

Generalizing Messiaen's Modes of Limited Transposition to a n -tone Equal Temperament

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ABSTRACT

Modes of limited transposition are musical modes originally conceived by the French composer Olivier Messiaen for a tempered system of 12 pitches per octave. They are defined on the base of symmetry-related criteria used to split an octave into a number of recurrent interval groups. This paper describes an algorithm to automatically compute the modes of limited transposition in a generic n -tone equal temperament. After providing a pseudo-code description of the process, a Web implementation will be proposed.

1. INTRODUCTION

Olivier Messiaen is considered one of the most important composers of the 20th Century. His production includes not only music pieces, but also theoretical works about his musical language. Concerning the latter aspect, the interest in ancient Greek music and exotic modes was already clear in his early compositions. For instance, while a student he experimented with his theories about new music modes in his first published works, *Eight Preludes for piano*, and throughout his life Messiaen continued to develop and evolve new composition techniques, always integrating them into his musical style. In [1] Messiaen's music has been described as outside the Western musical tradition, although growing out of that tradition and being influenced by it.

With respect to music scales, the most relevant innovation introduced by Messiaen is probably the definition and adoption of *modes of limited transposition* (in French: *modes à transpositions limitées*). In order to create new music resources for harmony, he determined all the ways to split an octave into recurrent groups of notes, where each group was internally formed by the same intervals and groups overlapped as regards their boundaries, namely the highest pitch of a group was the lowest of the following one. The way to compute note groups for each mode will be described in detail in next sections, and some clarifying

examples will be provided too. For a formal description of Messiaen's mode, please refer to [2].

In his theoretical works and music pieces, Messiaen was always referring to the equal division of an octave into 12 steps (12-EDO), commonly in use in Western music. Our goal is applying Messiaen's theories to a generalized n -tone equal temperament, where the original modes of limited transposition represent a special case. In order to achieve this goal, first we need to define some concepts, since Western music theory and score representation cannot be applied to this generalization in a straightforward way.

2. GENERALIZATION TO N -TONE EQUAL TEMPERAMENT

Even if Olivier Messiaen sought to overcome the limitations imposed by Western music system, his works had their roots in that musical culture and tradition. As regards melodic and harmonic aspects, Western music is largely based on 12-EDO. In this context, an octave is composed of 12 steps, where every pair of adjacent pitches has an identical frequency ratio, equal to $\sqrt[12]{2}$. In this way, 12-EDO divides the octave into 12 parts, known as *semitones* or *half tones*, which are the smallest musical interval commonly used in Western tonal music. Adopting equal temperament implies that all semitones are equal on a logarithmic scale. Since pitch is (roughly) perceived as the logarithm of frequency, the distance from every step to its nearest neighbor is the same for every step in the system.

In general terms, equal temperament is not the only possibility. Well-known even in ancient times¹ and far cultures,² this kind of temperament was extensively used in the European tradition only from the 16th Century, whereas Pythagorean tuning, Ptolemaic sequence and Zarlirian modality had been mainly adopted in earlier music [3]; 5-, 7- and 9-EDO are fairly common in ethnomusicology. For example, [4] discusses the issue of temperament in Thai music, whereas [5] analyses Javanese gamelan.

¹ One of the earliest descriptions of equal temperament is contained in the writing entitled *Elements of Harmony* by Aristoxenus, dating back to the 4th Century BC.

² For instance, in China an approximation for equal temperament was described by He Chengtian around 400 AD, whereas the *Complete Compendium of Music and Pitch* published by Zhu Zaiyu in 1584 contains a detailed discussion of this pitch theory with a precise numerical specification.

Another possible generalization is the application of equal temperament to non-octave intervals, thus passing from the concept of equal division of an octave into n subparts (n -EDO) to the n -tone equal temperament (n -TET). For instance, the equal-tempered version of the Bohlen-Pierce scale, described in [6] and [7], is based on the ratio 3:1. Such an interval, corresponding to a perfect fifth plus an octave in 12-EDO, is split into 13 equal parts. Consequently, every pair of adjacent pitches presents a frequency ratio equal to $\sqrt[13]{3}$.

