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On the response of the Earth to a fault system: its evaluation beyond the epicentral reference frame

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SUMMARY
Previous formalisms for determining the static perturbation of spherically symmetric self-gravitating elastic Earth models due to displacement dislocations deal with each infinitesimal element of the fault system in its epicentral reference frame. In this work we overcome this restriction and present novel and compact formulas for obtaining the perturbation due to the whole fault system in an arbitrary and common reference frame. Furthermore, we show that, even in an arbitrary reference frame, it is still possible to discriminate the contributions associated to the polar, bipolar and quadrupolar patterns of the seismic source response, as well as their relation with the along strike, along dip and tensile components of the displacement dislocation. These results allow a better understanding of the relation between the static perturbation and the whole fault system, and find direct applications in geodetic problems, like the modelling of long-wavelength geoid or gravity data from GRACE and GOCE space missions and of the perturbation of the deviatoric inertia tensor of the Earth.

1 INTRODUCTION

We review the static first-order perturbation theory for calculating the displacement field and the local incremental gravitational potential of spherically symmetric self-gravitating elastic Earth models due to displacement dislocations with the aim of overcoming some complexities of already developed formalisms for dealing with this problem (Smylie and Manshina, 1971; Takeuchi and Saito, 1972; Ben-Menahem and Singh, 1981; Dahlen and Tromp, 1998). At present, the method for calculating static perturbations due to displacement dislocations deviates somehow from that used for other forcings, like surface and internal loads and tidal forces (Farrell, 1972; Sabadini et al., 1982; Wu and Peltier, 1982), although both methods rely on the spherical harmonic expansion of the perturbations. Indeed, in the case of displacement dislocations, the reference frame with respect to which spherical harmonics are defined is the epicentral reference frame (Ben-Menahem and Singh, 1981), while there is no such a restriction in the other cases. This is essentially due to the fact that surface and internal loads and tidal forces can be dealt with as the superimposition of elementary problems which are axially symmetric. This advantage, instead, is lost when we face displacement dislocations and each infinitesimal element of the fault system is dealt with in its epicentral reference frame in order to take advantage of the fact that, in this reference frame, the seismic source response is a combination of polar, bipolar and quadrupolar patterns, i.e., it involves only spherical harmonics of order $m = 0, \pm 1$ and ± 2 (Ben-Menahem and Singh, 1981; Dahlen and Tromp, 1998). The seismic source response is thus obtained summing the perturbations due to each infinitesimal element of the fault system (Sun and Okubo, 1998; Xu et al., 2014), which implies a rotation in space (or, more laboriously in the spectral domain, the use of the Wigner's symbols) of each perturbation from its epicentral reference frame to a common reference frame, like the geographic one.

The goal of the present work consists in developing a novel formalism for dealing with displacement dislocations that preserves the interpretation of the seismic source response as a combination of polar, bipolar and quadrupolar patterns, and that overcomes the restriction of considering separately each infinitesimal surface element in its epicentral reference frame, obtaining directly the perturbation due to the fault system in an arbitrary reference frame. This formalism thus aims at improving our understanding of the relation between the static perturbation and the whole fault system, and to make easier the implementation of spherically symmetric Earth models in geodetic problems, like the modelling of long-wavelength gravity or geoid anomalies, nowadays detectable by Gravity Recovery and Climate Experiment (GRACE) and Gravity field and steady-state Ocean Circulation Explorer (GOCE) space missions (Han et al., 2006; Panet et al., 2007; de Linage et al., 2009, Cambiotti et al., 2011; Matsuo and Heki, 2011; Han et al., 2011; Wang et al., 2012; Cambiotti and Sabad ini, 2012, 2013; Fuchs et al., 2013).

In sections 2-5 we recall some basic results and introduce the notation necessary for discussing the seismic source response of spherically symmetric self-gravitating elastic Earth models. In sections 6-7 we discuss a thoughtful reorganization of the expression describing the discontinuity of the spheroidal vector solution due to an infinitesimal element of the fault system, and we introduce an alternative definition of the seismic Love numbers in order to obtain formulas of the static perturbation due to the whole fault system as simple as possible and in which the contributions associated to the polar, bipolar and quadrupolar patterns are still identifiable. In the end, in sections 8 and 9, we discuss some advantages of the present method, enlightening the differences with respect to previous works and, as example, applying it for obtaining the perturbation of the deviatoric inertia tensor of the Earth.

2 REFERENCE FRAMES

We consider a self-gravitating isotropic elastic Earth model that it is non-rotating, spherically symmetric and in hydrostatic equilibrium in the initial state (Wolf, 1991). In this respect, it is convenient to adopt a spherical reference frame with the Earth center as origin. We denote with θ, φ and r the colatitude, longitude and radial distance from the Earth center (radius from now on), and with $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\varphi}}$ and $\hat{\boldsymbol{r}}$ the spherical unit vectors in the direction of increasing colatitude, longitude and radius, respectively.

Since we shall both consider the observation points (at which the seismic source response is evaluated) and identify the infinitesimal elements of the fault system, we introduce the following convention to avoid confusion and simplify the notation. We will use unprimed symbols to denote the spherical coordinates, (θ, φ, r), and the vector position, $\boldsymbol{r}$, of an observation point, and we will reserve primed symbols to denote the spherical coordinates, (θ', φ', r'), and the vector position, $\boldsymbol{r}'$, of an infinitesimal element of the fault system. Primed spherical unit vectors shall be intended as evaluated at the angular coordinates (colatitude and longitude) of the infinitesimal surface element

$$\hat{\boldsymbol{\theta}}' = \hat{\boldsymbol{\theta}}(\theta', \varphi') \tag{1a}$$

$$\hat{\boldsymbol{\varphi}}' = \hat{\boldsymbol{\varphi}}(\theta', \varphi') \tag{1b}$$

$$\hat{\boldsymbol{r}}' = \hat{\boldsymbol{r}}(\theta', \varphi') \tag{1c}$$

and constitute the Cartesian unit vectors of the epicentral reference frame (Ben-Menahem and Singh, 1981).

Furthermore, we will denote with $\delta\boldsymbol{u}$ the displacement dislocation across an infinitesimal element of the fault system, and we will consider its decomposition into the along strike, δu_1, along dip, δu_2, and tensile, δu_3, components

$$\delta\boldsymbol{u} = \sum_{j=1}^{3} \delta u_j\, \boldsymbol{e}_j \tag{2}$$

where $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are the along strike and up-dip unit vectors, and $\boldsymbol{e}_3$ is the unit vector normal to the infinitesimal surface element. Their expressions in the epicentral reference frame read

$$\boldsymbol{e}_1 = -\cos(\zeta)\,\hat{\boldsymbol{\theta}}' + \sin(\zeta)\,\hat{\boldsymbol{\varphi}}' \tag{3a}$$

$$\boldsymbol{e}_2 = -\sin(\zeta)\,\cos(\alpha)\,\hat{\boldsymbol{\theta}}' - \cos(\zeta)\,\cos(\alpha)\,\hat{\boldsymbol{\varphi}}' + \sin(\alpha)\,\hat{\boldsymbol{r}}' \tag{3b}$$

$$\boldsymbol{e}_3 = \quad \sin(\zeta)\,\sin(\alpha)\,\hat{\boldsymbol{\theta}}' + \cos(\zeta)\,\sin(\alpha)\,\hat{\boldsymbol{\varphi}}' + \cos(\alpha)\,\hat{\boldsymbol{r}}' \tag{3c}$$

where α and ζ are the dip and strike angles (Aki and Richards, 2002).

3 VOLTERRA REPRESENTATION THEOREM

According to the Volterra representation theorem (Smylie and Manshina, 1971; Takeuchi and Saito, 1972; Ben-Menahem and Singh, 1981; Dahlen and Tromp, 1998), the perturbation of spherically symmetric self-gravitating elastic Earth models due to displacement dislocations can be interpreted as the superimposition of perturbations due to a continuous distribution of double couples and dipoles of point-like forces over the fault system. Then, rather than considering the boundary condition at the fault surface which describes the discontinuity of the displacement field, it is possible to account for its effects including in the momentum equation an equivalent volume force $\boldsymbol{f}$ (seismic force, from now on) defined as

$$\boldsymbol{f}(\boldsymbol{r}) = -\int_{\mathcal{S}} \mathrm{M}(\boldsymbol{r}') \cdot \boldsymbol{\nabla}\delta(\boldsymbol{r} - \boldsymbol{r}')\,\mathrm{d}S' \tag{4}$$

where $\boldsymbol{\nabla}$ stands for the gradient operator, δ is the Dirac delta distribution, $\mathcal{S}$ and $\mathrm{d}S'$ are the surface of the fault system and the area of the infinitesimal surface element, and $\mathbf{M}$ is the surface moment-density tensor (Dahlen and Tromp, 1998)

$$\mathbf{M} = \lambda\ (\delta\boldsymbol{u} \cdot \boldsymbol{e}_3)\ \mathbf{I} + \mu\ (\delta\boldsymbol{u}\,\boldsymbol{e}_3 + \boldsymbol{e}_3\,\delta\boldsymbol{u}) \tag{5}$$

with λ and μ being the two Lamé parameters, and $\mathbf{I}$ being the identity tensor. We note that, after substitution of eq. (2) into eq. (5), the surface moment-density tensor can be decomposed into the isotropic and deviatoric parts as follows

$$\mathbf{M} = \kappa\,\delta u_3\,\mathbf{I} + \mu \sum_{j=1}^{3} \delta u_j\,\mathbf{D}_j \tag{6}$$

where κ is the bulk modulus

$$\kappa = \lambda - \frac{2}{3}\,\mu \tag{7}$$

and $\mathbf{D}_j$ $(j = 1, 2, 3)$ are the following deviatoric tensors

$$\mathbf{D}_j = \boldsymbol{e}_j \otimes \boldsymbol{e}_3 + \boldsymbol{e}_3 \otimes \boldsymbol{e}_j \qquad \forall\, j = 1, 2 \tag{8a}$$

$$\mathbf{D}_3 = 2\,\left(\boldsymbol{e}_3 \otimes \boldsymbol{e}_3 - \tfrac{1}{3}\,\mathbf{I}\right) \tag{8b}$$

4 SPHERICAL HARMONIC EXPANSION

In order to take advantage of the spherical symmetry of the Earth model, it is customary to expand in spherical harmonics the scalar and vector fields describing the perturbation and the seismic force (Ben-Menahem and Singh, 1981; Dahlen and Tromp, 1998). Particularly, we shall consider the real spherical harmonic expansions of the local incremental gravitational potential, ϕ^{Δ}, the displacement, $\boldsymbol{u}$, the material incremental traction on a surface element of a sphere centered at the Earth center, $\boldsymbol{t}^{\delta}$, and the seismic force, $\boldsymbol{f}$,

$$\phi^{\Delta}(\boldsymbol{r}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \phi^{\Delta}_{\ell m}(r)\,Y_{\ell m}(\theta, \varphi) \tag{9a}$$

$$\boldsymbol{u}(\boldsymbol{r}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \big(U_{\ell m}(r)\,\boldsymbol{P}_{\ell m}(\theta, \varphi) + V_{\ell m}(r)\,\boldsymbol{B}_{\ell m}(\theta, \varphi) + W_{\ell m}(r)\,\boldsymbol{C}_{\ell m}(\theta, \varphi)\big) \tag{9b}$$

$$\boldsymbol{t}^{\delta}(\boldsymbol{r}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \big(R_{\ell m}(r)\,\boldsymbol{P}_{\ell m}(\theta, \varphi) + S_{\ell m}(r)\,\boldsymbol{B}_{\ell m}(\theta, \varphi) + T_{\ell m}(r)\,\boldsymbol{C}_{\ell m}(\theta, \varphi)\big) \tag{9c}$$

$$\boldsymbol{f}(\boldsymbol{r}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \big(P_{\ell m}(r)\,\boldsymbol{P}_{\ell m}(\theta, \varphi) + B_{\ell m}(r)\,\boldsymbol{B}_{\ell m}(\theta, \varphi) + C_{\ell m}(r)\,\boldsymbol{C}_{\ell m}(\theta, \varphi)\big) \tag{9d}$$

Here $Y_{\ell m}$, $\boldsymbol{P}_{\ell m}$, $\boldsymbol{B}_{\ell m}$ and $\boldsymbol{C}_{\ell m}$ are the real scalar spherical harmonics and the three real vector spherical harmonics of degree ℓ and order m that we define in appendix A, eqs (A.1) and (A.6). Also, $\phi^{\Delta}_{\ell m}$, $U_{\ell m}$, $V_{\ell m}$, $W_{\ell m}$, $R_{\ell m}$, $S_{\ell m}$, $T_{\ell m}$, $P_{\ell m}$, $B_{\ell m}$ and $C_{\ell m}$ are the coefficients of the spherical harmonic expansions, which are functions of the radius only. The expressions for the spherical harmonic coefficients of the seismic force ($P_{\ell m}$, $B_{\ell m}$ and $C_{\ell m}$) and of the material incremental traction ($R_{\ell m}$, $S_{\ell m}$ and $T_{\ell m}$) are given in appendices B and C, eqs (B.6) and (C.7) .

