Inverse problems for universal deformation rings of group representations

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Contents

Basic notation and conventions

All considered commutative rings are assumed to have an identity element. The morphisms between them are required to preserve the identity elements.

Given a positive integer n, we will denote by $[n]$ the set $\{1, 2, \ldots, n\}$.

For a positive integer n and an abelian group A the additive group of $n \times n$ matrices over A will be denoted by $M_n(A)$. We adopt the convention of writing $M(i, j)$ for the (i, j)-entry of $M \in M_n(A)$.

We reserve the symbol I_n for the $n \times n$ identity matrix. Given $i, j \in [n]$ we denote by e_{ij} the $n \times n$ matrix having only one non-zero entry, which is 1 at the (i, j)-th place.

Let F, G be functors between categories $\mathcal{D}$ and $\mathcal{E}$. If $\Phi : F \to G$ is a natural transformation then for $A \in \mathrm{Ob}(\mathcal{D})$ we denote the induced map $F(A) \to G(A)$ by Φ_A. Moreover, in case $\mathcal{E} = \mathrm{Sets}$ we say that Φ is injective (surjective) if and only if Φ_A is injective (surjective) for every $A \in \mathrm{Ob}(\mathcal{D})$.

If G, H are topological groups then $\mathrm{CHom}(G, H) := \{f \in \mathrm{Hom}(G, H) \mid f \text{ continuous}\}$.

Introduction and an overview of the main results

General introduction

Group representations

One of the most commonly studied algebraic structures is that of a group. Many groups have a "geometric flavour" and occur as groups of transformations of vector spaces. The dihedral, orthogonal and general linear groups serve here as examples. Representation theory deals with the problem of presenting abstract groups in such a geometric way.

Choosing a coordinate approach, one can say that representation theory aims at presenting abstract groups in the form of matrices with coefficients in a chosen field. More precisely, given a group G and a field k, by an n-dimensional group representation of G over k we understand a group homomorphism $G \to \mathrm{GL}_n(k)$.

Deformations of group representations

In this thesis we work with profinite groups and their continuous representations over finite fields. Every such representation can be "deformed" to representations over certain type of rings.

More specifically, suppose R is a complete, local and noetherian ring with finite residue field k and denote by $\pi : R \twoheadrightarrow k$ the reduction modulo the maximal ideal of R. For $n \in \mathbb{N}$, we denote by the same symbol the reduction $\pi : \mathrm{GL}_n(R) \to \mathrm{GL}_n(k)$. Given a profinite group G and a continuous representation $\bar{\rho} : G \to \mathrm{GL}_n(k)$, we define a *lift* of $\bar{\rho}$ to R as a continuous group homomorphism $\rho : G \to \mathrm{GL}_n(R)$ such that $\bar{\rho} = \pi \circ \rho$ (Definition 2.8).

$$\begin{tikzcd} & \mathrm{GL}_n(R) \arrow[d, "\pi"] \\ G \arrow[ur, dashed, "\rho"] \arrow[r, "\bar{\rho}"] & \mathrm{GL}_n(k) \end{tikzcd}$$

A *deformation* is a lift up to a conjugation by a matrix in the kernel of π (Definition 2.9).

Motivation

The described setup originates from number theory, where one studies representations of Galois groups ("Galois representations") over finite fields and their deformations ("Galois deformations") to, for example, p-adic representations. Such techniques were used by Andrew Wiles in his famous paper proving the Fermat's Last Theorem. However, in this thesis we work in a purely abstract setting, with groups and representations not necessarily coming from number theory.

Functoriality

The complete, local and noetherian rings with a given finite residue field k form a category, which we denote by $\hat{\mathcal{C}}$ (cf. Definition 1.2). Associating to $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ the set $\mathrm{Def}_{\bar{\rho}}(R)$ of all deformations of $\bar{\rho}$ to R we obtain the *deformation functor* $\mathrm{Def}_{\bar{\rho}} : \hat{\mathcal{C}} \to \mathrm{Sets}$.

It is a basic fact of deformation theory that under some mild assumptions $\mathrm{Def}_{\bar{\rho}}$ is representable (in the sense of category theory), cf. Proposition 2.23. If this is the case, the object representing it is called the universal deformation ring of $\bar{\rho}$.

The inverse problem

The question that is central in this thesis is the so called inverse problem for universal deformation rings of group representations:

For which $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ does there exist a profinite group G and a continuous group representation $\bar{\rho} : G \to \mathrm{GL}_n(k)$ such that R is the universal deformation ring of the resulting deformation functor?

This problem originates from a question asked by Matthias Flach, who wondered whether the universal deformation rings need to be complete intersections ([Chi]). This question can be motivated by the fact, that the universal deformation rings occurring in the context of Galois representations and arithmetic geometry in many cases do satisfy this condition (see for example results of Böckle in [Boe2, Boe3]).

The question of Flach was settled by Bleher and Chinburg ([BC1, BC2]) with a counterexample $\mathbb{Z}_2[X]/(X^2, 2X)$, for which Byszewski gave an alternative argument in [By]. The explicit formulation of the problem is due to Bleher, Chinburg and de Smit ([BCdS]), who also generalized the mentioned counterexample. Namely, they showed that, denoting by $\mathrm{W}(k)$ the ring of Witt vectors over the finite field k, the ring $\mathrm{W}(k)[X]/(X^2, p^n X)$ is a universal deformation ring of a group representation for every $n \in \mathbb{N}$. Another interesting class of universal deformation rings that are not complete intersections was obtained by Rainone: $\mathbb{Z}_p[[X]]/(p^n, p^m X)$, for $p > 3$ and $n, m \in \mathbb{N}$, $1 \leqslant m \leqslant n$ ([Ra]). This construction has also disproved some other conjectures on the structure of universal deformation rings.

The status of the problem at the beginning of the author's PhD project was as follows. On the one hand, no example of a ring $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ that is not a universal deformation ring was known. On the other hand, techniques for producing a representation with a given deformation ring were very limited.

As we will see in Chapter 5, actually every $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ can be realized as a universal deformation ring of some profinite group representation. The proof of this fact relies on a careful analysis of some natural representations of special linear groups. This is the most important result of the thesis.

A modification of the inverse problem

In this thesis we also discuss a variant of the inverse problem in which we wonder which rings occur as universal deformation rings of groups that are finite. The second most important result of the thesis is a non-trivial necessary condition for such rings of characteristic zero, presented in Chapter 6.

Content of the thesis

The first two chapters are almost exclusively devoted to recalling definitions and standard facts. The original contribution of the thesis is presented in the subsequent four chapters and its core are the results of the last two of them. We briefly describe now the content of each of the chapters.

Chapter 1

For the reader's convenience, we introduce the category $\hat{\mathcal{C}}$ and discuss properties of its objects and morphisms. We also discuss category theoretic results related to functors $\hat{\mathcal{C}} \to \text{Sets}$ and the problem of their representability.

A reader familiar with these topics will find the content of this chapter very standard, but should at least take a look at Theorem 1.16 and the following remarks, which seem to appear in the literature less frequently.

Chapter 2

In the second chapter we concentrate on deformation functors of group representations and introduce all basic notions needed in the rest of the thesis. Also the content of this chapter is rather standard, but, compared to other authors, we avoid making some customary finiteness assumptions (see section 2.3.2). We also devote slightly more attention to the concept of a versal deformation ring. For instance, we comment on some inconsistency of definitions used by different authors (section 2.3.1) and present examples of versal deformation rings that are not universal (Lemma 2.37).

Chapter 3

We vastly generalize the construction of the two-dimensional representation considered by Rainone in [Ra, Chapter 5] and analyze its deformation functor. The main result of the chapter is Theorem 3.11 from which we can conclude, among others, that for $k \neq \mathbb{F}_2, \mathbb{F}_3$, $m \in \mathbb{N}$ and arbitrary positive integers $k_0, k_1, \ldots, k_m$, the ring

$$R_u := \mathrm{W}(k)[[X_1, \ldots, X_m]]/(p^{k_0}, p^{k_1}X_1, \ldots, p^{k_m}X_m)$$

can be obtained as a universal deformation ring of a finite group representation (Corollary 3.14). This result links Chapter 3 with Chapter 6. Furthermore, some other results (see Remark 3.6) find their application in Chapter 5.

Chapter 4

This chapter does not contain a main result, but consists of a collection of several results that can be useful when studying the deformation rings; some of them will be applied in Chapter 5. The problems which we address in this chapter include the following questions:

- Given a deformation functor with a universal deformation ring R_u, how can one determine quotients of R_u without knowing this ring?
- How does the universal deformation ring change when passing to representations of subgroups or to representations of quotient groups?

A more detailed discussion of the obtained results can be found at the beginning of Chapter 4.

Chapter 5

This is the most important chapter of the thesis and it contains a solution to the inverse problem. Namely, we show that every $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ can be obtained as a universal deformation ring of some group representation.

More precisely, given $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ and $n \geqslant 2$ we consider the special linear group $G := \mathrm{SL}_n(R)$ together with its representation $\bar{\rho} : G \to \mathrm{GL}_n(k)$ induced by the reduction $R \twoheadrightarrow k$. Our analysis of the resulting deformation functors may be summarized as follows:

Theorem (Theorem 5.1)**.** *Under the above assumptions, R is the universal deformation ring of $\bar{\rho}$ if and only if $(n, k) \notin \{(2, \mathbb{F}_2), (2, \mathbb{F}_3), (2, \mathbb{F}_5), (3, \mathbb{F}_2)\}$.*

We also identify some universal deformation rings occurring in the exceptional cases, not covered by the above theorem (Proposition 5.17, Proposition 5.19). We conclude the chapter by discussing deformations of analogous representations of the closed subgroups of $\mathrm{GL}_n(R)$ containing $\mathrm{SL}_n(R)$ (Proposition 5.24, Corollary 5.25).

Chapter 6

In the last chapter we address a modification of the inverse problem and ask which rings occur as universal deformation rings of representations of *finite* groups.

It is relatively easy to observe that this problem has a different answer than the general inverse problem. We show that every finite $\hat{\mathcal{C}}$-ring can be obtained as a universal deformation ring of a finite group representation (Observation 6.2) and that some, but not all, infinite $\hat{\mathcal{C}}$-rings can be obtained in this context as well (see section 6.1). Hence, the interesting part of the problem is to distinguish the infinite rings that are universal deformation rings of finite group representations from the ones that are not. We do not solve this problem completely, but provide the following partial result.

Theorem (Theorem 6.30)**.** *Let* $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ *be of characteristic zero and a universal deformation ring of some finite group representation. Then* $R/\bigcup_{r=1}^{\infty} \mathrm{Ann}\, p^r$ *is reduced and has Krull dimension* 1.

In particular, the power series rings $\mathrm{W}(k)[[X_1, \ldots, X_m]]$, $m > 0$, over the ring of Witt vectors $\mathrm{W}(k)$ do not occur as universal deformation rings in the new setup.

It is interesting to note that while the initial inverse problem was solved using more group theoretic methods, our analysis of the second inverse problem is based on commutative algebra results.

Chapter 1

Complete noetherian local rings

The aim of this chapter is to recall several standard facts from commutative algebra and category theory that will be crucial for the rest of the thesis. We introduce the basic definitions, set the notation and recall the main properties that can be found in the literature. Proofs are omitted for brevity of the exposition.

1.1 Categories $\hat{\mathcal{C}}$ and $\mathcal{C}$

1.1.1 Definitions

The rings that are of main interest in this thesis are the complete, noetherian and local ones. Moreover, we will require their residue fields to be finite.

Notation 1.1. *The following notation will be widely used throughout the thesis:*

- *we reserve the symbols k and p for a finite field and its characteristic,*
- *the symbol $\mathrm{W}(k)$ stands for the ring of Witt vectors over k.*
- *whenever an element of some ring is denoted by ε, it is assumed that $\varepsilon \neq 0$ and $\varepsilon^2 = 0$. In particular, $k[\varepsilon] \cong k[X]/(X^2)$.*

Definition 1.2. Let k be a finite field. We will denote by $\hat{\mathcal{C}}$ the category of all complete noetherian local commutative rings with residue field k. Morphisms of $\hat{\mathcal{C}}$ are the local ring homomorphisms inducing the identity on k.

Definition 1.3. By $\mathcal{C}$ we will denote the full subcategory of artinian rings in $\hat{\mathcal{C}}$.

In what follows we will refer to the objects and morphisms of the category $\hat{\mathcal{C}}$ shortly as "$\hat{\mathcal{C}}$-rings" and "$\hat{\mathcal{C}}$-morphisms" (and analogously for the objects and morphisms of $\mathcal{C}$).

Remark 1.4. It is easy to check that, due to the finiteness of k, the category $\mathcal{C}$ coincides with the category of all finite $\hat{\mathcal{C}}$-rings.