A great number of equal divisions either of the octave or of other intervals have found use in microtonal music, ethnic cultures, theoretical experiences, etc. An in-depth discussion of tuning and temperament clearly goes beyond the goals of this work. For further details, please refer to [8] or [9].

Now we want to provide an extension of twelve-tone system, thus defining a generic n -TET where a given interval can be divided into n equally-spaced pitches. In order to avoid ambiguity with in-use terminology, we will define any available pitch of the equal temperament as a step. Each couple of adjacent steps presents a frequency ratio equal to $\sqrt[n]{r}$, where r is the frequency ratio of the interval to be subdivided and n is the number of equal steps. Since Messiaen defines his modes by splitting the octave, in the following we will focus on that interval, nonetheless our approach can be easily extended to any other interval.

Please note that reasoning in terms of steps instead of fixed frequencies allows an abstract description of the process. The modes defined in this way will be potentially instanced starting from any frequency, either available in the standard tuning system [10] or not.

3. INTRODUCTION TO MODES OF LIMITED TRANSPOSITION

According to many musicologists and experts, Messiaen's modes of limited transposition are the most relevant resource he used to create melody and harmony in music. The original idea was determining all the possible ways to split the tempered twelve-tone octave in a number of recurrent and non-overlapping note groups. Each group has to present the same internal pattern, made of a variable number of variable-size intervals. The smallest interval to build structures is the tempered semitone, but semitones can be aggregated to build bigger intervals.

In his theoretical works, Messiaen defines as *modes* the recurring note groupings which are limited in the amount of times they can be transposed, due to patterns within their structures. Based on a tempered system of 12 pitches, these modes are formed by several symmetrical groups, the last note of each group always being common with the first of the following group. At the end of a certain number of chromatic transpositions that varies with each mode, they are no longer transposable, giving exactly the same notes as the first [2].

In mathematics, this problem recalls the concept of *composition of an integer*. A composition of an integer n is a way of writing n as the sum of a sequence of (strictly) positive integers. Two sequences that differ in the order

of their terms define different compositions of their sum, while they are considered to define the same partition of that number. Each positive integer n has 2^{n-1} distinct compositions.

In mathematical terms, the process adopted by Messiaen allows to find all the compositions of 12 where 12 is the number of semitones in an octave that match an additional criterion, namely those presenting a pattern made of k repetitions, with $k > 1$. For example, both

$$12 = 1 + 3 + 1 + 3 + 1 + 3$$

and

$$12 = 3 + 1 + 3 + 1 + 3 + 1$$

satisfy this condition, since it is possible to recognize one group that is repeated 3 times: in the former case, the recurrent group is [1, 3], in the latter [3, 1]. On the contrary,

$$12 = 1 + 3 + 3 + 1 + 1 + 3,$$

which in mathematical terms would be another composition of 12, does not match the condition, since we cannot determine $k > 1$ repetitions of the same pattern inside the composition.

A relevant property of this redefinition of the concept of composition is cycle invariance. In other terms, if we consider the interval sequence in a circular way, left- and right-shifting do not introduce new models. Invariance can be applied to the complete sequence as well as to groups.

4. REPRESENTATION ISSUES

A problem to face is the textual and score representation of steps in a generic n -TET. In fact, both pitch names and their corresponding staff position have been originally conceived for a diatonic scale, namely a musical scale composed of seven pitches. The granularity of semitones can be textually and graphically rendered through the use of accidentals, a practice that however introduces ambiguity in the spelling of enharmonic equivalents. Going deeply into microtonal music language, commonly-accepted representations are available only for specific subdivisions, such as quarter tones in 12-EDO.

Messiaen had to manipulate semitones and their possible aggregations in the context of a 12-step chromatic scale. In this context, a staff view could be provided - and actually was provided by the author - but it would require spelling notes and solving enharmonic ambiguities, which is not strictly necessary.

In the generalized case we are addressing, the problem is assigning a name and providing an effective graphical representation to each of the n subdivisions of an arbitrary interval. This issue will be discussed in the following subsections.

4.1 Pitch Naming

A practical solution to naming problems is the adoption of pitch classes as defined in [11]. Pitch classes are an abstraction of pitches divorced from register, notation and compositional realization [12], and they can be effectively



Figure 1. An example of assignment of integers to pitch classes. Any enharmonically-equivalent note spelling would be represented by the same integer value.

