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SOME CONSIDERATIONS ON MEASURING THE PROGRESSIVE PRINCIPLE VIOLATIONS AND THE POTENTIAL EQUITY IN INCOME TAX SYSTEMS

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ABSTRACT

Kakwani and Lambert (1998) state three axioms which should be respected by an equitable tax system; then they propose a measurement system to evaluate at the same time the negative influences that axiom violations exert on the redistributive effect of taxes, and the potential equity of the tax system, which would be attained in absence of departures from equity. The authors calculate both the potential equity and the losses due to axiom violations, starting from the Kakwani (1977) progressivity index and the Kakwani (1984) decomposition of the redistributive effect. In this paper, we focus on the measure suggested by Kakwani and Lambert for the loss in potential equity, which is due to violations of the progressive principle: the authors' measure is based on the tax rate re-ranking index, calculated with respect to the ranking of pre-tax income distribution. The aim of the paper is to achieve a better understanding of what Kakwani and Lambert's measure actually represents, when it corrects the actual Kakwani progressivity index. The authors' measure is first of all considered under its analytical aspects and then observed in different simulated tax systems. In order to better highlight its behaviour, simulations compare Kakwani and Lambert's measure with the potential equity of a counterfactual tax distribution, which respects the progressive principle and preserves the overall tax revenue. The analysis presented in this article is performed by making use of the approach recently introduced by Pellegrino and Vernizzi (2013).

JEL Codes: C81, H23, H24

Key words: equity**,** personal income tax, progressive principle, redistribution, re-ranking.

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1. Introduction

Since their very beginning taxes have mainly been the way to gather by the state the resources necessary to ensure its proper functioning. Apart from performing the fiscal function, the state, by means of taxes, influences the 'fair' distribution of income, thus fulfilling the redistribution function. As Kakwani and Lambert (1998), thereafter KL, observe, the redistribution function through the tax system has to be performed respecting social equity principles; two basic commands of social equity are "*the equal treatment of equals and the appropriately unequal treatment of unequals*" (KL, p. 369). As Aronson, Johnson and Lambert (1994, p.262) stress, equity violations should be considered for given "*specifications of the utility/income relationship*".

The redistributive effect of the income tax system can be measured by the difference between the Gini coefficients for the pre-tax income distribution and the post-tax income one, respectively. The difference between these two indexes measures how the income tax system reduces inequality in income distribution. The potential equity in the tax system is a value of the redistributive effect which might be achieved if all inequities could be abolished, by a rearrangement of tax burdens which substantially maintains either the tax revenue or the tax schedule. Rearrangements are generally performed by means of tax credits, exemptions, allowances, income splitting or quotient. The assessment of the potential equity requires a definition of an equitable tax system.

KL propose an approach for measuring inequity in taxation. According to KL an equitable tax system should respect three axioms: (Axiom 1) tax should increase monotonically with respect to people's ability to pay; (Axiom 2) richer people should pay taxes at higher rates; (Axiom 3) no re-ranking should occur in people's living standards. In this paper we maintain the KL definition of equity in income taxation by means of the three axioms. Violations by an income tax system of each one of the three axioms provide the means to characterise the type of inequity present in an income tax system. A tax system is equitable if all axioms are satisfied.

Let X be the pre-tax income or living standard[1] T - the tax, and A - the tax-rate distribution, and Y - the disposable income. The three axioms ask that the ranking of T, A, and Y coincide with the ranking of X. It follows that, as KL suggest, the extent of each axiom violations can be measured by the Atkinson-Kakwani-Plotnick re-ranking index of each attribute T, A and Y, with respect to the X ordering.

By these re-ranking indexes, on the basis of the Kakwani (1977) progressivity index and of the Kakwani (1984) decomposition of the redistributive effect, KL evaluate the implicit or potential equity in the tax system, in the absence of inequities. In particular, by adding the tax re-ranking index to the Kakwani (1977)

[1] KL, page 372, use the term living standard for X, assuming that nominal incomes have been transformed by a proper equivalence scale.

progressivity index they evaluate the potential equity which the tax system would reach in the absence of Axiom 1 violations. Analogously, by adding the tax-rate re-ranking index to the Kakwani progressivity index, they estimate the potential equity which the tax system would reach in the absence of Axiom 2 violations, that is to say, in the absence of the progressive principle violations.

However, if the addition of the tax re-ranking index to the Kakwani progressivity index restores the progressivity which would be yielded without tax re-ranking, it is less simple to understand what happens when adding the tax-rate re-ranking index to the Kakwani progressivity index, as the next section illustrates.

The aim of this paper is first of all to contribute to a better understanding of what the KL measure of the potential equity implies. In so doing, we try to contribute to the definition of alternative measures for the potential equity, which the tax system would yield if violations in the progressive principle were eliminated. Our analysis is performed by making use of the approach recently introduced by Pellegrino and Vernizzi (2013).

In Section 2 the measure of the potential equity and the losses generated by axioms violations are presented, as suggested by KL. Next we present the potential equity for three different cases:

(*1*) both Axiom 1 and Axiom 2 are respected,
(*2*) Axiom 2 is violated, whilst Axiom 1 is respected,
(*3*) Axiom 1 is violated, which implies that Axiom 2 is violated too.

In fact, as KL observe,[1] "*a violation of minimal progression (Axiom 1) automatically entails a violation of the progressive principle*" (Axiom 2). Section 3 discusses KL's potential equity measure by analysing it at the level of income unit pairs' relations. This section also considers an alternative naïve measure for the potential equity. This measure is calculated by the Gini coefficient of the counterfactual tax distribution, which can be obtained by matching tax rates and pre-tax incomes, both ranked in non-decreasing order. Section 4 illustrates and completes the analytical considerations of previous sections by simulations performed on the income distribution of taxpayers from Wrocław (Poland). Section 5 concludes.

2. The loss due to axiom violations and the potential equity in the tax system

Let $x_1, x_2, ...x_K$ the pre-tax income levels of K income units, who are paying t_1, t_2, $...t_K$ in tax. Both incomes and taxes can be expressed either in nominal values or in equivalent values. Moreover, let $y_i = x_i - t_i$ and $a_i = t_i/x_i$ represent the disposable income and the tax rate, respectively, which result in unit i (i=1, 2,...,K), after having paid tax t_i.

[1] Kakwani and Lambert (1998), page 371.