Example 1.5. The ring $\mathrm{W}(k)$ is an object of $\hat{\mathcal{C}}$, but not of $\mathcal{C}$. The rings k and $k[\varepsilon]$ are examples of objects of both $\mathcal{C}$ and $\hat{\mathcal{C}}$.

The ring $k[\varepsilon]$ can be seen as a particular case of the following construction.

Example 1.6. We can identify the category $\mathfrak{V}$ of finite dimensional k-vector spaces with a full subcategory of $\mathcal{C}$. If $V \in \mathfrak{V}$, then we introduce the ring structure on the k-vector space $k[V] := k \oplus V$ by requiring $V^2 = 0$ and obtain an object of $\mathrm{Ob}(\mathcal{C})$. Moreover, for every V, $W \in \mathfrak{V}$ there is a bijective correspondence $f \leftrightarrow \mathrm{id} \oplus f$ between k-linear maps $f : V \to W$ and morphism $k[V] \to k[W]$ of $\mathcal{C}$.

Notation 1.7. *Let R be a $\hat{\mathcal{C}}$-ring. We will use the following notation:*

- $\mathfrak{m}_R$ *denotes the maximal ideal of R,*
- $R^\times$ *denotes the multiplicative group of R and $R^\times_{\equiv 1}$ denotes its subgroup $1 + \mathfrak{m}_R$,*
- μ_R *denotes the set $\{x \in R \mid x^{\#k-1} = 1\}$ of multiplicative representatives of the non-zero residue classes modulo $\mathfrak{m}_R$,*
- $\tau_R : k^\times \to \mu_R$ *denotes the Teichmüller lift of $k^\times$ to R.*

Remark 1.8. Note that, using the introduced notation, we have $R^\times \cong \mu_R \times R^\times_{\equiv 1}$.

The existence of the Teichmüller lift is a consequence of the following general and very useful property of complete rings.

Theorem 1.9 (Hensel's lemma). *Let R be a ring that is complete with respect to an ideal I and let $f \in R[X]$ be a polynomial. If $a \in R$ is such that $f'(a)$ is invertible and $f(a) \equiv 0 \pmod{I}$, then there exists a uniquely determined $b \in R$ such that $f(b) = 0$ and $b \equiv a \pmod{I}$.*

Proof. See [Ei, Theorem 7.3]. □

In some of our arguments we will also use the following easy and well-known result.

Lemma 1.10. *Every surjective endomorphism of a noetherian ring is an automorphism.*

Finally, since we will very often be working with reductions modulo ideals and with quotient rings, we also introduce the following convention.

Notation 1.11. *For $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ and a proper ideal $I \lhd R$ the symbol π_I will denote the reduction homomorphism $R \to R/I$.*

Remark 1.12. Note that π_I, defined as above, is always a $\hat{\mathcal{C}}$-morphism. Indeed, it is clear that R/I is a local noetherian ring and that π_I induces an isomorphism on the residue fields. It is only less obvious that R/I is complete. Observe that its $\mathfrak{m}_{R/I}$-adic completion $\widehat{R/I}$ is isomorphic to $R/\widehat{I}$ ([Ei, Lemma 7.15]) and that $\widehat{I} = I$ follows from Krull's intersection theorem. Hence, $R/I \cong \widehat{R/I}$ is complete.

1.1.2 Structure theorems

The structure of complete noetherian local rings (with arbitrary residue fields) was studied by I. S. Cohen already in 1940's in his paper [Coh]. For the reader's convenience we quickly present here the most important implications of Cohen's results for $\hat{\mathcal{C}}$-rings. We refer to the original paper, but an interested reader can learn this topic also from popular books on commutative algebra, like [Mat] or [Ei].

Theorem 1.13. *Every $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ is a quotient of a power series ring in finitely many variables over $\mathrm{W}(k)$. Moreover, it contains precisely one ring that is a homomorphic image of $\mathrm{W}(k)$.*

Proof. See [Coh, Theorems 9 and 10.(b)] for the case $\operatorname{char} R = p$ and [Coh, Theorems 11, 12 and 13] for the case $\operatorname{char} R \neq p$. $\square$

Note that Remark 1.12 implies a statement converse to the first claim: every quotient of a power series ring in finitely many variables over $\mathrm{W}(k)$ is in $\mathrm{Ob}(\hat{\mathcal{C}})$.

Corollary 1.14. *All $\hat{\mathcal{C}}$-rings have a natural $\mathrm{W}(k)$-algebra structure and $\hat{\mathcal{C}}$-morphisms coincide with local $\mathrm{W}(k)$-algebra homomorphisms.*

Remark 1.15. For a given $R \in \mathrm{Ob}(\hat{\mathcal{C}})$, the structure map $\mathrm{W}(k) \to R$ takes $\mu_{\mathrm{W}(k)}$ to μ_R. In some applications we will find it useful to identify these two groups, cf. for example Definition 3.2.

We will also need the following result, which can be interpreted as an analog of E. Noether's normalization theorem.

Theorem 1.16. *Let $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ be such that either $\operatorname{char} R = 0$ and $\operatorname{ht} pR = 1$ or $\operatorname{char} R = p$. Then there exists a subring R_0 of R such that R_0 is isomorphic to a power series ring over $\mathrm{W}(k)/(\operatorname{char} R)$ and R is a finite R_0-module.*

Proof. See [Coh, Theorem 16]. $\square$

Remark 1.17. The condition $\operatorname{ht} pR = 1$ is satisfied for example when p is not a zero-divisor in R (this is a consequence of Krull's "Hauptidealsatz").

Remark 1.18. Suppose R and R_0 are as in Theorem 1.16 and let $d := \dim R$ be the Krull dimension of R. By the properties of integral extensions, $\dim R = \dim R_0$, so $R_0 \cong k[[X_1, \ldots, X_d]]$ in the case $\operatorname{char} R = p$ and $R_0 \cong \mathrm{W}(k)[[X_1, \ldots, X_{d-1}]]$ in the case $\operatorname{char} R = 0$.

The structure of $\hat{\mathcal{C}}$-rings can also be better understood using the following observation connecting categories $\hat{\mathcal{C}}$ and $\mathcal{C}$.

Lemma 1.19. *Every $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ is an inverse limit of $\mathcal{C}$-rings.*

Proof. For every $r \in \mathbb{N}$ the ring $R/\mathfrak{m}_R^r$ is artinian and $R \cong \varprojlim_{r \in \mathbb{N}} R/\mathfrak{m}_R^r$. $\square$

Remark 1.20. Note that the converse statement is not true, i.e., not every limit of an inverse system of $\mathcal{C}$-rings is a $\hat{\mathcal{C}}$-ring. Indeed, such inverse limit need not be noetherian.

Corollary 1.21. *Every $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ is a profinite ring.*

Proof. Combine the above lemma with Remark 1.4. □

Corollary 1.22. *For every R, $S \in \mathrm{Ob}(\hat{\mathcal{C}})$ we have*

$$\mathrm{Hom}_{\hat{\mathcal{C}}}(R,S) \;=\; \varprojlim_{r\in\mathbb{N}} \mathrm{Hom}_{\mathcal{C}}(R/\mathfrak{m}_R^r, S/\mathfrak{m}_S^r).$$

Proof. It is sufficient to combine the following two facts: $\mathrm{Hom}_{\hat{\mathcal{C}}}(R,S) = \varprojlim_{r\in\mathbb{N}} \mathrm{Hom}_{\hat{\mathcal{C}}}(R, S/\mathfrak{m}_S^r)$ and $\mathrm{Hom}_{\hat{\mathcal{C}}}(R, S/\mathfrak{m}_S^r) \cong \mathrm{Hom}_{\mathcal{C}}(R/\mathfrak{m}_R^r, S/\mathfrak{m}_S^r)$ for every $r \in \mathbb{N}$. □

1.1.3 Some categorical constructions

Fiber products

Definition 1.23. Given two $\hat{\mathcal{C}}$-morphisms $\pi_1 : R_1 \to S$ and $\pi_2 : R_2 \to S$ let us define

$$R_1 \times_S R_2 := \{(r_1, r_2) \in R_1 \times R_2 \mid \pi_1(r_1) = \pi_2(r_2)\}.$$

We will consider this set with the subring structure inherited from the ring $R_1 \times R_2$. For $i = 1, 2$, the canonical projections $R_1 \times_S R_2 \to R_i$ will be denoted by p_i.

$$\begin{tikzcd}[column sep=small] & R_1 \times_S R_2 \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ R_1 \arrow[dr, "\pi_1"'] & & R_2 \arrow[dl, "\pi_2"] \\ & S & \end{tikzcd}$$

Example 1.24. If V, $W \in \mathfrak{V}$ then $k[V] \times_k k[W] \cong k[V \oplus W]$.

Lemma 1.25. *Consider the setup of Definition 1.23 and set $\tilde{R} := R_1 \times_S R_2$. Then:*

(i) If R_1, $R_2 \in \mathrm{Ob}(\mathcal{C})$ then $\tilde{R} \in \mathrm{Ob}(\mathcal{C})$.

(ii) If π_1, π_2 are surjective then $\tilde{R} \in \mathrm{Ob}(\hat{\mathcal{C}})$.

(iii) *If* $\tilde{R} \in \mathrm{Ob}(\hat{\mathcal{C}})$ *then it is the fiber product (in the category* $\hat{\mathcal{C}}$*) of* π_1 *and* π_2*. If* R_1*,* $R_2 \in \mathrm{Ob}(\mathcal{C})$ *then it is the fiber product of* π_1 *and* π_2 *also in* $\mathcal{C}$*.*

Sketch of the proof. Let $\mathfrak{m} := \mathfrak{m}_{R_1} \times \mathfrak{m}_{R_2}$ and $\tilde{\mathfrak{m}} := \mathfrak{m} \cap \tilde{R}$. We see that $\tilde{R}/\tilde{\mathfrak{m}} \cong k$, so $\tilde{\mathfrak{m}}$ is a maximal ideal of $\tilde{R}$. It is actually its only such ideal, since $\tilde{R}\backslash\tilde{\mathfrak{m}} \subseteq (R_1\backslash\mathfrak{m}_{R_1} \times R_2\backslash\mathfrak{m}_{R_2}) \cap \tilde{R} = \tilde{R}^\times$. Moreover, as a closed subring of the $\mathfrak{m}$-adically complete ring $R_1 \times R_2$, the ring $\tilde{R}$ is $\tilde{\mathfrak{m}}$-adically complete. We conclude that $\tilde{R}$ is in $\mathrm{Ob}(\hat{\mathcal{C}})$ if and only if it is noetherian.

If R_1 and R_2 are artinian, hence finite, then so is $\tilde{R}$ (see also Remark 1.4). Suppose now that π_1 and π_2 are surjective. Then so are p_1 and p_2. Let $K_i := \ker p_i$ $(i = 1, 2)$ and observe that $K_1 \cap K_2 = \{0\}$. Since $\tilde{R}/K_2 \cong R_2$ is a noetherian $\tilde{R}$-module, so is its submodule $(K_1 + K_2)/K_2 \cong K_1/K_1 \cap K_2 = K_1$. We conclude that both K_1 and $\tilde{R}/K_1 \cong R_1$ are noetherian $\tilde{R}$-modules, so $\tilde{R}$ is noetherian as well.

The above arguments prove the first two claims. The last one can be easily deduced from the following facts. Firstly, $\tilde{R}$ is the fiber product of R_1 and R_2 in the category of rings. Secondly, p_1 and p_2 are $\hat{\mathcal{C}}$-morphisms ($\mathcal{C}$-morphisms in case R_1 and R_2 are artinian). □

Remark 1.26. In general $R_1 \times_S R_2$ need not be an object of $\hat{\mathcal{C}}$. For example, Mazur presents in [Maz1, p. 270] the following example, accredited to Brian Conrad:

$$\pi_1 : k[[X, Y]] \xrightarrow{\text{mod } Y} k[[X]], \qquad \pi_2 : k \hookrightarrow k[[X]].$$

The resulting ring $k[[X, Y]] \times_{k[[X]]} k \cong k + Yk[[X, Y]]$ is not noetherian. Indeed, its ideal $(Y, YX, YX^2, \ldots)$ is not finitely generated.

Coproducts

Let R_1, $R_2 \in \mathrm{Ob}(\hat{\mathcal{C}})$ be given. It is known that, given a ring R, the coproduct in the category of commutative R-algebras is described by the tensor product. One can therefore expect the coproduct of R_1 and R_2 in $\hat{\mathcal{C}}$ to be related to $R_1 \otimes_{\mathrm{W}(k)} R_2$. Since this last ring does not necessarily belong to $\mathrm{Ob}(\hat{\mathcal{C}})$ (for example: it need not be complete), we make the following definition.

Definition 1.27. We define the completed tensor product $R_1 \widehat{\otimes}_{\mathrm{W}(k)} R_2$ of $R_1,\ R_2 \in \mathrm{Ob}(\hat{\mathcal{C}})$ as the completion of $R_1 \otimes_{\mathrm{W}(k)} R_2$ with respect to the maximal ideal $\mathfrak{m}_{R_1} \otimes R_2 + R_1 \otimes \mathfrak{m}_{R_2}$.