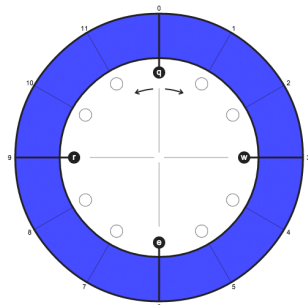


Figure 2. A ring diagram that provides a cyclic representation of pitch classes in 4-TET. The numbering of sectors is arbitrary as regards both the origin and the direction.

notated by assigning $pc = 0$ to a given step and consecutive integers to consecutive steps. Figure 1 shows one of the 12 possible correspondences among semitones in the 12-EDO and integer values, specifically the one where $pc = 0$ is assigned to C. In the definition of pitch classes, the octave is not relevant and the sequence of symbols can be read in a circular way, or in other words the system is modulo 12. This cyclic approach to pitch representation is coherent with Messiaen's one.

Please note that the adoption of pitch classes intrinsically solves the problem of score-spelling ambiguities: two spellings - like $E\sharp$ and F - that in an equal-tempered system produce the same sound, namely correspond to the same frequency, collapse into the same pitch class (e.g. $pc = 5$).

4.2 Ring Diagrams

As regards a graphical rendering suitable for our theoretical goals, we decided to represent tempered steps through a periodic tiling (or *tessellation*) of an annulus. Figure 2 provides an example where a generic interval has been divided into 4 equal steps. This diagram does not contain references to in-use note names: conventionally, each adjacent sector can be identified through consecutive numbers, and the origin can be set to any sector. Ring diagrams can be read (and their sectors can be numbered) both clockwise and counter clockwise.

A color code has been added to each sector, in order to visually mark both groups and intervals. Color combinations mainly have two purposes:

- Making visible the sequence of groups, all characterized by the same internal layout. Groups constitute the tessellation of the complete interval to be subdivided;
- Highlighting the internal composition of each group,

in terms of intervals, i.e. step aggregations. The same sequence of intervals can be found in any group.

This chromatic approach implies that each group is made of one (consecutive) block per color, and when the color sequence restarts a new instance of the group occurs, as shown in Figure 3.

Please note that, inside a group, also intervals with the same size have different colors. For example, let us consider one of the possible subdivisions of the global interval, say a specific subdivision of an octave into 12 steps which originates three 4-step groups [1, 2, 1]. In our representation, the two single-step intervals inside the group have different colors, whereas the color layout composed in this case by 3 different colors is repeated group by group. The letter sequence of a typical QWERTY layout has been adopted in order to identify blocks and to play the corresponding pitch, as explained in Section 8.

One of the advantages of ring diagrams is providing a cyclic representation of groups, in accordance with the concept of pitch class and Messiaen's theory about modes. Besides, this approach offers the possibility to read diagrams either clockwise or counter clockwise, provided that the same criterion is used for all diagrams. Finally, such a graphical representation allows the reader to choose the boundary of any colored block as the starting point, which intrinsically solves the issue of equivalent group spellings. For instance, a group made of 4 steps can be spelled as [3, 1] or equivalently as [1, 3], since the latter case simply implies building the group from the second pitch, as shown by the ring diagrams in Figure 4.

For the sake of clarity, in our diagrams the minimum

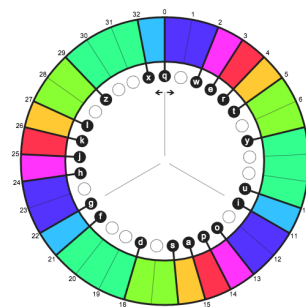


Figure 3. A ring diagram that highlights the [2,1,1,1,2,3,1] spelling of an 11-transposition mode in 33-TET.