Note that our definition of spherical harmonics agrees with the common convection used in geodesy (Chao and Gross, 1987; Wahr et al., 1998). Indeed, the local incremental gravitational potential outside the Earth can be expressed as

$$\phi^{\Delta}(\boldsymbol{r}) = -\frac{G\,M_{\mathrm{E}}}{r} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(\frac{a}{r}\right)^{\ell} \Delta X_{\ell m}\,Y_{\ell m}(\theta, \varphi) \qquad \forall\, r \geq a \tag{10}$$

where G is the universal gravitational constant, M_{E} and a are the Earth mass and radius, and $\Delta X_{\ell m}$ are the Stokes coefficients of cosine type for zero or negative spherical harmonic orders and of sine type for positive spherical harmonic orders. In other words, for brevity, we have set

$$\Delta X_{\ell m} = \begin{cases} \Delta C_{\ell |m|} & m \leq 0 \\ \Delta S_{\ell m} & m > 0 \end{cases} \tag{11}$$

where $\Delta C_{\ell m}$ ($m = 0, \cdots, \ell$) and $\Delta S_{\ell m}$ ($m = 1, \cdots, \ell$) are the Stokes coefficients of the cosine and sine type, respectively. The only difference with respect to Chao and Gross (1987) stands on the minus sign in the right-hand side (RHS) of eq. (10) that is due to the convention in physics for which a (positive) mass yields a negative gravitational potential.

5 SPHEROIDAL PERTURBATIONS

After spherical harmonic expansion, the momentum and Poisson equations (see Appendix C, eq. (C.1)) can be separated into two independent sets of differential equations governing the spheroidal and toroidal perturbations (Ben-Menahem and Singh, 1981; Dahlen and Tromp, 1998). Particularly, the spheroidal part can be recast in the form of a system of six linear differential equations

$$\frac{\mathrm{d}\mathbf{y}_{\ell m}(r)}{\mathrm{d}r} = \mathbf{A}_\ell(r)\,\mathbf{y}_{\ell m}(r) + \mathbf{q}_{\ell m}(r) \tag{12}$$

Here, $\mathbf{A}_\ell$ is the 6×6 matrix defining the linear differential system and is given by eq. (C.8), $\boldsymbol{q}_{\ell m}$ is the inhomogeneous term which accounts for the displacement dislocation over the fault system

$$\mathbf{q}_{\ell m} = \begin{pmatrix} 0 & 0 & -P_{\ell m} & -B_{\ell m} & 0 & 0 \end{pmatrix}^{\mathrm{T}} \tag{13a}$$

and $\mathbf{y}_{\ell m}$ is the spheroidal vector solution

$$\mathbf{y}_{\ell m} = \begin{pmatrix} U_{\ell m} & V_{\ell m} & R_{\ell m} & S_{\ell m} & \phi^{\Delta}_{\ell m} & Q_{\ell m} \end{pmatrix}^{\mathrm{T}} \tag{14}$$

with T standing for the transpose, and $Q_{\ell m}$ being the so-called potential stress

$$Q_{\ell m} = \frac{\mathrm{d}\phi^{\Delta}_{\ell m}}{\mathrm{d}r} + \frac{\ell+1}{r}\,\phi^{\Delta}_{\ell m} + 4\,\pi\,G\,\rho_0\,U_{\ell m} \tag{15}$$

The differential system, eq. (12), must be solved for each harmonic degree ℓ and order m. However, the cases of the harmonic degrees $\ell = 0, 1$ present some peculiarities that make their treatment slightly different from the case of higher harmonic degrees, $\ell \geq 2$, and they are thus discussed separately in appendix D. Also, the case of toroidal perturbations is discussed in appendix E.

The spheroidal vector solution must satisfy the free Earth surface boundary conditions and the continuity conditions at the solid-solid interfaces within the mantle and the lithosphere, eqs (C.4) and (C.6). After spherical harmonic expansion, they take the following form

$$\mathbf{P}_1\,\mathbf{y}_{\ell m}(a) = \mathbf{0} \tag{16a}$$
$$\mathbf{y}_{\ell m}(R^+) = \mathbf{y}_{\ell m}(R^-) \tag{16b}$$

where $\mathbf{P}_1$ is the projector for the third, fourth and sixth components of the spheroidal vector solution, $\mathbf{P}_1 = \begin{pmatrix} R_{\ell m} & S_{\ell m} & Q_{\ell m} \end{pmatrix}^{\mathrm{T}}$, and R denotes the radius of the internal interface. Furthermore, the conditions at the core-mantle boundary (CMB) can be expressed as

$$\mathbf{y}_{\ell m}(r_C) = \mathbf{C}_\ell\,\mathbf{c}_{\ell m} \tag{17}$$

where r_C is the core radius, $\mathbf{C}_\ell$ is the CMB 6×3 matrix given by eq. (C.9), and $\mathbf{c}_{\ell m}$ is an array collecting three constants of integrations that shall be determined using the free Earth surface boundary conditions, eq. (16a). Particularly, the first and third columns of the CMB matrix $\mathbf{C}_\ell$ describe the perturbation of the equipotential surface at the core radius and the buoyancy of the mantle into the core due to any departure of the CMB topography from the equipotential surface, respectively, while the second column accounts for the fact that the tangential displacement at the bottom of the mantle is not constrained by the inviscid core (Longman, 1963; Dahlen,1974; Chinnery, 1975; Cambiotti et al., 2013).

Within this framework, the spheroidal vector solution can be written in terms of an homogeneous solution, $\mathbf{y}^0_{\ell m}$, which satisfies the boundary conditions at the CMB and at the Earth surface, and a particular solution, $\mathbf{y}^1_{\ell m}$, which accounts for the displacement dislocation,

$$\mathbf{y}_{\ell m} = \mathbf{y}^0_{\ell m} + \mathbf{y}^1_{\ell m} \tag{18}$$

with

$$\mathbf{y}^0_{\ell m}(r) = \mathbf{Z}_\ell(r)\,\mathbf{y}^1_{\ell m}(a) \tag{19a}$$

$$\mathbf{y}^1_{\ell m}(r) = \int_{r_C}^{r} \mathbf{\Pi}_\ell(r,x)\,\mathbf{q}_{\ell m}(x)\,\mathrm{d}x \tag{19b}$$

Here, $\mathbf{Z}_\ell$ is given by

$$\mathbf{Z}_\ell(r) = -\mathbf{\Pi}_\ell(r,r_C)\,\mathbf{C}_\ell\left[\mathbf{P}_1\,\mathbf{\Pi}_\ell(a,r_C)\,\mathbf{C}_\ell\right]^{-1}\mathbf{P}_1 \tag{20}$$

and $\mathbf{\Pi}_\ell$ is the 6×6 propagator matrix which solves the following homogeneous differential system

$$\frac{\mathrm{d}}{\mathrm{d}r}\mathbf{\Pi}_\ell(r,x) = \mathbf{A}_\ell(r)\,\mathbf{\Pi}_\ell(r,x) \tag{21}$$

and satisfies the initial condition given by the 6×6 identity matrix $\mathbf{I}_6$ and, from eq. (16b), the continuity condition at any internal interfaces of the Earth model

$$\mathbf{\Pi}_\ell(x,x) = \mathbf{I}_6 \tag{22a}$$

$$\mathbf{\Pi}_\ell(R^+,x) = \mathbf{\Pi}_\ell(R^-,x) \tag{22b}$$

Note also that, in order to obtain the homogeneous solution, eq. (19a), we have applied the CMB conditions, eq. (17), and we have determined the constants of integrations by means of the free Earth surface boundary conditions, eq. (16a).

6 THE DISCONTINUITY OF THE SPHEROIDAL VECTOR SOLUTION

In order to further specify the particular solution which accounts for the displacement dislocation, eq. (19b), we have to consider the expression for the spherical harmonic coefficients of the seismic force (see appendix B, eq. (B.6)), and rearrange the inhomogeneous term of the spheroidal differential system, eq. (13a), as follows

$$\mathbf{q}_{\ell m}(r) = \int_{\mathcal{S}} \frac{1}{4\,\pi\,r^2}\left(\frac{\delta(r-r')}{r}\,\mathbf{a}_{\ell m}(\boldsymbol{r}') + \frac{\mathrm{d}\delta(r-r')}{\mathrm{d}r}\,\mathbf{d}_{\ell m}(\boldsymbol{r}')\right)\mathrm{d}S' \tag{23}$$

where

$$\mathbf{a}_{\ell m}(\boldsymbol{r}') = \begin{pmatrix} 0 & 0 & -\mathbf{M}(\boldsymbol{r}') : (\boldsymbol{\nabla}'_\Omega\,\boldsymbol{P}_{\ell m}(\theta',\varphi')) & -\mathbf{M}(\boldsymbol{r}') : (\boldsymbol{\nabla}'_\Omega\,\boldsymbol{B}_{\ell m}(\theta',\varphi')) & 0 & 0 \end{pmatrix}^{\mathrm{T}} \tag{24a}$$

$$\mathbf{d}_{\ell m}(\boldsymbol{r}') = \begin{pmatrix} 0 & 0 & \mathbf{M}(\boldsymbol{r}') : (\hat{\boldsymbol{r}}'\,\boldsymbol{P}_{\ell m}(\theta',\varphi')) & \mathbf{M}(\boldsymbol{r}') : (\hat{\boldsymbol{r}}'\,\boldsymbol{B}_{\ell m}(\theta',\varphi')) & 0 & 0 \end{pmatrix}^{\mathrm{T}} \tag{24b}$$

with : and $\boldsymbol{\nabla}'_\Omega$ standing for the double scalar product and the surface gradient operator with respect to the angular coordinates of the infinitesimal surface element, eq. (B.7). Then, substituting eq. (23) into eq. (19b), the particular solution becomes

$$\mathbf{y}^1_{\ell m}(r) = \int_{\mathcal{S}} \mathbf{\Pi}_\ell(r,r')\,\delta\mathbf{s}_{\ell m}(\boldsymbol{r}')\,H(r-r')\,\mathrm{d}S' \tag{25}$$

where H is the Heaviside function and $\delta\mathbf{s}_{\ell m}$ represents the discontinuity of the spheroidal vector solution due to an infinitesimal element of the fault system at the radius r' at which the same element is located

$$\delta\mathbf{s}_{\ell m}(\boldsymbol{r}') = \frac{1}{4\,\pi\,r'^2}\left(\frac{\mathbf{a}_{\ell m}(\boldsymbol{r}') + 2\,\mathbf{d}_{\ell m}(\boldsymbol{r}')}{r'} + \mathbf{A}_\ell(r')\,\mathbf{d}_{\ell m}(\boldsymbol{r}')\right) \tag{26}$$

Considering the decomposition of the surface moment-density tensor into the isotropic and deviatoric parts, eq. (6), and making use of the definitions of vector spherical harmonics and of their surface gradients (see appendix A, eqs (A.6) and (A.9)), eq. (26) can be reorganized as the following linear combination of four arrays, $\mathbf{s}^{\mathrm{I}}_\ell$ and $\mathbf{s}^{\mathrm{D}}_{\ell,p}$ $(p = 0,1,2)$,

$$\delta\mathbf{s}_{\ell m}(\boldsymbol{r}') = \mathbf{s}^{\mathrm{I}}_{\ell}(r')\,\Delta^{\mathrm{I}}_{\ell m}(\boldsymbol{r}') + \sum_{p=0}^{2}\mathbf{s}^{\mathrm{D}}_{\ell,p}(r')\,\Delta^{\mathrm{D}}_{\ell m,p}(\boldsymbol{r}') \tag{27}$$

The first and second terms in the RHS come from the isotropic and deviatoric parts of the surface moment-density tensor and are denoted by the subscripts I and D, respectively.