Lemma 1.28. *For every $R_1,\ R_2 \in \mathrm{Ob}(\hat{\mathcal{C}})$ the completed tensor product $R_1 \widehat{\otimes}_{\mathrm{W}(k)} R_2$ is an object of $\hat{\mathcal{C}}$ and the coproduct (in category $\hat{\mathcal{C}}$) of R_1 and R_2.*

Sketch of the proof. One can check that $R_1 \widehat{\otimes}_{\mathrm{W}(k)} R_2$ has also the following alternative descriptions (cf. [Maz1, §12]):

- $R_1 \widehat{\otimes}_{\mathrm{W}(k)} R_2 := \varprojlim_{k \in \mathbb{N}} \left(R_1/\mathfrak{m}_{R_1}^k \otimes_{\mathrm{W}(k)} R_2/\mathfrak{m}_{R_2}^k \right)$,
- If $R_1 \cong \mathrm{W}(k)[[X_1, \ldots, X_n]]/(f_1, \ldots, f_s)$, $R_2 \cong \mathrm{W}(k)[[Y_1, \ldots, Y_m]]/(g_1, \ldots, g_r)$ then:

$$R_1 \widehat{\otimes}_{\mathrm{W}(k)} R_2 \cong \mathrm{W}(k)[[X_1, \ldots, X_n, Y_1, \ldots, Y_m]]/(f_1, \ldots, f_s, g_1, \ldots, g_r).$$

It is clear from the definition that $R_1 \widehat{\otimes}_{\mathrm{W}(k)} R_2$ is complete and local with residue field $k \otimes_{\mathrm{W}(k)} k \cong k$. The second of the above alternative descriptions shows that $R_1 \widehat{\otimes}_{\mathrm{W}(k)} R_2$ is noetherian, while the first one, combined with Corollary 1.22, can be used for proving that $R_1 \widehat{\otimes}_{\mathrm{W}(k)} R_2$ is the coproduct in category $\hat{\mathcal{C}}$. □

1.1.4 Tangent space

Definition 1.29. We define the cotangent space to $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ as the k-vector space $t_R^* := \mathfrak{m}_R/(\mathfrak{m}_R^2, p)$ and the tangent space as $t_R := \mathrm{Hom}_k(t_R^*, k)$. Given a $\hat{\mathcal{C}}$-morphism $R \to S$ we denote by $t_f^* : t_R^* \to t_S^*$ the k-linear map induced by f.

Remark 1.30. Note that $R/(\mathfrak{m}_R^2, p) \cong k \oplus t_R^* = k[t_R^*]$.

One reason why this notion turns out to be very useful in the study of complete noetherian local rings is the following lemma.

Lemma 1.31. *A $\hat{\mathcal{C}}$-morphism $f : R \to S$ is surjective if and only if $t_f^* : t_R^* \to t_S^*$ is surjective.*

Proof. (cf. [Sch, Lemma 1.1]) Observe that f is surjective if and only if $\mathfrak{m}_S \subseteq \operatorname{im} f$, which by Nakayama's lemma holds true if and only if $\mathfrak{m}_S \subseteq (\operatorname{im} f, \mathfrak{m}_S^2)$. Using the fact that $p \in \mathfrak{m}_S$ and $p \in \operatorname{im} f$ we see that f is surjective if and only if the composition $R \xrightarrow{f} S \twoheadrightarrow S/(\mathfrak{m}_S^2, p)$ is surjective. It is sufficient to observe that this map factors via $R/(\mathfrak{m}_R^2, p)$ and apply Remark 1.30. □

As a consequence of Lemma 1.31, we can determine the minimal number of variables needed in the presentation described in Theorem 1.13.

Corollary 1.32. *A ring $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ can be presented as an epimorphic image of the ring $\mathrm{W}(k)[[X_1, \ldots, X_d]]$ if and only if $d \geqslant \dim_k t_R^*$.*

Proof. The tangent space to $\mathrm{W}(k)[[X_1, \ldots, X_d]]$ is d-dimensional, so $d \geqslant \dim_k t_R^*$ holds for every quotient ring R of $\mathrm{W}(k)[[X_1, \ldots, X_d]]$.

By Nakayama's lemma, $\dim_k \mathfrak{m}_R/\mathfrak{m}_R^2$ is equal to the minimal number of generators of the ideal $\mathfrak{m}_R$, so $\dim_k t_R^*$ is the minimal number of generators of its image in $R/(p)$. We conclude that for $d \geqslant \dim_k t_R^*$ there exist $x_1, \ldots, x_d \in \mathfrak{m}_R$ such that $\mathfrak{m}_R = (x_1, \ldots, x_d, p)$. Lemma 1.31 implies then that the map $\mathrm{W}(k)[[X_1, \ldots, X_d]] \xrightarrow{X_i \to x_i} R$ is a well-defined surjective $\hat{\mathcal{C}}$-morphism. □

1.2 Set valued functors on $\hat{\mathcal{C}}$

This thesis addresses several questions related to the problem of representability of some specific functors $\hat{\mathcal{C}} \to$ Sets, namely, the functors of deformations of group representations. Before introducing them (which will be done in the next chapter) we want to recall some standard results concerning the representability of (covariant) functors $\hat{\mathcal{C}} \to$ Sets in general.

To learn more about this topic, we recommend the paper [Sch] or [Maz1, §14- §20]. The reader might also find useful the short introduction on this topic contained in [By2, Chapter 1].

1.2.1 Tangent space

Definition 1.33. If F is a functor $F : \hat{\mathcal{C}} \to$ Sets then we define its tangent space as $t_F := F(k[\varepsilon])$.

Remark 1.34. This definition and the definition of the tangent space to a $\hat{\mathcal{C}}$-ring are closely connected. Namely, for $R \in \mathrm{Ob}(\hat{\mathcal{C}})$, the tangent spaces t_R and $t_{\mathrm{Hom}_{\hat{\mathcal{C}}}(R,-)}$ may be identified. See [Maz1, Proposition on p. 271].

We note that under some additional assumptions on F a natural k-vector space structure can be introduced on t_F ([Maz1, §15]).

Notation 1.35. *Let $k[\varepsilon] \times_k k[\varepsilon]$ denote the fiber product of two copies of the reduction map $\pi : k[\varepsilon] \twoheadrightarrow k$. We introduce the operation $+ : k[\varepsilon] \times_k k[\varepsilon] \to k[\varepsilon]$ defined by $(x + y_1\varepsilon,\ x + y_2\varepsilon) \mapsto x + (y_1 + y_2)\varepsilon$. Moreover, given $\alpha \in k$ we will denote by a_α the $\hat{\mathcal{C}}$-morphism $k[\varepsilon] \to k[\varepsilon]$ sending $x + y\varepsilon$ to $x + \alpha y\varepsilon$.*

Lemma 1.36. *Let us use the above notation and conventions introduced in Definition 1.23. Suppose $F : \hat{\mathcal{C}} \to \mathrm{Sets}$ is a covariant functor such that:*

(1) $F(k)$ is a one-element set.

(2) The map $\Phi := (F(p_1), F(p_2)) :\ F(k[\varepsilon] \times_k k[\varepsilon]) \to F(k[\varepsilon]) \times F(k[\varepsilon])$ is a bijection.

Then the following operations:

- *scalar multiplication $k \times t_F \to t_F$ defined as $(\alpha, \xi) \mapsto F(a_\alpha)(\xi)$,*
- *addition $t_F \times t_F \to t_F$ defined as $(\xi_1, \xi_2) \mapsto F(+)(\Phi^{-1}(\xi_1, \xi_2))$,*

define a structure of a k-vector space on $F(k[\varepsilon])$.

Proof. See [Maz1, §15] or [Sch, Lemma 2.10]. □

Remark 1.37. Note that this structure is natural, in the sense that for every natural transformation $\Phi : F \to G$ of functors F and G satisfying properties (1) and (2), the map $\Phi_{k[\varepsilon]} : F(k[\varepsilon]) \to G(k[\varepsilon])$ is k-linear with respect to the introduced structure.

Remark 1.38. Assuming that F satisfies the following slightly stronger assumption:

$$F(k[V] \times_k k[W]) \cong F(k[V]) \times F(k[W]) \text{ for every } V, W \in \mathfrak{V},$$

we obtain for every $V \in \mathfrak{V}$ a canonical k-vector space structure on $F(k[V])$, such that $F(k[V]) \cong t_F \otimes_k V$.

1.2.2 Continuous functors

The functors in which we will be interested are continuous in the following sense.

Definition 1.39. A functor $F : \hat{\mathcal{C}} \to \mathrm{Sets}$ is called continuous if and only if for every $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ the canonical map $F(R) \to \varprojlim_{l \in \mathbb{N}} F(R/\mathfrak{m}^l)$ is an isomorphism.

Since continuous functors are completely determined by their restrictions to $\mathcal{C}$, we could see them simply as functors defined on $\mathcal{C}$. Note that this subcategory has, for example, the advantage of being closed under fiber products, while $\hat{\mathcal{C}}$ does not have this property (see Lemma 1.25 and Remark 1.26).

On the other hand, there is a good reason to work in the full category $\hat{\mathcal{C}}$. Namely, we are interested in representability problems (see the next section) and a continuous functor that is representable in $\hat{\mathcal{C}}$ may restrict to a functor that is not representable in $\mathcal{C}$.

1.2.3 Representable functors and versal hulls

Notation 1.40. *Given a $\hat{\mathcal{C}}$-ring R, we will denote the functor* $\mathrm{Hom}_{\hat{\mathcal{C}}}(R, -) : \hat{\mathcal{C}} \to \mathrm{Sets}$ *by* h_R.

Definition 1.41. A functor $F : \hat{\mathcal{C}} \to \mathrm{Sets}$ is called representable if and only if there exists $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ representing it, i.e., $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ such that there exists a natural isomorphism $h_R \to F$.

Note that if a functor is representable and a natural isomorphism as in Definition 1.41 is fixed, then the object representing it is uniquely unique, i.e., unique up to a canonical isomorphism (this is a consequence of Yoneda's lemma). Observe also that h_R or, more generally, representable functors are continuous.

We introduce next a slightly weaker notion.

Definition 1.42. Let F and G be functors $\hat{\mathcal{C}} \to \mathrm{Sets}$. A natural transformation $F \to G$ is called *smooth* if for every surjection $B \twoheadrightarrow A$ in $\hat{\mathcal{C}}$ the induced map

$$F(B) \to F(A) \times_{G(A)} G(B)$$

is surjective. It is called *étale* if it is smooth and bijective on $k[\varepsilon]$.

Remark 1.43. Suppose F and G are continuous functors. Then the above definition is equivalent to the one in which we require the surjectivity property only for every surjection $B \twoheadrightarrow A$ in $\mathcal{C}$.

Remark 1.44. Suppose functors F, $G : \hat{\mathcal{C}} \to \mathrm{Sets}$ are such that $F(k)$ and $G(k)$ are one-element sets. If $\Phi : F \to G$ is a smooth transformation, then Φ is surjective on every $\hat{\mathcal{C}}$-ring R. Indeed, it is sufficient to apply the surjectivity property of Definition 1.42 to the reduction morphism $\pi_{\mathfrak{m}_R} : R \twoheadrightarrow k$.

Definition 1.45. We say that a ring $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ is a *versal hull* for a functor $F : \hat{\mathcal{C}} \to \mathrm{Sets}$ if there exists a natural transformation $h_R \to F$ that is étale.

Observe that if R represents some functor, then it is also its versal hull; the converse implication does not hold in general. The versal hull, if it exists, is uniquely determined up to isomorphism which, however, may be not canonical.

We finish this subsection showing how the notion of a tangent space can be useful in representability problems.

Proposition 1.46. *If R_v is a versal hull of a functor F, then R_v can be presented as a quotient of* $\mathrm{W}(k)[[X_1, \dots X_d]]$ *if and only if $d \geqslant \dim_k t_F$.*

Proof. Combine Corollary 1.32 with the definition of a versal deformation ring, by which t_F and t_{R_v} are isomorphic. □

1.2.4 Schlessinger criteria

The continuity assumption is very useful, since it allows us to use the criteria developed by Schlessinger in his paper [Sch].

Theorem 1.47 (Schlessinger Criteria)**.** *Let F be a continuous functor $\hat{\mathcal{C}} \to \mathrm{Sets}$ satisfying the following property (**H0**): $F(k)$ is a one-element set. Observe that for every $\mathcal{C}$-morphisms*† *$\pi_1 : R_1 \to S$, $\pi_2 : R_2 \to S$ we obtain an induced map*

$$\Psi : \quad F(R_1 \times_S R_2) \to F(R_1) \times_{F(S)} F(R_2)$$

and let us define the following conditions:

†Let us emphasize: morphisms of $\mathcal{C}$, not $\hat{\mathcal{C}}$. Recall that $\hat{\mathcal{C}}$ is not closed under fiber products.

