F}$ we have $\Sigma^\omega_{t-1} \in \mathcal{F}_{t-1}$ for every $1 \le t \le T$.

The random sets Θ_t represents a backward modification of the bid-ask spread. The intuition behind this operation is the following. Consider first $t = T$ and observe that Θ_T is simply C_T. The random set Θ_{T-1} is given by the intersection of the bid-ask spread at time $T-1$ and the set of all convex combination of elements with values in the bid ask-spread at time T. Consider now a probability measure $P \in \mathfrak{P}$ with finite support and suppose $P(\Sigma^\omega_{T-1}) > 0$. We note that if P is a martingale measure for some $(X_{T-1}, X_T) \in C_{T-1} \times C_T$ then the conditional expectation X_{T-1} needs to be a convex combination of X_T. We are therefore excluding from C_{T-1} those values that cannot represents a conditional expectation of an $\hat{\mathcal{F}}_T$-measurable random vector with values in C_T for any probability measure with finite support. This will lead us to the proof of the Fundamental Theorem of Asset Pricing. We begin with the following

Lemma 3.9. *For any $t = 0, \ldots, T+1$ the random set Θ_t as in* (87) *is $\mathcal{F}_t$-measurable.*

Proof. For $t = T+1$ the claim is obvious. Suppose now that the claim holds for any $s \in \{t, \ldots, T+1\}$, we show that Θ_{t-1} is $\mathcal{F}_{t-1}$-measurable. Observe first that $C_{t-1}(\omega)$ is the closed convex hull of the multifunction $\omega \mapsto p_1(\omega) \times \cdots \times p_d(\omega)$ where $p_j = \frac{1}{\pi^{j0}_{t-1}} \cup \pi^{0j}_{t-1}$ for $j = 1 \ldots d$. All the p_j are $\mathcal{F}_{t-1}$-measurable random sets being union of two $\mathcal{F}_{t-1}$-measurable random sets (whose values are singletons), by preservation of measurability through the operations of finite cartesian product, convexity and closure we have that $C_{t-1}(\omega)$ is also $\mathcal{F}_{t-1}$-measurable (see Proposition 3.27).

We turn now to the set $\Theta_t(\Sigma^\omega_{t-1})$. Denote by dom $\Theta_t := \{\omega \mid \Theta_t(\omega) \neq \varnothing\}$ Since, by hypothesis, Θ_t is $\mathcal{F}_t$-measurable it admits a Castaing representation, i.e. there exists a collection $\{\varphi_n\}$ of $\mathcal{F}_t$-measurable function $\varphi_n : \text{dom } \Theta_t \to \mathbb{R}^d$ such that $\overline{\{\varphi_n(\omega) \mid n \in \mathbb{N}\}} = \Theta_t(\omega)$ for any $\omega \in \Omega$. Define therefore for $n \in \mathbb{N}$ the multifunctions $G_n : \omega \mapsto \{\varphi_n(\widetilde{\omega}) \mid \widetilde{\omega} \in \Sigma^\omega_{t-1}\}$ which are $\mathcal{F}_{t-1}$-measurable since

$$\forall O \subseteq \mathbb{R}^d \text{ open} \quad \{\omega \in \Omega \mid G_n(\omega) \cap O \neq \varnothing\} = S^{-1}_{0:t-1}\left(S_{0:t-1}\left(\varphi_n^{-1}(O)\right)\right)$$

belong to $\mathcal{F}_{t-1}$. Recall indeed that image and counterimage of Borel sets through Borel measurable functions are analytic and that the Universal Filtration contains the class of analytic sets of $\mathcal{F}_{t-1}$(See for example Theorem III.18 and Theorem III.11 in [**DM82**]). Observe now that $\overline{\Theta}_t(\Sigma^\omega_{t-1}) = \overline{\cup_{n\in\mathbb{N}} G_n}$. The inclusion $\supseteq$ is obvious, while taking $\overline{x} \in \overline{\Theta}_t(\Sigma^\omega_{t-1})$ and a sequence $x_k \to \overline{x}$ we note that $x_k \in \overline{\cup_{n\in\mathbb{N}} G_n}$ for every k, since this set contains the collection $\{\varphi_n(\omega) \mid n \in \mathbb{N}, \omega, \in \Sigma^\omega_{t-1}\}$ induced by the Castaing representation of Θ_t on the level set Σ^ω_{t-1}. It therefore follows that $\overline{x} \in \overline{\cup_{n\in\mathbb{N}} G_n}$. We conclude that

$$\Theta_{t-1}(\omega) := C_{t-1}(\omega) \cap \overline{conv}\left(\Theta_t(\Sigma^\omega_{t-1})\right) = C_{t-1}(\omega) \cap \overline{conv}\left(\cup_{n\in\mathbb{N}} G_n\right) \tag{88}$$

is $\mathcal{F}_{t-1}$-measurable since the random sets C_{t-1} and $\{G_n\}_{n\in\mathbb{N}}$ share the same measurability property and the transformations involved in (88) preserve measurability (see Proposition 3.27). □

COROLLARY 3.10. *The random sets $C_t(\omega)$, $\Theta_{t+1}(\Sigma_t^\omega)$ and $\overline{conv}\,(\Theta_{t+1}(\Sigma_t^\omega))$ are $\mathcal{F}_t$-measurable for any $t = 0, \dots, T$.*

PROOF. Measurability of C_t follows from the first part of the proof of Lemma 3.9, measurability of $\Theta_{t+1}(\Sigma_t^\omega)$, and therefore of $\overline{conv}\,(\Theta_{t+1}(\Sigma_t^\omega))$, follows from (88) and the discussion right before. □

REMARK 3.11. *Note that with no loss of generality we may assume that if $\Theta_t(\omega) \neq \varnothing$ then $int(\Theta_t(\omega)) \neq \varnothing$. For $t = T$ this is true since, by construction, $\Theta_T = C_T$ and $int(C_T) \neq \varnothing$ by Assumption 3.3. If this is true up to time $t+1$ then it is true for time t by considering, if needed, a bid-ask spread with smaller transaction costs $\widetilde{\Pi}_t$. Indeed, since C_t and $\overline{conv}\left(\Theta_t(\Sigma_{t-1}^\omega)\right)$ have non empty interior by hypothesis, if the intersection has empty interior it is sufficient to consider an arbitrary small reduction of the bid-ask spread process to obtain $\Theta_t = \varnothing$. Take for example $\widetilde{\pi}_t^{0j} := \pi_t^{0j} - \varepsilon(\omega)$ and $1/\widetilde{\pi}_t^{j0} := 1/\pi_t^{j0} + \varepsilon(\omega)$ where $\varepsilon(\omega) := \varepsilon\left(\pi_t^{0j}(\omega) - 1/\pi_t^{j0}(\omega)\right) > 0$ for an arbitrary small $\varepsilon > 0$.*

LEMMA 3.12. *Let Θ_t for $t = 0 \dots T$ as defined in (87) then*

$$\{\omega \in \Omega \mid \Theta_t(\omega) \neq \varnothing \quad \forall t = 0 \dots T\} \neq \varnothing \Longrightarrow \mathcal{M}_\Pi \neq \varnothing$$

PROOF. Our aim is to build up a consistent price system iteratively. By definition of Θ_t and from the hypothesis, for any $\overline{y} \in \Theta_t(\omega) \neq \varnothing$ there exist $\lambda_1, \dots, \lambda_m > 0$ with $\sum_{i=1}^m \lambda_i = 1$ and $y_1, \dots y_m \subseteq \Theta_{t+1}(\Sigma_t^\omega) \subseteq C_{t+1}(\Sigma_t^\omega)$ such that $\overline{y} = \sum_{i=1}^m \lambda_i y_i$.
Start therefore with an arbitrary $x_0 \in int(\Theta_0(\omega))$ which is non-empty from the hypothesis and from Remark 3.11. Associate to x_0 the real number $p(x_0) = 1$. Suppose a set of trajectories $Z_t := \{x_{0:t} \in Mat(d \times (t+1))\}$ has been chosen up to time t with associated $p(x_{0:t}) > 0$ summing up to one. Here $Z_0 = \{x_0\}$. By applying the above procedure to x_t where x_t is the value at time t of a trajectory $x_{0:t} \in Z_t$, we can construct a new set

$$Z_{t+1} := \{[x_{0:t}, y_1(x_{0:t})], \dots, [x_{0:t}, y_m(x_{0:t})] \mid x_{0:t} \in Z_t\}$$

with associated $p([x_{0:t}, y_i(x_{0:t})]) = \lambda_i p(x_{0:t})$.
Observe that given the set Z_T for any $x_{0:T} \in Z_T$ there exists $\omega \in \Omega$ such that $\times x_{0:T} \in C_0(\omega) \times \dots \times C_T(\omega)$. Moreover, defining $\widetilde{S}_t(\omega) := x_t$ and the probability measure $Q(\omega) := p(x_{0:T})$ we have that $\widetilde{S}_t$ is $\mathcal{F}_t$-measurable for any $t = 0, \dots T$ and

$$E_Q[\widetilde{S}_t \mid \mathcal{F}_t] = \widetilde{S}_{t-1}, \quad \text{for t=1,... T} \tag{89}$$

Thus, Q is a martingale measure for $\widetilde{S}$ which by construction lays in the bid-ask spread. □

We are now able to prove the Fundamental Theorem of Asset Pricing.

PROOF OF THEOREM 3.8. The "if" part is easy and we prove it by contraposition. Suppose $\mathcal{M}_\Pi \neq \varnothing$, hence there exist $\widetilde{S} = (\widetilde{S}_t)_{t\in I}$ and $Q \in \mathfrak{P}$ such that $\widetilde{S}_t \in int(C_t)$ and $\widetilde{S}_t$ is a Q-martingale. Let $H \in \mathcal{H}$ such that $V_T(H) \geq 0$. By adding and subtracting $\widetilde{S}^t_j$ in (86) we note that the term

$$\sum_{t=0}^{T}\sum_{j=1}^{d}\left(H^j_t - H^j_{t+1}\right)\left((\overline{S}^j_t - \widetilde{S}^j_t)\mathbf{1}_{\{H^j_t \leq H^j_{t+1}\}} + (\underline{S}^j_t - \widetilde{S}^j_t)\mathbf{1}_{\{H^j_{t+1}\leq H^j_t\}}\right)$$

is always non positive and hence we get

$$0 \leq V_T(H) \leq \sum_{t=0}^{T}(H_t - H_{t+1})\cdot \widetilde{S}_t = \sum_{t=1}^{T} H_t \cdot (\widetilde{S}_t - \widetilde{S}_{t-1})$$

by also recalling that $H_0 = H_{T+1} = 0$. By taking now expectation on both sides we get $V_T(H) = 0$ for some $\omega \in \Omega$ from which No (Enhanceable) Model Independent Arbitrage is possible.

We prove now the "only if" through several steps.

Step 1: Define first the random time

$$\tau(\omega) := \inf\{0 \leq t \leq T \mid \Theta_t = \varnothing \text{ and } \overline{conv}\,(\Theta_{t+1}(\Sigma^\omega_t)) \neq \varnothing\}$$

Observe that τ is a stopping time: for any $t \in I$ the set $\{\tau \leq t\}$ coincides with the set $\cup_{s=1}^t(\{\omega : \Theta_s(\omega) = \varnothing\} \cap \{\omega : \overline{conv}\,(\Theta_{s+1}(\Sigma^\omega_s)) \neq \varnothing\})$ which belongs to $\mathcal{F}_t$ from Lemma 3.9 and Corollary 3.10. Observe now that under the assumption $\mathcal{M}_\Pi = \varnothing$, as a consequence of Lemma 3.12, for any ω there exists $s = s(\omega)$ such that $\Theta_s(\omega) = \varnothing$.
Moreover τ is a finite stopping time since straightforwardly from definition (87) it follows $\Theta_T(\omega) = C_T(\omega) \neq \varnothing$ and hence $\overline{conv}\left(\Theta_T(\Sigma^\omega_{T-1})\right) \neq \varnothing$. We can therefore deduce that $\tau(\omega) \leq T-1$ for any $\omega \in \Omega$.

Step 2: For any $t \in \{0,\dots T\}$ let $H = \{H_s \mid s \leq t\}$ with $H_0 = 0$ and $H_s \in \mathcal{L}^0(\mathcal{F}_{s-1};\mathbb{R}^d)$ be given. For $\xi := sgnH_t$, we introduce the following process $\hat{S}^\xi_t$ which take values at the boundary of the bid-ask spread.[2]

$$\hat{S}^\xi_t := \left(\underline{S}^1_t\mathbf{1}_{\{H^1_t\geq 0\}} + \overline{S}^1_t\mathbf{1}_{\{H^1_t<0\}}, \dots, \underline{S}^d_t\mathbf{1}_{\{H^d_t\geq 0\}} + \overline{S}^d_t\mathbf{1}_{\{H^d_t<0\}}\right) \tag{90}$$

We introduce also the sets A_t and B_t as follows:

$$A_t := \{\tau = t\} \cap \bigcap_{s=0}^{t}\{H_s = 0\}, \qquad B_t := \{H_t \neq 0\} \cap \{\hat{S}^\xi_t \notin \Theta_t\}$$

For an interpretation of these sets see Remark 3.13.

