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Dipartimento di Matematica Federigo Enriques

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Some applications of functional inequalities to semilinear elliptic equations

Ph.D. Thesis

Advisor
Prof. Bernhard Ruf

Candidate
Tommaso Passalacqua

Ph.D. Coordinator
Prof. Lambertus van Geemen

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Contents

Chapter 1

Introduction

In this thesis we study semilinear problems of the following type

$$\begin{cases} Lu = f(x,u) & \text{in } \Omega \\ B(x,u,D^\alpha u) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.0.1}$$

where Ω is a bounded domain (open set) of $\mathbb{R}^N$ with sufficiently smooth boundary, L is an elliptic differential operator of order two or four, $f(x,u)$ is the forcing term and $B(x,u,D^\alpha u)$ are the boundary conditions. We are interested in proving existence/non-existence of (weak) solutions of problem (1.0.1), regularity of the possible solutions and a priori estimates for solutions. The thesis is divided in two main parts, the first one is dedicated to the Hardy-Sobolev inequalities and some applications, while the second one is more focused on the Trudinger-Moser inequality.

In particular, the first part is divided in two main chapters. The first one is dedicated to the study of second order Hardy-Sobolev inequalities. We consider the second order case of the Hardy-Sobolev inequalities and, moreover, the case in which the origin is inside the domain. Then, there exists a positive constant C such that for all $u \in H_0^2(\Omega)$[1] the following inequality holds

$$\int_\Omega |\Delta u|^2 dx \geq C \left(\int_\Omega \frac{|u|^p}{|x|^\tau} dx \right)^{2/p},$$

with $N \geq 5$, $0 \leq \tau \leq 4$ and $2 \leq p \leq \sigma := 2^*(\tau) := \frac{2(N-\tau)}{N-4}$. We study the possibility to add to the preceding inequality a remainder term. Historically, the study of remainder terms for the Hardy-Sobolev inequalities has been a large field of research. We want to cite, among all the results, the well known results by H. Brezis and L. Nirenberg for the Sobolev inequality [14] and its counterpart for the Hardy inequality [16], by H. Brezis and J. L. Vázquez. We want to generalyze their results to the case of the

[1]The Sobolev space $W^{2,2}(\Omega)$ with Dirichlet boundary conditions.

biharmonic operator $(-\Delta)^2$. The main difficulty, in the fourth order case, is that some of the standard methods for second order elliptic equations are not available, namely symmetry argument and maximum/minimum principles, at least in the case of Dirichlet conditions.

The second chapter is devoted to the study of a specific biharmonic problem with Dirichlet boundary conditions. We consider the Hardy potential $\frac{1}{|x|^4}$ and the related biharmonic problem with Dirichlet conditions

$$\begin{cases} (-\Delta)^2 u = \dfrac{u^{p-1}}{|x|^4} & \text{in } \Omega \\ u = \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \setminus \{0\} \\ u > 0 & \text{in } \Omega, \end{cases}$$

where Ω is a star-shaped domain with respect to the origin $0 \in \partial\Omega$ and

$$1 < p - 1 < 2^* - 1 := \frac{2N}{N-4} - 1 = \frac{N+4}{N-4}.$$

Our aim is to generalize the result by J. Dávila and I. Peral Alonso [22] to the fourth order case. We are able to prove non-existence of (weak) solutions in star-shaped domain but we are not able to prove the existence of positive (weak) solutions in pathological domains, namely dumbbell domains, as done by Dávila and Peral in their paper. As before, the main difficulty here is the fact that maximum principles are not available. Moreover, we have to prove a priori regularity of the solutions. The key ingredient, in the proof of the non-existence, is an a priori estimate, in the spirit of the work of B. Gidas and J. Spruck [38]. Another difficulty here is the position of the origin, which is located on the boundary of Ω and not, as usual, in the interior of the domain Ω.

In the second part, we consider the limiting case of Sobolev embeddings for $W^{1,p}(\Omega)$, that is $p = N$. It is well known by the Trudinger-Moser inequality that the maximum order of integrability for functions in the Sobolev space $W^{1,N}(\Omega)$ is $e^{|u|^{\frac{N}{N-1}}}$. We study a priori estimates in L^∞ for (weak) solutions of the following problem

$$\begin{cases} -\Delta_N u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Δ_N is the $N-$laplacian operator and $f(s)$ has some specific growth conditions. We improve the results obtained in [44] considering more general functions f. Also in this case, we have to prove a priori regularity for solutions. The main problem is that we want to fill the gap in the work [44] between subexponential forcing terms and functions which behave like the exponential. We are able to improve their results but we are not still able to fill completely the gap.

1.1 Hardy-Sobolev inequalities

1.1.1 Hardy-Sobolev inequalities with remainder terms

In this chapter we present a joint work with Bernhard Ruf of Università degli Studi di Milano [53].

Let us consider $N \geq 3$ and the critical Sobolev embedding $W_0^{1,2}(\Omega) \subset L^{2^*}(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^N$ and $2^* = \frac{2N}{N-2}$ denotes the critical Sobolev exponent. It is known that the best embedding constant S_N in the corresponding Sobolev inequality

$$\int_\Omega |\nabla u|^2 dx \geq S_N \left(\int_\Omega |u|^{2^*} dx \right)^{2/2^*},$$

is independent of Ω and is never attained if $\Omega \neq \mathbb{R}^N$. A natural question is whether the preceding inequality remains valid if some suitable lower order terms (so-called remainder terms) are added. Brezis and Nirenberg showed in [14] that this is indeed the case. They proved the following result. Let $1 \leq q < \frac{N}{N-2}$. Then there exists a constant $C = C(\Omega, q) > 0$ such that for all $u \in W_0^{1,2}(\Omega)$

$$\int_\Omega |\nabla u|^2 dx \geq S_N \left(\int_\Omega |u|^{2^*} dx \right)^{2/2^*} + C\|u\|_{L^q}^2.$$

Brezis and Lieb proved in [11] a slightly stronger form with the weak Lebesgue norm in place of the strong Lebesgue norm. Similar results are obtained for the Hardy inequality by Brezis and Vázquez in [16] and for Hardy-Sobolev inequalities by many others authors. Similar questions can be asked about higher order Sobolev and Hardy inequalities. The best Sobolev and Hardy constants are given by

$$S_{k,p}(\Omega) := \inf_{W_0^{k,p}(\Omega)\setminus\{0\}} \frac{\|u\|_{W^{k,p}(\Omega)}^p}{\|u\|_{L^{p^*}(\Omega)}^p}, \quad p^* = \frac{Np}{N-kp},$$

and

$$H_{k,p}(\Omega) := \inf_{W_0^{k,p}(\Omega)\setminus\{0\}} \frac{\|u\|_{W^{k,p}(\Omega)}^p}{\|\frac{u}{|x|^k}\|_{L^p(\Omega)}^p}, \qquad p > 1.$$

The constants $S_{k,p}$ and $H_{k,p}$ are again independent of Ω, and are not attained if $\Omega \neq \mathbb{R}^N$ in the Sobolev case, and never attained in the Hardy case. It is a natural question whether the best embedding constants depend on all these traces or not. It is clear that the best constants computed in the space with Navier boundary conditions are less or equal than the constants computed in the space with Dirichlet boundary conditions, but the question is whether the opposite inequality holds. When $k = p = 2$ the question was answered positively, just for the Sobolev constant, by Van der Vorst in [76]. Also for general k and p the answer is positive, and it was given by Gazzola, Grunau and Sweers in [31]. Moreover Gazzola-Grunau-Sweers showed improvements of higher order Sobolev inequalities in [32].

Let us consider the following second order Hardy-Sobolev inequality

$$\int_\Omega |\Delta u|^2 dx \geq C_{HS}^\tau \left(\int_\Omega \frac{|u|^p}{|x|^\tau} dx \right)^{2/p},$$

for all $u \in W_0^{2,2}(\Omega)$, where $N \geq 5$, $0 \leq \tau \leq 4$ and $2 \leq p \leq \sigma := 2^*(\tau) := \frac{2(N-\tau)}{N-4}$. A priori, the fourth order critical Hardy-Sobolev constants may depend on the domain and on the boundary traces we are considering. By a simple scaling argument it is easy to see that the critical constants for $W^{2,2}(\Omega)$ do not depend on the domain, coherently with the second order case.

We consider the following questions.

1. Does the critical Hardy-Sobolev constant depend on all traces or not?

That is, if we define

$$C_{HS,\vartheta}^\tau(\Omega) := \inf_{W_{0,\vartheta}^{2,2}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\Delta u|^2 dx}{\left(\displaystyle\int_\Omega \frac{|u|^\sigma}{|x|^\tau} dx \right)^{\frac{2}{\sigma}}},$$

is it true that $C_{HS,\vartheta}^\tau(\Omega) = C_{HS}^\tau$ for all $0 \leq \tau \leq 4$?

The answer to this question is positive and is given by the following result.

Theorem 1. *Let $N \geq 5$, and $\Omega \subset \mathbb{R}^N$ a bounded domain containing the origin with boundary $\partial\Omega \in \mathcal{C}^4$, $0 \leq \tau \leq 4$. Then*

$$C_{HS,\vartheta}^\tau(\Omega) = C_{HS}^\tau(\Omega) = C_{HS}^\tau.$$

The second natural question is:

2. Can the preceding inequality be improved by adding some lower order terms, which can depend on the L^q-norm or on the weak L^q-norm of the function u?

The answer is, again, positive and it depends on the boundary conditions. In the case of Navier boundary conditions we have the following result.

Theorem 2. *Let us consider $N \geq 5$, $\Omega \subset \mathbb{R}^N$ a bounded domain containing the origin with $\partial\Omega \in \mathcal{C}^4$, $0 \leq \tau \leq 4$ and $1 \leq q < \frac{N}{N-4}$. Then there exists a constant $C > 0$, $C = C(\Omega, q, \tau)$, such that for any $u \in W_\vartheta^{2,2}(\Omega)$ the following inequality holds*

$$\int_\Omega |\Delta u|^2 dx \geq C_{HS}^\tau \left(\int_\Omega \frac{|u|^\sigma}{|x|^\tau} dx \right)^{2/\sigma} + C\|u\|_{L^q}^2.$$

For the Dirichlet boundary conditions we have a slightly stronger result.

Theorem 3. *Let $N \geq 5$, $\Omega \subset \mathbb{R}^N$ a bounded domain containing the origin with $\partial\Omega \in \mathcal{C}^4$, $0 \leq \tau \leq 4$ and $1 \leq q \leq \frac{N}{N-4}$. Then there exists a constant $C > 0$, $C = C(\Omega, q, \tau)$, such that for any $u \in W_0^{2,2}(\Omega)$ the following inequality holds*

$$\int_\Omega |\Delta u|^2 dx \geq C_{HS}^\tau \left(\int_\Omega \frac{|u|^\sigma}{|x|^\tau} dx \right)^{2/\sigma} + C\|u\|_{L_w^q}^2.$$

1.1.2 A supercritical semilinear biharmonic problem with Hardy potential

In this chapter we present a result in collaboration with María Medina of Universidad Autónoma de Madrid [46].

In the framework of Hardy-Sobolev inequalities, we consider the following particular biharmonic problem with Dirichlet boundary conditions and with a Hardy-type potential

$$\begin{cases} (-\Delta)^2 u = \dfrac{u^{p-1}}{|x|^4} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \setminus \{0\}. \end{cases} \tag{1.1.1}$$

We assume $\Omega \subset \mathbb{R}^N$ a smooth bounded domain with $0 \in \partial\Omega$, $N \geq 5$ and $p-1$ subcritical with respect to the Sobolev embedding and supercritical with respect to the Hardy weight, that is,

$$1 < p - 1 < 2^* - 1 := \frac{2N}{N-4} - 1 = \frac{N+4}{N-4}.$$

The main result we prove is the following.

Theorem 4. *Let Ω star-shaped with respect to the point $0 \in \partial\Omega$. Then the problem* (1.1.1) *has no positive (weak) solutions.*

This result is a generalization of the result in [22]. The key point in the proof is an a priori estimate in the spirit of [22, Lemma 2.2] and [38].

In the second order case the preceding problem with $N \geq 3$ is well studied. In the subcritical case, that is $0 < p - 1 < 1$, it is simple to prove existence of weak solutions, independently of the location of the origin. If $p = 2$, that is in the critical case, the problem was studied by Ghoussoub-Kang in [34] and by Ghoussoub-Robert in [36]. Finally J. Dávila and I. Peral studied in [22] the problem in the supercritical setting, that is $1 < p-1 < \frac{N+2}{N-2}$. Dávila and Peral proved in [22] non existence of positive weak solutions if the domain Ω is star-shaped. There are also some generalizations of problem (4.1.1), for example in the case of more general second order operators, that is in the case of p-Laplacian operator, as in the series of papers [47], [48] and [49].

1.2 A priori estimates for superlinear problems

In this chapter we present a joint work with Bernhard Ruf of Università degli Studi di Milano [54]. We consider the following problem

$$\begin{cases} -\Delta_N u = f(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2.1}$$

where Ω is a strictly convex, bounded and smooth domain in $\mathbb{R}^N$, $N \geq 2$ and $\Delta_N u := \operatorname{div}\left(|\nabla u|^{N-2}\nabla u\right)$ is the N-Laplacian operator. On the function f we assume the following conditions

$$f : \mathbb{R}_+ \to \mathbb{R}_+ \text{ is a locally Lipschitz function;} \tag{1.2.2}$$

$$\exists\, d > 0 \,:\, \liminf_{s\to+\infty} \frac{f(s)}{s^{N-1+d}} > 0; \tag{1.2.3}$$

and

$$\exists\, \sigma > 0, \quad \exists\, C, s_0 > 0 \,:\, f(s) \leq C e^{s/\log^{\sigma}(e+s)} \quad \forall\, s \geq s_0 \tag{1.2.4}$$

or

$$\exists\, 0 < \alpha < 1 \quad \exists\, C_1, C_2, s_0 > 0 \,:\, C_1 \frac{e^s}{(s+1)^{\alpha}} \leq f(s) \leq C_2 e^s \quad \forall\, s \geq s_0. \tag{1.2.5}$$

The main result is the following a priori estimate.

Theorem 5. *Under assumptions* (1.2.2)-(1.2.3)-(1.2.4) *or* (1.2.2)-(1.2.3)-(1.2.5) *there exists a constant $C > 0$ such that every positive weak solution $u \in W_0^{1,N}(\Omega)$ satisfies*

$$\|u\|_{L^\infty(\Omega)} \leq C.$$

The first general result for a priori estimates for superlinear elliptic equation is due to Brezis and Turner [15]. They considered a second order elliptic equation with nonlinearity $f = f(x, u)$ and they proved a priori bounds for positive weak solutions under the assumption

$$0 \leq f(x, s) \leq C s^{p-1} \quad 1 < p - 1 < 2_* - 1 := \frac{N+1}{N-1}.$$

If we restrict to the case of much more regular solutions, that is *classical solutions*, the Brezis-Turner exponent is not critical anymore. Indeed Gidas-Spruck proved in [38] that the a priori estimates hold for positive classical solutions, under the condition that there exists a continous function $a : \overline{\Omega} \to \mathbb{R}$ such that

$$\lim_{s\to+\infty} \frac{f(x,s)}{s^{p-1}} = a(x),$$

uniformly in $x \in \overline{\Omega}$, for $1 < p-1 < 2^* - 1$. A similar result was obtained by de Figueiredo, Lions and Nussbaum in [23].

All the preceding results are for $N \geq 3$ and are based on the fact that, thanks to the Sobolev embedding Theorem, we have $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ for all $1 < q \leq 2^*$, and the embedding is compact for all $q < 2^*$. For $N = 2$ we have the embedding $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ for all $q > 1$, but it is easy to prove that $H_0^1(\Omega) \not\hookrightarrow L^\infty(\Omega)$. Thus, one may ask which is the maximal growth function $g(s)$ such that

$$\int_\Omega g(u)dx < +\infty \quad \forall\, u \in H_0^1(\Omega).$$

This maximal growth is given by the *Trudinger-Moser inequality*, which says, for $N = 2$, that

$$\sup_{\|u\|_{H_0^1(\Omega)}\leq 1} \int_\Omega e^{\alpha u^2} dx \leq C(\Omega) \quad \forall\, \alpha \leq \alpha_N.$$

So, one can ask whether in dimension $N = 2$ it is possible to prove a priori estimates for nonlinearities with growth up to the Trudinger-Moser growth. This is not possible since Brezis and Merle provided in [13] examples of nonlinearities $f(x,s) = h(x)e^{|s|^\alpha}$ with $\alpha > 1$ for which there are no uniform estimates. Moreover, using the result of Brezis-Merle and the boundary estimates of de Figueiredo-Lions-Nussbaum for Ω a convex domain, it is possible to prove a priori estimates for nonlinearities f such that $C_1 e^s \leq f(x,s) \leq C_2 e^s$, $s \geq 0$. Recently, Lorca-Ubilla-Ruf proved in [44] an a priori result for the N-Laplacian in dimension N and for nonlinearities of maximal growth $e^{|s|^\alpha}$ for $\alpha < 1$ in the subcritical case or for $f \sim e^s$ in the critical case, $s \geq 0$. Our result is a direct improvement of their work, since we are able to prove a priori estimates also for nonlinearities f of maximal growth $e^{s/\log^\sigma(e+s)}$ with $\sigma > 0$ in the subcritical case or for nonlinearities f such that $C_1 \frac{e^s}{(s+1)^\alpha} \leq f(s) \leq C_2 e^s$ with $\alpha < 1$ in the critical case, $s \geq 0$.

Part I

Hardy-Sobolev inequalities

Chapter 2

Preliminaries

2.1 First order Hardy-Sobolev inequalities

Sobolev inequalities are well known functional inequalities. They estimate the integrability of a function u in terms of the integrability of its derivatives. They are often used, in the theory of PDEs, in the process of regularization of a solution. We want to consider here the *first order version of the Sobolev inequalities.*

Let $N \geq 2$ and Ω be a bounded domain in $\mathbb{R}^N$ with sufficiently smooth boundary $\partial\Omega$. The Sobolev embedding theorem asserts that if $1 < p < N$ then

$$W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \qquad 1 \leq q \leq p^* := \frac{Np}{N-p}, \tag{2.1.1}$$

that is there exists a positive constant $S_{N,p}$ such that for all $u \in W_0^{1,p}(\Omega)$ the following inequality holds

$$\|u\|_{L^q(\Omega)} \leq S_{N,p} \|\nabla u\|_{L^p(\Omega)} \qquad 1 \leq q \leq p^*. \tag{2.1.2}$$

Equivalently

$$\sup_{\substack{u \in W_0^{1,p}(\Omega) \\ \|\nabla u\|_{L^p}^p \leq 1}} \int_\Omega |u|^p dx < +\infty.$$

Moreover, the embedding in (2.1.1) is compact if $q < p^*$. The exponent p^* is called *Sobolev critical exponent* and the maximal growth $|u|^{p^*}$ for which (2.1.1) holds in the case $p < N$ is called *critical Sobolev growth.* A priori, the constant $S_{N,p}$ in (2.1.2) may also depend on the domain Ω. In [71] and [7] G. Talenti and T. Aubin computed the best constant $S_{N,p}$ for the Sobolev embedding in the whole $\mathbb{R}^N$ in the critical case $q = 2^*$ and they found that

$$S_{N,p}(\mathbb{R}^N) := \pi^{-\frac{1}{2}} N^{-\frac{1}{p}} \left(\frac{p-1}{N-p}\right)^{1-\frac{1}{p}} \left(\frac{\Gamma\left(1+\frac{N}{2}\right)\Gamma(N)}{\Gamma\left(\frac{N}{p}\right) + \Gamma\left(1+N-\frac{N}{p}\right)}\right)^{\frac{1}{N}}.$$

Moreover, they computed the explicit value of functions, often called *Talenti's functions*, for which the equality sign holds in the Sobolev inequalities. These functions have to be of the form

$$u(x) := \left(a + b|x|^{\frac{p}{p-1}}\right)^{1-\frac{N}{p}}.$$

By the Aubin-Talenti result, it is easy to prove that the best constant in the Sobolev inequality (2.1.2) in the critical case $q = p^*$ is independent of the domain Ω and it is never attained if $\Omega \neq \mathbb{R}^N$. Indeed, it is sufficient to note that the norms in (2.1.2) are invariant under the following scaling

$$u \mapsto u_\varepsilon(x) := \varepsilon^{-\frac{N}{q}} u\left(\frac{x}{\varepsilon}\right).$$

Hence, we have $S_{N,p}(\Omega) = S_{N,p}(\mathbb{R}^N)$ for each bounded domain Ω sufficiently smooth. Since in the next section we want to consider higher order Sobolev inequalities with constants depeding also on k[1], we drop the dependence on N and we denote the Sobolev constant with S_p.

In their groundbreaking article [14], H. Brezis and L. Nirenberg showed that the critical elliptic equations associated to the Sobolev embeddings have solutions if they are perturbed with suitable lower order terms. They proved the following result.

Let Ω be a smooth domain in $\mathbb{R}^N$ with $N \geq 3$ and let us consider the following semilinear problem

$$\begin{cases} -\Delta u = u^{2^*-1} + \lambda u & \text{in } \Omega \\ u > 0 & \text{in } \Omega \setminus \{0\} \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1.3}$$

If $N \geq 4$, then for any $\lambda \in (0, \lambda_1)$, with λ_1 the first eigenvalue of the Laplacian operator, there exists a positive weak solution of the preceding problem. If $N = 3$ then there exists a $\lambda^* \in [0, \lambda_1)$ such that for any $\lambda \in (\lambda^*, \lambda_1)$ there exists a positive weak solution. Moreover if $\Omega = B_1(0) \subset \mathbb{R}^3$ they computed $\lambda^* = \frac{\lambda_1}{4}$.

Furthermore, due to the Pohozaev identity [56], the corresponding critical elliptic equation with $\lambda \leq 0$ has no solution if Ω is starshaped. This result had an enormous impact on the study of critical equations, for understanding lack of compactness, for illuminating the phenomena of concentrating solutions, and leading eventually to the solution of the famous Yamabe problem.

A related question is whether critical Sobolev inequality for $p = 2$

$$\int_\Omega |\nabla u|^2 dx \geq S_N \left(\int_\Omega |u|^{2^*} dx\right)^{2/2^*}, \tag{2.1.4}$$

[1]The k denotes the number of derivatives.

remains valid if some suitable lower order terms (so-called *remainder terms*) are added. In [14], among the main result, Brezis and Nirenberg showed that this is indeed the case. They proved the following.

Let $1 \leq q < \frac{N}{N-2}$. *Then there exists a constant* $C = C(\Omega, q) > 0$ *such that*

$$\int_\Omega |\nabla u|^2 dx \geq S_N \left(\int_\Omega |u|^{2^*} dx \right)^{2/2^*} + C \left(\int_\Omega |u|^q dx \right)^{2/q} \quad \text{for all} \quad u \in W_0^{1,2}(\Omega).$$

The following slightly stronger form was obtained by H. Brezis and E. H. Lieb in [12].

Let $\|u\|_{p.w}$ *denote the weak* L^q*-norm of* u *and assume that* $q = \frac{N}{N-2}$. *Then there exists a constant* C *such that*

$$\int_\Omega |\nabla u|^2 dx \geq S_N \left(\int_\Omega |u|^{2^*} dx \right)^{2/2^*} + C \, \|u\|^2_{L^q_w(\Omega)} \quad \text{for all} \quad u \in W_0^{1,2}(\Omega).$$

In this result the remainder term is given by the *weak Lebesgue norm*, defined by

$$\|u\|_{L^p_w(\Omega)} := \left(\sup_{t>0} t^p \delta_u(t) \right)^{1/p},$$

with δ_u the *distribution function of* u

$$\delta_u(t) := \Big| \{x \, : \, |u(x)| > t\} \Big|.$$

We can also use the following equivalent definition of weak Lebesgue norm, due to Calderón, see [41, Equation 1.7],

$$\|u\|_{L^p_w(\Omega)} := \sup_{\substack{A \subset \Omega \\ |A| < +\infty}} \left\{ |A|^{-\frac{1}{q}} \int_A |u| dx \right\}.$$

Another fundamental inequality in mathematical analysis is the (generalized) *Hardy inequality*, see [28],

$$\int_\Omega |\nabla u|^p dx \geq H_p \int_\Omega \left| \frac{u}{|x|} \right|^p dx \quad \text{for all} \quad u \in W_0^{1,p}(\Omega).$$

The critical constant H_p is independent of Ω, and is never attained (not even in $\mathbb{R}^N$). Hardy inequality, sometimes called *Uncertainty Principle*, can be viewed as a weighted version of the Poincaré inequality.

So, it is natural to ask if the Hardy inequality may be improved by adding lower order terms. This is indeed possible, and again the name of Haïm Brezis is connected to the pioneering result; together with L. Vázquez they proved in [16]

Suppose that $0 \in \Omega$, *and let* $1 \le q < \frac{2N}{N-2}$. *Then there exists a constant* $C > 0$ *such that*

$$\int_\Omega |\nabla u|^2 dx \ge H_2 \int_\Omega \Big|\frac{u}{|x|}\Big|^2 dx + C \left(\int_\Omega |u|^q dx\right)^{2/q} \quad \textit{for all} \quad u \in W_0^{1,2}(\Omega).$$

In their paper, Brezis and Vázquez asked if it is possible to find an *infinite improvement* of the Hardy inequality, in the sense of adding an infinite sequence of remainder terms. This question was answered positively by Filippas-Tertikas in [27]. They proved

Suppose that $0 \in \Omega$ *and let* $D \ge \sup_{x\in\Omega} |x|$, *then for any* $u \in H_0^1(\Omega)$ *there holds*

$$\int_\Omega |\nabla u|^2 dx \ge H_2 \int_\Omega \Big|\frac{u}{|x|}\Big|^2 dx + \frac{1}{4}\sum_{i=1}^{\infty} \left(\int_\Omega \frac{|u|^2}{|x|^2} \prod_{k=1}^{k=i} X_k^2\left(\frac{|x|}{D}\right) dx\right)$$

where $X_1(t) = (1 - \log(t))^{-1}$ *for* $t \in (0, 1]$ *and* $X_k(t) = X_1(X_{k-1}(t)$ *for all* $k \in \mathbb{N}$.

Moreover, in [35] Ghoussoub and Moradifam gave a characterization of the optimal allowed perturbation term.

In the article [17] Caffarelli-Kohn-Nirenberg derived a family of sharp *first order interpolation inequalities with weights.* They proved

Let $N \ge 3$ *then there exists a positive constant* C *such that the following inequality holds for all* $u \in \mathcal{C}_0^\infty(\mathbb{R}^N)$

$$\left(\int_{\mathbb{R}^N} |x|^{\gamma r}|u|^r dx\right)^{1/r} \le C \left(\int_{\mathbb{R}^N} |x|^{\alpha p}|\nabla u|^p dx\right)^{a/p} \left(\int_{\mathbb{R}^N} |x|^{\beta q}|u|^q dx\right)^{(1-a)/q},$$

for $p, q, r, \alpha, \beta, \gamma, a$ *real parameters which satisfy some technical conditions.*

A special case are the so-called *Hardy-Sobolev inequalities*

$$\int_\Omega |\nabla u|^p dx \ge C(p, r) \left(\int_\Omega \Big|\frac{u}{|x|^\gamma}\Big|^r dx\right)^{p/r} \quad \text{for all} \quad u \in W_0^{1,p}(\Omega),$$

where $\frac{1}{r} + \frac{\gamma}{N} = \frac{1}{p}$, which are intermediate cases between the Sobolev and the Hardy inequalities. Also in these cases, improvements with lower order terms have recently been proved, see Rădulescu, Smets, and Willem in [60] and Z.-Q. Wang - Willem in [77]. In particular Rădulescu, Smets, and Willem proved the following result, which is a direct generalization of the result by Brezis-Lieb in the more general setting of Hardy-Sobolev inequalities.

For $0 < a < 1$ *there exists a positive constant* C *such that for every* $u \in H_0^1(\Omega)$ *the following inequalitiy holds*

$$\int_\Omega |\nabla u|^2 dx \ge S_a \left(\int_\Omega \frac{|u|^p}{|x|^{ap}} dx\right)^{2/p} + C\|u\|^2_{L^q_w(\Omega)}, \quad q = \frac{N}{N-2}.$$

2.2 Higher order Hardy-Sobolev inequalities

For a bounded domain $\Omega \subset \mathbb{R}^N$ let

$$W_0^{k,p}(\Omega) = \ cl \left\{ u \in \mathcal{C}_0^\infty(\Omega) \ : \ \|u\|_{W^{k,p}(\Omega)} < \infty \right\},$$

where

$$\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{|j| \le k} \|D^j u\|_{L^p(\Omega)}^p \right)^{1/p},$$

denotes the standard Sobolev norm. Moreover, it is well known that

$$\|u\|_{W^{k,p}(\Omega)} := \begin{cases} \|\Delta^h u\|_{L^p(\Omega)} & \text{for } k = 2h \\ \|\nabla(\Delta^h u)\|_{L^p(\Omega)} & \text{for } k = 2h+1, \end{cases}$$

denote equivalent norms on $W_0^{k,p}(\Omega)$. We denote by Δ^h the *polyharmonic operator of order* h, given by

$$(\Delta)^h \, u := \Delta \left(\Delta^{h-1} u \right).$$

In particular, we are interested in the *biharmonic operator* $(-\Delta)^2 = \Delta^2$.

Alternatively, if the boundary $\partial\Omega$ is sufficiently smooth, it is possible to define the traces of a function $u \in W^{k,p}(\Omega)$ as the continuous extensions to the space $W^{k,p}(\Omega)$ of the following linear operators defined on $\mathcal{C}^k(\overline{\Omega})$

$$T_j u := \left. \frac{\partial^j u}{\partial \nu^j} \right|_{\partial\Omega},$$

where ν denotes the unit outer normal to $\partial\Omega$. In this case, we can define equivalently

$$W_0^{k,p}(\Omega) := \bigcap_{j=0}^{k-1} \ker(T_j).$$

Note that for functions $u \in W_0^{k,p}(\Omega)$ all traces $\|D^j(u)\|_{L^p(\partial\Omega)}$ up to order $k-1$ are vanishing. These are the so-called *homogeneous Dirichlet boundary conditions.*

It is also possible to define other closed subspaces of $W^{k,p}(\Omega)$ with different types of boundary conditions. A natural choice is the space of functions with *homogeneous Navier boundary conditions* which is given by

$$W_\vartheta^{k,p}(\Omega) := \left\{ u \in W^{k,p}(\Omega) \ : \ \Delta^j u \,|_{\partial\Omega} = 0 \text{ in the sense of traces } \forall\, 0 \le j < \frac{k}{2} \right\}.$$

Moreover, if $k = p = 2$ it is well known that $H_\vartheta^2(\Omega) = H^2(\Omega) \cap H_0^1(\Omega)$. In this work, we consider only Dirichlet and Navier boundary conditions. For more general boundary conditions and also for a more precise introduction to polyharmonic elliptic problems we refer to the book of Gazzola-Grunau-Sweers [32].