(***H1***) Ψ *is surjective whenever* π_2 *is a surjection.*

(***H2***) Ψ *is bijective whenever* π_2 *is the reduction* $k[\varepsilon] \twoheadrightarrow k$*.*

(***H3***) $\dim_k t_F$ *is finite.*

(***H4***) Ψ *is bijective whenever* $\pi_2 = \pi_1$ *is a surjection.*

*Then F has a versal hull if and only if it satisfies properties (**H1**)–(**H3**) and is representable if and only if it satisfies properties (**H1**)–(**H4**).*

Proof. See [Sch, Theorem 2.11]. □

Remark 1.48. Compared to Schlessinger's original formulation, there are some minor changes in the statement of this theorem. Firstly, the theorem was originally stated for functors $\mathcal{C} \to \text{Sets}$. Secondly, Schlessinger requires the properties described in conditions (**H1**) and (**H4**) only for the so called "small surjections" ([Sch, Definition 1.2]). However, it is easy to check that these formulations are equivalent. See also [Sch, Remark 2.14]. Finally, Schlessinger does not require the residue field to be finite.

Remark 1.49. Property (**H0**) coincides with condition (1) of Lemma 1.36 and property (**H2**) implies condition (2) of the same lemma (it even implies the stronger condition of Remark 1.38), which makes the symbol $\dim_k t_F$ appearing in property (**H3**) well-defined. Alternatively, to avoid recurring to the definition of the vector space structure on t_F, we could phrase (**H3**) simply as "t_F is finite", relying on the fact that k is finite.

Chapter 2

Deformation functors

In this chapter we introduce the deformation functors of group representations and present all fundamental definitions necessary to understand the rest of the thesis. This way we also provide a concrete example showing how the theory and notions introduced in the previous chapter can be used.

A more detailed introduction to deformations of group representations can be found in [Go], [Maz1], [Maz2], [Ble] or [Boe1]. A characteristic feature of our exposition is that we try to make clear relations between the Schlessinger criteria appearing in Theorem 1.47 and certain widely used assumptions on groups and representations (see our discussion in section 2.3.2). In particular, opting for greater generality, we do not require our groups to satisfy the so called p-finiteness condition (see Definition 2.27). We also pay a bit more attention to the concept of a versal deformation ring: we comment on some apparent inconsistency of its definitions used by different authors (section 2.3.1) and present an example of a versal deformation ring that is not a universal deformation ring (Lemma 2.37).

2.1 Deformations of group representations

We recall that here and elsewhere in the thesis we use Notation 1.1. In particular, k stands for a finite field of characteristic p.

2.1.1 Group representations

Let G be a group and R be a ring. The term "an n-dimensional representation of G over R" is used in the literature for two slightly different, but related concepts. It can refer to either of the following:

- a group homomorphism $G \to \mathrm{GL}_n(R)$, or
- an RG-module V that is free and of rank n over R.

To make a clear distinction, we refer to V as in the second definition as to a "representation module" ("representation space", in case R is a field). Note that a representation module can be equivalently defined as a free R-module of finite rank over R, equipped with an R-linear action of G.

The relation between the two definitions is as follows. A representation space can be seen as a pair consisting of a free R-module V of rank n and a group homomorphism $G \to \mathrm{Aut}_R(V)$. Choosing an R-basis, one identifies V with R^n and $G \to \mathrm{Aut}_R(V)$ with $G \to \mathrm{GL}_n(R)$. Since the basis can be chosen in different ways, representation modules correspond to equivalence classes $\{K\rho K^{-1} | K \in \mathrm{GL}_n(R)\}$ of representations $\rho : G \to \mathrm{GL}_n(R)$. Conversely, it is easy to find a representation module corresponding to a group representation. We will use the following notation.

Notation 2.1. *Given a group representation $\rho : G \to \mathrm{GL}_n(R)$, we denote by V_ρ the R-module of $n \times 1$ column vectors over R, on which $g \in G$ acts via multiplication by $\rho(g)$.*

By $\mathrm{Ad}(\rho)$ we mean the free R-module $M_n(R)$ on which $g \in G$ acts via conjugation with $\rho(g)$. Thus $\mathrm{Ad}(\rho) \cong \mathrm{End}_R(V_\rho)$. By $\mathrm{Ad}(\rho)^G$ we denote the submodule of G-invariants.

When G is a topological group and R is a topological ring, it is natural to focus on representations that are continuous. The corresponding representation modules can be relatively easily characterized if G and R are profinite. Note that this extra assumption is always satisfied in this thesis, since we only study representations of profinite groups over objects of $\hat{\mathcal{C}}$ (which are profinite by Corollary 1.21).

Definition 2.2. Given a topological ring R and a profinite group G, we define the completed group algebra $R[[G]]$ as the inverse limit of usual group algebras $R[G/N]$, where N ranges over all open normal subgroups of G.

Remark 2.3. If G is finite then we have $R[[G]] \cong R[G]$.

One checks that equivalence classes of continuous finite dimensional representations of a profinite group G over a profinite ring R correspond bijectively with topological $R[[G]]$-modules that are free and of finite rank over R.

2.1.2 Basic definitions

We present two ways of defining deformation functors, corresponding to the discussed two points of view on group representations.

The module-theoretic approach

Let G be a profinite group and V be a topological $k[[G]]$-module that is of finite dimension over k.

Definition 2.4. We define a *lift* of V to $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ as a pair (W, ϕ), where W is a topological $R[[G]]$-module, free, of finite rank over R and $\phi : k \otimes_R W \to V$ is an isomorphism of topological $k[[G]]$-modules.

We will say that two lifts, (W_1, ϕ_1) and (W_2, ϕ_2), of V to $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ are isomorphic if and only if there exists an isomorphism $\Phi : W_1 \to W_2$ of topological $R[[G]]$-modules such that $\phi_2 \circ (\mathrm{id}_k \otimes \Phi) = \phi_1$, i.e., such that the following diagram commutes:

$$\begin{tikzcd} k \otimes_R W_1 \arrow[rr, "\mathrm{id}_k \otimes \Phi"] \arrow[dr, "\phi_1"'] & & k \otimes_R W_2 \arrow[dl, "\phi_2"] \\ & V & \end{tikzcd}$$

Definition 2.5. A *deformation* of V is, by definition, an isomorphism class of lifts of V. Given $R \in \mathrm{Ob}(\hat{\mathcal{C}})$, we define $\mathrm{Def}_V(R)$ to be the set of deformations of V to R.

Definition 2.6. Let a $\hat{\mathcal{C}}$-morphism $f : R \to R'$ be given and denote by ι the canonical isomorphism $k \otimes_{R'} (R' \otimes_R V) \cong k \otimes_R V$. We define $\mathrm{Def}_V(f)$ as the map $\mathrm{Def}_V(R) \to \mathrm{Def}_V(R')$ that takes (W, Φ) to $(R' \otimes_R W, \Phi \circ \iota)$.

Observe that using the above definitions we obtain a covariant functor $\mathrm{Def}_V : \hat{\mathcal{C}} \to \mathrm{Sets}$, the *deformation functor* of V.

The matrix approach

Let G be a profinite group and $\overline{\rho} : G \to \mathrm{GL}_n(k)$ be a continuous representation (where k and, consequently, $\mathrm{GL}_n(k)$ are considered with discrete topology).

Notation 2.7. *For a $\hat{\mathcal{C}}$-morphism $f : R \to S$ and $A \in M_n(R)$ let $fA \in M_n(S)$ be the matrix obtained applying f to every entry of A. We denote the continuous group homomorphism $\mathrm{GL}_n(R) \ni A \mapsto fA \in \mathrm{GL}_n(S)$ either by the same symbol f or, if this might lead to confusion, by $\mathrm{GL}_n(f)$.*

Definition 2.8. We define a lift of $\overline{\rho}$ to $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ as a continuous group homomorphism $\rho : G \to \mathrm{GL}_n(R)$ such that $\overline{\rho} = \mathrm{GL}_n(\pi_{\mathfrak{m}_R}) \circ \rho$. The set of all lifts of $\overline{\rho}$ to ρ will be denoted by $\mathrm{Lift}_{\overline{\rho}}(R)$.

$$\begin{tikzcd}
 & \mathrm{GL}_n(R) \arrow[d, two heads, "\pi_{\mathfrak{m}_R}"] \\
G \arrow[ur, dashed, "\rho"] \arrow[r, "\overline{\rho}"] & \mathrm{GL}_n(k)
\end{tikzcd}$$

Definition 2.9. We call $\rho, \rho' \in \mathrm{Lift}_{\overline{\rho}}(R)$ strictly equivalent if and only if there exists $K \in \ker \mathrm{GL}_n(\pi_{\mathfrak{m}_R})$ such that $\rho' = K\rho K^{-1}$. The set of resulting equivalence classes will be denoted by $\mathrm{Def}_{\overline{\rho}}(R)$ and any of its elements will be called a deformation of $\overline{\rho}$ to R.

Notation 2.10. *The strict equivalence class of $\rho \in \mathrm{Lift}_{\overline{\rho}}(R)$ will be denoted by $[\rho]$.*

Observe that every morphism $f : R \to S$ of $\hat{\mathcal{C}}$ induces a map $\mathrm{Lift}_{\overline{\rho}}(R) \ni \rho \to \mathrm{GL}_n(f) \circ \rho \in \mathrm{Lift}_{\overline{\rho}}(S)$, which we will denote by $\mathrm{Lift}_{\overline{\rho}}(f)$. Since this map preserves strict equivalence classes, we also obtain an induced map $\mathrm{Def}_{\overline{\rho}}(f) : \mathrm{Def}_{\overline{\rho}}(R) \to \mathrm{Def}_{\overline{\rho}}(S)$.

Observation 2.11. *The above definitions define covariant functors* $\mathrm{Lift}_{\overline{\rho}}$ *and* $\mathrm{Def}_{\overline{\rho}}$ *from $\hat{\mathcal{C}}$ to* Sets, *the* lift *and* deformation functor *of $\overline{\rho}$.*

The interplay between the definitions

Suppose G is a profinite group. Let V be a topological $k[[G]]$-module, n-dimensional over k. Choosing a k-basis $e_1, \ldots, e_n$ we identify V with k^n and obtain an associated continuous group homomorphism $\bar{\rho} : G \to \mathrm{GL}_n(k)$.

If (W, ϕ) is a lift of V to $R \in \mathrm{Ob}(\hat{\mathcal{C}})$, we can choose an R-basis $f_1, \ldots, f_n$ of W that lifts $e_1, \ldots, e_n$, i.e., such that $\phi(1 \otimes f_i) = e_i$. The corresponding continuous group homomorphism $\rho : G \to \mathrm{GL}_n(R)$ is a lift of $\bar{\rho}$. The definition of ρ depends on the choice of vectors $f_1, \ldots, f_n$, but its deformation class does not. Hence, the deformations of V and the deformations of $\bar{\rho}$ can be identified. It is easy to conclude that we obtain an isomorphism of functors $\mathrm{Def}_V \cong \mathrm{Def}_{\bar{\rho}}$.

Remark 2.12. The definition of a deformation functor given in the module-theoretic setting seems to be more elegant and better motivated. However, in the rest of the thesis we will work with the second definition, which we find more convenient for explicit computations.

Remark 2.13. The lift functor can be interpreted in the module-theoretic setting as the *framed deformation functor*, see [Boe1, § 1.1].

2.2 Basic properties of deformation functors

We assume that G is a profinite group and $\bar{\rho} : G \to \mathrm{GL}_n(k)$ is a continuous representation.

Proposition 2.14. *The functor* $\mathrm{Def}_{\bar{\rho}}$ *is continuous (in the sense of Definition 1.39) and satisfies properties (**H0**)-(**H2**) of Theorem 1.47.*

Proof. See [Maz2, Proposition 1] or a more detailed discussion in [Go, Lemma 2.3, Lemma 3.4, Lemma 3.6]. □

The lift functor plays for us only an auxiliary role, but we briefly comment also on its properties.