We now show that A_t and B_t are $\mathcal{F}_t$-measurable. The measurability of A_t is obvious from τ being a stopping time and the measurability of H_s for $s \leq t$. Now, observe that $sgn(H_t)$ is $\mathcal{F}_{t-1}$-measurable since for any $x \in \Xi := \{x \in \mathbb{R}^d \mid x^i \in \{-1,0,1\}\}$, $sgn(H_t)^{-1}(x) = H_t^{-1}(x_1(0,\infty)\times,\dots\times x_d(0,\infty))$ where with a slight abuse of notation $x_i(0,\infty)$ is either $(0,\infty)$, $(-\infty,0)$ or $\{0\}$ according to x being respectively 1, -1 or 0.

[2]The choice for the event $\{H^i_t = 0\}$ can be actually arbitrary without affecting the value of the strategy, for the sake of simplicity it is included here in the positive case.

$\hat{S}_t^\xi$ is instead $\mathcal{F}_t$-measurable since for any Borel set of the form $O := O_1 \times \ldots \times O_d \subseteq \mathbb{R}^d$ with O_i open for $i = 1, \ldots d$, we have

$$(\hat{S}_t^\xi)^{-1}(O) = \bigcap_{i=1}^d \quad (\underline{S}^i)^{-1}(O_i) \cap (\xi)^{-1}[0,\infty) \quad \cup \\ (\overline{S}^i)^{-1}(O_i) \cap (\xi)^{-1}(-\infty,0)$$

The set $\{\hat{S}_t^\xi \in \Theta_t\}$ is $\mathcal{F}_t$-measurable since it is the projection on Ω of the intersection of $Graph(\hat{S}^\xi)$ and $Graph(\Theta_t)$. We easily conclude that B_t is $\mathcal{F}_t$-measurable.

Step 3: Consider the sets A_t as in step 2. We show that for any $t = 1, \ldots T$ and for any $\varepsilon > 0$, there exists an $\mathcal{F}_{t-1}$-random variable H_t^A such that $\forall \omega \in A_{t-1}$

$$H_t^A(\omega) \cdot (y - x) \geq \varepsilon \qquad \forall y \in \Theta_t(\Sigma_{t-1}^\omega),\ \forall x \in C_{t-1}(\omega) \tag{91}$$

For any $\omega \in A_{t-1}$ since $\Theta_{t-1} = \varnothing$, by (87) the random sets $C_{t-1}(\omega)$ and $\overline{conv}\left(\Theta_t(\Sigma_{t-1}^\omega)\right)$ are closed, convex and disjoint. In particular Hahn-Banach Theorem applies and for every $\omega \in A_{t-1}$ there exists $\varphi \in \mathbb{R}^d$ such that $\varphi \cdot x > \varepsilon > \varphi \cdot y$ for any $x \in \Theta_t(\Sigma_{t-1}^\omega)$, $y \in C_{t-1}(\omega)$. We have therefore that the random set $(\Theta_t(\Sigma_{t-1}^\omega) - C_{t-1}(\omega))^\varepsilon$ (see Notation 3.4) is non-empty on A_{t-1} and $\mathcal{F}_{t-1}$-measurable by Corollary 3.10 and Lemma 3.25. Take therefore H_t^A a measurable selector of this set.
Let us stress that the value ε can be arbitrary.

Step 4: We are now ready to construct iteratively the strategy that will realize an arbitrage opportunity and, in particular, with an arbitrary $\delta > 0$, it will satisfy the following

$$V_{t-1}(H) + H_t \cdot y \geq \frac{\delta}{2^{t-1}} \text{ for any } y \in \Theta_t(\Sigma_{t-1}^\omega) \text{ and for any } \omega \in A_{t-1} \cup B_{t-1}. \tag{92}$$

with $H_t = 0$ otherwise. For $t = 1$ Equation (92) is trivially satisfied by H_1^A as in (91) with $\varepsilon = \delta$ arbitrary: we have indeed that $B_0 = \varnothing$ and from (86), $V_0(H_1^A) + H_1^A \cdot y = H_1^A \cdot (y - \hat{x})$ for $\hat{x} := \overline{S}_t^j \mathbf{1}_{\{0 \leq H_1^j\}} + \underline{S}_t^j \mathbf{1}_{\{H_1^j \leq 0\}} \in C_0(\omega)$ Define therefore $H_1 := H_1^A$ and suppose we are given a strategy $H = (H_u)_{u=1}^t$ satisfying (92).

Recall $\xi := sgn H_t$. For any $\eta \in \Xi$ denote the partial order relation on $\mathbb{R}^d$ given by

$$h_1 \preceq_\eta h_2 \quad \text{ iff } \quad h_1 - h_2 \in \eta^1[0,\infty) \times \cdots \times \eta^d[0,\infty)$$

and consider now

$$f^\eta := \omega \mapsto \left\{ h \in \mathbb{R}^d \mid H_t(\omega) \preceq_\eta h \text{ and } V_t^h(H) + h \cdot y \geq \frac{\delta}{2^t} \quad \forall y \in \Theta_{t+1}(\Sigma_t^\omega) \right\}$$

where $V_t^h(H) := V_{t-1}(H) + (H_t - h) \cdot \hat{S}_t^\eta(\omega)$ is the value of the strategy $H = H_1, \ldots, H_t$ extended with $H_{t+1}(\omega) = h$ (cfr Equation (86)). We first show that for any $\eta \in \Xi$, f^η is $\mathcal{F}_t$-measurable. Then we show that for any $\omega \in A_t \cup B_t$, for at least one $\eta \in \Xi$, the set f^η is non-empty so that by choosing a measurable selector of $\cup_{\eta \in \Xi} f^\eta$ (which exists by Proposition 3.27 and Theorem 3.28) we get the desired inequality (92) for time t.

For the sake of measurability we consider the $(\delta/2^t)$-dual of the $\mathcal{F}_t$-measurable set $[\Theta_{t+1}(\Sigma_t^\omega) - \hat{S}_t^\eta(\omega); V_{t-1}(H) + H_t \cdot \hat{S}_t^\eta]$ (see Corollary 3.10 and recall Notation 3.4) i.e.

$$\left\{(h, h_{d+1}) \in \mathbb{R}^d \times \mathbb{R} \mid h \cdot (y - \hat{S}_t^\eta(\omega)) + h_{d+1}(V_{t-1}(H) + H_t \cdot \hat{S}_t^\eta) \geq \frac{\delta}{2^t} \quad \forall y \in \Theta_{t+1}(\Sigma_t^\omega)\right\}$$

and take the intersection with the $\mathcal{F}_t$-measurable random set

$$\eta_1(-\infty, H_t^1(\omega)]\times, \ldots \times \eta_d(-\infty, H_t^d(\omega)] \times \{1\}$$

f_2^η coincides with the projection on the first d components of the resulting set and it is thus $\mathcal{F}_t$-measurable (see again Proposition 3.27).

We now show that for any $\omega \in A_t \cup B_t$ fixed, the set $\cup_{\eta \in \Xi} f^\eta(\omega)$ is always non-empty. On A_t we consider again H_{t+1}^A as in (91) with $\varepsilon = \delta/2^t$ and the conclusion follows as above.
We now turn to $\omega \in B_t$. Observe first that if $\hat{S}_t^\xi(\omega) \in \Theta_t(\omega)$ then the position can be closed with a strictly positive gain. Indeed with $h = 0$ we get from (86) and the iterative hypothesis (92)

$$V_t^h(H) = V_{t-1}(H) + \sum_{j=1}^d \left(H_t^j - 0\right) \left(\overline{S}_t^j \mathbf{1}_{\{H_t^j \leq 0\}} + \underline{S}_t^j \mathbf{1}_{\{0 \leq H_t^j\}}\right) \geq \frac{\delta}{2^{t-1}} > \frac{\delta}{2^t}$$

If $\hat{S}_t^\xi(\omega) \notin \Theta_t(\omega)$. The position cannot be closed without a loss at time t. We show that nevertheless it is possible to rebalance the portfolio in order to maintain a positive wealth, namely we show that (92) holds also at time t.

Consider set of vertices of $C_t(\omega)$

$$V := \bigcup \left\{ [\underline{S}_t^j \mathbf{1}_{\{\eta^j \geq 0\}} + \overline{S}_t^j \mathbf{1}_{\{\eta^j < 0\}}]_{j=1}^d \mid \eta \in \{-1, 0, 1\}^d \right\}$$

and the set

$$L := \{y \in \mathbb{R}^d \mid V_{t-1}(H) + H_t \cdot y < 0\} \cap V$$

From the inductive hypothesis we have: i) $B_t \subseteq A_{t-1} \cup B_{t-1}$ since $H_t(\omega) \neq 0$ only on $A_{t-1} \cup B_{t-1}$ and ii) $\Theta_t(\omega) \cap L(\omega) = \varnothing$. Moreover, since $\hat{S}_t^\xi(\omega)$ as in (90) is a vertex and $\hat{S}_t^\xi(\omega) \notin \Theta_t(\omega)$, we thus have $\hat{S}_t^\xi(\omega) \in L(\omega)$. Consider now the set

$$F := \left\{h \in \mathbb{R}^d \mid h \cdot (y - l) \geq 0 \quad \forall y \in \Theta_{t+1}(\Sigma_t^\omega),\ \forall l \in L(\omega)\right\}$$

which is non-empty for $\omega \in B_t$: by definition $\Theta_t(\omega) := C_t(\omega) \cap \overline{conv}(\Theta_{t+1}(\Sigma_t^\omega))$ (see (87) above) from which the sets $\overline{conv}(L(\omega))$ and $\overline{conv}(\Theta_{t+1}(\Sigma_t^\omega))$ are disjoint and applying Hyperplane separating Theorem we obtain the assertion. Note, moreover, that since the separation is strict for any $h \neq 0$ there exists $\varepsilon > 0$ such that $h \cdot (y - l) \geq \varepsilon \quad \forall y \in \Theta_{t+1}(\Sigma_t^\omega),\ \forall l \in L(\omega)$.

For any $h \in \mathbb{R}^d$ define now

$$[\hat{S}_t^h]^j := \overline{S}_t^j \mathbf{1}_{\{H_t^j \leq h^j\}} + \underline{S}_t^j \mathbf{1}_{\{h^j \leq H_t^j\}} \tag{93}$$

where $[\cdot]^j$ denotes the j^{th} component of a vector. We can distinguish two cases:

(1) there exists $h \in F$ such that $\hat{S}_t^h \in L$.
(2) for all $h \in F$, $\hat{S}_t^h \in V \setminus L$.

In case (1) there exists $h \in F$ and $\varepsilon > 0$ such that $h \cdot (y - \hat{S}_t^h) \geq \varepsilon$ for all $y \in \Theta_{t+1}(\Sigma_t^\omega)$. Define now

$$\alpha_1 := \max\left\{\frac{1}{\varepsilon}\left(-V_{t-1}(H) - H_t \cdot \hat{S}_t^h + \frac{\delta}{2^{t-1}}\right), 1 + \frac{\delta}{2^{t-1}}\right\} \geq 1 + \frac{\delta}{2^{t-1}}, \qquad \bar{h} := \alpha_1 h \in F \quad (94)$$

and observe that

$$V_{t-1}(H) + H_t \cdot \hat{S}_t^h + \bar{h} \cdot (y - \hat{S}_t^h) \geq \frac{\delta}{2^{t-1}} \qquad \forall y \in \Theta_{t+1}(\Sigma_t^\omega) \quad (95)$$

In order to retrieve the value $V_t^{\bar{h}}(H)$ in (95) we need to replace $\hat{S}_t^h$ with $\hat{S}_t^{\bar{h}}$. By showing that $(H_t - \bar{h}) \cdot \hat{S}_t^{\bar{h}} \geq (H_t - \bar{h}) \cdot \hat{S}_t^h$, it will follow from (95) that

$$\begin{aligned} V_t^{\bar{h}}(H) + \bar{h} \cdot y = V_{t-1}(H) + (H_t - \bar{h}) \cdot \hat{S}_t^{\bar{h}} + \bar{h} \cdot y &\geq \\ V_{t-1}(H) + (H_t - \bar{h}) \cdot \hat{S}_t^h + \bar{h} \cdot y &\geq \frac{\delta}{2^{t-1}} \qquad \forall y \in \Theta_{t+1}(\Sigma_t^\omega) \end{aligned}$$

and hence the desired inequality. To show the claim let $j \in \{1, \ldots, d\}$. If $h^j H_t^j \leq 0$ or $|h^j| \geq |H_t^j|$ then from (93) and $\alpha_1 > 1$ we get $[\hat{S}_t^{\bar{h}}]^j = [\hat{S}_t^h]^j$. Suppose now $H_t^j \leq h^j < 0$ then again from (93) and $\alpha_1 > 1$ we obtain $[\hat{S}_t^{\bar{h}}]^j \leq [\hat{S}_t^h]^j$ from which

$$(H_t^j - \bar{h}^j)[\hat{S}_t^{\bar{h}} - \hat{S}_t^h]^j \geq 0$$

One can easily check that the same is true for $0 < h^j \leq H_t^j$.