The four arrays $\mathbf{s}^{\mathrm{I}}_{\ell}$ and $\mathbf{s}^{\mathrm{D}}_{\ell,p}$ ($p = 0,1,2$) have been defined in such a way to include the dependences on the radius at which the infinitesimal surface element is located and on the elastic parameters of the Earth model evaluated at this radius

$$\mathbf{s}^{\mathrm{I}}_{\ell}(r') = N_{\ell 0}\,\frac{2\,\ell+1}{4\,\pi\,r'^3}\,\frac{\kappa'}{\lambda'+2\,\mu'}\begin{pmatrix} r' & 0 & -4\,\mu' & 2\,L\,\mu' & 0 & 0\end{pmatrix}^{\mathrm{T}} \tag{28a}$$

$$\mathbf{s}^{\mathrm{D}}_{\ell,0}(r') = N_{\ell 0}\,\frac{2\,\ell+1}{8\,\pi\,r'^3}\,\frac{\mu'}{\lambda'+2\,\mu'}\begin{pmatrix} 2\,r' & 0 & 6\,\kappa' & -3\,L\,\kappa' & 0 & 0\end{pmatrix}^{\mathrm{T}} \tag{28b}$$

$$\mathbf{s}^{\mathrm{D}}_{\ell,1}(r') = N_{\ell 1}\,\frac{2\,\ell+1}{4\,\pi\,r'^2}\,\frac{1}{L}\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0\end{pmatrix}^{\mathrm{T}} \tag{28c}$$

$$\mathbf{s}^{\mathrm{D}}_{\ell,2}(r') = N_{\ell 2}\,\frac{2\,\ell+1}{4\,\pi\,r'^3}\,\frac{\mu'}{L}\begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0\end{pmatrix}^{\mathrm{T}} \tag{28d}$$

with

$$L = \sqrt{\ell\,(\ell+1)} \tag{29}$$

The factors $\Delta^{\mathrm{I}}_{\ell m}$ and $\Delta^{\mathrm{D}}_{\ell m,p}$ ($p = 0,1,2$), instead, depend on the angular coordinates, geometry and displacement dislocation of the infinitesimal element of the fault system. Due to their lengthy expressions, we introduce hereinafter some definitions to make evident their physical meaning. We then write

$$\Delta^{\mathrm{I}}_{\ell m}(\boldsymbol{r}') = \delta u_3(\boldsymbol{r}')\,F^{0}_{\ell m}(\theta',\varphi') \tag{30a}$$

$$\Delta^{\mathrm{D}}_{\ell m,p}(\boldsymbol{r}') = \sum_{j=1}^{3}\delta u_j(\boldsymbol{r}')\,\frac{D^{p}_{j}(\boldsymbol{r}')\,F^{p}_{\ell m}(\theta',\varphi') + D^{-p}_{j}(\boldsymbol{r}')\,F^{-p}_{\ell m}(\theta',\varphi')}{1+\delta_{p,0}} \qquad \forall\,p=0,1,2 \tag{30b}$$

with $\delta_{i,j}$ being the Kronecker delta. Here, $F^{p}_{\ell m}$ ($p = -2,\cdots,2$) are the following combinations of spherical harmonics and their partial derivatives

$$F^{0}_{\ell m} = \frac{Y_{\ell m}}{(2\,\ell+1)\,N_{\ell 0}} \tag{31a}$$

$$F^{-1}_{\ell m} = \frac{\partial_\theta Y_{\ell m}}{(2\,\ell+1)\,N_{\ell 1}} \tag{31b}$$

$$F^{1}_{\ell m} = \frac{\csc\theta\,\partial_\varphi Y_{\ell m}}{(2\,\ell+1)\,N_{\ell 1}} \tag{31c}$$

$$F^{-2}_{\ell m} = \frac{\left(2\,\partial^2_\theta + L^2\right)Y_{\ell m}}{(2\,\ell+1)\,N_{\ell 2}} \tag{31d}$$

$$F^{2}_{\ell m} = \frac{2\,\partial_\theta\left(\csc\theta\,\partial_\varphi\right)Y_{\ell m}}{(2\,\ell+1)\,N_{\ell 2}} \tag{31e}$$

and they enter eq. (30) once evaluated at the angular coordinates of the infinitesimal surface element. Also, the factors D^{p}_{j} ($p = -2,\cdots,2$) are the components (or combination of components) of the tensor $\mathbf{D}_j$, eq. (8), in the epicentral reference frame

$$D^{0}_{j}(\boldsymbol{r}') = \hat{\boldsymbol{r}}'\cdot\mathbf{D}_j(\boldsymbol{r}')\cdot\hat{\boldsymbol{r}}' \qquad \forall\,j=1,2,3 \tag{32a}$$

$$D^{-1}_{j}(\boldsymbol{r}') = \hat{\boldsymbol{r}}'\cdot\mathbf{D}_j(\boldsymbol{r}')\cdot\hat{\boldsymbol{\theta}}' \qquad \forall\,j=1,2,3 \tag{32b}$$

$$D^{1}_{j}(\boldsymbol{r}') = \hat{\boldsymbol{r}}'\cdot\mathbf{D}_j(\boldsymbol{r}')\cdot\hat{\boldsymbol{\varphi}}' \qquad \forall\,j=1,2,3 \tag{32c}$$

$$D^{-2}_{j}(\boldsymbol{r}') = \tfrac{1}{2}\left(\hat{\boldsymbol{\theta}}'\cdot\mathbf{D}_j(\boldsymbol{r}')\cdot\hat{\boldsymbol{\theta}}' - \hat{\boldsymbol{\varphi}}'\cdot\mathbf{D}_j(\boldsymbol{r}')\cdot\hat{\boldsymbol{\varphi}}'\right) \qquad \forall\,j=1,2,3 \tag{32d}$$

$$D^{2}_{j}(\boldsymbol{r}') = \hat{\boldsymbol{\theta}}'\cdot\mathbf{D}_j(\boldsymbol{r}')\cdot\hat{\boldsymbol{\varphi}}' \qquad \forall\,j=1,2,3 \tag{32e}$$

and they thus depend only on the geometry of the infinitesimal surface element, i.e., the dip and strike angles. Particularly, substituting eqs (3) and (8) into eq. (32), we have

$$D_1^0 = 0 \tag{33a}$$
$$D_1^{-1} = -\cos(\zeta)\,\cos(\alpha) \tag{33b}$$
$$D_1^1 = \sin(\zeta)\,\cos(\alpha) \tag{33c}$$
$$D_1^{-2} = -\sin(2\,\zeta)\,\sin(\alpha) \tag{33d}$$
$$D_1^2 = -\cos(2\,\zeta)\,\sin(\alpha) \tag{33e}$$
$$D_2^0 = \sin(2\,\alpha) \tag{33f}$$
$$D_2^{-1} = -\sin(\zeta)\,\cos(2\,\alpha) \tag{33g}$$
$$D_2^1 = -\cos(\zeta)\,\cos(2\,\alpha) \tag{33h}$$
$$D_2^{-2} = \tfrac{1}{2}\cos(2\,\zeta)\,\sin(2\,\alpha) \tag{33i}$$
$$D_2^2 = -\tfrac{1}{2}\sin(2\,\zeta)\,\sin(2\,\alpha) \tag{33j}$$
$$D_3^0 = \tfrac{1}{3} + \cos(2\,\alpha) \tag{33k}$$
$$D_3^{-1} = \sin(\zeta)\,\sin(2\,\alpha) \tag{33l}$$
$$D_3^1 = \cos(\zeta)\,\sin(2\,\alpha) \tag{33m}$$
$$D_3^{-2} = -\cos(2\,\zeta)\,\sin^2(\alpha) \tag{33n}$$
$$D_3^2 = \sin(2\,\zeta)\,\sin^2(\alpha) \tag{33o}$$
$$\tag{33p}$$

The physical meaning of eq. (27) can be enlightened resorting to the classical approach dealing with point-like seismic sources, that is bringing the infinitesimal surface element to the polar axis of the spherical reference frame ($\theta' \to 0$ and $\varphi' \to 0$), and keeping constant all the other characteristics, i.e., the displacement dislocation and the geometry (Takeuchi and Saito, 1972; Ben-Menahem and Singh, 1981). From eq. (A.10), the factors $F_{\ell m}^p$ obey to the following limit

$$\lim_{\substack{\theta \to 0 \\ \varphi \to 0}} F_{\ell m}^p(\theta, \varphi) = \delta_{p,m} \tag{34}$$

that can be proved using eq. (4.5.5) of Ben-Menahem and Singh (1981), taking care of the different definition of spherical harmonics. We then can write the discontinuity of the spheroidal vector solution due to an infinitesimal surface element at the north pole as follows

$$\lim_{\substack{\theta' \to 0 \\ \varphi' \to 0}} \delta \mathbf{s}_{\ell m} = \left(\mathbf{s}_\ell^{\mathrm{I}}(r')\,\delta u_3 + \mathbf{s}_{\ell,0}^{\mathrm{D}}(r') \sum_{j=2}^{3} \delta u_j\, D_j^0 \right) \delta_{0,m} + \mathbf{s}_{\ell,1}^{\mathrm{D}}(r') \left(\delta_{1,m} \sum_{j=1}^{3} \delta u_j\, D_j^1 + \delta_{-1,m} \sum_{j=1}^{3} \delta u_j\, D_j^{-1} \right)$$
$$+ \mathbf{s}_{\ell,2}^{\mathrm{D}}(r') \left(\delta_{2,m} \sum_{j=1}^{3} \delta u_j\, D_j^2 + \delta_{-2,m} \sum_{j=1}^{3} \delta u_j\, D_j^{-2} \right) \tag{35}$$

This result coincides with those obtained by previous works (Takeuchi and Saito, 1972; Sun and Okubo, 1993), taking care of the different definition of spherical harmonics (non-normalized complex spherical harmonics rather than real spherical harmonics normalized to $4\,\pi$ here considered). Furthermore, it shows how the discontinuity of the spheroidal vector solution at the radius of the infinitesimal surface element can be decomposed into contributions from the isotropic and deviatoric parts of the surface moment-density tensor. As expected, the contribution from the isotropic part yields only polar patterns (spherical harmonic order $m = 0$) and is associated to the tensile component of the displacement dislocation. The contributions from the deviatoric parts, instead, yield polar, bipolar and quadrupolar patterns (spherical harmonic orders $m = 0, \pm 1$ and ± 2) and are associated to each component of the displacement dislocation.