Proposition 2.15. *(i) The functor* $\mathrm{Lift}_{\bar{\rho}}$ *is continuous and satisfies properties (**H0**), (**H1**), (**H2**) and (**H4**).*

*(ii) Property (**H3**) is satisfied by* $\mathrm{Lift}_{\bar{\rho}}$ *if and only if it is satisfied by* $\mathrm{Def}_{\bar{\rho}}$*.*

(iii) The canonical natural transformation $\mathrm{Lift}_{\bar{\rho}} \to \mathrm{Def}_{\bar{\rho}}$ *is smooth.*

Proof. We only comment on part (ii), the other claims can be verified by a straightforward check. Of course finiteness of $\mathrm{Lift}_{\bar{\rho}}(k[\varepsilon])$ implies finiteness of $\mathrm{Def}_{\bar{\rho}}(k[\varepsilon])$. The converse statement follows from the fact that, since $k[\varepsilon]$ is a finite ring, every deformation class of $\bar{\rho}$ to $k[\varepsilon]$ contains only finitely many lifts. □

2.2.1 Tangent space

Proposition 2.14 and Remark 1.38 (see also Remark 1.49) imply that for every $V \in \mathfrak{V}$ there is a natural k-vector space structure on $\mathrm{Def}_{\bar{\rho}}(k[V])$. We present now a cohomological interpretation for the tangent space $t_{\mathrm{Def}_{\bar{\rho}}} = \mathrm{Def}_{\bar{\rho}}(k[\varepsilon])$.

Observe that $\mathrm{Def}_{\bar{\rho}}(k[\varepsilon])$ is non-empty, since we obtain a lift of $\bar{\rho}$ composing it with the inclusion $k \hookrightarrow k[\varepsilon]$. We will denote the resulting lift by the same symbol $\bar{\rho}$.

Notation 2.16. *If G is a profinite group and M is a discrete G-module then we will use the symbol $H^r(G, M)$ to denote the continuous cochain cohomology group. Similarly, $Z^r(G, M)$ stands for the group of continuous cocycles (see [Se2, §2] for definitions).*

Lemma 2.17. *Given $\rho \in \mathrm{Lift}_{\bar{\rho}}(k[\varepsilon])$, let $f_\rho : G \to M_n(k)$ be such that $\rho = (I_n + \varepsilon f_\rho) \cdot \bar{\rho}$. Then for every $\rho \in \mathrm{Lift}\,\bar{\rho}(k[\varepsilon])$ we have that $f_\rho \in Z^1(G, \mathrm{Ad}(\bar{\rho}))$ and*

$$t_{\mathrm{Def}_{\bar{\rho}}} \ni [\rho] \ \mapsto \ [f_\rho] \in H^1(G, \mathrm{Ad}(\bar{\rho}))$$

is an isomorphism of k-vector spaces.

Proof. See [Maz1, §21]. □

Remark 2.18. We easily deduce from Lemma 2.17 the following slightly more general result: *The functors $\mathfrak{V} \to \mathrm{Vect}_k$ defined by $V \mapsto \mathrm{Def}_{\bar{\rho}}(k[V])$ and $V \mapsto H^1(G, \mathrm{End}_k(V))$ are naturally isomorphic.*

2.3 Representability of deformation functors

We assume that G is a profinite group and $\bar{\rho} : G \to \mathrm{GL}_n(k)$ is a continuous representation.

2.3.1 Versal and universal deformation rings

Definition 2.19. If $\mathrm{Def}_{\bar\rho}$ is representable, then the object representing it will be called the *universal deformation ring* of $\bar\rho$. If $\mathrm{Def}_{\bar\rho}$ has a versal hull, then it will be called the *versal deformation ring* of $\bar\rho$.

Working out the definitions and using Yoneda's lemma, one sees that representability of $\mathrm{Def}_{\bar\rho}$ is equivalent to the following statement:

There exists $R_u \in \mathrm{Ob}(\hat{\mathcal{C}})$ and $\rho_u \in \mathrm{Lift}_{\bar\rho}(R_u)$ such that for every $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ each element of $\mathrm{Def}_{\bar\rho}(R)$ is of the form $[f \circ \rho_u]$, for a uniquely determined $f \in \mathrm{Hom}_{\hat{\mathcal{C}}}(R_u, R)$.

$$\begin{tikzcd}
\mathrm{GL}_n(R_u) \arrow[rrd, dotted, "f"] & & \\
& & \mathrm{GL}_n(R) \arrow[d, "\pi"] \\
& G \arrow[luu, "\rho_u"] \arrow[ru, dashed, "\rho"] \arrow[r] & \mathrm{GL}_n(k)
\end{tikzcd}$$

Definition 2.20. Suppose R_u is a universal deformation ring of $\bar\rho$. Every $\rho_u \in \mathrm{Lift}_{\bar\rho}(R_u)$ for which the map $f \to [f \circ \rho_u]$ defines an isomorphism $h_R \cong \mathrm{Def}_\rho$ will be called a *universal lift* of $\bar\rho$. Similarly, if R_u is only a versal hull and ρ_u is such that the mentioned transformation is étale, we will say that it is a *versal lift* of $\bar\rho$.

Remark 2.21. Remark 1.44 shows that, given a versal lift of ρ_v of $\bar\rho$, the transformation $h_{R_v}(S) \ni f \to [f\rho_u] \in \mathrm{Def}_{\bar\rho}(S)$ is surjective for every $S \in \mathrm{Ob}(\hat{\mathcal{C}})$.

This observation is actually used by some authors to define the versal deformation ring in the following, seemingly less restrictive way.

Definition 2.22. (Alternative definition of a versal def. ring) We will say that R_v is a versal deformation ring of $\bar\rho$ if there exists a natural transformation $h_{R_v} \to \mathrm{Def}_{\bar\rho}$ that is surjective on every $\hat{\mathcal{C}}$-ring and bijective on $k[\varepsilon]$.

It is clear that a versal ring in the sense of Definition 2.19 is also a versal ring in the sense of Definition 2.22. The converse implication is also true, but not completely trivial, as shown in Example 4.14. Both definitions appear often in the literature, the first one for example in [Maz1, By, Go], the second one for example in [Ble, BCdS, Maz2], but the author has never seen any comment on this apparent inconsistency of definitions. We fill this small gap in the literature in section 4.1.3, see Corollary 4.13.

2.3.2 Existence of versal and universal deformation rings

Proposition 2.23. *(i) The versal deformation ring of $\bar{\rho}$ exists if and only if* $\mathrm{Def}_{\bar{\rho}}$ *satisfies property* (***H3***) *of Theorem 1.47.*

(ii) The universal deformation ring of $\bar{\rho}$ exists if and only if $\mathrm{Def}_{\bar{\rho}}$ *satisfies properties* (***H3***) *and* (***H4***).

(iii) Property (***H3***) *of* $\mathrm{Def}_{\bar{\rho}}$ *is implied if* $\mathrm{CHom}(\ker\bar{\rho}, \mathbb{Z}/p\mathbb{Z})$ *is finite.*

(iv) Property (***H4***) *of* $\mathrm{Def}_{\bar{\rho}}$ *is implied if* $\mathrm{Ad}(\bar{\rho})^G = kI_n$ *holds.*

Proof. Part (*i*) and (*ii*) is a direct consequence of Theorem 1.47 and Proposition 2.14. For part (*iii*) see the argument of [Go, Lemma 3.7] and for part (*iv*) see [Go, Lemma 3.9] or prove the claim using Lemma 4.39. □

Remark 2.24. (1) Using Property 2.15 we conclude that $\mathrm{Def}_{\bar{\rho}}$ has a versal hull if and only if $\mathrm{Lift}_{\bar{\rho}}$ is representable.

(2) In Remarks 2.31 and 2.36 we show that implications converse to those presented in parts (*iii*) and (*iv*) of Proposition 2.23 do not hold true. On the other hand, in Proposition 2.33 we present a necessary condition for property (**H3**).

(3) The condition $\mathrm{Ad}(\bar{\rho})^G = kI_n$ holds when $\bar{\rho}$ is absolutely irreducible (see [Maz1, §4, Corollary]).

It is worth noting that the sufficient condition appearing in Proposition 2.23.(*iii*) can be formulated in several equivalent ways using the lemma below.

Definition 2.25. Given a profinite group G, the symbols G^p and $G^{ab,p}$ stand for the pro-p (abelianized pro-p) completion of G, i.e., $\varprojlim_N G/N$, with the limit taken over all open normal subgroups N such that G/N is a p-group (an abelian p-group).

Lemma 2.26. *Let G be a profinite group and define $\Gamma := G^{ab,p}$. Then* $\mathrm{CHom}(G, \mathbb{Z}/p\mathbb{Z})$ *is finite if and only if any of the following equivalent conditions is satisfied:*

- *G^p is topologically finitely generated.*
- *Γ is a finitely generated $\mathbb{Z}_p$-module.*
- *$\Gamma/p\Gamma$ is finite.*

Proof. See [Go, Lemma 2.1]. □

The finiteness condition

Parts (i) and (iii) of Proposition 2.23 motivate the following definition.

Definition 2.27. We say that G satisfies the "p-finiteness condition" (Φ_p) if and only if for every open subgroup $J \leqslant G$, the set $\mathrm{CHom}(J,\ \mathbb{Z}/p\mathbb{Z})$ is finite.

Note that finiteness of $\mathrm{CHom}(G, \mathbb{Z}/p\mathbb{Z})$ alone does not imply (Φ_p) .

Example 2.28. Suppose $p \neq 2$, let $H := \varprojlim_{n \in \mathbb{N}} (\mathbb{Z}/p\mathbb{Z})^n$ and define $G := C_2 \ltimes H$, with the action of $C_2 = \langle \varepsilon \rangle$ defined as $\forall h \in H :\ \varepsilon.h = -h$. Then G is a profinite group and $G^{ab,p}$ is trivial, but for the open subgroup H we have that $\mathrm{CHom}(H, \mathbb{Z}/p\mathbb{Z})$ is infinite.

If a group G satisfies the condition (Φ_p), it follows from Proposition 2.23 that every continuous representation of G has a versal deformation ring. We note that also the converse statement holds (a hint: if $U \leqslant G$ is an open subgroup of G for which $\mathrm{CHom}(U, \mathbb{Z}/p\mathbb{Z})$ is infinite, consider the $[G : U]$-dimensional representation of G induced from the trivial one-dimensional representation of U and use Lemma 2.29).

A comment on the usage of the introduced conditions in the literature

Almost all authors writing on deformations of group representations formulate their results only for profinite groups satisfying (Φ_p). This is quite understandable, since the theory has originally been developed in order to be applied to representations of Galois groups, which automatically satisfy this condition (cf. [Go, Theorem 1.6 and Problem 1.26], [Boe1, Examples 1.2.2]). On the other hand, such a restriction turns out to be an obstacle and an unnecessary complication, weakening the results when working in a fully abstract setting (as we do in this thesis). consciously avoid assuming condition (Φ_p), in contrast to other authors.

The reader should be aware of this difference and keep in mind that we often refer to standard results which, formally speaking, have only been proved in the literature for groups satisfying (Φ_p). However, in such a situation the reader will easily check that the proofs to which we refer are valid in a fuller generality.

Similarly, we note that some results are stated in the literature only for $\bar{\rho}$ satisfying the condition $\mathrm{Ad}(\bar{\rho})^G = kI_n$ (or even: only for absolutely irreducible $\bar{\rho}$), whereas their proofs go through in any case in which the universal deformation ring exists.

2.3.3 Non-noetherian setting

We would like to comment also on our choice to work in the noetherian setting. It is naturally motivated, sufficient for many applications (like Galois deformations) and simplifies many arguments. Its big advantage is that it allows us to use the Schlessinger criteria and structure theorems presented in the first chapter.

On the other hand, we have to admit that it sometimes turns out to be an unnatural obstacle. This will be seen for example in the awkward formulations of Proposition 2.30, Proposition 4.40 or Corollary 5.25.

The reader will observe that many times we obtain results regarding representability not using the existence criteria provided by Schlessinger, but explicitly determining or constructing the universal deformation ring (cf. for example the proof of Theorem 5.10). Looking closer, the reader will notice that some of our proofs hold true (or can be easily modified to hold true) for arbitrary inverse limits of artinian rings, also the non-noetherian ones (cf. Remark 1.20).

We would like to mention that some authors have approached the problem of representability of deformation functors $\mathrm{Def}_{\bar{\rho}}$ in a category of (not necessarily noetherian) local $\mathrm{W}(k)$-algebras, arising as inverse limits of artinian rings. It is worth noting that the condition $\mathrm{Ad}(\bar{\rho})^G = kI_n$ appearing in Proposition 2.23.(iv) guarantees representability of $\mathrm{Def}_{\bar{\rho}}$ in this larger category, by an object which is noetherian if and only if $\dim_k t_{\mathrm{Def}_{\bar\rho}} < \infty$. For details, see for example [dSL], [Hi, §2.3] or [Go, Appendix 1].

2.4 Examples

We present various examples with the aim of illustrating the introduced notions and properties. Some of the results presented here will also serve us in the next chapters.

2.4.1 One-dimensional representations

Let G be a profinite group and $\bar\rho : G \to \mathrm{GL}_n(k)$ be a continuous representation. In this subsection we will assume that $n = 1$. Since for every ring R we can identify $\mathrm{GL}_1(R)$ with $R^\times$, this means $\bar\rho$ will simply be a continuous character $G \to k^\times$. This is the easiest case to consider and it has been worked out in [Maz2, §1.4] or, in greater detail, in [Go, Proposition 3.13]. However, note that these authors work with the assumption that G satisfies the p-finiteness condition (Φ_p) of Definition 2.27.