Suppose now we are in case (2). Recall that $\hat{S}_t^\xi \in L(\omega)$. For any $h \in F$ there exists $\varepsilon > 0$ such that for any $y \in \Theta_{t+1}(\Sigma_t^\omega)$,

$$h \cdot (y - \hat{S}_t^h) + h \cdot (\hat{S}_t^h - \hat{S}_t^\xi) \geq \varepsilon \Longrightarrow h \cdot (y - \hat{S}_t^h) \geq \varepsilon - h \cdot (\hat{S}_t^h - \hat{S}_t^\xi)$$

There exists $\alpha_2 > 0$ such that $\alpha_2(\varepsilon - h \cdot (\hat{S}_t^h - \hat{S}_t^\xi)) \geq -\delta/2^t$. Denote

$$\alpha_2 := \min\left\{\frac{\delta}{2^t \varepsilon |1 - h \cdot (\hat{S}_t^h - \hat{S}_t^\xi)|}, 1\right\} \qquad \bar{h} := \alpha_2 h \in F \quad (96)$$

Similarly as above $\bar{h} \cdot (\hat{S}_t^h - \hat{S}_t^{\bar{h}}) \geq 0$ and hence

$$\bar{h} \cdot (y - \hat{S}_t^{\bar{h}}) = \bar{h} \cdot (y - \hat{S}_t^h) + \bar{h} \cdot (\hat{S}_t^h - \hat{S}_t^{\bar{h}}) \geq \bar{h} \cdot (y - \hat{S}_t^h) \geq -\delta/2^t$$

Observe now that in case (2), $V_{t-1}(H) + H_t \cdot \hat{S}_t^{\bar{h}} \geq \delta/2^{t-1}$ and hence

$$V_{t-1}(H) + H_t \cdot \hat{S}_t^{\bar{h}} + \bar{h} \cdot (y - \hat{S}_t^{\bar{h}}) \geq \delta/2^t$$

as desired.

Step 5: Let $(H_u)_{u=1}^T$ the iterative strategy constructed in step 4. For every $\omega \in \Omega$ we have $\tau(\omega) \leq T - 1$ and $H_{\tau(\omega)+1} \neq 0$. Moreover, since $\Theta_T = C_T$ we obviously have $B_T = \varnothing$ and hence there exists $\bar{t}(\omega) \leq T$ such that $\hat{S}_t^{sgn H_{\bar{t}}}(\omega) \in \Theta_t(\omega)$. From step 4, $V_T(H)(\omega) > \delta/2^{\bar{t}}$. Since $\omega \in \Omega$ is arbitrary we have the conclusion.

□

REMARK 3.13. *The sets A_t and B_t represents two different actions that must be undertaken in order to realize a Model Independent Arbitrage opportunity. Note indeed that $A_t \cap B_t = \varnothing$. Fix $\omega \in \Omega$ and t. If $\omega \in A_t$, a new position is taken. No strategy has been open before t since we are working on $\bigcap_{s=0}^t \{H_s = 0\}$ and this is the first time where we can make a model independent gain by trading in S since $\tau(\omega) = t$. At this stage we are not concerned about liquidating the position.*

Suppose that at time t we already have an open position (so $\omega \in A_s$ for some $s \leq t$). If $\omega \notin B_t$ then it can be liquidated at this time producing a strictly positive wealth with zero initial cost. If $\omega \in B_t$ then it is not possible to liquidate the position at this time and we need to keep (or modify) the position and close it at subsequent times. By noting that B_T is always the empty set, either because the position is closed before T or because by (92) $\{\hat{S}_T^{sgn H_T} \notin \Theta_T\} = \varnothing$ *on* $\{H_T \neq 0\}$ *we see that it is always possible to close the position opened on A_s.*

3. On Superhedging

Recall the definition of the class $\mathcal{M}_\Pi$ of price systems consistent with the bid-ask spread Π (see Definition 3.7) and the definition of C_t in (85). Consider the following

$$\mathcal{Q} := \left\{ Q \in \mathfrak{P} \mid \exists S \text{ a } Q\text{-martingale with } S_t \in \mathcal{L}^0(\mathcal{F}_t; C_t) \text{ for } t = 0, \dots, T \right\} \tag{97}$$

or, in other words, the projection of $\mathcal{M}_\Pi$ on the set of probability measures and

$$\mathcal{S} := \left\{ S = (S_t)_{t \in I} \mid S_t \in \mathcal{L}^0(\mathcal{F}_t; C_t) \text{ and } \exists Q \in \mathfrak{P} \text{ s.t. } S \text{ is a } Q\text{-martingale} \right\} \tag{98}$$

i.e. the projection of $\mathcal{M}_\Pi$ on the set of $\mathbb{F}$-adapted process. For any $S \in \mathcal{S}$ define also the section of $\mathcal{M}_\Pi$ as

$$\mathcal{Q}_S := \{ Q \in \mathcal{Q} \mid S \text{ is a } Q\text{-martingale} \} \tag{99}$$

The maximal $\mathcal{Q}_S$-polar set has been characterized in [**BFM14**] and denoted as $(\Omega_*)^c$. We here adapt the definition of Ω_* in this market with frictions. In particular let

Definition 3.14. *Let $\mathcal{Q}$ as in* (97)*. We define the support of the consistent price system* Π *as*

$$\Omega_* := \{ \omega \in \Omega \mid \exists Q \in \mathcal{Q} \textit{ such that } Q(\{\omega\}) > 0 \}$$

The aim of this section is to prove the following version of the superhedging Theorem:

Theorem 3.15. *Let $g : \Omega \mapsto \mathbb{R}$ $\mathcal{F}$-measurable*

$$\sup_{Q \in \mathcal{Q}} \mathbb{E}_Q[g] = \inf\{ x \in \mathbb{R} \mid \exists H \in \mathcal{H} \textit{ s.t. } x + V_T(H) \geq g \quad \forall \omega \in \Omega_* \} =: \overline{p}(g) \tag{100}$$

where $\mathcal{Q}$ is defined in (97) *and Ω_* in Definition 3.14.*

proof of ($\leq$). Let $S = (S_t)_{t \in I}$ be a process in $\mathcal{S}$. Take $x \in \mathbb{R}$, $H \in \mathcal{H}$ such that $x + V_T(H) \geq g$. Consider now equation (86) from which we add and substract S_t^j for any $t \in I$ and for any $j = 1, \dots d$ as follows:

$$\begin{aligned}
V_T(H) &= \sum_{t=0}^{T} \sum_{j=1}^{d} \left(H_t^j - H_{t+1}^j \right) \left(\overline{S}_t^j \mathbf{1}_{\{H_t^j \leq H_{t+1}^j\}} + \underline{S}_t^j \mathbf{1}_{\{H_{t+1}^j \leq H_t^j\}} - S_t^j + S_t^j \right) \\
&= \sum_{t=0}^{T} \sum_{j=1}^{d} \left(H_t^j - H_{t+1}^j \right) S_t^j + \\
&= \sum_{t=0}^{T} \sum_{j=1}^{d} \left(H_t^j - H_{t+1}^j \right) \left((\overline{S}_t^j - S_t^j) \mathbf{1}_{\{H_t^j \leq H_{t+1}^j\}} + (\underline{S}_t^j - S_t^j) \mathbf{1}_{\{H_{t+1}^j \leq H_t^j\}} \right) \\
&\leq \sum_{t=0}^{T} \sum_{j=1}^{d} \left(H_t^j - H_{t+1}^j \right) S_t^j = \sum_{j=1}^{d} \left(\sum_{t=1}^{T} H_t^j S_t^j - \sum_{t=1}^{T} H_t^j S_{t-1}^j \right) = (H \circ S)_T
\end{aligned}$$

where $(H \circ S)_T$ is the usual (discrete time) stochastic integral. Note that the previous inequality follows from the fact that $S^j_t \in (\underline{S}^j_t, \overline{S}^j_t)$ for any j and for any t, while all the equalities are simply rearrangement of the terms. Recall also $H_0 = H_{T+1} = 0$. We have therefore obtained that for any strategy H and for any $S \in \mathcal{S}$

$$V_T(H) \leq (H \circ S)_T \tag{101}$$

Observe now that by taking expectation with respect to a martingale measure Q for the process S in both sides we get $E_Q[V_T(H)] \leq 0$. Since (101) holds true for an arbitrary couple (S, Q) and by recalling that Ω_* is the support of the consistent price system (see Definition 3.14) we have

$$g(\omega) \leq x + V_T(H)(\omega) \quad \forall \omega \in \Omega_* \quad \Longrightarrow \quad E_Q[g] \leq x \quad \forall Q \in \mathcal{Q}$$

Take now the supremum over $Q \in \mathcal{Q}$ in both sides and then the infimum over $x \in \mathbb{R}$ to obtain

$$\sup_{Q \in \mathcal{Q}} \mathbb{E}_Q[g] \leq \overline{p}(g)$$

as desired. □

As usual one implication is easy. In order to prove the opposite we need some preliminary results. We first rewrite the explicit expression of the wealth process (86) in the following way

$$V_T(H) = \sum_{j=1}^{d} V^j_T(H) \tag{102}$$

where

$$\begin{aligned} V^j_T(H) &= \sum_{t=0}^{T} \left(H^j_t - H^j_{t+1}\right) \left(\overline{S}^j_t \mathbf{1}_{\{H^j_t \leq H^j_{t+1}\}} + \underline{S}^j_t \mathbf{1}_{\{H^j_{t+1} < H^j_t\}}\right) \\ &= \sum_{t=0}^{T} H^j_t \left(\overline{S}^j_t \mathbf{1}_{\{H^j_t \leq H^j_{t+1}\}} + \underline{S}^j_t \mathbf{1}_{\{H^j_{t+1} < H^j_t\}}\right) \\ &+ \sum_{t=0}^{T} -H^j_{t+1} \left(\overline{S}^j_t \mathbf{1}_{\{H^j_t \leq H^j_{t+1}\}} + \underline{S}^j_t \mathbf{1}_{\{H^j_{t+1} < H^j_t\}}\right) \end{aligned}$$

By recalling the convention $H_0 = H_{T+1} = 0$ and by changing the index in the second sum we get $V_T(H) = \sum_{t=0}^{T} H_t \cdot \Delta S^H_t$ where

$$\begin{aligned} [\Delta S^H_t]_j &= \left(\overline{S}^j_t - \overline{S}^j_{t-1}\right) \mathbf{1}_{\{H^j_{t-1} \leq H^j_t \leq H^j_{t+1}\}} + \left(\overline{S}^j_t - \underline{S}^j_{t-1}\right) \mathbf{1}_{\{H^j_t < H^j_{t-1}\} \cap \{H^j_t \leq H^j_{t+1}\}} \\ &+ \left(\underline{S}^j_t - \underline{S}^j_{t-1}\right) \mathbf{1}_{\{H^j_{t+1} < H^j_t < H^j_{t-1}\}} + \left(\underline{S}^j_t - \overline{S}^j_{t-1}\right) \mathbf{1}_{\{H^j_t \geq H^j_{t-1}\} \cap \{H^j_t > H^j_{t+1}\}} \end{aligned}$$

We will construct now an auxiliary superhedging problem which involves a family of processes in $\mathcal{S}$ where $\mathcal{S}$ is defined in (98).

Introduce first,

$$F_T(\omega, x) = g(\omega) \quad \text{ and } \quad \mathbb{S}_T(\omega) := C_T(\omega) \tag{103}$$

Define iteratively for $t = T-1, \ldots, 1$ the random sets

$$\mathbb{S}_t(\omega) := \overline{conv}\{\mathbb{S}_{t+1}(\widetilde{\omega}) \mid \widetilde{\omega} \in \Sigma^{\omega}_t\} \cap C_t \tag{104}$$

Note that for every t, $\mathbb{S}_t$ is $\mathcal{F}_t$-measurable. Indeed, for $t = T$ it is obvious; suppose this is true for $t+1 \leq s \leq T$. Consider $\mathbb{S}_{t+1}(\Sigma_t^\omega)$ which is obviously an $\mathcal{F}_t$-measurable map since for any open set $O \subseteq \mathbb{R}^d$ we have

$$\{\omega \in \Omega \mid \mathbb{S}_{t+1}(\Sigma_t^\omega) \cap O \neq \varnothing\} = S_{0:t}^{-1}\left(S_{0:t}(\{\omega \in \Omega \mid \mathbb{S}_{t+1}(\omega) \cap O \neq \varnothing\}\right) \in \mathcal{F}_t$$

the measurability of $\mathbb{S}_t$ follows therefore from Proposition 3.27 in the Appendix.

Definition 3.16. *We call the set-valued superhedging problem the following backward procedure. For any $t = T, \dots, 1$, for any $y \in \mathbb{R}$ define*

$$\mathcal{H}_t^y(\omega, x) = \left\{H \in \mathbb{R}^d \mid y + H \cdot (s - x) \geq F_t(\omega, s) \quad \forall s \in \mathbb{S}_t(\widetilde{\omega}),\ \forall \widetilde{\omega} \in \Sigma_{t-1}^\omega\right\}$$

and

$$F_{t-1}(\omega, x) := \inf\{y \in \mathbb{R} \mid \mathcal{H}_t^y(\omega, x) \neq \varnothing\}$$

We simply denote by $\mathcal{H}_t(\omega, x) := \mathcal{H}_t^{F_{t-1}(\omega,x)}(\omega, x)$ the set of optimal strategies at time $t \in I$.
$F_0(x_0)$ will be called the set-valued superhedging price for the initial value $x_0 \in \mathbb{R}^d$.

The next Proposition is crucial for the well-posedness of the prescribe procedure. It provides fundamental measurability properties for the whole scheme. Its proof, as well as the proof of the subsequent results, is technical and hence they are all postponed at Section 3.1.