For $k = 1$ the Sobolev inequality has an underlying geometric meaning which is situated in the deep relation between isoperimetric inequalities and best constants for Sobolev embeddings. This is more clear if we consider the case $p = 1$, but is also true for $1 < p < N$. For higher order Sobolev spaces, that is $W^{k,p}(\Omega)$ for $k > 1$, this geometric connection between Sobolev inequalities and isoperimetric inequalities is less clear but the Sobolev embedding theorem still holds. The generic formulation of the Sobolev embeddings theorem says that if $N \geq 2$, Ω a bounded domain in $\mathbb{R}^N$ with sufficiently smooth boundary $\partial\Omega$ and $1 < p < N$ then

$$W_0^{k,p}(\Omega) \hookrightarrow L^q(\Omega) \qquad 1 \leq q \leq p^* := \frac{Np}{N - kp},$$

and the embedding is compact for $q < p^*$. In terms of inequalities this means that there exists a constant $S_{k,p}$ such that for all $u \in W_0^{k,p}(\Omega)$ the following inequality holds

$$\|u\|_{L^q(\Omega)} \leq S_{k,p} \|D^k u\|_{L^p(\Omega)} \qquad 1 \leq q \leq p^*.$$

The same natural extension to higher order Sobolev spaces can be done for the Hardy inequality. The best Sobolev and Hardy constants are then given by

$$S_{k,p}(\Omega) := \inf_{W_0^{k,p}(\Omega)\setminus\{0\}} \frac{\|u\|^p_{W^{k,p}(\Omega)}}{\|u\|^p_{L^{p^*}(\Omega)}},$$

and

$$H_{k,p}(\Omega) := \inf_{W_0^{k,p}(\Omega)\setminus\{0\}} \frac{\|u\|^p_{W^{k,p}(\Omega)}}{\|\frac{u}{|x|^k}\|^p_{L^p(\Omega)}}.$$

The constants $S_{k,p}$ and $H_{k,p}$ are again independent of Ω, and are not attained if $\Omega \neq \mathbb{R}^N$ in the Sobolev case, and never attained in the Hardy case. This is, as in the first order case, a consequence of the invariance under scaling of the norms considered in the inequalities.

It is a natural question whether the best embedding constants depend on *all* these traces or not. In other words, if we consider the Sobolev space with Navier conditions $W_\vartheta^{k,p}(\Omega)$ and we define the Sobolev and Hardy constants in the same way as before, that is

$$S_{k,p,\vartheta}(\Omega) := \inf_{W_\vartheta^{k,p}(\Omega)\setminus\{0\}} \frac{\|u\|^p_{W^{k,p}(\Omega)}}{\|u\|^p_{L^{p^*}(\Omega)}} \ ,$$

and

$$H_{k,p,\vartheta}(\Omega) := \inf_{W_\vartheta^{k,p}(\Omega)\setminus\{0\}} \frac{\|u\|^p_{W^{k,p}(\Omega)}}{\|\frac{u}{|x|^k}\|^p_{L^p(\Omega)}} \ .$$

then it is clear that

$$S_{k,p,\vartheta}(\Omega) \leq S_{k,p}, \quad H_{k,p,\vartheta} \leq H_{k,p} \ ,$$

but the question is whether the opposite inequality holds. When $k = p = 2$ the question was answered positively, just for the Sobolev constant, by Van der Vorst in [76]. Also for general k and p the answer is positive, and was given by Gazzola-Grunau-Sweers in [31]. The result says that for any dimension $N \in \mathbb{N}$ and $1 < p < \frac{N}{k}$, then for a bounded domain Ω with sufficiently smooth boundary we have

$$S_{k,p,\vartheta}(\Omega) = S_{k,p}, \quad H_{k,p,\vartheta}(\Omega) = H_{k,p}.$$

Concerning improvements of higher order Sobolev inequalities with lower remainder terms, we mention recent results by Gazzola-Grunau-Sweers who proved such improvements for the polyharmonic Sobolev inequality, [32, Theorem 7.58, Corollary 7.59 and Theorem 7.60], both for Dirichlet and Navier boundary conditions. They proved the following two results, which generalize both the result of Brezis-Lieb and Brezis-Nirenberg to the case of higher order Sobolev inequalities with both Dirichlet and Navier conditions.

Let $k \in \mathbb{N}^+$ and let Ω a bounded domain in $\mathbb{R}^N$, $N > 2k$. Then there exists a constant $C = C(\Omega, N, k)$ such that

$$\|u\|^2_{H^k_0(\Omega)} \geq S_{k,2}\|u\|^2_{L^{2^*}(\Omega)} + C\|u\|^2_{L^q_w(\Omega)}, \quad q = \frac{N}{N-2k}.$$

Let $k \in \mathbb{N}^+$ and let Ω a bounded $\mathcal{C}^m$-smooth domain in $\mathbb{R}^N$, $N > 2k$. Then for all $p \in [1, q)$, there exists a constant $C = C(\Omega, N, k)$ such that

$$\|u\|^2_{H^k_\vartheta(\Omega)} \geq S_{k,2}\|u\|^2_{L^{2^*}(\Omega)} + C\|u\|^2_{L^p(\Omega)}, \quad q = \frac{N}{N-2k}.$$

There are also analogous improvements of the Hardy inequality for the biharmonic operator, see Tertikas-Zographopoulos [73], Gazzola-Grunau-Mitidieri [30] and Yao-Shen-Chen [79].

A generalization of the result by Caffarelli-Kohn-Nirenberg done by C. S. Lin [43] is the following.

Let $N \geq 3$ then there exists a positive constant C such that the following inequality holds for all $u \in \mathcal{C}^\infty_0(\mathbb{R}^N)$

$$\left(\int_{\mathbb{R}^N} |x|^{\gamma r}|D^j u|^r dx\right)^{1/r} \leq C\left(\int_{\mathbb{R}^N} |x|^{\alpha p}|D^m u|^p dx\right)^{a/p}\left(\int_{\mathbb{R}^N} |x|^{\beta q}|u|^q dx\right)^{(1-a)/q},$$

for $p, q, r, \alpha, \beta, \gamma, a$ real parameters and j, m integers such that they satisfied some technical conditions.

As a Corollary of the result by Lin we can derive, in particular, the following second order Hardy-Sobolev inequalities

$$\int_\Omega |\Delta u|^2 dx \geq C \left(\int_\Omega \frac{|u|^p}{|x|^\tau} dx \right)^{2/p}, \quad u \in W_0^{2,2}(\Omega), \tag{2.2.1}$$

where $N \geq 5$, $0 \leq \tau \leq 4$ and

$$2 \ \leq \ p \ \leq \ \sigma \ := \ 2^*(\tau) \ := \ \frac{2(N-\tau)}{N-4} \ .$$

Inequality (2.2.1) is an interpolation inequality between the Sobolev and the Hardy inequality, and appeared in this form in Yao-Shen-Chen [79] and Yao-Wang-Shen [80].

We can define the *critical Hardy-Sobolev constant* as the largest constant such that (2.2.1) holds for any $u \in W_0^{2,2}(\Omega)$ in the critical case $p = \sigma = 2^*(\tau)$ or, equivalently, as

$$C_{HS}^\tau(\Omega) := \inf_{W_0^{2,2}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\Delta u|^2 dx}{\left(\displaystyle\int_\Omega \frac{|u|^\sigma}{|x|^\tau} dx \right)^{2/\sigma}},$$

The critical Hardy-Sobolev constant does *not* depend on the domain Ω (see below) and so if we define

$$C_{HS}^\tau := \inf_{\mathcal{D}^{2,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^N} |\Delta u|^2 dx}{\left(\displaystyle\int_{\mathbb{R}^N} \frac{|u|^\sigma}{|x|^\tau} dx \right)^{2/\sigma}},$$

with

$$\mathcal{D}^{2,2}(\mathbb{R}^N) := cl \left\{ u \in C_0^\infty(\mathbb{R}^N) \ : \ \int_{\mathbb{R}^N} |\Delta u|^2 < \infty \right\},$$

we have $C_{HS}^\tau(\Omega) = C_{HS}^\tau$.

We consider then the following two natural questions.

1. Does the critical Hardy-Sobolev constant depend on all traces or not, that is, if we define

$$C_{HS,\vartheta}^\tau(\Omega) := \inf_{W_{0,\vartheta}^{2,2}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\Delta u|^2 dx}{\left(\displaystyle\int_\Omega \frac{|u|^\sigma}{|x|^\tau} dx \right)^{2/\sigma}},$$

is it true that

$$C_{HS,\vartheta}^\tau(\Omega) = C_{HS}^\tau \ , \quad \text{for all } \ 0 \leq \tau \leq 4?$$

2. Can inequality (2.2.1) be improved by adding some lower order terms, which can depend on the L^q-norm or on the weak L^q-norm of the function u?

2.3 A conjecture by Pucci and Serrin

The problem of adding remainder terms to functional inequalities like Hardy-Sobolev inequalities is not only interesting by itself but it is also related to a conjecture of P. Pucci and J. Serrin. Let us denote with $\lambda_{1,k}$ the first eigenvalue of $(-\Delta)^k$ under homogeneous Dirichlet boundary conditions. As we said before, H. Brezis and L. Nirenberg proved that in the case $k = 1$ if Ω is a ball then there exists a positive radial solution of (2.1.3) for every $\lambda \in (0, \lambda_{1,1})$ if $N \geq 4$ and for every $\lambda \in \left(\frac{\lambda_{1,1}}{4}, \lambda_{1,1}\right)$ if $N = 3$. Moreover, in this second case, they proved that problem (2.1.3) has no nontrivial radial solution if $\lambda \leq \frac{\lambda_{1,1}}{4}$. We can consider the polyharmonic version of problem (2.1.3) as the following.

$$\begin{cases} (-\Delta)^k u = |u|^{2^*-2}u + \lambda u & \text{in } \Omega \\ u \not\equiv 0 & \text{in } \Omega \\ D^\alpha u = 0 & \text{on } \partial\Omega, \quad \forall\, |\alpha| \leq k-1. \end{cases} \tag{2.3.1}$$

P. Pucci and J. Serrin in [58] raised the question in which way this critical behavior of certain dimensions depends on the order $2k$ of the semilinear problems (2.3.1). Let $\Omega \subset \mathbb{R}^N$ be a ball. The dimension N is called *critical* with respect to problem (2.3.1) if there exists a positive λ^* such that if there exists a nontrivial radial solution of problem (2.3.1) then $\lambda > \lambda^*$, or equivalently if $\lambda \leq \lambda^*$ then there are no nontrivial radial solutions of problem (2.3.1). Brezis and Nirenberg proved in [14] that $N = 3$ is critical for second order problems, so for $k = 1$. Pucci and Serrin proved that for all $k \in \mathbb{N}$ the dimension $N = 2k + 1$ is critical and moreover if we consider $k = 2$ then also dimensions $N = 6, 7$ are critical. So, they conjectured:

the critical dimensions for problem (2.3.1) *are exactly* $N = 2k + 1, ..., 4k - 1$.

Hence, the conjecture is proved completely for $k = 1, 2$ by the work of Brezis-Nirenberg and Pucci-Serrin. For a generic $k > 2$, only the fact that $N = 2k+1$ is critical is known.

By the well known result by Gidas-Ni-Nirenberg [37], for $k = 1$ and $\lambda \geq 0$ if the domain Ω is radial it is equivalent to consider positive solutions or positive radial solutions of problem (2.3.1). A generalization of the result by Gidas-Ni-Nirenberg to polyharmonic operators is given by [32, Theorem 7.1] and hence also for a generic k it is true that it is equivalent to consider positive solutions or positive radial solutions of problem (2.3.1). Hence, Gazzola-Grunau-Sweers proposed a weakened version of the Pucci-Serrin conjecture. The dimension N is called *weakly critical* with respect to problem (2.3.1) if there exists a positive λ^* such that if there exists a positive solution of problem (2.3.1) then $\lambda > \lambda^*$.

In this weakened formulation, the conjecture is proved by Gazzola-Grunau-Sweers and it is strictly related with the existence of remainder terms in the Sobolev inequalities.

For more details about the Pucci-Serrin conjecture and its weakened version by Gazzola-Grunau and Sweers we refer to [32] and [58].

Since symmetry results, like Gidas-Ni-Nirenberg, hold also for semilinear problems with Hardy-Sobolev type potentials, which is opposite to what happens for Hénon type potentials in which symmetry breaking phenoma may occur, the question to add remainder terms to Hardy-Sobolev inequalities in any order k is possibly related to a weakened version of the preceding conjectures for more generic polyharmonic problems.

Chapter 3

Hardy-Sobolev inequalities for the biharmonic operator with remainder terms

The results written in this chapter are collected in the paper [53]. The techniques we use to prove the following results are a combination of old techniques, coming from the paper of Brezis-Nirenberg [14] and Brezis-Lieb [12] and some adaptations to polyharmonic problems, see for example [32]. In particular it is important to remark that the situation with Navier boundary conditions is easier than the case with Dirichlet boundary conditions. Indeed, polyharmonic problems with homogeneous Navier boundary conditions can be treated as systems of coupled harmonic problems, leading to a substantially easier argument. In the Dirichlet conditions case, on the opposite, the problem is inherently a fourth order problem and some of the well known techniques of second order elliptic problems, like maximum principles and symmetrization, are not true.

3.1 A Talenti comparison principle

We recall here a few basic concepts about *symmetrization* and *rearrangements*. A more detailed treament of this argument can be found in the book of S. Kesavan [40].

Given a bounded measurable set $\Omega \subset \mathbb{R}^N$ and a measurable function $u : \Omega \to \mathbb{R}$, we define the *distribution function* of u as

$$\delta_u(t) := |\{x \in \Omega \ : \ u(x) > t\}| \,.$$

The *(unidimensional) decreasing rearrangement* of u is hence defined as the function

$u^\sharp : [0, |\Omega|] \to \mathbb{R}$ such that

$$u^\sharp(t) := \begin{cases} \operatorname{esssup}_\Omega (u) & \text{for } t = 0 \\ \inf\{s \ : \ \delta_u(s) < t\} & \text{for } t > 0. \end{cases}$$

Given a set Ω with finite measure, we denote with Ω^* the open ball centered at the origin and having the same measure of Ω, i.e $|\Omega^*| = |\Omega|$. Moreover, we denote with ω_N the measure of the unit ball in $\mathbb{R}^N$. Finally, the *Schwarz symmetrization* or *spherically symmetric and decreasing rearrangement* of u is the function $u^* : \Omega^* \to \mathbb{R}$ such that

$$u^*(x) := u^\sharp\left(\omega_N |x|^N\right) \quad \forall\, x \in \Omega^*.$$

Among all the well known properties of rearrangements, we recall here two of the most important. Given a non-negative Borel measurable function $F : \mathbb{R} \to \mathbb{R}$, then

$$\int_{\Omega^*} F(u^*(x))dx = \int_\Omega F(u(x))dx.$$

In particular this implies that

$$\|u^*\|_{L^p(\Omega^*)} = \|u\|_{L^p(\Omega)} \quad \forall\, 1 \le p < +\infty.$$

The second one is the famous inequality by G. Pólya and G. Szegő, see [57].
Let $1 \le p < +\infty$ and let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $u \in W_0^{1,p}(\Omega)$ such that $u \ge 0$. Then

$$\int_{\Omega^*} |\nabla u^*(x)|^p dx \le \int_\Omega |\nabla u(x)|^p dx.$$

A crucial tool in the proof of our results about Hardy-Sobolev inequalities is the following result due to G. Talenti [72, Theorem 1]. Altough the result presented here is not original, we include the higher order version given in [31, Proposition 3] with a detailed proof, in order to make the dissertation more self-contained.

Proposition 3.1.1. *Let $N \ge 2$, $\Omega \subset \mathbb{R}^N$ be a bounded domain with $\partial\Omega \in \mathcal{C}^k$ such that $|\Omega| = \omega_N$. Let $r \ge \frac{2N}{N+2}$ and $k = 2h$. Let $f \in L^r(\Omega)$ and $u \in W_\vartheta^{k,r}(\Omega)$ be the unique strong solution to*

$$\begin{cases} (-\Delta)^h u = f & \text{in } \ \Omega, \\ \Delta^j u = 0 & \text{on } \partial\Omega,\ \forall\, j = 0, ..., h-1. \end{cases} \tag{3.1.1}$$

Let $f^* \in L^r(\Omega^*)$, $u^* \in W_0^{1,r}(\Omega^*)$ and let $v \in W_\vartheta^{k,r}(\Omega^*)$ be the unique strong solution to

$$\begin{cases} (-\Delta)^h v = f^* & \text{in } \Omega^*, \\ \Delta^j v = 0 & \text{on } \partial\Omega^*,\ \forall\, j = 0, ..., h-1. \end{cases} \tag{3.1.2}$$

Then $v \ge u^*$ a.e. in Ω^*.

Proof. We proceed by finite induction. For $h = 1$ the result is exactly [72, Theorem 1]. Hence we consider $h \geq 2$. As we said before, with Navier boundary conditions is often convenient to rewrite a polyharmonic problem as a system of coupled harmonic problems. Hence, we may rewrite problems (3.1.1) and (3.1.2) as

$$\begin{cases} -\Delta u_1 = f & \text{in } \Omega \\ u_1 = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} -\Delta u_i = u_{i-1} & \text{in } \Omega \\ u_i = 0 & \text{on } \partial\Omega, \end{cases} \qquad i = 2, ..., h, \tag{3.1.3}$$

$$\begin{cases} -\Delta v_1 = f^* & \text{in } \Omega^* \\ v_1 = 0 & \text{on } \partial\Omega^*, \end{cases} \qquad \begin{cases} -\Delta v_i = v_{i-1} & \text{in } \Omega^* \\ v_i = 0 & \text{on } \partial\Omega^*, \end{cases} \qquad i = 2, ..., h. \tag{3.1.4}$$

Clearly $u_h = u$ and $v_h = v$. Applying the result for $i = 1$, we know that $v_1 \geq u_1^*$ in Ω^*. Assume that $v_i \geq u_i^*$ for $i = h - 1$. Then, by (3.1.3) and (3.1.4), we have that

$$\begin{cases} -\Delta u_{i+1} = u_i & \text{in } \Omega \\ u_{i+1} = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} -\Delta v_{i+1} = v_i & \text{in } \Omega^* \\ v_{i+1} = 0 & \text{on } \partial\Omega^*. \end{cases} \tag{3.1.5}$$

By combining equation (3.1.5) with the maximum principle for the Laplacian and a further application of the Talenti result we have that $v \geq u^*$ a.e. in Ω^*. □

As pointed out in [31, Remark 4], to apply the Talenti original result we need only that the boundary of Ω is $\mathcal{C}^{1,1}$ and not $\mathcal{C}^k$. In this case, the solution of the problem with right hand side $f \in L^r(\Omega)$ is in $W^{2,r}_\vartheta(\Omega)$. Nevertheless, if we do not have more regularity on $\partial\Omega$, the solution of (3.1.1) is not in $W^{k,r}_\vartheta(\Omega)$.

3.2 The dual cone decomposition of Moreau

We discuss here an abstract result by J. J. Moreau in [50] about the decomposition of a generic Hilbert space into dual cones. Altough this result is well known, we report here the full detailed proof in order to make the work more self-contained. We recall that a *cone* in a real Hilbert space is defined as a subset $\mathcal{X} \subset H$ such that if $u \in \mathcal{X}$ and $a \geq 0$ is a scalar then $au \in \mathcal{X}$. Given a cone $\mathcal{X}$ in a real Hilbert space H we define its *dual cone* as

$$\mathcal{X}^* := \{w \in H \,:\, (w, v)_H \leq 0 \quad v \in \mathcal{X}\}\,.$$

Then we have the following decomposition of an Hilbert space into dual cones, due to Moreau. See [50] and also [32] for more considerations about the dual cone decomposition and its applications.

Proposition 3.2.1. *Let H a Hilbert space. Let $\mathcal{X} \subset H$ be a closed convex nonempty cone and let $\mathcal{X}^*$ be its dual cone. Then for any $u \in H$ there exists a unique pair $(u_1, u_2) \in \mathcal{X} \times \mathcal{X}^*$ such that*

$$u = u_1 + u_2, \quad (u_1, u_2)_H = 0.$$

Moreover if we decompose $u, v \in H$ in $u = u_1 + u_2$ and $v = v_1 + v_2$ then we have

$$\|u - v\|_H^2 \ \geq \|u_1 - v_1\|_H^2 + \|u_2 - v_2\|_H^2.$$

In particular, the projection onto $\mathcal{X}$ is Lipschitz-continuous with constant 1.

Proof. Let $u \in H$ fixed. Let u_1 the projection of u onto $\mathcal{X}$ defined as

$$\|u - u_1\|_H := \min_{v \in \mathcal{X}} \|u - v\|_H,$$

and let $u_2 := u - u_1$. Then for all $t \geq 0$ and $v \in \mathcal{X}$ we have, by the definition of a cone and of u_1, that

$$\|u - u_1\|_H^2 \leq \|u - (u_1 + tv)\|_H^2 = \|u - u_1\|_H^2 - 2t(u - u_1, v)_H + t^2\|v\|_H^2,$$

so that

$$2t(u_2, v)_H \leq t^2\|v\|_H^2.$$

Then, dividing the preceding expression by t and letting $t \searrow 0$, we obtain that

$$(u_2, v)_H \leq 0 \quad \forall\, v \in \mathcal{X},$$

and hence $u_2 \in \mathcal{X}^*$. Choosing $v = u_1$ allows us to take $t \in (-1, 0]$ and then, dividing by $t < 0$ and letting $t \nearrow 0$, we have that

$$(u_2, u_1)_H \geq 0.$$

Hence, we have that

$$(u_2, u_1) = 0,$$

and this proves the existence.

Now we prove the Lipschitz continuity of the projection. We take $u, v \in H$ and we consider $u = u_1 + u_2$ and $v = v_1 + v_2$. Then by the inequalities

$$(u_1, v_2)_H \leq 0, \quad (v_1, u_2)_H \leq 0,$$

and by the orhtogonality, we obtain

$$\begin{aligned}
\|u - v\|_H^2 &= (u_1 + u_2 - v_1 - v_2, u_1 + u_2 - v_1 - v_2)_H \\
&= ((u_1 - v_1) + (u_2 - v_2), (u_1 - v_1) + (u_2 - v_2))_H \\
&= \|u_1 - v_1\|_H^2 + \|u_2 - v_2\|_H^2 + 2(u_1 - v_1, u_2 - v_2)_H \\
&= \|u_1 - v_1\|_H^2 + \|u_2 - v_2\|_H^2 - 2(u_1, v_2)_H - 2(v_1, u_2)_H \\
&\geq \|u_1 - v_1\|_H^2 + \|u_2 - v_2\|_H^2,
\end{aligned}$$

and then we have the Lipschitz continuity. By the Lipschitz continuity, taking $u = v$, we obtain the uniqueness of the decomposition. $\square$

The dual cone decomposition by Moreau is a generalization of the standard decomposition of a function u in its positive and negative part, u_+ and u_- respectively. To see this fact it is sufficient to consider as $\mathcal{X}$ the positive cone in H, that is

$$\mathcal{X} := \Big\{ u \in H \,:\, u \geq 0 \text{ a.e } \Big\},$$

for $H = \big\{L^2(\Omega), H_0^1(\Omega), ..., H_0^k(\Omega)\big\}$. If $H = L^2(\Omega)$ then $\mathcal{X}^* = -\mathcal{X}$ and then the dual cone decomposition is the standard decomposition

$$u = u_+ - u_-.$$

If $H = H_0^1(\Omega)$ then $v \in \mathcal{X}^*$ if and only if

$$\int_\Omega \nabla u \cdot \nabla v dx \leq 0 \quad \forall\, u \in \mathcal{X}.$$

Therefore

$$\mathcal{X}^* = \Big\{ v \in H_0^1(\Omega) \,:\, v \text{ is weakly subharmonic} \Big\} \subsetneq -\mathcal{X}.$$

Then, altough

$$\int_\Omega \nabla u_+ \cdot \nabla u_- dx \leq 0,$$

the decomposition obtained in $H_0^1(\Omega)$ is different from the decomposition in positive and negative part. In higher order Sobolev spaces the decomposition in u_+ and u_- is no longer admissible since if $u \in H^k(\Omega)$ then a priori $u_+, u_- \notin H^k(\Omega)$. In general, if $H = H_0^k(\Omega)$ then

$$\mathcal{X}^* = \Big\{ v \in H_0^k(\Omega) \,:\, (-\Delta)^k v \leq 0 \text{ weakly} \Big\}.$$

3.3 The Boggio formula

Here we present the well known result by T. Boggio in [10]. We do not report the proof of this result because it is extremely technical. We refer to the book [32] for a complete analysis of the Boggio formula and its consequences.

Let us consider a generic polyharmonic boundary problem with homogeneous Dirichlet boundary conditions

$$\begin{cases} (-\Delta)^k u = f & \text{in } \Omega \\ D^\alpha u = 0 & \text{on } \partial\Omega \quad \forall\, |\alpha| \leq k-1, \end{cases} \tag{3.3.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain and the datum f is in a suitable functional space. In order to give an explicit solution of the problem (3.3.1) we have to compute the *fundamental solution* of $(-\Delta)^k$ in $\mathbb{R}^N$. We define

$$\Phi_{k,N}(x) := \begin{cases} A_{k,N}|x|^{2k-N} & \text{if } N > 2k \text{ or N is odd} \\ B_{k,N}|x|^{2k-N}(-\log|x|) & \text{if } N \leq 2k \text{ is even,} \end{cases}$$

with $A_{k,N}$ and $B_{k,N}$ two constants, so that, in distributional sense,

$$(-\Delta)^k \Phi_{k,N} = \delta_0.$$

Thanks to the fundamental solution one may define the *Green function* $G_{k,N}(x,y)$ for a domain Ω. Formally, the unique solution of problem (3.3.1) is given by

$$u(x) = \int_\Omega G_{k,N}(x,y) f(y) dy.$$

If the datum f is in a suitable functional space then the preceding equation is well defined and it gives the explicit formula of the unique solution. As in the second order case, the explicit representation of the Green function is not easily determined. T. Boggio computed the Green function for the unitary ball in $\mathbb{R}^N$.

Proposition 3.3.1. *The Green function for the Dirichlet problem* (3.3.1)*, with* Ω *the unitary ball, is positive and given by*

$$G_{k,N}(x,y) := C_{k,N}|x-y|^{2k-N} \int_1^{\left||x|y-\frac{x}{|x|}\right|/|x-y|} (v^2-1)^{k-1} v^{1-N} dv.$$

The positive constants $C_{k,N}$ *are defined by*

$$C_{K,N} := \frac{\Gamma\left(1+\frac{N}{2}\right)}{N\pi^{\frac{N}{2}} 4^{k-1}((k-1)!)^2} \; .$$

The Boggio formula is important for two facts. The first one is that it gives us the explicit formula of the Green function in the unitary ball. We use this fact to show the improved Hardy-Sobolev inequalities with Dirichlet boundary conditions. Moreover, the Boggio formula says also that the Green function in the ball is positive. This is crucial when we want to consider maximum principles. As we said before, polyharmonic boundary problems with Navier conditions can be solved as systems of coupled harmonic problems. This is important because for harmonic problems maximum principles are well known and hence they also hold for polyharmonic problems with Navier boundary conditions. In the Dirichlet boundary conditions case, this is not true anymore.

We say that problem (3.3.1) has the *positivity preserving property* when for all u and f satistying (3.3.1) we have that

$$f \geq 0 \quad \Longrightarrow \quad u \geq 0.$$

In the second order case for regular domain this is given by the maximum principle. In general, verifying the positivity preserving of a domain is an hard problem. Nevertheless, in case that the Green function exists, the positivity preserving property of the domain Ω holds true if and only if the Green function for the domain Ω is non-negative. Then, the Boggio formula says that the positivity preserving property holds also in the polyharmonic case with Dirichlet boundary conditions for the balls in $\mathbb{R}^N$.

3.4 Independence of the critical optimal Hardy-Sobolev constants on the domain and on the traces

We recall that we are interested in the following second order Hardy-Sobolev inequality

$$\int_\Omega |\Delta u|^2 dx \geq C \left(\int_\Omega \frac{|u|^p}{|x|^\tau} dx \right)^{2/p}, \quad u \in W_0^{2,2}(\Omega), \tag{3.4.1}$$

where $N \geq 5$, $0 \leq \tau \leq 4$ and

$$2 \leq p \leq \sigma := 2^*(\tau) := \frac{2(N-\tau)}{N-4}. \tag{3.4.2}$$

We remember also the definition of the critical Hardy-Sobolev constant with Dirichlet boundary conditions, that is the largest constant such that inequality (3.4.1) holds

$$C_{HS}^\tau(\Omega) := \inf_{W_0^{2,2}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\Delta u|^2 dx}{\left(\displaystyle\int_\Omega \frac{|u|^\sigma}{|x|^\tau} dx \right)^{2/\sigma}}, \tag{3.4.3}$$

and also the definition of the Hardy-Sobolev constant with Navier boundary conditions

$$C_{HS,\vartheta}^\tau(\Omega) := \inf_{W_{0,\vartheta}^{2,2}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\Delta u|^2 dx}{\left(\displaystyle\int_\Omega \frac{|u|^\sigma}{|x|^\tau} dx \right)^{2/\sigma}}, \tag{3.4.4}$$

and finally the definition of the Hardy-Sobolev constant on $\mathbb{R}^N$

$$C_{HS}^\tau := \inf_{\mathcal{D}^{2,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^N} |\Delta u|^2 dx}{\left(\displaystyle\int_{\mathbb{R}^N} \frac{|u|^\sigma}{|x|^\tau} dx \right)^{2/\sigma}}, \tag{3.4.5}$$

with

$$\mathcal{D}^{2,2}(\mathbb{R}^N) := cl\left\{u \in \mathcal{C}_0^\infty(\mathbb{R}^N)\ :\ \int_{\mathbb{R}^N} |\Delta u|^2 < \infty\right\}.$$

A priori, the second order critical Hardy-Sobolev constants may depend on the domain and on the boundary traces we are considering; but this is not the case, as mentioned in Chapter 2. We prove first that the critical constants for $W^{2,2}(\Omega)$ do *not* depend on the domain.