Let G_X , G_T , G_A and G_Y be the Gini coefficients for attributes *X*, *T*, *A* and *Y*, respectively. Let $C_{Z|X}$ be the concentration coefficient for an attribute *Z* of income units (*Z*= *T*, *A*, *Y*) when the attribute *Z* is ranked by *X*, which refers to the pre-tax incomes lined up in ascending order. KL detect axiom violations via the three following Atkinson-Plotnick-Kakwani re-ranking indexes: $R_{T|X} = \left(G_T - C_{T|X}\right)$, $R_{A|X} = \left(G_A - C_{A|X}\right)$ and $R_{Y|X} = \left(G_Y - C_{Y|X}\right)$.[1] According to KL, if $R_{T|X} > 0$, Axiom 1 is violated; analogously, if $R_{A|X} - R_{T|X} > 0$ Axiom 2 is violated, and if $R_{Y|X} > 0$ it is Axiom 3 which is violated.[2]

KL use the three re-ranking indexes to evaluate the loss in the potential redistributive effect of the tax system. They represent the redistributive effect $RE = G_X - G_Y$, which, on the basis of the Kakwani decomposition, can be written as [3]

$$RE = \tau P - R_{Y|X} , \qquad (1)$$

where τ is the ratio between the tax average - μ_T, and the disposable income average - μ_Y. *P* is the Kakwani progressivity index, [4] which is defined as follows:

$$P = C_{T|X} - G_X . \qquad (2)$$

If no tax-re-ranking occurred, then $C_{T|X} = G_T$, and $P = G_T - G_X$. Therefore, KL can define $\tau R_{T|X}$ as the loss of redistributive effect due to tax re-ranking.[5] The potential equity after the correction for income and tax re-ranking can then be defined as:

$$PE_T = \tau P + \tau R_{T|X} = \tau\left(C_{T|X} - G_X + G_T - C_{T|X}\right) = \tau\left(G_T - G_X\right). \qquad (3)$$

KL evaluate the loss due tax-rate re-ranking by the quantity $\tau\left(R_{A|X} - R_{T|X}\right)$. In analogy to (3), after having corrected also for the tax-rate re-ranking, KL define the potential equity as:

$$PE_A = \tau P + \tau R_{T|X} + \tau\left(R_{A|X} - R_{T|X}\right) = \tau\left(C_{T|X} + R_{A|X} - G_X\right). \qquad (4)$$

In order to understand how the corrections yield the potential equity as per formulae (3) and (4), we now gather income unit pairs into three different groups, according to Axiom 1 and 2 violations.

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[1] Being $G_Z \geq C_{Z|X}$, $R_{Z|X} \geq 0$.

[2] As KL observe (page 372), Axiom 3 can be violated only if Axiom 2 (and consequently Axiom 1) holds.

[3] See, e.g., Lambert (2001), pp. 238–242.

[4] Lambert (2001), ibidem. We observe that the progressivity index P does not contain any information on the incidence of taxation; τ is an indicator of the taxation incidence, which, conversely, does not contain any information on the progressivity: intuitively the redistributive effect is a function both of progressivity and of incidence.

[5] More details about equations (1) and (2) can be found in Appendix 3.

Group (*1*) includes income unit pairs presenting neither tax re-ranking nor tax rate re-ranking: for these pairs both Axiom 1 and Axiom 2 hold.

Group (*2*) includes income unit pairs presenting tax rate re-ranking but no tax re-ranking: for these pairs Axiom 1 holds, whilst Axiom 2 is violated.

Group (*3*) includes income unit pairs presenting both tax rate re-ranking and tax re-ranking: here, Axiom 1, and consequently Axiom 2, are both violated.

According to this classification, G_T, $C_{T|X}$, $R_{A|X}$ and G_X can split into three different components, $G_T^{(g)}$, $C_{T|X}^{(g)}$, $R_{A|X}^{(g)}$ and $G_X^{(g)}$ ($g = 1,2,3$), respectively, each related to one of the three different groups. This can be done by making use of the notation adopted by Pellegrino and Vernizzi (2013, page 3), and by a group selector function. Pellegrino and Vernizzi express the Gini coefficient in the following form:

$$G_Z = \frac{1}{2\mu_Z N^2}\sum_{i=1}^{K}\sum_{j=1}^{K}\left(z_i - z_j\right) p_i p_j I_{i-j}^{Z}, \quad \text{where} \quad I_{i-j}^{Z} = \begin{cases} 1: & z_i \ge z_j \\ -1: & z_i < z_j \end{cases}. \tag{5}$$

In (5) p_i and p_j are weights associated to z_i and z_j, respectively; $\sum_{i=1}^{K} p_i = N$; μ_Z is the average of *Z*, and I_{i-j}^{Z} is an indicator function.

When income units are lined up by ascending order of *X*, Pellegrino and Vernizzi write the concentration coefficient of attribute *Z* as:

$$C_{Z|X} = \frac{1}{2\mu_Z N^2}\sum_{i=1}^{K}\sum_{j=1}^{K}\left(z_i - z_j\right) p_i p_j I_{i-j}^{Z|X}, \qquad I_{i-j}^{Z|X} = \begin{cases} 1: & x_i > x_j \\ -1: & x_i < x_j \\ I_{i-j}^{Z}: & x_i = x_j \end{cases}. \tag{6}$$

Consequently, they formulate the re-ranking index of *Z* with respect to *X* as:

$$R_{Z|X} = \frac{1}{2\mu_Z N^2}\sum_{i=1}^{K}\sum_{j=1}^{K}\left(z_i - z_j\right) p_i p_j \left(I_{i-j}^{Z} - I_{i-j}^{Z|X}\right). \tag{7}$$

Let us introduce the indicator function $I_{i-j}^{(g)}$, which is 1 when the income unit pair $\{i, j\}$ is classified into group *g*, and it is zero otherwise: $I_{i-j}^{(g)}$ is then a group selector, which classifies each income unit pair into one of the three groups.[1] For each of the three different groups we can write:

$$G_T^{(g)} = \frac{1}{2\mu_T N^2}\sum_{i=1}^{K}\sum_{j=1}^{K}\left(t_i - t_j\right) p_i p_j I_{i-j}^{T} I_{i-j}^{(g)}, \tag{8}$$

[1] More details on the indicator function $I_{i-j}^{(g)}$ are in Appendix 3.

$$C_{T|X}^{(g)} = \frac{1}{2\mu_T N^2} \sum_{i=1}^{K}\sum_{j=1}^{K} \left(t_i - t_j\right) p_i p_j I_{i-j}^{T|X} I_{i-j}^{(g)}, \tag{9}$$

$$R_{A|X}^{(g)} = \frac{1}{2\mu_A N^2} \sum_{i=1}^{K}\sum_{j=1}^{K} \left(a_i - a_j\right) p_i p_j \left(I_{i-j}^{A} - I_{i-j}^{A|X}\right) I_{i-j}^{(g)}, \tag{10}$$

$$G_{X}^{(g)} = \frac{1}{2\mu_X N^2} \sum_{i=1}^{K}\sum_{j=1}^{K} \left(x_i - x_j\right) p_i p_j I_{i-j}^{X} I_{i-j}^{(g)}. \tag{11}$$