Proposition 3.17. *Let $F_t(\cdot,\cdot) : \Omega \times \mathbb{R}^d \mapsto \mathbb{R} \cup \{\pm\infty\}$ for $t = 0, \dots T$ as in Definition 3.16. Denote by $D_{F_t}(\omega) := \{x \in \mathbb{R}^d \mid F_t(\omega, x) > -\infty\}$ the effective domain.*
We have that

1. *For every $x \in \mathbb{R}^d$ fixed, the map $F_t(\cdot, x)$ is $\mathcal{F}_t$-measurable.*
   Moreover, when finite, $F_t(\cdot, x)$ is a minimum.
2. *For every $\omega \in \Omega$ the map $F_t(\omega, \cdot)$ restricted to $\overline{D_{F_t}(\omega)}$ is continuous.*
3. *For every $\omega \in \Omega$, $\overline{D_{F_t}(\omega)}$ is convex.*

Items 1 and 2 imply that $F_t(\cdot,\cdot)$ is a so-called Charatéodory map in its effective domain.

Proof. We postpone the proof at Section 3.1. □

For any initial value $x_0 \in \mathbb{R}$ the set-valued superhedging price $F_0(x_0)$, from Definition 3.16, represents (when finite) the minimum amount of cash needed for superhedging $F_t(\omega, s)$, for any time $t \in I$, for any $\omega \in \Omega_*$ and for any intermediate value $s \in \mathbb{S}_t(\widetilde{\omega})$. This value looks too conservative since it consider many possible intermediate values for S_t. We nevertheless now show the existence of $\bar{x}_0 \in C_0$ such that: i) there exists a process $(S_t)_{t \in I}$ with $S_0 = \bar{x}_0$ and with values in the bid-ask spread such that the superhedging price of g with no frictions is $F_0(x_0)$. ii) the collection of strategies provided by the solution of the set-valued problem compose a self-financing trading strategy such that

$$F_0(\bar{x}_0) + V_T(H) \geq g \qquad \forall \omega \in \Omega_*$$

We prove this in a constructing way. More precisely we need the following step-forward iteration: suppose that at time $t \geq 1$ the random variables $S_{t-1} \in \mathcal{L}^0(\mathcal{F}_{t-1})$ and $H_t \in \mathcal{L}^0(\mathcal{F}_{t-1}; \mathcal{H}_t)$ are given and define

$$X_{t-1}(\omega) := F_{t-1}(\omega, S_{t-1}(\omega)),$$

LEMMA 3.18. *Suppose $X_{t-1}(\omega) < \infty$ for any $\omega \in \Omega$. There exists a random vector $S_t \in \mathcal{L}^0(\mathcal{F}_t; C_t)$ such that, for all $\omega \in \Omega$,*

$$X_{t-1}(\omega) = \inf\{y \in \mathbb{R} \mid \exists H \in \mathbb{R}^d \ s.t. \ y + H \cdot \Delta S_t(\widetilde{\omega}) \geq F_t(\widetilde{\omega}, S_t(\widetilde{\omega})) \quad \forall \widetilde{\omega} \in \Sigma^{\omega}_{t-1}\}$$

where $\Delta S := S_t - S_{t-1}$. Moreover, if $X_{t-1}(\omega) > -\infty$, $H_t(\omega)$ is an optimal strategy.

PROOF. We postpone the proof at Section 3.1. □

PROPOSITION 3.19. *For every $x_0 \in C_0$ there exists a price process $S = (S_t)_{t \in I}$ such that:*

- *$S_0 = x_0$, $S_t \in \mathcal{L}^0(\mathcal{F}_t; C_t)$ for every $0 \leq t \leq T$.*
- *Let $\Omega_*(S) := \{\omega \in \Omega \mid \exists Q \in \mathcal{Q}_S \ s.t. \ Q(\{\omega\}) > 0\}$. Then,*

$$\inf\{x \in \mathbb{R} \mid \exists H \in \mathcal{H} \ s.t. \ x + (H \circ S)_T(\omega) \geq g(\omega) \quad \forall \omega \in \Omega_*(S)\} = F_0(x_0)$$

  where $\mathcal{Q}_S$ is defined in (99).

PROOF. We postpone the proof at Section 3.1. □

We now construct, for a given initial value $x_0 \in C_0$, a strategy $H := (H_1, \dots H_T)$ whose terminal payoff dominates g. We first need the following step-forward iteration. Recall from Definition 3.16 that $\mathcal{H}_{t+1}(\cdot, \cdot)$ is the set of optimal strategies for the (conditional) set-valued superhedging problem.

PROPOSITION 3.20. *There exist a random vector $\widehat{S}_t \in \mathcal{L}_t(\mathcal{F}_t; C_t)$ and a trading strategy $H_{t+1} \in \mathcal{L}^0(\mathcal{F}_t)$ such that, for every $\omega \in \{X_{t-1} < \infty\}$, we have $H_{t+1}(\omega) \in \mathcal{H}_{t+1}(\omega, \widehat{S}_t(\omega))$,*

$$X_{t-1}(\omega) + H_t \cdot (\widehat{S}_t(\widetilde{\omega}) - S_{t-1}(\widetilde{\omega})) \geq F_t(\widetilde{\omega}, \widehat{S}_t(\widetilde{\omega})) \quad \forall \widetilde{\omega} \in \Sigma^{\omega}_{t-1}, \tag{105}$$

and the following properties:

- *if $H^i_t(\omega) < H^i_{t+1}(\omega)$ then $\widehat{S}^i_t(\omega) = \overline{S}^i(\omega)$*
- *if $H^i_t(\omega) > H^i_{t+1}(\omega)$ then $\widehat{S}^i_t(\omega) = \underline{S}^i(\omega)$*

In particular if $\hat{S}^i_t \in \left(\underline{S}^i(\omega), \overline{S}^i(\omega)\right)$ we necessarily have $H^i_t(\omega) = H^i_{t+1}(\omega)$.

PROOF. We postpone the proof at Section 3.1. □

REMARK 3.21. *With a slight abuse of notation, when $X_{t-1}(\omega) = -\infty$ we intend that there exists a sequence $\{(y_n, H_n)\} \subseteq \mathbb{R} \times \mathcal{L}^0(\mathcal{F}_t)$ with $y_n \to -\infty$, such that for every $n \in \mathbb{N}$ the conditions of Proposition 3.20 are satisfied. The same apply to Corollary 3.22 when $F_0(x_0) = -\infty$.*

COROLLARY 3.22. *For every $x_0 \in C_0$ with $F_0(x_0) < \infty$ there exists a predictable process $H := (H_1, \dots H_T)$ such that*

$$F_0(x_0) + (0 - H_1 \cdot x_0) + \sum_{t=1}^{T} \sum_{j=1}^{d} \left(H^j_t - H^j_{t+1}\right) \left(\overline{S}^j_t \mathbf{1}_{\{H^j_t \leq H^j_{t+1}\}} + \underline{S}^j_t \mathbf{1}_{\{H^j_{t+1} < H^j_t\}}\right) \geq g \ on \ \Omega_*$$

PROOF. Note first that if $F_0(x_0) < \infty$ then $X_t < \infty$ $\forall t = 0, \dots T$. Applying iteratively Proposition 3.20, there exists a process $\hat{S}$ with $\hat{S}_0 = x_0$ which satisfy the following inequalities

$$\begin{aligned} F_0(x_0) + H_1 \cdot \Delta\hat{S}_1 &\geq X_1 \\ F_0(x_0) + H_1 \cdot \Delta\hat{S}_1 + H_2 \cdot \Delta\hat{S}_2 &\geq X_2 \\ &\dots \\ F_0(x_0) + \sum_{t=1}^{T} H_t \cdot \Delta\hat{S}_t &\geq X_T = g \end{aligned}$$

on $A := \{\omega \in \Omega \mid X_t(\omega) > -\infty \ \forall t = 0, \dots T\}$. Note that, by construction, $X_t(\omega) = -\infty$ for some $t = 0, \dots T$ if and only if $Q(\{\omega\}) = 0$ for every $Q \in \mathcal{Q}$, so that $A = \Omega_*$. Rearranging the terms in the summation as

$$\sum_{t=1}^{T} H_t \cdot \Delta\hat{S}_t = \sum_{t=1}^{T} (H_t - H_{t+1}) \cdot \hat{S}_t - H_1 \cdot x_0$$

the properties of $\hat{S}$ yield the desired inequality. □

For any starting point x_0 we can therefore find the desired process by iterative application of Lemma 3.18. This lead us to the proof of Theorem 3.15 as follows:

PROOF OF $(\geq)$ IN (100) OF THEOREM 3.15. Let $F_0(x)$ the solution of the superhedging problem in Definition 3.16. Take

$$m := \sup_{x \in C_0} F_0(x)$$

Suppose first that $m = \infty$. There exists a sequence $x_n \in C_0$ such that $F_0(x_n) \to \infty$. From Proposition 3.19 there exists a sequence of processes $S^n := (S^n_t)_{t \in I} \subseteq \mathcal{S}$ whose superhedging price explode to ∞ and hence the inequality is trivial. If $m = -\infty$ then by Corollary 3.22 and $(\leq)$ in (100) the equality follows again trivially as a degenerate case. If m is finite then $m = \sup_{x \in \overline{D_{F_0}}} F_0(x)$. By Proposition 3.17 F_0 is non-random, continuous and $\overline{D_{F_0}}$ is a closed subset of a compact set C_0. Thus m is a maximum and we denote by $\bar{x}_0$ a maximizer. By Proposition 3.19 there exists a process $(\widetilde{S}_t)_{t \in I}$ with $\widetilde{S}_0 = \bar{x}_0$ whose superhedging price is m, namely,

$$m = \inf \left\{ x \in \mathbb{R} \mid \exists H \in \mathcal{H} \text{ s.t. } x + (H \circ \widetilde{S})_T(\omega) \geq g(\omega) \quad \forall \omega \in \Omega_*(\widetilde{S}) \right\} = \sup_{Q \in \mathcal{Q}_{\widetilde{S}}} E_Q[g] \tag{106}$$

where the last equality derives from Theorem 1.1 in [**BFM15**].

On the other hand by adding a fictitious node $t = -1$ to the set-valued problem in Definition 3.16, with $\mathbb{S}_{-1} = \bar{x}_0$, we have that the minimization

$$\inf \left\{ y \in \mathbb{R} \mid H \in \mathbb{R}^d \text{ s.t. } y + H \cdot (s - \bar{x}_0) \geq F_0(s) \quad \forall s \in \mathbb{S}_0, \right\}$$

has the obvious solution $X_{-1} = m$, with corresponding optimal strategy $H_0 = 0$. By applying Proposition 3.20 we obtain H_1 such that

$$H_1 \cdot x_0 = \sum_{j=1}^{d} H^j_{t+1} \left(\overline{S}^j_t \mathbf{1}_{\{0 \leq H^j_{t+1}\}} + \underline{S}^j_t \mathbf{1}_{\{H^j_{t+1} < 0\}} \right)$$

Apply now Corollary 3.22, with $x_0 = \bar{x}_0$, to get the existence of a trading strategy $(H_t)_{t \in I}$ such that (cfr equation (102))

$$m + V_T(H)(\omega) \geq g(\omega) \quad \forall \omega \in \Omega_* \tag{107}$$

The desired inequality follows from (106) and (107):

$$\sup_{Q\in\mathcal{Q}} \mathbb{E}_Q[g] = \sup_{S\in\mathcal{S}} \sup_{Q\in\mathcal{Q}_S} \mathbb{E}_Q[g] \geq \sup_{Q\in\mathcal{Q}_{\widetilde{S}}} \mathbb{E}_Q[g] = m \geq \overline{p}(g)$$

□

3.1. Proofs.

PROOF OF PROPOSITION 3.17. For $t = T$ the claim is trivial. Suppose it is true for all $t+1 \leq s \leq T-1$.

1. We first show that $\mathbb{S}_{t+1}$ takes value in the effective domain of $F_{t+1}(\omega, s)$. From (104), any $\bar{s} \in \mathbb{S}_{t+1}(\omega)$ is limit of convex combinations of elements in $\mathbb{S}_{t+2}(\Sigma_t^\omega)$. Let $s_n \to \bar{s}$. Therefore, for any $n \in \mathbb{N}$, there exist:

- $\omega_1, \ldots, \omega_{k(n)}$ with $\omega_i \in \Sigma_t^\omega$ for every i;
- $z_1, \ldots, z_{k(n)}$ with $z_i \in \mathbb{S}_{t+2}(\omega_i)$ for every i;
- $\lambda_1, \ldots \lambda_{k(n)}$, with $0 < \lambda_i < 1$ for every i;

such that $s_n := \sum_{i=1}^{k(n)} \lambda_i z_i$. Consider a frictionless, one-period model, on $\{z_1, \ldots, z_n\}$ with $\widetilde{S}_0 = x_n$, $\widetilde{S}_1(z_i) = z_i$ for every i. $Q(\{z_i\}) := \lambda_i$ define a martingale measure for the process $\widetilde{S}$.
Denote by $\mathcal{M}(\widetilde{S})$ the set of martingale measures for $\widetilde{S}$ and $\pi_{\widetilde{S}}(g)$ the superhedging price for $g(z_i) := F_{t+2}(\omega_i, z_i)$ in the one-period model. From the classical theory

$$-\infty < x_n \leq \sup_{Q\in\mathcal{M}(\widetilde{S})} E_Q[g] = \pi_{\widetilde{S}}(g) \leq F_{t+1}(\omega, x_n)$$

where the last inequality follows from F_{t+1} being the solution of the set-valued problem. We thus have that $s_n \in D_{F_{t+1}}(\omega)$ for every n and hence $\bar{s} \in \overline{D_{F_{t+1}}(\omega)}$.