Lemma 2.29. *If $n = 1$ then:*

(i) $\mathrm{Def}_{\bar\rho}$ *depends only on the group G and not on the character $\bar\rho$. More precisely, the functors* $\mathrm{Def}_{\bar\rho}$, $\mathrm{Lift}_{\bar\rho}$ *and* $R \mapsto \mathrm{CHom}(G, R^\times_{\equiv 1})$ *are naturally isomorphic.*

(ii) $\mathrm{Def}_{\bar\rho}$ *satisfies property* (***H3***) *if and only if* $\mathrm{CHom}(G, \mathbb{Z}/p\mathbb{Z})$ *is finite.*

(iii) $\mathrm{Def}_{\bar\rho}$ *satisfies property* (***H4***).

Proof. (*i*) The fact that functors $\mathrm{Def}_{\bar\rho}$ and $\mathrm{Lift}_{\bar\rho}$ coincide is an easy consequence of the commutativity of $\mathrm{GL}_1(k) \cong k^\times$. Observe that for every $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ there exists at least one lift of $\bar\rho$ to R, namely $\rho_0 := \tau_R \circ \bar\rho$. Every lift of $\bar\rho$ to R is therefore of the form $\lambda \cdot \rho_0$, where $\lambda : G \to R^\times_{\equiv 1}$ is a continuous group homomorphism.

(*ii*) By the first claim, $t_{\mathrm{Def}_{\bar\rho}} \cong \mathrm{CHom}(G, k[\varepsilon]^\times_{\equiv 1})$. The multiplicative group $k[\varepsilon]^\times_{\equiv 1}$ is isomorphic to the additive group of k, which (due to the finiteness of k) is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^m$ for some $m \in \mathbb{N}$. It follows that $\mathrm{CHom}(G, k[\varepsilon]^\times_{\equiv 1}) \cong \mathrm{CHom}(G, (\mathbb{Z}/p\mathbb{Z})^m)$ is finite if and only if $\mathrm{CHom}(G, \mathbb{Z}/p\mathbb{Z})$ is finite.

(*iii*) The claim follows from the first one and Proposition 2.15. □

We conclude, using Proposition 2.23, that either $\mathrm{CHom}(G, \mathbb{Z}/p\mathbb{Z})$ is finite, in which case $\mathrm{Def}_{\bar{\rho}}$ is representable, or $\bar{\rho}$ does not even have a versal deformation ring. It is also clear that property (Φ_p) implies finiteness of $\mathrm{CHom}(G, \mathbb{Z}/p\mathbb{Z})$.

Proposition 2.30. *Suppose* $\mathrm{CHom}(G, \mathbb{Z}/p\mathbb{Z})$ *is finite and define* $\Gamma := G^{ab,p}$. *Then* $R_u := \mathrm{W}(k)[[\Gamma]]$ *is an object of* $\hat{\mathcal{C}}$ *and the universal deformation ring of* $\bar{\rho}$. *Denoting the Teichmüller lift* $k^\times \to \mathrm{W}(k)^\times$ *by* τ *and the image of* $g \in G$ *in* Γ *by* $\bar{g}$ *we have that*

$$G \ni \; g \mapsto \tau\big(\bar{\rho}(g)\big) \cdot [\bar{g}] \;\in R_u^\times$$

is a universal lift of $\bar{\rho}$.

Proof. See the argument of [Go, Proposition 3.13] (the result to which we refer is actually stated for groups satisfying (Φ_p), but its proof uses only the fact that Γ is finitely generated as a $\mathbb{Z}_p$-module). □

Remark 2.31. Using the notation from Example 2.28, consider the one-dimensional continuous representation defined as the composition $\bar{\rho} : G \twoheadrightarrow C_2 \xrightarrow{\varepsilon \mapsto (-1)} k^\times$. In this case G does not satisfy (Φ_p) , but property (**H3**) holds. Moreover, since $\ker \bar{\rho} = H$ and $\mathrm{CHom}(H, \mathbb{Z}/p\mathbb{Z})$ is infinite, we see that the sufficient condition for property (**H3**), described in Proposition 2.23.(*iii*), is not a necessary one. On the other hand, $\bar{\rho}|_H$ is a representation for which (**H3**) does not hold.

Remark 2.32. Observe that requiring the universal deformation ring to be noetherian is in this subsection only a complication (cf. our remarks in section 2.3.3). As one can check, if $\Gamma/p\Gamma$ is not finite, the ring $\mathrm{W}(k)[[\Gamma]]$ is not noetherian. However, working not in $\hat{\mathcal{C}}$, but in a bigger category comprising also non-noetherian inverse limits of local artinian rings we could simply say in Proposition 2.30 that $\mathrm{Def}_{\bar{\rho}}$ is represented by $\mathrm{W}(k)[[\Gamma]]$.

An application: twisting by one-dimensional representations

Let us remark that one-dimensional characters occur naturally also when studying higher dimensional representations, mainly due to the following observation.

Let G be a profinite group and $\bar{\rho} : G \to \mathrm{GL}_n(k)$ be a continuous representation, with n not necessarily equal one. If $\bar{\rho}_1 : G \to \mathrm{GL}_1(k) \cong k^\times$ is the trivial map then $\mathrm{Lift}_{\bar{\rho}_1} \cong \mathrm{Def}_{\bar{\rho}_1}$ is a group functor acting on $\mathrm{Lift}_{\bar{\rho}}$ as follows:

$$\forall\, S \in \mathrm{Ob}(\hat{\mathcal{C}}) : \quad \mathrm{Lift}_{\bar{\rho}_1}(S) \times \mathrm{Lift}_{\bar{\rho}}(S) \ni (\lambda, \rho) \mapsto \lambda \cdot \rho \in \mathrm{Lift}_{\bar{\rho}}(S).$$

This action is, in particular, faithful for every $S \in \mathrm{Ob}(\hat{\mathcal{C}})$. We also obtain the induced action of $\mathrm{Lift}_{\bar{\rho}_1}$ on $\mathrm{Def}_{\bar{\rho}}$. This fact has some consequences for the structure of universal deformation rings, which we shall not discuss here – see for example [Maz2, §1.4] and [Go, Problem 6.6]. Instead, we present the following corollary, giving a necessary condition for property (**H3**).

Proposition 2.33. *If $\bar{\rho}$ has a versal deformation ring then* $\mathrm{CHom}(G, \mathbb{Z}/p\mathbb{Z})$ *is finite.*

Proof. If $\bar{\rho}$ has a versal deformation ring then $\mathrm{Lift}_{\bar{\rho}}(k[\varepsilon])$ is finite by Propositions 2.15 and 2.23. In view of the above observation, $\mathrm{Lift}_{\bar{\rho}_1}(k[\varepsilon])$, acting faithfully on $\mathrm{Lift}_{\bar{\rho}}(k[\varepsilon])$, is finite and the claim follows from Lemma 2.29. □

2.4.2 Projective modules

Proposition 2.34. *If G is finite and $\bar{\rho}$ is such that the kG-module $V_{\bar{\rho}}$ is projective then* $\mathrm{W}(k)$ *is the universal deformation ring of $\bar{\rho}$. In particular, this applies to every representation of a finite group G of order coprime to p.*

Proof. Let R_v be the versal deformation ring of $\bar{\rho}$ (it exists, since G obviously satisfies (Φ_p)). Since $V_{\bar{\rho}}$ is kG-projective, $\mathrm{Ad}(\bar{\rho})$ is kG-projective as well, hence cohomologically trivial. The tangent space $H^1(G, \mathrm{Ad}(\bar{\rho}))$ to $D_{\bar{\rho}}$ is therefore zero-dimensional and so R_v is a quotient of $\mathrm{W}(k)$ by Corollary 1.32. Hence, there is at most one deformation of $\bar{\rho}$ to every $S \in \mathrm{Ob}(\hat{\mathcal{C}})$; in particular: R_v is universal. On the other hand, by Prop. 42, §14.4 in [Se], the kG-module $V_{\bar{\rho}}$ can be lifted to a $\mathrm{W}(k)G$-module that is free over $\mathrm{W}(k)$. This implies that $R_v = \mathrm{W}(k)$. In case G is finite and $p \nmid \#G$, every kG-module of finite k-dimension is projective by Maschke's theorem. □

Example 2.35. Let G be the cyclic group C_2 with generator g and define $\bar{\rho} : G \to \mathrm{GL}_2(k)$ by $\bar{\rho}(g) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Then $V_{\bar{\rho}} \cong kG$ and, by Proposition 2.34, $\mathrm{W}(k)$ is the universal deformation ring of $\bar{\rho}$. Note that here we do not require $|G|$ to be coprime to p.

Remark 2.36. Proposition 2.34 provides an easy way of constructing a representation $\bar{\rho}$ satisfying (**H4**), for which $\mathrm{Ad}(\bar{\rho})^G = kI_n$ does not hold. For instance, consider a trivial non-one-dimensional representation of an arbitrary group of order coprime to p.

2.4.3 Trivial representations

In this subsection we want to provide an example of a representation for which the versal deformation ring of $\bar{\rho}$ exists, but is not the universal deformation ring. According to Proposition 2.23, this is precisely the case when property (**H3**) is satisfied, but (**H4**) is not. Other examples with this property can be constructed using Lemma 3.9.

Lemma 2.37. *Let G be a profinite group and $\bar{\rho}$ be its trivial n-dimensional representation $\bar{\rho} : G \twoheadrightarrow \{I_n\} \hookrightarrow \mathrm{GL}_n(k)$. Then:*

(i) The canonical natural transformation $\mathrm{Lift}_{\bar{\rho}} \to \mathrm{Def}_{\bar{\rho}}$ is étale.

(ii) The functor $\mathrm{Lift}_{\bar{\rho}}$ is representable if and only if $\mathrm{CHom}(G, \mathbb{Z}/p\mathbb{Z})$ is finite.

(iii) If $\mathrm{CHom}(G, \mathbb{Z}/p\mathbb{Z})$ is finite and the canonical map $\mathrm{Lift}_{\bar{\rho}} \to \mathrm{Def}_{\bar{\rho}}$ is not as isomorphism then $\bar{\rho}$ has a versal deformation ring which is not universal.

Proof. (*i*) The transformation is smooth by Proposition 2.15. If $\rho \in \mathrm{Lift}_{\bar{\rho}}(k[\varepsilon])$ then $\operatorname{im}\rho \subseteq I_n + \varepsilon M_n(k)$, so for every $K \in I_n + \varepsilon M_n(k)$ we have $K\rho K^{-1} = \rho$. This implies that the canonical map $\mathrm{Lift}_{\bar{\rho}}(k[\varepsilon]) \to \mathrm{Def}_{\bar{\rho}}(k[\varepsilon])$ is an isomorphism.

(*ii*) By Proposition 2.15 and Theorem 1.47, the functor $\mathrm{Lift}_{\bar{\rho}}$ is representable if and only if $\mathrm{Lift}_{\bar{\rho}}(k[\varepsilon])$ is finite. Here we have $\mathrm{Lift}_{\bar{\rho}}(k[\varepsilon]) \cong \mathrm{CHom}(G, 1 + \varepsilon M_n(k)) \cong \mathrm{CHom}(G, M_n(k))$. Due to the finiteness of k, the additive group $M_n(k)$ is a product of a finite number of copies of $\mathbb{Z}/p\mathbb{Z}$, so $\mathrm{CHom}(G, M_n(k))$ is finite if and only if $\mathrm{CHom}(G, \mathbb{Z}/p\mathbb{Z})$ is finite.

(*iii*) Suppose that both listed conditions are satisfied. Part (*ii*) implies that $\mathrm{Lift}_{\bar{\rho}}$ is representable. Due to part (*i*), the object R representing $\mathrm{Lift}_{\bar{\rho}}$ is also a versal hull for $\mathrm{Def}_{\bar{\rho}}$. On the other hand, R does not represent $\mathrm{Def}_{\bar{\rho}}$, since $\mathrm{Lift}_{\bar{\rho}} \to \mathrm{Def}_{\bar{\rho}}$ is not an isomorphism. □

We present an easy application of Lemma 2.37:

Example 2.38. For every $n \in \mathbb{N}_{>1}$, the trivial n-dimensional representation of $G := \mathbb{Z}_p$ has a versal deformation ring R_v which is not universal.

It is actually easy to describe R_v (defined in the above example) explicitly, namely: $R_v \cong \mathrm{W}(k)[[X_{ij}]]_{1 \leqslant i,j \leqslant n}$. Indeed, if $A \in M_n(R_v)$ is the matrix with (i,j)-th entry equal to X_{ij}, then the continuous group homomorphism $\rho : \mathbb{Z}_p \to \mathrm{GL}_n(R_u)$ that is uniquely defined by $\rho(1) = I_n + A$, is a versal lift of $\bar{\rho}$.