Proposition 3.4.1. *Let $N \geq 5$, and $\Omega \subset \mathbb{R}^N$ a bounded domain containing the origin with boundary $\partial\Omega \in \mathcal{C}^4$, $0 \leq \tau \leq 4$. Then*

$$C_{HS}^\tau(\Omega) = C_{HS}^\tau.$$

Proof. By the Dirichlet boundary conditions we can extend any function $u \in W_0^{2,2}(\Omega)$ by zero outside Ω obtaining a function in $\mathcal{D}^{2,2}(\mathbb{R}^N)$. So we have that $C_{HS}^\tau(\Omega) \geq C_{HS}^\tau$. Conversely, if $\{u_j\}$ is a minimizing sequence for the critical Hardy-Sobolev constant in $\mathcal{D}^{2,2}(\mathbb{R}^N)$, that is

$$C_{HS}^\tau(\mathbb{R}^N) = \liminf_{j\to+\infty} \frac{\displaystyle\int_{\mathbb{R}^N} |\Delta u_j|^2 dx}{\left(\displaystyle\int_{\mathbb{R}^N} \frac{|u_j|^\sigma}{|x|^\tau} dx\right)^{2/\sigma}}$$

we can suppose by density that $\{u_j\}$ is in $\mathcal{C}_0^\infty(\mathbb{R}^N)$. Now let us consider the scaling

$$u \mapsto u_\varepsilon(x) := \varepsilon^{-\frac{4-\tau}{\sigma-2}} u\left(\frac{x}{\varepsilon}\right). \tag{3.4.6}$$

Scaling the sequence $\{u_j\}$ for sufficiently small ε_j we find that the sequence of rescaled function $\{v_j\} := \left\{u_{\varepsilon_j}\right\}$ is in $\mathcal{C}_0^\infty(\Omega)$. But the quotient

$$Q(u) := \frac{\displaystyle\int |\Delta u|^2 dx}{\left(\displaystyle\int \frac{|u|^\sigma}{|x|^\tau} dx\right)^{2/\sigma}}$$

is invariant under the scaling (3.4.6). Indeed

$$\begin{aligned}
\int_\Omega |\Delta u_\varepsilon(x)|^2 dx &= \int_\Omega \left|\Delta\left(\varepsilon^{-\frac{4-\tau}{\sigma-2}} u\left(\frac{x}{\varepsilon}\right)\right)\right|^2 dx \\
&= \varepsilon^{-2\frac{4-\tau}{\sigma-2}} \int_\Omega \varepsilon^{N-4} |\Delta u(y)|^2 dy \\
&= \varepsilon^{-2\frac{4-\tau}{\sigma-2}+N-4} \int_\Omega |\Delta u|^2 dx,
\end{aligned}$$

and

$$\begin{aligned}\left(\int \frac{|u|^\sigma}{|x|^\tau}dx\right)^{2/\sigma} &= \varepsilon^{-2\frac{4-\tau}{\sigma-2}}\left(\int_\Omega \frac{|u(y)|^\sigma}{|\varepsilon y|^\tau}\varepsilon^N dy\right)^{2/\sigma}\\ &= \varepsilon^{-2\frac{4-\tau}{\sigma-2}+\frac{2}{\sigma}(N-\tau)}\left(\int_\Omega \frac{|u|^\sigma}{|x|^\tau}dx\right)^{2/\sigma}.\end{aligned}$$

Hence

$$\begin{aligned}Q(u_\varepsilon) &= \varepsilon^{N-4-\frac{2}{\sigma}(N-\tau)}Q(u)\\ &= Q(u),\end{aligned}$$

since

$$\sigma = \frac{2(N-\tau)}{N-4}.$$

Then we have that

$$C^\tau_{HS}(\Omega) \le \liminf_{j\to\infty} Q(v_j) = C^\tau_{HS}.$$

□

Hence, we can drop the dependence on the domain in the critical Hardy-Sobolev constants, writing C^τ_{HS}.

The second question we want to answer is about the dependence on the traces. We prove that the critical Hardy-Sobolev constant do not depend on all the traces in the space $W^{2,2}(\Omega)$, in particular we prove that the constant with Navier conditions coincides with the constant with Dirichlet boundary conditions. This is a generalization of [31, Theorem 1 and Theorem 2] concerning the best Sobolev and Hardy constants. Indeed, the Sobolev case corresponds to $\tau = 0$ and the Hardy case to $\tau = 4$ in the following Theorem.

Theorem 3.4.1. *Let $N \ge 5$, and $\Omega \subset \mathbb{R}^N$ a bounded domain containing the origin with boundary $\partial\Omega \in \mathcal{C}^4$, $0 \le \tau \le 4$. Then*

$$C^\tau_{HS,\vartheta}(\Omega) = C^\tau_{HS}(\Omega) = C^\tau_{HS}. \tag{3.4.7}$$

To prove Theorem 3.4.1 we need the following result, which is a Corollary of the Talenti comparison principle, Proposition 3.1.1.

Proposition 3.4.2. *Let $N \ge 5$, $\Omega \subset \mathbb{R}^N$ be a bounded domain containing the origin with $\partial\Omega \in \mathcal{C}^4$ and $|\Omega| = \omega_N$, $0 \le \tau \le 4$ and let $u \in W^{2,2}_\vartheta(\Omega)$. Then there exists a positive radial function $v \in W^{2,2}_\vartheta(\Omega^*)$ such that*

(i) $r \mapsto -\Delta v(r)$ is positive and radially decreasing;

(ii) the following inequality holds

$$\frac{\displaystyle\int_{\Omega} |\Delta u|^2 dx}{\left(\displaystyle\int_{\Omega} \frac{|u|^\sigma}{|x|^\tau} dx\right)^{2/\sigma}} \geq \frac{\displaystyle\int_{\Omega^*} |\Delta v|^2 dx}{\left(\displaystyle\int_{\Omega^*} \frac{|v|^\sigma}{|x|^\tau} dx\right)^{2/\sigma}}.$$

Proof. We are in the even case of [31, Lemmata 5-6]. Moreover we are considering a weighted Lebesgue space with weight $|x|^{-\tau}$. Let us consider the function $v \in W^{2,2}_{\vartheta}(\Omega^*)$ defined as the unique strong solution of

$$\begin{cases} -\Delta v = (-\Delta u)^* & \text{in } \Omega^*, \\ v = 0 & \text{on } \partial\Omega^*. \end{cases}$$

Then, by definition of v we have that $-\Delta v$ is positive, radially symmetric and radially decreasing in Ω^*. Then, using Proposition 3.1.1, we can conclude that $v \geq u^*$ a.e. in Ω^*. Using the fact that Schwarz symmetrization non decreases L^p-norms with singular weight [6, Theorem 2.2], we can conclude that

$$\left(\int_{\Omega^*} \frac{|v|^\sigma}{|x|^\tau} dx\right)^{2/\sigma} \geq \left(\int_{\Omega^*} \frac{|u^*|^\sigma}{|x|^\tau} dx\right)^{2/\sigma} \geq \left(\int_{\Omega} \frac{|u|^\sigma}{|x|^\tau} dx\right)^{2/\sigma}.$$

Finally, it is clear that

$$\int_{\Omega^*} |\Delta v|^2 dx = \int_{\Omega^*} |(\Delta u)^*|^2 dx = \int_{\Omega} |\Delta u|^2 dx.$$

Hence the quotient

$$Q(u) := \frac{\displaystyle\int |\Delta u|^2 dx}{\left(\displaystyle\int \frac{|u|^\sigma}{|x|^\tau} dx\right)^{2/\sigma}},$$

is non increasing if we replace u with v. □

We are now able to prove Theorem 3.4.1.

Proof of Theorem 3.4.1.
Let us consider a minimizing sequence for $C^{\tau}_{HS,\vartheta}(\Omega)$ in $W^{2,2}_{\vartheta}(\Omega)$. Since Ω is bounded and sufficiently regular, then smooth functions are dense in $W^{2,2}_{\vartheta}(\Omega)$. We remark that

$$2 > \frac{2N}{N+2} =: \widetilde{r},$$

which is the exponent we have in Proposition 3.1.1. So we can reduce, without loss of generality, to minimizing sequences in $W^{2,2}_{\vartheta}(\Omega) \cap W^{2,\widetilde{r}}_{\vartheta}(\Omega)$. If we define

$$\mathcal{R}^{2,2}_{\vartheta}(B_1) := \Big\{\text{convex positive cone of } W^{2,2}_{\vartheta}(B_1) \text{ containing positive radially symmetric functions } v \text{ s.t. } r \mapsto -\Delta v(r) \text{ is radially decreasing}\Big\},$$

then we know, by Proposition 3.4.2, that

$$C^{\tau}_{HS,\mathcal{R}} := \inf_{R^{2,2}_{\vartheta}(B_1)\setminus\{0\}} \frac{\displaystyle\int_{B_1} |\Delta u|^2 dx}{\left(\displaystyle\int_{B_1} \frac{|u|^{\sigma}}{|x|^{\tau}} dx\right)^{2/\sigma}} \leq C^{\tau}_{HS,\vartheta}(\Omega) \leq C^{\tau}_{HS}.$$

So, the proof is complete if we can prove that

$$C^{\tau}_{HS,\mathcal{R}} \geq C^{\tau}_{HS}. \tag{3.4.8}$$

Suppose by contradiction that

$$C^{\tau}_{HS,\mathcal{R}} < C^{\tau}_{HS}.$$

Then, we can assume that there exists a function $u \in \mathcal{R}^{2,2}_{\vartheta}(B_1)$ such that

$$\frac{\displaystyle\int_{B_1} |\Delta u|^2 dx}{\left(\displaystyle\int_{B_1} \frac{|u|^{\sigma}}{|x|^{\tau}} dx\right)^{2/\sigma}} < C^{\tau}_{HS}. \tag{3.4.9}$$

But the function $u : B_1(0) \to \mathbb{R}$ is radial, so we can denote it with $u(r) : [0,1] \to \mathbb{R}$. Without loss of generality we can assume that $u'(1) \neq 0$, since $u'(1) = 0$ would imply $u \in W^{2,2}_0(B_1)$ but this is in contradiction with (3.4.9) and (3.4.3). So we can suppose that $u'(1) < 0$, by the decreasing property. Now we apply the extension argument of Gazzola-Grunau-Sweers, [31, Section 3]. Starting from the function u we construct another radial function w with the same laplacian but with a larger Lebesgue norm weighted with $|x|^{-\tau}$: we add a constant in the ball $B_1(0)$ and a multiple of the fundamental solution outside of the ball, namely we define

$$w(r) := \begin{cases} u(r) + \dfrac{1}{N-2}|u'(1)| & \forall\, r \in (0,1], \\[2ex] \dfrac{r^{2-N}}{N-2}|u'(1)| & \forall\, r \in [1,+\infty), \end{cases}$$

We remark that w, as a function from $\mathbb{R}^N \to \mathbb{R}$, is in $\mathcal{D}^{2,2}(\mathbb{R}^N)$ and is in $\mathcal{C}^{1,1}((0,+\infty))$, as a real function. Moreover, by construction, we have

$$\|\Delta w\|_{L^2(B_1)} = \|\Delta u\|_{L^2(B_1)}.$$

Outside of the ball w is a multiple of the fundamental solution and then the laplacian vanishes. Moreover, by adding a positive quantity, results that

$$\left(\int_{B_1} \frac{|w|^\sigma}{|x|^\tau} dx\right)^{2/\sigma} > \left(\int_{B_1} \frac{|u|^\sigma}{|x|^\tau} dx\right)^{2/\sigma},$$

and then we have, by (3.4.9) and by the preceding estimates, that

$$C_{HS}^\tau < \frac{\displaystyle\int_{B_1} |\Delta u|^2 dx}{\left(\displaystyle\int_{B_1} \frac{|u|^\sigma}{|x|^\tau} dx\right)^{2/\sigma}} < C_{HS}^\tau,$$

which is a contradiction. □

The fact that the critical Hardy-Sobolev constant does not depend on all the traces in the critical case is in contrast with the *subcritical* embeddings where the best constant does depend on the traces. as proved for the Sobolev case by A. Ferrero, F. Gazzola and T. Weth in [26]. Moreover, as we said in Chapter 2, in the case $p = 1$ the Sobolev embeddings behave in a different way and Theorem 3.4.1 is false, as proved by D. Cassani, B. Ruf and C. Tarsi in [19].

As a final comment, we want to remark that the proof of Theorem 3.4.1 follows the lines of [30, Theorem 1] and it is a generalization of [30, Theorem 1 and Theorem 2] . The main difference is that we have to recall that symmetrization increases also L^p-norms with singular radial weight in the origin and not only L^p-norms with no weights.

3.5 Improved Hardy-Sobolev inequalities with Navier boundary conditions

The result we prove is an L^q-norm improvement of the critical Hardy-Sobolev inequality (2.2.1) with Navier boundary conditions for a smooth bounded domain Ω. In the Navier conditions case we can use the generalization of the Talenti comparison principle and then we can use an argument by symmetrization. So we can reduce to the case of u and $-\Delta u$ positive and radially symmetric decreasing. Moreover, the biharmonic problem

$$\begin{cases} (-\Delta)^2 u = f(u, x) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.5.1}$$

is equivalent, with the substitution $w := -\Delta u$, to the system of harmonic problems

$$\begin{cases} -\Delta w = f(u, x) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} -\Delta u = w & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

So, we can use standard methods for harmonic problems, such as maximum principle and Hopf Lemma, and then the proof is an argument of constrained minimization of the energy functional

$$\tilde{E}(u) := \int_\Omega |\Delta u|^2 dx - C \left(\int_\Omega |u|^q dx \right)^{2/q},$$

on the constraint

$$F(u) := \left(\int_\Omega \frac{|u|^\sigma}{|x|^\tau} dx \right)^{2/\sigma} = 1.$$

Thus, in the Navier case, we can basically follow the proof in [14], by adapting the arguments to the singular potential $|x|^{-\tau}$. We also have to compute a Pohozaev type identity in the biharmonic case for problem (3.5.1). Pohozaev type identities are well known, even in the polyharmonic cases and even for Dirichlet conditions, [32, Theorem 7.27, Theorem 7.29], but with f depending only on the solution u. Here, we need to consider functions $f = f(u, x)$, in particular

$$f(u, x) := A \frac{|u|^{\sigma-1}}{|x|^\tau} + B|u|^{q-1},$$

with A, B positive constants.

We remark here that, with respect to the original proof of Brezis-Nirenberg, we do not need that the domain Ω is star-shaped. Indeed, by the symmetrization argument we can reduce, starting from a general bounded domain Ω containing the origin, to the case of the ball of radius one centered in the origin. It is also possible to give a proof of the following result by unconstrained minimization, following the argument in [33] and using the mountain-pass-type geometry of the free energy functional.

Theorem 3.5.1. *Let us consider $N \geq 5$, $\Omega \subset \mathbb{R}^N$ a bounded domain containing the origin with $\partial\Omega \in \mathcal{C}^4$, $0 \leq \tau \leq 4$ and $1 \leq q < \frac{N}{N-4}$. Then there exists a constant $C > 0$, $C = C(\Omega, q, \tau)$, such that for any $u \in W^{2,2}_\vartheta(\Omega)$ the following inequality holds*

$$\int_\Omega |\Delta u|^2 dx \geq C^\tau_{HS} \left(\int_\Omega \frac{|u|^\sigma}{|x|^\tau} dx \right)^{2/\sigma} + C \left(\int_\Omega |u|^q dx \right)^{2/q}. \tag{3.5.2}$$

We want to use the well known argument by Brezis-Nirenberg, [14, Equation 1.53], that is a proof by contradiction or, equivalently, a non-existence Theorem. For this, we need a Pohozaev type identity for our problem. As said before, similar identities for the polyharmonic cases are well known. We need to adapt them to the case of a given datum f which depends both on x and u and which is, moreover, singular in the origin.

Proposition 3.5.1. *Let $N \geq 5$, $\Omega = B_1(0) \subset \mathbb{R}^N$, $0 \leq \tau \leq 4$ and $1 \leq q < \frac{N}{N-4}$. Let $u \in W^{2,2}_{\vartheta}(\Omega)$ be a weak solution of the following problem with Navier boundary conditions*

$$\begin{cases} (-\Delta)^2 u = A\dfrac{u^{\sigma-1}}{|x|^{\tau}} + Bu^{q-1} & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega \setminus \{0\}, \end{cases} \tag{3.5.3}$$

where A, B are given positive constants. Then u satisfies the following Pohozaev-type identity

$$\int_{\Omega} |\Delta u|^2 dx = \alpha \int_{\Omega} \frac{u^{\sigma}}{|x|^{\tau}} dx + \beta \int_{\Omega} u^q dx + \gamma \int_{\partial\Omega} \frac{\partial(\Delta u)}{\partial \nu} \frac{\partial u}{\partial \nu} (x \cdot \nu)\, d\mathcal{H}^{N-1}, \tag{3.5.4}$$

with

$$\alpha := A\frac{2(N-\tau)}{\sigma(N-4)}, \quad \beta := B\frac{2N}{q(N-4)}, \quad \gamma := \frac{2}{N-4}.$$

Proof. We set

$$\Omega_{\varepsilon} := \Omega \setminus B_{\varepsilon}(0) = B_1(0) \setminus B_{\varepsilon}(0).$$

We can localize the solution u to the set Ω_{ε}, obtaining that u is a weak solution of the same problem but in the smaller set Ω_{ε}. Now, viewing the problem as a system of harmonic and elliptic problems and using elliptic regularity, we can say that $u \in C^4(\Omega_{\varepsilon})$. We can choose $x \cdot \nabla u$ as test function in Ω_{ε} to obtain

$$\int_{\Omega_{\varepsilon}} ((-\Delta)^2 u)(x \cdot \nabla u) dx = A \int_{\Omega_{\varepsilon}} \frac{u^{2^{\sigma-1}}}{|x|^{\tau}} (x \cdot \nabla u) dx + B \int_{\Omega_{\varepsilon}} u^{q-1} (x \cdot \nabla u) dx.$$

Being the function $u \in C^4(\Omega_{\varepsilon})$, we can do all the computations in a classical way and we find

$$\begin{aligned} \int_{\Omega_{\varepsilon}} ((-\Delta)^2 u)(x \cdot \nabla u) dx = \int_{\partial\Omega_{\varepsilon}} \Big(\nabla \Delta u (x \cdot \nabla u) - \Delta u \nabla u \\ -\Delta u (x, D^2 u) + \frac{x}{2} |\Delta u|^2 \Big) \cdot \nu \, d\mathcal{H}^{N-1} + \frac{4-N}{2} \int_{\Omega_{\varepsilon}} |\Delta u|^2 dx, \end{aligned} \tag{3.5.5}$$

Doing the same for the right hand side. we obtain

$$\int_{\Omega_{\varepsilon}} u^{q-1} (x \cdot \nabla u) dx = -\frac{1}{q} \int_{\partial B_{\varepsilon}} u^q |x| \, d\mathcal{H}^{N-1} - \frac{N}{q} \int_{\Omega_{\varepsilon}} u^q dx, \tag{3.5.6}$$

and

$$\int_{\Omega_{\varepsilon}} \frac{u^{\sigma-1}}{|x|^{\tau}} (x \cdot \nabla u) dx = -\frac{1}{\sigma} \int_{\partial B_{\varepsilon}} u^{\sigma} \frac{|x|}{|x|^{\tau}} \, d\mathcal{H}^{N-1} - \frac{N-\tau}{\sigma} \int_{\Omega_{\varepsilon}} \frac{u^{\sigma}}{|x|^{\tau}} dx. \tag{3.5.7}$$

Now, we use in (3.5.5), (3.5.6) and (3.5.7) the facts that $u = \Delta u = 0$ on $\partial\Omega$, that the limits for $\varepsilon \to 0$ of the integrals over Ω_{ε} converge to the same integrals over Ω by the

Dominated Convergence Theorem, and that the limits for $\varepsilon \to 0$ of the surface integrals over ∂B_ε converge to zero, using the same argument as in [18][Pag. 121-122], to obtain the thesis. $\square$

Next, we prove a Brezis-Lieb type result as [11]. Again the problem is to take care of the singularity due to the weight $|x|^{-\tau}$ in the origin.

Proposition 3.5.2. *Let $\Omega \subseteq \mathbb{R}^N$ a domain containing the origin, $1 \le p < +\infty$, $\{f_j\} \subset L^p(\Omega)$ such that*

(i) $\|f_j\|_{L^p(\Omega)} \le C < +\infty$ *for all* j;

(ii) $f_j \to f$ *for a.e.* $x \in \Omega$.

Then

$$\left(\int_\Omega \frac{|f|^p}{|x|^p}dx\right)^{2/p} = \lim_{j\to+\infty}\left(\left(\int_\Omega \frac{|f_j(x)|^p}{|x|^p}dx\right)^{2/p} - \left(\int_\Omega \frac{|f(x)-f_j(x)|^p}{|x|^p}dx\right)^{2/p}\right). \tag{3.5.8}$$

Proof. First of all we observe that for all $\varepsilon > 0$ there exists a constant $C_\varepsilon = C(\varepsilon) > 0$ such that for all $s \in \mathbb{R}$ the following inequality in $\mathbb{R}$ holds

$$\big||s+1|^p - |s|^p - 1\big| \le \varepsilon |s|^p + C_\varepsilon. \tag{3.5.9}$$

Indeed the function

$$s \mapsto s^p$$

is convex for all $1 \le p < +\infty$. Then, from (3.5.9), we obtain that for all $a, b \in \mathbb{R}$ the following inequality holds in $\mathbb{R}$

$$\big||a+b|^p - |a|^p - |b|^p\big| \le \varepsilon|a|^p + C_\varepsilon|b|^p. \tag{3.5.10}$$

Now we define

$$\begin{cases} u_j := \big||f_j|^p - |f_j - f|^p - |f|^p\big|, \\ v_j := (u_j - \varepsilon|f_j - f|^p)_+ = \sup\{u_j - \varepsilon|f_j - f|^p, 0\}. \end{cases} \tag{3.5.11}$$

Since $f_j \to f$ a.e then $u_j \to 0$, $v_j \to 0$ for a.e. $x \in \mathbb{R}^N$. Moreover using (3.5.10) in the first and the second line of (3.5.11), respectively, we find

$$0 \le v_j \le C_\varepsilon |f|^p,$$

and

$$0 \le u_j \le C_\varepsilon|f|^p + \varepsilon|f_j - f|^p.$$

Now passing to the integral and using the fact that $u_j \to 0$, $v_j \to 0$ a.e., we obtain

$$\lim_{j\to+\infty} \int_\Omega \frac{v_j(x)}{|x|^p} dx = \int_\Omega \lim_{j\to+\infty} \frac{v_j(x)}{|x|^p} dx = 0,$$

and

$$\begin{aligned}\int_\Omega \frac{u_j(x)}{|x|^p} dx &\le \varepsilon \int_\Omega \frac{|f_j(x) - f(x)|^p}{|x|^p} dx + \int_\Omega \frac{v_j(x)}{|x|^p} dx \\ &\le \varepsilon \left(\left(\int_\Omega \frac{|f_j(x)|^p}{|x|^p} dx \right)^{1/p} + \left(\int_\Omega \frac{|f(x)|^p}{|x|^p} dx \right)^{1/p} \right)^p + \int_\Omega \frac{v_j(x)}{|x|^p} dx \\ &\le \varepsilon \, (2M)^p + \int_\Omega \frac{v_j(x)}{|x|^p} dx,\end{aligned}$$

with

$$M := \sup \left\{ \left(\int_\Omega \frac{|f_j(x)|^p}{|x|^p} dx \right)^{1/p} \right\}.$$

Then

$$\limsup_{j\to+\infty} \int_\Omega \frac{u_j(x)}{|x|^p} dx \le \varepsilon \, (2M)^p + \limsup_{j\to+\infty} \int_\Omega \frac{v_j(x)}{|x|^p} dx \le \varepsilon \, (2M)^p .$$

and so

$$\lim_{j\to+\infty} \int_{\mathbb{R}^N} \frac{u_j(x)}{|x|^p} dx = 0,$$

which implies (3.5.8). □

Proof of Theorem 3.5.1.
We need to prove (3.5.2) only for any $u \in W^{2,2}_\vartheta(\Omega^*)$ where Ω^* is the ball centered in the origin such that $|\Omega^*| = |\Omega|$. Indeed if we suppose that (3.5.2) holds for any $u \in W^{2,2}_\vartheta(\Omega^*)$ then for any $u \in W^{2,2}_\vartheta(\Omega)$ we can choose a $v \in W^{2,2}_\vartheta(\Omega^*)$ such that v is the unique strong solution of

$$\begin{cases} -\Delta v = (-\Delta u)^* & \text{in } \Omega^*, \\ v = 0 & \text{on } \partial\Omega^*. \end{cases}$$

Then by Proposition 3.1.1 and a rescaling argument we can conclude that $v \ge u^*$ a.e. By standard properties of Schwarz symmetrization, we obtain

$$\begin{aligned}\int_\Omega |\Delta u|^2 dx = \int_{\Omega^*} |\Delta v|^2 dx \\ &\ge C^\tau_{HS} \left(\int_{\Omega^*} \frac{|v|^\sigma}{|x|^\tau} dx \right)^{2/\sigma} + C \left(\int_{\Omega^*} |v|^q dx \right)^{2/q} \\ &\ge C^\tau_{HS} \left(\int_{\Omega^*} \frac{|u|^\sigma}{|x|^\tau} dx \right)^{2/\sigma} + C \left(\int_\Omega |u|^q dx \right)^{2/q} .\end{aligned}$$

So we can reduce without loss of generality to the problem with the further assumptions that Ω is the unitary ball centered in the origin, $u \in W^{2,2}_{\vartheta}(\Omega)$ is positive and radially symmetric decreasing and that $-\Delta u$ is positive and radially symmetric decreasing.

Step 1. We prove that the following statements are equivalent

(i) there exists a constant $C(N,q,\tau) > 0$ such that (3.5.2) holds for any $u \in \mathcal{X}$ with

$$\mathcal{X} := \left\{ v \in W^{2,2}_{\vartheta}(\Omega) \,:\, v \geq 0 \text{ a.e.} \right\};$$

(ii) there exists a constant $C(N,q,\tau) > 0$ such that $\widetilde{C}^{\tau}_{HS} = C^{\tau}_{HS}$ with

$$\widetilde{C}^{\tau}_{HS} := \inf_{u \in \mathcal{X} \setminus \{0\}} \frac{\displaystyle\int_{\Omega} |\Delta u|^2 dx - C \left(\int_{\Omega} |u|^q dx \right)^{2/q}}{\displaystyle\left(\int_{\Omega} \frac{|u|^{\sigma}}{|x|^{\tau}} dx \right)^{2/\sigma}}.$$

We are considering here also C^{τ}_{HS} as an infimum over $\mathcal{X}$. If (i) holds then we have

$$\int_{\Omega} |\Delta u|^2 dx - C \left(\int_{\Omega} |u|^q dx \right)^{2/q} \geq C^{\tau}_{HS} \left(\int_{\Omega} \frac{|u|^{\sigma}}{|x|^{\tau}} dx \right)^{2/\sigma}$$

and so $\widetilde{C}^{\tau}_{HS} \geq C^{\tau}_{HS}$. The opposite inequality is true by definition and then $\widetilde{C}^{\tau}_{HS} = C^{\tau}_{HS}$. Conversely we have for any $u \in \mathcal{X}$

$$\begin{aligned} \int_{\Omega} |\Delta u|^2 dx &\geq \widetilde{C}^{\tau}_{HS} \left(\int_{\Omega} \frac{|u|^{\sigma}}{|x|^{\tau}} dx \right)^{2/\sigma} + C \left(\int_{\Omega} |u|^q dx \right)^{2/q} \\ &= C^{\tau}_{HS} \left(\int_{\Omega} \frac{|u|^{\sigma}}{|x|^{\tau}} dx \right)^{2/\sigma} + C \left(\int_{\Omega} |u|^q dx \right)^{2/q}. \end{aligned}$$

And then (3.5.2) holds for any $u \in \mathcal{X}$.

Step 2. We prove that if $\widetilde{C}^{\tau}_{HS} < C^{\tau}_{HS}$ then $\widetilde{C}^{\tau}_{HS}$ is attained in $\mathcal{X} \setminus \{0\}$, which means that there exists a function $u \in \mathcal{X} \setminus \{0\}$ such that

$$\widetilde{C}^{\tau}_{HS} = E(u),$$

with the energy functional E defined as

$$E(u) := \frac{\displaystyle\int_{\Omega} |\Delta u|^2 dx - C \left(\int_{\Omega} |u|^q dx \right)^{2/q}}{\displaystyle\left(\int_{\Omega} \frac{|u|^{\sigma}}{|x|^{\tau}} dx \right)^{2/\sigma}}.$$

Let us consider a minimizing sequence in $\mathcal{X}$ for $\widetilde{C}^{\tau}_{HS}$, that is a sequence $\{u_j\} \subset \mathcal{X}$ such that

$$E(u_j) \to \widetilde{C}^{\tau}_{HS}.$$

We can suppose without loss of generality that

$$\int_\Omega \frac{|u_j|^\sigma}{|x|^\tau} dx = 1.$$

If not, we can simply consider

$$\tilde{u}_j := \frac{u_j}{\left(\int_\Omega \frac{|u_j|^\sigma}{|x|^\tau} dx\right)^{1/\sigma}}.$$

Using Sobolev and compactness of embeddings, we have, up to a subsequence,

$$\begin{cases} u_j \to u & \text{in } L^r(\Omega) \quad \forall\, r < 2^* \\ u_j \rightharpoonup u & \text{in } W^{2,2}_\vartheta(\Omega) \\ u_j \to u & \text{for a.e } x \in \Omega. \end{cases}$$

Moreover, using Fatou Lemma

$$\int_\Omega \frac{|u|^\sigma}{|x|^\tau} dx \le \liminf_{j\to+\infty} \int_\Omega \frac{|u_j|^\sigma}{|x|^\tau} dx = 1.$$

Let $v_j := u_j - u$, then

$$\begin{cases} v_j \to 0 & \text{in } L^r(\Omega) \quad \forall\, r < 2^* \\ v_j \rightharpoonup 0 & \text{in } H^2_\vartheta(\Omega) \\ v_j \to 0 & \text{for a.e } x \in \Omega, \end{cases}$$

We want to prove that $u \not\equiv 0$. By the minimizing property of $\{u_j\}$, we have that $E(u_j) \to \widetilde{C}^\tau_{HS}$ which means

$$\lim_{j\to+\infty} \left(\int_\Omega |\Delta u_j|^2 dx - C \left(\int_\Omega |u_j|^q dx \right)^{2/q} \right) = \widetilde{C}^\tau_{HS}. \tag{3.5.12}$$

Now, we can use the fact that $v_j \rightharpoonup 0$ and [11, Theorem 1] to conclude

$$\lim_{j\to+\infty} \left(\int_\Omega |\Delta u_j|^2 dx - C \left(\int_\Omega |u_j|^q dx \right)^{2/q} \right) \ge C^\tau_{HS} - C \left(\int_\Omega |u|^q dx \right)^{2/q},$$

and then

$$\widetilde{C}^\tau_{HS} \ge C^\tau_{HS} - C \left(\int_\Omega |u|^q dx \right)^{2/q}. \tag{3.5.13}$$

Taking into account $\widetilde{C}^\tau_{HS} < C^\tau_{HS}$ in (3.5.13), we find

$$C \left(\int_\Omega |u|^q dx \right)^{2/q} \ge C^\tau_{HS} - \widetilde{C}^\tau_{HS} > 0,$$

and then $u \not\equiv 0$. Using Proposition 3.5.2, we know

$$1 = \int_\Omega \frac{|v_j + u|^\sigma}{|x|^\tau} dx = \int_\Omega \frac{|v_j|^\sigma}{|x|^\tau} dx + \int_\Omega \frac{|u|^\sigma}{|x|^\tau} dx + o(1),$$

and then

$$1 \leq \left(\int_\Omega \frac{|v_j|^\sigma}{|x|^\tau} dx \right)^{2/\sigma} + \left(\int_\Omega \frac{|u|^\sigma}{|x|^\tau} dx \right)^{2/\sigma} + o(1).$$

Now, we want to prove the following inequality

$$\int_\Omega |\Delta u|^2 dx - C \left(\int_\Omega |u|^q dx \right)^{2/q} \leq \widetilde{C}^\tau_{HS} \left(\int_\Omega \frac{|u|^\sigma}{|x|^\tau} dx \right)^{2/\sigma}. \tag{3.5.14}$$

(a) Let $\widetilde{C}^\tau_{HS} > 0$, then

$$\begin{aligned} \widetilde{C}^\tau_{HS} &\leq \widetilde{C}^\tau_{HS} \left(\int_\Omega \frac{|u|^\sigma}{|x|^\tau} dx \right)^{2/\sigma} + \widetilde{C}^\tau_{HS} \left(\int_\Omega \frac{|v_j|^\sigma}{|x|^\tau} dx \right)^{2/\sigma} + o(1) \\ &\leq \widetilde{C}^\tau_{HS} \left(\int_\Omega \frac{|u|^\sigma}{|x|^\tau} dx \right)^{2/\sigma} + \frac{\widetilde{C}^\tau_{HS}}{C^\tau_{HS}} \int_\Omega |\Delta v_j|^2 dx + o(1), \end{aligned}$$

So, we obtain (3.5.14) in the following way

$$\begin{aligned} \int_\Omega |\Delta u|^2 dx - C \left(\int_\Omega |u|^q dx \right)^{\frac{2}{q}} &\leq \widetilde{C}^\tau_{HS} \left(\int_\Omega \frac{|u|^\sigma}{|x|^\tau} dx \right)^{2/\sigma} \\ &\quad + \left(\frac{\widetilde{C}^\tau_{HS}}{C_{HS}} - 1 \right) \int_\Omega |\Delta v_j|^2 dx + o(1) \\ &\leq \widetilde{C}^\tau_{HS} \left(\int_\Omega \frac{|u|^\sigma}{|x|^\tau} dx \right)^{2/\sigma}. \end{aligned}$$

(b) Let $\widetilde{C}^\tau_{HS} \leq 0$, then we find

$$\widetilde{C}^\tau_{HS} \leq \widetilde{C}^\tau_{HS} \left(\int_\Omega \frac{|u|^\sigma}{|x|^\tau} dx \right)^{2/\sigma}.$$

With a similar argument to case (a) we find again (3.5.14).

