Magnetic vector potential in secondary school: a teachers' path

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Abstract

The magnetic vector potential is traditionally presented only at university level and is widely considered as a pure mathematical tool to calculate the magnetic and the electric fields, i.e., a device without any (or at least with very poor) physical meaning. Even if, also in recent literature, many papers can be found which, on the contrary, clarify the physical meaning of the vector potential, to the best of our knowledge a clear and complete educational path on it is still missing. Our experience and some pilot experimentations with secondary school students and teachers, however, have driven us to seriously consider the opportunity to introduce the magnetic vector potential also at secondary school as a way both to better understand some fundamental aspects of classical electromagnetism and to open a door for a simple and a direct way of introducing important aspects of modern physics (i.e. the notion of the photon and the London equation of superconductivity). In this paper we'll discuss the motivations that led us to develop an educational path for the introduction of magnetic vector potential in upper secondary school, some considerations in order to clarify general aspects of its physical meaning with examples, and the framework of our course on magnetic vector potential for pre-service teachers training at the Milano TFA (Formative Active Training) course.

Keywords: secondary education, magnetic vector potential, classical electromagnetism

Introduction

The Physics Education Research Group of the University of Milano (UMIL-PERG) has been studying for many years' ways and paths for a meaningful introduction of modern physics in secondary school. For this purpose, the general framework of field theory appears more and more fruitful, and therefore an educational reconstruction of many physics topics has become a necessity. At the moment, UMIL-PERG is especially developing paths on normal modes of oscillations [1], superconductivity [2,3] and basic aspects of electromagnetism [4,5]. The educational path for the magnetic vector potential **A** is aimed at pre-service physics teachers and is part of this general research [4,5].

The magnetic vector potential **A** is very useful to treat many important quantum physics aspects, for instance the quantization of the electromagnetic field, the electromagnetic gauge theories and the Aharonov-Bohm effect [6]. Moreover it is also fundamental to understand classical phenomena, for instance the Maxwell-Lodge effect [7,8].

In dealing with superconductivity one has to face the phenomenological London equation [9]:

$$\mathbf{B} + \lambda^2 \nabla \times \nabla \times \mathbf{B} = 0, \tag{1}$$

where **B** is the magnetic field and λ is a constant called *penetration depth*. Due to the presence of differential operators, eq. (1) is too much difficult to be presented in secondary school. With the use of the vector potential **A**, we can write the substantially equivalent equation:

$$\mathbf{J}_s = -\mathrm{k}\mathbf{A}, \tag{2}$$

where $\mathbf{J_s}$ is the superconductive current density and k is a constant related to λ. To be noted that eq. (2) is much simpler than eq. (1) and is formally equal to the well-known Ohm law:

$$\mathbf{J} = \sigma\mathbf{E}.\ (3) \tag{3}$$

Therefore, the introduction of the magnetic vector potential **A** can give the opportunity to discuss London equation at secondary school level.

It is also interesting to point out that Maxwell, in his "Treatise" on electromagnetism [10], widely used the potentials, not only the electric scalar potential, but also the magnetic vector one and that, before him, also Faraday often reasoned in terms of a so called "electrotonic state" [11] that is the actual vector potential.

But nowadays **A** is generally presented only at university level and only as a useful mathematical tool, disregarding its physical meaning. In fact it is generally said that the electric and magnetic fields, **E** and **B**, and the electric scalar potential φ do have a clear physical meaning (even if φ is not uniquely defined). On the contrary, the vector potential **A** is regarded only as a simple mathematical devise, useful to perform calculations, but without a physical interpretation, even if some papers can be found in the literature [12-14] that try to clarify that vector potential does indeed have a physical meaning. What, to the best of our knowledge, is still missing, is a clear educational path on vector potential.

In the following we will present an educational path on the magnetic vector potential addressed to pre-service physics teachers with examples of teaching sequences for secondary school.

