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Small horizons

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Abstract: All near-horizon geometries of supersymmetric black holes in a $N = 2, D = 5$ higher-derivative supergravity theory are classified. Depending on the choice of near-horizon data we find that either there are no regular horizons, or horizons exist and the spatial cross-sections of the event horizons are conformal to a squashed or round S^3, $S^1 \times S^2$, or T^3. If the conformal factor is constant then the solutions are maximally supersymmetric. If the conformal factor is not constant, we find that it satisfies a non-linear vortex equation, and the horizon may admit scalar hair.

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Contents

1 Introduction

In recent years it has become evident that there are exotic black hole solutions in higher dimensional gravitational theories. The most notable examples are the five-dimensional black rings [1–7]. These are solutions where the spatial cross-sections of the event horizon have $S^1 \times S^2$ topology. Moreover, the black hole uniqueness theorems, originally formulated in four dimensional general relativity [8–13], do not generalize straightforwardly to higher dimensions. However, uniqueness theorems have been formulated for static solutions in higher dimensions in [14–16], and for solutions with extra rotational Killing vectors, in [17–19]. One method to investigate the structure of extremal higher dimensional black objects with regular horizons is to study their near-horizon limit. In such a limit, information about the asymptotic behaviour of the black hole is removed and only information concerning the structure of the horizon is retained. If one considers supersymmetric black holes, then further conditions on the near-horizon geometry are obtained due to supersymmetry. Supersymmetric near-horizon geometries for the ungauged five-dimensional minimal supergravity were first considered in [20]. The case with vector multiplets was considered in [21]. Later in [22], the results of [20] were generalized to the minimal gauged supergravity with negative cosmological constant. In this case, one obtains weaker conditions and as such a complete classification of the near-horizon geometries was not possible. However, new solutions were found which were subsequently generalized in [23]. Also supersymmetric near-horizon geometries, with two commuting rotational Killing vectors, in the theory with a negative cosmological constant, were considered in [24]. The near-horizon analysis was also performed for ten-dimensional heterotic supergravity in [25]. We note that the near-horizon geometries of the so called five-dimensional de-Sitter supergravity theory coupled to vector multiplet was performed recently in [26].

In the present work we shall investigate the near-horizon geometries of supersymmetric extremal black hole solutions in higher derivative $N = 2$, $D = 5$ supergravity, coupled to a number of abelian vector multiplets [27]. The higher-derivative theory has, in addition to the spacetime metric, real scalars X^I, abelian 2-form field strengths F^I, and two auxiliary fields consisting of an auxiliary 2-form H and a real auxiliary scalar D. The solutions found are either the maximally supersymmetric near-horizon solutions found in [20, 28], or solutions for which the spatial cross-sections of the event horizon are conformal to a squashed or round S^3, $S^1 \times S^2$ or T^3. The function defining the conformal factor satisfies a nonlinear partial differential equation.

This work is organised as follows. In section two, necessary and sufficient conditions for the existence of a supersymmetric near-horizon geometry associated with the event horizon of a supersymmetric extremal black hole in our theory are examined. In sections three and four the local conditions satisfied by our geometries are obtained via the analysis of the gravitino, gaugino and auxiliary Killing spinor equations. In section five we perform the global analysis by demanding that the spatial cross-section of the event horizon $\mathcal{S}$ is compact without boundary. It is demonstrated that $\mathcal{S}$ must be conformal to one of these spaces: squashed S^3, round S^3, $S^1 \times S^2$ and T^3. In section six, we consider the auxiliary D-field equation. It turns out that if this equation is satisfied, together with the conditions obtained from the Killing spinor equations, then all the remaining equations of motion are satisfied. The D-equation of motion implies that either there are no solutions, or the solutions reduce to those found in [20, 21, 28], or the conformal factor satisfies a vortex-like nonlinear partial differential equation. In section 7, we introduce local co-ordinates and list all of the solutions. We conclude in section 8.

2 Supersymmetry and near-horizon geometries

We shall examine the necessary and sufficient conditions for there to be a supersymmetric near-horizon geometry associated with the event horizon of a supersymmetric extremal

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black hole in higher derivative ungauged $N = 2, D = 5$ supergravity coupled to an arbitrary number of abelian vector multiplets. After taking the near-horizon limit, the metric on the near-horizon geometry is [20, 29, 30]

$$ds^2 = 2du\left(dr + rh - \frac{1}{2}r^2\Delta du\right) + ds^2_{\mathcal{S}} \ . \tag{2.1}$$

Here $\frac{\partial}{\partial u}$ is a Killing vector; it is assumed that the event horizon is a Killing horizon of $\frac{\partial}{\partial u}$. This has been shown to hold for a large class of 2-derivative supergravity theories coupled to Maxwell fields and scalars [31], modulo certain technical assumptions, however it has not been proven for the higher derivative theory we consider here.

The horizon is located at $r = 0$, and $\mathcal{S}$ denotes the spatial cross-sections of the event horizon, which is taken to be compact and without boundary. The metric $ds^2_{\mathcal{S}}$ does not depend on u or r, Δ and h are a scalar and 1-form on $\mathcal{S}$ respectively, which also do not depend on u or r. We remark that the near-horizon limit corresponds to setting

$$r = \lambda r', \quad u = \lambda^{-1}u' \tag{2.2}$$

and then taking the limit $\lambda \to 0$ and dropping the primes.

We shall mostly use the conventions of [32], however we denote the scalars M^I as X^I, and rescale the auxiliary 2-form field v as $v = \frac{3}{4}H$ in order to simplify some coefficients. We also work in a mostly plus signature $(-,+,+,+,+)$. With these modified conventions, the gravitino, gaugino and auxiliary Killing spinor equations (KSEs) are

$$\nabla_\mu \epsilon - \frac{i}{8}\Gamma_\mu H_{\nu_1\nu_2}\Gamma^{\nu_1\nu_2}\epsilon + \frac{3i}{4}H_\mu{}^\nu\Gamma_\nu\epsilon = 0 \tag{2.3}$$

and

$$\left(\left(F^I + X^I H\right)_{\nu_1\nu_2}\Gamma^{\nu_1\nu_2} + 2i\Gamma^\nu\nabla_\nu X^I\right)\epsilon = 0 \tag{2.4}$$

and

$$\left(D - \frac{3}{2}H_{\nu_1\nu_2}H^{\nu_1\nu_2} - \frac{i}{2}dH_{\nu_1\nu_2\nu_3}\Gamma^{\nu_1\nu_2\nu_3} + \frac{3i}{2}\star\left(d\star H + H\wedge H\right)_\nu\Gamma^\nu\right)\epsilon = 0 \tag{2.5}$$

where ϵ is a Dirac Killing spinor whose structure will be investigated in greater detail later, and

$$\nabla_\mu\epsilon = \left(\partial_\mu + \frac{1}{4}\omega_{\mu,\nu_1\nu_2}\Gamma^{\nu_1\nu_2}\right)\epsilon \tag{2.6}$$

is the supercovariant derivative, where ω is the spin connection. It will be convenient to work with a light-cone basis $\{\mathbf{e}^+, \mathbf{e}^-, \mathbf{e}^i\}$ for $i = 1, 2, 3$ such that $\mathbf{e}^i$ is a (u, r-independent) basis for $\mathcal{S}$ and

$$\mathbf{e}^+ = du, \qquad \mathbf{e}^- = dr + rh - \frac{1}{2}r^2\Delta du \tag{2.7}$$

and

$$ds^2 = 2\mathbf{e}^+\mathbf{e}^- + \delta_{ij}\mathbf{e}^i\mathbf{e}^j \ . \tag{2.8}$$

The non-vanishing components of the spin connection associated with this basis are listed in appendix A. In addition to the metric (2.1) being regular in the near-horizon limit, we shall furthermore assume that all of the other bosonic fields are also regular in this limit, including the auxiliary fields H, D. In terms of the scalars, this means that after taking the near-horizon limit, X^I and D are smooth functions on $\mathcal{S}$ which are independent of u and r. Furthermore, the 2-forms H and F^I can be written as

$$H = \Phi\mathbf{e}^+ \wedge \mathbf{e}^- + r\mathbf{e}^+ \wedge \mathcal{B} + \tilde{H} \tag{2.9}$$

and

$$F^I = \Phi^I\mathbf{e}^+ \wedge \mathbf{e}^- + r\mathbf{e}^+ \wedge \mathcal{B}^I + \tilde{F}^I \tag{2.10}$$

where Φ, Φ^I are smooth u, r-independent scalars on $\mathcal{S}$; $\mathcal{B}, \mathcal{B}^I$ are smooth u, r-independent 1-forms on $\mathcal{S}$; and $\tilde{H}, \tilde{F}^I$ are smooth u, r-independent 2-forms on $\mathcal{S}$.