Then

$$E(u) \leq \widetilde{C}^\tau_{HS} = \inf_{\mathcal{X} \setminus \{0\}} E(u),$$

and then u is a minimum.

Step 3. We prove (3.5.2). Suppose on the contrary that for any $C > 0$ we have that $C^\tau_{HS} \neq \widetilde{C}^\tau_{HS}$, which means $\widetilde{C}^\tau_{HS} < C^\tau_{HS}$. By the preceding step, we have that there exists a function $u \in \mathcal{X}$, $u > 0$, such that

$$\widetilde{C}^\tau_{HS} := E(u).$$

Now, $\widetilde{C}^{\tau}_{HS}$ is a minimum obtained by unconstrained minimization of the functional E. At the same time. we can obtain $\widetilde{C}^{\tau}_{HS}$ as a minimum by constrained minimization of the functional $\widetilde{E}$ defined as

$$\widetilde{E}(u) := \int_\Omega |\Delta u|^2 dx - C\left(\int_\Omega |u|^q dx\right)^{2/q},$$

on the constraint $F(u) = 1$ with

$$F(u) = \left(\int_\Omega \frac{|u|^\sigma}{|x|^\tau} dx\right)^{2/\sigma}.$$

The functional $\widetilde{E}$ is Frechet differentiable and then by the Lagrange multiplier theorem we have

$$\widetilde{E}'[u](v) = \widetilde{C}^{\tau}_{HS}\, F'[u](v) \quad \forall\, v \in W^{2,2}_{\vartheta}(\Omega). \tag{3.5.15}$$

with

$$\widetilde{E}'[u](v) = 2\int_\Omega \Delta u \Delta v dx - 2C\|u\|^{2-q}_{L^q(\Omega)} \int_\Omega |u|^{q-1} v dx, \tag{3.5.16}$$

and

$$F'[u](v) = 2\left(\int_\Omega \frac{|u|^\sigma}{|x|^\tau} dx\right)^{(2-\sigma)/\sigma} \int_\Omega \frac{|u|^{\sigma-1}}{|x|^\tau} v dx. \tag{3.5.17}$$

Finally, using the boundary conditions of v, we find

$$\int_\Omega \Delta u \Delta v dx = \int_\Omega \Delta^2 u\, v dx. \tag{3.5.18}$$

Using (3.5.16), (3.5.17) and (3.5.18) in (3.5.15), we obtain

$$2\int_\Omega \Delta^2 u\, v dx = 2\widetilde{C}^{\tau}_{HS}\left(\int_\Omega \frac{|u|^\sigma}{|x|^\tau} dx\right)^{(2-\sigma)/\sigma} \int_\Omega \frac{|u|^{\sigma-1}}{|x|^\xi} v dx + 2C\|u\|^{2-q}_{L^q(\Omega)} \int_\Omega |u|^{q-1} v dx,$$

for any $v \in W^{2,2}_{\vartheta}(\Omega)$. Using the fact that $F(u) = 1$, we find that u is a weak solution in $W^{2,2}_{\vartheta}(\Omega)$ of the problem

$$\begin{cases} \Delta^2 u = A\dfrac{u^{\sigma-1}}{|x|^\tau} + Bu^{q-1} & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega \\ u > 0 & \text{in } \Omega \setminus \{0\}, \end{cases} \tag{3.5.19}$$

with

$$A := \widetilde{C}^{\tau}_{HS}, \quad B := C\|u\|^{2-q}_{L^q(\Omega)}.$$

We can apply Proposition 3.5.1 to conclude that

$$\int_\Omega |\Delta u|^2 dx = \alpha \int_\Omega \frac{u^\sigma}{|x|^\tau} dx + \beta \int_\Omega u^q dx + \gamma \int_{\partial\Omega} (\Delta u)_\nu \frac{\partial u}{\partial \nu}(x\cdot\nu)\, d\mathcal{H}^{N-1}. \tag{3.5.20}$$

Moreover, integrating the equation in (3.5.19) and using (3.5.18) with $v = u$, we obtain

$$\int_\Omega |\Delta u|^2 dx = \widetilde{C}^\tau_{HS} \int_\Omega \frac{u^\sigma}{|x|^\tau} dx + C\|u\|^{2-q}_{L^q(\Omega)} \int_\Omega u^q dx. \tag{3.5.21}$$

Taking (3.5.21) minus (3.5.20), we find

$$0 = \tilde\alpha \int_\Omega \frac{u^\sigma}{|x|^\tau} dx + \tilde\beta \left(\int_\Omega u^q dx\right)^{2/q} - \gamma \int_{\partial\Omega} (\Delta u)_\nu \frac{\partial u}{\partial \nu}(x\cdot\nu)\, d\mathcal{H}^{N-1}. \tag{3.5.22}$$

with

$$\tilde\alpha := \widetilde{C}^\tau_{HS}\left(1 - \frac{2(N-\tau)}{\sigma(N-4)}\right), \quad \tilde\beta := C\|u\|^{2-q}_{L^q(\Omega)}\left(1 - \frac{2N}{q(N-4)}\right), \quad \gamma := \frac{2}{N-4}.$$

Using (3.4.2) and $0 \le \tau \le 4$, we obtain $\tilde\alpha = 0$ and then, inserting this in (3.5.22),

$$0 = \tilde\beta \left(\int_\Omega u^q dx\right)^{2/q} - \gamma \int_{\partial\Omega} (\Delta u)_\nu \frac{\partial u}{\partial \nu}(x\cdot\nu)\, d\mathcal{H}^{N-1}. \tag{3.5.23}$$

By $q < \frac{N}{N-4}$, we have $\tilde\beta < 0$. We want to find the sign of the second term.
We have $x \cdot \nu = 1$, $(\Delta u)_\nu = (\Delta u)'$, and $\frac{\partial u}{\partial \nu} = u'$. With the substitution $w := -\Delta u$, we can split problem (3.5.19) in

$$\begin{cases} -\Delta w = f(u,x) & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} -\Delta u = w & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $f(u,x) > 0$. Then, applying the maximum principle, we obtain $-\Delta u \ge 0$. Hence, by the Hopf Lemma, we can conclude

$$u' < 0, \quad (-\Delta u)' < 0, \quad \text{on } \partial\Omega.$$

Then

$$-\gamma \int_{\partial\Omega} (\Delta u)_\nu \frac{\partial u}{\partial \nu}(x\cdot\nu)\, d\mathcal{H}^{N-1} > 0,$$

so we have two terms with different signs. But u and Δu are radially symmetric and then

$$\begin{aligned} \int_{\partial\Omega} (\Delta u)' u'\, d\mathcal{H}^{N-1} &= \omega_{N-1} u'(1)(\Delta u)'(1) \\ &= \frac{1}{\omega_{N-1}} \left(\int_{\partial\Omega} u' dS\right)\left(\int_{\partial\Omega} (\Delta u)'\, d\mathcal{H}^{N-1}\right) \\ &= \frac{1}{\omega_{N-1}} \left(\int_{\partial\Omega} \nabla u \cdot \nu\, d\mathcal{H}^{N-1}\right)\left(\int_{\partial\Omega} \nabla\Delta u \cdot \nu\, d\mathcal{H}^{N-1}\right) \\ &= \frac{1}{\omega_{N-1}} \left(\int_\Omega \Delta u dx\right)\left(\int_\Omega \Delta^2 u dx\right). \end{aligned}$$

Now, let φ the unique radial positive and smooth solution to the problem

$$\begin{cases} -\Delta\varphi = 1 & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega, \end{cases}$$

then

$$-\int_\Omega \Delta u dx = \int_\Omega \varphi \Delta^2 u dx.$$

Hence

$$\begin{aligned} -\int_{\partial\Omega} (\Delta u)' u' \, d\mathcal{H}^{N-1} &= -\frac{1}{\omega_{N-1}} \left(\int_\Omega \Delta u dx \right) \left(\int_\Omega \Delta^2 u dx \right) \\ &= \frac{1}{\omega_{N-1}} \left(\int_\Omega -\Delta u dx \right) \left(\int_\Omega \Delta^2 u dx \right) \\ &= \frac{1}{\omega_{N-1}} \left(\int_\Omega \varphi \Delta^2 u dx \right) \left(\int_\Omega \Delta^2 u dx \right). \end{aligned}$$

Finally

$$\begin{aligned} \int_\Omega \Delta^2 u dx &= \int_{|x|\le\frac12} \Delta^2 u dx + \int_{\frac12<|x|\le 1} \Delta^2 u dx \\ &\le \int_{|x|\le\frac12} \Delta^2 u dx + K(N) \Delta^2 u \left(\frac12\right) \\ &\le \int_{|x|\le\frac12} \Delta^2 u dx + K(N) \int_{|x|\le\frac12} \Delta^2 u dx \\ &\le K(N) \int_\Omega \varphi \Delta^2 u dx \ ; \end{aligned}$$

indeed φ is radial symmetric decreasing and then

$$\int_{|x|\le\frac12} \varphi \Delta^2 u dx \ge K(N) \int_{|x|\le\frac12} \Delta^2 u dx.$$

Using the Sobolev inequality, we have

$$\begin{aligned} -\gamma \int_{\partial\Omega} (\Delta u)' u' \, d\mathcal{H}^{N-1} &\ge K(N) \left(\int_\Omega \Delta^2 u dx \right)^2 \\ &\ge K(N) \left(\int_\Omega u^q dx \right)^{2/q} \end{aligned}$$

Coming back to (3.5.23), we find

$$0 \ge (K - \tilde\beta) \int_\Omega u^q dx > 0 \quad \forall\, C < \widetilde{C},$$

with

$$\widetilde{C} := \frac{qK(N-4)}{2N\|u\|^{2-q}_{L^q(\Omega)}}.$$

So, we find a contradiction for any $0 < C < \widetilde{C}$. □

3.6 Improved Hardy-Sobolev inequalities with Dirichlet boundary conditions

The third result we prove is an improvement of the critical Hardy-Sobolev inequality with Dirichlet conditions with a weak L^q remainder term. The proof of the following Theorem follows the argument of [12, Inequality 1.4]. The main difference is we cannot use a symmetrization argument and moreover we cannot use the maximum principle, in the case of Dirichlet conditions for a biharmonic problem. So, we can not reduce ourselves to the case of radial and positive functions, as in the Navier conditions case. However, using Proposition 3.2.1, we can reduce to the case of fixed sign functions. Then, we need an extension argument to apply the Hardy-Sobolev inequality in $\mathbb{R}^N$. This is also the strategy used in the proof of the L^q-remainder term for the polyharmonic Sobolev inequality [29] or [32, Theorem 7.58, Corollary 7.59 and Theorem 7.60].

Theorem 3.6.1. *Let $N \geq 5$, $\Omega \subset \mathbb{R}^N$ a bounded domain containing the origin with $\partial\Omega \in \mathcal{C}^4$, $0 \leq \tau \leq 4$ and $1 \leq q \leq \frac{N}{N-4}$. Then there exists a constant $C = C(\Omega, q, \tau) > 0$, such that for any $u \in W_0^{2,2}(\Omega)$ the following inequality holds*

$$\int_\Omega |\Delta u|^2 dx \geq C_{HS}^\tau \left(\int_\Omega \frac{|u|^\sigma}{|x|^\tau} dx \right)^{2/\sigma} + C\|u\|^2_{L^q_w(\Omega)}. \tag{3.6.1}$$

Proof. Ω is bounded, then there exists a radius $R > 0$ such that $\Omega \subset B_R(0)$. Now by the Dirichlet conditions we can extend any function $u \in W_0^{2,2}(\Omega)$ by zero outside Ω obtaining a function $\widetilde{u} \in W_0^{2,2}(B_R)$. So we have to prove the Theorem only for functions $u \in W_0^{2,2}(B_R(0))$.
Consider the closed convex cone in $W_0^{2,2}(B_R)$ of non-negative functions

$$\mathcal{X} := \left\{ v \in W_0^{2,2}(B_R) \,:\, v \geq 0 \text{ a.e. in } B_R(0) \right\}.$$

Let $g \in L^\infty(B_R)$ and let $v \in W_0^{2,2}(B_R)$ a solution of the following problem with Dirichlet boundary conditions

$$\begin{cases} (-\Delta)^2\, v = g & \text{in } B_R(0) \\ v = |\nabla v| = 0 & \text{on } \partial B_R(0). \end{cases}$$

Then, $v \in W_0^{2,2}(B_R) \cap L^\infty(B_R)$. We take a function $u \in \mathcal{X} \setminus \{0\}$ and define the auxiliary function

$$\phi := \begin{cases} u - v + \|v\|_{L^\infty(B_R)} & \text{in } B_R \\ \|v\|_{L^\infty(B_R)} \left[\left(\frac{R}{|x|}\right)^{N-3} - \left(\frac{R}{|x|}\right)^{N-4} \right] & \text{in } B_R^c. \end{cases}$$

So ϕ is a function in $W_0^{2,2}(\mathbb{R}^N)$ and we can apply the Hardy-Sobolev inequality to ϕ in all $\mathbb{R}^N$ obtaining

$$\int_{\mathbb{R}^N} |\Delta\phi|^2 dx \geq C_{HS}^{\tau} \left(\int_{\mathbb{R}^N} \frac{|\phi|^{\sigma}}{|x|^{\tau}} dx \right)^{2/\sigma}. \tag{3.6.2}$$

With straightforward computations, we obtain

$$\left| \Delta \left(\left(\frac{R}{|x|} \right)^{N-3} - \left(\frac{R}{|x|} \right)^{N-4} \right) \right|^2 = \frac{(2-N)^2(3-N)^2 R^{2N-6}}{|x|^{2N-2}}$$
$$- \frac{(2-N)(3-N)^2(4-N)R^{2N-7}}{|x|^{2N-3}} + \frac{(3-N)^2(4-N)^2 R^{2N-8}}{|x|^{2N-4}},$$

$$\begin{aligned}\int_{\mathbb{R}^N} |\Delta\phi|^2 dx &= \int_{B_R} |\Delta\left(u - v + \|v\|_{L^\infty}\right)|^2 dx \\ &\quad + \|v\|_{L^\infty}^2 \int_{B_R^c} \left| \Delta \left(\left(\frac{R}{|x|} \right)^{N-3} - \left(\frac{R}{|x|} \right)^{N-4} \right) \right|^2 dx \\ &= \int_{B_R} |\Delta(u-v)|^2 dx + k\|v\|_{L^\infty}^2,\end{aligned}$$

$$\begin{aligned} k :&= \int_{B_R^c} \left| \Delta \left(\left(\frac{R}{|x|} \right)^{N-3} - \left(\frac{R}{|x|} \right)^{N-4} \right) \right|^2 dx \\ &= (2-N)^2(3-N)^2 R^{2N-6} \int_{B_R^c} \frac{dx}{|x|^{2N-2}} \\ &\quad - (2-N)(3-N)^2(4-N)R^{2N-7} \int_{B_R^c} \frac{dx}{|x|^{2N-3}} \\ &\quad + (3-N)^2(4-N)^2 R^{2N-8} \int_{B_R^c} \frac{dx}{|x|^{2N-4}} \\ &= C_1 I_1 + C_2 I_2 + C_3 I_3. \end{aligned}$$

Using polar coordinates, we find

$$I_1 = \frac{\omega_N}{N-2} R^{2-N}, \quad I_2 = \frac{\omega_N}{N-3} R^{3-N}, \quad I_3 = \frac{\omega_N}{N-4} R^{4-N},$$

so that

$$k = C R^{N-4}. \tag{3.6.3}$$

Finally, we can conclude

$$\int_{\mathbb{R}^N} |\Delta\phi|^2 dx = \int_{B_R} |\Delta(u-v)|^2 dx + k\|v\|_{L^\infty}^2. \tag{3.6.4}$$

But

$$\int_{B_R} |\Delta(u-v)|^2 dx = \int_{B_R} |\Delta u|^2 dx + \int_{B_R} |\Delta v|^2 dx - 2\int_{B_R} \Delta u \, \Delta v dx,$$

and integrating by parts twice and using $u = 0$ and $|\nabla u| = 0$ on ∂B_R we find

$$2\int_{B_R} \Delta u \, \Delta v dx = 2\int_{B_R} ugdx. \tag{3.6.5}$$

Moreover, using $u > 0$ and $-v + \|v\|_{L^\infty} \geq 0$, we can conclude

$$\int_{\mathbb{R}^N} \frac{|\phi|^\sigma}{|x|^\tau} dx \geq \int_{B_R} \frac{|u|^\sigma}{|x|^\tau} dx. \tag{3.6.6}$$

Now, using (3.6.4)-(3.6.5)-(3.6.6) in (3.6.2) we find

$$\int_{B_R} |\Delta u|^2 dx + \int_{B_R} |\Delta v|^2 dx + 2\int_{B_R} ugdx + k\|v\|^2_{L^\infty} \geq C^\tau_{HS} \left(\int_{B_R} \frac{|u|^\sigma}{|x|^\tau} dx \right)^{2/\sigma}. \tag{3.6.7}$$

Substituting $g \mapsto \lambda g$, $v \mapsto \lambda v$, $\lambda > 0$ in (3.6.7), we have

$$\begin{aligned} E(\lambda) := \int_{B_R} |\Delta u|^2 dx + \lambda^2 \int_{B_R} |\Delta v|^2 + 2\lambda \int_{B_R} ugdx + \lambda^2 k \|v\|^2_{L^\infty} \\ - C^\tau_{HS} \left(\int_{B_R} \frac{|u|^\sigma}{|x|^\tau} dx \right)^{2/\sigma} \geq 0. \end{aligned}$$

Minimizing in λ, we find

$$\frac{d}{d\lambda} E(\lambda) = 0 \quad \text{iff} \quad \lambda = -\frac{\displaystyle\int_{B_R} ugdx}{\displaystyle\int_{B_R} |\Delta v|^2 dx + k\|v\|^2_{L^\infty}},$$

then

$$\int_{B_R} |\Delta u|^2 dx \geq C^\tau_{HS} \left(\int_{B_R} \frac{|u|^\sigma}{|x|^\tau} dx \right)^{2/\sigma} + \frac{\left(\displaystyle\int_{B_R} ugdx \right)^2}{\displaystyle\int_{B_R} |\Delta v|^2 dx + k\|v\|^2_{L^\infty}}, \tag{3.6.8}$$

for any $u \in \mathcal{X} \setminus \{0\}$, for any $g \in L^\infty(B_R)$ and for any $v \in W^{2,2}_0(B_R) \cap L^\infty(B_R)$ solution of

$$\begin{cases} (-\Delta)^2 v = g & \text{in } B_R \\ v = |\nabla v| = 0 & \text{on } \partial B_R. \end{cases}$$

Consider $g = \chi_A$, with $A \subset \Omega \subset B_R(0)$ a measurable set. Then

$$\int_{B_R} ugdx = \int_{B_R} udx. \tag{3.6.9}$$

$$\int_{B_R} |\Delta v|^2 dx \leq C|A|^{1+\frac{4}{N}}. \tag{3.6.10}$$

Indeed, using the fact that v is a solution of the preceding problem with Dirichlet boundary conditions and integrating by parts we find

$$\int_{B_R} |\Delta v|^2 dx = \int_{B_R} vg dx = \int_A v dx.$$

Then, using Hölder inequality Sobolev inequality with $p = \frac{2N}{N-4}$ and $q = \frac{2N}{N+4}$,

$$\begin{aligned}
\int_{B_R} |\Delta v|^2 dx &= \int_A v dx \\
&\leq \int_A |v| dx \\
&\leq \|v\|_{L^p(A)} |A|^{\frac{1}{q}} \\
&\leq \|v\|_{L^p(B_R)} |A|^{\frac{1}{q}} \\
&\leq C\|v\|_{W_0^{2,2}(B_R)} |A|^{\frac{N+4}{2N}}.
\end{aligned}$$

Then (3.6.10) is proved, just dividing by $\|v\|_{W_0^{2,2}}$ and taking the square power. Finally, using Proposition 3.3.1 and computations as in [31, Theorem 7.58], we find

$$\|v\|_{L^\infty} \leq C|A|^{\frac{4}{N}}. \tag{3.6.11}$$

Using (3.6.9)-(3.6.10)-(3.6.11), the definitions of k in (3.6.3) and that $A \subset \Omega \subset B_R$,[1] we find

$$\int_{B_R} |\Delta v|^2 dx + k\|v\|^2_{L^\infty} \leq CR^{N-4}|A|^{\frac{8}{N}}. \tag{3.6.12}$$

Using (3.6.12) in (3.6.8), we obtain,

$$\begin{aligned}
\int_{B_R} |\Delta u|^2 dx &\geq C_{HS}^\tau \left(\int_{B_R} \frac{|u|^\sigma}{|x|^\tau} dx \right)^{2/\sigma} + \frac{\left(\int_{B_R} ugdx \right)^2}{\int_{B_R} |\Delta v|^2 dx + k\|v\|^2_{L^\infty}} \\
&\geq C_{HS}^\tau \left(\int_{B_R} \frac{|u|^\sigma}{|x|^\tau} dx \right)^{2/\sigma} + C(\Omega)|A|^{-\frac{8}{N}} \left(\int_A u dx \right)^2 \\
&\geq C_{HS}^\tau \left(\int_{B_R} \frac{|u|^\sigma}{|x|^\tau} dx \right)^{2/\sigma} + C(\Omega) \sup_{A \subset \Omega} \left\{ |A|^{-\frac{8}{N}} \int_A u dx \right\}^2 \\
&\geq C_{HS}^\tau \left(\int_{B_R} \frac{|u|^\sigma}{|x|^\tau} dx \right)^{2/\sigma} + C(\Omega) \|u\|^2_{L_w^{\frac{N}{N-4}}(B_R)},
\end{aligned}$$

[1] In this way $|A| \leq CR^N$.

for any $A \subset B_R$, where in the last line we use the following characterization of the weak norm

$$\|u\|_{L^p_w(\Omega)} := \sup_{A \subset \Omega} \left\{ |A|^{-\frac{1}{q}} \int_A u dx \right\}, \quad \frac{1}{q} + \frac{1}{p} = 1,$$

and with the constant

$$C(\Omega) := \frac{1}{CR^{N-4}}.$$

This concludes the proof of the theorem in the positive cone.
Let u a generic changing sign function in $W_0^{2,2}(B_R)$. We can just apply the dual cone decomposition choosing

$$\mathcal{X} := \{v \in W_0^{2,2}(B_R) \,:\, v \geq 0 \text{ a.e in } B_R\},$$

$$\mathcal{X}^* := \{v \in W_0^{2,2}(B_R) \,:\, v \leq 0 \text{ a.e in } B_R\},$$

to conclude, using [31, Proposition 3.6], that

$$|u_1 + u_2| \leq \max\{|u_1|, |u_2|\},$$

and then for any $r > 0$ we have

$$|u_1 + u_2|^r \leq \max\{|u_1|^r, |u_2|^r\} \leq |u_1|^r + |u_2|^r \text{ for a.e. } x \in B_R.$$

If $r \geq 2$, we fix $r = \sigma$, $\rho = \frac{\tau}{\sigma}$ and we have

$$\begin{aligned}
\left(\int_{B_R} \frac{|u|^\sigma}{|x|^\tau} dx \right)^{2/\sigma} &= \left(\int_{B_R} \left(\frac{|u|}{|x|^\rho} \right)^\sigma dx \right)^{2/\sigma} \\
&= \left\| \frac{u}{|x|^\rho} \right\|^2_{L^\sigma(B_R)} \\
&= \left\| \frac{u_1 + u_2}{|x|^\rho} \right\|^2_{L^\sigma(B_R)} \\
&\leq \left(\int_{B_R} \left(\frac{|u_1|}{|x|^\rho} \right)^\sigma + \left(\frac{|u_2|}{|x|^\rho} \right)^\sigma dx \right)^{2/\sigma} \\
&\leq \left\| \frac{u_1}{|x|^\rho} \right\|^2_{L^\sigma(B_R)} + \left\| \frac{u_2}{|x|^\rho} \right\|^2_{L^\sigma(B_R)}.
\end{aligned}$$

Finally, we know that the thesis holds separately for u_1 and u_2 and then the theorem is proved. □

Using the properties of the weak Lebesgue norm, namely $\|u\|_{L^{\bar{q}}} \leq \|u\|_{L^q_w}$ for $\bar{q} < q$ (see [39, Pag. 255]), we have the following Corollary, which is the counterpart of Theorem 3.5.1 for the Dirichlet case.

Corollary 3.6.1. *Let $N \geq 5$, $\Omega \subset \mathbb{R}^N$ a bounded domain containing the origin with $\partial\Omega \in \mathcal{C}^4$, $0 \leq \tau < 4$ and $1 \leq q < \frac{N}{N-4}$. Then there exists a constant $C > 0$, $C = C(\Omega, q, \tau)$, such that for any $u \in W_0^{2,2}(\Omega)$ the following inequality holds*

$$\int_\Omega |\Delta u|^2 dx \geq C_{HS}^{\tau} \left(\int_\Omega \frac{|u|^\sigma}{|x|^\tau} dx \right)^{2/\sigma} + C \left(\int_\Omega |u|^q dx \right)^{2/q}. \tag{3.6.13}$$

From Chebyshev's inequality we obtain for all $1 \leq q < +\infty$

$$\|u\|_{L^q_w(\Omega)} \leq \|u\|_{L^q(\Omega)} \ ,$$

and then it is clear that for all $1 \leq q < \frac{N}{N-4}$ Corollary 3.6.1 implies also Theorem 3.6.1. But in the limit case $q = \frac{N}{N-4}$ inequality (3.6.13) fails, because the remainder term with the L^q-norm is too big, while the inequality (3.6.1) still holds, since the remainder term with the weak L^q-norm is slightly smaller.

Chapter 4

A supercritical semilinear biharmonic problem with Hardy potential

The results written in this Chapter are obtained in collaboration with María Medina of Universidad Autónoma de Madrid [46].

4.1 A brief history of the problem

Let us consider the following fourth order problem with Hardy potential

$$\begin{cases} (-\Delta)^2 u = \dfrac{u^{p-1}}{|x|^4} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $0 < p - 1 < \frac{N+4}{N-4}$ and $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, with $N \geq 5$.

In the Hardy case we have that $\tau = 4$, $\sigma = 4$ and hence $p = 2$ in (3.4.2). When $p - 1 > 1$ or equivalently when $p > 2$ we are in the supercritical case of the Hardy inequality. We want to prove a non-existence result for the preceding problem when the origin is located on the boundary of Ω.

In the second order case the problem

$$\begin{cases} -\Delta u = \dfrac{u^{p-1}}{|x|^2} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1.1}$$

with $N \geq 3$ is well studied. In the subcritical case, that is $0 < p-1 < 1$, it is simple to prove existence of weak solutions, independently of the location of the origin, by using the Hardy inequality as in [1]. If $p = 2$, that is in the critical case, the problem was studied by Ghoussoub-Kang in [34] and by Ghoussoub-Robert in [36]. To be more precise, they studied the problem in the more general setting of critical Hardy-Sobolev potential

$$\begin{cases} -\Delta u = \dfrac{u^{\sigma-1}}{|x|^\tau} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \setminus \{0\}, \end{cases}$$

for $0 \leq \tau \leq 2$ and $\sigma = \frac{2(N-\tau)}{N-2}$. They gave sufficient local conditions on the boundary at 0, precisely a condition of negativity of the curvatures and the mean curvature at 0, for the best constant in the corresponding embedding to be attained, which yields existence of a solution to the preceding problem.