Magnetic vector potential: our educational path

Teachers' path

It was through an integral relation that Maxwell introduced the notion of vector potential ([10], p. 405). Given a magnetic field **B**, the magnetic vector potential **A** is a vector such that the flux of the magnetic field **B** through any surface Σ is equal to the circulation of **A** around the boundary ∂Σ of Σ, that is with modern symbology:

$$\int_{\Sigma} \mathbf{B}\cdot\mathbf{n}d\Sigma = \oint_{\partial\Sigma}\mathbf{A}\cdot d\mathbf{s}. \tag{4}$$

Most of the textbooks introduce **A** through the following local relation:

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}. \tag{5}$$

Therefore, as it is well-known, **A** is not univocally defined by **B**, as can be inferred by eqs. (4, 5). This fact is at the basis of the so-called gauge invariance and is one of the main reasons of the difficulties in understanding the physical meaning of **A**.

When dealing with slowly varying time-dependent fields, that is when we can neglect terms multiplied by $1/c^2$, we can express the magnetic field **B** at position **r** and time *t*, in vacuum, in terms of the conduction current density **J** at position **r'** and time *t*:

$$\mathbf{B}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int_{\tau'}\frac{\mathbf{J}(\mathbf{r}',t)\times\Delta\mathbf{r}}{(\Delta r)^3}d\tau', \tag{6}$$

where τ' is the region containing the currents, $\Delta\mathbf{r} \equiv \mathbf{r}-\mathbf{r}'$ and $\Delta r \equiv |\mathbf{r}-\mathbf{r}'|$.

With rather simple calculations [5], eq. (6) can be rearranged as:

$$\mathbf{B}(\mathbf{r},t) = \nabla \times \left(\frac{\mu_0}{4\pi}\int_{\tau'}\frac{\mathbf{J}(\mathbf{r}',t)}{\Delta r}d\tau'\right), \tag{7}$$

that clearly shows that the vector:

$$\mathbf{A}(\mathbf{r},t) \equiv \frac{\mu_0}{4\pi}\int_{\tau'}\frac{\mathbf{J}(\mathbf{r}',t)}{\Delta r}d\tau' \tag{8}$$

is a magnetic vector potential because it satisfies eq. (5). Moreover, in the framework of our slowly varying time-dependent approximation, it is *the* magnetic vector potential to which one is naturally led. As it is seen in eq. (8), the potential **A** is explicitly expressed in terms of its empirical references, that are the conduction currents, and its behavior follows that of the currents.

It is interesting to observe that eq. (8) has the same structure of:

$$\varphi(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\int_{\tau'}\frac{\rho(\mathbf{r}',t)}{\Delta r}d\tau', \tag{9}$$

where φ is the electric scalar potential and ρ is the charge density.

Eq. (8) tells us that, once the currents are known, **A** is univocally determined. Instead, if we would start from eq. (5), **A** would not be unique. To completely determine **A** one has to choose a gauge condition and this is generally done by arbitrary fixing the divergence of **A**. In our approach, on the contrary, we have no necessity of fixing a gauge. By directly calculating the divergence of **A** given by eq. (8), always in the limit of slowly varying fields, we find that we are in the Coulomb gauge, that is:

$$\nabla \cdot \mathbf{A} = 0. \tag{10}$$

The Coulomb gauge can therefore be seen as the ‘natural’ gauge for slowly varying fields.

It is now important to give a physical meaning to the magnetic vector potential. Suppose that in a certain space region a slow varying magnetic field, generated by a distribution of currents, is acting. If $\nabla\varphi = 0$, as for instance when no free charges are present, the electric field is linked to **A** by the following relation:

$$\mathbf{E}(\mathbf{r},t) = -\frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t}. \tag{11}$$