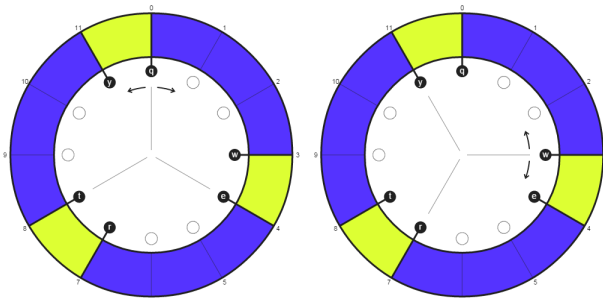


Figure 4. Ring diagrams for two equivalent spellings: [3,1] and [1,3].

interval (namely the step) is always surrounded by radial lines that delimit its extension. These lines are drawn with strong strokes where a step aggregation starts or ends, whereas boundaries inside an aggregation are thin. However, colors and thin lines simply provide graphic hints to the user: the diagram’s semantics resides in the specific tessellation of the annulus, which is different from a mode to another. As mentioned above, different spellings of the same mode can be obtained through suitable rotations of the diagram.

5. AN ALGORITHM TO CALCULATE GENERALIZED MODES OF LIMITED TRANSPOSITION

From a historical point of view, only some temperaments have been considered, due to their application to specific context (e.g. in ethnomusicology) or to theoretical reasons (for instance, the adherence of a given interval in a temperament to its theoretical value in terms of frequency ratio). Our goal is investigating the generalized equal temperaments by following an automatic approach.

In this section we will describe an algorithm to compute all the possible modes emerging from a given subdivision of an arbitrary interval. In order to perform calculations, the only required input is the number of steps we want to consider, i.e. the minimum granularity to build aggregations. If we need an audio rendering too, two more inputs are necessary, namely the frequencies of the pitches that delimit the global interval to be divided.

The algorithm can be decomposed into 3 steps:

1. *Calculating all the integer divisors of the global interval.* The key requirement by Messiaen is covering such an interval through a number $k > 1$ of occurrences of the same pattern. This implies that each group is made of an equal (integer) number of steps, say s . Consequently, the size n of the global interval is split into smaller groupings according to the following equation: $n = k \cdot s$. Since $k > 1$, n is not considered a divisor of itself, which adheres to Messiaen’s theories. Provided that n has been set, the purpose of this step is finding all suitable values for k , and consequently for s . Please note that s is also the number of transpositions for a given mode, since after s 1-step transpositions pitches are repeated. For example, a chromatic scale in 12-EDO ($n = 12$, $k = 12$, $s = 1$) is a mode presenting only one transposition, whereas the whole-tone scale ($n = 12$, $k = 6$, $s = 2$) has two transpositions, misaligned by one semitone;
2. For each grouping size s , *finding all the compositions of s* , i.e. any way to write s as a sorted sum of positive integers. If the algorithm goes from smaller to bigger values, covering not only the mentioned divisors but any integer in the range $[1 \dots (s-1)]$, each iteration can benefit from already available compositions. For instance, the fifth iteration aims at finding the compositions of 5. One of them is $5 = 4 + 1$, but all the compositions of 4 have been already computed during the fourth iteration and can be reused

Key	Value
s = 1	[1]
s = 2	[2] [1,1]
s = 3	[3] [2,1] [1,2] [1,1,1]
s = 4	[4] [3,1] [2,1,1] [1,3] [1,2,1] [1,1,2] [1,1,1,1]
s = 6	[6] [5,1] [4,2] [4,1,1] [3,2,1] [3,1,1,1] [2,3,1] [2,2,1,1] [2,1,1,1,1] /

Table 1. The results for the computation of all available modes in 12-EDO. Slashed values are the ones removed by pruning. Intentionally the last row does not contain unwanted values since their number would be too high.

here. A well-known programming technique to implement this behavior is recursion;

3. *Pruning*, i.e. removing unwanted values from data structures. The algorithm does not produce wrong results, nonetheless some values need to be purged. First, as some compositions directly come from aggregations of more atomic ones, in this context they are redundant. For instance, [2, 2] is a spelling of 4, but the mode built through the repetition of [2, 2] is indistinguishable from the mode made of single [2], already available inside the data structures. Besides, we have to manage the mentioned equivalent spellings, like [1, 2, 3], [2, 3, 1] and [3, 1, 2], corresponding to different ways to read the same interval pattern.