We shall also find it convenient to decompose spinors into positive and negative chirality parts

$$\epsilon = \epsilon_+ + \epsilon_-, \qquad \Gamma_\pm\epsilon_\pm = 0, \qquad \Gamma_\pm\epsilon_\pm = \pm\epsilon_\pm \tag{2.11}$$

and note the useful identities

$$\Gamma_{ij}\epsilon_\pm = \mp i\epsilon_{ij}{}^k\Gamma_k\epsilon_\pm, \quad \Gamma_{ijk}\epsilon_\pm = \mp i\epsilon_{ijk}\epsilon_\pm \tag{2.12}$$

where ϵ_{ijk} denotes the volume form of $\mathcal{S}$. Various spinorial geometry conventions are listed in appendix B

3 Analysis of gravitino KSE

To begin with, we analyse the gravitino KSE (2.3). As all of the dependence on the u, r components in the bosonic fields is known explicitly, we begin by solving the $+$ and the $-$ components. In particular, from the $-$ component one finds that

$$\epsilon_+ = \phi_+, \qquad \epsilon_- = \phi_- + r\Gamma_-\left(\frac{1}{4}(h + \star_3\tilde{H})_i\Gamma^i + \frac{i}{2}\Phi\right)\phi_+ \tag{3.1}$$

where $\phi_\pm$ do not depend on r, and $\star_3$ denotes the Hodge dual on $\mathcal{S}$. The $+$ component of the KSE implies that

$$\partial_u\epsilon_+ + \left(\frac{1}{2}r\Delta + \frac{i}{4}r(\star_3 dh)_i\Gamma^i + \frac{i}{4}r\mathcal{B}_i\Gamma^i\right)\epsilon_+ + \Gamma_+\left(-\frac{1}{4}(h - \star_3\tilde{H})_i\Gamma^i + \frac{i}{2}\Phi\right)\epsilon_- = 0 \tag{3.2}$$

and

$$\begin{aligned}\partial_u\epsilon_- + &\left(-\frac{1}{2}r\Delta - \frac{i}{4}r(\star_3 dh)_i\Gamma^i + \frac{3i}{4}r\mathcal{B}_i\Gamma^i\right)\epsilon_- \\ &+ r^2\Gamma_-\left(\frac{1}{4}(\Delta h_i - \partial_i\Delta)\Gamma^i + \frac{1}{2}\Delta\left(\frac{1}{4}(h + \star_3\tilde{H})_i\Gamma^i + \frac{i}{2}\Phi\right)\right)\epsilon_+ = 0 \ .\end{aligned} \tag{3.3}$$

Note that (3.2) and (3.3) imply that

$$\phi_- = \eta_- , \qquad \phi_+ = \eta_+ + u\Gamma_+ \left(\frac{1}{4}(h - \star_3 \tilde{H})_i \Gamma^i - \frac{i}{2}\Phi\right)\eta_- \tag{3.4}$$

where $\eta_\pm$ do not depend on u and r, and $\eta_\pm$ must also satisfy a number of algebraic conditions.

Before considering these algebraic conditions in further detail, it is useful to compute the 1-form spinor bilinear

$$Z_\mu = -\frac{1}{2}B(\epsilon, \Gamma_\mu \epsilon) \tag{3.5}$$

where B is the Spin(4, 1) invariant inner product defined in (B.6). It is known that this 1-form is dual to a Killing vector, which is a symmetry of the full solution [33]. We shall require that this 1-form spinor bilinear be proportional to

$$V = -\frac{1}{2}r^2 \Delta \mathbf{e}^+ + \mathbf{e}^- \tag{3.6}$$

where V is the 1-form dual to the Killing vector $\frac{\partial}{\partial u}$. We remark that this condition is not a priori necessary. In particular, it need not hold in the case for which the near-horizon geometry is supersymmetric, but the bulk black hole solution is not. Such solutions are known to exist in the 2-derivative theory [34, 35], and may also exist in the higher derivative theory as well. However, in this work we shall assume that both the bulk and the near-horizon geometry are supersymmetric, and therefore take Z to be proportional to V.

Recalling that at $r = u = 0$, $\epsilon = \eta_+ + \eta_-$ as a consequence of (3.1) and (3.4), it is straightforward to show that requiring that $Z_+ = 0$ at $r = u = 0$ implies that

$$\eta_- = 0 \; . \tag{3.7}$$

Furthermore, η_+ can be simplified further by making use of a r, u-independent SU(2) gauge transformation to write

$$\eta_+ = \alpha(1 - e_1) \tag{3.8}$$

for some r, u independent function α, $\alpha \in \mathbb{R}$. It follows that at $r = 0$, $\epsilon = \eta_+ = \alpha(1 - e_1)$. Also note that at $r = 0$,

$$Z_- = \sqrt{2}\alpha^2 \tag{3.9}$$

and so comparing with V_- we require that α^2 be constant. Without loss of generality set $\alpha = 1$, so

$$\eta_+ = 1 - e_1 \; . \tag{3.10}$$

Next, on imposing the conditions $Z_i = 0$, one finds that

$$\tilde{H} = - \star_3 h \; . \tag{3.11}$$

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The Killing spinor therefore simplifies further, and one finds that

$$\epsilon_+ = \eta_+, \qquad \epsilon_- = \frac{i}{2} r \Phi \Gamma_- \eta_+ \tag{3.12}$$

where η_+ is given by (3.10). Finally, we compute the ratio

$$\frac{Z_+}{Z_-} = -\frac{r^2}{2}\Phi^2 \, . \tag{3.13}$$

On requiring that this be equal to $\frac{V_+}{V_-}$ one finds that

$$\Delta = \Phi^2 \, . \tag{3.14}$$

On substituting the Killing spinor (3.12) back into (3.2) and (3.3) and making use of the conditions (3.11) and (3.14) one finds two additional conditions

$$\mathcal{B} = - \star_3 dh - 2\Phi h \tag{3.15}$$

and

$$d\Delta + 2\Delta h + 2\Phi \star_3 dh = 0 \, . \tag{3.16}$$

It remains to evaluate the components of (2.3) along the directions of $\mathcal{S}$, with the spinor ϵ given by (3.12). One finds the following conditions

$$\hat{\nabla}_i \eta_+ + \left(\frac{i}{4} \Phi \Gamma_i - \frac{i}{2} (\star_3 h)_i{}^j \Gamma_j \right) \eta_+ = 0 \tag{3.17}$$

where $\hat{\nabla}$ is the supercovariant derivative of $\mathcal{S}$, and

$$d\Phi + \Phi h + \star_3 dh = 0 \, . \tag{3.18}$$

Observe that (3.18) together with (3.14) imply (3.16). Furthermore, observe that the integrability condition of (3.17) implies that the Ricci tensor of $\mathcal{S}$ is

$$\hat{R}_{ij} = \left(\frac{1}{2}\Phi^2 + h^2 - \hat{\nabla}^n h_n \right) \delta_{ij} - \hat{\nabla}_{(i} h_{j)} - h_i h_j \, . \tag{3.19}$$

To summarize, the gravitino KSE implies that one can take the Killing spinor ϵ as in (3.12), with η_+ in (3.10), and in addition, (3.11), (3.14), (3.15), (3.17) and (3.18) are obtained, which in turn imply that the Ricci tensor of $\mathcal{S}$ is given by (3.19). This exhausts the content of (2.3).