Differently from previous formulations, however, we have paid peculiar attention at decomposing the discontinuity of the spheroidal vector solution in such a way that the contributions from the isotropic and deviatoric parts of the surface moment-density tensor, as well as their relation with the along strike, along dip and tensile components of the displacement dislocation, are clearly identifiable. Particularly, this decomposition shows that the along strike, along dip and tensile components of the displacement dislocation contribute to the discontinuity of the spheroidal vector solution in a similar way. The only difference stands on the way in which the geometry (dip and strike angles) of the infinitesimal surface element affects the factors $\Delta_{\ell m,p}^{\mathrm{D}}$ of the second term in the RHS of eq. (27) and that the tensile component is also responsible for the first term related to the isotropic part of the surface moment-density tensor. Moreover, we note that this decomposition is not restricted by any requirement about the reference frame. It can be performed in whatever reference frame and, thus, even in a reference frame that is kept common for all the infinitesimal element of the fault system, like the geographic reference frame or any other that fits the goal of the application of the displacement dislocation theory.

7 SEISMIC LOVE NUMBERS

Recalling the expressions just obtained for the homogeneous and particular solutions, eqs (19a) and (25), the spheroidal vector solution, eq. (18), can be rearranged in the following compact form

$$\mathbf{y}_{\ell m}(r) = \int_{\mathcal{S}} \Big(\mathbf{X}_\ell(r,r')\, \mathbf{s}^{\mathrm{I}}_\ell(r')\, \Delta^{\mathrm{I}}_{\ell m}(\boldsymbol{r}') + \sum_{p=0}^{2} \mathbf{X}_\ell(r,r')\, \mathbf{s}^{\mathrm{D}}_{\ell,p}(r')\, \Delta^{\mathrm{D}}_{\ell m,p}(\boldsymbol{r}') \Big) \mathrm{d}S' \tag{36}$$

with

$$\mathbf{X}_\ell(r,r') = H(r-r')\, \mathbf{\Pi}(r,r') + \mathbf{Z}_\ell(r)\, \mathbf{\Pi}_\ell(a,r') \tag{37}$$

In the perspective of estimating the (radial and tangential) spheroidal components of the displacement, $U_{\ell m}$ and $V_{\ell m}$, and the local incremental gravitational potential, $\phi^{\Delta}_{\ell m}$, it is convenient to introduce the following four seismic Love numbers

$$\begin{pmatrix} h^{\mathrm{I}}_\ell(r,r') \\ l^{\mathrm{I}}_\ell(r,r') \\ k^{\mathrm{I}}_\ell(r,r') \end{pmatrix} = \mathbf{k}^{\mathrm{I}}_\ell(r,r') = \mathbf{N}^{-1}\, \mathbf{P}_2\, \mathbf{X}_\ell(r,r')\, \mathbf{s}^{\mathrm{I}}_\ell(r') \tag{38a}$$

$$\begin{pmatrix} h^{\mathrm{D}}_{\ell,p}(r,r') \\ l^{\mathrm{D}}_{\ell,p}(r,r') \\ k^{\mathrm{D}}_{\ell,p}(r,r') \end{pmatrix} = \mathbf{k}^{\mathrm{D}}_{\ell,p}(r,r') = \mathbf{N}^{-1}\, \mathbf{P}_2\, \mathbf{X}_\ell(r,r')\, \mathbf{s}^{\mathrm{D}}_{\ell,p}(r') \qquad \forall\, p = 0,1,2 \tag{38b}$$

where $\mathbf{P}_2$ is the projector for the first, second and fifth components of the spheroidal vector solution, $\mathbf{P}_2 = \begin{pmatrix} U_{\ell m} & V_{\ell m} & \phi^{\Delta}_{\ell m} \end{pmatrix}^{\mathrm{T}}$, and the symbols h, l and k denote the radial, tangential and gravitational components of the static perturbation, accordingly to the classic notation. Also, $\mathbf{N}$ is a diagonal matrix that we use to make non-dimensional these components

$$\mathbf{N} = a^{-2}\, \mathrm{Diag}\big[\; 1 \quad 1 \quad -g_0(a) \;\big] \tag{39}$$

Note that the seismic Love numbers depend both on the radius r at which the perturbation is calculated and on the radius r' at which the infinitesimal element of the fault system is located.

With these definitions and once applied the projector $\mathbf{P}_2$ to eq. (36), we thus can write

$$\begin{pmatrix} U_{\ell m}(r) \\ V_{\ell m}(r) \\ \phi^{\Delta}_{\ell m}(r) \end{pmatrix} = \mathbf{P}_2\, \mathbf{y}_{\ell m}(r) = \mathbf{N} \int_{\mathcal{S}} \Big(\mathbf{k}^{\mathrm{I}}_\ell(r,r')\, \Delta^{\mathrm{I}}_{\ell m}(\boldsymbol{r}') + \sum_{p=0}^{2} \mathbf{k}^{\mathrm{D}}_{\ell,p}(r,r')\, \Delta^{\mathrm{D}}_{\ell m,p}(\boldsymbol{r}') \Big) \mathrm{d}S' \tag{40}$$

Eq. (40) provides a method for computing the spherical harmonic coefficients of the static perturbation due to the whole fault system directly in an arbitrary and common reference frame, and, thus, overcomes the need of considering each infinitesimal surface element in its epicentral reference frame as it was in previous formulations. This has been made possible by the specific decomposition of the discontinuity of the spheroidal vector solution that we have presented in section 6, eq. (27), and, particularly, by the fact that the factors $\Delta^{\mathrm{I}}_{\ell m}$ and $\Delta^{\mathrm{D}}_{\ell m,p}$, eqs (30), include the information about the angular coordinates of the infinitesimal element of the fault system in the chosen reference frame via the functions $F^{p}_{\ell m}$, eqs (31). In any case, our seismic Love numbers can still be used for obtaining the static perturbation due to an infinitesimal surface element in its epicentral reference frame. Particularly, from, eqs (34) and (40), we obtain

$$\begin{pmatrix} U_{\ell m}(r) \\ V_{\ell m}(r) \\ \phi^{\Delta}_{\ell m}(r) \end{pmatrix} = \mathbf{N} \Big\{ \Big[\mathbf{k}^{\mathrm{I}}_\ell(r,r')\, \delta u_3 + \mathbf{k}^{\mathrm{D}}_{\ell,0} \left(\delta u_2\, D^0_2 + \delta u_3\, D^0_3 \right) \Big]\, \delta_{m0} + \sum_{p=1}^{2} \mathbf{k}^{\mathrm{D}}_{\ell,p}(r,r') \sum_{j=1}^{3} \delta u_j \left(D^p_j\, \delta_{mp} + D^{-p}_j\, \delta_{m-p} \right) \Big\}\, \mathrm{d}S' \tag{41}$$

and, as expected in this case, only the coefficients of spherical harmonic order $m = -2, \cdots, 2$ differ from zero.

The seismic Love numbers here defined can be calculated by means of whatever computational method already designed to this purpose, like (i) the integration of the homogeneous differential system given by eq. (21) in order to obtain the propagator matrix to be implemented in eq. (38), (ii) the reciprocity theorem by Okubo (1993) or (iii) the minor vector method by Attar and Woodhouse (2008). The first method is the simplest one but becomes numerically unstable at harmonic degrees greater than about 400 (Boerse et al., 2014). Differently, the other two methods perform well even at very high harmonic degrees (Sun and Okubo, 1993; Sun and Okubo, 1998; Attar and Woodhouse, 2008).

8 PERTURBATION OF THE EARTH INERTIA

A validation of the present formalism and a proof of its usefulness in geodetic problems can be shown by deriving the analytical expression of the perturbation of the deviatoric inertia tensor of the Earth. In this perspective, we first note that, from eqs (10), (38) and (40), the perturbation of the Stokes coefficients in the geographic reference frame can be directly obtained as follows

$$\Delta X_{\ell m} = a^{-3} \int_{\mathcal{S}} \Big(k^{\mathrm{I}}_{\ell}(a, r')\, \Delta^{\mathrm{I}}_{\ell m}(\boldsymbol{r}')\, \mathrm{d}S' + \sum_{p=0}^{2} k^{\mathrm{D}}_{\ell,p}(a, r')\, \Delta^{\mathrm{D}}_{\ell m,p}(\boldsymbol{r}') \Big) \mathrm{d}S' \tag{42}$$

Then, following Chao and Gross (1987), the deviatoric inertia tensor can be related to the Stokes coefficients of harmonic degree 2 as follows

$$\Delta\mathbf{I} = M\, a^2 \sum_{m=-2}^{2} \mathbf{Q}_m\, \Delta X_{2m} \tag{43}$$

where $\mathbf{Q}_m$ ($m = -2 \cdots, 2$) are the following tensors

$$\mathbf{Q}_0 = \tfrac{\sqrt{5}}{3}\, (\hat{\boldsymbol{x}}_1\, \hat{\boldsymbol{x}}_1 + \hat{\boldsymbol{x}}_2\, \hat{\boldsymbol{x}}_2 - 2\, \hat{\boldsymbol{x}}_3\, \hat{\boldsymbol{x}}_3) \tag{44a}$$

$$\mathbf{Q}_{-1} = -\sqrt{\tfrac{5}{3}}\, (\hat{\boldsymbol{x}}_1\, \hat{\boldsymbol{x}}_3 + \hat{\boldsymbol{x}}_3\, \hat{\boldsymbol{x}}_1) \tag{44b}$$

$$\mathbf{Q}_{1} = -\sqrt{\tfrac{5}{3}}\, (\hat{\boldsymbol{x}}_2\, \hat{\boldsymbol{x}}_3 + \hat{\boldsymbol{x}}_3\, \hat{\boldsymbol{x}}_2) \tag{44c}$$

$$\mathbf{Q}_{-2} = \sqrt{\tfrac{5}{3}}\, (\hat{\boldsymbol{x}}_2\, \hat{\boldsymbol{x}}_2 - \hat{\boldsymbol{x}}_1\, \hat{\boldsymbol{x}}_1) \tag{44d}$$

$$\mathbf{Q}_{2} = -\sqrt{\tfrac{5}{3}}\, (\hat{\boldsymbol{x}}_1\, \hat{\boldsymbol{x}}_2 + \hat{\boldsymbol{x}}_2\, \hat{\boldsymbol{x}}_1) \tag{44e}$$

with $\hat{\boldsymbol{x}}_j$ ($j = 1, 2, 3$) being the Cartesian unit vectors of the geographic reference frame ($\hat{\boldsymbol{x}}_3$ points towards the north pole, and $\hat{\boldsymbol{x}}_1$ and $\hat{\boldsymbol{x}}_2$ are along the equatorial plane, with $\hat{\boldsymbol{x}}_1$ pointing towards the Greenwich meridian).

This result shows that, when our alternative formalism is used, the calculation of the Stokes coefficients is straightforward, eq. (42), and so the calculation of the perturbation of the deviatoric inertia tensor of the Earth, eq. (43). Differently, previous formalisms, which adopt the epicentral reference frame for each infinitesimal element of the fault system, require additional considerations to rotate the seismic source response in the geographic reference frame that make this issue even more complicated (see, for instance, Xu and Jiang (1964), Sabadini et al. (2007) and Xu et al. (2014)).