The data structure used to contain final results is a dictionary whose elements are dynamic arrays of dynamic arrays. The dictionary is made of couples $\langle K, V \rangle$, i.e. key-value associations. In this case, the keys are the collection of divisors s identified during Step 1. Each key is associated to a number of corresponding compositions, cleaned from unwanted duplicates and redundant spellings at Step 3. The adoption of nested dynamic data structures for values comes from the fact that each s_i potentially presents a different number of compositions $c_{i,j}$, and compositions themselves are made of a variable number of addends.

Such an algorithm can be implemented in different programming languages. An implementation based on HTML5 and JavaScript will be described in Section 8 and has been made publicly available.

6. COMPUTATION AND REPRESENTATION OF MESSIAEN’S MODES

In this section, we will apply the proposed algorithm to the well-known domain discussed in Messiaen’s works, namely modes of limited transposition in a 12-EDO. Table 1 illustrates the results obtained through the algorithm, which perfectly fit those described by Messiaen in his theoretical works. The corresponding ring diagrams are shown in Figure 5.

From the contents of the data structure it is possible to reconstruct the complete subdivision of the global interval. For any value of s , it is necessary to replicate the interval

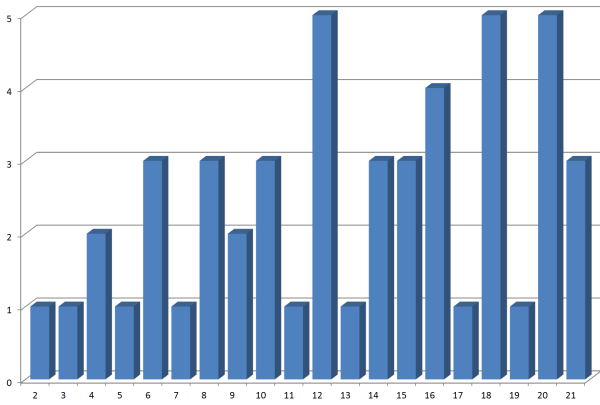


Figure 6. Divisors available in n -TET for $n \in [2..21]$.

pattern k times, where $k = n/s = 12/s$. In the following we will adopt the pitch class naming convention.

7. COMPUTATION AND REPRESENTATION OF GENERALIZED MODES

The approach used to validate the algorithm in 12-EDO can be easily extended to any other temperament and interval.

First, it is possible to represent (and listen to) specific temperaments that are relevant in musicology, ethnomusicology and microtonal composition, such as 5-, 7-, 29-, 31-, 41- and 53-EDO. In all these cases, we are choosing a *prime*³ as the number of steps to divide the global interval. Consequently, in order to obtain a tessellation through repetitive patterns, only 1-step groupings are allowed. Other temperaments, e.g. 24-TET, support multiple modes and mode spellings, since they have many divisors of the original step number and divisors are great enough to allow many compositions.

The main advantage of an algorithmic approach is automatically obtaining modes of limited transpositions even in cases where their definition is difficult to obtain by hand. As shown in Figure 6, the number of divisors available in n -TET for $n \in [2..21]$ belongs to the range [1..5]; but Figure 7 demonstrates that the number of corresponding spellings rapidly grows.

8. WEB PROTOTYPE

The algorithm described above has been implemented in HTML5 and JavaScript, and a Web prototype has been released. The application is available at <http://www.lim.di.unimi.it/messiaen>.

The main goal of this prototype is showing generalized modes both from a graphical and from an audio point of view. Colored ring diagrams are produced on-the-fly depending by user's inputs, and both groupings and internal step aggregations are highlighted through the already mentioned graphical devices: colors selected from a chromatic space and different line strokes. In order to produce a ring

³ A *prime* is a natural number greater than 1 that has no positive divisors other than 1 and itself.