Observe now that, from the inductive hypothesis, $s \mapsto (s - x, F_{t+1}(\omega, s))$ is a Charatéodory map in the domain of F_{t+1}. Since $\mathbb{S}_{t+1}$ takes value in $D_{F_{t+1}}$, Lemma 3.29 in the Appendix, implies that the multifunction

$$\psi(\omega) : \omega \mapsto (\mathbb{S}_{t+1}(\omega) - x, F_{t+1}(\omega, \mathbb{S}_{t+1}(\omega))$$

is $\mathcal{F}_t$-measurable. Applying now Lemma 3.31 with $\Delta\widetilde{S}_t = \psi$, $X = 0$ $C = \mathbb{R}^d \times \{-1\}$. This particular choice yields

$$A_C(\omega) = \left\{(H, y) \in \mathbb{R}^{d+1} \mid y + H\cdot(s - x) \geq F_{t+1}(\omega, s) \quad \forall s \in \mathbb{S}_{t+1}(\widetilde{\omega}),\ \forall\widetilde{\omega} \in \Sigma_t^\omega\right\},$$

so that the resulting X_t, $\mathcal{H}_{t+1}$ from Lemma 3.31 represents, for any $\omega \in \Omega$ the minimum amount of cash needed for superhedging $F_{t+1}(\omega, s)$ for any intermediate value $s \in \mathbb{S}_{t+1}(\widetilde{\omega})$, and the corresponding optimal strategies as desired.

3. We first show item 3.
Fix $\omega \in \Omega$. If $D_F = \varnothing$ there is nothing to show. Denote by

$$A(x) := \{H \in \mathbb{R}^d \mid H\cdot(s - x) \geq 0 \quad \forall s \in \mathbb{S}_{t+1}(\Sigma_t^\omega) \ \text{ with} > 0 \text{ for some } \bar{s}\}$$

We show that the set $C := \{x \in D_{F_t}(\omega) \mid A(x) = \varnothing\}$ is convex and $\overline{D_{F_t}(\omega)} = \overline{C}$ from which the thesis follows. Denote by

$$\Gamma := conv\{\mathbb{S}_{t+1}(\Sigma_t^\omega)\}$$

Take now $x_1, x_2 \in C$ and recall that, from Hyperplane separation Theorem, $A(x_i) = \varnothing$ if and only if $x_i \in ri(\Gamma)$. As Γ is a convex set for any $0 \le \lambda \le 1$, $\lambda x_1 + (1-\lambda)x_2 \in ri(\Gamma)$ and hence $A(\lambda x_1 + (1-\lambda)x_2)) = \varnothing$.
We now show that if $x \in D_F(\omega)$ then there exists a sequence $x_k \in C$ such that $x_k \to x$. Take $x \notin C$ otherwise is trivial. Note first that $x \in \overline{\Gamma}$ otherwise by Hyperplane separation Theorem there would exists $v \in \mathbb{R}^d$ and $\varepsilon > 0$ such that $v \cdot (\mathbb{S}_{t+1}(\Sigma_t^\omega) - x) \ge \varepsilon$ which would give $x \notin D_F(\omega)$. Take now $\widetilde{x} \in ri(\Gamma)$, for every $k \in \mathbb{N}$

$$x_k := \left(1 - \frac{1}{k}\right) x + \frac{\widetilde{x}}{k} \in ri(\Gamma)$$

clearly $x_k \to x$ as $k \to \infty$ and again from Hyperplane separation Theorem $x_k \in C$.

2. First observe that if there exists $\widetilde{x}$ such that $F_t(\omega, \widetilde{x}) = +\infty$ then $F_t(\omega, \cdot) \equiv +\infty$ and hence: $D_{F_t}(\omega) = \mathbb{R}^d$ and $F_t(\omega, \cdot)$ is trivially continuous. Indeed, since $F_t(\omega, \widetilde{x}) = +\infty$, for any $H \in \mathbb{R}^d$ there exists a sequence $\{(\omega_n, s_n)\}$ such that $H \cdot (s_n - \widetilde{x}) - F_{t+1}(\omega_n, s_n) \to -\infty$. Therefore the same holds true for the sequence $H \cdot (s_n - x) - F_{t+1}(\omega_n, s_n)$ with x arbitrary. Thus, $F_t(\omega, x) = +\infty$.

We may now suppose that $F_t(\omega, \cdot) < +\infty$. We first show that $F(\omega, \cdot)$ is upper semi-continuous at $x \in \overline{D_{F_t}(\omega)}$.

For $x \in D_{F_t}(\omega)$, Lemma 3.31 implies that there exists an optimal strategy H such that

$$F_t(\omega, x) + H \cdot (s - x) \ge F_{t+1}(\omega, s) \quad \forall s \in \mathbb{S}_{t+1}(\widetilde{\omega}),\ \forall \widetilde{\omega} \in \Sigma_t^\omega \tag{108}$$

Let now $\{x_k\}_{k=1}^\infty$ such that $x_k \to x$ for $k \to \infty$. Observing that $H \cdot (s - x) = H \cdot (s - x_k) + H \cdot (x_k - x)$ we get, from (108), $F_t(\omega, x_k) \le F_t(\omega, x) + H \cdot (x_k - x)$. By taking limits in both sides we can conclude that $F_t(\omega, \cdot)$ is upper semi-continuous:

$$\limsup_{k \to \infty} F_t(\omega, x_k) \le F_t(\omega, x) \tag{109}$$

The case of $x \notin D_{F_t}(\omega)$ is similar. Since $F_t(\omega, x) = -\infty$ there exists a sequence $\{H_n\}$ such that (108) is satisfied with $(-n, H_n)$ replacing $(F_t(\omega, x), H)$. We analogously obtain $F_t(\omega, x_k) \le -n + H_n \cdot (x_k - x)$. By taking the limit in k in both sides we get $\limsup_{k \to \infty} F_t(\omega, x_k) \le -n$ for any $n \in \mathbb{N}$, from which the upper semi-continuity follows.

We now turn to the lower semi-continuity. Let $x \in \overline{D_{F_t}(\omega)}$. If $x \notin D_{F_t}(\omega)$, i.e. $F_t(\omega, x) = -\infty$, from the previous step we already have continuity. Suppose therefore $x \in D_{F_t}(\omega)$. Lemma 3.31 implies that there exists an optimal strategy H such that (108) is satisfied.

case a) If the inequality in (108) is actually an equality we have perfect replication and hence for any $\widetilde{x} \in D_{F_t}(\omega)$ we have $F_t(\omega, \widetilde{x}) = F_t(\omega, x) + H \cdot (\widetilde{x} - x)$: if indeed there exists (z, H_z) a superhedging strategy with $l := z - F_t(\omega, x) + H \cdot (\widetilde{x} - x) < 0$ then it is easy to see that

$$(H_z - H) \cdot (\mathbb{S}_{t+1}(\Sigma_t^\omega) - \widetilde{x}) \geq -l > 0$$

from which $\widetilde{x} \notin D_{F_t}(\omega)$.
By considering $\{x_k\}_{k=1}^\infty$ such that $x_k \to x$ we obtain

$$\lim_{k\to\infty} F_t(\omega, x_k) = \lim_{k\to\infty} (F_t(\omega, x) + H \cdot (x_k - x)) = F_t(\omega, x)$$

as desired.

case b) Define

$$G_t(\omega, x) := \sup \left\{ y \in \mathbb{R} \mid \exists\, H \in \mathbb{R}^d : \ y + H \cdot (s - x) \leq F_{t+1}(\widetilde{\omega}, s), \ \forall s \in \mathbb{S}_{t+1}(\widetilde{\omega}), \ \forall \widetilde{\omega} \in \Sigma_t^\omega \right\}.$$

and, for all $y \in \mathbb{R}$, the set

$$\Gamma_y(x) := co\,(conv\,\{[s - x\ ;\ y - F_{t+1}(\widetilde{\omega}, s)] \mid s \in \mathbb{S}_{t+1}(\widetilde{\omega}), \ \widetilde{\omega} \in \Sigma_t^\omega\}) \subseteq \mathbb{R}^d \times \mathbb{R}$$

Note that $F_t(\omega, x) > G_t(\omega, x)$ and $int(\Gamma_y(x)) \neq \varnothing$ otherwise there is perfect replication and we are back to case a). Take therefore $y \in (G_t(\omega, x), F_t(\omega, x))$.

If $0 \in int(\Gamma_y(x))$ there exists $\bar{\varepsilon} > 0$ such that for every $\varepsilon \leq \bar{\varepsilon}$, $B_{2\varepsilon}(0) \subseteq int(\Gamma_y(x))$. For any $(0, \widetilde{x}) \in B_\varepsilon(0)$ with $\widetilde{x} \in \mathbb{R}^d$, we have $0 \in int(\Gamma_y(\widetilde{x}))$, hence, there is no non-zero $(H, h) \in \mathbb{R}^d \times \mathbb{R}$, such that either

$$h(y - F_{t+1}(\widetilde{\omega}, s)) + H \cdot (s - \widetilde{x}) \geq 0 \ \text{ or } \ h(y - F_{t+1}(\widetilde{\omega}, s)) + H \cdot (s - \widetilde{x}) \leq 0 \tag{110}$$

for every $s \in \mathbb{S}_{t+1}(\widetilde{\omega})$ and $\widetilde{\omega} \in \Sigma_t^\omega$. In particular there is no $H \in \mathbb{R}^d$ such that $y + H \cdot (s - \widetilde{x}) \geq F_{t+1}(\widetilde{\omega}, s)$ for every $s \in \mathbb{S}_{t+1}(\widetilde{\omega})$ and $\widetilde{\omega} \in \Sigma_t^\omega$. Thus, $F_t(\omega, \widetilde{x}) > y$. Since the same holds true for every $\widetilde{x}$ such that $\|\widetilde{x} - x\| < \varepsilon$, we get

$$\liminf F_t(\omega, x_k) \geq F_t(\omega, x) - \varepsilon \tag{111}$$

Since ε is arbitrary small we obtain the thesis.

If $0 \notin int(\Gamma_y(x))$ there exists a separator $(H, h) \in \mathbb{R}^d \times \mathbb{R}$ such that (110) holds true but since $y \in (G_t(\omega, x), F_t(\omega, x))$ we necessarily have $h = 0$. Consider now a separator $\hat{H} := (H, 0)$ with $H \in \mathbb{R}^d$ and denote by $\hat{H}^{++}$, $\hat{H}^+$ the positive and non-negative half-spaces associated to $\hat{H}$. Analogously $\hat{H}^{--}$, $\hat{H}^-$. Define

$$A := \{z \in \mathbb{R}^{d+1} \mid \hat{H} \cdot z = 0\} \cap \overline{\Gamma_y(x)}$$

Claim 3.23. $0 \in ri(A)$.

Observe that since $\Gamma_y(x) \subseteq \hat{H}^+$ and $0 \in ri(A)$, there exists $\bar{\varepsilon} > 0$ such that for every $\varepsilon \leq \bar{\varepsilon}$, we have $B_{2\varepsilon}(0) \cap \hat{H}^{++} \subseteq int(\Gamma_y(x))$. As in case a) for every $(0, \widetilde{x}) \in B_\varepsilon(0) \cap \hat{H}^{++}$ we have $0 \in int(\Gamma_y(\widetilde{x}))$. This implies $F_t(\omega, \widetilde{x}) > y$. In order to conclude observe that if $(0, \widetilde{x}) \in B_\varepsilon(0) \cap H^-$ then $\widetilde{x} \notin ri(D_{F_t}(\omega))$. If indeed $\widetilde{x}$ is such that $H \cdot (\widetilde{x} - x) \leq 0$ then

$$H \cdot (s - \widetilde{x}) \geq 0 \qquad \forall s \in \mathbb{S}_{t+1}(\widetilde{\omega}), \ \widetilde{\omega} \in \Sigma_t^\omega \tag{112}$$

It is easy to see that in every neighbourhood of $\widetilde{x}$ there exists an element $\bar{x}$ for which, replacing $\widetilde{x}$ with $\bar{x}$ in (112) the inequality is satisfied with a lower bound. Thus $\bar{x}$ is not in $D_{F_t}(\omega)$.

We have therefore obtained that if $x_k \to x$ with $x_k \in ri(D_{F_t}(\omega))$ then (111) holds true and hence, also in case b), the thesis.

We are only left to show Claim 3.23. Suppose by contradiction that there exists $r \in \mathbb{R}^{d+1}$ such that $\hat{H} \cdot r = 0$ and $\alpha r \notin A$ for every $\alpha > 0$. Note that from $r \notin A$ we have $dist(r, \overline{\Gamma_y(x)}) > 0$ so that there exists $\delta > 0$ such that $B_\delta(r) \cap \overline{\Gamma_y(x)} = \varnothing$. Since $\overline{\Gamma_y(x)}$ is a cone we can conclude that the segment $[0, \widetilde{r}]$ with $\widetilde{r} \in B_\delta(r)$ has empty intersection with $\overline{\Gamma_y(x)}$. Since obviously $0 \in \cup_{0 \le \alpha \le 1} \alpha B_\delta(r)$ we can infer that there exists $(\widetilde{H}, \widetilde{h})$ with $\widetilde{h} \neq 0$ such that

$$\widetilde{h}(y - F_{t+1}(\widetilde{\omega}, s)) + \widetilde{H} \cdot (s - x) \geq 0 \qquad \forall s \in \mathbb{S}_{t+1}(\widetilde{\omega}),\ \widetilde{\omega} \in \Sigma_t^\omega$$

which is a contradiction since $y \in (G_t(\omega, x), F_t(\omega, x))$.