Finally J. Dávila and I. Peral studied in [22] the problem in the supercritical setting, that is $1 < p-1 < \frac{N+2}{N-2}$. We recall that $u \in H_0^1(\Omega)$ is a positive *weak solution* of the problem (4.1.1) if $u > 0$ a.e. in Ω and satisfies

$$\int_\Omega \nabla u \cdot \nabla \varphi \, dx = \int_\Omega \frac{u^{p-1}}{|x|^2} \varphi dx, \qquad \left| \int_\Omega \frac{u^{p-1}}{|x|^2} \varphi \, dx \right| < +\infty,$$

for every $\varphi \in H_0^1(\Omega)$. Dávila and Peral proved in [22] the following result

Let $N \geq 3$ and $1 < p-1 < \frac{N+2}{N-2}$. Then problem (4.1.1) *has no positive weak solutions if $0 \in \partial\Omega$ and Ω is a smooth domain star-shaped with respect to* 0*.*

Moreover, they were also able to prove existence of weak solutions for problem (4.1.1) for domains with specific geometry, in particular for *dumbbell domains*. A dumbbell domain Ω_δ is a bounded domain with smooth boundary of the form $\Omega_\delta = \Omega_1 \cup \Omega_2 \cup C_\delta$ where Ω_1 and Ω_2 are smooth bounded domains such that $\overline{\Omega}_1 \cap \overline{\Omega}_2 = \emptyset$ and C_δ is a region contained in a tubolar neighborhood of radius less than $\delta > 0$ around a curve joining Ω_1 and Ω_2. In this case, they proved the following existence result.

Let $N \geq 3$ and $1 < p-1 < \frac{N+2}{N-2}$. Assume that Ω_δ is a dumbbell domain with $0 \in \partial\Omega_1 \cap \partial\Omega_\delta$ then there exists a $\delta_0 > 0$ such that if $\delta < \delta_0$ then there exists a positive weak solution of problem (4.1.1) *in Ω_δ.*

There are some generalizations of problem (4.1.1), for example in the case of more generic second order operators, that is in the case of p-Laplacian operator, as in the series of papers [47], [48] and [49].

We want to generalize the non existence result of [22] to the setting of the biharmonic problem with Dirichlet boundary conditions, that is we want to prove non existence of positive weak solutions of the following problem

$$
(4.1.2) \qquad \begin{cases} (-\Delta)^2 u = \dfrac{u^p}{|x|^4} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}
$$

with $N \geq 5$, Ω a bounded smooth domain star-shaped with respect to $0 \in \partial\Omega$ and $1 < p < \frac{N+4}{N-4}$. [1]

Still, we are not able to generalize the second part of [22] to the problem (4.1.2). Indeed, the original proof of Dávila-Peral is heavily based on maximun principles and comparison principles which are not available in the setting of biharmonic problems with Dirichlet boundary conditions for dumbbell domains.

4.2 Regularity of solutions

Concerning regularity, in this Chapter we distinguish two different kinds of solutions. More precisely, if we consider a general problem

$$
(4.2.1) \qquad \begin{cases} (-\Delta)^2 u = f \text{ in } \Omega, \\ u = \dfrac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega, \end{cases}
$$

by *weak* and *strong* solutions we mean the following.

Definition 4.2.1. We say that $u \in H_0^2(\Omega)$ is a positive *weak solution* of the problem (4.2.1) if $u > 0$ a.e. in Ω and satisfies

$$
\int_\Omega \Delta u \Delta \varphi \, dx = \int_\Omega f \varphi dx, \qquad \left| \int_\Omega f \varphi \, dx \right| < +\infty,
$$

for every $\varphi \in H_0^2(\Omega)$.

Definition 4.2.2. We say that u is a positive *strong solution* of the problem (4.2.1) if $u \in H_0^2(\Omega) \cap W^{4,q}(\Omega)$ for some $q > 1$, $u > 0$ a.e. in Ω, and the equation and the boundary conditions in (4.2.1) are satisfied a.e.

The purpose of this section is to prove that positive weak solutions of problem (4.1.2) are actually much more regular. Indeed we will see that solutions of (4.1.2) are positive

[1] We notice that here the notation is changed. Indeed, in problem (4.1.2) we denote the exponent by p while in the notation of the preceding chapter it was denoted by $p-1$. Since we are only interested in the differential problem and not in its variational counterpart it is more comfortable to work with the simpler form p and not with $p-1$.

strong solutions of the same problem and, moreover, that they are smooth away from the origin.

From [32, Theorem 2.20 and Corollary 2.21], we know that if we consider $f = f(u) \in L^q(\Omega)$ with $q > 1$ in the general problem (4.2.1), then every weak solution of problem (4.2.1) is also a positive strong solution of the same problem. Moreover, there exists a constant $C = C(\Omega, N) > 0$ such that

$$\|u\|_{W^{4,q}(\Omega)} \leq C\|f(u)\|_{L^q(\Omega)}.$$

First of all, we prove that a priori a positive weak solution of (4.1.2) is also in $W^{4,q}(\Omega)$ for some $q > 1$. And moreover u satisfies the equation a.e, that is weak solutions are also strong solutions.

Proposition 4.2.1. *Let $u \in H_0^2(\Omega)$ be a positive weak solution of problem* (4.1.2). *Then u is a positive strong solution of the same problem, and in particular, $u \in W^{4,q}(\Omega)$ for every $1 < q < \frac{N(p+1)}{Np+4}$.*

Proof. Consider $f(x,u) := \dfrac{u^p}{|x|^4}$. We want to check if $f(x,u) \in L^q(\Omega)$ for some $q > 1$. Let us take in particular $q \in (1, \frac{N(p+1)}{Np+4})$. Thus, applying Hölder inequality we find that

$$\int_\Omega |f(x,u)|^q\, dx = \int_\Omega \frac{u^{pq}}{|x|^{4q}}\, dx \leq \left(\int_\Omega \frac{u^{p+1}}{|x|^4}\, dx\right)^{1/r} \left(\int_\Omega |x|^{-\frac{4q}{p+1-pq}}\, dx\right)^{1/s},$$

with $r = \frac{p+1}{pq}$ and $s = \frac{p+1}{p+1-pq}$. Since $q < \frac{N(p+1)}{Np+4}$, then $\frac{4q}{p+1-pq} < N$ and the second integral is finite. Moreover, since u is a weak solution of (4.1.2), testing in Definition 4.2.1 with $\varphi := u$ we know that necessarily

$$\int_\Omega \frac{u^{p+1}}{|x|^4}\, dx < +\infty,$$

and therefore $f \in L^q(\Omega)$ provided that $q \in (1, \frac{N(p+1)}{Np+4})$. $\square$

We notice here that, in order to have $u \in \mathcal{C}^{0,\gamma}(\overline{\Omega})$ for some $0 < \gamma < 1$ directly by Sobolev embeddings, we need that

$$\left\lfloor \frac{N}{q} \right\rfloor \leq 3,$$

and hence $q > \frac{N}{4}$.[2] Since by Proposition 4.2.1 $q < \frac{N(p+1)}{Np+4}$ and by assumptions $1 < p < \frac{N+4}{N-4}$ and $N \geq 5$, we have that $q < \frac{N}{4}$. So, we can not bootstrap directly from a strong solution to a continuous solution.

[2]For Morrey embeddings if $u \in W^{k,p}(\Omega)$ then $u \in \mathcal{C}^{k-\lfloor \frac{N}{p} \rfloor -1}(\overline{\Omega})$.

Notice that in the previous result no assumption on the position of the origin is made. We are assuming that 0 can be either inside or on the boundary of Ω. Let us suppose now that $0 \in \partial\Omega$, and define, for fixed $\varepsilon > 0$,

$$\Gamma_\varepsilon := B_\varepsilon(0) \cap \Omega, \qquad \Omega_\varepsilon := \Omega \setminus \overline{\Gamma_\varepsilon}. \tag{4.2.2}$$

The boundary of Ω and the boundary of Γ_ε are smooth but the boundary of Ω_ε, which is exactly

$$\partial\Omega_\varepsilon := \partial\Omega_\varepsilon^1 \cup \partial\Omega_\varepsilon^2 = (\partial\Omega \setminus (\partial\Omega \cap \Gamma_\varepsilon)) \cup (\partial\Gamma_\varepsilon \cap \Omega)\,, \tag{4.2.3}$$

is not smooth in the intersection of the two components $\partial\Omega_\varepsilon^1$ and $\partial\Omega_\varepsilon^2$. Nevertheless, we may assume that $\partial\Omega_\varepsilon$ is smooth, since locally it is a Lipschitz graph and hence it can be smoothened by a standard mollification argument. Therefore, without loss of generality, we will assume that Ω_ε is a smooth domain for every $\varepsilon > 0$.

We want to prove that solutions of (4.1.2) are indeed smooth far from the origin. The idea is to consider a cut-off function and to prove that the solution multiplied by the cut-off function is regular far from the origin.

Proposition 4.2.2. *Suppose $0 \in \partial\Omega$. If u is a weak solution of problem* (4.1.2)*, then for every $\varepsilon > 0$ we have that $u \in \mathcal{C}^{0,\gamma}(\overline{\Omega_\varepsilon})$ for some $0 < \gamma < 1$, where Ω_ε is defined in* (4.2.2)*.*

Proof. Fix $\varepsilon > 0$ and consider a smooth function η such that $0 \le \eta \le 1$ and

$$\eta(x) := \begin{cases} 0 & |x| \le \varepsilon \\ > 0 & \varepsilon < |x| < 2\varepsilon \\ 1 & |x| \ge 2\varepsilon. \end{cases}$$

Let us define also the following function,

$$w(x) := \eta(x)u(x), \qquad x \in \overline{\Omega}. \tag{4.2.4}$$

First of all we observe that w is well defined in Ω_ε and w has zero Dirichlet values on the boundary $\partial\Omega_\varepsilon$. Indeed u has zero Dirichlet conditions on $\partial\Omega_\varepsilon^1$ by assumption, while $\eta \equiv 0$ on $\partial\Omega_\varepsilon^2$. Using the cut-off η we are avoiding the origin, where the problem is singular, obtaining a semilinear problem for the function w. Indeed, let us take $\varphi \in C_0^\infty(\Omega_\varepsilon)$. Thus, using that u is a weak solution of (4.1.2),

$$\begin{aligned} \int_{\Omega_\varepsilon} \Delta w \Delta\varphi dx = \int_{\Omega_\varepsilon} \varphi\eta f\, dx + \int_{\Omega_\varepsilon} \varphi \big(&2\Delta u\Delta\eta + u\Delta^2\eta + 2\nabla\Delta u \cdot \nabla\eta \\ &+ 2\nabla\Delta\eta \cdot \nabla u + 2\operatorname{div}(D^2u \cdot \nabla\eta) + 2\operatorname{div}(D^2\eta \cdot \nabla u)\big)\, dx, \end{aligned} \tag{4.2.5}$$

where $f = f(x,u) := \dfrac{u^p}{|x|^4}$. Notice that, by Proposition 4.2.1, we know that $u \in W^{4,q}(\Omega)$ with $q > 1$, and therefore all the previous computations are justified in the weak sense. Moreover, by a density argument, we can say that (4.2.5) holds for every $\varphi \in H_0^2(\Omega_\varepsilon)$, and thus we can conclude that w is a positive weak solution of the problem

$$\begin{cases} \Delta^2 w = F(x,u,\eta) & \text{in } \Omega_\varepsilon \\ w = \dfrac{\partial w}{\partial \nu} = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{4.2.6}$$

$$\begin{aligned} F(x,\eta,u) :=& u\Delta^2\eta + \eta f + 2\Delta u \Delta\eta + 2\nabla\Delta u \cdot \nabla\eta + 2\nabla\Delta\eta \cdot \nabla u \\ &+ 2\operatorname{div}(D^2 u \cdot \nabla\eta) + 2\operatorname{div}(D^2\eta \cdot \nabla u). \end{aligned} \tag{4.2.7}$$

Now we want to compute r such that $F(x,\eta,u) \in L^r(\Omega_\varepsilon)$. First of all we observe that

$$\begin{aligned} |\operatorname{div}(D^2 u \cdot \nabla\eta)| &= \left|\sum_l \partial_l (D^2 u \cdot \nabla\eta)_l\right| = \left|\sum_l \partial_l \left(\sum_j u_{lj}\eta_j\right)\right| = \left|\sum_{l,j} u_{llj}\eta_j + \sum_{l,j} u_{lj}\eta_{jl}\right| \\ &\le C_1 \sum_{l,j} |u_{llj}| + C_2 \sum_{l,j} |u_{lj}|.^{3} \end{aligned}$$

Here we used the fact that for weak derivatives the symmetry of mixed derivatives always holds. The use of formal integration by parts to define weak differentiation puts the symmetry question back onto the test functions which are smooth and satisfy the Schwartz theorem. Then we have

$$\begin{aligned} \|F\|_{L^r(\Omega_\varepsilon)} \le C\Bigg(& \|u\|_{L^r(\Omega_\varepsilon)} + \|f\|_{L^r(\Omega_\varepsilon)} + \|\nabla u\|_{L^r(\Omega_\varepsilon)} + \|\Delta u\|_{L^r(\Omega_\varepsilon)} \\ &+ \|\nabla\Delta u\|_{L^r(\Omega_\varepsilon)} + \left(\int_{\Omega_\varepsilon} \sum_{i,j} |u_{iij}|^r dx\right)^{1/r} + \left(\int_{\Omega_\varepsilon} \sum_{i,j} |u_{ij}|^r dx\right)^{1/r}\Bigg). \end{aligned} \tag{4.2.8}$$

Notice that, since $x \in \Omega_\varepsilon$, the term f is not singular anymore. In particular, $f(x,u) = \frac{u^p}{|x|^4} \le Cu^p$. Therefore, using that $u \in H^2(\Omega) \cap W^{4,q}(\Omega_\varepsilon)$, and the Sobolev embeddings associated to every order of derivatives, we can assure that F belongs to $L^r(\Omega_\varepsilon)$ if

$$\begin{aligned} r &:= \min\left\{\max\left\{\frac{2N}{(N-4)p}, \frac{Nq}{(N-4q)p}\right\}, \frac{Nq}{N-4q}, \frac{Nq}{N-3q}, \frac{Nq}{N-2q}, \frac{Nq}{N-q}\right\} \\ &= \min\left\{\max\left\{\frac{2N}{(N-4)p}, \frac{Nq}{(N-4q)p}\right\}, \frac{Nq}{N-q}\right\}.^{4} \end{aligned}$$

By [32, Corollary 2.21] we get that $u \in W^{4,r}(\Omega_\varepsilon)$. Now if $r \ge \frac{N}{4}$ we conclude. Indeed, if $r > \frac{N}{4}$ we can directly apply the Sobolev embedding theorem to conclude that w,

[3] We are using the following notation. If $A := (a_{(1)}, ..., a_{(d)})^T$ is a vector in $\mathbb{R}^d$ then $(A)_l := a_{(l)}$ is his lth coordinate. If $u : \mathbb{R}^d \to \mathbb{R}$ is a function in x then $u_l := \frac{\partial u}{\partial x_l}$. Hence $\operatorname{div}(A) := \sum_l \partial_l (A)_l = \sum_l \partial_l a_{(l)}$.

[4] The numbers in the definition of r come from the Sobolev embeddings. Since $D^j u \in W^{4-j,q}$ then $D^j \in L^{\frac{Nq}{N-(4-j)q}}$ for $j = 0,1,2,3$. Moreover, since $u \in H^2 \cap W^{4,q}$ then $u^p \in L^\alpha$ with $\alpha := \max\left\{\frac{2N}{(N-4)p}, \frac{Nq}{(N-4q)p}\right\}$.

and therefore u, belongs to $\mathcal{C}^{0,\gamma}(\overline{\Omega_\varepsilon})$ for some $0 < \gamma < 1$. If $r = \frac{N}{4}$ we are in the limit case of the Sobolev embeddings and we can conclude that $u, w \in L^q(\Omega_\varepsilon)$ for every $q \geq 1$. Repeating the preceding argument, we obtain that there exists an r such that $r = r(q) > \frac{N}{4}$.

Suppose now that $r < \frac{N}{4}$. The idea is to perform an iterative argument, via a bootstrapping technique, to reach the desired integrability. More precisely, let us define

$$\begin{aligned} &q_0 := 0, \\ &q_k := \min\left\{\max\left\{\frac{2N}{(N-4)p}, \frac{Nq_{k-1}}{(N-4q_{k-1})p}\right\}, \frac{Nq_{k-1}}{N-q_{k-1}}\right\},\ k \in \mathbb{N}. \end{aligned} \tag{4.2.9}$$

Firstly, notice that if $\{q_k\}$ is non constant, that is $q_k \not\equiv \frac{2N}{(N-4)p}$, since $q_{k-1} < \frac{N}{4}$ because if not we can stop the iterative argument in $k-1$, then $\{q_k\}_{k\in\mathbb{N}}$ is a non decreasing sequence. This is not enough to assure that q_k overcome the value $\frac{N}{4}$ for some k. Indeed, we need to study the behavior of the increments $q_k - q_{k-1}$, to exclude the case where this quantity tends to 0. Suppose first that

$$q_k = \frac{Nq_{k-1}}{N-q_{k-1}}.$$

Thus,

$$q_k - q_{k-1} = \frac{q_{k-1}^2}{N-q_{k-1}}, \qquad q_{k-1} - q_{k-2} = \frac{q_{k-2}^2}{N-q_{k-2}}.$$

Let us define

$$g(x) := \frac{x^2}{N-x}, \qquad x < \frac{N}{4}.$$

It is easy to check that for this range of values of x, $g'(x) > 0$, that is, g is increasing. Therefore, since $\{q_k\}_{k\in\mathbb{N}}$ is non decreasing, we can assure that for every $k \in \mathbb{N}$,

$$q_k - q_{k-1} \geq q_1 - q_0 =: c_1 > 0, \tag{4.2.10}$$

where c_1 is independent of k. On the other hand, let us suppose that

$$q_k = \max\left\{\frac{2N}{(N-4)p}, \frac{Nq_{k-1}}{(N-4q_{k-1})p}\right\}.$$

In particular, it can be checked that

$$\frac{2N}{(N-4)p} \geq \frac{Nq_{k-1}}{(N-4q_{k-1})p} \text{ if and only if } q_{k-1} \leq \frac{2N}{N+4}.$$

Thus, assume first

$$q_k = \frac{2N}{(N-4)p},$$

and consider the function

$$h(x) = \frac{2N}{(N-4)p} - x, \qquad x \leq \frac{2N}{N+4}.$$

By definition,

$$h(x) \geq \frac{2N}{(N-4)p} - \frac{2N}{N+4} =: c_2,$$

with c_2 independent of x. Notice that $c_2 > 0$ if and only if $p < \frac{N+4}{N-4}$, that is true by the general assumptions on p. Therefore, we can conclude that also in this case

$$q_k - q_{k-1} \geq =: c_2 > 0, \tag{4.2.11}$$

with c_2 independent of k. Finally, suppose that

$$q_k = \frac{Nq_{k-1}}{(N-4q_{k-1})p}, \text{ i.e., } q_k > \frac{2N}{N+4}.$$

Consider

$$\tau(x) := \frac{Nx}{(N-4x)p} - x, \qquad x > \frac{2N}{N+4}.$$

Hence,

$$\tau(x) = \frac{x(N-Np+4xp)}{(N-4x)p} > \frac{x(N-Np+4xp)}{Np}.$$

Moreover, since $x > \frac{2N}{N+4}$,

$$N - Np + 4xp > N - Np + 4\frac{2N}{N+4}p > 0 \text{ if and only if } p < \frac{N+4}{N-4}.$$

Therefore, there exists $c_3 > 0$ such that $\tau(x) > c_3$ for all $x > \frac{2N}{N+4}$. Thus, in this case

$$q_k - q_{k-1} \geq =: c_3 > 0, \tag{4.2.12}$$

with c_3 independent of k. Finally, putting together (4.2.10), (4.2.11) and (4.2.12), we can assure that

$$q_k - q_{k-1} > c := \min\{c_1, c_2, c_3\} > 0,$$

with c independent of k.
That is, in every iteration we increase at least by a fixed quantity. Therefore, in a finite number of steps we obtain necessarily that $q_k > \frac{N}{4}$ and we finish. More precisely, if we start with $F \in L^{q_0}(\Omega)$, after k iterations we know $F \in L^{q_k}(\Omega)$, and therefore, once $q_k > \frac{N}{4}$, we conclude again by the Sobolev embeddings theorem. □

Proposition 4.2.3. *Suppose* $0 \in \partial\Omega$. *If* u *is a weak solution of problem* (4.1.2)*, then for every* $\varepsilon > 0$ $u \in \mathcal{C}^{\alpha,\gamma}(\overline{\Omega_\varepsilon}) \cap W^{\alpha+4,q}(\Omega_\varepsilon)$*, where* $\alpha := \lfloor p \rfloor$*,* $q > \frac{N}{4}$ *and* $0 < \gamma < 1$.

Proof. We prove first that $u \in C^{1,\gamma}(\overline{\Omega_\varepsilon})$. Consider w defined in (4.2.4), satisfying problem (4.2.6). In Proposition 4.2.2 we obtained that w, and therefore u, belong to $\mathcal{C}^{0,\gamma}(\overline{\Omega_\varepsilon})$

by means of an interative argument. More precisely, we proved that F, defined in (4.2.7), is in $L^{q_k}(\Omega_\varepsilon)$ with $q_k > \frac{N}{4}$.
The idea now is to reproduce this argument to see that F belongs not only to $L^{q_k}(\Omega_\varepsilon)$, but also to $W^{1,q_k}(\Omega_\varepsilon)$. Therefore, we have to compute the gradient of F, and check its integrability. By straightforward computations we find that

$$|\nabla F(u)| \leq C_0|\nabla u| + C_1 p u^{p-1}|\nabla u| + C_2\Delta u + C_3|\nabla \Delta u| + C_4\Delta^2 u.$$

Since $F \in L^{q_k}(\Omega_\varepsilon)$, we knew $u \in W^{4,q_k}(\Omega_\varepsilon)$, that is, every derivative up to order 4 belongs to $L^{q_k}(\Omega_\varepsilon)$. Moreover, since $u \in \mathcal{C}^{0,\gamma}(\overline{\Omega_\varepsilon})$ and $p > 1$, we can bound uniformly the term u^{p-1}. Thus, $\nabla F \in L^{q_k}(\Omega_\varepsilon)$, and therefore $F \in W^{1,q_k}(\Omega_\varepsilon)$. Hence, by [32, Corollary 2.21] we obtain that $u \in W^{5,q_k}(\Omega_\varepsilon)$.
Finally, since $q_k > \frac{N}{4}$, by [25, Theorem 6] we conclude $u \in \mathcal{C}^{1,\gamma}(\overline{\Omega_\varepsilon})$ for some $0 < \gamma < 1$. Likewise, if $p > 2$, we can reproduce this argument to check if $F \in W^{2,q_k}(\Omega_\varepsilon)$. In such a case, $u \in W^{6,q_k}(\Omega_\varepsilon)$ and finally $u \in C^{2,\gamma}(\overline{\Omega_\varepsilon})$. But again, $D^2F \in L^{q_k}$ it is just a consequence of the fact that $u \in W^{5,q_k}(\Omega_\varepsilon) \cap \mathcal{C}^{1,\gamma}(\overline{\Omega_\varepsilon})$, since

$$|u^{p-2}| \leq C, \qquad |\nabla u| \leq C, \qquad \text{and} \qquad D^\beta u \in L^{q_k}(\Omega_\varepsilon),\ \beta \leq 5.$$

Iterating this argument, if $p \geq \alpha$, we obtain $u \in C^{\alpha,\gamma}(\overline{\Omega_\varepsilon}) \cap W^{\alpha+4,q_k}(\Omega_\varepsilon)$ as a consequence of $u \in C^{\alpha-1,\gamma}(\overline{\Omega_\varepsilon}) \cap W^{\alpha+3,q_k}(\Omega_\varepsilon)$ with $q_k > \frac{N}{4}$ and $0 < \gamma < 1$. □

Finally, we can prove that the solutions are indeed smooth in Ω_ε.

Proposition 4.2.4. *Suppose* $0 \in \partial\Omega$. *If* u *is a weak solution of problem* (4.1.2), *then for every* $\varepsilon > 0$ *and for every* $k \in \mathbb{N}$, $u \in \mathcal{C}^\infty(\Omega_\varepsilon) \cap W^{k,q}(\Omega_\varepsilon)$.

Proof. By Proposition 4.2.3, we know $u \in \mathcal{C}^{\alpha,\gamma}(\overline{\Omega_\varepsilon}) \cap W^{\alpha+4,q}(\Omega_\varepsilon)$ for $\alpha \leq p$, $q > \frac{N}{4}$ and $0 < \gamma < 1$. Let $x^0 \in \Omega_\varepsilon$ and let $r > 0$ such that $B_r(x^0) \subset \Omega_\varepsilon$. Taking ε small enough in Proposition 4.2.3, we know that $u \in \mathcal{C}^{\alpha,\gamma}(\overline{B_r(x^0)}) \cap W^{\alpha+4,q}(B_r(x^0))$. In particular, since u is positive and continuous in B_r,

$$\text{there exists a constant } \rho > 0 \text{ such that } u(x) \geq \rho,\ \forall\, x \in B_r(x^0). \tag{4.2.13}$$

Suppose then that $\alpha > p$. Let us define now $w(x) := \eta(x)u(x)$, where in this case η is a smooth cut-off function supported in B_r. Thus, one can reproduce the proof of Proposition 4.2.2 to check that w satisfies the problem (4.2.6) but in B_r instead of Ω_ε. Hence, as in the proof of Proposition 4.2.3, we reduce the problem to control the $W^{\alpha+4,q}(B_r)$ regularity. But the conclusion follows in a very similar way, only taking into account that in this case $p - \alpha < 0$ and thus, to control the term $u^{p-\alpha}$ instead of the continuity we need to use (4.2.13). Then we obtained that the function u is $\mathcal{C}^\infty$ in every point x^0 of Ω_ε and then we get that $u \in \mathcal{C}^\infty(\Omega_\varepsilon)$. □

4.3 A priori estimates

For second order equations an important tool is the Gidas-Spruck estimate [38, Theorem 1.1]. There is also a polyharmonic version of the same result [62, Theorem 1] and [63, Theorem 1]. To apply these kind of results we need to have some informations about the boundary regularity of our solutions. But a priori we can not assure that positive weak solutions of (4.1.2) are continuous up to the boundary of Ω, due to the presence of the singularity in the origin, which is located on the boundary. However we can use a scaling argument, as in the proof of [22, Lemma 2.2], to reduce ourselves to a situation in which we can apply a *blow-up* argument to reduce the problem to a Liouville equation, [78, Theorem 1.4] for the entire space, or [68] for the half-space with Dirichlet conditions. The key point of the proof is to avoid the possibility that the positive blow-ups of the function are exploding in the origin. This kind of argument is heavily based on the fact that the weight we have in the forcing term $f(x,u)$ is coherent with the number of derivatives we are considering. So, for second order problem, we have the weight $\frac{1}{|x|^2}$, [22], and for our fourth order problem we have $\frac{1}{|x|^4}$.

Proposition 4.3.1. *Suppose* $0 \in \partial\Omega$. *If* u *is a weak solution of problem* (4.1.2), *then* $u \in L^\infty(\Omega)$.

Proof. Step 1. Suppose $x_0 \in \Omega$. Define

$$r = r(x_0) := \frac{1}{2}\operatorname{dist}(x_0, \partial\Omega) := \frac{1}{2}\inf_{y\in\partial\Omega}|x_0 - y|,$$

Then $B_r(x_0) \subset \Omega$ and $0 \notin B_r(x_0)$. Finally notice that

$$|x_0| \geq 2r \quad \forall\, x_0 \in \Omega. \tag{4.3.1}$$

Indeed suppose by contradiction that there exists a $\tilde{x}_0 \in \Omega$ such that $|\tilde{x}_0| < 2r$ then we have

$$r = \frac{1}{2}\inf_{y\in\partial\Omega}|\tilde{x}_0 - y| = \frac{1}{2}|\tilde{x}_0| < r.$$

Now we define the *rescaled function*

$$v_{x_0}(y) := u(x_0 + ry) \quad \forall\, y \in \frac{\Omega - x_0}{r} = B_1(0), \tag{4.3.2}$$

that satisfies

$$\Delta^2 v_{x_0} = \frac{r^4}{|x_0 + ry|^4} v_{x_0}^p \quad \text{in } B_1(0). \tag{P_{x_0}}$$

Since $0 \notin B_r(x_0)$ we have that u is smooth in $B_r(x_0)$ and hence the rescaled function v_{x_0} is also smooth in $B_1(0)$. So v_{x_0} satisfies equation (P_{x_0}) pointwise. Thus, noticing that $u(x_0) = v_{x_0}(0)$, if we prove the existence of a constant such that $|v_{x_0}(0)| \le C$ for every x_0 and for every solution v_{x_0} to (P_{x_0}), the result follows.

Step 2. Assume by contradiction that for every constant $C > 0$, there exists a point x_0 and a solution v_{x_0} to the problem (P_{x_0}) satisfying $v_{x_0}(0) > C$.
Thus, we can build a sequence of pairs $\{(x_{0,k}, v_{x_{0,k}})\}$, where $v_{x_{0,k}}$ is a solution of problem

$$(P_{x_{0,k}}) \qquad \Delta^2 v_{x_0,k} = \frac{r_k^4}{|x_{0,k} + r_k y|^4} v_{x_0,k}^p \quad \text{in } B_1(0),$$

with $r_k := r(x_{x_0,k})$, so that

$$M_k := \|v_{x_{0,k}}\|_{L^\infty(B_1(0))} \to \infty, \text{ as } k \to \infty.$$

For simplicity, we denote $v_k := v_{x_{0,k}}$, but recalling that for every k, v_k can be a solution to a different problem.