We observe that:[1]

$$C_{T|X}^{(1)} = G_T^{(1)},\ C_{T|X}^{(2)} = G_T^{(2)},\ C_{T|X}^{(3)} = -G_T^{(3)};$$

$$C_{A|X}^{(1)} = G_A^{(1)},\ C_{A|X}^{(2)} = -G_A^{(2)},\ C_{A|X}^{(3)} = -G_A^{(3)}; \tag{12}$$

from which

$$R_{T|X}^{(1)} = 0,\ R_{T|X}^{(2)} = 0,\ R_{T|X}^{(3)} = 2G_T^{(3)};$$

$$R_{A|X}^{(1)} = 0,\ R_{A|X}^{(2)} = 2G_A^{(2)},\ R_{A|X}^{(3)} = 2G_A^{(3)} \tag{13}$$

By making use of the expressions (8)-(11), we can split P, PE_T and PE_A defined by formulae (2), (3) and (4), into three components, $P^{(g)}$, $PE_T^{(g)}$ and $PE_A^{(g)}$ ($g = 1,2,3$), respectively, each related to one of the three groups. In addition, using the observations (12) we have the following relations:

- group (*1*)

$$P^{(1)} = G_T^{(1)} - G_X^{(1)},\ PE_T^{(1)} = \tau\, P^{(1)}, \tag{14}$$

$$PE_A^{(1)} = PE_T^{(1)}; \tag{15}$$

- group (*2*)

$$P^{(2)} = G_T^{(2)} - G_X^{(2)},\ PE_T^{(2)} = \tau\, P^{(2)} \tag{16}$$

$$PE_A^{(2)} = \tau\left(P^{(2)} + R_{A|X}^{(2)}\right) \tag{17}$$

- group (*3*)

$$P^{(3)} = C_{T|X}^{(3)} - G_X^{(3)},\ PE_T^{(3)} = \tau\left(P^{(3)} + R_{T|X}^{(3)}\right) = \tau\left(G_T^{(3)} - G_X^{(3)}\right) \tag{18}$$

$$PE_A^{(3)} = PE_T^{(3)} + \tau\left(R_{A|X}^{(3)} - R_{T|X}^{(3)}\right) = \tau\left(C_{T|X}^{(3)} + R_{A|X}^{(3)} - G_X^{(3)}\right) \tag{19}$$

[1] For more details on equations (12), see Appendix 3.

We can see that, according to the KL method, in case (*2*), where only tax-rate re-ranking is present, $R_{A|X}^{(2)}$ corrects just for the loss due to tax-rate re-ranking (expression 17). In case (*3*), $R_{A|X}^{(3)}$ corrects simultaneously both for tax and for tax-rate re-ranking (expression 19). Having in mind (15), KL's potential equity can then be written as:

$$PE_A = PE_A^{(1)} + PE_A^{(2)} + PE_A^{(3)} = PE_T^{(1)} + PE_A^{(2)} + PE_A^{(3)}. \tag{20}$$

However, expression (19) and, as a consequence, expression (20) are measures which are not strictly faithful to what KL state (page. 372) "*Axiom 2 is violated if the rankings by X and by A of income units pairs {i, j} for which Axiom 1 holds differ [........]*". According to this statement, as all income units pairs, classified in case (*3*), present tax re-ranking and violate Axiom 2, they cannot be considered as violating Axiom 3, even if, as a consequence, they present tax-rate re-ranking too. In fact, the KL command specifies that tax-rate re-ranking should be considered only for income unit pairs classified in group (*2*).

Then, if we want to observe literally the KL command, the potential equity should be written as:[1]

$$PE_{A,T} = PE_T^{(1)} + PE_A^{(2)} + PE_T^{(3)}. \tag{21}$$

Even if Pellegrino and Vernizzi (page 242) specify that their new measure should be adopted "*If we want to observe literally the KL command*", they do not exclude adopting the original KL measure; they just stress that one should be aware that the KL measure "*does not involve only income units pairs for which Axiom 1 holds*". Actually, as it will appear clearer in the pursue, the tax rate re-ranking index often results to be greater that the tax re-ranking index, and this is coherent with the matter of fact that even after having eliminated tax re-ranking, tax-rate re-ranking can still persist.[2] So in this article we shall consider both the original KL measure and the one introduced by Pellegrino and Vernizzi.

In the next section we discuss how the potential equity measures act at the level of income pairs, in particular we will focus on (17) and (19).

3. The potential equity at the microscope

If we consider expression (3) for PE_T, after having added $R_{T|X}$ to $C_{T|X}$, one yields the Gini coefficient of the tax liability distribution. Conversely, for what concerns PE_A, as per expression (4), we need more considerations in order to

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[1] Expression (21) can also be obtained by adding τP and S_2^*, defined in Pellegrino and Vernizzi (2013), at formula (9).

[2] Consider the following simple example: gross incomes {*X*: 1000, 2000, 3000, 5000}, taxes {*T*: 450, 400, 300, 200}. If we align taxes in ascending order and match them with incomes, the ratios {*A*: 200/1000, 300/2000, 400/3000, 450/5000} still remain in decreasing order, with respect to incomes.

interpret what one yields by adding $R_{A|X}$ to $C_{T|X}$. We shall now consider the effects of this addition at the level of the addends which constitute the sums in (9) and (10), which enter $PE_A^{(2)}$ and $PE_A^{(3)}$, given at (17) and (19), respectively.

The sum of $C_{T|X}$ and $R_{A|X}$, can be expressed as:

$$C_{T|X} + R_{A|X} = \frac{1}{2\mu_T N^2}\sum_{i=1}^{K}\sum_{j=1}^{K} d_{i-j} p_i p_j \, , \tag{22}$$

having defined

$$d_{i-j} = \left(t_i - t_j\right) I_{i-j}^{T|X} + \left(a_i - a_j\right)\lambda\mu_X \left(I_{i-j}^{A} - I_{i-j}^{A|X}\right), \tag{23}$$

with $\lambda = (\mu_T / \mu_X)/\mu_A$.

In (23) the tax difference $\left(t_i - t_j\right) I_{i-j}^{T|X}$ is "corrected" by adding a component which is either 0 or $2\left|a_i - a_j\right|\lambda\mu_X$:[1] that is to say, when a tax rate pair does not respect the *X* ordering, the absolute difference of the two tax rates is transformed into a monetary value by the constant factor $\lambda\mu_X$.