In order to better understand how the present formalism works and, particularly, the role of the functions $F^{p}_{\ell m}$ entering the factors $\Delta^{\mathrm{I}}_{\ell m}$ and $\Delta^{\mathrm{D}}_{\ell m,p}$, we substitute eqs (30) and (42) into eq. (43) and, after some straightforward algebra, we obtain

$$\Delta\mathbf{I} = M_E\, a^2 \sum_{m=-2}^{2} \int_{\mathcal{S}} \mathbf{O}_m(\theta', \varphi)\, \delta X_{2m}(\boldsymbol{r}') \tag{45}$$

with

$$\delta X_{\ell m} = a^{-3} \Big\{ \Big[k^{\mathrm{I}}_{\ell}(a, r')\, \delta u_3 + k^{\mathrm{D}}_{\ell,0}(a, r')\, \big(\delta u_2\, D^0_2 + \delta u_3\, D^0_3\big) \Big]\, \delta_{m0} + \sum_{p=1}^{2} k^{\mathrm{D}}_{\ell,p}(a, r') \sum_{j=1}^{3} \delta u_j\, \big(D^p_j\, \delta_{mp} + D^{-p}_j\, \delta_{m-p} \big) \Big\}\, \mathrm{d}S' \tag{46a}$$

$$\mathbf{O}_m(\theta', \varphi') = \sum_{m=-2}^{2} F^{m}_{2p}(\theta', \varphi')\, \mathbf{Q}_m \tag{46b}$$

We note that $\delta X_{\ell m}$ corresponds to the perturbation of the Stokes coefficients due to an infinitesimal element of the fault system located at the north pole, i.e, in its epicentral reference system. This can be understood from eq. (41) or by considering the limit for $\theta' \to 0$ and $\varphi' \to 0$ of the integrand of eq. (42) and using eq. (34). Furthermore, it is straightforward to show that the tensors $\mathbf{O}_m$ coincide with the tensors $\mathbf{Q}_m$ once evaluated at the north pole

$$\mathbf{O}_m(0, 0) = \mathbf{Q}_m \tag{47}$$

and, in general, take the following form

$$\mathbf{O}_0(\theta',\varphi') = \frac{\sqrt{5}}{3}\left(\hat{\boldsymbol{\theta}}'\,\hat{\boldsymbol{\theta}}' + \hat{\boldsymbol{\varphi}}'\,\hat{\boldsymbol{\varphi}}' - 2\,\hat{\boldsymbol{r}}'\,\hat{\boldsymbol{r}}'\right) \tag{48a}$$

$$\mathbf{O}_{-1}(\theta',\varphi') = -\sqrt{\tfrac{5}{3}}\left(\hat{\boldsymbol{\theta}}'\,\hat{\boldsymbol{r}}' + \hat{\boldsymbol{r}}'\,\hat{\boldsymbol{\theta}}'\right) \tag{48b}$$

$$\mathbf{O}_{1}(\theta',\varphi') = -\sqrt{\tfrac{5}{3}}\left(\hat{\boldsymbol{\varphi}}'\,\hat{\boldsymbol{r}}' + \hat{\boldsymbol{r}}'\,\hat{\boldsymbol{\varphi}}'\right) \tag{48c}$$

$$\mathbf{O}_{-2}(\theta',\varphi') = \sqrt{\tfrac{5}{3}}\left(\hat{\boldsymbol{\varphi}}'\,\hat{\boldsymbol{\varphi}}' - \hat{\boldsymbol{\theta}}'\,\hat{\boldsymbol{\theta}}'\right) \tag{48d}$$

$$\mathbf{O}_{2}(\theta',\varphi') = -\sqrt{\tfrac{5}{3}}\left(\hat{\boldsymbol{\theta}}'\,\hat{\boldsymbol{\varphi}}' + \hat{\boldsymbol{\varphi}}'\,\hat{\boldsymbol{\theta}}'\right) \tag{48e}$$

In this respect, eq. (45) can be interpreted as the application of eq. (43) for each infinitesimal surface element in its epicentral reference frame and the integration of these contributions over the whole surface of the fault system. Particularly, we note that the orientation of each contribution to the deviatoric inertia tensor is taken into account by the dependence of the tensors $\mathbf{O}_m$ on the Cartesian unit vectors of the epicentral reference frame, rather than on the Cartesian unit vectors of the geographic reference frame as it is in the case of the tensors $\mathbf{Q}_m$.

9 CONCLUSIONS

At present, methods for calculating the static perturbation of spherically symmetric self-gravitating elastic Earth models are laborious because all of them adopt an *ad hoc* reference frame for each infinitesimal element of the fault system, namely the epicentral reference frame. These methods thus make difficult to clearly understand the relation between the static perturbation and the fault system itself, just because the latter is never seen as a whole. Reviewing the displacement dislocation theory for spherically symmetric Earth models and keeping arbitrary the reference frame, we have presented a thoughtful reorganization of the expression for the discontinuity of the spheroidal vector solution at the seismic source radius, eq. (27), separating and making clearly identifiable the dependence on the shear and tensile components of the displacement dislocation, the dip and strike angles and the angular coordinates of the infinitesimal element of the fault system. Particularly, although this reorganization is based on the possibility of decomposing the seismic source response into polar, bipolar and quadrupolar patterns in the epicentral reference frame of each infinitesimal element of the fault system, we have shown how it is possible to take advantage of this decomposition even in reference frames different from the epicentral one.

These considerations have lead us to define one seismic Love number, $\mathbf{k}^{\mathrm{I}}_{\ell}$, for the isotropic part (or centre of compression) of the surface moment-density tensor (associated to the only polar pattern) and three seismic Love numbers, $\mathbf{k}^{\mathrm{D}}_{\ell,0}$, $\mathbf{k}^{\mathrm{D}}_{\ell,1}$ and $\mathbf{k}^{\mathrm{D}}_{\ell,2}$, for the deviatoric part (associated to polar, bipolar and quadrupolar patterns, respectively), that can be used to obtain the spherical harmonic coefficients of the static perturbation due to the displacement dislocation over the whole fault system in a common reference frame, eq. (40). In this respect, our seismic Love numbers (or Green's functions) are organized in a different way compared to the treatment proposed by Sun and Okubo (1993) and their following works (Sun and Okubo, 1998; Tanaka et al., 2006; Sun et al., 2009). These previous treatments, indeed, introduce four Green's functions (for strike and dip slips on a vertical fault and tensile fracturing on horizontal and vertical faults), and the spheroidal perturbation due to general tensile and shear dislocations are obtained combining three and all four Green's functions, respectively (see eqs (115) and (117) of Sun and Okubo (1993)). The Green's function for tensile fracturing on a vertical plane is composed of both polar and quadrupolar perturbations (order $m = 0$ and ± 2), while the Green's functions here presented do not mix the different types of perturbation, being organized just for separating them. Our method, in addition to shed light on the relation between the static perturbation and the whole fault system, discriminates in a more effective way the contributions from tensile and shear displacement dislocations and makes evident their relation with the polar, bipolar and quadrupolar patterns.

In the end, we note that the possibility of computing the spherically harmonic coefficients of the static perturbation due to the whole fault system in a common reference frame can be useful especially in problems which require the spherical harmonic expansion of the perturbation, like the evaluation of the perturbation of the deviatoric inertia tensor of the Earth (which only involve perturbations of spherical harmonic degree 2) and the modelling of long-wavelength geoid or gravity data form GRACE and GOCE space missions. Particularly, it simplifies the implementation of anisotropic filters for dealing with the peculiar noise at equatorial latitudes (the so-called 'stripes') of the Stokes coefficients from GRACE and the spatial localization of seismic source response in a polar cup including the near field of the earthquake using the Slepian functions. Indeed, both the anisotropic filters and the spatial localization are usually defined in terms of linear combinations of spherical harmonic coefficients of the perturbation in a fixed reference frame: the geographic reference frame for anisotropic filtering (Sweanson and Wahr, 2006; Kusche et al., 2007, 2009) and the spherical reference frame with the polar axis along the center of the cup for the spatial localization (Simons, 2006; Wang et al., 2012; Cambiotti and Sabaini, 2012; 2013).

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APPENDIX A: SPHERICAL HARMONICS

In order to simplify the application of the present formalism to geodetic problems, we decide to rely on real, rather than complex, spherical harmonics and to agree with the common convection used in geodesy (Chao and Gross, 1987; Wahr et al., 1998). The real spherical harmonics, $Y_{\ell m}$ ($m = -\ell, \cdots, \ell$ and $\ell = 0, \cdots, \infty$), are then defined as follows

$$Y_{\ell m}(\theta, \varphi) = \begin{cases} \hat{P}_{\ell|m|}(\cos\theta)\,\cos(m\,\varphi) & m \leq 0 \\ \hat{P}_{\ell m}(\cos\theta)\,\sin(m\,\varphi) & m > 0 \end{cases} \tag{A.1}$$

where ℓ and m are the spherical harmonic degree and order, and $\hat{P}_{\ell m}$ is the normalized associated Legendre function

$$\hat{P}_{\ell m} = \frac{P_{\ell m}}{N_{\ell m}} \qquad m \geq 0 \tag{A.2}$$

Here $P_{\ell m}$ is the non-normalized associated Legendre function

$$P_{\ell m}(x) = \frac{(1-x^2)^{m/2}}{2^\ell\,\ell!}\,\frac{d^{(\ell+m)}(x^2-1)^\ell}{dx^{(\ell+m)}} \qquad m \geq 0 \tag{A.3}$$

and $N_{\ell m}$ is the factor given by

$$N_{\ell m} = \sqrt{\frac{(\ell+m)!}{(2-\delta_{0m})\,(2\,\ell+1)\,(\ell-m)!}} \qquad m \geq 0 \tag{A.4}$$

Note that the factor $N_{\ell m}$ has been chosen so that the real spherical harmonics satisfy the following condition of orthogonality

$$\int_\Omega Y_{\ell m}\,Y_{\ell' m'}\,\mathrm{d}\Omega = 4\,\pi\,\delta_{\ell\ell'}\,\delta_{mm'} \tag{A.5}$$

where Ω denotes the unit sphere and $\mathrm{d}\Omega = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi$ is the infinitesimal solid angle.