Let us now consider a point-like charge q, in the position **r** at a time $t = -\infty$, when the currents, and consequently both the magnetic field and the magnetic vector potential are zero. When we slowly switch the currents on, they will generate a magnetic field **B,** a magnetic vector potential **A** and therefore, according to eq. (11), an electric field **E** which will act on the charge q. If we want to keep the charge fixed in **r**, we must apply an impulse against the field forces, given by:

$$\mathbf{I}_q(\mathbf{r},t) = -\int_{-\infty}^{t} q\,\boldsymbol{E}(\mathbf{r},t')dt' = -\int_{-\infty}^{t} -q\frac{\partial \mathbf{A}(\mathbf{r},t')}{\partial t'}dt' = q\mathbf{A}(\mathbf{r},t). \tag{12}$$

The magnetic vector potential can be thus interpreted as the total momentum per unit charge that must be transferred to a charge, during the time interval $(-\infty, t)$, in order to keep it at rest at the point **r**, while the field slowly varies from zero to the value **B**.

The analogy between the structures of the scalar and the vector potentials, as given by eqs. (8, 9), is also reflected in their physical interpretations. When the magnetic vector potential is time-independent, the potential energy $U_q(\mathbf{r}, t)$ of a point-like charge q set at position **r** at time t, is given by:

$$U_q(\mathbf{r},t) = -\int_{\infty}^{\mathbf{r}} q\,\mathbf{E}(\mathbf{r}',t)\cdot d\mathbf{r}' = q\varphi(\mathbf{r},t). \tag{13}$$

Therefore the electric scalar potential represents the work (independent of the chosen path), per unit charge, done to move the charge from infinity, where the electric field is zero, to the point **r**, against the forces of the electric field, at a given time t.

Eqs. (12, 13) have a dual structure, in the sense that in the former the integration is performed over time at a fixed position, while in the latter the integration is performed over space at a fixed time. In other words, the roles of space and time are interchanged. The magnetic vector potential can therefore be seen as a 'momentum vector' per unit charge, while the electric scalar potential is an energy component per unit charge. As $q\varphi$ is called potential energy, $q\mathbf{A}$ can be called potential momentum (of the charge q, at point **r** and time t).

The vector potential gives also the possibility to write some physical relations in a more understandable way, deepening the physical meaning previously discussed. For example, let's consider an electromagnetic, linearly polarized, harmonic, plane wave of amplitude E_0 and angular frequency ω, propagating in vacuum. Its intensity is usually written as:

$$I = \frac{1}{2}\varepsilon_0 E_0^2 c. \tag{14}$$

In terms of the vector potential, eq. (14) can also be written as:

$$I = \frac{1}{2}\varepsilon_0 A_0^2 \omega^2 c, \tag{15}$$

where A_0 is the amplitude of the vector potential. This last equation reflects in a complete way that giving the intensity of a mono-dimensional mechanical wave:

$$I = \frac{1}{2}\rho S_0^2 \omega^2 \mathrm{v}, \tag{16}$$

where ρ is the linear mass density, S_0 is the wave amplitude and v the propagation velocity. Eq. (16) shows that **A** plays for the electromagnetic field the same role of the displacement S for a mechanical wave. Moreover, in this simple case, eq. (15) gives also a way of measuring the vector potential; in fact since the intensity, the frequency and the velocity of a wave are measurable quantities, from eq. (15) we can determine A_0 [12].

Students' path

In secondary school, the scalar potential plays an important role but, of course, only a simplified version of eq. (9) is presented. In general, with obvious symbology, in the case of a continuous charge distribution, the electric scalar potential (that we assume to go to zero at infinity) can be written as:

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{\Delta Q_1}{|r_1 - r|} + \frac{\Delta Q_2}{|r_2 - r|} + \cdots + \frac{\Delta Q_N}{|r_N - r|} + \cdots\right), \tag{17}$$

where it is clear that the sources of the potential are the electric charges. In complete analogy with eq. (17), we propose to introduce the new vector, related to magnetic effects:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left(\frac{i_1\Delta\mathbf{l}_1}{|r_1 - r|} + \frac{i_2\Delta l_2}{|r_2 - r|} + \cdots + \frac{i_N\Delta l_N}{|r_N - r|} + \cdots\right), \tag{18}$$

where i_k is the current circulating in the section $\Delta\mathbf{l}_k$ of the circuit and $\Delta\mathbf{l}_k$ is a vector oriented as the current.