In $PE_A^{(3)}$, the term $\left(t_i - t_j\right) I_{i-j}^{T|X}$ is negative; then d_{i-j} has to express the redistributive effect between income units *i* and *j*, which would be potentially yielded after compensation both for tax and for tax rate re-ranking. As a consequence, d_{i-j} should be not lower, at least, than the term $\left(t_i - t_j\right) I_{i-j}^{T} = \left|t_i - t_j\right|$, which expresses the potential equity, when the correction is limited to tax re-ranking. This implies that:

$$\left|a_i - a_j\right|\lambda\mu_X \geq \left|t_i - t_j\right|, \quad \text{i.e.} \quad \frac{\left|a_i - a_j\right|}{\mu_A} \geq \frac{\left|t_i - t_j\right|}{\mu_T}. \tag{24}$$

From the analysis reported in Appendix 2, there are more chances that inequality (24) is verified, than it is not. According to our simulations, reported in section 4, despite the fact that there is a not insignificant number of cases which do not verify (24), $PE_A^{(3)}$ happens to be always significantly greater than $PE_T^{(3)}$. KL "*found* $\left[R_{A|X} - R_{T|X}\right]$ *always to be non-negative in extensive simulations*".[2] The results of our simulations reinforce KL's findings. In fact, according to our experiments, not only the overall potential equity PE_A is greater than the corresponding PE_T: even when considering only pairs in case (*3*), where expression (24) is not necessarily satisfied, $PE_A^{(3)}$ is generally greater than $PE_T^{(3)}$.

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[1] In case (*1*) $\left(a_i - a_j\right)\lambda\mu_X\left(I_{i-j}^{A} - I_{i-j}^{A|X}\right)$ is equal to 0; in cases (*2*) and (*3*) it is $2\left|a_i - a_j\right|\lambda\mu_X$, as $\left(I_{i-j}^{A} - I_{i-j}^{A|X}\right)$ is 2, when $\left(a_i - a_j\right)$ is positive, and it is -2 when the difference is negative.

[2] Kakwani and Lambert (1998, page 373).

We could try to measure the loss due to Axiom 2 violations by introducing a counterfactual tax distribution, which respects Axiom 2. This counterfactual tax distribution could be obtained by matching tax rates and pre-tax incomes, both aligned in ascending order. Tax rates are then rescaled in order to maintain the same tax revenue. However, we have to be aware that, in so doing, we would not strictly follow KL's command which asks that Axiom 2 violations should be *"confined to those income unit pairs for which Axiom 1 {i, j} holds"*. KL themselves do not fully respect their command as in measuring the extent of Axiom 2 violations by $\tau\left(R_{A|X}-R_{T|X}\right)$, as a matter of fact, they consider all unit pairs for which Axiom 2 does not hold.

In our opinion there are some reasons which could lead to considering all income unit pairs in evaluating the extent of Axiom 2 violations; in fact, after having matched both taxes and pre-tax incomes in ascending order, the tax rates derived from this matching do not necessarily become aligned in ascending order, as the example in footnote 11 illustrates. Consequently, the loss due to Axiom 2 violations can go further than the loss due to Axiom 1 violations.

If we denote this counterfactual tax distribution by T^{CF}, and the Gini index for the counterfactual tax distribution by G_T^{CF}, the loss due to Axiom 2 violations could then be measured by $\tau\left(G_T^{CF}-G_T\right)$, and the expression for the potential equity would become:

$$PE_A^{CF}=\tau\left(G_T^{CF}-G_X\right). \tag{25}$$

In the case of $G_T^{CF}<G_T$, further progressive counterfactual tax distributions could be generated by exploiting the progressivity reserve implicit in the tax system. In fact there are several counterfactual tax distributions which respect Axiom 2. For example, a more progressive counterfactual tax distribution could be generated by matching the pre-tax and income distribution and a modified tax-rate distribution which *a*) maintains the same tax rates in the upper queue of *A* distribution, and *b*) lowers tax rates in the lower queue of *A* distribution. Obviously, both distributions being in ascending order. Operations *a*) and *b*) could be calibrated in such a way that the tax revenue remains the same as that of *T*.

In the next section, we will evaluate the potential equity measures PE_T, PE_A, $PE_{A,T}$, and PE_A^{CF}, together with the incidence of cases which verify (24).

4. Simulation results

The measures of the potential equity, described in the previous section, were calculated for a Polish data set, by applying sixteen different hypothetical tax systems. The data come from the Lower-Silesian tax offices, 2001. The data set contains information on gross income for individual residents in the Municipality of Wrocław (Poland). After deleting observations with non-positive gross income, the whole population consists of 37,080 individuals. For the analysis we used a random sample with size 10 000. The summary statistics for the sample of gross income distribution are: mean income = 18,980 PLN; standard deviation = 23,353 PLN; skewness = 13.29; kurtosis = 424.05. The Gini coefficient for the pre-tax income distribution is 0.45611.

The sixteen tax systems were constructed on the basis of four tax structures actually applied or widely discussed in Poland, from the simplest flat tax system to a more progressive tax system with four income brackets. In order to implement the “iniquity”, within each tax structure net incomes were “disturbed” by introducing four different types of random errors (more details are reported in Appendix 1).

The resulting sixteen tax systems present *RE* indexes ranging from 0.141% to 3.584 %.

Table 1 reports basic indexes for the 16 tax systems, whereas Table 2 reports the potential equity measures for each tax system.

From Table 1, columns (3), (4) and (5), we can see that in each tax system the most unit pairs belong to group (*1*), where neither tax re-ranking nor tax rate re-ranking occurs. In four tax systems which derive from the Basic System 4, that is to say *T-S_4*, *T-S_8*, *T-S_12* and *T-S_16*, the percentage of income unit pairs belonging to group (*3*) is slightly greater than that of pairs belonging to group (*2*). Group (*2*) is much more crowded than group (*3*) in the remaining tax systems.

The percentage of pairs in group (*3*), which verify (24) (see Table 1, column 6), is never lower than 37.95% and greater than 83.6%. This percentage is changeable. However, if we consider column (14) in Table 1, we can observe that the ratios $\left(R_{A|X}^{(3)} / G_T^{(3)}\right)$ are stable at around 4 and a half. These empirical findings reinforce KL’s simulation results: whenever $PE_A^{(3)}$ is greater than $PE_T^{(3)}$, a fortiori it is verified that $PE_A > PE_T$, as necessarily $PE_A^{(2)} \geq PE_T^{(2)}$.