□

REMARK 3.24. *Observe that from the proof of Proposition 3.17 we actually obtained that $F_t(\omega, \cdot)$ is upper semi-continuous in the whole space $\mathbb{R}^d$ and note only on $\overline{D_{F_t}(\omega)}$. Note, moreover, that for showing the lower semi-continuity one could argue that $F_t(\omega, x) \le F_t(\omega, x_k) + H_k \cdot (x - x_k)$, where H_k is an optimal strategy associated to $F_t(\omega, x_k)$, and then take the limit. Nevertheless in order to conclude that $F_t(\omega, \cdot)$ is lower semi-continuous we would need, for instance, that the sequence $\{H_k\}$ is bounded, which in general cannot be guaranteed.*

PROOF OF LEMMA 3.18. For any $m \in \mathbb{N}$ let

$$\{r_{m,n}\}_{n=1}^\infty := \left\{ r \in \mathbb{Q}^{d+1} \mid \|r\| = \frac{1}{m} \right\} \cap (-\infty, 0) \times \mathbb{R}^d \tag{113}$$

denoting $r_{m,n} = (r^1_{m,n}, \bar{r}_{m,n}) \in \mathbb{Q} \times \mathbb{Q}^d$ we define for any $m, n \in \mathbb{N}$,

$$X_{m,n}(\omega) := X_{t-1}(\omega) + r^1_{m,n}, \qquad H_{m,n}(\omega) := H_t(\omega) + \bar{r}_{m,n}$$

Note that for any $m, n \in \mathbb{N}$ these functions are obviously $\mathcal{F}_{t-1}$-measurable and we can extend the definiton with $X_{0,0} := X_{t-1}$, $H_{0,0} := H_t$. Observe now that for any $m, n \in \mathbb{N}^2 \cup \{(0,0)\}$ the function $G_{m,n} : \Omega \times \mathbb{R}^d \mapsto \mathbb{R}$ defined by

$$G_{m,n}(\omega, x) := [X_{m,n}(\omega) + H_{m,n}(\omega) \cdot (x - S_{t-1}(\omega)) - F(\omega, x)] \mathbf{1}_{\{|X_{t-1}(\omega)| < \infty\}} \tag{114}$$

is a Charathéodory map. Since $\mathbb{S}_t$ is a closed valued $\mathcal{F}_t$-measurable set, from Theorem 3.30 in the Appendix, the set

$$E_{m,n} := \{\omega \in \Omega \mid \exists x \in \mathbb{S}_t(\omega) \text{ with } G_{m,n}(\omega, x) \in (-\infty, 0]\} \tag{115}$$

is $\mathcal{F}_t$-measurable and there exists a measurable function $S_{m,n} : E_{m,n} \mapsto \mathbb{R}^d$ such that

$$S_{m,n}(\omega) \in \mathbb{S}_t(\omega), \qquad G_{m,n}(\omega, S_{m,n}(\omega)) \le 0$$

Note that $E_{m,n}$ is non-empty since X_{t-1}, when finite, is a minimum and when infinite $G_{m,n}(\omega, x) \equiv 0$.

Define, for any $m \in \mathbb{N}$, $S_m := \sum_{n=1}^\infty S_{m,n} \mathbf{1}_{E_{m,n} \setminus \cup_{j=1}^{n-1} E_{m,j}}$; by denoting $E_m := \cup_{n \in \mathbb{N}} E_{m,n}$ and $E := \cup_{m \in \mathbb{N}} E_m$ we define

$$S_t := \left(\sum_{m=1}^{\infty} S_m \mathbf{1}_{E_m \setminus \cup_{k=1}^{m-1} E_k} \right) \mathbf{1}_E + \widehat{S}_t \mathbf{1}_{E^c}$$

where $\widehat{S}_t \in \mathcal{L}^0(\mathcal{F}_t)$ is an arbitrary measurable selector of $\mathbb{S}_t$.
As (X_{t-1}, H_t) solves the set-valued problem in Definition 3.16 in particular it satisfies, when finite,

$$X_{t-1}(\omega) + H_t(\omega) \cdot (S_t(\omega) - S_{t-1}(\omega)) - F(\omega, S_t(\omega)) \geq 0 \qquad \forall \omega \in \Omega \tag{116}$$

We now show that this is also optimal.
If $X_{t-1}(\omega) = -\infty$ equation (116) holds by replacing $(X_{t-1}(\omega), H_t(\omega))$ with (y_n, H_n) and $y_n \to -\infty$. The desired inequality is therefore obvious.
Let $\omega \in \Omega$ such that $|X_{t-1}(\omega)| < \infty$. Suppose by contradiction that for some $\omega \in \Omega_*$ there exists and optimal value $y < X_{t-1}(\omega)$ with optimal strategy $H_y \in \mathbb{R}^d$. Consider the acceptance set given by

$$A := \bigcap_{\widetilde{\omega} \in \Sigma^{\omega}_{t-1}} \left\{ (x, h) \in \mathbb{R}^{1+d} \mid x + h \cdot \Delta S_t(\widetilde{\omega}) \geq F_t(\widetilde{\omega}, S_t(\widetilde{\omega})) \right\} \tag{117}$$

Clearly the optimal value satisfy $a_1 := (y, H_y) \in A$. Moreover, since $a_2 := (X_{t-1}(\omega), H_t(\omega))$ is optimal for the set-valued problem we also have $a_2 \in A$. Observe now that A is a closed convex cone in $\mathbb{R}^{d+1}$ and hence for any $\varepsilon > 0$, the ball with radius ϵ with center in a_1, denoted by $B_\varepsilon(a_1)$, intersects the relative interior of A. Take now $\varepsilon < d(a_1, a_2)$. Note that since $y < X_{t-1}(\omega)$, there exist $r := (r^1, \bar{r}) \in \mathbb{Q}^{1+d}$ with $r^1 < 0$, and $\lambda > 0$, such that $a := a_2 + \lambda r \in B_\varepsilon(a_1) \cap ri(A)$. With no loss of generality we can choose $\|r\| \leq 1$ and $\lambda > 1$ so that, by construction, $r = r_{m,n}$ for some $m, n \in \mathbb{N}^2$ as defined in (113). We claim that for some $\widetilde{\omega} \in \Sigma^{\omega}_{t-1}$,

$$a \cdot [1; \Delta S_t(\widetilde{\omega})] \leq F_t(\widetilde{\omega}, S_t(\widetilde{\omega}))$$

but since $a \in ri(A)$ we necessarily have $a \cdot [1; \Delta S_t(\widetilde{\omega})] > F_t(\widetilde{\omega}, S_t(\widetilde{\omega}))$ which is a contradiction.

We are only left to show the claim. Note first that $G_{m,n}(\widetilde{\omega}, S_t(\widetilde{\omega})) \leq 0$ for some $\widetilde{\omega} \in \Sigma^{\omega}_{t-1}$ which can be rewritten as $(a_2 + r) \cdot [1; \Delta S_t(\widetilde{\omega})] \leq F_t(\widetilde{\omega}, S_t(\widetilde{\omega}))$. Consider the half-line $R := \{a_2 + \lambda(a - a_2) \mid \lambda \in \mathbb{R}^+\}$ and the hyperplane $L := \{(x, h) \in \mathbb{R}^{1+d} \mid (x, h) \cdot [1; \Delta S_t(\widetilde{\omega})] = F_t(\widetilde{\omega}, S_t(\widetilde{\omega}))\}$. Denote by L^+, L^- the non-negative and non-positive half-spaces associated to L and observe that $a_2 + r \in L^-$. Since $a_2 \in L^+$ we have that $L \cap R \neq \varnothing$ and there exists (x_0, h_0) in the intersection such that $(x_0, h_0) = a_2 + \bar{\lambda} r$ for some $\bar{\lambda} \leq 1$. By a change of coordinate, we can rewrite $L = \{(x, h) \in \mathbb{R}^{1+d} \mid (x - x_0, h - h_0) \cdot [1; \Delta S_t(\widetilde{\omega})] = 0\}$. Recall now that $a = a_2 + \lambda r$ with $\lambda > 1$ and $a_2 + r \in L^-$, hence,

$$(1 - \bar{\lambda}) r \cdot [1; \Delta S_t(\widetilde{\omega})] \leq 0 \quad \Rightarrow (\lambda - \bar{\lambda}) r \cdot [1; \Delta S_t(\widetilde{\omega})] \leq 0$$

which is the desired inequality in the new coordinate system. □

PROOF OF PROPOSITION 3.19. Start with $S_0 := x_0$ and suppose first $F_0(x_0) < \infty$. From Lemma 3.18 there exists S_1 such that $F_0(x_0) + H_1 \Delta S_1 \geq X_1$. Applying iteratively Lemma 3.18

we get the inequalities

$$\begin{aligned} F_0(x_0) + H_1 \cdot \Delta S_1 &\geq X_1 \\ F_0(x_0) + H_1 \cdot \Delta S_1 + H_2 \cdot \Delta S_2 &\geq X_2 \\ &\dots \\ F_0(x_0) + \sum_{t=1}^{T} H_t \cdot \Delta S_t &\geq X_T = g \end{aligned}$$

and hence the cheapest super-hedge from the minimality of X_t for $t = 0, \dots T$. Obviously S belongs to the bid-ask spread since $S_t \in \mathbb{S}_t$ for every t.

Suppose now that $F_0(x_0) = \infty$. Note that if $F_s(\omega, x) = \infty$ for some $s \in I$, $x \in \mathbb{R}^d$ then $F_s(\omega, \cdot) \equiv \infty$. Let $t := \min\{s \in I \mid F_s(\omega, \cdot) < \infty \ \forall \omega \in \Omega\} \geq 1$. For all $y \in \mathbb{R}$, consider the set

$$\Gamma_y(\mathbb{S}_t) := co\left(conv\left\{[s - x \ ; \ y - F_t(\widetilde{\omega}, s)] \mid s \in \mathbb{S}_t(\widetilde{\omega}), \ \widetilde{\omega} \in \Sigma_{t-1}^{\omega}\right\}\right) \subseteq \mathbb{R}^d \times \mathbb{R}$$

Observe first that if for a finite set $\{\omega_1, \dots \omega_k\}$ (or for the empty set) we have $0 \notin int(\Gamma_y(U))$ with $U := \mathbb{S}_t(\Sigma_{t-1}^{\omega}) \setminus \{\mathbb{S}_t(\omega_1), \dots \mathbb{S}_t(\omega_k)\}$ then there exists $(H, h) \setminus (0, 0) \in \mathbb{R}^d \times \mathbb{R}$, such that

$$h(y - F_t(\widetilde{\omega}, s)) + H \cdot (s - x) \geq 0 \tag{118}$$

If $h > 0$ then $y + H/h \cdot (s - x) \geq F_t(\widetilde{\omega}, s)$ for all such s. From the continuity of $F_t(\omega, \cdot)$ (see Proposition 3.17) and from $\mathbb{S}_t$ being closed and bounded we have that the quantities

$$l_j := \min\{y + H/h \cdot (s - x) - F_t(\omega_j, s) \mid s \in \mathbb{S}_t(\omega_j)\} < 0, \qquad l := -\min_j l_j$$

are well defined and finite. Observe now that $(y + l, H/h)$ solves the set-valued superhedging problem of Definition 3.16 which is a contradiction since $F_{t-1}(\omega, x) = \infty$.
If $h < 0$ then $y + H/h \cdot (s - x) \leq F_t(\widetilde{\omega}, s)$ for every $s \in \mathbb{S}_t(\widetilde{\omega})$ and $\widetilde{\omega} \in \Sigma_{t-1}^{\omega} \setminus \{\Sigma_t^{\omega_1}, \dots \Sigma_t^{\omega_k}\}$. In particular for an arbitrary measurable random variable S_t with values in C_t satisfies

$$\begin{aligned} &\inf\{x \in \mathbb{R} \mid x + H \cdot \Delta S_t(\widetilde{\omega}) \geq F_t(\widetilde{\omega}, S_t(\widetilde{\omega})) \quad \forall \widetilde{\omega} \in \Sigma_{t-1}^{\omega}\} && \geq \\ &\inf\{x \in \mathbb{R} \mid x + H \cdot \Delta S_t(\widetilde{\omega}) \geq F_t(\widetilde{\omega}, S_t(\widetilde{\omega})) \quad \forall \widetilde{\omega} \in \Sigma_{t-1}^{\omega} \setminus \{\omega_1, \dots \omega_k\}\} && \geq \quad y \end{aligned} \tag{119}$$

If now the set of $Y := \{y \in \mathbb{R}$ such that (119) holds$\}$ is unbounded from above then an arbitrary S_t satisfies the desired equality.