We also define the real vector spherical harmonics $\boldsymbol{P}_{\ell m}$, $\boldsymbol{B}_{\ell m}$ and $\boldsymbol{C}_{\ell m}$ as follows (Ben-Menahem and Singh, 1981; Dahlen and Tromp, 1998)

$$\boldsymbol{P}_{\ell m} = Y_{\ell m}\,\hat{\boldsymbol{r}} \tag{A.6a}$$

$$L\,\boldsymbol{B}_{\ell m} = \boldsymbol{\nabla}_\Omega Y_{\ell m} = \hat{\boldsymbol{\theta}}\,\frac{\partial Y_{\ell m}}{\partial\theta} + \hat{\boldsymbol{\varphi}}\,\csc\theta\,\frac{\partial Y_{\ell m}}{\partial\varphi} \tag{A.6b}$$

$$L\,\boldsymbol{C}_{\ell m} = -\hat{\boldsymbol{r}}\times\boldsymbol{\nabla}_\Omega Y_{\ell m} = \hat{\boldsymbol{\theta}}\,\csc\theta\,\frac{\partial Y_{\ell m}}{\partial\varphi} - \hat{\boldsymbol{\varphi}}\,\frac{\partial Y_{\ell m}}{\partial\theta} \tag{A.6c}$$

where L is given by eq. (29) and $\boldsymbol{\nabla}_\Omega$ is the surface gradient operator in the spherical reference frame

$$\boldsymbol{\nabla}_\Omega = \hat{\boldsymbol{\theta}}\,\frac{\partial}{\partial\theta} + \hat{\boldsymbol{\varphi}}\,\csc\theta\,\frac{\partial}{\partial\varphi} \tag{A.7}$$

They satisfy the following conditions of orthogonality

$$\boldsymbol{P}_{\ell m} \cdot \boldsymbol{B}_{\ell' m'} = \boldsymbol{P}_{\ell m} \cdot \boldsymbol{C}_{\ell' m'} = \int_\Omega \boldsymbol{B}_{\ell m} \cdot \boldsymbol{C}_{\ell' m'} \, \mathrm{d}\Omega = 0 \tag{A.8a}$$

$$\int_\Omega \boldsymbol{P}_{\ell m} \cdot \boldsymbol{P}_{\ell' m'} \, \mathrm{d}\Omega = \int_\Omega \boldsymbol{B}_{\ell m} \cdot \boldsymbol{B}_{\ell' m'} \, \mathrm{d}\Omega = \int_\Omega \boldsymbol{C}_{\ell m} \cdot \boldsymbol{C}_{\ell' m'} \, \mathrm{d}\Omega = 4\,\pi\, \delta_{\ell\ell'}\, \delta_{mm'} \tag{A.8b}$$

In order to obtain the spherical harmonic expansion of the seismic force equivalent to the displacement dislocation (see section 3), we shall consider the surface gradients of the vector spherical harmonics

$$\boldsymbol{\nabla}_\Omega \boldsymbol{P}_{\ell m} = \left(\hat{\boldsymbol{\theta}}\,\hat{\boldsymbol{\theta}} + \hat{\boldsymbol{\varphi}}\,\hat{\boldsymbol{\varphi}}\right) Y_{\ell m} + \hat{\boldsymbol{\theta}}\,\hat{\boldsymbol{r}}\,\frac{\partial Y_{\ell m}}{\partial \theta} + \hat{\boldsymbol{\varphi}}\,\hat{\boldsymbol{r}}\,\csc\theta\,\frac{\partial Y_{\ell m}}{\partial \varphi} \tag{A.9a}$$

$$\begin{aligned}\boldsymbol{\nabla}_\Omega \boldsymbol{B}_{\ell m} = -L^{-1}\left[\hat{\boldsymbol{\theta}}\,\hat{\boldsymbol{r}}\,\frac{\partial Y_{\ell m}}{\partial \theta} + \hat{\boldsymbol{\varphi}}\,\hat{\boldsymbol{r}}\,\csc\theta\,\frac{\partial Y_{\ell m}}{\partial \varphi} + \left(\hat{\boldsymbol{\varphi}}\,\hat{\boldsymbol{\varphi}} - \hat{\boldsymbol{\theta}}\,\hat{\boldsymbol{\theta}}\right)\left(\frac{\partial^2}{\partial\theta^2} + \frac{L^2}{2}\right) Y_{\ell m} - \left(\hat{\boldsymbol{\theta}}\,\hat{\boldsymbol{\varphi}} + \hat{\boldsymbol{\varphi}}\,\hat{\boldsymbol{\theta}}\right)\frac{\partial}{\partial\theta}\left(\csc\theta\,\frac{\partial Y_{\ell m}}{\partial\varphi}\right)\right] \\ - \frac{L}{2}\left(\hat{\boldsymbol{\theta}}\,\hat{\boldsymbol{\theta}} + \hat{\boldsymbol{\varphi}}\,\hat{\boldsymbol{\varphi}}\right) Y_{\ell m}\end{aligned} \tag{A.9b}$$

$$\begin{aligned}\boldsymbol{\nabla}_\Omega \boldsymbol{C}_{\ell m} = L^{-1}\left[\hat{\boldsymbol{\varphi}}\,\hat{\boldsymbol{r}}\,\frac{\partial Y_{\ell m}}{\partial \theta} - \hat{\boldsymbol{\theta}}\,\hat{\boldsymbol{r}}\,\csc\theta\,\frac{\partial Y_{\ell m}}{\partial \varphi} + \left(\hat{\boldsymbol{\theta}}\,\hat{\boldsymbol{\theta}} - \hat{\boldsymbol{\varphi}}\,\hat{\boldsymbol{\varphi}}\right)\frac{\partial}{\partial\theta}\left(\csc\theta\frac{\partial Y_{\ell m}}{\partial\varphi}\right) - \left(\hat{\boldsymbol{\theta}}\,\hat{\boldsymbol{\varphi}} + \hat{\boldsymbol{\varphi}}\,\hat{\boldsymbol{\theta}}\right)\left(\frac{\partial^2}{\partial\theta^2} + L^2/2\right) Y_{\ell m}\right] \\ + \frac{L}{2}\left(\hat{\boldsymbol{\theta}}\,\hat{\boldsymbol{\varphi}} - \hat{\boldsymbol{\varphi}}\,\hat{\boldsymbol{\theta}}\right) Y_{\ell m}\end{aligned} \tag{A.9c}$$

Furthermore, the following limits of the spherical harmonics and of their partial derivatives for the angular coordinates going to the polar axis of the spherical reference frame hold

$$\lim_{\theta,\varphi\to 0} Y_{\ell m} = (2\,\ell + 1)\, N_{\ell 0}\, \delta_{0m} \tag{A.10a}$$

$$\lim_{\theta,\varphi\to 0} \frac{\partial Y_{\ell m}}{\partial\theta} = (2\,\ell + 1)\, N_{\ell 1}\, \delta_{-1m} \tag{A.10b}$$

$$\lim_{\theta,\varphi\to 0} \csc\theta \frac{\partial Y_{\ell m}}{\partial\varphi} = (2\,\ell + 1)\, N_{\ell 1}\, \delta_{1m} \tag{A.10c}$$

$$\lim_{\theta,\varphi\to 0} \left(2\,\frac{\partial^2}{\partial\theta^2} + L^2\right) Y_{\ell m} = (2\,\ell + 1)\, N_{\ell 2}\, \delta_{-2m} \tag{A.10d}$$

$$\lim_{\theta,\varphi\to 0} 2\,\frac{\partial}{\partial\theta}\left(\csc\theta\frac{\partial Y_{\ell m}}{\partial\varphi}\right) = (2\,\ell + 1)\, N_{\ell 2}\, \delta_{2m} \tag{A.10e}$$

These limits can be proved using eq. (4.5.5) of Ben-Menahem and Singh (1981), taking care of the different definition of spherical harmonics (non-normalized complex spherical harmonics rather than real spherical harmonics normalized to $4\,\pi$ here considered).

APPENDIX B: SEISMIC FORCE

In view of the orthogonality relations of the vector spherical harmonics, eq. (A.8), the coefficients $P_{\ell m}$, $B_{\ell m}$ and $C_{\ell m}$ of the seismic force $\boldsymbol{f}$ equivalent to displacement dislocation, eqs (4) and (9d), are given by

$$X(r) = \frac{1}{4\,\pi} \int_\Omega \boldsymbol{X}(\theta,\varphi) \cdot \boldsymbol{f}(\boldsymbol{r})\,\mathrm{d}\Omega \tag{B.1}$$

with $X = P_{\ell m}, B_{\ell m}, C_{\ell m}$ for $\boldsymbol{X} = \boldsymbol{P}_{\ell m}, \boldsymbol{B}_{\ell m}, \boldsymbol{C}_{\ell m}$, respectively.

Considering the definition of the gradient operator and of the Dirac delta distribution in spherical coordinates

$$\boldsymbol{\nabla} = \hat{\boldsymbol{r}}\,\frac{\partial}{\partial_r} + r^{-1}\,\boldsymbol{\nabla}_\Omega \tag{B.2a}$$

$$\delta\left(\boldsymbol{r} - \boldsymbol{r}'\right) = \frac{\delta(r - r')\,\delta(\theta - \theta')\,\delta(\varphi - \varphi')}{r^2\,\sin\theta} \tag{B.2b}$$

we can write the gradient of the Dirac delta distribution entering eq. (4) as follows

$$\boldsymbol{\nabla}\delta(\boldsymbol{r} - \boldsymbol{r}') = \frac{\delta(r - r')}{r^3}\left[\boldsymbol{\nabla}_\Omega\left(\frac{\delta(\theta - \theta')\,\delta(\varphi - \varphi')}{\sin\theta}\right) - 2\,\hat{\boldsymbol{r}}\,\frac{\delta(\theta - \theta')\,\delta(\varphi - \varphi')}{\sin\theta}\right] + \frac{1}{r^2}\,\frac{\partial\delta(r - r')}{\partial r}\,\frac{\delta(\theta - \theta')\,\delta(\varphi - \varphi')}{\sin\theta}\,\hat{\boldsymbol{r}} \tag{B.3}$$

Then, substituting eq. (4) into eq. (B.1) and making use of the following identities,

$$\int_\Omega a(\boldsymbol{r})\,\frac{\delta(\theta-\theta')\,\delta(\varphi-\varphi')}{\sin\theta}\,\mathrm{d}\Omega = a(\boldsymbol{r})\Big|_{\substack{\theta=\theta'\\ \varphi=\varphi'}} \tag{B.4a}$$

$$\int_\Omega \boldsymbol{a}(\boldsymbol{r})\cdot\left[\boldsymbol{\nabla}_\Omega\left(\frac{\delta(\theta-\theta')\,\delta(\varphi-\varphi')}{\sin\theta}\right)-2\,\hat{\boldsymbol{r}}\,\frac{\delta(\theta-\theta')\,\delta(\varphi-\varphi')}{\sin\theta}\right]\mathrm{d}\Omega = -\,\boldsymbol{\nabla}_\Omega\cdot\boldsymbol{a}(\boldsymbol{r})\Big|_{\substack{\theta=\theta'\\ \varphi=\varphi'}} \tag{B.4b}$$

which hold for any (sufficiently smooth) scalar, a, and vector, $\boldsymbol{a}$, field, we obtain

$$X(r)=\frac{1}{4\,\pi}\int_{\mathcal{S}}\left[\frac{\delta(r-r')}{r^3}\,\boldsymbol{\nabla}_\Omega\cdot\big(\boldsymbol{X}(\theta,\varphi)\cdot\mathbf{M}(\boldsymbol{r}')\big)-\frac{\partial_r\delta(r-r')}{r^2}\,\hat{\boldsymbol{r}}\cdot\big(\boldsymbol{X}(\theta,\varphi)\cdot\mathbf{M}(\boldsymbol{r}')\big)\right]\Bigg|_{\substack{\theta=\theta'\\ \varphi=\varphi'}}\mathrm{d}S' \tag{B.5}$$

Then, considering that the surface moment-density tensor does not depend on the angular coordinates of the observation points, eq. (B.5) can be arranged as follows

$$X(r)=\frac{1}{4\,\pi}\int_{\mathcal{S}}\mathbf{M}(\boldsymbol{r}'):\left[\frac{\delta(r-r')}{r^3}\,\boldsymbol{\nabla}'_\Omega\,\boldsymbol{X}(\theta',\varphi')-\frac{\partial_r\delta(r-r')}{r^2}\,\hat{\boldsymbol{r}}'\,\boldsymbol{X}(\theta',\varphi')\right]\mathrm{d}S' \tag{B.6}$$

where : stands for the double scalar product and $\boldsymbol{\nabla}'_\Omega$ stands for the surface gradient operator with respect to the angular coordinates of the infinitesimal surface element

$$\boldsymbol{\nabla}'_\Omega=\hat{\boldsymbol{\theta}}'\,\partial_{\theta'}+\hat{\boldsymbol{\varphi}}'\,\csc\theta'\,\partial_{\varphi'} \tag{B.7}$$