Beyond any discussion about how measuring the loss in potential equity due to Axiom 2 violations, the role of this Axiom appears evident from Table 2. Considering the differences between PE_A (column 6) and PE_T (column 5), the marginal contribution to potential equity yielded in the absence of Axiom 2 violations is greater, and, in some cases, incomparably greater than that yielded by Axiom 1, which is given by the difference between PE_T and τP (column 4). This finding is not invalidated if we consider $PE_{A,T}$ (column 7), instead of PE_A, that is to say KL’s measure, corrected as per formula (20). The counterfactual PE_A^{CF} potential equity gives different measures from PE_A, and in general the values estimated by the former are lower than those yielded by the latter; however, if one considers columns (6) and (8), distances present a much lower extent than those existing between column (6) and (5).

These results confirm the relevance of KL's contribution on distinguishing the different sources of inequity, and so, under this aspect, in spite of their conceptual and empirical differences, PE_A, PE_A^{CF} and $PE_{A,T}$ seem to give a coherent signal.

In any case, in our opinion, further research and discussion is still needed to conceive a measure which can solve some remaining ambiguities.

5. Concluding remarks

In this paper we have reconsidered the problem of measuring loss due to the progressive principle violations in a personal tax system. Our results confirm first of all the relevance of the contribution of Kakwani and Lambert's article (1998). The violations of minimal progression (KL's Axiom 1) and of the progressive principle (KL's Axiom 2) produce different effects and these effects have to be kept distinct. According to our simulations, on the whole, violations of the progressive principle appear to be even more relevant than those regarding minimal progression.

This note argues whether KL's measure of the progressive principle needs some refinements. The authors' measure implicitly transforms tax rate differences into tax differences by a factor which is constant, irrespective of income levels: this factor depends only on overall effects of a tax system, i.e. on the average tax rate and the average liability.

We air the idea of measuring the potential equity by introducing counterfactual tax distributions. As an example, we have simulated the behaviour of a naïve tax distribution, obtained by matching tax rates and incomes, both aligned in ascending order. Tax rates have been rescaled in order to maintain the same tax revenue. On the one hand this naïve measure produces different values from those obtained through KL's approach, on the other hand, it confirms the relevance of potential equity losses, due to Axiom 2 violations.

However, the counterfactual measures here outlined can be applied if one does not exclude income unit pairs which do not respect minimal progression. We observed that, even after having restored minimal progression by matching both the tax and the pre-tax income distribution in ascending order, the resulting counterfactual tax rates are not necessarily in ascending order too.

In conclusion, the discussion presented in this paper, which is based on the cornerstone represented by Kakwani and Lambert's (1998) article, intends to be an initial contribution towards a satisfying measure for the potential equity when the progressivity principle is violated.

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Table 1. Basic indexes: (1) neither tax re-ranking nor tax rate re-ranking, (2) tax rate re-ranking, without tax re-ranking, (3) tax and tax rate re-ranking. $G_X \times 100$=45.611; N=10,000

Tax system	λ $(\mu_T/\mu_X)/\mu_A$	Percentage of pairs in groups: Group (*1*)	Group (*2*)	Group (*3*)	Percentage of pairs in group (*3*) for which $d_{i-j} \geq \lvert t_i - t_j \rvert$	$\frac{G_X^{(1)}}{G_X} \cdot 100$	$\frac{G_X^{(2)}}{G_X} \cdot 100$	$\frac{G_X^{(3)}}{G_X} \cdot 100$	$C_{T\mid X}^{(1)} \cdot 100$ / $G_T^{(1)} \cdot 100$	$C_{T\mid X}^{(2)} \cdot 100$ / $G_T^{(2)} \cdot 100$	$\frac{R_{A\mid X}^{(2)}}{G_T^{(2)}}$	$C_{T\mid X}^{(3)} \cdot 100$ / $-G_T^{(3)} \cdot 100$	$\frac{R_{A\mid X}^{(3)}}{G_T^{(3)}}$
(1)	**(2)**	**(3)**	**(4)**	**(5)**	**(6)**	**(7)**	**(8)**	**(9)**	**(10)**	**(11)**	**(12)**	**(13)**	**(14)**
T-S_1	1.114	59.933	35.185	4.882	81.257	60.363	39.156	0.481	31.389	15.385	0.321	-0.205	4.634
T-S_2	1.368	85.259	10.759	3.982	80.476	94.105	5.558	0.337	55.536	1.807	0.606	-0.148	4.705
T-S_3	1.310	84.297	11.692	4.012	79.782	91.795	7.853	0.351	52.102	2.818	0.449	-0.161	4.458
T-S_4	1.753	93.242	3.094	3.665	64.617	98.945	0.774	0.281	69.337	0.206	1.128	-0.093	4.712
T-S_5	1.118	59.920	34.066	6.013	83.596	61.818	37.435	0.747	32.910	14.160	0.419	-0.326	4.664
T-S_6	1.359	83.084	12.089	4.828	83.199	92.368	7.127	0.505	54.816	2.277	0.674	-0.234	4.672
T-S_7	1.304	82.120	13.041	4.840	82.374	89.989	9.490	0.521	51.466	3.297	0.525	-0.250	4.448
T-S_8	1.758	92.363	3.450	4.187	68.812	98.643	1.000	0.358	69.517	0.262	1.182	-0.140	4.674
T-S_9	1.116	59.655	38.360	1.985	64.081	60.586	39.333	0.081	29.521	17.159	0.105	-0.021	4.650
T-S_10	1.367	93.507	4.760	1.733	59.907	98.716	1.213	0.070	56.725	0.440	0.362	-0.015	4.720
T-S_11	1.306	93.159	5.113	1.728	59.224	98.198	1.731	0.071	53.902	0.701	0.250	-0.016	4.496
T-S_12	1.749	96.678	1.269	2.053	37.949	99.745	0.119	0.136	69.381	0.032	0.973	-0.010	4.730
T-S_13	1.118	60.025	34.802	5.174	82.180	61.906	37.558	0.536	32.330	14.677	0.353	-0.230	4.713
T-S_14	1.364	84.746	11.056	4.198	81.301	93.755	5.870	0.375	55.349	1.897	0.634	-0.168	4.698
T-S_15	1.304	83.707	12.095	4.198	80.416	91.412	8.206	0.382	51.744	2.939	0.466	-0.179	4.464
T-S_16	1.743	93.063	3.164	3.774	65.526	98.877	0.827	0.296	69.186	0.223	1.111	-0.104	4.630