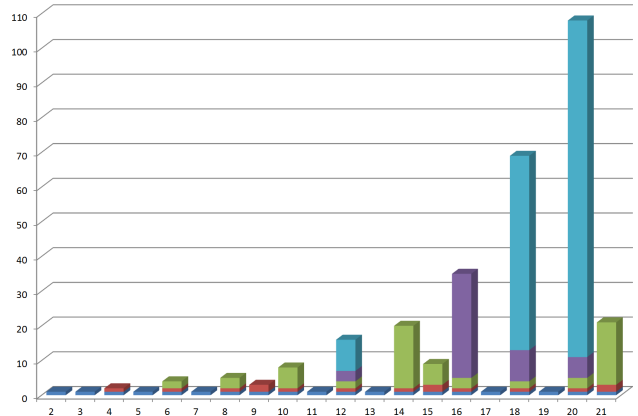


Figure 7. Spellings of the divisors in n -TET for $n \in [2..21]$.

diagram, it is sufficient to select one of the step number values (for computational reasons $s \in [2 \dots 52]$), then one of the available divisors for that number, and finally one of the proposed groupings. Arrow-shaped controls are provided to show different spellings of each group, which virtually correspond to suitable rotations of the diagram.

The audio rendering of generalized modes requires some additional inputs. Specifically, two controls let the user set the start and end frequencies of the global interval to be split.⁴ Default values are 220 Hz and 440 Hz, corresponding to a 2:1 ratio, namely to the octave interval. Changing this preset allows to subdivide any interval, thus implementing a first degree of generalization with respect to Messiaen's theories. The resulting frequencies for any pitch in the mode are shown to the side of the ring diagram. A sort of circular keyboard has been implemented: little circles can be mouse-clicked to produce the corresponding frequency, and they have been associated also to the keystrokes listed inside the black circles. Play, stop and BPM controls let the user listen to the selected mode as a perpetual scale.

A screenshot of the interface at the moment of writing is shown in Figure 8.

9. CONCLUSIONS

In this paper we presented a generalized approach to the theoretical work on modes by Olivier Messiaen. An algorithm has been designed and implemented in order to compute all possible groupings and interval patterns coming from a subdivision of a given interval into a given number of steps. Under this perspective, Olivier Messiaen's modes of limited transposition are one of the possible instances, as well as Nicholas Mercator researches on 53-EDO and other extensions of Pythagorean tuning. This work may present multiple implications, ranging from music performance to microtonal music theory, from tuning practice to composition.

As regards future work, the prototype will be improved

⁴ Please note that timbre is also relevant to the dissonance levels for intervals within different scales [13]. As a consequence, an additional control to choose among different timbres would be desirable.