APPENDIX C: MOMENTUM AND POISSON EQUATIONS AND BOUNDARY CONDITIONS

The momentum and Poisson equations describing the infinitesimal perturbations of spherically symmetric self-gravitating elastic Earth model with respect to the initial state of hydrostatic equilibrium reads (Wolf, 1991)

$$\boldsymbol{\nabla}\cdot\mathbf{S}^\delta-\boldsymbol{\nabla}\left(\rho_0\,g_0\,\hat{\boldsymbol{r}}\cdot\boldsymbol{u}\right)-\rho_0\,\boldsymbol{\nabla}\phi^\Delta-\rho^\Delta\,g_0\,\hat{\boldsymbol{r}}+\boldsymbol{f}=\boldsymbol{0} \tag{C.1a}$$

$$\nabla^2\phi^\Delta=4\,\pi\,G\,\rho^\Delta \tag{C.1b}$$

where $\boldsymbol{\nabla}$ and ∇^2 stand for the gradient and Laplace operators, G is the universal gravitational constant, ρ_0 and g_0 are the initial density and gravity, ϕ^Δ and ρ^Δ are the local incremental gravitational potential and density, $\boldsymbol{u}$ is the displacement field and $\mathbf{S}^\delta$ is the material incremental stress tensor that, for an elastic isotropic medium, reads

$$\mathbf{S}^\delta=\left(\kappa-\tfrac{2}{3}\right)\,\boldsymbol{\nabla}\cdot\boldsymbol{u}\,\mathbf{I}+\mu\,\left[\boldsymbol{\nabla}\boldsymbol{u}+(\boldsymbol{\nabla}\boldsymbol{u})^{\mathrm{T}}\right] \tag{C.2}$$

In view of the equation of the conservation of mass, the local incremental density reads

$$\rho^\Delta=-\boldsymbol{\nabla}\cdot(\rho_0\,\boldsymbol{u}) \tag{C.3}$$

Note that, in the momentum equation, eq. (C.1a), we have included the seismic force $\boldsymbol{f}$ equivalent to the displacement dislocation over the fault system, eq. (4).

The momentum and Poisson equations must be solved within the whole Earth together with the following continuity conditions at the solid-solid internal interfaces of the Earth model (Wolf, 1991)

$$\big[\boldsymbol{u}\big]_-^+=\boldsymbol{0} \tag{C.4a}$$

$$\big[\boldsymbol{t}^\delta\big]_-^+=\boldsymbol{0} \tag{C.4b}$$

$$\big[\phi^\Delta\big]_-^+=0 \tag{C.4c}$$

$$\big[\boldsymbol{\nabla}\phi^\Delta+4\,\pi\,G\,\rho_0\,\boldsymbol{u}\big]_-^+=0 \tag{C.4d}$$

where, for brevity, we have introduced the notation

$$[f]_{-}^{+} = \lim_{\epsilon \to 0} f(\boldsymbol{r} + \epsilon\, \hat{\boldsymbol{r}}) - f(\boldsymbol{r} - \epsilon\, \hat{\boldsymbol{r}}) \tag{C.5}$$

for any (scalar and vector) field f, and $\boldsymbol{t}^{\delta}$ is the material incremental traction on a surface element of a sphere centered at the Earth center, $\boldsymbol{t}^{\delta} = \mathbf{S}^{\delta} \cdot \hat{\boldsymbol{r}}$. These continuity conditions hold also at the core-mantle boundary (CMB), which is a fluid-solid interface, but for the fact that the tangential components of the displacement, $\boldsymbol{u} \cdot \hat{\boldsymbol{\theta}}$ and $\boldsymbol{u} \cdot \hat{\boldsymbol{\varphi}}$, are not constrained and that the tangential components of the material incremental traction, $\boldsymbol{t}^{\delta} \cdot \hat{\boldsymbol{\theta}}$ and $\boldsymbol{t}^{\delta} \cdot \hat{\boldsymbol{\varphi}}$, are zero. Furthermore, the solution must satisfy the free Earth surface boundary conditions

$$\boldsymbol{t}^{\delta}\Big|_{r=a} = \mathbf{0} \tag{C.6a}$$

$$\boldsymbol{\nabla}\phi^{\Delta} + 4\,\pi\, G\, \rho_0\, \boldsymbol{u}\Big|_{r=a} = 0 \tag{C.6b}$$

As discussed in the main text (sections 4 and 5), the momentum and Poisson equations are customarily solved expanding in spherical harmonics the scalar and vector fields describing the static perturbation, eqs (9). Hereinafter we provide a few definitions of quantities that we shall use in the main text. The spherical harmonic coefficients of the material incremental traction, $R_{\ell m}$, $S_{\ell m}$ and $T_{\ell m}$, can be expressed in terms of those of the displacement field, $U_{\ell m}$, $V_{\ell m}$ and $W_{\ell m}$, as follows

$$R_{\ell m} = \lambda \left(\frac{\partial U_{\ell m}}{\partial r} + \frac{2\, U_{\ell m} - L\, V_{\ell m}}{r} \right) + 2\,\mu\, \frac{\partial U_{\ell m}}{\partial r} \tag{C.7a}$$

$$S_{\ell m} = \mu \left(\partial_r V_{\ell m} + \frac{L\, U_{\ell m} - V_{\ell m}}{r} \right) \tag{C.7b}$$

$$T_{\ell m} = \mu \left(\partial_r W_{\ell m} - \frac{W_{\ell m}}{r} \right) \tag{C.7c}$$

The matrix $\mathbf{A}$ defining the spheroidal differential system, eq. (12), takes the following form

$$\mathbf{A}_{\ell} = \begin{pmatrix} -\dfrac{2\,\lambda}{r\,(\lambda + 2\,\mu)} & \dfrac{L\,\lambda}{r\,(\lambda + 2\,\mu)} & \dfrac{1}{\lambda + 2\,\mu} & 0 & 0 & 0 \\ -\dfrac{L}{r} & \dfrac{1}{r} & 0 & \dfrac{1}{\mu} & 0 & 0 \\ \dfrac{4}{r^2}\left(\dfrac{3\,\kappa\,\mu}{\lambda + 2\,\mu} - \rho_0\, g_0\, r \right) & \dfrac{L}{r^2}\left(\rho_0\, g_0\, r - \dfrac{6\,\kappa\,\mu}{\lambda + 2\,\mu} \right) & -\dfrac{4\,\mu}{r\,(\lambda + 2\,\mu)} & \dfrac{L}{r} & -\dfrac{\rho_0\,(\ell+1)}{r} & \rho_0 \\ \dfrac{L}{r^2}\left(\rho_0\, g_0\, r - \dfrac{6\,\kappa\,\mu}{\lambda + 2\,\mu} \right) & \dfrac{2\,\mu}{r^2}\left[L^2 \left(1 + \dfrac{\lambda}{\lambda + 2\,\mu} \right) - 1 \right] & -\dfrac{L\,\lambda}{r\,(\lambda + 2\,\mu)} & -\dfrac{3}{r} & \dfrac{L\,\rho_0}{r} & 0 \\ -4\,\pi\, G\, \rho_0 & 0 & 0 & 0 & -\dfrac{\ell+1}{r} & 1 \\ -\dfrac{4\,\pi\, G\, \rho_0\,(\ell+1)}{r} & \dfrac{4\,\pi\, G\, \rho_0\, L}{r} & 0 & 0 & 0 & \dfrac{\ell - 1}{r} \end{pmatrix} \tag{C.8}$$

with ρ_0 and g_0 being the initial density and gravity. Within the assumption of an inviscid core, the core-mantle boundary (CMB) matrix is given by (Longman, 1963; Farrell, 1972; Dahlen,1974; Saito, 1974; Chinnery, 1975; Cambiotti et al., 2013)

$$\mathbf{C}_{\ell} = \left. \begin{pmatrix} -\dfrac{f_\ell}{g_0} & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & g_0\, \rho_0 \\ 0 & 0 & 0 \\ f_\ell & 0 & 0 \\ q_\ell & 0 & 4\,\pi\, G\, \rho_0 \end{pmatrix} \right|_{r=r_C^-} \tag{C.9}$$

where all the quantities are evaluated at $r = r_C^-$ (i.e., just below the CMB), and f_ℓ is the regular solution (up to a constant of integration) of the local incremental gravitational potential that solves, after spherical harmonic expansion, the momentum and Poisson equations within the inviscid core, that is

$$\partial_r^2 f_\ell + \frac{2}{r}\, \partial_r f_\ell - \left(\frac{4\,\pi\, G\, \partial_r \rho_0}{g_0} + \frac{L^2}{r^2} \right) f_\ell = 0 \tag{C.10}$$

with $\lim_{r\to 0} f_\ell\, r^{-\ell} = 1$. Furthermore, q_ℓ is defined in terms of the function f_ℓ as

$$q_\ell = \partial_r f_\ell + \left(\frac{\ell+1}{r} - \frac{4\,\pi\,G\,\rho_0}{g_0}\right) f_\ell \tag{C.11}$$

We refer the reader to Takeuchi and Saito (1972) and Tanaka et al. (2006) for the case in which only the outer core is taken as an inviscid layer while the inner core is taken as a solid layer (with non zero shear modulus). The matrices defining the toroidal differential system and the relevant CMB matrix will be introduced in last section of this supplementary online material (SOM).

APPENDIX D: THE CASES OF THE HARMONIC DEGREES 0 AND 1

The cases of spheroidal perturbations of harmonic degrees $\ell = 0, 1$ need some specific considerations and their treatment deviate somehow from that for higher harmonic degrees, $\ell \geq 2$. However, once redefined some of the quantities and formulas introduced in sections 5-7, all the remaining equations still apply to the cases of $\ell = 0, 1$. In this respect, hereinafter, we just discuss the necessary changes and redefinitions, without reporting all those equations which are formally equivalent to the case of higher harmonic degrees, $\ell \geq 2$ and leaving to the reader the straightforward task to translate their meaning.

D1 Harmonic degree 0

Spheroidal perturbations of harmonic degree 0 mainly differ from those of higher harmonic degrees because, by definition, the tangential components of the displacement and of the radial traction are zero, $V_{00} = S_{00} = 0$. This reduces the spheroidal differential system, eq. (12), to only four linear differential equations.

In light of this, for $\ell = 0$, we redefine the 4×4 matrix of the spheroidal differential system, $\mathbf{A}_0$, the inhomogeneous term which accounts for the displacement dislocation, $\mathbf{q}_{00}$, and the spheroidal vector solution, $\mathbf{y}_{00}$,

$$\mathbf{A}_0 = \begin{pmatrix} -\dfrac{2\,\lambda}{r\,(\lambda+2\,\mu)} & \dfrac{1}{(\lambda+2\,\mu)} & 0 & 0 \\ \dfrac{4}{r^2}\left(\dfrac{3\,\kappa\,\mu}{\lambda+2\,\mu} - \rho_0\,g_0\,r\right) & -\dfrac{4\,\mu}{r\,(\lambda+2\,\mu)} & -\dfrac{\rho_0}{r} & \rho_0 \\ -4\,\pi\,G\,\rho_0 & 0 & -\dfrac{1}{r} & 1 \\ -\dfrac{4\,\pi\,G\,\rho_0}{r} & 0 & 0 & -\dfrac{1}{r} \end{pmatrix} \tag{D.1a}$$

$$\mathbf{q}_{00} = \begin{pmatrix} 0 & -P_{00} & 0 & 0 \end{pmatrix}^{\mathrm{T}} \tag{D.1b}$$

$$\mathbf{y}_{00} = \begin{pmatrix} U_{00} & R_{00} & \Phi_{00} & Q_{00} \end{pmatrix}^{\mathrm{T}} \tag{D.1c}$$

and the projectors $\mathbf{P}_1$ and $\mathbf{P}_2$ for the second and fourth components, and first and third components of the spheroidal vector solution, $\mathbf{P}_1\,\mathbf{y}_{00} = \begin{pmatrix} R_{00} & Q_{00} \end{pmatrix}^{\mathrm{T}}$ and $\mathbf{P}_2\,\mathbf{y}_{00} = \begin{pmatrix} U_{00} & \Phi_{00} \end{pmatrix}^{\mathrm{T}}$, respectively. As it concerns the CMB boundary conditions, they can be still expressed in the form of eq. (17), taking care of a few considerations for which we refer the reader to Smylie and Mansinha (1971).