Let us now define the *blow-up functions*

$$w_k(y) := \frac{1}{M_k} v_k \left(M_k^{\frac{1-p}{4}} y \right). \tag{4.3.3}$$

Thus, w_k is well defined in $B_{\rho_k}(0)$, where $\rho_k := M_k^{\frac{p-1}{4}}$, and

$$\|w_k\|_{L^\infty(B_{\rho_k}(0))} = 1, \text{ and } w_k(0) \le 1 \text{ for every } k \in \mathbb{N}. \tag{4.3.4}$$

In such a case, clearly

$$\lim_{k \to +\infty} \rho_k = +\infty.$$

Moreover,

$$D^\alpha w_k(y) = M_k^{\frac{1-p}{4}|\alpha| - 1} D^\alpha v_k \left(M_k^{\frac{1-p}{4}} y \right),$$

and thus,

$$\Delta^2 w_k(y) = M_k^{-p} \Delta^2 v_k \left(M_k^{\frac{1-p}{4}} y \right) \quad \text{in } B_{\rho_k}(0),$$

Now from $(P_{x_{0,k}})$ we deduce

$$\begin{aligned}
\Delta^2 w_k(y) &= M_k^{-p} \Delta^2 v_k \left(M_k^{\frac{1-p}{4}} y \right) \\
&= M_k^{-p} \frac{r_k^4}{\left| x_{0,k} + r_k \left(M_k^{\frac{1-p}{4}} y \right) \right|^4} v_{x_0,k}^p \left(M_k^{\frac{1-p}{4}} y \right) \\
&=: f_k(y)
\end{aligned}$$

where

$$f_k(y) := M_k^{-p} \frac{r_k^4}{|x_{0,k} + r_k z_k|^4} v_k^p(z_k) = \frac{r_k^4}{|x_{0,k} + r_k z_k|^4} w_k^p(y) \quad z_k := M_k^{\frac{1-p}{4}} y.$$

Then, since $y \in B_{\rho_k(0)}$,

$$|z_k| \le M_k^{\frac{1-p}{4}} |y| \le 1,$$

and hence, recalling the definition of r_k and (4.3.1),

$$|x_{0,k} + r_k z_k| \ge |x_{0,k}| - r_k |z_k| \ge 2r_k - r_k = r_k. \tag{4.3.5}$$

Thus,

$$f_k(y) = \frac{r_k^4}{|x_{0,k} + r_k z_k|^4} w_k^p(y) \le w_k^p(y) \quad \forall\, k \in \mathbb{N},$$

and by (4.3.4) we conclude that

$$\|f_k\|_{L^\infty(B_{\rho_k}(0))} \le 1 \quad \forall\, k \in \mathbb{N}. \tag{4.3.6}$$

Therefore, since $\rho_k \to +\infty$, using [62, Corollary 6] on the balls $B_R(0)$ with $R > 0$ and (4.3.6), we find that for every $q \in (1, +\infty)$

$$\begin{aligned} \|w_k\|_{W^{4,q}(B_R)} &\le C(q,R) \left(\|f_k\|_{L^q(B_R)} + \|w_k\|_{L^q(B_R)} \right) \\ &\le C(q,R) \left(\|f_k\|_{L^\infty(B_R)} + \|w_k\|_{L^\infty(B_R)} \right) \\ &\le C(q,R). \end{aligned}$$

[62, Corollary 6] is a direct consequence of the general regularity results by Agmon-Douglis-Nirenberg, [4], using a cut-off function. Now using the Morrey Immersion and choosing $q > N$ we find that also

$$\|w_k\|_{\mathcal{C}^{3,\gamma}(\overline{B_R})} \le C(q,R,N) \quad \text{for some } \gamma \in (0,1),$$

so we may extract a subsequence, which we still denote by $\{w_k\}_{k\in\mathbb{N}}$ such that

$$w_k \to w \text{ in } \mathcal{C}^{3,\gamma}(\overline{B_R}) \text{ for every } R > 0, \tag{4.3.7}$$

so that $w \in \mathcal{C}^{3,\gamma}_{loc}(\mathbb{R}^N)$. Moreover by (4.3.7) $\|w\|_{L^\infty(\mathbb{R}^N)} = 1$. Furthermore, since for every compact set $K \subset \mathbb{R}^N$ the sequence $\{f_k\}_{k\in\mathbb{N}}$ is uniformly bounded in $L^\infty(K)$ by (4.3.6), we can say by Banach-Alaoglu Theorem, that, up to a subsequence, $f_k \overset{*}{\rightharpoonup} F$ in $L^\infty(K)$. Now we want to compute the limit function F. Since by (4.3.5) we deduce that the weight is uniformly bounded

$$0 \le \frac{r_k^4}{|x_{0,k} + r_k z_k|^4} \le 1,$$

and by (4.3.7), we can say that, up to a subsequence, there exists a positive constant $C > 0$, not depeding on k, such that f_k is converging to

$$F(y) = C w^p(y).$$

Since we may assume that $w_k \to w$ in $W^{2,q}_{loc}(\mathbb{R}^N)$, we have that w is a bounded weak solution of the problem

$$\Delta^2 w = C w^p(y) \text{ in } \mathbb{R}^N. \tag{4.3.8}$$

Furthermore, since $F = Cw^p \in L^\infty(\mathbb{R}^N)$, by bootstraping we obtain that

$$w \in W^{4,q}_{loc}(\mathbb{R}^N) \cap \mathcal{C}^{3,\gamma}_{loc}(\mathbb{R}^N),$$

and therefore, w is a bounded strong solution of (4.3.8). Using now that the right hand side is much more regular, indeed $F \in \mathcal{C}^{3,\gamma}_{loc}$, and Schauder estimates for w, we can conclude that w is also a bounded classical solution of (4.3.8). Now by [62, Lemma A] we can conclude that w is also non negative. Thus, by [78, Theorem 1.4], necessarily $w \equiv 0$, a contradiction with the fact that $\|w\|_{L^\infty(\mathbb{R}^N)} = 1$.

Therefore, we have proved the existence of a constant $C > 0$ such that for every $x_0 \in \Omega$,

$$u(x_0) = v_{x_0}(0) \leq C,$$

and hence $u \in L^\infty(\Omega)$. $\square$

Since by Proposition 4.2.3 weak solutions of problem (4.1.2) are at least C^1 up to the boundary far from the origin, and by Proposition 4.2.1 they satisfy the equation and the boundary conditions almost everywhere, we can say, being $u = 0$ on the boundary of Ω, that there exists a constant $C > 0$ such that

$$u(x) \leq C \quad \forall\, x \in \overline{\Omega} \setminus \{0\}.$$

By Proposition 4.3.1 we can prove, by scaling, a priori estimates on the derivatives of u up to order three.

Corollary 4.3.1. *Suppose* $0 \in \partial\Omega$. *If* u *is a weak solution of problem* (4.1.2) *then there exists a constant* $C > 0$ *such that*

$$|\nabla u(x)| \leq \frac{C}{|x|}, \quad |D^2 u(x)| \leq \frac{C}{|x|^2}, \quad |\nabla \Delta u(x)| \leq \frac{C}{|x|^3} \quad \forall\, x \in \overline{\Omega} \setminus \{0\}.$$

Proof. Let us consider $x_0 \in \overline{\Omega} \setminus \{0\}$. Then we define

$$r := \frac{|x_0|}{2}$$

and $\Omega_r := B_r(x_0) \cap \overline{\Omega}$. From the choice of r we have that $0 \notin \Omega_r$. Now we define, as before, the *rescaled function* $v_{x_0}(y) := u(x_0 + ry)$. for every $y \in \frac{\Omega - x_0}{r} \subset B_1(0)$. Since $|\nabla v_{x_0}(0)| = r|\nabla u(x_0)|$, we have that the thesis is equivalent to prove that there exists a constant $C > 0$ such that

$$|\nabla v_{x_0}(0)| \leq C,$$

for each $x_0 \in \overline{\Omega} \setminus \{0\}$ and for each v_{x_0}. We notice that the rescaled function v_0 satisfies in a classically way the problem (P_{x_0}) in $\frac{\Omega - x_0}{r}$ eventually coupled with the Dirichlet boundary conditions, since we are far from the origin

$$v_{x_0} = \frac{\partial v_{x_0}}{\partial \nu} = 0 \text{ on } \partial\left(\frac{\Omega - x_0}{r}\right) \cap B_1(0). \tag{4.3.9}$$

By the same argument as before and since $|u| \leq C$ we can prove that the right hand side of problem (P_{x_0}) is uniformly bounded by a constant. Indeed

$$\left\| \frac{r^4}{|x_0 + ry|^4} v_{x_0}^p \right\|_{L^\infty\left(\frac{\Omega - x_0}{r}\right)} \leq \sup \frac{r^4}{|x_0 + ry|^4} \left| v_{x_0}^p \right| \leq C,$$

with C which is not depending on x_0, by Proposition 4.3.1. [5]

Now v_{x_0} is a solution of the more general fourth order problem

$$\begin{cases} L v_{x_0} = \dfrac{r^4}{|x_0 + ry|^4} v_{x_0}^p \text{ in } B_1^+ \\ v_{x_0} = \dfrac{\partial v_{x_0}}{\partial \nu} = 0 \text{ on } \{x_1 = 0\}, \end{cases}$$

where the coefficients of the elliptic operator L are smooth. [6] Now using [62, Corollary 6 part (ii)] we can conclude that

$$\|v_{x_0}\|_{W^{4,q}(B_1^+)} \leq C\left(\|u\|_{L^q} + \|f\|_{L^q}\right) \leq C.$$

Using Morrey immersion and choosing $q > N$, we can say that

$$v_{x_0} \in \mathcal{C}^{3,\gamma}(\overline{B_1^+}) \quad \text{ for some } 0 < \gamma < 1.$$

Moreover the bound on the norm is uniform since it depends on the bound of $f := \frac{r^4}{|x_0 + ry|^4} v_{x_0}^p$ which is uniform. So we have that there exists a constant $C > 0$ such that

$$\|v_{x_0}\|_{\mathcal{C}^{3,\gamma}(\overline{B_1^+})} \leq C.$$

Hence we have that $|\nabla v_{x_0}(0)|$, $|D^2 v_{x_0}(0)|$ and $|\nabla \Delta v_{x_0}(0)|$ are uniformly bounded by C. Then by scaling we have the thesis which holds for every $x_0 \in \overline{\Omega} \setminus \{0\}$ since x_0 is arbitrary. □

[5]We notice here that the set $\frac{\Omega - x_0}{r} \subset B_1(0)$ is not a priori connected. Indeed it is possible, for pathological domains, that Ω_r is given by two or more connected components, and the same holds for $\frac{\Omega - x_0}{r}$. Restricting to a small connected neighborhood $\mathcal{U} \subset \frac{\Omega - x_0}{r}$ of 0 we can say that $v_{x_0} \in W^{4,q}(\mathcal{U})$ for every $q \in (1, +\infty)$. Indeed by a diffeomorfism φ we can say that $\mathcal{U}$ is diffeomorphic to the half-ball $B_1^+(0)$ together with zero Dirichlet boundary conditions on $\{x \in B_1 : x_1 = 0\}$.

[6]The coefficients of the elliptic operator L depend also on the diffeomorphism φ of the preceding note.

4.4 Nonexistence of positive solutions in star-shaped domains

The second tool we need to prove non existence of positive weak solutions of problem (4.1.2) is a Pohozaev type identity. The original result [56], is for the classical Laplace operator. Moreover versions of the same identity are well known also for polyharmonic operators , [32, Theorem 7.27] and [32, Theorem 7.29], both for Dirichlet and Navier boundary conditions. In Chapter 3 we proved a Pohozaev type identity, in the easier case with Navier boundary conditions and in which the origin is in the interior of the domain. Here, we need to consider the situation in which the boundary conditions are of Dirichlet type and in which the origin is located on the boundary.

Proposition 4.4.1. *Suppose* $0 \in \partial\Omega$. *If* u *is a weak solution of problem* (4.1.2)*, then the following Pohozaev type identity holds*

$$\int_\Omega |\Delta u|^2\, dx = \frac{2}{p+1}\int_\Omega \frac{u^{p+1}}{|x|^4}\, dx - \frac{1}{N-4}\int_{\partial\Omega} u_{\nu\nu}^2 (x\cdot\nu)\, d\mathcal{H}^{N-1}.$$

Proof. We define Ω_ε as in (4.2.2) for any ε positive. If u is a weak solution of (4.1.2), by Proposition 4.2.2 we know that u is smooth in Ω_ε, and thus, multiplying both sides of the equation by $(x\cdot\nabla u)$, we get

$$\int_{\Omega_\varepsilon} \Delta^2 u (x\cdot\nabla u)\, dx = \int_{\Omega_\varepsilon} \frac{u^{p+1}}{|x|^4}(x\cdot\nabla u)\, dx. \tag{4.4.1}$$

We analyze first the left hand side.

$$\begin{aligned}
\int_{\Omega_\varepsilon} \Delta^2 u (x\cdot\nabla u)\, dx &= \int_{\Omega_\varepsilon} \operatorname{div}(\nabla\Delta u)(x\cdot\nabla u)\, dx\\
&= \int_{\Omega_\varepsilon} \operatorname{div}(\nabla\Delta u\cdot(x\cdot\nabla u))\, dx - \int_{\Omega_\varepsilon}\nabla\Delta u\cdot\nabla(x\cdot\nabla u)\, dx\\
&= \int_{\Omega_\varepsilon} \operatorname{div}(\nabla\Delta u\cdot(x\cdot\nabla u))\, dx - \int_{\Omega_\varepsilon}\nabla\Delta u\cdot\nabla u\, dx\\
&\quad -\int_{\Omega_\varepsilon}\nabla\Delta u\cdot\langle x, D^2u\rangle\, dx\\
&= \int_{\Omega_\varepsilon} \operatorname{div}(\nabla\Delta u\cdot(x\cdot\nabla u))\, dx - \int_{\Omega_\varepsilon}\operatorname{div}(\nabla u\Delta u)\, dx + \int_{\Omega_\varepsilon}|\Delta u|^2\, dx\\
&\quad -\int_{\Omega_\varepsilon}\operatorname{div}(\Delta u\cdot\langle x, D^2u\rangle)\, dx + \int_{\Omega_\varepsilon}\Delta u\operatorname{div}(\langle x, D^2u\rangle)\, dx.
\end{aligned}$$

Moreover,

$$\int_{\Omega_\varepsilon}\Delta u\operatorname{div}(\langle x, D^2u\rangle)\, dx = \int_{\Omega_\varepsilon}|\Delta u|^2\, dx + \int_{\Omega_\varepsilon}\Delta u(x\cdot\nabla\Delta u)\, dx,$$

$$\begin{aligned}\int_{\Omega_\varepsilon} \Delta u(x\cdot\nabla\Delta u)\,dx &= \int_{\Omega_\varepsilon} x\cdot\nabla\left(\frac{|\Delta u|^2}{2}\right)dx\\ &= \int_{\Omega_\varepsilon} \operatorname{div}\left(\frac{x}{2}|\Delta u|^2\right)dx - \frac{N}{2}\int_{\Omega_\varepsilon}|\Delta u|^2\,dx.\end{aligned}$$

Thus, putting all together and applying the Divergence Theorem,

$$\begin{aligned}\int_{\Omega_\varepsilon} \Delta^2 u(x\cdot\nabla u)\,dx &= \int_{\Omega_\varepsilon}\operatorname{div}(\nabla\Delta u\cdot(x\cdot\nabla u))\,dx - \int_{\Omega_\varepsilon}\operatorname{div}(\nabla u\Delta u)\,dx\\ &\quad -\int_{\Omega_\varepsilon}\operatorname{div}(\Delta u\cdot\langle x, D^2u\rangle)\,dx + \int_{\Omega_\varepsilon}\operatorname{div}\left(\frac{x}{2}|\Delta u|^2\right)dx\\ &\quad +\frac{4-N}{2}\int_{\Omega_\varepsilon}|\Delta u|^2\,dx\\ &= \int_{\partial\Omega_\varepsilon}\nabla\Delta u(x\cdot\nabla u)\cdot\nu\,d\mathcal{H}^{N-1} - \int_{\partial\Omega_\varepsilon}\nabla u\Delta u\cdot\nu\,d\mathcal{H}^{N-1}\\ &\quad -\int_{\partial\Omega_\varepsilon}\Delta u\langle x, D^2u\rangle\cdot\nu\,d\mathcal{H}^{N-1} + \int_{\partial\Omega_\varepsilon}\frac{x}{2}|\Delta u|^2\cdot\nu\,d\mathcal{H}^{N-1}\\ &\quad +\frac{4-N}{2}\int_{\Omega_\varepsilon}|\Delta u|^2\,dx.\end{aligned}$$

Finally, applying the boundary conditions and reminding (4.2.3)we reach that

$$\begin{aligned}\int_{\Omega_\varepsilon} \Delta^2 u(x\cdot\nabla u)\,dx =&\int_{\partial\Omega_\varepsilon^2}\nabla\Delta u\cdot(x\cdot\nabla u)\cdot\nu\,d\mathcal{H}^{N-1} - \int_{\partial\Omega_\varepsilon^2}\nabla u\Delta u\cdot\nu\,d\mathcal{H}^{N-1}\\ &-\int_{\partial\Omega_\varepsilon}\Delta u\cdot\langle x, D^2u\rangle\cdot\nu\,d\mathcal{H}^{N-1} + \int_{\partial\Omega_\varepsilon}\frac{x}{2}|\Delta u|^2\cdot\nu\,d\mathcal{H}^{N-1}\\ &+\frac{4-N}{2}\int_{\Omega_\varepsilon}|\Delta u|^2\,dx.\end{aligned}\tag{4.4.2}$$

On the other hand, attending to the right hand side of (4.4.1), we get

$$\begin{aligned}\int_{\Omega_\varepsilon}\frac{u^{p+1}}{|x|^4}(x\cdot\nabla u)\,dx &= \frac{1}{p+1}\int_{\Omega_\varepsilon}\nabla(u^{p+1})\cdot\frac{x}{|x|^4}\,dx\\ &= \frac{1}{p+1}\int_{\Omega_\varepsilon}\operatorname{div}\left(u^{p+1}\frac{x}{|x|^4}\right)dx - \frac{1}{p+1}\int_{\Omega_\varepsilon}u^{p+1}\operatorname{div}\left(\frac{x}{|x|^4}\right)dx\\ &= \frac{1}{p+1}\int_{\Omega_\varepsilon}\operatorname{div}\left(u^{p+1}\frac{x}{|x|^4}\right)dx - \frac{N-4}{p+1}\int_{\Omega_\varepsilon}\frac{u^{p+1}}{|x|^4}\,dx,\end{aligned}$$

and again, applying the Divergence Theorem and the boundary conditions, we deduce

$$\int_{\Omega_\varepsilon}\frac{u^{p+1}}{|x|^4}(x\cdot\nabla u)\,dx = \frac{1}{p+1}\int_{\partial\Omega_\varepsilon^2}u^{p+1}\frac{x}{|x|^4}\cdot\nu\,d\mathcal{H}^{N-1} - \frac{N-4}{p+1}\int_{\Omega_\varepsilon}\frac{u^{p+1}}{|x|^4}\,dx.\tag{4.4.3}$$

Thus, joining (4.4.2) and (4.4.3), we obtain

$$
\begin{aligned}
\frac{N-4}{2}\int_{\Omega_\varepsilon}|\Delta u|^2\,dx =& \int_{\partial\Omega_\varepsilon^2}\nabla\Delta u(x\cdot\nabla u)\cdot\nu\,d\mathcal{H}^{N-1} - \int_{\partial\Omega_\varepsilon^2}\nabla u\Delta u\cdot\nu\,d\mathcal{H}^{N-1}\\
&- \int_{\partial\Omega_\varepsilon}\Delta u\cdot\langle x, D^2u\rangle\cdot\nu\,d\mathcal{H}^{N-1} + \int_{\partial\Omega_\varepsilon}\frac{x}{2}|\Delta u|^2\cdot\nu\,d\mathcal{H}^{N-1}\\
&- \frac{1}{p+1}\int_{\partial\Omega_\varepsilon^2}u^{p+1}\frac{x}{|x|^4}\cdot\nu\,d\mathcal{H}^{N-1} + \frac{N-4}{p+1}\int_{\Omega_\varepsilon}\frac{u^{p+1}}{|x|^4}\,dx
\end{aligned}
\tag{4.4.4}
$$

Finally, we want to pass to the limit when $\varepsilon \to 0$ in this identity. Since u is a weak solution of problem (4.1.2) we have

$$
\int_\Omega|\Delta u|^2dx < +\infty, \qquad \int_\Omega\frac{u^{p+1}}{|x|^4}dx < +\infty.
$$

But then, applying the Dominated Convergence Theorem, we have

$$
\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon}|\Delta u|^2\,dx = \lim_{\varepsilon\to 0}\int_\Omega|\Delta u|^2\chi_{\Omega_\varepsilon}\,dx = \int_\Omega|\Delta u|^2\,dx,
\tag{4.4.5}
$$

and

$$
\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon}\frac{u^{p+1}}{|x|^4}\,dx = \lim_{\varepsilon\to 0}\int_\Omega\frac{u^{p+1}}{|x|^4}\chi_{\Omega_\varepsilon}\,dx = \int_\Omega\frac{u^{p+1}}{|x|^4}\,dx.
\tag{4.4.6}
$$

Moreover using Proposition 4.3.1 and Corollary 4.3.1 we know that

$$
|u(x)| \le C, \quad |\nabla u(x)| \le \frac{C}{|x|}, \quad |\Delta u(x)| \le \frac{C}{|x|^2}, \quad |D^2u(x)| \le \frac{C}{|x|^2}, \quad |\nabla\Delta u(x)| \le \frac{C}{|x|^3},
$$

for all $x \in \Omega$. Then we have that

$$
\lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon^2}\nabla\Delta u(x\cdot\nabla u)\cdot\nu\,d\mathcal{H}^{N-1} = 0 \quad\Longleftrightarrow\quad \left|\lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon^2}\nabla\Delta u(x\cdot\nabla u)\cdot\nu\,d\mathcal{H}^{N-1}\right| = 0.
$$

But then

$$
\begin{aligned}
\left|\lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon^2}\nabla\Delta u(x\cdot\nabla u)\cdot\nu\,d\mathcal{H}^{N-1}\right| &\le \lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon^2}|\nabla\Delta u||x||\nabla u||\nu|\,d\mathcal{H}^{N-1}\\
&\le C\lim_{\varepsilon\to 0}\frac{|\partial\Omega_\varepsilon^2|_{\mathcal{H}^{N-1}}}{\varepsilon^3}\\
&\le C\lim_{\varepsilon\to 0}\varepsilon^{N-4} = 0.
\end{aligned}
$$

So we have that

$$
\lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon^2}\nabla\Delta u(x\cdot\nabla u)\cdot\nu\,d\mathcal{H}^{N-1} = 0.
\tag{4.4.7}
$$

Moreover

$$\left|\lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon^2} u^{p+1}\frac{(x\cdot\nu)}{|x|^4}\,d\mathcal{H}^{N-1}\right| \le \lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon^2}\frac{u^{p+1}}{|x|^3}\,d\mathcal{H}^{N-1} \le C\lim_{\varepsilon\to 0}\varepsilon^{N-4}=0,$$

and

$$\left|\lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon^2}\nabla u\Delta u\cdot\nu\,d\mathcal{H}^{N-1}\right| \le \lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon^2}|\nabla u||\Delta u|\,d\mathcal{H}^{N-1} \le C\lim_{\varepsilon\to 0}\varepsilon^{N-4}=0.$$

So we have

$$\lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon^2}\nabla u\Delta u\cdot\nu\,d\mathcal{H}^{N-1}=0, \tag{4.4.8}$$

and

$$\lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon^2} u^{p+1}\frac{(x\cdot\nu)}{|x|^4}\,d\mathcal{H}^{N-1}=0. \tag{4.4.9}$$

Using Proposition 4.3.1 and Corollary 4.3.1 we can prove that

$$\lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon^2}\frac{x}{2}|\Delta u|^2\cdot\nu\,d\mathcal{H}^{N-1}=0.$$

But then

$$\begin{aligned}
\lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon}\frac{x}{2}|\Delta u|^2\cdot\nu\,d\mathcal{H}^{N-1} &= \lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon^1}\frac{x}{2}|\Delta u|^2\cdot\nu\,d\mathcal{H}^{N-1}+\lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon^2}\frac{x}{2}|\Delta u|^2\cdot\nu\,d\mathcal{H}^{N-1}\\
&= \lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon^1}\frac{x}{2}|\Delta u|^2\cdot\nu\,d\mathcal{H}^{N-1}\\
&= \lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon^1}\frac{|\Delta u|^2}{2}(x\cdot\nu)\,d\mathcal{H}^{N-1}\\
&= \lim_{\varepsilon\to 0}\int_{\partial\Omega}\frac{|\Delta u|^2}{2}(x\cdot\nu)\chi_{\partial\Omega_\varepsilon^1}\,d\mathcal{H}^{N-1}\\
&= \int_{\partial\Omega}\frac{|\Delta u|^2}{2}(x\cdot\nu)\,d\mathcal{H}^{N-1}.
\end{aligned}$$

We can put the limit inside the integral using the Dominated Convergence Theorem. Indeed by Corollary 4.3.1 we have

$$|\Delta u(x)|\le\frac{C}{|x|^2}\quad\forall\,x\in\overline{\Omega}\setminus\{0\}.$$

But then

$$\int_{\partial\Omega}\frac{|x|}{2}|\Delta u|^2\,d\mathcal{H}^{N-1}\le C\int_{\partial\Omega}\frac{1}{|x|^3}\,d\mathcal{H}^{N-1}$$

Now let $\mathcal{O}(0) := B_r(0) \cap \partial\Omega$ a spherical neighborhood of 0 in $\partial\Omega$. By the regularity up to the boundary of the function $\frac{1}{|x|^3}$ far from the origin we can say that

$$\int_{\partial\Omega} \frac{1}{|x|^3}\, d\mathcal{H}^{N-1} \leq \int_{\partial\Omega\setminus\mathcal{O}} \frac{1}{|x|^3}\, d\mathcal{H}^{N-1} + \int_{\mathcal{O}} \frac{1}{|x|^3}\, d\mathcal{H}^{N-1} \leq C + \int_{\mathcal{O}} \frac{1}{|x|^3}\, d\mathcal{H}^{N-1}.$$

Consider a point $x \in \mathcal{O}(0) \setminus \{0\}$, $x = (x_1, x_2, \ldots, x_N)$. Since Ω is smooth $\mathcal{O}$ is a smooth graph. Thus, after an appropriate rotation, this graph can be written as

$$x_N = h(x_1, x_2, \ldots, x_{N-1}),$$

with h smooth and in a way such that Ω lies locally above this graph, that is $\mathcal{O}$ lies above this graph. So we have, using the fact that h is smooth, that

$$\sqrt{1 + |\nabla h(x')|^2} \leq C \quad \forall\, x' \,\text{such that}\; x \in \mathcal{O}.$$

But then

$$\begin{aligned}
\int_{\mathcal{O}} |x|^{-3}\, d\mathcal{H}^{N-1} &= \int_{\mathcal{O}} \left(x_1^2 + \ldots + x_{N-1}^2 + (h(x'))^2\right)^{-\frac{3}{2}} \sqrt{1 + |\nabla h(x')|^2}\, dx' \\
&\leq C \int_{\mathcal{O}} \left(x_1^2 + \ldots + x_{N-1}^2 + (h(x'))^2\right)^{-\frac{3}{2}}\, dx' \\
&\leq C \int_{\mathcal{O}} \left(x_1^2 + \ldots + x_{N-1}^2\right)^{-\frac{3}{2}}\, dx' \qquad (4.4.10)\\
&\leq C \int_{\mathcal{O}} |x'|^{-3}\, dx' \\
&= C \int_0^r \rho^{N-5}\, d\rho = C \quad \forall\, N > 4.
\end{aligned}$$

But this implies that the functions $g(x) := \frac{x}{2}|\Delta u|^2 \in L^1(\partial\Omega)$ and then, by the Dominated Convergence Theorem, that

$$\lim_{\varepsilon\to 0} \int_{\partial\Omega_\varepsilon} \frac{x}{2}|\Delta u|^2 \cdot \nu\, d\mathcal{H}^{N-1} = \int_{\partial\Omega} \frac{x}{2}|\Delta u|^2 \cdot \nu\, d\mathcal{H}^{N-1}. \qquad (4.4.11)$$

Finally, we can apply the same argument to the last term. Indeed

$$\begin{aligned}
\lim_{\varepsilon\to 0} \int_{\partial\Omega_\varepsilon} \Delta u \langle x, D^2u, \nu\rangle\, d\mathcal{H}^{N-1} = &\lim_{\varepsilon\to 0} \int_{\partial\Omega_\varepsilon^1} \Delta u \langle x, D^2u, \nu\rangle\, d\mathcal{H}^{N-1} \\
&+ \lim_{\varepsilon\to 0} \int_{\partial\Omega_\varepsilon^2} \Delta u \langle x, D^2u, \nu\rangle\, d\mathcal{H}^{N-1}.
\end{aligned}$$

As before

$$\begin{aligned}
\lim_{\varepsilon\to 0} \int_{\partial\Omega_\varepsilon^2} \Delta u \langle x, D^2u, \nu\rangle\, d\mathcal{H}^{N-1} &\leq \lim_{\varepsilon\to 0} \int_{\partial\Omega_\varepsilon^2} \Delta u |D^2u||x|\, d\mathcal{H}^{N-1} \\
&\leq C \lim_{\varepsilon\to 0} \varepsilon^{N-4} = 0.
\end{aligned}$$

Finally

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\partial\Omega^1_\varepsilon}\Delta u\langle x, D^2u,\nu\rangle\,d\mathcal{H}^{N-1} &= \lim_{\varepsilon\to 0}\int_{\partial\Omega}\Delta u\langle x, D^2u,\nu\rangle\chi_{\partial\Omega^1_\varepsilon}\,d\mathcal{H}^{N-1}\\ &= \int_{\partial\Omega}\Delta u\langle x, D^2u,\nu\rangle\,d\mathcal{H}^{N-1}.\end{aligned}$$

if we are able to put the limit under the integral. To do that we need

$$\int_{\partial\Omega}\Delta u\langle x, D^2u,\nu\rangle\,d\mathcal{H}^{N-1} < +\infty.$$

But using the regularity up to the boundary of the solution u and the computation in (4.4.10) we have

$$\begin{aligned}\int_{\partial\Omega}\Delta u\langle x, D^2u,\nu\rangle\,d\mathcal{H}^{N-1} &\le \int_{\partial\Omega}\frac{C}{|x|^3}\,d\mathcal{H}^{N-1}\\ &\le C\int_{\partial\Omega\setminus\mathcal{O}(0)}|x|^{-3}\,d\mathcal{H}^{N-1} + C\int_{\mathcal{O}}|x|^{-3}\,d\mathcal{H}^{N-1}\\ &\le C.\end{aligned}$$

So we find

$$\lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon}\Delta u\langle x, D^2u,\nu\rangle\,d\mathcal{H}^{N-1} = \int_{\partial\Omega}\Delta u\langle x, D^2u,\nu\rangle\,d\mathcal{H}^{N-1}. \tag{4.4.12}$$

Joining (4.4.5), (4.4.6), (4.4.7), (4.4.8), (4.4.9), (4.4.11), (4.4.12) in (4.4.4) we find

$$\begin{aligned}\frac{N-4}{2}\int_\Omega|\Delta u|^2\,dx &= \frac{N-4}{p+1}\int_\Omega\frac{u^{p+1}}{|x|^4}\,dx + \int_{\partial\Omega}\frac{x}{2}|\Delta u|^2\cdot\nu\,d\mathcal{H}^{N-1}\\ &\quad - \int_{\partial\Omega}\Delta u\cdot\langle x, D^2u,\nu\rangle\,d\mathcal{H}^{N-1}.\end{aligned} \tag{4.4.13}$$

Since $u\equiv 0$ on $\partial\Omega$ we have that $\nabla u = \frac{\partial u}{\partial\nu}$ on $\partial\Omega$. Since

$$D^2u := \begin{pmatrix}\nabla(u_1)\\ \vdots\\ \nabla(u_N)\end{pmatrix},[7]$$

and the fact that

$$\nabla(u_i) = \frac{\partial u_i}{\partial\nu}\nu,$$

we have

$$(D^2u)_{ij} := \frac{\partial^2 u}{\partial x_i\partial x_j} = \frac{\partial^2 u}{\partial\nu^2}\nu_i\nu_j.$$

[7] We recall the notation used before, that is $u_l = \frac{\partial u}{\partial x_l}$.