For any $y > \sup Y$ we are left with two cases: i) $0 \in int(\Gamma_y(\mathbb{S}_t))$ or ii) $0 \notin int(\Gamma_y(\mathbb{S}_t))$ and (118) is satisfied iff $h = 0$.
Start with $y_1 > \sup Y$. Observe first that there exist a finite number of vectors $U_1 := \{s_1, \dots, s_{k_1}\} \subseteq \overline{\Gamma_{y_1}(\mathbb{S}_t)}$ such that in case i) $0 \in int(conv(U_1))$, in case ii) $0 \notin int(conv(U_1))$ but (118) is satisfied for any $s \in U_1$ iff $h = 0$. If $0 \in int(\Gamma_{y_1}(\mathbb{S}_t))$ it is obvious. In case ii) it follows from Claim 3.23 which implies implies $0 \in ri\{A\}$ where $A := \{z \in \mathbb{R}^{d+1} \mid \hat{H} \cdot z = 0\} \cap \overline{\Gamma_y(x)}$.

For any $j = 1, \dots, k_1$, $s_j = \lim_{n \to \infty} s_j^n$ for some $s_j^n \in \Gamma_{y_1}(\mathbb{S}_t)$. If s_j^n eventually belong to $\mathbb{S}_t(\omega_j)$ for some ω_j, the sequence s_j^n can be taken constantly equal to s_j since $\mathbb{S}_t(\omega_j)$ is closed. Moreover, with no loss of generality, if $s_i, s_j \in \Gamma_{y_1}(\mathbb{S}_t)$ we may suppose that the corresponding ω_i, ω_j satisfy

$\mathbb{S}_t(\omega_i) \neq \mathbb{S}_t(\omega_j)$ for $i \neq j$. Indeed, by the previous considerations, having $s_1, \ldots, s_l$ it is possible to find s_{l+1} in $\mathbb{S}_t(\Sigma^{\omega}_{t-1}) \backslash \{\mathbb{S}_t(\omega_1), \ldots \mathbb{S}_t(\omega_l)\}$. If $s_i \in \overline{\Gamma_{y_1}(\mathbb{S}_t)} \backslash \Gamma_{y_1}(\mathbb{S}_t)$ me may suppose that $s_i^n \in \mathbb{S}_t(\omega_i^n)$ with $\omega_i^n \neq \omega_j^m$ for any $m \neq n$, $j \neq i$.
Let $E_1 := \cup_{j=1}^{k_1} \cup_{n=1}^{\infty} \{\omega_j^n \mid s_j^n \in \mathbb{S}_t(\omega_j^n)\}$ and set

$$S_t^1(\omega) := \begin{cases} s_j^n & \text{if } \omega \in \Sigma_t^{\omega_j^n} \\ \hat{S}_t(\omega) & \text{otherwise} \end{cases}$$

where $\hat{S}_t$ is an arbitrary measurable random variable with values in C_t. S_t^1 has the same measurability of $\hat{S}_t$ since they coincide up to an union of countably many measurable sets. Note that by construction

$$\inf\{x \in \mathbb{R} \mid x + H \cdot (S_t^1(\widetilde{\omega}) - S_{t-1}(\widetilde{\omega})) \geq F_t(\widetilde{\omega}, S_t(\widetilde{\omega})) \quad \forall \widetilde{\omega} \in \Sigma^{\omega}_{t-1}\} \geq y_1 \tag{120}$$

Define now $y_n := y_1 + n$. For any $n \in \mathbb{N}$ we can apply the same procedure which yields a collection $\{S_t^n\}_{n\in\mathbb{N}}$ with the property that (120) is satisfied with S_t^n and y_n. Moreover with no loss of generality we can choose $U_{n+1} \supseteq U_n$ and hence $E_{n+1} \supseteq E_n$ in order to have $S_t^{n+1} = S_t^n$ on E_n. We therefore have that $S_t := \lim_{n\to\infty} S_t^n$ is well defined and

$$\inf\{x \in \mathbb{R} \mid x + H \cdot (S_t(\widetilde{\omega}) - S_{t-1}(\widetilde{\omega})) \geq F_t(\widetilde{\omega}, S_t(\widetilde{\omega})) \quad \forall \widetilde{\omega} \in \Sigma^{\omega}_{t-1}\} \geq \sup_n y_n = \infty$$

□

PROOF OF PROPOSITION 3.20. Similarly as in the proof of Proposition 3.18 the function $G : \Omega \times \mathbb{R}^d \mapsto \mathbb{R}$ defined by

$$G(\omega, x) := X_{t-1}(\omega) + H_t(\omega) \cdot (x - \widetilde{S}_{t-1}(\omega)) - F_t(\omega, x) \tag{121}$$

is a Charathéodory map and since $\mathbb{S}_t$ is a closed valued $\mathcal{F}_t$-measurable set, the set

$$Y_t(\omega) := \inf\left\{X_{t-1}(\omega) + H_t(\omega) \cdot (s - \widetilde{S}_{t-1}(\omega)) - F_t(\omega, s) \mid s \in \mathbb{S}_t\right\} \tag{122}$$

is $\mathcal{F}_t$-measurable. Note that since $\mathbb{S}_t(\omega)$ is a bounded set for every $\omega \in \Omega_*$, the infimum is equal to $-\infty$ if and only if $F_t(\omega, s) = \infty$ for every $s \in \mathbb{S}_t(\omega)$. In such a case $X_{t-1}(\omega) = \infty$ and H_t, H_{t+1} can be chosen arbitrarily. We may therefore suppose, without loss of generality, that $Y_t(\omega)$ is a minimum for every $\omega \in \Omega_*$. From Theorem 3.30 in the Appendix, the set $E := \{\omega \in \Omega \mid \exists x \in \mathbb{S}_t(\omega) \text{ with } G(\omega, x) = Y_t(\omega)\}$ is $\mathcal{F}_t$-measurable and there exists a measurable function $m : E \mapsto \mathbb{R}^d$ such that

$$m(\omega) \in \mathbb{S}_t(\omega), \qquad G(\omega, m(\omega)) = Y_t(\omega), \qquad \forall \omega \in E \subseteq \Omega_*. \tag{123}$$

We now show that there exists $H_{t+1} \in \mathcal{L}^0(\mathcal{F}_t)$ such that, for any $\omega \in \Omega_*$, $H_{t+1}(\omega) \in \mathcal{H}_{t+1}(m(\omega))$ and

- if $H_t^i(\omega) < H_{t+1}^i(\omega)$ then $m^i(\omega) = \overline{S}^i(\omega)$
- if $H_t^i(\omega) > H_{t+1}^i(\omega)$ then $m^i(\omega) = \underline{S}^i(\omega)$

and hence the desired random vector is $\widehat{S}_t := m$.

Fix a level set Σ_t^ω with $\omega \in \Omega_*$. For simplicity we set $m := m(\omega)$, as no confusion arise here. We also use the following shorthand: $\bar{\mathbb{S}}_{t+1}$ is the set of $\bar{s} := (\omega_s, s) \in \Sigma_t^\omega \times \mathbb{S}_{t+1}(\Sigma_t^\omega)$ with $s \in \mathbb{S}_{t+1}(\omega_s)$ and, for $\bar{s} \in \bar{\mathbb{S}}_{t+1}$, we denote $\mathbb{X}_{t+1}(\bar{s}) := F_{t+1}(\omega_s, s)$.

Step 1. Observe that for any $\widetilde{H} \in \mathcal{H}_{t+1}(m)$

$$\inf \left\{ F_t(\omega, m) + \widetilde{H} \cdot (\bar{s} - m) - \mathbb{X}_{t+1}(\bar{s}) \mid \bar{s} \in \bar{\mathbb{S}}_{t+1} \right\} = 0 \tag{124}$$

otherwise $\widetilde{H}$ would not be optimal. Since the inner product is continuous there exist a minimizing sequence $\{y_n\}_{n=1}^\infty \subseteq \bar{\mathbb{S}}_{t+1}$ with $y := \lim_{n\to\infty} y_n$ and $\mathbb{X}(y) := \lim_{n\to\infty} \mathbb{X}_{t+1}(y_n)$ such that the minimum is attained i.e.

$$F_t(\omega, m) + \widetilde{H} \cdot (y - m) = \mathbb{X}(y) \tag{125}$$

Let

$$Y := \left\{ \lim_{n\to\infty} y_n \mid \{y_n\}_{n=1}^\infty \subseteq \bar{\mathbb{S}}_{t+1} \text{ and (125) is satisfied } \right\} \tag{126}$$

In a first step we show that, for any $y \in conv(Y)$, $\widetilde{H}$ is still optimal for the price process $(y, \mathbb{S}_{t+1})$, that is, $\widetilde{H} \in \mathcal{H}_{t+1}(y)$.

Take $y := \sum_{i=1}^n \lambda_i y_i \in Y$. The set-valued superhedging price $F_t(\omega, y)$ must satisfy, in particular, the constraints

$$x + \alpha \cdot (y_i - y) \geq \mathbb{X}(y_i) \qquad \forall i = 1, \dots, n$$

and hence $F_t(\omega, y) \geq \sum_{i=1}^n \lambda_i \mathbb{X}(y_i)$. Note however that $\widetilde{H}$ satisfies

$$\begin{aligned} F_t(\omega, m) + \widetilde{H} \cdot (\bar{s} - m) &\geq \mathbb{X}_{t+1}(\bar{s}) \qquad \forall \bar{s} \in \bar{\mathbb{S}}_{t+1} &(127)\\ F_t(\omega, m) + \widetilde{H}(\bar{s} - y) + \widetilde{H} \cdot (y - m) &\geq \mathbb{X}_{t+1}(\bar{s}) \qquad \forall \bar{s} \in \bar{\mathbb{S}}_{t+1} &(128)\\ \sum_{i=1}^n \lambda_i \mathbb{X}(y_i) + \widetilde{H} \cdot (\bar{s} - y_t) &\geq \mathbb{X}_{t+1}(\bar{s}) \qquad \forall \bar{s} \in \bar{\mathbb{S}}_{t+1} &(129) \end{aligned}$$

where the last inequality follows from the fact that (125) holds for every y_i with $i = 1, \dots, n$ and hence

$$F_t(\omega, m) + \widetilde{H} \cdot (y - m) = \sum_{i=1}^n \lambda_i \left(F_t(\omega, m) + \widetilde{H} \cdot (y_i - m) \right) = \sum_{i=1}^n \lambda_i \mathbb{X}(y_i)$$

We have therefore that $\widetilde{H} \in \mathcal{H}_{t+1}(y)$.

Step 2 We now prove that for any $y_0, y_1 \in \mathbb{R}^d$, for any $0 \leq \lambda \leq 1$

$$\mathcal{H}_{t+1}(y_0) \cap \mathcal{H}_{t+1}(y_1) \subseteq \mathcal{H}_{t+1}((1-\lambda)y_0 + \lambda y_1) \tag{130}$$

Denote $y_\lambda := (1-\lambda)y_0 + \lambda y_1$. Let $\widetilde{H} \in \mathcal{H}_{t+1}(y_0) \cap \mathcal{H}_{t+1}(y_1)$. We need to show that $\widetilde{H}$ is optimal for any price process $(y_\lambda, \mathbb{S}_{t+1})$. For $\lambda = 0, 1$ the claim is trivial. Note that in analogy with (128), for any $0 \leq \lambda \leq 1$, the payoff of $\widetilde{H} \cdot (\bar{\mathbb{S}}_{t+1} - y_\lambda)$ dominates $\mathbb{X}_{t+1}(\bar{\mathbb{S}}_{t+1})$ by adding $F_t(\omega, y_0) + \widetilde{H}(y_{\bar{\lambda}} - y_0)$. Suppose that for some $\bar{\lambda} \in (0, 1)$ this is not optimal and hence there exists a dominating strategy $H_{\bar{\lambda}}$ with

$$F_t(\omega, y_{\bar{\lambda}}) < F_t(\omega, y_0) + \widetilde{H}(y_{\bar{\lambda}} - y_0) \tag{131}$$

From

$$\begin{aligned} F_t(\omega, y_{\bar\lambda}) + H_{\bar\lambda}(y_0 - y_{\bar\lambda}) + H_{\bar\lambda}(\bar s - y_0) &\geq \mathbb{X}_{t+1}(\bar s) \qquad \forall \bar s \in \bar{\mathbb{S}}_{t+1} \\ F_t(\omega, y_{\bar\lambda}) + H_{\bar\lambda}(y_1 - y_{\bar\lambda}) + H_{\bar\lambda}(\bar s - y_1) &\geq \mathbb{X}_{t+1}(\bar s) \qquad \forall \bar s \in \bar{\mathbb{S}}_{t+1} \end{aligned}$$

we get

$$F_t(\omega, y_0) \leq F_t(\omega, y_{\bar\lambda}) + H_{\bar\lambda}(y_0 - y_{\bar\lambda}) \tag{132}$$

$$F_t(\omega, y_1) = F_t(\omega, y_0) + \widetilde{H}(y_1 - y_0) \leq F_t(\omega, y_{\bar\lambda}) + H_{\bar\lambda}(y_1 - y_{\bar\lambda}) \tag{133}$$

From (131) and (132) we have $(\widetilde{H} - H_{\bar\lambda})(y_{\bar\lambda} - y_0) > 0$. As $y_{\bar\lambda} - y_0 = \lambda(y_1 - y_0)$ we thus obtain

$$(\widetilde{H} - H_{\bar\lambda})(y_1 - y_0) > 0 \tag{134}$$

Now, from (131) and (133) we get
$\widetilde{H}(y_1 - y_0) < \widetilde{H}(y_{\bar\lambda} - y_0) + H_{\bar\lambda}(y_1 - y_{\bar\lambda})$ from which $\widetilde{H}(y_1 - y_{\bar\lambda}) < H_{\bar\lambda}(y_1 - y_{\bar\lambda})$. Since $y_1 - y_{\bar\lambda} = (1-\lambda)(y_1 - y_0)$ we thus obtain

$$(\widetilde{H} - H_{\bar\lambda})(y_1 - y_0) < 0 \tag{135}$$

Equation (135) clearly contradicts (134).