Furthermore, following the same steps discussed for higher harmonic degrees, we obtain that the discontinuity of the spheroidal vector solution due to an infinitesimal surface element of the fault plane can be expressed as follows

$$\delta\mathbf{s}_{00}(\boldsymbol{r}') = \mathbf{s}_0^{\mathrm{I}}(r')\,\Delta_{00}^{\mathrm{I}}(\boldsymbol{r}') + \mathbf{s}_{0,0}^{\mathrm{D}}(r')\,\Delta_{00,0}^{\mathrm{D}}(\boldsymbol{r}') \tag{D.2}$$

with

$$\mathbf{s}_\ell^{\mathrm{I}}(r') = \frac{1}{4\,\pi\,r'^3}\,\frac{\kappa'}{\lambda'+2\,\mu'} \begin{pmatrix} r' & -4\,\mu' & 0 & 0 \end{pmatrix}^{\mathrm{T}} \tag{D.3a}$$

$$\mathbf{s}_{\ell,0}^{\mathrm{D}}(r') = \frac{1}{4\,\pi\,r'^3}\,\frac{1}{\lambda'+2\,\mu'} \begin{pmatrix} r' & 3\,\kappa' & 0 & 0 \end{pmatrix}^{\mathrm{T}} \tag{D.3b}$$

Note that, in this case, only polar perturbations are involved. This also implies that only tensile and along dip components of the displacement dislocation affects the perturbation of harmonic degree 0, while along strike component does not.

With these definitions and similarly to eq. (40), the radial spheroidal components of the displacement and the local incremental gravitational potential can be expressed as follows

$$\begin{pmatrix} U_{00}(r) \\ \Phi_{00}(r) \end{pmatrix} = \mathbf{N} \int_{\mathcal{S}} \left(\mathbf{k}_0^{\mathrm{I}}(r,r')\, \Delta_{00}^{\mathrm{I}}(\boldsymbol{r}') + \mathbf{k}_{0,0}^{\mathrm{D}}(r,r')\, \Delta_{00,0}^{\mathrm{D}}(\boldsymbol{r}') \right) \mathrm{d}S' \tag{D.4}$$

where $\mathbf{N}$ is now given by

$$\mathbf{N} = a^{-2}\, \mathrm{Diag}\left[\begin{array}{cc} 1 & -g_0(a) \end{array}\right] \tag{D.5}$$

and only the radial, h, and gravitational, k, seismic Love numbers are defined. We note also that, in this case, there are no seismic Love numbers associated to the bipolar ($p = 1$) and quadrupolar ($p = 2$) patterns.

D2 Harmonic degree 1

The treatment for the case of the harmonic degree $\ell = 1$ differs from that for other harmonic degrees because, in addition to deformations of the Earth, it also describes the shift of the center of mass of the Earth (Farrell, 1972, Okubo and Endo, 1986, Okubo, 1993). However, it is customary to choose the origin of the reference frame fixed to the center of mass of the Earth and make the minor approximation that this reference frame is inertial.

It can be shown that the shift of the center of mass, which we denote with $\boldsymbol{R}$, is related to the perturbations of the Stokes coefficients (or the local incremental gravitational potential at the Earth surface) of harmonic degree $\ell = 1$ by means of the following formula (Klemann and Martinec, 2009)

$$\boldsymbol{R} = a\sqrt{3}\, \left(\hat{\boldsymbol{x}}_1\, \Delta X_{1-1} + \hat{\boldsymbol{x}}_2\, \Delta X_{11} + \hat{\boldsymbol{x}}_3\, \Delta X_{10}\right) \tag{D.6}$$

Then, consistently with the assumption that the origin of the reference frame coincides with the center of mass, $\boldsymbol{R} = \boldsymbol{0}$, we must require that the local incremental gravitational potential of harmonic degree $\ell = 1$ is zero at the Earth surface, $\phi_{1m}^{\Delta}(a) = 0$. Furthermore, as discussed in Farrell (1972), Okubo and Endo (1986) and Okubo (1993), spheroidal perturbations of harmonic degree $\ell = 1$ satisfy the following consistency relation.

$$R_{1m}(a) + 2\, S_{1m}(a) + \frac{4\,\pi\, G}{g_0(a)}\, Q_{1m}(a) = 0 \tag{D.7}$$

which is equivalent to the constraint that the total force should vanish within the Earth.

In light of these considerations, spheroidal perturbations of harmonic degree $\ell = 1$ can be obtained by means of the same formula for greater harmonic degrees $\ell \geq 2$ (see sections 5-7), once redefined the projector $\mathbf{P}_1$ for the third, fourth and fifth (rather than sixth) components of the spheroidal vector solution, $\mathbf{P}_1\, \mathbf{y}_{1m} = \begin{pmatrix} R_{1m} & S_{1m} & \Phi_{1m} \end{pmatrix}^{\mathrm{T}}$. With these definitions and similarly to eq. (40), the radial and tangential spheroidal components of the displacement and the local incremental gravitational potential can be expressed as follows

$$\begin{pmatrix} U_{1m}(r) \\ V_{1m}(r) \\ \Phi_{1m}(r) \end{pmatrix} = \mathbf{N} \int_{\mathcal{S}} \left(\mathbf{k}_1^{\mathrm{I}}(r,r')\, \Delta_{1m}^{\mathrm{I}}(\boldsymbol{r}') + \sum_{p=0}^{1} \mathbf{k}_{1,p}^{\mathrm{D}}(r,r')\, \Delta_{1m,0}^{\mathrm{D}}(\boldsymbol{r}') \right) \mathrm{d}S' \tag{D.8}$$

We note that, in this case, there are no seismic Love numbers associated to the quadrupolar ($p = 2$) pattern. Also, the CMB matrix is the same as that for higher harmonic degree, $\ell \geq 2$. We note, however, that the solution of the momentum and Poisson equations within the core is already known. Indeed, it is straightforward to show that the function f_1 entering eq. (C.9), is proportional to the initial gravity of the Earth model

$$f_1(r) = \frac{3\, g_0(r)}{4\,\pi\, G\, \rho_0(0)} \tag{D.9}$$

APPENDIX E: TOROIDAL PERTURBATIONS

The case of toroidal perturbations can be dealt with similarly to the case of spheroidal perturbations. In fact, adapting the matrices and arrays used to dealt with the spheroidal case, it is possible to discuss toroidal perturbations with the same formalism and equations. Particularly, after spherical harmonic expansion, the toroidal component of the momentum equation can be cast in the form of a system of two linear differential equations

$$\frac{\mathrm{d}\mathbf{y}_{\ell m}(r)}{\mathrm{d}r} = \mathbf{A}_\ell(r)\,\mathbf{y}_{\ell m}(r) + \mathbf{q}_\ell(r) \tag{E.1}$$

where $\mathbf{A}_\ell$ is the 2×2 matrix defining the toroidal differential system, $\mathbf{q}_{\ell m}$ is the inhomogeneous term which accounts for the displacement dislocation, and $\mathbf{y}_{\ell m}$ is the toroidal vector solution which collects the toroidal components of the displacement, $W_{\ell m}$, and material incremental traction, $T_{\ell m}$,

$$\mathbf{A}_\ell = \begin{pmatrix} \dfrac{1}{r} & \dfrac{1}{\mu} \\ \dfrac{L^2-2}{r^2}\,\mu & -\dfrac{3}{r} \end{pmatrix} \tag{E.2a}$$

$$\mathbf{q}_{\ell m} = \begin{pmatrix} 0 & -C_{\ell m} \end{pmatrix}^{\mathrm{T}} \tag{E.2b}$$

$$\mathbf{y}_{\ell m} = \begin{pmatrix} W_{\ell m} & T_{\ell m} \end{pmatrix}^{\mathrm{T}} \tag{E.2c}$$

Furthermore, we introduce the projectors $\mathbf{P}_1$ and $\mathbf{P}_2$ for the second and first components of the toroidal vector solution, $\mathbf{P}_1\,\mathbf{y}_{\ell m} = T_{\ell m}$ and $\mathbf{P}_2\,\mathbf{y}_{\ell m} = W_{\ell m}$, respectively, and we define the CMB 2×1 matrix as

$$\mathbf{C}_\ell = \begin{pmatrix} 1 & 0 \end{pmatrix}^{\mathrm{T}} \tag{E.3}$$

which simply expresses the fact that the toroidal component of the displacement is not constrained at the CMB, while the toroidal components of the material incremental traction is zero.

With these definitions, following the same steps discussed for the spheroidal case, we obtain that the discontinuity of the toroidal vector solution due to an infinitesimal element of the fault system can be expressed as follows

$$\delta\mathbf{s}_{\ell m}(\boldsymbol{r}') = \sum_{p=1}^{2} \mathbf{s}^{\mathrm{D}}_{\ell p}(r')\,\Delta^{\mathrm{D}}_{\ell m,p}(\boldsymbol{r}') \tag{E.4}$$

Here, $\mathbf{s}^{\mathrm{D}}_{\ell p}$ $(p = 1, 2)$ are the following arrays

$$\mathbf{s}^{\mathrm{D}}_{\ell,1} = N_{\ell 1}\,\frac{2\,\ell+1}{4\,\pi\,r'^2}\,\frac{1}{L}\begin{pmatrix} -1 & 0 \end{pmatrix}^{\mathrm{T}} \tag{E.5}$$

$$\mathbf{s}^{\mathrm{D}}_{\ell,2} = N_{\ell 2}\,\frac{2\,\ell+1}{4\,\pi\,r'^3}\,\frac{\mu'}{L}\begin{pmatrix} 0 & 1 \end{pmatrix}^{\mathrm{T}} \tag{E.6}$$

and $\Delta^{\mathrm{D}}_{\ell m,p}$ $(p = 1, 2)$ are the following factors

$$\Delta^{\mathrm{D}}_{\ell m,1} = \sum_{j=1,2} \delta u_j\,\left(D^1_j\,F^{-1}_{\ell m} - D^{-1}_j\,F^1_{\ell m}\right) \tag{E.7a}$$

$$\Delta^{\mathrm{D}}_{\ell m,2} = \sum_{j=1,2} \delta u_j\,\left(D^2_j\,F^{-2}_{\ell m} - D^{-2}_j\,F^2_{\ell m}\right) \tag{E.7b}$$

Furthermore, the toroidal component of the displacement, $W_{\ell m}$, can be expressed in terms of the bipolar and quadrupolar seismic Love numbers, $w^{\mathrm{D}}_{\ell,1}$ and $w^{\mathrm{D}}_{\ell,2}$, as follows

$$W_{\ell m}(r) = a^{-2}\int_{\mathcal{S}} \sum_{p=1,2} w^{\mathrm{D}}_{\ell,p}(r,r')\,\Delta^{\mathrm{D}}_{\ell m,p}(\boldsymbol{r}')\,\mathrm{d}S' \tag{E.8}$$

with

$$w^{\mathrm{D}}_{\ell,p}(r,r') = a^2\,\mathbf{P}_2\,\mathbf{X}_\ell(r,r')\,\hat{\mathbf{s}}^{\mathrm{D}}_{\ell,p}(r') \qquad \forall p = 1,2 \tag{E.9}$$

We note that there is no contribution associated to the tensile component of the displacement dislocation and to the polar pattern of the seismic source response.