But then
$$\Delta u := \sum_i \frac{\partial^2 u}{\partial x_i^2} = \sum_i \frac{\partial^2 u}{\partial \nu^2}\nu_i^2 = u_{\nu\nu},$$
and
$$\langle x, D^2u, \nu\rangle = \sum_{ij} x_i \frac{\partial^2 u}{\partial x_i \partial x_j}\nu_j = \sum_{ij} x_i \frac{\partial^2 u}{\partial \nu^2}\nu_i\nu_j^2 = u_{\nu\nu}^2(x\cdot\nu).$$
Finally joining the preceding computations in (4.4.13) we find the thesis. □

We have now all the tools to prove the nonexistence of positive (weak) solutions of problem (4.1.2). We need one more assumption, that is Ω has to be *star-shaped with respect to the origin*. We remember that the origin is on the boundary of Ω. We need to give carefully the definition of a domain star-shaped with respect to a point in its closure. In general a set $S \subseteq \mathbb{R}^N$ is star-shaped with respect to a point $x_0 \in S$ if for every $x \in S$ the line segment $[x, x_0] \subset S$. If Ω is a domain, that is a bounded subset of $\mathbb{R}^N$, we define Ω star-shaped with respect to $x_0 \in \partial\Omega$ if the closure of Ω is star-shaped with respect to x_0, with respect to the definition of starshaped set. So, we give the following definition.

Definition 4.4.1. Let $\Omega \subset \mathbb{R}^N$ a set. Ω is *star-shaped with respect to a point* $x_0 \in \overline{\Omega}$ if for every $x \in \overline{\Omega}$ the line segment $[x, x_0]$ is entirely contained in $\overline{\Omega}$.

Lemma 4.4.1. *Let Ω strictly star-shaped with respect to the point $0 \in \partial\Omega$. Then $x\cdot\nu > 0$ for every $x \in \partial\Omega$, $x \neq 0$.*

Proof. Suppose by contradiction that there exists a point $x \neq 0$, $x \in \partial\Omega$ such that $x\cdot\nu \leq 0$. We remark that $x\cdot\nu = |x|\cos\vartheta$ where ϑ is the angle between x and ν. Since Ω is star-shaped with respect to the origin we have that for every $x \in \partial\Omega$ the line segment $(x, 0) = x \subset \overline{\Omega}$. Moreover if $x\cdot\nu \leq 0$ then $\cos\vartheta \leq 0$ and then $\vartheta \in [\frac{\pi}{2}, \frac{3}{2}\pi]$. Since ν is the exterior normal vector in the point $x \in \partial\Omega$ this implies that the segment $x \not\subset \overline{\Omega}$ which is a contradiction. □

Theorem 4.4.1. *Let Ω star-shaped with respect to the point $0 \in \partial\Omega$. Then the problem (4.1.2) has no positive (weak) solutions.*

Proof. If u is a positive weak solution of problem (4.1.2) we have by Proposition 4.4.1
$$\int_\Omega |\Delta u|^2\,dx = \frac{2}{p+1}\int_\Omega \frac{u^{p+1}}{|x|^4}\,dx - \frac{1}{N-4}\int_{\partial\Omega} u_{\nu\nu}^2(x\cdot\nu)\,d\mathcal{H}^{N-1}.$$
Integrating the equation in (4.1.2) against the function u we obtain
$$\int_\Omega |\Delta u|^2\,dx = \int_\Omega \frac{u^{p+1}}{|x|^4}\,dx.$$

Substracting the first equation minus the second we obtain

$$0 = \left(\frac{2}{p+1} - 1\right) \int_{\Omega} \frac{u^{p+1}}{|x|^4}\, dx - \frac{1}{N-4} \int_{\partial\Omega} u_{\nu\nu}^2 (x \cdot \nu)\, d\mathcal{H}^{N-1}.$$

But since $(x \cdot \nu) > 0$ and $\frac{2}{p+1} - 1 < 0$ we have that $u \equiv 0$ which is a contradiction. $\square$

Part II

A priori estimates for superlinear problems

Chapter 5

Preliminaries

5.1 Trudinger-Moser inequality

In Chapter 2 we considered the classical case of the Sobolev embeddings, that is $p < N$, and we said that

$$W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \quad \forall\, 1 \leq q \leq p^* := \frac{Np}{N-p}.$$

Now, we want to consider the limiting case. Let $\Omega \subset \mathbb{R}^N$ and $N \geq 2$ and let $p = N$. Then we have that

$$W_0^{1,N} \hookrightarrow L^q(\Omega) \quad \forall\, q \geq 1.$$

Hence, every polynomial growth is allowed, in contrast with the subcritical case in which the maximal growth is given by $|u|^{p^*}$. Since formally

$$p^* := \frac{Np}{N-p} \to +\infty \quad \text{as } p \to N,$$

one may expect that a function $u \in W_0^{1,N}(\Omega)$ is in $L^\infty(\Omega)$, but it is a well known fact that

$$W_0^{1,N}(\Omega) \not\hookrightarrow L^\infty(\Omega).$$

Indeed, one may consider the counterexample given by

$$u(x) := \begin{cases} \log|\log|x|| & 0 < |x| < \frac{1}{\varepsilon} \\ 0 & \text{elsewhere.} \end{cases}$$

It is easy to see that for any domain $\Omega \subset \mathbb{R}^N$ containing the unit ball centered in the origin

$$\|\nabla u\|^N_{L^N(\Omega)} = \frac{\omega_{N-1}}{N-1},$$

but clearly $u \notin L^\infty(\Omega)$. Still, one may look for the maximal growth function $g : \mathbb{R} \to \mathbb{R}_+$ such that

$$\sup_{\substack{u\in W_0^{1,N}(\Omega)\\ \|\nabla u\|_{L^N(\Omega)}\le 1}} \int_\Omega g(u)dx < +\infty.$$

V. I. Yudovich [81], S. I. Pohozaev [55] and N. S. Trudinger [75] proved independently that the maximal growth is of exponential type. More precisely, they proved that there exists a positive constant α_N, depending only on the dimension N, such that

$$\sup_{\substack{u\in W_0^{1,N}(\Omega)\\ \|\nabla u\|_{L^N(\Omega)}\le 1}} \int_\Omega e^{\alpha_N |u|^{\frac{N}{N-1}}} dx < +\infty.$$

The original arguments by Yudovich-Pohozaev-Trudinger relied on the same idea. They devoloped the exponential function in a power series and proved that the series of L^p-norms converges. However, this argument does not produce the optimal exponent α_N.

Later, J. Moser [51], using a different approach, found the best exponent α_N, proving the following result.

There exists a constant $C_N > 0$ such that

$$\sup_{\substack{u\in W_0^{1,N}(\Omega)\\ \|\nabla u\|_{L^N(\Omega)}\le 1}} \int_\Omega e^{\alpha |u|^{\frac{N}{N-1}}} dx < +\infty \quad \forall\, \alpha \le \alpha_N, \tag{5.1.1}$$

where $\alpha_N := N\omega_{N-1}^{1/(N-1)}$. Furthermore, inequality (5.1.1) *is sharp, that is if $\alpha > \alpha_N$ then the supremum in* (5.1.1) *is infinite.*

As for the Sobolev embeddings theorem, we may want to consider the general higher order case $k > 1$. In this case, the result is given by D. R. Adams in [2].

Let $k \in \mathbb{N}$ and Ω a domain in $\mathbb{R}^N$ with $k < N$. Then there exists a constant $C_{k,N} > 0$ such that

$$\sup_{\substack{u\in W_0^{k,N/k}(\Omega)\\ \|\boldsymbol{D}^k u\|_{L^{N/k}(\Omega)}\le 1}} \int_\Omega e^{\beta |u|^{\frac{N}{N-k}}} dxdx < +\infty \quad \forall\, \beta \le \beta_{k,N}, \tag{5.1.2}$$

where

$$\beta_{k,N} := \frac{N}{\omega_{N-1}} \begin{cases} \left[\dfrac{\pi^{\frac{N}{2}} 2^k \Gamma\left(\frac{k}{2}\right)}{\Gamma\left(\frac{N-k}{2}\right)}\right]^{N/(N-k)} & m \textit{ is even} \\[2ex] \left[\dfrac{\pi^{\frac{N}{2}} 2^k \Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{N-k+1}{2}\right)}\right]^{N/(N-k)} & m \textit{ is odd.} \end{cases}$$

Furthermore, inequality (5.1.2) *is sharp.*

We can see the Adams-Trudinger-Moser inequality as the counterpart of the Sobolev embeddings theorem in the critical case $kp = N$. In this work, we are interested only in the embedding of the space $W_0^{1,N}(\Omega)$ with Ω a bounded domain.

We refer to the monograph of F. Sani [67] and to the related papers [45] and [65] for more details about Adams-Trudinger-Moser inequalities, their generalizations and some of their applications to elliptic problems.

5.2 Classical results about a priori estimates

What do we mean by *a priori* estimates ? Let us consider the following semilinear elliptic problem

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u > 0 & \text{in } \Omega, \end{cases} \tag{5.2.1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 3$ and $f = f(x,u)$ is a continuous function (more generally a locally Lipschitz function) in the second variable with a superlinear growth at infinity. By an a priori estimate we mean a uniform L^∞-estimate for solutions of problem (5.2.1), that is we want to prove that there exists a constant C such that

$$\|u\|_{L^\infty(\Omega)} \leq C \quad \forall\, u \text{ solution of (5.2.1)}.$$

The possibility to obtain such a kind of estimate depends both on the term f and on the type of solution we are considering.

We want to recall here the basic definitions for solutions of problem (5.2.1). We assume that the function f is a *Carathéodory function*, that is $f(\cdot,t)$ is measurable and $f(x,\cdot)$ is continuous.

If $f(\cdot,u(\cdot)) \in \mathcal{C}^0(\Omega)$ and $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$ we say that u is a *classical solution* of problem (5.2.1) if u satisfies pointwise the equation and the boundary conditions in (5.2.1).

If $f(\cdot,u(\cdot))$ is not a regular function then the definition of classical solutions does not make sense. Hence we may define weakened formulation of solutions. If $f(\cdot,u(\cdot)) \in L^2(\Omega)$ then we say that $u \in H_0^1(\Omega)$ is a *weak solution* of problem (5.2.1) if

$$\int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega f(x,u) v dx \quad \forall\, v \in H_0^1(\Omega).$$

It is well known that it is sufficient to check the preceding equality for all $v \in \mathcal{C}_0^\infty(\Omega)$ by density. Weak solutions are often called also *variational solutions* since they are critical points of the energy functional

$$E(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \int_\Omega F(x, u(x))dx, \quad F(x,t) := \int_0^t f(x,s)ds.$$

If we accept to relax further the definition of solution we can consider $f(\cdot, u(\cdot)) \in L^1(\Omega)$. In this case case we call a L^1*-solution* of problem (5.2.1) a function $u \in L^1(\Omega)$ such that

$$\int_\Omega -u\Delta v dx = \int_\Omega f(x,u)v dx \quad \forall\, v \in C^2(\overline{\Omega}), \quad v\big|_{\partial\Omega} = 0.$$

L^1-solutions are also called *distributional solutions*.

Now let us denote with $\delta(x) := \mathrm{dist}(x, \partial\Omega)$ for all $x \in \Omega$. For all $1 \le p \le +\infty$ we define

$$L^p_\delta(\Omega) := L^p(\Omega, \delta(x)dx),$$

endowed with the norm

$$\|u\|_{L^p_\delta(\Omega)} := \left(\int_\Omega |u|^p \delta(x)dx\right)^{1/p}.$$

If $f \in L^1_\delta(\Omega)$ we say that u is a L^1_δ*-solution* of problem (5.2.1) if $u \in L^1_\delta$ is such that

$$\int_\Omega -u\Delta v dx = \int_\Omega f(x,u)v dx \quad \forall\, v \in C^2(\overline{\Omega}), \quad v\big|_{\partial\Omega} = 0.$$

L^1_δ-solutions generalize L^1-solution in case of singularity on the boundary of Ω.

A priori estimates for solutions of elliptic equations have been a deep focus of research in the last three decades. We present here some of the classical results about this theme.

The first general result for a priori estimates for superlinear elliptic equation is due to H. Brezis and R. E. L. Turner [15]. They considered a second order elliptic equation with nonlinearity $f = f(x,u)$ and proved a priori bounds for positive weak solutions under the assumption

$$0 \le f(x,s) \le Cs^p \quad 1 < p < 2_* - 1 := \frac{N+1}{N-1}.$$

The critical exponent

$$2_* := \frac{2N}{N-1} \tag{5.2.2}$$

is called *Brezis-Turner exponent* and notice that it is smaller than the critical Sobolev exponent $2^* := \frac{2N}{N-2}$ for all $N \ge 3$. The proof of Brezis-Turner is heavily based on the Hardy-Sobolev inequality.

If $f(x,u) \in L^1_\delta(\Omega)$ the exponent $2_* - 1$ is crucial if we want to consider L^1_δ-solutions. Quittner and Souplet in [59], generalized the result of Brezis-Turner to L^1_δ-solutions for nonlinearities with $p < 2_* - 1$. Moreover Souplet proved in [69], that the Brezis-Turner exponent is critical for L^1_δ-solutions, that is if the growth of the nonlinearity is greater than u^p for $p > 2_* - 1$, then there exist examples of unbounded L^1_δ-solutions. Del Pino, Musso and Pacard proved a counterexample also for the case $p = 2_* - 1$ in [24].

If we consider slightly more regular solutions, namely L^1-solutions, then the critical growth is not anymore given by the Brezis-Turner exponent, but by $\tilde{2} - 1 := \frac{N}{N-2}$. A priori estimates for L^1-solutions were proved in this case by extending the result of Quittner and Souplet for L^1_δ-solutions. Combining two different results by Ni-Sacks and by Aviles, [52] and [8], it is possible to show that $\tilde{2} - 1$ is critical, finding unbounded L^1-solutions for $p \geq \tilde{2} - 1$.

If we restrict to the case of much more regular solutions, that is classical solutions, the Brezis-Turner exponent is not critical anymore and same for $\tilde{2} - 1$. Indeed, Gidas-Spruck proved in [38] that under the condition that there exists a continous function $a : \overline{\Omega} \to \mathbb{R}$ such that

$$\lim_{s \to +\infty} \frac{f(x, s)}{s^p} = a(x),$$

uniformly in $x \in \overline{\Omega}$, for $1 < p < 2^* - 1$, then for positive classical solutions the a priori estimates hold. Their technique is based on a blow-up argument and a Liouville Theorem on $\mathbb{R}^N$. A similar result was obtained by de Figueiredo, Lions and Nussbaum, [23]. They proved a priori bounds under the condition that the domain Ω is convex and the nonlinearity f is superlinear at infinity and satisfies

$$f(x, s) \leq cs^p \quad 1 < p < 2^* - 1.$$

Their argument is based on moving plane techniques and hence on maximum principle.

5.3 A priori estimates for the m-Laplacian operator

Concerning the $m-$Laplacian case, Azizieh and Clément proved in [9] a priori bounds for positive solutions of the problem

$$\begin{cases} -\Delta_m u = f(x, u) \text{ in } \Omega \\ u > 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

for $1 < m < 2$ and $f(x, u) = f(u)$ with $C_1 u^p \leq f(u) \leq C_2 u^p$ and $1 < p < \frac{N(m-1)}{N-m}$ and the domain Ω is convex. We recall here that the *m-Laplacian operator* is defined as

$$\Delta_m u := \operatorname{div}(|\nabla u|^{m-2} \nabla u).$$

It is trivial that if $m = N = 2$ then $\Delta_m u = \Delta$u.

A more general case was studied by Ruiz, [66]. If $1 < m \leq 2$, Ω does not need to be convex and f can also depend on x. All the preceding results are for $N \geq 3$ and are based on the Sobolev embeddings. Since for $m = N$ we are in the limiting case of Sobolev embeddings, one may ask whether it is possible to prove a priori estimates for nonlinearities with growth up to the Trudinger-Moser growth for the operator $-\Delta_N$.

If we restrict ourselves to the case $N = 2$ then this is not possible since Brezis and Merle gave in [13] examples of nonlinearities $f(x,s) = h(x)e^{s^\alpha}$ with $\alpha > 1$ for which there exists a sequence of unbounded solutions.

Nevertheless, using the result of Brezis-Merle and the boundary estimates of de Figueiredo-Lions-Nussbaum for Ω convex, it is possible to prove a priori estimates for nonlinearities f such that $C_1 e^s \leq f(x,s) \leq C_2 e^s$. Recently, Lorca-Ubilla-Ruf proved in [44] a priori results for the N-Laplacian in dimension N and for nonlinearities of maximal growth e^{s^α} for $\alpha < 1$ or for $f \sim e^s$. However, notice that their result leaves open the small range between e^{s^α} and e^s, for example nonlinearities of the form $\frac{e^s}{(s+1)^\alpha}$ or $e^{s/\log^\sigma(e+1)}$. Our result below narrows this gap. Indeed, we are able to prove a priori estimates also for nonlinearities f of maximal growth $e^{s/\log^\sigma(e+s)}$ with $\sigma > 0$ in the subcritical case or for nonlinearities f such that $C_1 \frac{e^s}{(s+1)^\alpha} \leq f(s) \leq C_2 e^s$ with $\alpha < 1$ in the critical case. Still we are not able to prove the result for nonlinearities $f \sim \frac{e^s}{(s+1)^\alpha}$ for $\alpha \geq 1$. It it not clear to the authors if this is a matter of techniqualities or whether the case $\alpha \geq 1$ is intrinsically different from the other cases.

Chapter 6

Superlinear problems for the N-laplacian

The results written in this chapter are collected in the paper [54]. The argument of our proof is inspired by the original work [44]. By the same argument of de Figueiredo-Lions-Nussbaum it is possible to obtain uniform boundary estimates, for convex domain. The boundary estimates lead to uniform L^1-estimates of the right hand side $f(u)$. Finally we can use this uniform bound on the forcing term to obtain uniform L^∞-bounds on u. We use the Trudinger-Moser inequality and some more subtle considerations on the estimates in [44]. Still, in the case $\frac{e^s}{(s+1)^\alpha}$ for $\alpha \geq 1$ our argument does not seem to work.

Let us consider the following problem

$$\begin{cases} -\Delta_N u = f(u) \text{ in } \Omega \\ u > 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{6.0.1}$$

where Ω is a strictly convex, bounded and smooth domain in $\mathbb{R}^N$, $N \geq 2$ and

$$\Delta_N u := \operatorname{div}\left(|\nabla u|^{N-2}\nabla u\right), \tag{6.0.2}$$

is the N-Laplacian operator. On the function f we are assuming the following conditions:

$$f : \mathbb{R}_+ \to \mathbb{R}_+ \text{ is a locally Lipschitz function;} \tag{f_0}$$

$$\exists\, d > 0 \,:\, \liminf_{s\to+\infty} \frac{f(s)}{s^{N-1+d}} > 0; \tag{f_1}$$

$$\exists\, \sigma > 0, \quad \exists\, C, s_0 > 0 \,:\, f(s) \leq C e^{s/\log^\sigma(e+s)} \quad \forall\, s \geq s_0 \tag{f_2}$$

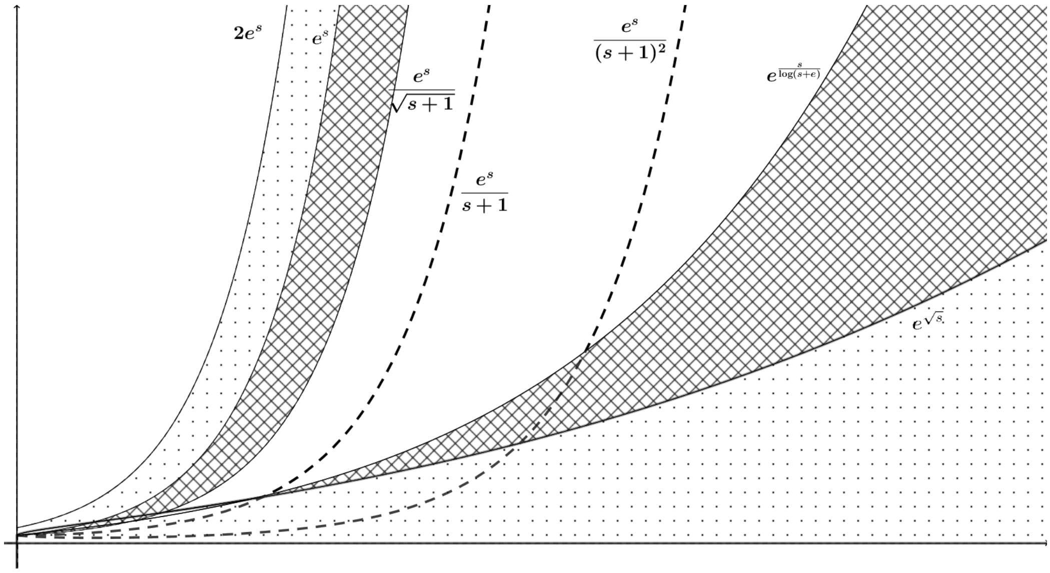

Figure 6.1: Improvements of conditions in [44] choosing $\sigma = 1$ and $\alpha = \frac{1}{2}$ in assumptions (f_2) and (f_2'). The limit case $\frac{e^s}{(s+1)^\alpha}$ for $\alpha \geq 1$ is still outside from the set of admissible nonlinearities.

or

$$(f_2') \qquad \exists\, 0 < \alpha < 1 \quad \exists\, C_1, C_2, s_0 > 0 \,:\, C_1 \frac{e^s}{(s+1)^\alpha} \leq f(s) \leq C_2 e^s \quad \forall\, s \geq s_0;$$

We remark here that conditions (f_2) and (f_2') generalize both conditions [44, (f_3), (f_4)] respectively. In case (f_2') the condition (f_1) is unnecessary because condition (f_2') implies the superlinearity at infinity.

We consider positive weak solutions for problem (6.0.1) in the following sense.

Definition 6.0.1. We say that u in $W_0^{1,N}(\Omega)$ is a *positive weak solution* of problem (6.0.1) if u is positive a.e and

$$\int_\Omega |\nabla u|^{N-2} \nabla u \cdot \nabla \varphi dx = \int_\Omega f(u) \varphi dx \quad \forall\, \varphi \in W_0^{1,N}(\Omega).$$

The main result is the following a priori estimate.

Theorem 6.0.1. *Under assumptions* (f_0)-(f_1)-(f_2) *or* (f_0)-(f_1)-(f_2') *there exists a constant* $C > 0$ *such that every positive weak solution* $u \in W_0^{1,N}(\Omega)$ *satisfies*

$$\|u\|_{L^\infty(\Omega)} \leq C.$$

We divide the proof of Theorem 6.0.1 in two cases. First, we consider the forcing term f satisfying assumption (f_2). Then, we consider f satisfying condition (f_2').

6.1 Orcliz spaces

We recall here some basic facts about Orlicz spaces, for more details see for example [3] and [61]. A continuous function $\phi : \mathbb{R} \to \mathbb{R}_+$ is called a *N-function*[1], if it is convex, even, $\phi(t) = 0$ iff $t = 0$ and

$$\lim_{t\to 0} \frac{\phi(t)}{t} = 0, \quad \lim_{t\to +\infty} \frac{\phi(t)}{t} = +\infty.$$

Given a N-function ϕ we can define its *conjugate function* $\widetilde{\phi}$ as

$$\widetilde{\phi}(s) := \sup_{t>0} \{st - \phi(t)\}.$$

Associated to a N-function ϕ and a domain $\Omega \subset \mathbb{R}^N$ we introduce the *Orlicz class of functions* defined by

$$K_\phi(\Omega) := \left\{ u : \Omega \to \mathbb{R} \,:\, u \text{ is measurable and } \int_\Omega \phi(u(x))dx < +\infty \right\}.$$

In general Orlicz classes are convex sets but not linear spaces. We define the *Orlicz space* as

$$L_\phi(\Omega) := \{\text{the vector space generated by } K_\phi(\Omega)\}.$$

On the Orlicz space $L_\phi(\Omega)$ we can define the following norm, called *Luxemburg norm*

$$\|u\|_\phi := \inf\left\{ \lambda > 0 \,:\, \int_\Omega \phi\left(\frac{|u|}{\lambda}\right) dx \le 1 \right\}. \tag{6.1.1}$$

finding that $(L_\phi(\Omega), \|\cdot\|_\phi)$ is a Banach space.

Given a N-function ϕ and its conjugate $\widetilde{\phi}$ it is clear that $\widetilde{\widetilde{\phi}} = \phi$. Moreover they satisfy the *Young inequality*

$$st \le \phi(t) + \widetilde{\phi}(s) \quad \forall\, s,t \in \mathbb{R}, \tag{6.1.2}$$

with equality when $s = \phi'(t)$ or $t = \widetilde{\phi}'(s)$.

Moreover in the spaces L_ϕ and $L_{\widetilde{\phi}}$ the *Hölder inequality* holds

$$\left| \int_\Omega u(x)v(x)dx \right| \le 2\|u\|_\phi \|v\|_{\widetilde{\phi}}. \tag{6.1.3}$$

Orlic spaces are a generalization of the classical Lebesgue spaces in the following sense. If we choose as N-function $\phi(t) = t^p$ for $1 \le p < +\infty$ we have that its conjugate function is given exactly by $\widetilde{\phi} = s^q$ with $\frac{1}{p} + \frac{1}{q} = 1$. In this sense the Young inequality and the Hölder inequality are generalizations of the standard properties of Lebesgue norms.

[1] N is not related to the dimension of the space $\mathbb{R}^N$.

The definition of spaces in the sense of Orlicz is useful when we consider functions with exponential or logarithmic growth. For example if

$$\varphi(t) := t\left(e^{t^\gamma} - 1\right) \quad \gamma \in \mathbb{R},$$

then its conjugate function is given by

$$\tilde{\varphi}(s) := s(\log(s+1))^{1/\gamma}.$$

6.2 Regularity of solutions

A priori we are considering weak solutions of problem (6.0.1), that is functions in $W_0^{1,N}(\Omega)$. Howewer, we are able to prove that they are in $\mathcal{C}^{1,\gamma}(\Omega)$ for some $0 < \gamma < 1$. If $N = 2$, that is when the $N-$Laplacian operator coincides with the classical Laplacian operator, it is easy to prove more regularity, that is weak solutions are also classical solutions in $\mathcal{C}^2(\Omega)$ while in the case $N > 2$ the $\mathcal{C}^{1,\gamma}$ regularity is optimal. The idea of our argument is similar to [20, Proposition 3.1]. To prove the regularity result we need the following result by G. Stampacchia [70] .

Proposition 6.2.1. *Assume φ is a nonnegative, nonincreasing function defined in $[0, +\infty)$. Suppose that there exist positive constants C, δ, β with $\beta > 1$ such that*

$$\varphi(h) \leq \frac{C}{(h-k)^\delta}\varphi(k)^\beta \quad \forall\, h > k \geq 0.$$

Then there exists $k_0 \geq 0$ such that $\varphi(h) = 0$ for all $h \geq k_0$.

Proposition 6.2.2. *Let $u \in W_0^{1,N}(\Omega)$ be a weak solution of problem* (6.0.1)*. Then $u \in \mathcal{C}^{1,\gamma}(\Omega)$ for some $0 < \gamma < 1$.*

Proof. By the Trudinger-Moser inequality (5.1.1) we have that

$$\int_\Omega |f(u)|^q dx \leq C(u) < +\infty, \text{[2]} \tag{6.2.1}$$

for all $q > 1$. Now we define

$$A_k(u) := \{x \in \Omega \,:\, u(x) > k\} \quad \forall\, k \in R_+,$$

and the function $G_k : \mathbb{R}_+ \to \mathbb{R}_+$ as

$$G_k(s) := \begin{cases} s - k & s > k \\ 0 & |s| \leq k. \end{cases}$$

[2] We use the notation here $C = C(u)$ to stress the fact that the estimate is not uniform in u.

Let us consider $v := G_k(u)$. Then v is an admissible test function in the definition of weak solution. Hence using (6.2.1) and Hölder inequality we obtain

$$\begin{aligned}\int_\Omega |\nabla G_k(u)|^N dx &= \int_\Omega |\nabla u|^{N-2}\nabla G_k(u)dx \\ &= \int_\Omega f(u)G_k(u)dx \\ &\le \|f(u)\|_{L^q(\Omega)}\|G_k(u)\|_{L^{q'}(\Omega)},\end{aligned}$$

with

$$1 = \frac{1}{q} + \frac{1}{q'}.$$

We can choose also $q' := \frac{N}{N-1}$. Now we take $r < N$ such that

$$r^* := \frac{rN}{N-r} > q'.$$

Then, we have

$$\begin{aligned}\|\nabla G_k(u)\|^N_{L^N(\Omega)} &\le \|f(u)\|_{L^q(\Omega)}\|G_k(u)\|_{L^{q'}(\Omega)} \\ &\le \|f(u)\|_{L^q(\Omega)}\|G_k(u)\|_{L^{r^*}(\Omega)}|A_k(u)|^{\frac{1}{q'}-\frac{1}{r^*}} \\ &\le C\|f(u)\|_{L^q(\Omega)}\|\nabla G_k(u)\|_{L^r(\Omega)}|A_k(u)|^{\frac{1}{q'}-\frac{1}{r^*}} \\ &\le C\|f(u)\|_{L^q(\Omega)}\|\nabla G_k(u)\|_{L^N(\Omega)}|A_k(u)|^{\frac{1}{q'}}.\end{aligned}$$

Hence we obtain

$$\|\nabla G_k(u)\|^{N-1}_{L^N(\Omega)} \le C\|f(u)\|_{L^q(\Omega)}|A_k(u)|^{\frac{1}{q'}}. \tag{6.2.2}$$

Approximating $A_k(u)$ by open sets we find also

$$\|\nabla G_k(u)\|^{N-1}_{L^N(\Omega)} \ge |A_h(u)|^{\frac{N-1}{N}}(h-k)^{N-1}. \tag{6.2.3}$$

Now we choose as φ in Lemma 6.2.1

$$\varphi(k) := A_k(u)^{\frac{N-1}{N}}.$$

Then we have by (6.2.2) and (6.2.3) that

$$\varphi(h)\varphi(k)^{-1}(h-k)^{N-1} \le C\|f(u)\|_{L^q(\Omega)}\varphi(k),$$

hence

$$\varphi(k) \le \frac{C}{(h-k)^\delta}\varphi(k)^2.$$

Hence by Proposition 6.2.1 we have that there exists k_0 such that $\varphi(h) = 0$ for all $h \ge k_0$ but hence $u \in L^\infty(\Omega)$. Now that we have that every weak solutions of problem (6.0.1) is bounded we can use the result of Tolksdorf [74], to conclude that $u \in \mathcal{C}^{1,\gamma}(\Omega)$ for some $0 < \gamma < 1$. □

6.3 Boundary estimate and uniform bound

In this section we consider f satisfying assumptions (f_0)-(f_1)-(f_2) or (f_0)-(f_1)-(f_2').