4 Analysis of gaugino and auxiliary KSE

Next, we examine the gaugino KSE (2.4). On using the conditions obtained in the previous section, one finds that the positive chirality part of the gaugino KSE implies

$$\Phi^I = -\Phi X^I, \qquad \tilde{F}^I = X^I \star_3 h + \star_3 dX^I \tag{4.1}$$

and the negative chirality part of the gaugino KSE implies

$$\mathcal{B}^I = -X^I \mathcal{B} - \Phi dX^I \tag{4.2}$$

and hence

$$F^I = -du \wedge d\big(r\Phi X^I\big) + X^I \star_3 h + \star_3 dX^I \ . \tag{4.3}$$

Note that the Bianchi identity $dF^I = 0$ implies that

$$\hat{\nabla}^2 X^I + h^i \hat{\nabla}_i X^I + X^I \hat{\nabla}^i h_i = 0 \ . \tag{4.4}$$

Next, consider the auxiliary KSE (2.5). To evaluate the condition obtained from this equation, note that

$$H = du \wedge d(r\Phi) - \star_3 h \tag{4.5}$$

so

$$dH = -d \star_3 h \tag{4.6}$$

and also note that

$$\star \left(d \star H + H \wedge H\right) = -2r\big(\hat{\nabla}^i \hat{\nabla}_i \Phi + h^i \hat{\nabla}_i \Phi\big) \mathbf{e}^+ \ . \tag{4.7}$$

On substituting these conditions into (2.5), one finds that the auxiliary KSE is equivalent to

$$D = 3h^2 - 3\Phi^2 - 3\hat{\nabla}^i h_i \ . \tag{4.8}$$

5 Global analysis

Having extracted all of the local conditions from the KSE, we proceed to obtain additional conditions by making use of the fact that $\mathcal{S}$ is compact without boundary. In particular, the condition on the Ricci tensor (3.19) implies that $\mathcal{S}$ admits a Gauduchon-Tod structure [36, 37]. There exists a regular, positive function Ω, such that on making a conformal re-scaling and setting

$$ds^2_{\tilde{\mathcal{S}}} = \Omega^2 ds^2_{\mathcal{S}}, \qquad h' = h + \Omega^{-1} d\Omega \tag{5.1}$$

one can choose Ω such that

$$\tilde{\nabla}^i h'_i = 0 \tag{5.2}$$

where $\tilde{\nabla}$ denotes the Levi-Civita connection on $\tilde{\mathcal{S}}$ equipped with the conformally rescaled metric $ds^2_{\tilde{\mathcal{S}}}$. Note that (5.2) can be rewritten as

$$\hat{\nabla}^2 \Omega + h^i \hat{\nabla}_i \Omega + \Omega \hat{\nabla}^i h_i = 0 \ . \tag{5.3}$$

Furthermore, observe that (3.18) implies that

$$\hat{\nabla}^2\Phi + h^i\hat{\nabla}_i\Phi + \Phi\hat{\nabla}^i h_i = 0 \; . \tag{5.4}$$

Then (5.3) and (5.4) imply

$$\tilde{\nabla}^2(\Phi\Omega^{-1}) + (h')^i\tilde{\nabla}_i(\Phi\Omega^{-1}) = 0 \tag{5.5}$$

where in the above expression, the frame indices are taken w.r.t the conformally rescaled frame. Compactness of $\tilde{\mathcal{S}}$, then implies that

$$\Phi\Omega^{-1} = k \tag{5.6}$$

for constant k. So there are two cases to consider. If $k = 0$ then $\Phi = 0$. If $k \neq 0$ then without loss of generality one can set $\Phi = \Omega$. We shall consider these two cases separately.

5.1 Solutions with $\Phi \neq 0$

On setting the conformal factor $\Omega = \Phi$, one finds that the Ricci tensor of the rescaled metric is

$$\tilde{R}_{ij} = \left((h')^2 + \frac{1}{2}\right)\delta_{ij} - \tilde{\nabla}_{(i}h'_{j)} - h'_i h'_j \tag{5.7}$$

and moreover (3.18) can be rewritten as

$$\tilde{\star}_3 dh' = -h' \tag{5.8}$$

where $\tilde{\star}_3$ denotes the Hodge dual on $\tilde{\mathcal{S}}$. It is then straightforward to show that

$$\tilde{\nabla}^2(h')^2 + (h')^i\tilde{\nabla}_i(h')^2 = 2\tilde{\nabla}^{(i}(h')^{j)}\tilde{\nabla}_{(i}(h')_{j)} \; . \tag{5.9}$$

Then compactness of $\tilde{\mathcal{S}}$ implies that $(h')^2$ is constant, and moreover $\tilde{\nabla}_{(i}(h')_{j)} = 0$. So the Ricci tensor of $\tilde{\mathcal{S}}$ simplifies to

$$\tilde{R}_{ij} = \left((h')^2 + \frac{1}{2}\right)\delta_{ij} - h'_i h'_j \; . \tag{5.10}$$

It follows that if $h' \neq 0$, then $\tilde{\mathcal{S}}$ is a squashed S^3, whereas if $h' = 0$, $\tilde{\mathcal{S}}$ is a round S^3. Also, note that the Bianchi identity (4.4) can be rewritten as

$$\tilde{\nabla}^2\big(\Phi^{-1}X^I\big) + (h')^i\tilde{\nabla}_i\big(\Phi^{-1}X^I\big) = 0 \tag{5.11}$$

and hence, compactness of $\tilde{\mathcal{S}}$ implies that

$$X^I = \Phi Z^I \tag{5.12}$$

for constant Z^I.

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5.2 Solutions with $\Phi = 0$

In this case, the Ricci tensor of the conformally rescaled metric is

$$\tilde{R}_{ij} = (h')^2\delta_{ij} - \tilde{\nabla}_{(i}h'_{j)} - h'_i h'_j \tag{5.13}$$

and moreover (3.18) can be rewritten as

$$dh' = 0 \ . \tag{5.14}$$

Again, one finds that

$$\tilde{\nabla}^2(h')^2 + (h')^i\tilde{\nabla}_i(h')^2 = 2\tilde{\nabla}^{(i}(h')^{j)}\tilde{\nabla}_{(i}(h')_{j)} \tag{5.15}$$

so compactness of $\tilde{\mathcal{S}}$ implies that $(h')^2$ is constant, and moreover $\tilde{\nabla}_{(i}(h')_{j)} = 0$, and hence h' is covariantly constant $\tilde{\nabla}h' = 0$. So the Ricci tensor of $\tilde{\mathcal{S}}$ simplifies to

$$\tilde{R}_{ij} = (h')^2\delta_{ij} - h'_i h'_j \ . \tag{5.16}$$

It follows that if $h' \neq 0$, then $\tilde{\mathcal{S}}$ is $S^1 \times S^2$, whereas if $h' = 0$, $\tilde{\mathcal{S}}$ is T^3. Also, note that the Bianchi identity (4.4) can be rewritten as

$$\tilde{\nabla}^2\big(\Omega^{-1}X^I\big) + (h')^i\tilde{\nabla}_i\big(\Omega^{-1}X^I\big) = 0 \tag{5.17}$$

and hence, compactness of $\tilde{\mathcal{S}}$ implies that

$$X^I = \Omega Z^I \tag{5.18}$$

for constant Z^I.