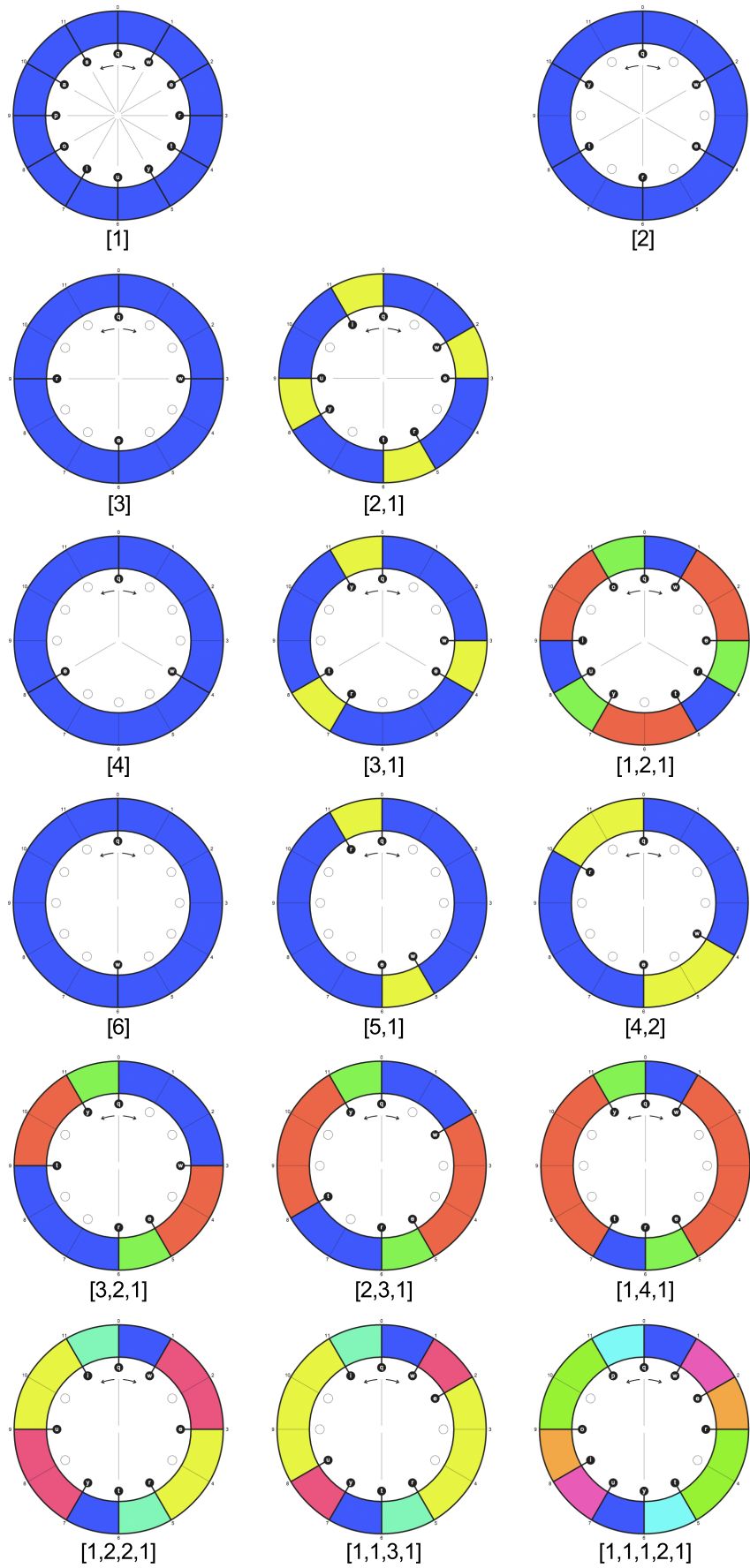


Figure 5. Ring diagrams for Messiaen's modes: *Row i. (left)* – Modes with 1 transposition; *Row i. (right)* – Modes with 2 transpositions; *Row ii.* – Modes with 3 transpositions; *Row iii.* – Modes with 4 transpositions; *Rows iv-vi.* – Modes with 6 transpositions.

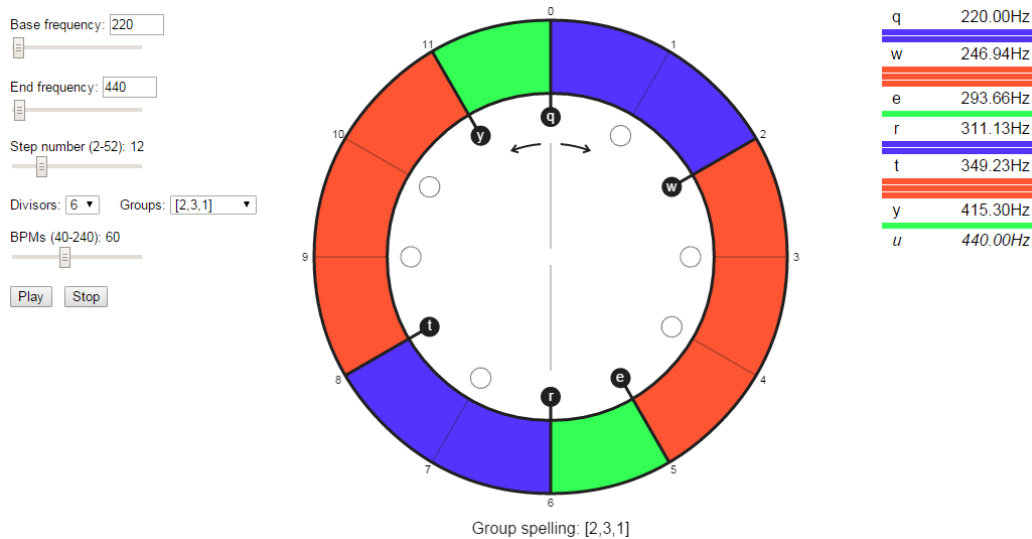


Figure 8. A Web interface to compute generalized Messiaen's modes.

by implementing controls of the timbre. Besides, the project will include a Max/MSP and a PureData porting, so that user-defined timbre generators will be able to interface with the scales defined by the algorithm.

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