Proposition 6.3.1. *There exist positive constants C and r such that every weak solution $u \in W_0^{1,N}(\Omega)$ of problem (6.0.1) satisfies*

$$u \in \mathcal{C}^{1,\gamma}(\overline{\Omega_r}), \quad \|u\|_{\mathcal{C}^{1,\gamma}(\overline{\Omega_r})} \leq C,$$

for some $0 < \gamma < 1$ with $\Omega_r := \{x \in \Omega \,:\, \operatorname{dist}(x, \partial\Omega) \leq r\}$.

Proof. For $x \in \partial\Omega$ let $\nu(x)$ denote the outward normal vector to $\partial\Omega$ in the point x. By [21, Theorem 1.5] there exists a $t_0 > 0$ such that $u(x - t\nu(x))$ is nondecreasing for $t \in [0, t_0]$. The number t_0 depends only on the geometry of Ω. Now following the proof of [23, Theorem 1.1] we can prove that there exists an angle $\vartheta > 0$, depending again only on Ω, such that $u(z - t\sigma)$ is nondecreasing for all $t \in [0, t_1]$, where $|\sigma| = 1$ is such that $\sigma \cdot \nu(z) \geq \alpha$, $z \in \partial\Omega$ and t_1 depending only on Ω. We remark here that a priori t_1 is smaller than t_0. Since $u(z - t\sigma)$ is nondecreasing in t for all $z \in \partial\Omega$ and σ as above we can find positive numbers δ and ε, depeding on Ω, and a measurable set $I_x = I(x)$ such that for all $x \in \Omega_\varepsilon$ we have

(i) $|I_x| \geq \delta$,

(ii) $I_x \subset \{x \in \Omega \,:\, \operatorname{dist}(x, \partial\Omega) \geq \frac{\varepsilon}{2}\}$,

(iii) $u(y) \geq u(x)$ for all $y \in I_x$.

By the Picone Identity for the N-Laplacian [5], we know that for all u, v differentiable with $u > 0$ and $v \geq 0$ the following inequality holds

$$|\nabla v|^N \geq |\nabla u|^{N-2} \nabla \left(\frac{v^N}{u^{N-1}} \right) \cdot \nabla u.$$

Now we choose as $v = e_1$, that is the first (positive) eigenfunction of the N-Laplacian in Ω and u a (positive) weak solution of problem (6.0.1). We may assume also that e_1 is normalized. Since by the Hopf Lemma u has nonzero outward normal derivative on the boundary of Ω and since u is positive then $\frac{e_1^N}{u^{N-1}} \in W_0^{1,N}(\Omega)$. So we have, using the Picone Identity and the fact that the function $\frac{e_1^N}{u^{N-1}}$ is a test function

$$C \geq \int_\Omega |\nabla e_1|^N dx \geq \int_\Omega |\nabla u|^{N-2} \nabla u \cdot \nabla \left(\frac{e_1^N}{u^{N-1}} \right) dx = \int_\Omega f(u) \frac{e_1^N}{u^{N-1}} dx.$$

We split our domain Ω in Ω_1 and Ω_2 with

$$\Omega_1 := \{x \in \Omega \,:\, u(x) < s_0\}, \quad \Omega_2 := \{x \in \Omega \,:\, u(x) \geq s_0\}.$$

Now using the condition (f_1) we find that there exists a constant $d > 0$ such that

$$\begin{aligned}\int_\Omega u^d e_1^N dx &= \int_{\Omega_1} u^d e_1^N dx + \int_{\Omega_2} u^d e_1^N dx\\ &\le s_0^d \int_{\Omega_1} e_1^N dx + C_1 \int_{\Omega_2} \frac{f(u)}{u^{N-1}} e_1^N dx\\ &\le C + C_1 \int_{\Omega_2} f(u) \frac{e_1^N}{u^{N-1}} dx\\ &\le C.\end{aligned}$$

But this implies

$$\eta^N \int_{\Omega \setminus \Omega_{\frac{\varepsilon}{2}}} u^d dx \le C,$$

where $e_1(z) \ge \eta > 0$ for all $z \in \Omega \setminus \Omega_{\frac{\varepsilon}{2}}$. By (ii) we have also that

$$\eta^N \int_{I_x} u^d \le C.$$

Since

$$u^d(x)|I_x| \le \int_{I_x} u^d,$$

we have that, by (i) and (ii), $u^d(x) \le \frac{C}{\delta \eta^N}$ and hence $u(x) \le C$ for all $x \in \Omega_\varepsilon$. Finally by [42, Theorem 2] we have the thesis for $r := \frac{\varepsilon}{2}$. □

Proposition 6.3.2. *There exists a positive constant C such that for every positive weak solutions of problem* (6.0.1) *it holds*

$$\int_\Omega f(u)dx \le C.$$

Proof. Let ψ be a cut-off function in $\Omega \setminus \Omega_r$, that is $\psi \in \mathcal{C}_0^\infty(\Omega)$ such that $\psi \equiv 1$ in $\Omega \setminus \Omega_r$, where r is chosen such that the preceding Proposition holds. Since ψ is an admissible test function we have that

$$\int_\Omega |\nabla u|^{N-2} \nabla u \cdot \nabla \psi dx = \int_\Omega f(u) \psi dx.$$

Now using the boundary estimates of Proposition 6.3.1 and the properties of ψ we have

$$\begin{aligned}\int_\Omega f(u)dx &= \int_{\Omega_r} f(u)dx + \int_{\Omega \setminus \Omega_r} f(u)dx\\ &\le \int_{\Omega_r} f(u)\psi dx + \int_{\Omega \setminus \Omega_r} f(u)\psi dx + \int_{\Omega_r} f(u)(1-\psi)dx\\ &\le \int_\Omega f(u)\psi dx + C\\ &= \int_\Omega |\nabla u|^{N-2}\nabla u \cdot \nabla \psi dx + C = \int_{\Omega_r} |\nabla u|^{N-2} \nabla u \cdot \nabla \psi dx + C \le C.\end{aligned}$$

□

6.4 Case (f_2)

In this section we assume that f satisties assumptions (f_0), (f_1) and (f_2). Moreover, let us consider the N-function

$$\varphi(t) := t\left(e^{t^\gamma} - 1\right), \tag{6.4.1}$$

and its conjugate N-function

$$\tilde{\varphi}(s) := s(\log(s+1))^{1/\gamma}, \tag{6.4.2}$$

with $\gamma \in \mathbb{R}$ to be defined. It is easy to see that there exists $d_\gamma > 0$ such that

$$\varphi(t) \leq e^{d_\gamma t^\gamma} - 1.$$

Proof of Theorem 6.0.1. Since $u \in W_0^{1,N}(\Omega)$ is a weak solution of problem (6.0.1) we have that

$$\int_\Omega |\nabla u|^N dx = \int_\Omega f(u)u dx.$$

But then, since $u \neq 0$, we can multiply and divide by $u^\beta \log^\alpha(e+u)$ for some α and β to be defined and conclude that

$$\int_\Omega |\nabla u|^N dx = \int_\Omega \frac{f(u)\log^\alpha(e+u)}{u^\beta \log^\alpha(e+u))} u^{1+\beta} dx \leq \int_\Omega \frac{f(u)\log^\alpha(e+u)}{\log^\alpha(e+u)u^\beta} \chi_u u^{1+\beta} dx + C,$$

where

$$\chi_u := \begin{cases} 1 & \forall\, x : u(x) > s_0 \\ 0 & \forall\, x : u(x) \leq s_0, \end{cases}$$

with s_0 to be defined. Hence, we can conclude by (6.1.3) that

$$\int_\Omega |\nabla u|^N dx \leq C \left\| \frac{f(u)\log^\alpha(e+u)}{u^\beta} \chi_u \right\|_{\tilde{\varphi}} \left\| \frac{u^{1+\beta}}{\log^\alpha(e+u)} \right\|_{\varphi} + C,$$

with φ and $\tilde{\varphi}$ as in (6.4.1) and (6.4.2) respectively. We have to estimate the two Luxemburg norms in the preceding inequality.

$$\begin{aligned} \left\| \frac{u^{1+\beta}}{\log^\alpha(e+u)} \right\|_{\varphi} &:= \inf\left\{ k > 0 \,:\, \int_\Omega \varphi\left(\frac{u^{1+\beta}}{\log^\alpha(e+u)k} \right) dx \leq 1 \right\} \\ &= \inf\left\{ k > 0 \,:\, \int_\Omega \frac{u^{1+\beta}}{\log^\alpha(e+u)k} \left(e^{\frac{u^{(1+\beta)\gamma}}{(k\log^\alpha(e+u))^\gamma}} - 1 \right) dx \leq 1 \right\} \\ &\leq \inf\left\{ k > 0 \,:\, \int_\Omega \left(e^{d_\gamma \frac{u^{(1+\beta)\gamma}}{(k\log^\alpha(e+u))^\gamma}} - 1 \right) dx \leq 1 \right\}. \end{aligned}$$

First of all we can observe that for all $\varepsilon > 0$

$$\int_\Omega \left(e^{d_\gamma \frac{u^{(1+\beta)\gamma}}{(k \log^\alpha (e+u))^\gamma}} - 1 \right) dx = \int_{\Omega_\varepsilon} \left(e^{d_\gamma \frac{u^{(1+\beta)\gamma}}{(k \log^\alpha (e+u))^\gamma}} - 1 \right) dx + \int_{\Omega_\varepsilon^c} \left(e^{d_\gamma \frac{u^{(1+\beta)\gamma}}{(k \log^\alpha (e+u))^\gamma}} - 1 \right) dx,$$

where

$$\Omega_\varepsilon := \left\{ x \in \Omega \,:\, \log^{\alpha\gamma}(e+u) \geq \frac{1}{\varepsilon} \right\} \quad \Longleftrightarrow \quad \Omega_\varepsilon = \left\{ x \in \Omega \,:\, u(x) \geq e^{-\varepsilon^{\frac{1}{\alpha\gamma}}} - e \right\}.$$

It is easy to see that

$$\int_{\Omega_\varepsilon^c} \left(e^{d_\gamma \frac{u^{(1+\beta)\gamma}}{(k \log^\alpha (e+u))^\gamma}} - 1 \right) dx \leq C_{\varepsilon,k}.$$

It is important to remark that the constant $C_{\varepsilon,k}$ is indepedent of u, since in Ω_ε^c we have a uniform bound on u in term of ε, while it is still depending on ε and k. In Ω_ε we have

$$\int_{\Omega_\varepsilon} \left(e^{d_\gamma \frac{u^{(1+\beta)\gamma}}{(k \log^\alpha (e+u))^\gamma}} - 1 \right) dx \leq \int_\Omega \left(e^{\varepsilon d_\gamma \frac{u^{(1+\beta)\gamma}}{k^\gamma}} - 1 \right) dx.$$

If we want to apply the Trudinger-Moser inequality we have to choose γ and β such that

$$(1+\beta)\gamma = \frac{N}{N-1}. \tag{6.4.3}$$

With this choice, using the Trudinger-Moser inequality, we find that for all $\varepsilon > 0$

$$\left\| \frac{u^{1+\beta}}{\log^\alpha(e+u)} \right\|_\varphi \leq C\varepsilon^{\frac{1}{\gamma}} \|\nabla u\|_{L^N(\Omega)}^{\frac{N}{N-1}\frac{1}{\gamma}} + C_\varepsilon. \tag{6.4.4}$$

For the second norm we have

$$\begin{aligned}
&\left\| \frac{f(u)\log^\alpha(e+u)}{u^\beta} \chi_u \right\|_{\tilde\varphi} := \inf \left\{ k > 0 \,:\, \int \tilde\varphi \left(\frac{f(u)\log^\alpha(e+u)\chi_u}{u^\beta k} \right) dx \leq 1 \right\} \\
&= \inf \left\{ k > 0 \,:\, \int \frac{f(u)\log^\alpha(e+u)}{u^\beta k} \chi_u \left(\log \left(\frac{f(u)\log^\alpha(e+u)}{u^\beta k} \chi_u + 1 \right) \right)^{1/\gamma} dx \leq 1 \right\} \\
&\leq \inf \left\{ k \geq 1 \,:\, \int \frac{f(u)\log^\alpha(e+u)}{u^\beta k} \chi_u \left(\log \left(\frac{f(u)\log^\alpha(e+u)}{u^\beta k} \chi_u + 1 \right) \right)^{1/\gamma} dx \leq 1 \right\} \\
&\leq \inf \left\{ k \geq 1 \,:\, \int \frac{f(u)\log^\alpha(e+u)}{u^\beta k} \chi_u \left(\log \left(f(u) + 1 \right) \right)^{1/\gamma} dx \leq 1 \right\} \\
&\leq \inf \left\{ k \geq 1 \,:\, \int \frac{f(u)\log^\alpha(e+u)}{u^\beta k} \chi_u \left(\log \left(C f(u) \right) \right)^{1/\gamma} dx \leq 1 \right\} \\
&= \inf \left\{ k \geq 1 \,:\, \int \frac{f(u)\log^\alpha(e+u)}{u^\beta k} \chi_u \left(\log \left(C e^{\frac{u}{\log^\sigma(e+u)}} \right) \right)^{1/\gamma} dx \leq 1 \right\} \\
&\leq \inf \left\{ k \geq 1 \,:\, C \int \frac{f(u)\log^\alpha(e+u)}{u^\beta k} \chi_u \frac{u^{\frac{1}{\gamma}}}{\log^{\frac{\sigma}{\gamma}}(e+u)} dx \leq 1 \right\} \\
&= \inf \left\{ k \geq 1 \,:\, \frac{C}{k} \int f(u)(u)^{\frac{1}{\gamma}-\beta} \log^{\alpha - \frac{\sigma}{\gamma}}(e+u)\chi_u dx \leq 1 \right\}.
\end{aligned}$$

Choosing

$$\frac{1}{\gamma} - \beta = 0, \quad \alpha - \frac{\sigma}{\gamma} \le 0, \tag{6.4.5}$$

we find that

$$\left\| \frac{f(u)\log^{\alpha}(e+u)}{u^{\beta}} \chi_u \right\|_{\tilde{\varphi}} \le \inf \left\{ k \ge 1 \, : \, \frac{C}{k} \int_{\Omega} f(u)\chi_u dx \le 1 \right\} \le C. \tag{6.4.6}$$

By (6.4.4) and (6.4.6) we have for all $\varepsilon > 0$

$$\|\nabla u\|^{N}_{L^N(\Omega)} \le C\varepsilon^{1/\gamma} \|\nabla u\|^{\frac{N}{N-1}\frac{1}{\gamma}}_{L^N(\Omega)} + C_{\varepsilon}. \tag{6.4.7}$$

By (6.4.3) and (6.4.5) we have that

$$\gamma = \frac{N}{N-1} - 1 = \frac{1}{N-1},$$

and

$$0 < \alpha \le \sigma(N-1),$$

which is always possible since $\sigma > 0$. Since $\frac{1}{\gamma} = N-1$ in (6.4.7) we obtain that for all $\varepsilon > 0$

$$\|\nabla u\|^{N}_{L^N(\Omega)} \le C\varepsilon^{N-1} \|\nabla u\|^{N}_{L^N(\Omega)} + C_{\varepsilon}, \tag{6.4.8}$$

for some C and C_{ε} which are not depending on u. Then, for ε sufficiently small, we can conclude that

$$\|\nabla u\|_{L^N(\Omega)} \le C_N \tag{6.4.9}$$

with

$$C_N := \frac{C_{\varepsilon}}{1 - C\varepsilon^{N-1}},$$

which is not depending on u. From the uniform energy estimate (6.4.9) we obtain a uniform L^{∞}- bound in the following way. Let $p > 1$, then for given $\varepsilon > 0$ there exists a $C = C(\varepsilon)$ such that

$$ps \le \varepsilon s^{\frac{N}{N-1}} + C(\varepsilon).$$

Thus we can estimate

$$\begin{aligned}
\int_{\Omega} |f(u)|^p dx &\le C \int_{\Omega} \left(e^{\frac{u}{\log^{\alpha}(e+u)}} \right)^p dx + C \\
&\le C \int_{\Omega} e^{\varepsilon|u|^{\frac{N}{N-1}} + C(\varepsilon)} dx + C \\
&\le C(\varepsilon) \int_{\Omega} e^{\varepsilon|u|^{\frac{N}{N-1}}} dx + C.
\end{aligned}$$

Now, choosing $\varepsilon > 0$ such that $\varepsilon C_N^{N/(N-1)} \leq \alpha_N$, the estimate (6.4.9) and the Trudinger-Moser inequality imply

$$\int_\Omega |f(u)|^p dx \leq C(\varepsilon) \int_\Omega e^{\varepsilon C_N^{N/(N-1)} \left| \frac{u}{\|\nabla u\|_{L^N(\Omega)}} \right|^{N/(N-1)}} dx \leq C(\Omega).$$

So, since

$$\int_\Omega |f(u)|^p dx \leq C,$$

we have by [44, Lemma 3.2]

$$\|u\|_{L^\infty(K)} \leq C,$$

for every K compact set of Ω. Since by Proposition 6.3.1 we have the uniform estimate near the boundary we have proved the thesis. □

6.5 Case (f_2')

We assume in this section that f satisfies (f_0), (f_1) and (f_2'). Following the idea in [44] we introduce the following number

$$d_N := \inf_{\substack{x,y\in\mathbb{R}^N \\ x\neq y}} \frac{\langle |x|^{N-2}x - |y|^{N-2}y, x-y\rangle}{|x-y|^N}. \tag{6.5.1}$$

It is easy to see that $d_N \leq 1$. Moreover by [64, Proposition 4.6] we know that

$$d_N \geq \frac{2}{N}\left(\frac{1}{2}\right)^{N-2}.$$

In case $N = 2$ it is trivial that $d_2 = 1$ and hence the following argument is substantially easier.

Proof of Theorem 6.0.1. (The proof follows ideas of [15] and [44])
First Step: suppose by contradiction that there is no a priori estimate, then there would exist a sequence $\{u_n\} \subset W_0^{1,N}(\Omega) \cap \mathcal{C}^{1,\alpha}(\Omega)$ of weak solutions of problem (6.0.1) such that

$$\|u_n\|_{L^\infty(\Omega)} \to +\infty \quad \text{as } n \to +\infty.$$

Moreover we know by Proposition 6.3.2 that

$$\int_\Omega f(u_n)dx \leq C \quad \forall\, n \in \mathbb{N}.$$

We may assume that $f(u_n) \to \mu$ on Ω in the sense of measures, where μ is a nonnegative bounded measure, that is

$$\int_\Omega f(u_n)\psi dx \to \int_\Omega \psi d\mu \quad \forall\, \psi \text{ simple function in } \Omega.$$

We say that a point $x_0 \in \Omega$ is *regular with respect to* μ if there exists an open neighborhood $\mathcal{V} \subset \Omega$ of x_0 such that

$$\int_\Omega \chi_{\mathcal{V}} d\mu = \int_{\mathcal{V}} d\mu < N^{N-1}\omega_N.$$

Next we define the set A in the following way: a point $x \in A$ iff there exists an open neighborhood $\mathcal{U} \subset \Omega$ of x such that

$$\int_{\mathcal{U}} d\mu < N^{N-1}\omega_N d_N.$$

Since $d_N \leq 1$ we have that if $x \in A$ then x is a regular point. Moreover we define $B := \Omega \setminus A$ and we observe that

$$\#(B) < +\infty.$$

Indeed if $x \in B$ then

$$\int_{B_R(x)} d\mu \geq N^{N-1}\omega_N d_N \quad \forall\, R \,:\, B_R(x) \subset \Omega,$$

which implies

$$\mu(\{x\}) \geq N^{N-1}\omega_N d_N.$$

Hence, since μ is bounded,

$$\sum_{x\in B} \mu(\{x\}) \leq \mu(\Omega) \leq C,$$

then if $\#(B) = +\infty$ we would have a contradiction.
Second step: we claim that if x_0 is a regular point then there exist a constant C, and a radius R such that for all $n \in \mathbb{N}$ there holds

$$\|u_n\|_{L^\infty(B_R(x_0))} \leq C.$$

First of all we consider the case in which $x_0 \in A$. By the definition of points in A we have that there exists R such that

$$\mu(B_R(x_0)) < N^{N-1}\omega_N d_N.$$

Since $f(u_n)$ converges to μ in the sense of measures then there exist δ and n_0 such that for all $n > n_0$

$$\left(\int_{B_R(x_0)} f(u_n)dx\right)^{\frac{1}{N-1}} \leq \left(N\omega_N^{\frac{1}{N-1}} - \delta\right) d_N^{\frac{1}{N-1}}.$$

Let φ_n be a solution of

$$\begin{cases} -\Delta_N \varphi_n = 0 \text{ in } B_R(x_0) \\ \varphi_n = u_n \text{ on } \partial B_R(x_0). \end{cases} \tag{6.5.2}$$

By the weak maximum principle for the N-Laplacian we have that $\varphi_n \le u_n$ in $B_R(x_0)$. Since by Proposition 6.3.2 and assumption (f_2'), there exists a $n_1 \in \mathbb{N}$ such that for $d > 1$

$$C \ge \int_\Omega f(u_n)dx \ge C_1 \int_\Omega \frac{e^{u_n}}{(u_n+1)^\alpha} dx \ge C \int_\Omega u_n^d dx \quad \forall n > n_1,$$

then

$$\int_\Omega \varphi_n^N dx \le C \quad \forall n > n_1.$$

Now applying [44, Lemma 5.3], see also [64, Lemma 4.1 and 4.3], we have

$$\int_{B_R(x_0)} e^{q|u_n-\varphi_n|} dx \le \frac{N\omega_N^{\frac{1}{N-1}} R^N C}{\delta'},$$

with

$$q := \frac{N\omega_N^{\frac{1}{N-1}} - \delta'}{\|f(u_n)\|_{L^1(B_R)}^{\frac{1}{N-1}}} d_N^{\frac{1}{N-1}}, \tag{6.5.3}$$

for all $\delta' \in (0, N\omega_N^{1/(N-1)})$. Taking δ' small enough we have that $q > 1$ and hence

$$\int_{B_{R/2}} e^{q|u_n-\varphi_n|} dx \le \int_{B_R} e^{q|u_n-\varphi_n|} dx \le C.$$

Using [44, Lemma 3.2] we can conclude that

$$\|\varphi_n\|_{L^\infty(B_{R/2})} \le CR^{-1}(\|\varphi_n\|_{L^N(B_R)} + c) \le C,$$

and hence

$$\int_{B_{R/2}} e^{qu_n} \le C.$$

Then using (f_2') we have that for large n

$$\int_{B_{R/2}} |f(u_n)|^q dx \le C.$$

Again by [44, Lemma 3.2] we can infer that for large n

$$\|u_n\|_{L^\infty(B_{R/4})} \le C,$$

and hence

$$\|u_n\|_{L^\infty(B_{R/4})} \le C \quad \forall n \in \mathbb{N}.$$

Now let us suppose that $x_0 \notin A$ but still x_0 is regular. Since B is finite we can choose $R > 0$ such that $\partial B_R(x_0) \subset A$. Taking $x \in \partial B_R(x_0)$ we have by the preceding point that there exists $r = r(x)$ such that for all $n \in \mathbb{N}$

$$\|u_n\|_{L^\infty(B_{r(x)})} \le C(x).$$

This implies, by the compactness of ∂B_R, that for some $k \in \mathbb{N}$

$$\partial B_r \subseteq \bigcup_{i=1}^{k} B_{r(x_i)}(x_i).$$

If $y \in \partial B_r$ then $y \in B_{r(x_{i_0})}(x_{i_0})$ for some $1 \le i \le k$. Hence

$$\|u_n\|_{L^\infty(\partial B_R)} \le \max_{i=1,\dots,k} C(x_i) =: K \quad \forall\, n \in \mathbb{N}.$$

Let U_n be the solution of

$$\begin{cases} -\Delta_N U_n = f(u_n) \text{ in } B_R \\ U_n = K \text{ on } \partial B_R, \end{cases}$$

which is equivalent to

$$\begin{cases} -\Delta_N (U_n - K) = f(u_n) \text{ in } B_R \\ U_n - K = 0 \text{ on } \partial B_R. \end{cases}$$

Therefore $U_N \ge u_n$ on B_R by the weak maximum principle. Thus, by [44, Lemma 5.2], for any $\delta' \in (0, N\omega_N^{1/(N-1)})$ we have

$$\int_{B_R} e^{q|U_n - K|} dx \le \frac{N\omega_N^{1/(N-1)} C R^N}{\delta'},$$

with q defined as in (6.5.3). Since x_0 is regular there exists $R_1 < R$ and $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$ we have for some $\delta > 0$

$$\left(\int_{B_{R_1}(x_0)} f(u_n) dx \right)^{\frac{1}{N-1}} < N\omega_N^{\frac{1}{N-1}} - \delta.$$

Again taking δ' sufficiently small we obtain that $q > 1$ and hence

$$\int_{B_{R_1}} e^{qU_n} dx \le C,$$

which implies

$$\int_{B_{r_1}} e^{qu_n} dx \le C.$$

Now, using assumption (f_2') we have that also

$$\int_{B_{R_1}} f(u_n) dx \le C,$$

and then also

$$\|u_n\|_{L^N(B_{R_1})} \le C.$$

Hence by [44, Lemma 3.2]

$$\|u_n\|_{L^\infty(B_{R_{1/2}})} \le C.$$

Third step: Now we define $\Sigma := \{x \in \Omega \,:\, x \text{ is not regular}\}$ and we prove that $\Sigma = \emptyset$ and then by the preceding step we have the thesis. We notice that since $\Sigma \subset B$ and B has a finite number of elements then also Σ is finite. Suppose by contradiction that there exists $x_0 \in B$ and $R > 0$ such that

$$B_R(x_0) \cap \Sigma = \{x_0\}.$$

Since all the points in $B_R(x_0) \setminus \{x_0\}$ are regular we have that, up to subsequence, $u_n \to u$ in $C^1(K)$ for all K compact subsets of $B_R(x_0) \setminus \{x_0\}$. Consider

$$w(x) := N \log \frac{R}{|x - x_0|},$$

such that

$$\begin{cases} -\Delta_N w = N^{N-1}\omega_N \delta_{x_0} \text{ in } B_R(x_0) \\ w = 0 \text{ on } \partial B_R(x_0). \end{cases}$$

For $k > 0$ define

$$T_k(s) := \begin{cases} 0 & s < 0 \\ s & 0 \le s \le k \\ k & s \ge k, \end{cases}$$

and

$$z_n^{(k)} := T_k(w - u_n).$$

Then $z_n^{(k)} \in W_0^{1,N}(B_R)$. Moreover $z_n^{(k)}(x_0) = k$ for all n. Also $z_n^{(k)} \to z^{(k)}$ with

$$z^{(k)} := \begin{cases} T_k(w - u) & x \ne x_0 \\ k & x = x_0, \end{cases}$$

with $z^{(k)}$ measurable. We have, by the fact that w and u_n are weak solutions of their respective problems,

$$\int_{B_R} (|\nabla w|^{N-2}\nabla w - |\nabla u_n|^{N-2}\nabla u_n) \cdot \nabla z_n^{(k)} dx = N^{N-1}\omega_N k - \int_{B_R} f(u_n) z_n^k dx.$$

Now, setting $d\mu_n := f(u_n)dx$, we have, by [44, Proposition pag. 2052],

$$\begin{aligned} \liminf_{n\to\infty} \int_{B_R} f(u_n) z_n^{(k)} dx &= \liminf_{n\to\infty} \int_{B_R} z_n^{(k)} d\mu_n \\ &\ge \int_{B_R} z^{(k)} d\mu \\ &\ge N^{N-1}\omega_N k. \end{aligned}$$

Hence
$$\int_{B_R} (|\nabla w|^{N-2}\nabla w - |\nabla u|^{N-2}\nabla u)\cdot \nabla z^{(k)} dx \leq 0 \quad \forall\, k,$$
and then
$$\int_{B_R\cap\{0\leq w-u\leq k\}} (|\nabla w|^{N-2}\nabla w - |\nabla u|^{N-2}\nabla u)\cdot \nabla (w-u) dx \leq 0 \quad \forall\, k,$$
Finally, for $k\to\infty$ and using (6.5.1)
$$d_N \int_{B_R} |\nabla (w-u)^+|^N dx \leq 0 \quad \forall\, k,$$
and so $w\leq u$ by the weak maximum principle. Now we observe that since $\alpha<1$ the function $\frac{e^s}{(s+1)^\alpha}$ is monotone increasing for all $s\in R_+$, indeed
$$\frac{d}{ds}\left(\frac{e^s}{(s+1)^\alpha}\right) = \frac{e^s}{(s+1)^\alpha}\left(1-\frac{\alpha}{s+1}\right) > 0 \text{ iff } s>\alpha-1,$$
which is true for all $s>0$. Then we have, since $w\leq u$, and assumption (f_2')
$$\begin{aligned}
C &\geq \liminf_{n\to\infty} \int_{B_R} f(u_n)dx \\
&\geq \liminf_{n\to\infty} C_1 \int_{B_R} \frac{e^{u_n}}{(u_n+1)^\alpha} dx \\
&\geq C\int_{B_R} \frac{e^{u}}{(u+1)^\alpha} dx \\
&\geq C\int_{B_R} \frac{e^{w}}{(w+1)^\alpha} dx \\
&\geq C\int_{B_R} \frac{e^{w}}{w^\alpha} dx \\
&= C\int_{B_R} \frac{1}{|x-x_0|^N}\left(\log\left(\frac{1}{|x-x_0|^N}\right)\right)^{-\alpha} dx \\
&= C\int_0^R \frac{1}{\rho\left(\log\left(\frac{1}{\rho^N}\right)\right)^\alpha} d\rho \\
&= C\frac{\left(\log\left(\frac{1}{\rho^N}\right)\right)^{1-\alpha}}{N(\alpha-1)}\Bigg|_0^R = +\infty,
\end{aligned}$$
since $0<\alpha<1$. Hence we have a contradiction and then $B=\emptyset$. To conclude the proof, we may assume that there exists a sequence of points x_n in Ω such that
$$\|u_n\|_{L^\infty(\Omega)} = u_n(x_n).$$

By the compactness of $\overline{\Omega}$ and up to subsequence we may assume that $x_n \to x_0 \in \overline{\Omega}$. Since by Proposition 6.3.1 we have an a priori estimate near the boundary of Ω we can conclude that $x_0 \in \Omega$. It is easy to see that

$$\lim_{n\to+\infty} \|u_n\|_{L^\infty(B_R(x_0))} = +\infty \quad \forall\, R > 0,$$

and hence $x_0 \in B$. But this is a contradiction and then we have the thesis. $\square$

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