6 Analysis of field equations

To proceed, we analyse the auxiliary D-field equation, which is

$$\frac{1}{6}C_{IJK}X^IX^JX^K - 1 = -\frac{1}{72}c_{2I}\left(\frac{3}{4}H_{\mu\nu}F^{I\mu\nu} + DX^I\right) \ . \tag{6.1}$$

Again, we treat the cases $\Phi \neq 0$ and $\Phi = 0$ separately. In all cases, it is possible to check directly, using a computer calculation, that the conditions obtained in the previous sections from the analysis of the Killing spinor equations, together with the D-field equation (6.1) are sufficient to imply that the Einstein, scalar, gauge, and auxiliary 2-form equations are satisfied.[1] It is therefore sufficient to consider the conditions imposed on the solution by (6.1).

[1]Due to the length of these field equations, we do not list them here; however they can be found in the appendix of [38].

6.1 Solutions with $\Phi \neq 0$

After some manipulation, one can rewrite (6.1) as

$$\Phi^3\left(\frac{1}{6}C_{IJK}Z^IZ^JZ^K+\frac{1}{48}c_{2I}Z^I\big((h')^2-1\big)\right)-1$$
$$=-\frac{1}{72}c_{2I}Z^I\left(-\frac{3}{2}\Phi^2(h')^i\tilde{\nabla}_i\Phi+3\Phi^2\tilde{\nabla}^2\Phi-3\Phi\tilde{\nabla}^i\Phi\tilde{\nabla}_i\Phi\right)\,. \tag{6.2}$$

Observe that if $c_{2I}Z^I=0$ then this expression implies that Φ is constant, and the conditions on the spacetime geometry are then equivalent to those found by [20] for the 2-derivative theory. This solution is the maximally supersymmetric near-horizon BMPV geometry [39].

Suppose instead that $c_{2I}Z^I \neq 0$. On setting $\Phi=e^{-\frac{V}{3}}$, (6.1) can be further simplified to

$$\tilde{\nabla}^2V-\frac{1}{2}(h')^i\tilde{\nabla}_iV=ae^V+b \tag{6.3}$$

where

$$a=-\frac{72}{c_{2I}Z^I},\qquad b=\frac{12}{c_{2I}Z^I}C_{MNP}Z^MZ^NZ^P+\frac{3}{2}((h')^2-1)\,. \tag{6.4}$$

This type of equation has been considered in appendix C. If $a>0, b\geq 0$, or $a<0, b\leq 0$ then it admits no solutions, and if $a>0, b<0$ then V is constant. If V is constant, then the solution is the maximally supersymmetric near-horizon BMPV geometry.

For the remaining case $a<0, b>0$, one also finds that (6.3) can be further simplified to

$$(h')^i\tilde{\nabla}_iV=0\qquad \tilde{\nabla}^2V=ae^V+b\,. \tag{6.5}$$

6.2 Solutions with $\Phi = 0$

After some manipulation, one can rewrite (6.1) as

$$\Omega^3\left(\frac{1}{6}C_{IJK}Z^IZ^JZ^K+\frac{1}{48}c_{2I}Z^I(h')^2\right)-1$$
$$=-\frac{1}{72}c_{2I}Z^I\left(-\frac{3}{2}\Omega^2(h')^i\tilde{\nabla}_i\Omega+3\Omega^2\tilde{\nabla}^2\Omega-3\Omega\tilde{\nabla}^i\Omega\tilde{\nabla}_i\Omega\right)\,. \tag{6.6}$$

Again, if $c_{2I}Z^I=0$ then this expression implies that Ω is constant, and the conditions on the spacetime geometry are then equivalent to those found by [20] for the 2-derivative theory. In particular, in this case, the solution is either $AdS_3\times S^2$ if $h\neq 0$, or $\mathbb{R}^{4,1}$ if $h=0$, and these solutions are maximally supersymmetric.

Suppose instead that $c_{2I}Z^I\neq 0$. On setting $\Omega=e^{-\frac{V}{3}}$, (6.1) can be further simplified to

$$\tilde{\nabla}^2V-\frac{1}{2}(h')^i\tilde{\nabla}_iV=ae^V+b \tag{6.7}$$

where

$$a=-\frac{72}{c_{2I}Z^I},\qquad b=\frac{12}{c_{2I}Z^I}C_{MNP}Z^MZ^NZ^P+\frac{3}{2}(h')^2\,. \tag{6.8}$$

From the results of appendix C, if $a > 0, b \geq 0$, or $a < 0, b \leq 0$ then (6.7) admits no solutions. If $a > 0, b < 0$ then V is constant, and the solution is $AdS_3 \times S^2$ if $h \neq 0$, and $\mathbb{R}^{4,1}$ if $h = 0$.

For the remaining case $a < 0, b > 0$, one also finds that (6.7) can be further simplified to

$$(h')^i \tilde{\nabla}_i V = 0 \qquad \tilde{\nabla}^2 V = ae^V + b \ . \tag{6.9}$$

7 Summary of solutions

In this section, we collate our results and summarise the near-horizon geometries. In addition, as we have obtained the Ricci tensor for $\tilde{\mathcal{S}}$ in (5.10) and (5.16), it is straightforward to introduce local co-ordinates on $\tilde{\mathcal{S}}$ in order to write the solutions explicitly. The details for this calculation can be found in [20].

We remark that we have proven that either $c_{2I}X^I$ vanishes identically, or is never zero. In the former case, the contribution from the higher derivative terms vanishes, and the solutions reduce to the maximally supersymmetric near-horizon geometries found in [20, 21]. Hence, for the remainder of this section we shall assume that $c_{2I}X^I \neq 0$.

7.1 Timelike solutions with event horizon topology S^3

If $\Delta \neq 0$ then the spatial cross sections of the event horizon are conformal to a squashed, or round, S^3, with metric

$$ds^2_{\mathcal{S}} = e^{\frac{2V}{3}} \left(\lambda \big((\sigma^1)^2 + (\sigma^2)^2 \big) + \lambda^2 \big(\sigma^3\big)^2 \right) \tag{7.1}$$

where $0 < \lambda \leq 1$ is constant,[2] V is a function on $\mathcal{S}$, and

$$\begin{aligned} \sigma^1 &= \sin\phi d\theta - \cos\phi \sin\theta d\psi \\ \sigma^2 &= \cos\phi d\theta + \sin\phi \sin\theta d\psi \\ \sigma^3 &= d\phi + \cos\theta d\psi \end{aligned} \tag{7.2}$$

are left-invariant 1-forms on SU(2) satisfying

$$d\sigma^i = -\frac{1}{2}\epsilon^{ijk}\sigma^j \wedge \sigma^k \ . \tag{7.3}$$

The metric on $\tilde{\mathcal{S}}$ is

$$ds^2_{\tilde{\mathcal{S}}} = \lambda \big((\sigma^1)^2 + (\sigma^2)^2 \big) + \lambda^2 \big(\sigma^3\big)^2 \tag{7.4}$$

with volume form

$$\tilde{\epsilon}^{(3)} = \lambda^2 \sigma^1 \wedge \sigma^2 \wedge \sigma^3 \ . \tag{7.5}$$

[2]Solutions with $\lambda > 1$ might be expected to correspond to the higher derivative generalisation of over-rotating BMPV black holes. However, as such solutions do not have regular horizons, these do not appear in our classification.