Table 2. Potential equity measures. $G_X \times 100$=45.611

Tax system	$RE \cdot 100$	τ	$(\tau P/\mathrm{RE}) \cdot 100$	$(PE_T/\mathrm{RE}) \cdot 100$	$(PE_A/\mathrm{RE})\,100$	$(PE_{A,T}/\mathrm{RE})\,100$	$(PE_A^{CF}/\mathrm{RE})\,100$	$(R_{Y\vert X}/\mathrm{RE})\,100$
(1)	**(2)**	**(3)**	**(4)**	**(5)**	**(6)**	**(7)**	**(8)**	**(9)**
T-S_1	0.154	0.174	108.375	154.811	774.165	713.011	642.279	8.351
T-S_2	2.504	0.218	100.641	103.207	116.198	112.727	112.387	0.640
T-S_3	1.720	0.190	100.789	104.344	122.649	118.281	122.095	0.788
T-S_4	3.584	0.151	100.196	100.977	103.011	101.952	102.540	0.196
T-S_5	0.141	0.135	108.596	171.068	822.886	739.662	645.733	8.578
T-S_6	1.844	0.165	100.794	104.982	124.321	118.727	119.574	0.792
T-S_7	1.278	0.145	100.963	106.638	133.230	126.285	131.053	0.962
T-S_8	2.817	0.118	100.220	101.386	104.240	102.680	103.502	0.220
T-S_9	0.169	0.163	100.772	104.802	283.570	278.231	250.409	0.765
T-S_10	2.322	0.201	100.062	100.330	102.073	101.708	101.746	0.062
T-S_11	1.573	0.175	100.076	100.441	102.851	102.395	103.391	0.076
T-S_12	3.323	0.140	100.023	100.106	100.352	100.238	100.290	0.023
T-S_13	0.154	0.140	106.070	147.973	675.609	618.762	537.408	6.055
T-S_14	1.964	0.172	100.580	103.525	118.039	114.067	114.364	0.579
T-S_15	1.324	0.150	100.713	104.756	125.240	120.258	123.472	0.711
T-S_16	2.848	0.120	100.173	101.053	103.259	102.102	102.886	0.173

APPENDIXES

Appendix 1. The simulated tax systems

Four basic tax structures are hypothesised as follows:

BASIC SYSTEM 1. One 15 per cent tax rate is applied to all incomes. All taxpayers benefit from 556.02 PLN tax credit.

BASIC SYSTEM 2. A system with three income brackets: *i*) 19 per cent from 0 to 44,490 PLN, *ii*) 30 per cent from 44,490 to 85,528 PLN, *iii*) 40 per cent over 85,528 PLN. All taxpayers benefit from 586.85 PLN tax credit.

BASIC SYSTEM 3. A system with two income brackets: *i*) 18 per cent from 0 to 85,528 PLN, *ii*) 32 per cent over 85,528 PLN. All taxpayers benefit from 556.02 PLN tax credit.

BASIC SYSTEM 4. A system with four income brackets*: i)* 10 per cent from 0 to 20,000 PLN, *ii*) 20 per cent from 20,000 to 40,000 PLN, *iii*) 30 per cent from 40,000 to 90,000 PLN, iv) 40 per cent over 90,000 PLN. All taxpayers benefit from 500.00 PLN tax credit.

For each taxpayer, the tax $T(x_i)$ that results after the application of a basic tax system is then modified by a random factor, so that net income becomes $y_i = x_i - T(x_i) + z_i \cdot T(x_i)$; the factor z_i is drawn:

(*a*) from the uniform distributions:

(*a*1) $Z \sim U(-0.2 \div 0.2)$, (*a*2) $Z \sim U(0 \div 0.4)$;

(*b*) from the normal distributions:

(*b*1) $Z \sim N(0\,;0.0133)$, (*b*2) $Z \sim N(0\,;0.12)$;

Then, each basic system generates four sub-systems. When the normal distribution is applied, the random factor z_i is considered in absolute value; the programme did not allow incomes to become either negative or greater that $2x_i$.

In this way we receive the following sixteen hypothetical tax systems:

T-S_1: BASIC SYSTEM 1 modified by a random factor (*a*1)

T-S_2: BASIC SYSTEM 2 modified by a random factor (*a*1)

T-S_3: BASIC SYSTEM 3 modified by a random factor (*a*1)

T-S_4: BASIC SYSTEM 4 modified by a random factor (*a*1)

T-S_5: BASIC SYSTEM 1 modified by a random factor (*a*2)

T-S_6: BASIC SYSTEM 2 modified by a random factor (*a*2)

T-S_7: BASIC SYSTEM 3 modified by a random factor (*a*2)

T-S_8: BASIC SYSTEM 4 modified by a random factor (*a*2)

T-S_9: BASIC SYSTEM 1 modified by a random factor (*a*1)

T-S_10: BASIC SYSTEM 2 modified by a random factor (*b*1)

T-S_11: BASIC SYSTEM 3 modified by a random factor (*b*1)

T-S_12: BASIC SYSTEM 4 modified by a random factor (*b*1)

T-S_13: BASIC SYSTEM 1 modified by a random factor (*a*2)

T-S_14: BASIC SYSTEM 2 modified by a random factor (*b*2)

T-S_15: BASIC SYSTEM 3 modified by a random factor (*b*2)

T-S_16: BASIC SYSTEM 4 modified by a random factor (*b*2)

Appendix 2. On the sign of ($PE_A - PE_T$) for income unit pairs from group (*3*)

From expressions (18), (19) and (22), the pair {*i*, *j*} contributes to $PE_A^{(3)}$ at a greater extent than to $PE_T^{(3)}$ if

$$\left(t_i - t_j\right) I_{i-j}^{T|X} + \left(a_i - a_j\right) \lambda \mu_X \left(I_{i-j}^{A} - I_{i-j}^{A|X}\right) \geq \left(t_i - t_j\right) I_{i-j}^{T} . \qquad \text{(A.1)}$$

For the sake of simplicity and without any lack of generality, let us consider only the differences corresponding to incomes $x_i < x_j$; for group (*3*), when the difference $\left(x_i - x_j\right)$ is negative, $\left(t_i - t_j\right)$, $\left(a_i - a_j\right)$, $\left(I_{i-j}^{A} - I_{i-j}^{A|X}\right)$, and I_{i-j}^{T} are positive, whilst $I_{i-j}^{T|X}$ is negative. Then, for group (*3*), inequality (A.1) is verified if

$$\left(a_i - a_j\right) \lambda \mu_X \geq \left(t_i - t_j\right), \text{ for } x_i < x_j, a_i > a_j, \text{ and } t_i > t_j . \qquad \text{(A.2)}$$

As $t_i = a_i x_i$ and, analogously, $t_j = a_j x_j$, (A.2) can be rearranged as

$$a_i \left(\lambda \mu_X - x_i\right) \geq a_j \left(\lambda \mu_X - x_j\right). \qquad \text{(A.3)}$$

Being $a_i > a_j$, and $x_i < x_j$, we can conclude that, whenever $x_i < \lambda \mu_x$, strict inequality holds.