One of the most important theorem about the electric field is the Gauss theorem that is widely used to calculate the electric field in presence of particular symmetries. However, in secondary school this theorem is generally only stated without demonstration, except for the case of a point-like charge in the centre of a spherical surface. For what concerns the magnetic vector potential, eq. (4) may be written with a notation more suited to secondary school. In fact, if we denote with $C_\gamma(\mathbf{A})$ the circulation of the magnetic vector potential **A** along the closed line γ and with $\Phi_\Sigma(\mathbf{B})$ the flux of the magnetic field **B** through any surface Σ with boundary γ and oriented like γ, we can write:

$$\Phi_\Sigma(\boldsymbol{B}) = C_\gamma(\mathbf{A}), \tag{19}$$

and this equation allows to calculate **A** in some simple cases.

As an example, let us consider an infinite solenoid of radius a, carrying a current I that generates an internal magnetic field of intensity $B = \mu_0 nI$, where n is the number of turns per unit length, while the external magnetic field is zero. We want to calculate **A** both inside and outside the solenoid.

Eq. (18) indicates that the field lines of **A** follow the current direction, therefore, for symmetry reasons they are circumferences co-axial with the solenoid (Figure 1).

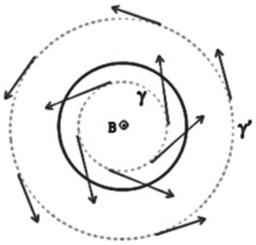

Figure 1. Section of an infinite solenoid, carrying a current, with the field lines of **A**

Inside the solenoid, if r is the distance from the symmetry axis, from eq. (19), taking the field lines of **A** as lines to calculate circulation, we get:

$$\pi r^2 B = 2\pi r A, \tag{20}$$

giving:

$$A = \frac{1}{2}Br = \frac{1}{2}\mu_0 nIr. \tag{21}$$

Outside the solenoid, the same procedure leads to:

$$\pi a^2 B = 2\pi r A, \tag{22}$$

giving

$$A = \frac{1}{2}\frac{a^2 B}{r} = \frac{1}{2}\frac{a^2}{r}\,\mu_0 nI. \tag{23}$$

Eqs. (21, 23) show that the magnetic vector potential grows linearly, while approaching the current, inside the solenoid, whereas it decreases as $1/_r$ going away from the current, outside the solenoid. Figure 2 displays the r-behaviour of **A** both inside and outside the solenoid.

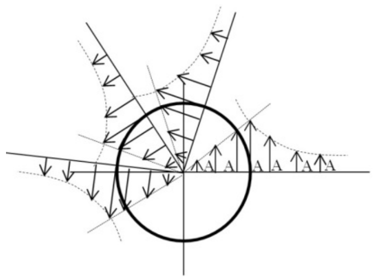

Figure 2. Behaviour of **A** both inside and outside the solenoid

Other examples of calculation of the magnetic vector potential can be found in [3-5].

Conclusions

Two main facts hinder the comprehension of the magnetic vector potential:

- the non-univocity of **A** implied by its definition (eq. (5));
- the lack of discussion traditionally devoted to its physical meaning.

Convinced of the educational value of the vector potential in dealing with many physical situations, we have developed a path which, in our opinion, can overcome the above stated difficulties.

We attained a particular expression of **A** in terms of its empirical referent, i.e. the conduction current density, for slowly time-dependent electric and magnetic fields.

We found a privileged gauge (the Coulomb gauge) and the physical meaning of **A** was discussed in a way similar to that of the electric scalar potential, another fact which can help comprehension.

In some circumstances, the use of the vector potential allowed us to highlight interesting parallelisms with mechanical situations.

To conclude, we firmly consider the introduction of the magnetic vector potential in electromagnetism not only a good tool for making calculations, but also a useful way to better understand many physical phenomena.

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