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It is also convenient to define a new radial co-ordinate ρ as

$$\rho = e^{\frac{V}{3}} r \ . \tag{7.6}$$

With these conventions, the five-dimensional near-horizon geometry is

$$ds^2 = 2e^{-\frac{V}{3}} du \Big(d\rho \pm \sqrt{\lambda - \lambda^2} \rho \sigma^3 - \frac{1}{2} \rho^2 e^{-V} du \Big) + e^{\frac{2V}{3}} \Big(\lambda \big((\sigma^1)^2 + (\sigma^2)^2 \big) + \lambda^2 (\sigma^3)^2 \Big) \tag{7.7}$$

and the scalars X^I and 2-form gauge field strengths F^I are given by

$$\begin{aligned} X^I &= e^{-\frac{V}{3}} Z^I \\ F^I &= Z^I \Big(d(e^{-V} \rho du) \pm \sqrt{\lambda - \lambda^2} \sigma^1 \wedge \sigma^2 \Big) \end{aligned} \tag{7.8}$$

for constants Z^I. The auxiliary 2-form H is

$$H = -d\Big(e^{-\frac{2V}{3}} \rho^2 du \Big) - e^{\frac{V}{3}} \tilde{\star}_3 \Big(\pm \sqrt{\lambda - \lambda^2} \sigma^3 + \frac{1}{3} dV \Big) \tag{7.9}$$

and the auxiliary scalar is

$$D = 3e^{-\frac{2V}{3}} \Big(\lambda^{-1} - 2 - \frac{1}{3} \tilde{\nabla}^2 V \Big) \ . \tag{7.10}$$

The function V satisfies

$$\tilde{\nabla}^2 V = a e^V + b \tag{7.11}$$

where $\tilde{\nabla}^2 = \tilde{\nabla}^i \tilde{\nabla}_i$ is the Laplacian on $\tilde{\mathcal{S}}$, and

$$a = -\frac{72}{c_{2I} Z^I}, \qquad b = \frac{12}{c_{2I} Z^I} C_{MNP} Z^M Z^N Z^P + \frac{3}{2}(\lambda^{-1} - 2) \tag{7.12}$$

are constants. If $\lambda \neq 1$, then as a consequence of the compactness arguments presented in appendix C, V is a function on S^2, i.e. is independent of ϕ, whereas if $\lambda = 1$ then V is a function on the (round) S^3.

If $a > 0, b \geq 0$, or $a < 0, b \leq 0$ then there are no regular horizons. If $a > 0, b < 0$ then V is constant and the solution is the maximally supersymmetric near-horizon (higher-derivative) BMPV geometry found in [28]. Observe that if $a < 0$, then by choosing a sufficiently small value of λ, one can obtain a positive value for b. The status of such solutions remains to be determined.

7.2 Null solutions

The null solutions, which have $\Delta = 0$, split into two sub-cases, according to whether $h' \neq 0$ or $h' = 0$ corresponding to event horizon cross-sections with topology $S^1 \times S^2$ and T^3 respectively.

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7.2.1 Null solutions with event horizon topology $S^1 \times S^2$

For these solutions, the spatial cross-sections of the horizon are conformal to $S^1 \times S^2$. One can introduce local co-ordinates on $\mathcal{S}$, $\{\phi, \theta, \psi\}$ such that

$$ds^2_{\mathcal{S}} = \lambda e^{\frac{2V}{3}} \Big(d\phi^2 + d\theta^2 + \sin^2\theta d\psi^2 \Big) \tag{7.13}$$

where λ is a positive constant, and V is a function on S^2 (i.e. $V = V(\theta, \psi)$). The metric on $\tilde{\mathcal{S}}$ is

$$ds^2_{\tilde{\mathcal{S}}} = \lambda \Big(d\phi^2 + d\theta^2 + \sin^2\theta d\psi^2 \Big) \tag{7.14}$$

with volume form

$$\tilde{\epsilon}^{(3)} = \lambda^{\frac{3}{2}} \sin\theta d\phi \wedge d\theta \wedge d\psi \ . \tag{7.15}$$

Again, it is convenient to define a new radial co-ordinate as

$$\rho = e^{\frac{V}{3}} r \ . \tag{7.16}$$

With these conventions, the five-dimensional near-horizon geometry is

$$ds^2 = 2e^{-\frac{V}{3}} du \big(d\rho + \rho d\phi\big) + \lambda e^{\frac{2V}{3}} \Big(d\phi^2 + d\theta^2 + \sin^2\theta d\psi^2 \Big) \tag{7.17}$$

and the scalars X^I and 2-form gauge field strengths F^I are given by

$$\begin{aligned} X^I &= e^{-\frac{V}{3}} Z^I \\ F^I &= \lambda^{\frac{1}{2}} Z^I \sin\theta d\theta \wedge d\psi \end{aligned} \tag{7.18}$$

for constants Z^I. The auxiliary 2-form H is

$$H = -e^{\frac{V}{3}} \tilde{\star}_3 \Big(d\phi + \frac{1}{3} dV \Big) \tag{7.19}$$

and the auxiliary scalar is

$$D = 3e^{-\frac{2V}{3}} \Big(\lambda^{-1} - \frac{1}{3} \tilde{\nabla}^2 V \Big) \ . \tag{7.20}$$

The function V satisfies

$$\tilde{\nabla}^2 V = a e^V + b \tag{7.21}$$

where $\tilde{\nabla}^2 = \tilde{\nabla}^i \tilde{\nabla}_i$ is the Laplacian on S^2 equipped with metric

$$ds^2(S^2) = \lambda \Big(d\theta^2 + \sin^2\theta d\psi^2 \Big) \tag{7.22}$$

and

$$a = -\frac{72}{c_{2I} Z^I}, \qquad b = \frac{12}{c_{2I} Z^I} C_{MNP} Z^M Z^N Z^P + \frac{3}{2}\lambda^{-1} \tag{7.23}$$

are constants.

If $a > 0, b \geq 0$, or $a < 0, b \leq 0$ then there are no regular horizons. If $a > 0, b < 0$ then V is constant and the solution is the maximally supersymmetric (higher-derivative) $AdS^3 \times S^2$ solution of [28]. Observe that if $a < 0$, then by choosing a sufficiently small value of λ, one can obtain a positive value for b. The status of such solutions remains to be determined.

7.2.2 Null solutions with event horizon topology T^3

For these solutions the spatial cross-sections of the event horizon are conformal to T^3. One can introduce local co-ordinates on T^3, $\{\phi, \theta, \psi\}$ such that

$$ds^2_{\mathcal{S}} = e^{\frac{2V}{3}} \left(d\phi^2 + d\theta^2 + d\psi^2 \right) \tag{7.24}$$

where V is a function on T^3. The metric on $\tilde{\mathcal{S}}$ is

$$ds^2_{\tilde{\mathcal{S}}} = d\phi^2 + d\theta^2 + d\psi^2 \tag{7.25}$$

with volume form

$$\tilde{\epsilon}^{(3)} = d\phi \wedge d\theta \wedge d\psi \ . \tag{7.26}$$

Again, it is convenient to define a new radial co-ordinate as

$$\rho = e^{\frac{V}{3}} r \ . \tag{7.27}$$

With these conventions, the five-dimensional near-horizon geometry is

$$ds^2 = 2e^{-\frac{V}{3}} du d\rho + e^{\frac{2V}{3}} \left(d\phi^2 + d\theta^2 + d\psi^2 \right) \tag{7.28}$$

and the scalars X^I and 2-form gauge field strengths F^I are given by

$$\begin{aligned} X^I &= e^{-\frac{V}{3}} Z^I \\ F^I &= 0 \end{aligned} \tag{7.29}$$

for constants Z^I. The auxiliary 2-form H is

$$H = \frac{1}{3} e^{\frac{V}{3}} \tilde{\star}_3 dV \tag{7.30}$$

and the auxiliary scalar is

$$D = -e^{-\frac{2V}{3}} \tilde{\nabla}^2 V \ . \tag{7.31}$$

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The function V satisfies

$$\tilde{\nabla}^2 V = ae^V + b \tag{7.32}$$

where $\tilde{\nabla}^2 = \tilde{\nabla}^i \tilde{\nabla}_i$ is the Laplacian on T^3, and

$$a = -\frac{72}{c_{2I} Z^I}, \qquad b = \frac{12}{c_{2I} Z^I} C_{MNP} Z^M Z^N Z^P \tag{7.33}$$

are constants.

If $a > 0, b \geq 0$, or $a < 0, b \leq 0$ then there are no regular horizons. If $a > 0, b < 0$ then V is constant and the solution is $\mathbb{R}^{1,4}$. The status of solutions with $a < 0, b > 0$ remains to be determined.