Chapter 3

A generalization of Rainone's construction

In this chapter we analyze deformations of some particular two-dimensional representations, defined in Definition 3.2. The presented construction generalizes the one considered by Rainone in [Ra, Chapter 5] and has several interesting features. The main result of the chapter is Theorem 3.11 in which we simultaneously obtain:

- a family of versal, but not universal, deformation rings;
- for every finite field $k \neq \mathbb{F}_2, \mathbb{F}_3$, a way of realizing every ring of the form
$$\mathrm{W}(k)[[X_1, \dots, X_m]]/(a_0, a_1 X_1, \dots, a_m X_m),$$
where $m \in \mathbb{N}$, $a_0, \dots, a_m \in \mathrm{W}(k)$, as a universal deformation ring of some group representation.

In particular, we show in Example 3.13 that for every finite field $k \neq \mathbb{F}_2, \mathbb{F}_3$ and every natural r the power series ring $k[[X_1, \dots, X_r]]$ is a universal deformation ring of some finite group representation. As we will discuss in section 6.6.4, this observation contrasts the main result of Chapter 6.

We also note that the analysis performed in this chapter will be used in a particular case considered in Chapter 5 – see Remark 3.6 and Theorem 5.14.

3.1 The construction

In what follows we use the notation introduced in Chapter 1.

Notation 3.1. *We denote by* $\mathcal{M}$ *the category of topological profinite* $\mathrm{W}(k)$*-modules (i.e., inverse limits of finite* $\mathrm{W}(k)$*-modules) and continuous homomorphisms.*

For example, every $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ considered as a topological $\mathrm{W}(k)$-module is an object of $\mathcal{M}$. Note that for $k = \mathbb{F}_p$ the category $\mathcal{M}$ coincides with the category of abelian pro-p groups. We will be mainly interested in the modules that are finitely generated over $\mathrm{W}(k)$, to which we devote Lemmas 3.7 and 3.8.

Definition 3.2. Let $M \in \mathrm{Ob}(\mathcal{M})$ be given and let us write μ for $\mu_{\mathrm{W}(k)}$.

- We will denote by $\chi : \mu \times \mu \to \mu$ the homomorphism $\chi : (u, v) \mapsto \frac{u}{v}$. The same symbol will stand also for the homomorphism $\mu \times \mu \to \mathrm{Aut}_{\mathcal{M}}(M)$ sending $g \in \mu \times \mu$ to the automorphism $M \xrightarrow{x \mapsto \chi(g)x} M$.
- We define $G_M := M \rtimes_\chi (\mu \times \mu)$ and for $R \in \mathrm{Ob}(\hat{\mathcal{C}})$, $\alpha \in \mathrm{Hom}_{\mathcal{M}}(M, R)$ we denote by ρ_α the representation
$$G_M \ni \big(m, (u, v)\big) \mapsto \begin{pmatrix} 1 & \alpha(m) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \in \mathrm{GL}_2(R).$$
(note that here and later in this chapter we identify the groups $\mu_{\mathrm{W}(k)}$ and μ_R, cf. Remark 1.15).

It is easy to check that every group G_M defined as above is profinite and every ρ_α is a well-defined continuous representation. This chapter is devoted to studying deformation functors of representations of this particular type. The original example of Rainone corresponds to the case $M = \mathbb{Z}/p^n\mathbb{Z} \oplus \mathbb{Z}/p^m\mathbb{Z}$, $1 \leqslant m \leqslant n$, and $\alpha : M \to \mathbb{F}_p$ defined by $\alpha(a, b) = a \pmod p$.

3.2 Lifts and deformations of $\mathcal{M}$-morphisms

Mimicking the definitions of lifts and deformations of group representations, we define similar notions for $\mathcal{M}$-morphisms. These ad hoc definitions are not used in the literature and we introduce them only in order to facilitate our analysis of the described deformation functors.

Definition 3.3. Let $M \in \mathrm{Ob}(\mathcal{M})$ and $\alpha \in \mathrm{Hom}_{\mathcal{M}}(M,k)$ be given. For $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ we define:

- if $\beta, \gamma \in \mathrm{Hom}_{\mathcal{M}}(M,R)$ then $\beta \sim \gamma :\Leftrightarrow \exists\, u \in R_1^\times : \beta = u\gamma$,
- $\mathrm{Lift}_\alpha(R) := \{\beta \in \mathrm{Hom}_{\mathcal{M}}(M,R) \mid \pi_{\mathfrak{m}_R} \circ \beta = \alpha\}$,
- $\mathrm{Def}_\alpha(R) := \mathrm{Lift}_\alpha(R)/\sim$.

$$\begin{tikzcd} & R \arrow[d, two heads, "\pi_{\mathfrak{m}_R}"] \\ M \arrow[ur, dashed, "\beta"] \arrow[r, "\alpha"] & k \end{tikzcd}$$

One extends these definitions in an obvious way and obtains functors Lift_α, $\mathrm{Def}_\alpha : \hat{\mathcal{C}} \to \mathrm{Sets}$. We note that they are connected by the following property.

Lemma 3.4. *If $M \in \mathrm{Ob}(\mathcal{M})$ and $\alpha \in \mathrm{Hom}_{\mathcal{M}}(M,k)$ then the canonical transformation $\mathrm{Lift}_\alpha \to \mathrm{Def}_\alpha$ is smooth.*

Proof. Pick a $\hat{\mathcal{C}}$-surjection $\pi : B \twoheadrightarrow A$. We have to prove that the induced map $\mathrm{Lift}_\alpha(B) \to \mathrm{Lift}_\alpha(A) \times_{\mathrm{Def}_\alpha(A)} \mathrm{Def}_\alpha(B)$ is surjective. Let $\beta \in \mathrm{Lift}_\alpha(A)$ and $\gamma \in \mathrm{Lift}_\alpha(B)$ be such that $[\beta] = [\pi \circ \gamma]$. By definition, there exists $u \in A_{\equiv 1}^\times$ such that $\beta = u(\pi \circ \gamma)$. Choose any $v \in \pi^{-1}(u^{-1})$. Then $v\gamma \in \mathrm{Lift}_\alpha(B)$ is such that $\pi(v\gamma) = \beta$ and $[v\gamma] = [\gamma]$. □

The connection between the introduced deformation functors and the deformation functors of group representations described in Definition 3.2 is explained in the following proposition.

Proposition 3.5. *Let $M \in \mathrm{Ob}(\mathcal{M})$ and $\alpha \in \mathrm{Hom}_{\mathcal{M}}(M,k)$ be given. If $\alpha \equiv 0$, assume that $p > 3$, otherwise assume $k \neq \mathbb{F}_2, \mathbb{F}_3$. Then the correspondence $[\beta] \to [\rho_\beta]$ defines a natural isomorphism between Def_α and $\mathrm{Def}_{\rho_\alpha}$.*

Proof. The assumption $k \neq \mathbb{F}_2, \mathbb{F}_3$ implies that there exists $\lambda \in k^\times$ such that $\lambda^2 \neq 1$. In case $p > 3$ we can even (and will) choose such λ in the prime subfield $\mathbb{F}_p \subseteq k$. Given a ring $R \in \mathrm{Ob}(\hat{\mathcal{C}})$, we denote by the same symbol λ

the Teichmüller lift of λ to μ_R. Observe that $\lambda^2 - 1$ is then invertible in R and in case $p > 3$ we also have that λ lies in the image of $\mathbb{Z}_p$ in R.

We will denote the matrix $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1 \end{smallmatrix}\right)$ by A. It is easy to check that, for every $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ and $M \in M_2(R)$, the matrices A and M commute if and only if M is diagonal (for this property it is actually only important that $\lambda - 1$ is invertible).

We clearly have a natural transformation $\mathrm{Lift}_\alpha \to \mathrm{Lift}_{\rho_\alpha}$ defined by $\beta \mapsto \rho_\beta$. In order to prove that the transformation $[\beta] \to [\rho_\beta]$ is well-defined and injective, we verify the following property: *if $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ and $\beta, \gamma \in \mathrm{Lift}_\alpha(R)$ then $[\rho_\beta] = [\rho_\gamma]$ holds if and only if $\beta = u\gamma$ for some $u \in R_1^\times$.* Suppose first $[\rho_\beta] = [\rho_\gamma]$. Then there exists a matrix $K \in \ker \pi_{\mathfrak{m}_R}$ such that $K\rho_\beta K^{-1} = \rho_\gamma$. Evaluating both sides at A we obtain that K and A commute, so K is diagonal. If $a, b \in R_1^\times$ are such that $K = \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$, then $\gamma = \frac{a}{b}\beta$. Conversely, if $\gamma = u\beta$ for some $u \in R_1^\times$, then $\left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right) \rho_\beta \left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1} = \rho_\gamma$.

It remains to check the surjectivity of the transformation. Consider $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ and $\xi \in \mathrm{Def}_{\rho_\alpha}(R)$. Since the order of $\mu^2 \leqslant G_M$ is coprime to p, Proposition 2.34 implies that $\rho_\alpha|_{\mu^2}$ has precisely one deformation to R. It follows easily that $\xi = [\rho]$ for some $\rho \in \mathrm{Lift}_{\rho_\alpha}(R)$ such that $\rho|_{\mu^2} = id|_{\mu^2}$. In particular, $\rho \left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

We claim that for every $m \in M$ the matrix $\rho \left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right)$ is upper triangular. Pick $m \in M$ and let $a, d \in 1 + \mathfrak{m}_R$, $b \equiv \alpha(m) \pmod{\mathfrak{m}_R}$, $c \in \mathfrak{m}_R$ be such that $\rho \left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Then $\rho \left(\begin{smallmatrix} 1 & \lambda m \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1 \end{smallmatrix}\right) \rho \left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} a & \lambda b \\ \frac{1}{\lambda} c & d \end{smallmatrix}\right)$. We proceed now in two cases.

If α is not trivial: The subset $M \backslash \ker \alpha$ additively generates M, so it is sufficient to prove the claim for all $m \notin \ker \alpha$. If $\alpha(m) \neq 0$, then b is invertible. Since $\left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & \lambda m \\ 0 & 1 \end{smallmatrix}\right)$ commute, so do their images, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} a & \lambda b \\ \frac{1}{\lambda} c & d \end{smallmatrix}\right)$; in particular: $a + \frac{1}{\lambda} bc = a + \lambda bc$. Consequently, $(\lambda^2 - 1)bc = 0$ and $c = 0$ because $(\lambda^2 - 1)b$ is invertible.

If α is trivial: By our choice, λ lies in the image of $\mathbb{Z}_p$ in R, so there exists a sequence $(f_n)_{n \in \mathbb{N}}$ of integers converging to λ. For $n \in \mathbb{Z}$, let a_n, b_n, c_n, $d_n \in R$ be such that $\rho \left(\begin{smallmatrix} 1 & nm \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^n = \left(\begin{smallmatrix} a_n & b_n \\ c_n & d_n \end{smallmatrix}\right)$. Easy induction on $n \in \mathbb{N}$ shows that $c_n = ct_n$ for some $t_n \in R$ satisfying $t_n \equiv n \pmod{\mathfrak{m}_R}$; this property immediately extends to all $n \in \mathbb{Z}$. By continuity,

$$\lim_{n \to \infty} \begin{pmatrix} a_{f_n} & b_{f_n} \\ c t_{f_n} & d_{f_n} \end{pmatrix} = \lim_{n \to \infty} \rho \begin{pmatrix} 1 & f_n m \\ 0 & 1 \end{pmatrix} = \rho \begin{pmatrix} 1 & \lambda m \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & \lambda b \\ \frac{1}{\lambda} c & d \end{pmatrix},$$

so $c(t_{f_n} - \frac{1}{\lambda})$ converges to zero. For sufficiently large n we have $t_{f_n} \equiv f_n \equiv \lambda \pmod{\mathfrak{m}_R}$, which implies that $t_{f_n} \not\equiv \frac{1}{\lambda} \pmod{\mathfrak{m}_R}$ and $t_{f_n} - \frac{1}{\lambda}$ is invertible. Hence, $c = 0$.

To finish the proof (in both cases), pick $m \in M$ and let $\tilde{m} := \frac{m}{\lambda - 1}$. From the relation $\rho\left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right) = [\left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1 \end{smallmatrix}\right), \rho\left(\begin{smallmatrix} 1 & \tilde{m} \\ 0 & 1 \end{smallmatrix}\right)]$ and the fact that $\rho\left(\begin{smallmatrix} 1 & \tilde{m} \\ 0 & 1 \end{smallmatrix}\right)$ is upper-triangular, one concludes that the diagonal entries of $\rho\left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right)$ are identities. There exists therefore a function $\beta : M \to R$ such that $\rho\left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right) = \begin{pmatrix} 1 & \beta(m) \\ 0 & 1 \end{pmatrix}$ for every $m \in M$. It follows easily that $\beta \in \mathrm{Lift}_\alpha(R)$ and we obtain $\xi = [\rho_\beta]$, as required. □

We mention a possible generalization of the argument presented in the proof of Proposition 3.5. It will find its application in Chapter 5 – see Theorem 5.14.