Income distributions are, in general, positive skew, so more than 50% of incomes are lower than μ_x and even more incomes are lower than $\lambda \mu_x$, as we expect that λ>1. To understand why λ is greater than 1, observe that if the tax rate schedule can be approximated by a strictly concave function of pre-tax incomes, (*i*) due to Jensen's inequality[1], it results in $\mu_A < \left[t\left(\mu_X\right) / \mu_X \right]$; moreover, (*ii*) if the

[1] See e.g. Lambert (2001), page 11.

tax distribution can be approximated by a strictly convex function, still due to Jensen's inequality[1], it results in $t\left(\mu_X\right) < \mu_T$. As in general both (*i*) and (*ii*) happen, a fortiori, we can expect that $\mu_A < \left(\mu_T / \mu_X\right)$, from which $\lambda > 1$.

If $x_i > \lambda\mu_X$, strict inequality holds in (A.3), if $a_i\left(x_i - \lambda\mu_x\right) < a_j\left(x_j - \lambda\mu_x\right)$, or, which is the same, if

$$\frac{x_i - \lambda\mu_x}{x_j - \lambda\mu_x} < \frac{a_j}{a_i} \tag{A.4}$$

which can be either verified or not.

Our simulations (Table 1, column 6) confirm that in group (*3*) there are more income unit pairs which verify inequality (24) than those which do not: the share of pairs which verify the inequality is never less than 57.9 %.

Less immediate is interpreting the effect of *λ*. If one considers that keeping constant all the remaining components in (A.4), the left hand side of the inequality is a decreasing function of *λ*, it can be surprising to observe that the percentage of pairs in group (*3*), which verify (A.3), appears to be inversely related to *λ*. We can try to explain this by observing that the progressivity of a tax system does not act only on λ[2]: by its interactions with different sources of unfairness, it acts also on the actual tax rates and, consequently on the ratio $\left(a_j / a_i\right)$. It is then difficult foreseeing the final outcomes concerning the inequality at (A.4). Moreover there is no reason to believe that the distribution of incomes lower than $\lambda\mu_X$ should remain equally distributed through the three groups.

As *λ* increases the percentage of violations of both Axiom 1 and Axiom 2 decreases as we can see from Table 1, columns (2), (4) and (5), progressivity should augment the theoretical distance of net incomes. Another not surprising result is that, for what concerns Axiom 2 violations (column 12), the relative correction, expressed by the ratio $\frac{R^{(2)}_{A|X}}{G^{(2)}_T}$, appears to be a direct function of *λ*.

We conclude observing that in column (14) the ratio $\frac{R^{(3)}_{A|X}}{G^{(3)}_T}$ assumes nearly constant values in all the simulated tax systems, ranging from 4.448 to 4.720.

[1] See e.g. Lambert (2001), page 223.

[2] From Table 1, column (2), we can see that *λ* is highest for tax systems *T-S_4*, *T-S_8*, *T-S_12* and *T-S_16*, which derive from the most progressive basic system, the fourth one. Conversely it is lowest for *T-S_1*, *T-S_5*, *T-S_9* and *T-S_13*, which derive from basic system 1, which has only one and quite low tax rate (15%).

Appendix 2. The decomposition of the redistributive effect

The Kakwani progressivity index, $P = C_{T|X} - G_X$, is based on the Jakobsson-Fellman and the Jakobsson-Kakwani theorems.[1]

First of all, from the Jakobsson-Fellman theorem it follows that if the derivative of the tax rate is non-negative, i.e. $a'(x) \geq 0$, then $P \geq 0$.

Let us now consider two different tax systems, $t_1(x)$ and $t_2(x)$, applied to a same income distribution. If the tax elasticities the two tax systems, $LP_1(x) = \left[t_1'(x)/a_1(x)\right]$ and $LP_2(x) = \left[t_2'(x)/a_2(x)\right]$, are such that $LP_1(x) \leq LP_2(x)$, then, from the Jakobsson-Kakwani theorem, it follows that $P_1 \leq P_2$.

The redistributive effect of a tax system, which is defined as the difference between $RE = G_X - G_Y$ [2], can be represented in terms of the Kakwani progressivity index and of the Atkinson-Plotnick-Kakwani re-ranking index, as per formula (1):

$$RE = \tau P - \left(G_Y - C_{Y|X}\right) = \tau P - R_{Y|X} .$$

By this expression one can immediately evaluate how much of the redistributive effect depends on the tax progressivity and how much is lost due to the re-ranking of incomes; it follows that τP represents the redistributive effect which would be achieved if no income re-ranking were introduced by taxes.[3]

If the tax ordering coincides with the pre-tax income ordering, we have that $C_{T|X} = G_T$, and, consequently, $P = G_T - G_X$. If the two orderings do not coincide, being $C_{T|X} < G_T$, it results that $P < G_T - G_X$, and, consequently, $\tau P < \tau(G_T - G_X)$: $\tau(G_T - G_X)$ is the potential redistributive effect which would be achieved if neither post-tax income re-ranking nor tax re-ranking occurred, with respect to the pre-tax income ordering.

We introduce the indicator function $I_{i-j}^{(g)}$ ($g=1, 2, 3$):

- $I_{i-j}^{(1)}$ is 1 when both the sign of $(t_i - t_j)$ and the sign of $(a_i - a_j)$ are not opposite to that of $(x_i - x_j)$, $I_{i-j}^{(1)}$ is 0 otherwise;

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[1] Lambert (2001), pp.190-191, 199-200.

[2] Lambert (2001), pp.37-41.

[3] As already stressed in footnote 3, $R_{Y|X}$ is non-negative; it is zero when $G_Y = C_{Y|X}$.

- $I_{i-j}^{(2)}$ is 1 when the sign of $(t_i - t_j)$ is not opposite to that of $(x_i - x_j)$, and, conversely, the sign of $(a_i - a_j)$ is opposite to that of $(x_i - x_j)$; $I_{i-j}^{(2)}$ is 0 otherwise;
- $I_{i-j}^{(3)}$ is 1 when both the sign of $(t_i - t_j)$ and the sign of $(a_i - a_j)$ are opposite to that of $(a_i - a_j)$, $I_{i-j}^{(3)}$ is 0 otherwise.