8 Conclusions

In this paper, we have classified all supersymmetric extremal near-horizon geometries of supersymmetric black hole solutions in the higher derivative $N = 2, D = 5$ supergravity constructed in [27]. We have proven that either $c_{2I} X^I$ vanishes identically on the horizon, or it never vanishes. In the former case, the near-horizon solutions are the maximally supersymmetric near-horizon solutions already known in the two-derivative theory [20, 21]. In the latter case, we have found all possible supersymmetric near-horizon solutions, and we have shown that the spatial cross-sections of the event horizon are conformal to either a squashed or round S^3, $S^1 \times S^2$, or T^3. In all cases, the conformal factor is determined in terms of a function V satisfying a non-linear PDE of the form

$$\tilde{\nabla}^2 V = ae^V + b \tag{8.1}$$

where a, b are constants determined in terms of the near-horizon data, as described in the previous section, with $a \neq 0$. The sign of a is identical to that of $c_{2I} X^I$. The function V is either defined on the (round) S^3, or S^2, or T^3, and $\tilde{\nabla}^2$ is the appropriate Laplacian in each case. Equation (8.1) has been examined in [40–43].

If $a > 0, b \geq 0$, or $a < 0, b \leq 0$ then there is no solution to the equation (8.1), and there are no regular horizons. If $a > 0, b < 0$ then V is constant and the solution reduces to one of the maximally supersymmetric solutions found in [28].

However, the most interesting case arises when $a < 0, b > 0$. In particular, for the solutions with event horizon topology S^3 or $S^1 \times S^2$, we have shown that provided $c_{2I} X^I$ is negative, one can always arrange for $b > 0$ by choosing another parameter (corresponding to the angular momentum associated with h') to be sufficiently small. It is clear that if $a < 0, b > 0$ then (8.1) admits a solution for which V is constant, and for which the geometry is again one of the maximally supersymmetric near-horizon solutions of [28]. It is however far from clear that this solution to (8.1) is unique, when a and b have this choice of sign. Notwithstanding this, we are also not aware of any explicit globally well-defined and regular non-constant solutions to (8.1) when $a < 0, b > 0$. If such solutions were to exist, then they would describe supersymmetric black holes with scalar hair on the horizon.

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They would also lie outside of the classification of solutions given in [44], because solutions with non-constant V would not have horizons with two commuting rotational isometries.

Another interesting issue is the possible extension of the analysis to construct a uniqueness theorem for supersymmetric black holes. In the two-derivative theory, it has been shown in [20, 21] that the only supersymmetric black holes with event horizon topology S^3 are the BMPV black holes. We remark that no corresponding uniqueness theorem exists for black rings with event horizon topology $S^1 \times S^2$. Although no analytic solution is currently known for black holes with event horizon topology S^3 in the higher derivative theory, it is reasonable to expect that such solutions exist. One can use supersymmetry to constrain the bulk geometries of these solutions.

The uniqueness proof for the 2-derivative theory considers the case for which the Killing vector $\frac{\partial}{\partial u}$ constructed from the Killing spinor is timelike, both in the near-horizon limit solution, and in the bulk geometry. The analysis proceeds by recalling that one can write the 5-dimensional solution as a U(1) fibration over a 4-dimensional hyper-Kähler base space HK

$$ds^2 = -f^2(du+\omega)^2 + f^{-1} ds^2_{HK} \tag{8.2}$$

where f is a u-independent function, ds^2_{HK} is a u-independent hyper-Kähler metric on the base space, and ω is a u-independent 1-form on HK. First, it was shown that for the near-horizon geometry of solutions with S^3 horizon topology, HK is $\mathbb{R}^4$. Given this, together with sufficient assumptions of regularity outside the event horizon, it was then shown that the base space for the bulk black hole solution must be $\mathbb{R}^4$. Following on from this, one can also write the gauge field strengths as

$$F^I = -d\big(f^2 X^I (du+\omega)\big) + \Theta^I \tag{8.3}$$

where Θ^I are harmonic 2-forms on HK, and for the near-horizon geometry one finds $\Theta^I = 0$. This can be extended into the bulk black hole solution, for which one must have $\Theta^I = 0$.

It is straightforward to show that exactly the same results hold for the higher derivative theory considered here. In particular, as a consequence of the classification of the timelike supersymmetric solutions in [32], it is known that the metric can be written as a U(1) fibration over a 4-dimensional hyper-Kähler base space HK as in (8.2). For the near-horizon solutions constructed here, one finds that

$$f = e^{-\frac{2V}{3}}\rho, \qquad \omega = -\rho^{-2} e^{V}\left(d\rho \pm \sqrt{\lambda-\lambda^2}\,\rho(d\phi + \cos\theta d\psi)\right) \tag{8.4}$$

and

$$\begin{aligned} ds^2_{HK} = \frac{1}{\rho}&\left(d\rho \pm \sqrt{\lambda-\lambda^2}\,\rho(d\phi+\cos\theta d\psi)\right)^2 \\ &+\rho\left(\lambda^2(d\phi+\cos\theta d\psi)^2 + \lambda(d\theta^2 + \sin^2\theta d\psi^2)\right) . \end{aligned} \tag{8.5}$$

This metric is flat, and so just as in the two derivative theory, one finds that the hyper-Kähler base space for the full black hole solution must be $\mathbb{R}^4$. Furthermore, one can also write the gauge field strengths for the timelike solutions as in (8.3), where Θ^I are harmonic self-dual 2-forms on HK. For the near-horizon geometry constructed here, one finds that $\Theta^I = 0$, and hence using exactly the same reasoning as in the analysis for the 2-derivative theory, one finds that $\Theta^I = 0$ for the full black hole solution.

The requirement that $HK = \mathbb{R}^4$ and $\Theta^I = 0$ imposes significant constraints on any possible black hole solutions. It remains to analyse the remaining field equations, which in spite of the simplification described above, remain somewhat non-trivial. The extent to which this analysis will produce a uniqueness theorem will depend on the existence (or non-existence) of non-constant solutions to (8.1) when $a < 0, b > 0$. Work on this is in progress.

A Spin connection

The non-vanishing components of the spin connection associated with the basis (2.7) are

$$\begin{aligned}
&\omega_{+,+-} = -r\Delta, \qquad \omega_{+,+i} = \frac{r^2}{2}\big(\Delta h_i - \partial_i \Delta\big), \qquad \omega_{+,-i} = -\frac{1}{2}h_i, \qquad \omega_{+,ij} = -\frac{r}{2}(dh)_{ij}\\
&\omega_{-,+i} = -\frac{1}{2}h_i\\
&\omega_{i,+-} = \frac{1}{2}h_i, \qquad \omega_{i,+j} = -\frac{r}{2}(dh)_{ij}, \qquad \omega_{i,jk} = \tilde{\omega}_{i,jk}
\end{aligned} \tag{A.1}$$

where $\tilde{\omega}_{i,jk}$ is the spin connection of $\mathcal{S}$.