Step 3 We now conclude the proof of the Proposition. As $H \in \mathcal{H}_t(\omega)$ is fixed, for simplicity, we can translate H in the origin. Denote by

$$\begin{aligned} I_u &:= \{i \in \{1, \dots d\} \mid m^i = \overline{S}^i(\omega)\} \\ I_d &:= \{i \in \{1, \dots d\} \mid m^i = \underline{S}^i(\omega)\} \\ \xi_i &:= \mathbf{1}_{I_u}(i) - \mathbf{1}_{I_d}(i) \end{aligned}$$

and define

$$R := \xi_1[0, \infty) \times, \dots \times \xi_d[0, \infty)$$

where with a slight abuse of notation $\xi_i[0,\infty)$ is either $[0,\infty)$, $(-\infty, 0]$ or $\{0\}$ according to ξ_i being respectively 1,-1 or 0.
Suppose that there is no $\widetilde{H} \in \mathcal{H}_{t+1}(m)$ that meets the requirement that is

$$\mathcal{H}_{t+1}(m) \cap R = \varnothing$$

As $\mathcal{H}_{t+1}(m)$ and R are both closed convex sets in $\mathbb{R}^d$, by Hahn Banach Theorem, there exists $\eta \in \mathbb{R}^d$, $\gamma \in \mathbb{R}$ such that

$$\eta \cdot \widetilde{H} \geq \gamma > \sup_{r \in R} \eta \cdot r \qquad \forall \widetilde{H} \in \mathcal{H}_{t+1}(m)$$

Note that $\forall i \in I_u$ and $\forall \alpha \geq 0$ we have that $\alpha e_i \in R$ where e_i is the i^{th} element of the canonical basis of $\mathbb{R}^d$. Since $\sup_{r \in R} \eta \cdot r$ is bounded from above we infer that $\eta_i \leq 0$ if $i \in I_u$. Similarly $\eta_i \geq 0$ if $i \in I_d$. Any separator η must therefore satisfy

$$\eta_i \leq 0 \quad \text{if} \quad i \in I_u \tag{136}$$

$$\eta_i \geq 0 \quad \text{if} \quad i \in I_d \tag{137}$$

Note moreover that as $0 \in R$

$$\eta \cdot \widetilde{H} > 0 \qquad \forall \widetilde{H} \in \mathcal{H}_{t+1}(m) \tag{138}$$

Denote by $l := d(\mathcal{H}_{t+1}(m), R)$ the distance between the two sets and denote by $\widehat{H}$ the minimizing strategy which exists since $\mathcal{H}_{t+1}(m)$ is closed.
Let $Y = Y(\widehat{H})$ as in (126) in Step 1 and introduce the convex cone $V := co\,(conv\{y - m \mid y \in Y\})$. We show that that the dual cone

$$V^* = co(\widetilde{H} - \widehat{H} \mid \widetilde{H} \in \mathcal{H}_{t+1}(m))$$

satisfies $w \cdot (y - m) \geq 0$ The inclusion $\supseteq$ is obvious since, from (125), any $y \in Y$ satisfies

$$F_t(\omega, m) + \widetilde{H} \cdot (y - m) \geq \mathbb{X}(y) = F_t(\omega, m) + \widehat{H} \cdot (y - m)$$

from which $(\widetilde{H} - \widehat{H}) \cdot (y - m) \geq 0$. For the converse inclusion observe that any $y \in Y$ defines a supporting hyperplane for the set $\mathcal{H}_{t+1}(m) - \widehat{H}$ at 0. Since $w \in V^*$ is in the positive half-space generated by Y there exists $\alpha > 0$ such that $\alpha(w - \widehat{H}) \in \mathcal{H}_{t+1}(m) - \widehat{H}$ from which the claim follows.

Observe now that $\eta \in V^{**} = V$ and hence $\eta = y - m$, for some $y \in Y$. Equations (136) and (137) imply that

$$y_t^i \leq m^i \quad \text{if } i \text{ is such that} \quad m^i = \overline{S}^i(\omega) \tag{139}$$

$$y_t^i \geq m^i \quad \text{if } i \text{ is such that} \quad m^i = \underline{S}^i(\omega) \tag{140}$$

Since $\widehat{H} \in \mathcal{H}_{t+1}(m)$, from Step 1, we have $\widehat{H} \in \mathcal{H}_{t+1}(y)$. Thus, from Step 2, $\widehat{H} \in \mathcal{H}_{t+1}(\lambda m + (1 - \lambda)y)$ is also true for every $0 \leq \lambda \leq 1$. From (139) and (140) there exists λ sufficiently close to 1 such that $y_\lambda := (1 - \lambda)m + \lambda y \in C_t$ and

$$F_t(\omega, y_\lambda) = F_t(\omega, m) + \widehat{H}(y_\lambda - y_0) \tag{141}$$

Note moreover that, by construction, $y_\lambda \in \mathbb{S}_t(\omega)$. By recalling that $\eta = \lambda(y_\lambda - m)$ and by translating back 0 in H, equation (138) implies $\widehat{H} \cdot (y_\lambda - m) > H \cdot (y_\lambda - m)$. In combination with (141) and the fact that $F_t(\omega, m) = X_{t-1}(\omega) - Y_t(\omega) + H \cdot (m - S_{t-1}(\omega))$ from equations (122) and (123), it yields

$$\begin{aligned}
F_t(\omega, y_\lambda) &= F_t(\omega, m) + \widehat{H} \cdot (y_\lambda - m) \\
&> F_t(\omega, m) + H \cdot (y_\lambda - m) \\
&= X_{t-1}(\omega) - Y_t(\omega) + H \cdot (m - S_{t-1}(\omega)) + H \cdot (y_\lambda - m) \\
&= X_{t-1}(\omega) - Y_t(\omega) + H \cdot (y_\lambda - S_{t-1}(\omega))
\end{aligned}$$

which is a contradiction since $y_\lambda \in \mathbb{S}_t(\omega)$ and $Y_t(\omega)$ is a minimum in (122). $\square$

4. Appendix

Let $(\Omega, \mathcal{A})$ a measurable space.

LEMMA 3.25. *Let* $\Psi : \Omega \mapsto 2^{\mathbb{R}^d}$ *a* $\mathcal{A}$*-measurable multifunction. Let* $\varepsilon > 0$ *then*

$$\Psi^\varepsilon : \omega \mapsto \left\{ v \in \mathbb{R}^d \mid v \cdot s \geq \varepsilon \quad \forall s \in \Psi(\omega) \setminus \{0\} \right\}$$

is a $\mathcal{A}$*-measurable multifunction.*

PROOF. see Appendix of [**BFM14**] □

THEOREM 3.26. *[Theorem 14.5* [**RW98**]*] The following are equivalent*

- $\Psi : \Omega \mapsto 2^{\mathbb{R}^d}$ *is a closed valued, $\mathcal{A}$-measurable multifunction*
- Ψ *admits a Castaing representation: there is a countable family $\{\psi_n\}_{n\in\mathbb{N}}$ of $\mathcal{A}$-measurable function $\psi_n : dom\Psi \mapsto \mathbb{R}^d$ such that for any $\omega \in \Omega$*

$$\Psi(\omega) = cl\,\{\psi_n(\omega) \mid n \in \mathbb{N}\}$$

PROPOSITION 3.27. *[Proposition 14.2-11-12* [**RW98**]*] Consider a class of $\mathcal{A}$-measurable set-valued functions. The following operations preserve $\mathcal{A}$-measurability: countable unions, countable intersections (if the functions are closed-valued), finite linear combination, convex/linear/affine hull, generated cone, polar set, closure, cartesian product of a finite number of $\mathcal{A}$-measurable multifunctions.*

THEOREM 3.28. *[Corollary 14.6* [**RW98**]*] A closed-valued measurable mapping always admits a measurable selector.*

LEMMA 3.29. *[Example 14.15 in* [**RW98**]*] Let $F : \Omega \times \mathbb{R}^n \mapsto \mathbb{R}^m$ be a Charatéodory map and let $X(\omega) \subseteq \mathbb{R}^n$ be closed-valued and $\mathcal{A}$-measurable then the following map are $\mathcal{A}$-measurable*

- $\omega \mapsto F(\omega, X(\omega))$
- $\omega \mapsto (X(\omega), F(\omega, X(\omega)))$

THEOREM 3.30. *[Theorem 14.16 in* [**RW98**]*] Let $F : \Omega \times \mathbb{R}^n \mapsto \mathbb{R}^m$ be a Charatéodory map and let $X(\omega) \subseteq \mathbb{R}^n$ and $D(\omega) \subseteq \mathbb{R}^m$ be closed sets that depends measurably on ω. Then the set*

$$E := \{\omega \in \Omega \mid \exists x \in X(\omega) \text{ with } F(\omega, x) \in D(\omega)\}$$

is $\mathcal{F}_t$-measurable and there exists a measurable function $x : E \mapsto \mathbb{R}^d$ such that

$$x(\omega) \in X(\omega) \quad \text{and} \quad F(\omega, x(\omega)) \in D(\omega)\ \forall \omega \in E$$

LEMMA 3.31. *Let $1 \le u \le T$ and Σ^{ω}_{u-1} be the level sets specified by S. Let $X : \Omega \rightrightarrows [-\infty, +\infty]$ and $\Delta\widetilde{S}_u : \Omega \rightrightarrows \mathbb{R}^d$ be multi-functions measurable with respect to $\mathcal{F}$, $\mathcal{F}_u$, respectively. Given a closed valued, $\mathcal{F}_{t-1}$-measurable random set of constraints $C \subseteq \mathbb{R}^d$, the following multi-function is $\mathcal{F}_{u-1}$ measurable*

$$A_C(\omega) = \left\{(H, y) \in C \times \mathbb{R} \mid y + \sum_{i=1}^{d} H^i \Delta\widetilde{S}^i_u(\widetilde{\omega}) \ge X(\widetilde{\omega}) \quad \forall \widetilde{\omega} \in \Sigma^{\omega}_{u-1}\right\},$$

Moreover, denoting with $\Pi_{x_1,\dots,x_d}(\cdot)$ and $\Pi_{x_{d+1}}(\cdot)$ the canonical projection on the first d components and on the $(d+1)^{th}$ component, respectively, we have that

$$X_{u-1} = \min \Pi_{x_{d+1}}(A_C), \qquad \mathcal{H}_u = \Pi_{x_1,\dots,x_d}\left(A_C \cap \left\{\mathbb{R}^d \times X_{u-1}\right\}\right)$$

are also $\mathcal{F}_{u-1}$-measurable multi-functions.

PROOF. First consider the multifunction

$$\psi : \omega \mapsto \left\{\Delta\widetilde{S}_u(\widetilde{\omega}) \times 1 \times X(\widetilde{\omega}) \mid \widetilde{\omega} \in \Sigma^{\omega}_{u-1}\right\} \subseteq \mathbb{R}^{d+2}$$

which is $\mathcal{F}_{u-1}$-measurable multifunction since for any open set $O \subseteq \mathbb{R}^d \times \mathbb{R}^2$

$$\{\omega \in \Omega \mid \psi(\omega) \cap O \neq \varnothing\} = S^{-1}_{0:u-1}\left(S_{0:u-1}(B)\right) \in \mathcal{F}_{u-1}$$

where $B = \left\{\omega \in \Omega \mid \{\Delta\widetilde{S}_u(\omega) \times 1 \times X(\omega)\} \cap O \neq \varnothing\right\} \in \mathcal{F}$ from Proposition 3.27. By preservation of measurability (again Proposition 3.27) the multifunction

$$\psi^*(\omega) := \left\{H \in \mathbb{R}^{d+2} \mid H \cdot y \leq 0 \quad \forall y \in \psi(\omega)\right\}$$

is also $\mathcal{F}_{u-1}$-measurable and thus, the same holds true for $-\psi^* \cap C \times \mathbb{R} \times \{-1\}$. It is easy to see now that $A_C = \Pi_{x_1,\dots,x_{d+1}}(-\psi^* \cap C \times \mathbb{R} \times \{-1\})$ which is measurable from the continuity of the projection maps.

Observe now that the measurability of A_C implies now those of X_{u-1} and $\mathcal{H}_u$. $\bar{A} := \Pi_{x_{d+1}}(A_C)$ is again measurable by the continuity of projections. We have now that by taking the infimum of the real random set $\bar{A}$ the measurability is preserved since, for any $y \in \mathbb{R}$, it easily follows that

$$\left\{\omega \in \Omega \mid \inf\{a \mid a \in \bar{A}(\omega)\} < y\right\} = \left\{\omega \in \Omega \mid \bar{A}(\omega) \cap (-\infty, y) \neq \varnothing\right\} \in \mathcal{F}_{u-1}$$

As in the classical case, the infimum is actually a minimum by repeating the same arguments as in Proposition 2.3 in Chapter 2. Finally $\mathcal{H}_u$ is again $\mathcal{F}_{u-1}$-measurable by preservation of measurability. $\square$

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