Therefore, $I_{i-j}^{(g)}$ selects income unit pairs in relation to their behaviour in fulfilling Axiom 1 and Axiom 2.
In more general terms we can write as follows:

$$I_{i-j}^{(g)} = \begin{cases} 1 & \text{for income unit pairs belonging to group } (g) \\ 0 & \text{otherwise} \end{cases}, \quad (g=1, 2, 3).$$

By applying the indicator function $I_{i-j}^{(g)}$, we can decompose $C_{T|X}$ as:

$$C_{T|X} = C_{T|X}^{(1)} + C_{T|X}^{(2)} + C_{T|X}^{(3)}$$

Having in mind expression (6), $C_{T|X}^{(1)}$ can be written as

$$C_{T|X}^{(1)} = \frac{1}{2\mu_T N^2} \sum_{i=1}^{K} \sum_{j=1}^{K} (t_i - t_j) p_i p_j I_{i-j}^{T|X} I_{i-j}^{(1)} \tag{A.5}$$

When $I_{i-j}^{(1)} = 1$, that is when income unit pairs are selected from group (*1*), the equality $I_{i-j}^{T|X} = I_{i-j}^{T}$ holds, so, as (A.5) yields

$$C_{T|X}^{(1)} = \frac{1}{2\mu_T N^2} \sum_{i=1}^{K} \sum_{j=1}^{K} (t_i - t_j) p_i p_j I_{i-j}^{T} I_{i-j}^{(1)} = G_T^{(1)}. \tag{A.6}$$

Analogously, also when $I_{i-j}^{(2)} = 1$, that is when income unit pairs are selected from group (*2*), it results that $C_{T|X}^{(2)} = G_T^{(2)}$, because, for all income unit pairs from group (*2*), we have $I_{i-j}^{T|X} = I_{i-j}^{T}$.

Differently for income unit pairs from group (*3*), which are selected by $I_{i-j}^{(3)} = 1$, we have $C_{T|X}^{(3)} = -G_T^{(3)}$. This is due to the fact that in the expression

$$C_{T|X}^{(3)} = \frac{1}{2\mu_T N^2} \sum_{i=1}^{K} \sum_{j=1}^{K} (t_i - t_j) p_i p_j I_{i-j}^{T|X} I_{i-j}^{(3)}, \tag{A.7}$$

the indicator function $I_{i-j}^{T|X}$ has sign opposite to I_{i-j}^{T}; using $I_{i-j}^{T|X} = (-1) \cdot I_{i-j}^{T}$, (A.7) yields

$$C_{T|X}^{(3)} = (-1) \frac{1}{2\mu_T N^2} \sum_{i=1}^{K} \sum_{j=1}^{K} (t_i - t_j) p_i p_j I_{i-j}^{T} I_{i-j}^{(3)} = -G_T^{(3)}. \tag{A.8}$$

From $C_{T|X}^{(1)} = G_T^{(1)}$, $C_{T|X}^{(2)} = G_T^{(2)}$, and $C_{T|X}^{(3)} = -G_T^{(3)}$, it follows that

$$R_{T|X}^{(1)} = G_T^{(1)} - C_{T|X}^{(1)} = 0\,,\ R_{T|X}^{(2)} = G_T^{(2)} - C_{T|X}^{(2)} = 0\,, \text{ and } R_{T|X}^{(3)} = G_T^{(3)} - C_{T|X}^{(3)} = 2G_T^{(3)}\,. \tag{A.9}$$

Also the re-ranking index $R_{A|X}$ can be decomposed into three components:

$$R_{A|X} = R_{A|X}^{(1)} + R_{A|X}^{(2)} + R_{A|X}^{(3)}\,, \tag{A.10}$$

with

$$R_{A|X}^{(1)} = 0,\ R_{A|X}^{(2)} = 2G_{A|X}^{(2)}\,, \text{ and } R_{A|X}^{(3)} = 2G_A^{(3)}\,. \tag{A.11}$$

The first element of the sum in equation (A.10), $R_{A|X}^{(1)} = G_A^{(1)} - C_{A|X}^{(1)}$, equals zero, because $C_{A|X}^{(1)} = G_A^{(1)}$. Using formulae (5) and (6) we can write:

$$G_A^{(1)} = \frac{1}{2\mu_A N^2}\sum_{i=1}^{K}\sum_{j=1}^{K}\left(a_i - a_j\right) p_i p_j I_{i-j}^{A} I_{i-j}^{(1)}\,, \tag{A.12}$$

$$C_{A|X}^{(1)} = \frac{1}{2\mu_A N^2}\sum_{i=1}^{K}\sum_{j=1}^{K}\left(a_i - a_j\right) p_i p_j I_{i-j}^{A|X} I_{i-j}^{(1)}\,. \tag{A.13}$$

When income unit pairs are selected from group (*1*), that is when $I_{i-j}^{(1)} = 1$, the differences $\left(a_i - a_j\right)$ and $\left(x_i - x_j\right)$ have the same sign and, consequently, in (A.12) and in (A.13) the equality $I_{i-j}^{A|X} = I_{i-j}^{A}$ holds, from which it follows that $C_{A|X}^{(1)} = G_A^{(1)}$.

Conversely, both $R_{A|X}^{(2)} = G_A^{(2)} - C_{A|X}^{(2)}$ and $R_{A|X}^{(3)} = G_A^{(3)} - C_{A|X}^{(3)}$ are greater than 0.

For income units pairs from group (*2*), that is when $I_{i-j}^{(2)} = 1$, having the differences $\left(a_i - a_j\right)$ and $\left(x_i - x_j\right)$ opposite sign, it follows that $I_{i-j}^{A|X} = \left(-1\right)\cdot I_{i-j}^{A}$. Therefore, we have

$$C_{A|X}^{(2)} = \frac{1}{2\mu_A N^2}\sum_{i=1}^{K}\sum_{j=1}^{K}\left(a_i - a_j\right) p_i p_j I_{i-j}^{A|X} I_{i-j}^{(2)}$$

$$= \left(-1\right)\frac{1}{2\mu_A N^2}\sum_{i=1}^{K}\sum_{j=1}^{K}\left(a_i - a_j\right) p_i p_j I_{i-j}^{A} I_{i-j}^{(2)} = -G_A^{(2)}\,. \tag{A.14}$$

As $C_{A|X}^{(2)}$ has opposite sign with respect to $G_A^{(2)}$, it is verified that $R_{A|X}^{(2)} = 2G_A^{(2)}$.

Analogously, for what concerns $R_{A|X}^{(3)}$, as $C_{A|X}^{(3)} = -G_A^{(3)}$, we can verify that $R_{A|X}^{(3)} = 2G_A^{(3)}$.

Equations (A.6)−(A.14) illustrate equivalences and relations reported in (12) and (13).