B Spinorial geometry conventions

Spinorial geometry techniques were originally developed to analyse supersymmetric solutions of ten and eleven dimensional supergravity [45, 46]. Here we apply them to five-dimensional supergravity. The space of Dirac spinors consists of the space of complexified forms on $\mathbb{R}^2$, which has basis $\{1, e_1, e_2, e_{12} = e_1 \wedge e_2\}$. We define the action of the Clifford algebra generators on this space via

$$\gamma_i = -e_i \wedge -i_{e_i}, \qquad \gamma_{i+2} = i\big(-e_i \wedge +i_{e_i}\big) \quad i = 1, 2 \tag{B.1}$$

and set

$$\gamma_0 = i\gamma_{1234} \tag{B.2}$$

which acts as

$$\gamma_0 1 = i1, \quad \gamma_0 e_{12} = ie_{12}, \quad \gamma_0 e_i = -ie_i\ . \tag{B.3}$$

We then define generators adapted to the frame (2.8) as

$$\Gamma_\pm = \frac{1}{\sqrt{2}}(\gamma_3 \pm \gamma_0), \quad \Gamma_1 = \gamma_1, \quad \Gamma_2 = \sqrt{2}e_2\wedge, \qquad \Gamma_{\bar{2}} = \sqrt{2}i_{e_2} \tag{B.4}$$

where we take a basis $\{\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^{\bar{2}}\}$ for $\mathcal{S}$ such that $\mathbf{e}^{\bar{2}} = (\mathbf{e}^2)^*$ and

$$ds^2_{\mathcal{S}} = (\mathbf{e}^1)^2 + 2\mathbf{e}^2\mathbf{e}^{\bar{2}} \ . \tag{B.5}$$

With these conventions, the space of positive chirality spinors is spanned by $\{1-e_1, e_2+e_{12}\}$, and the space of negative chirality spinors is spanned by $\{1+e_1, e_2-e_{12}\}$ and we remark that Spin(3), with generators $i\Gamma_{2\bar{2}}, \Gamma_1(\Gamma_2+\Gamma_{\bar{2}}), i\Gamma_1(\Gamma_2-\Gamma_{\bar{2}})$ form a representation of SU(2) acting on $\{1-e_1, e_2+e_{12}\}$.

A Spin(4, 1) invariant inner product B on the space of spinors is then given by

$$B(\epsilon_1, \epsilon_2) = \langle \gamma_0 \epsilon_1, \epsilon_2 \rangle \tag{B.6}$$

where $\langle , \rangle$ denotes the canonical inner product on $\mathbb{C}^4$ equipped with basis $\{1, e_1, e_2, e_{12}\}$.

C A vortex equation

Suppose M is a compact manifold without boundary, κ is an isometry of M, and V is a smooth function on M satisfying

$$\nabla^2 V - \frac{1}{2}\kappa^i \nabla_i V = a e^V + b \tag{C.1}$$

for constants a, b, $a \neq 0$, where ∇ is the Levi-Civita connection, and $\nabla^2 V = \nabla_i \nabla^i V$.

First consider the cases for which $a > 0$ and $b \geq 0$, or $a < 0$ and $b \leq 0$. Note that

$$\int_M a e^V + b = \int_M \nabla^2 V - \frac{1}{2}\kappa^i \nabla_i V = 0 \tag{C.2}$$

on integrating by parts. However, as $ae^V + b$ is either everywhere positive or negative, this leads to a contradiction. Hence there are no solutions in these two cases.

Suppose instead that $a > 0$ and $b < 0$. Note that V attains a global minimum at $p \in M$. At p, $\kappa^i \nabla_i V = 0$, and $\nabla^2 V = \alpha \geq 0$. It follows that at p,

$$e^V = \frac{\alpha - b}{a} \tag{C.3}$$

and hence

$$e^V \geq \frac{\alpha - b}{a} \tag{C.4}$$

everywhere on M. In particular, one then finds $ae^V + b \geq 0$ everywhere on M. It then follows as a consequence of (C.2) that $ae^V + b = 0$, i.e. V is constant.

Next suppose that $a < 0$ and $b > 0$. We shall consider two cases which are of particular importance in the context of the black hole solutions, and in both cases we take M to be 3-dimensional.

In the first case, suppose that

$$R_{ij} = \left(\kappa^2 + \frac{1}{2}\right)\delta_{ij} - \kappa_i \kappa_j, \qquad \nabla_i \kappa_j = -\frac{1}{2}\epsilon_{ij}{}^k \kappa_k \ . \tag{C.5}$$

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Note that this expression for the Ricci tensor implies that

$$\nabla_i\big(\nabla^2 V\big) = -\Big(\kappa^2 + \frac{1}{2}\Big)\nabla_i V + \kappa_i \kappa^\ell \nabla_\ell V + \nabla^2 \nabla_i V \ . \tag{C.6}$$

It follows that

$$\begin{aligned}\int_M \kappa^i \nabla_i V \nabla^2 V &= -\int_M V\kappa^i \nabla_i(\nabla^2 V) \\ &= -\int_M V\kappa^i\Big(-\Big(\kappa^2+\frac{1}{2}\Big)\nabla_i V + \kappa_i\kappa^\ell\nabla_\ell V + \nabla^2\nabla_i V\Big) \\ &= -\int_M V\kappa^i \nabla^2 \nabla_i V \end{aligned} \tag{C.7}$$

where we have made use of the fact that (C.5) implies that κ^2 is constant. However, note also that

$$\begin{aligned}\int_M \kappa^i \nabla_i V \nabla^2 V &= \int_M V\nabla^2\big(\kappa^i\nabla_i V\big) \\ &= \int_M V\kappa^2\nabla^2\nabla_i V + 2V\nabla^\ell\kappa^i\nabla_\ell\nabla_i V + V\nabla^2\kappa^i\nabla_i V \\ &= \int_M V\kappa^i\nabla^2\nabla_i V \end{aligned} \tag{C.8}$$

where again (C.5) has been used to rewrite $\nabla^2\kappa_i = -\frac{1}{2}\kappa_i$. On comparing, (C.7) with (C.8) one finds

$$\int_M \kappa^i \nabla_i V \nabla^2 V = 0 \ . \tag{C.9}$$

Next, note that (C.1) implies

$$\int_M \kappa^i\nabla_i V\Big(\nabla^2 V - \frac{1}{2}\kappa^j\nabla_j V\Big) = \int_M \kappa^i\nabla_i V\big(ae^V + b\big) \tag{C.10}$$

and note that on partially integrating, the contribution from the r.h.s. vanishes, and also (C.9) implies that the contribution from the first term on the l.h.s. also vanishes. Hence

$$\int_M (\kappa^i\nabla_i V)^2 = 0 \tag{C.11}$$

so

$$\kappa^i\nabla_i V = 0 \ . \tag{C.12}$$

Hence, if $\kappa \neq 0$, then one finds that (C.1) simplifies further to

$$\mathcal{L}_\kappa V = 0, \qquad \Box V = ae^V + b \tag{C.13}$$

where $\Box$ denotes the Laplacian on S^2.

In the second case, suppose that

$$R_{ij} = \kappa^2\delta_{ij} - \kappa_i\kappa_j, \qquad \nabla\kappa = 0 \ . \tag{C.14}$$

Using essentially the same reasoning used to treat the previous case, one again finds that (C.1) can be simplified to (C.13).

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References

[1] H. Elvang, R. Emparan, D. Mateos and H.S. Reall, *A supersymmetric black ring*, *Phys. Rev. Lett.* **93** (2004) 211302 [hep-th/0407065] [INSPIRE].

[2] R. Emparan and H.S. Reall, *A rotating black ring solution in five-dimensions*, *Phys. Rev. Lett.* **88** (2002) 101101 [hep-th/0110260] [INSPIRE].

[3] I. Bena and N.P. Warner, *One ring to rule them all... and in the darkness bind them?*, *Adv. Theor. Math. Phys.* **9** (2005) 667 [hep-th/0408106] [INSPIRE].

[4] H. Elvang, R. Emparan, D. Mateos and H.S. Reall, *Supersymmetric black rings and three-charge supertubes*, *Phys. Rev.* **D 71** (2005) 024033 [hep-th/0408120] [INSPIRE].

[5] G.T. Horowitz and H.S. Reall, *How hairy can a black ring be?*, *Class. Quant. Grav.* **22** (2005) 1289 [hep-th/0411268] [INSPIRE].

[6] J.P. Gauntlett and J.B. Gutowski, *Concentric black rings*, *Phys. Rev.* **D 71** (2005) 025013 [hep-th/0408010] [INSPIRE].

[7] J.P. Gauntlett and J.B. Gutowski, *General concentric black rings*, *Phys. Rev.* **D 71** (2005) 045002 [hep-th/0408122] [INSPIRE].

[8] W. Israel, *Event horizons in static vacuum space-times*, *Phys. Rev.* **164** (1967) 1776 [INSPIRE].

[9] B. Carter, *Axisymmetric black hole has only two degrees of freedom*, *Phys. Rev. Lett.* **26** (1971) 331 [INSPIRE].

[10] S. Hawking, *Black holes in general relativity*, *Commun. Math. Phys.* **25** (1972) 152 [INSPIRE].

[11] D. Robinson, *Uniqueness of the Kerr black hole*, *Phys. Rev. Lett.* **34** (1975) 905 [INSPIRE].

[12] W. Israel, *Event horizons in static electrovac space-times*, *Commun. Math. Phys.* **8** (1968) 245 [INSPIRE].

[13] P. Mazur, *Proof of uniqueness of the Kerr-Newman black hole solution*, *J. Phys.* **A 15** (1982) 3173 [INSPIRE].

[14] G.W. Gibbons, D. Ida and T. Shiromizu, *Uniqueness and nonuniqueness of static black holes in higher dimensions*, *Phys. Rev. Lett.* **89** (2002) 041101 [hep-th/0206049] [INSPIRE].

[15] M. Rogatko, *Uniqueness theorem of static degenerate and nondegenerate charged black holes in higher dimensions*, *Phys. Rev.* **D 67** (2003) 084025 [hep-th/0302091] [INSPIRE].

[16] M. Rogatko, *Classification of static charged black holes in higher dimensions*, *Phys. Rev.* **D 73** (2006) 124027 [hep-th/0606116] [INSPIRE].

[17] P. Figueras and J. Lucietti, *On the uniqueness of extremal vacuum black holes*, *Class. Quant. Grav.* **27** (2010) 095001 [arXiv:0906.5565] [INSPIRE].

[18] S. Tomizawa, Y. Yasui and A. Ishibashi, *Uniqueness theorem for charged rotating black holes in five-dimensional minimal supergravity*, *Phys. Rev.* **D 79** (2009) 124023 [arXiv:0901.4724] [INSPIRE].

[19] S. Hollands and S. Yazadjiev, *A uniqueness theorem for* 5*-dimensional Einstein-Maxwell black holes*, *Class. Quant. Grav.* **25** (2008) 095010 [arXiv:0711.1722] [INSPIRE].

[20] H.S. Reall, *Higher dimensional black holes and supersymmetry*, *Phys. Rev.* **D 68** (2003) 024024 [*Erratum ibid.* **D 70** (2004) 089902] [hep-th/0211290] [INSPIRE].

[21] J.B. Gutowski, *Uniqueness of five-dimensional supersymmetric black holes*, *JHEP* **08** (2004) 049 [hep-th/0404079] [INSPIRE].

[22] J.B. Gutowski and H.S. Reall, *Supersymmetric AdS_5 black holes*, *JHEP* **02** (2004) 006 [hep-th/0401042] [INSPIRE].

[23] Z. Chong, M. Cvetič, H. Lü and C. Pope, *Five-dimensional gauged supergravity black holes with independent rotation parameters*, *Phys. Rev.* **D 72** (2005) 041901 [hep-th/0505112] [INSPIRE].

[24] H.K. Kunduri, J. Lucietti and H.S. Reall, *Do supersymmetric Anti-de Sitter black rings exist?*, *JHEP* **02** (2007) 026 [hep-th/0611351] [INSPIRE].

[25] J. Gutowski and G. Papadopoulos, *Heterotic black horizons*, *JHEP* **07** (2010) 011 [arXiv:0912.3472] [INSPIRE].

[26] J. Gutowski and W. Sabra, *Towards cosmological black rings*, *JHEP* **05** (2011) 020 [arXiv:1012.2120] [INSPIRE].

[27] K. Hanaki, K. Ohashi and Y. Tachikawa, *Supersymmetric completion of an R^2 term in five-dimensional supergravity*, *Prog. Theor. Phys.* **117** (2007) 533 [hep-th/0611329] [INSPIRE].

[28] B. de Wit and S. Katmadas, *Near-horizon analysis of $D = 5$ BPS black holes and rings*, *JHEP* **02** (2010) 056 [arXiv:0910.4907] [INSPIRE].

[29] J. Isenberg and V. Moncrief, *Symmetries of cosmological Cauchy horizons*, *Commun. Math. Phys.* **89** (1983) 387.

[30] H. Friedrich, I. Racz and R.M. Wald, *On the rigidity theorem for space-times with a stationary event horizon or a compact Cauchy horizon*, *Commun. Math. Phys.* **204** (1999) 691 [gr-qc/9811021] [INSPIRE].

[31] S. Hollands and A. Ishibashi, *On the 'stationary implies axisymmetric' theorem for extremal black holes in higher dimensions*, *Commun. Math. Phys.* **291** (2009) 403 [arXiv:0809.2659] [INSPIRE].

[32] A. Castro, J.L. Davis, P. Kraus and F. Larsen, *String theory effects on five-dimensional black hole physics*, *Int. J. Mod. Phys.* **A 23** (2008) 613 [arXiv:0801.1863] [INSPIRE].

[33] J.P. Gauntlett, J.B. Gutowski, C.M. Hull, S. Pakis and H.S. Reall, *All supersymmetric solutions of minimal supergravity in five- dimensions*, *Class. Quant. Grav.* **20** (2003) 4587 [hep-th/0209114] [INSPIRE].

[34] I. Bena, G. Dall'Agata, S. Giusto, C. Ruef and N.P. Warner, *Non-BPS black rings and black holes in Taub-NUT*, *JHEP* **06** (2009) 015 [arXiv:0902.4526] [INSPIRE].

[35] K. Goldstein and S. Katmadas, *Almost BPS black holes*, *JHEP* **05** (2009) 058 [arXiv:0812.4183] [INSPIRE].

[36] K.P. Tod, *Compact* 3*-dimensional Einstein-Weyl structures*, *J. London Math. Soc.* **45** (1992) 341.

[37] P. Gauduchon, *La* 1*-forme de torsion d'une variété hermitienne compacte*, *Math. Ann.* **267** (1984) 495.

[38] F. Bonetti, J.B. Gutowski, D. Klemm, W.A. Sabra and P. Sloane, *All supersymmetric solutions of $N = 2$, $D = 5$ supergravity with R^2 corrections*, in preparation.

[39] J. Breckenridge, R.C. Myers, A. Peet and C. Vafa, *D-branes and spinning black holes*, *Phys. Lett.* **B 391** (1997) 93 [hep-th/9602065] [INSPIRE].

[40] C.H. Taubes, *Arbitrary N: vortex solutions to the first order Landau-Ginzburg equations*, *Commun. Math. Phys.* **72** (1980) 277 [INSPIRE].

[41] J.L. Kazdan and F.W. Warner, *Curvature functions for compact* 2*-manifolds*, *Ann. Math. Second Ser.* **99** (1974) 14.

[42] S. Bradlow, *Vortices in holomorphic line bundles over closed Kähler manifolds*, *Commun. Math. Phys.* **135** (1990) 1 [INSPIRE].

[43] N.S. Manton and P. Sutcliffe, *Topological solitons*, Cambridge University Press, Cambridge U.K. (2004).

[44] H.K. Kunduri, J. Lucietti and H.S. Reall, *Near-horizon symmetries of extremal black holes*, *Class. Quant. Grav.* **24** (2007) 4169 [arXiv:0705.4214] [INSPIRE].

[45] U. Gran, J. Gutowski and G. Papadopoulos, *The spinorial geometry of supersymmetric IIB backgrounds*, *Class. Quant. Grav.* **22** (2005) 2453 [hep-th/0501177] [INSPIRE].

[46] J. Gillard, U. Gran and G. Papadopoulos, *The spinorial geometry of supersymmetric backgrounds*, *Class. Quant. Grav.* **22** (2005) 1033 [hep-th/0410155] [INSPIRE].

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