Remark 3.6. Given $\alpha \in \mathrm{Hom}_{\mathcal{M}}(M, k)$ and $H \leqslant \mu \times \mu$, one can consider the group $G_{M,H} := M \rtimes_\chi H \leqslant G_M$ and its representation $\rho_{\alpha,H} := \rho_\alpha|_{G_{M,H}}$. If $\alpha \not\equiv 0$, define $X_H := \chi(H)\backslash\{\pm 1\}$, otherwise let $X_H := (\chi(H)\backslash\{\pm 1\}) \cap \mathbb{Z}_p$. If X_H is non-empty, the analysis carried out in the proof of Proposition 3.5 can be almost entirely applied to studying $\mathrm{Def}_{\rho_{\alpha,H}}$. Indeed, one only needs to alter the proof by substituting $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1 \end{smallmatrix}\right)$ with an arbitrary element of the set $H \cap \chi^{-1}(X_H)$ to conclude: *Every deformation class of $\rho_{\alpha,H}$ to $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ contains a lift ρ such that $\rho|_H = \mathrm{id}_H$; for every ρ with this property there exists a function $\beta : M \to R$ such that $\rho\left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right) = \begin{pmatrix} 1 & \beta(m) \\ 0 & 1 \end{pmatrix}$ for every $m \in M$.*

The only significant difference is that, depending on $\chi(H)$, the conditions imposed on β may be different than in the general case (more specifically, in some cases it does not have to be W(k)-linear). However, this problem does not occur if the ring generated by $\mathbb{Z}_p$ and $\chi(H)$ coincides with W(k).

3.3 Finitely generated W(k)-modules

We are mostly interested in studying the representations introduced in Definition 3.2 in case $M \in \mathrm{Ob}(\mathcal{M})$ is finitely generated over W(k). Before proceeding further with our analysis, we present therefore two technical lemmas related to properties of such modules.

Lemma 3.7. *Let M be a finitely generated* W(k)*-module. Then:*

(i) *If $M \in \mathrm{Ob}(\mathcal{M})$ then the topology on M is the p-adic one (the one in which $p^r M$, $r \in \mathbb{N}$, forms the base of topology). Conversely, M together with p-adic topology is an object of $\mathcal{M}$.*

(ii) *If $N \in \mathcal{M}$ then every $\mathrm{W}(k)$-linear morphism $f : M \to N$ is continuous.*

(iii) *There exist $e_1, \dots, e_m \in M$ such that $M = \oplus_{i=1}^m W(k)e_i$.*

(iv) *If $\alpha \in \mathrm{Hom}_{\mathcal{M}}(M, k)$ is a non-zero morphism then there exist $e_0 \in 1 + \ker \alpha$ and $e_1, \dots, e_m \in \ker \alpha$ such that $M = \oplus_{i=0}^m \mathrm{W}(k)\, e_i$.*

Proof. (i) If M is finitely generated then for every $r \in \mathbb{N}$, the module $M/p^r M$ is finitely generated over $\mathrm{W}(k)/(p^r)$, hence finite. This proves that M with the p-adic topology is an object of $\mathcal{M}$. On the other hand, if $M \in \mathrm{Ob}(\mathcal{M})$ then for every $r \in \mathbb{N}$, the submodule $p^r M$ of M is closed. Since it is of finite index, it is also open. For every open submodule U of M we have that M/U is a p-group of finite index, hence $p^r M \subseteq U$ for some $r \in \mathbb{N}$. This shows that $p^r M$ is the basis for the topology on M.

(ii) If U is an open submodule of N then $p^r N \subseteq U$ for some $r \in \mathbb{N}$. Hence, $p^r M \subseteq f^{-1}(U)$, so $f^{-1}(U)$ is open.

(iii) This is just a direct application of the structure theorem for finitely generated modules over principal ideal domains.

(iv) We use the previous claim to obtain $e_0, e_1, \dots, e_m \in M$ such that $M = \oplus_{i=0}^m W(k)e_i$. The set $Z := \{e_0, \dots, e_m\} \cap (M \backslash \ker \alpha)$ is non-empty and without loss of generality we may assume that $e_0 \in Z$ and $\#W(k)e_0 = \min\{\#W(k)e \mid e \in Z\}$. Rescaling, we can furthermore assume that $e_0 \in 1 + \ker \alpha$. If $e_i \in Z$ for some $i > 0$ then there exists $\lambda_i \in \mathrm{W}(k)$ such that $e_i' := e_i - \lambda_i e_0$ belongs to $\ker \alpha$. Changing e_i to e_i' for all such $i > 0$, we obtain e_i with all required properties. □

The following lemma can be seen as an extension of Lemma 2.26.

Lemma 3.8. *Given $M \in \mathrm{Ob}(\mathcal{M})$, the k-vector space $\mathrm{Hom}_{\mathcal{M}}(M, k)$ is finite dimensional if and only if M is a finitely generated $\mathrm{W}(k)$-module.*

Proof. The "if" part is obvious. For the other implication, suppose that $\mathrm{Hom}_{\mathcal{M}}(M,k) = \mathrm{Hom}_{\mathcal{M}}(M/pM,k)$ is finite dimensional. Then there exist only finitely many open subspaces of M/pM of codimension one. Since every open subspace of M/pM is an intersection of such subspaces, we conclude that M/pM has only finitely many open subspaces, hence is finite dimensional.

To finish the proof, it suffices to apply the following variant of Nakayama's lemma ([Ei, Exercise 7.2]): *Suppose that M is a module over a ring R that is complete with respect to an ideal $\mathfrak{m}$. If M is separated (that is: $\bigcap_{r\in\mathbb{N}} \mathfrak{m}^r M = \{0\}$) and the images of $m_1,\ldots,m_n \in M$ generate $M/\mathfrak{m}M$, then $m_1,\ldots,m_n$ generate M.* Note that in our case $R = \mathrm{W}(k)$ is complete with respect to the ideal (p) and M is separated, since it is an inverse limit of finite $\mathrm{W}(k)$-modules (each of which is separated). □

3.4 Representability

Our goal is to examine the existence of universal deformation rings of the representations defined in Definition 3.2. In cases in which they exist, we are also interested in their explicit determination.

Keeping in mind Proposition 3.5, we will look closer at properties of deformation functors Def_α introduced in Definition 3.3. We divide our analysis into two cases.

Lemma 3.9. *If $M \in \mathrm{Ob}(\mathcal{M})$ and $\alpha \in \mathrm{Hom}_{\mathcal{M}}(M,k)$ is the zero morphism then:*

(i) Lift_α *coincides with the functor* $R \mapsto \mathrm{Hom}_{\mathcal{M}}(M,\mathfrak{m}_R)$.

(ii) The canonical transformation $\mathrm{Lift}_\alpha \to \mathrm{Def}_\alpha$ *is étale.*

(iii) $\mathrm{Def}_\alpha(k[\varepsilon])$ *is finite dimensional if and only if M is finitely generated over* $\mathrm{W}(k)$.

(iv) Suppose M is finitely generated, $M = \oplus_{i=1}^m W(k)e_i$ (cf. Lemma 3.7) and $a_i \in \mathrm{W}(k)$, $i \in \{1,\ldots,m\}$, are such that $\mathrm{W}(k)\,e_i \cong \mathrm{W}(k)/(a_i)$. Then

$$R_v := \mathrm{W}(k)[[X_1,\ldots,X_m]]/(a_1X_1,\ldots,a_mX_m)$$

is the versal hull of Def_α. *However, R_v does not represent* Def_α, *unless $M = 0$.*

Proof. (*i*) This claim follows directly from definition.

(*ii*) The transformation $\mathrm{Lift}_\alpha \to \mathrm{Def}_\alpha$ is smooth by Lemma 3.4. The fact that the tangent spaces to Lift_α and Def_α are isomorphic follows easily from the first part of the lemma and the following observation: if $R = k[\varepsilon]$, then $ux = x$ for every $u \in R^\times_{\equiv 1}$ and $x \in \mathfrak{m}_R$.

(*iii*) By the first two parts of the lemma, $\mathrm{Def}_\alpha(k[\varepsilon]) \cong \mathrm{Lift}_\alpha(k[\varepsilon]) \cong \mathrm{Hom}_{\mathcal{M}}(M, k)$. The claim follows directly from Lemma 3.8.

(*iv*) Given $R \in \mathrm{Ob}(\hat{\mathcal{C}})$, every $\beta \in \mathrm{Hom}_{\mathrm{W}(k)}(M, \mathfrak{m}_R)$ is fully determined by its values on $e_1, \ldots, e_m$. We obtain thus a natural isomorphism $h_{R_v} \cong \mathrm{Lift}_\alpha$, defined by assigning to each $\varphi \in h_{R_v}(R)$ the only $\beta \in \mathrm{Hom}_{\mathrm{W}(k)}(M, \mathfrak{m}_R)$ for which $\beta(e_i) = \varphi(X_i)$. Since the canonical transformation $\Phi : \mathrm{Lift}_\alpha \to \mathrm{Def}_\alpha$ is étale, R_v is a versal hull of Def_α.

To finish the proof, we have to show that Φ is not injective when $M \neq 0$. Indeed, choose R for which there exists a non-zero $\gamma \in \mathrm{Hom}_{\mathcal{M}}(M, \mathfrak{m}_R)$, for example: $R = \mathrm{W}(k)/(pa_1)$. Then Φ is not injective on $R[[X]]$. Indeed, $\gamma \sim (1 + X)\gamma$ and $\gamma \neq (1 + X)\gamma$. $\square$

Lemma 3.10. *If $M \in \mathrm{Ob}(\mathcal{M})$ and $\alpha \in \mathrm{Hom}_{\mathcal{M}}(M, k)$ is a non-zero morphism then:*

(*i*) *For every $e_0 \in 1 + \ker\alpha$ we have a natural isomorphism of functors Def_α and $R \mapsto \{\beta \in \mathrm{Lift}_\alpha(R) \mid \beta(e_0) = 1\}$.*

(*ii*) *The canonical transformation $\mathrm{Lift}_\alpha \to \mathrm{Def}_\alpha$ is smooth, but not étale.*

(*iii*) *$\mathrm{Def}_\alpha(k[\varepsilon])$ is finite dimensional if and only if M is finitely generated over $\mathrm{W}(k)$.*

(*iv*) *Suppose M is finitely generated, $e_0 \in 1 + \ker\alpha$, $e_1, \ldots, e_m \in \ker\alpha$ are such that $M = \bigoplus_{i=0}^m \mathrm{W}(k)\, e_i$ (cf. Lemma 3.7) and let $a_i \in \mathrm{W}(k)$, $i \in \{0, \ldots, m\}$, be such that $\mathrm{W}(k)\, e_i \cong \mathrm{W}(k)/(a_i)$. Then*

$$R_v := \mathrm{W}(k)[[X_1, \ldots, X_m]]/(a_0, a_1X_1, \ldots, a_mX_m)$$

represents Def_α.

Proof. (*i*) The claim follows easily from the following observation: if $e_0 \in 1 + \ker\alpha$ and $R \in \mathrm{Ob}(\hat{\mathcal{C}})$, then each $\beta \in \mathrm{Lift}_\alpha(R)$ is in relation $\sim$ with precisely one $\beta_0 \in \mathrm{Lift}_\alpha(R)$ such that $\beta_0(e_0) = 1$, namely with $\frac{1}{\beta(e_0)} \cdot \beta$.

(*ii*) The transformation $\mathrm{Lift}_\alpha \to \mathrm{Def}_\alpha$ is smooth by Lemma 3.4. However, the induced map $\mathrm{Lift}_\alpha(k[\varepsilon]) \to \mathrm{Def}_\alpha(k[\varepsilon])$ is not injective, since it has a one-dimensional kernel (an easy corollary from the first part of the lemma).

(*iii*) Choose $e_0 \in 1+\ker\alpha$. By the first part $\mathrm{Def}_\alpha(k[\varepsilon]) \cong \{\beta \in \mathrm{Lift}_\alpha(k[\varepsilon]) \mid \beta(e_0) = 1\}$. We have $M = W(k)e_0 + \ker\alpha$ and $\mathrm{W}(k)e_0 \cap \ker\alpha = pe_0$, so $\beta \in \mathrm{Lift}_\alpha(k[\varepsilon])$ such that $\beta(e_0) = 1$ correspond bijectively with $\beta \in \mathrm{Hom}_{\mathcal{M}}(\ker\alpha/\mathrm{W}(k)pe_0, k\varepsilon)$. We conclude, using Lemma 3.8, that $\mathrm{Def}_\alpha(k[\varepsilon])$ is finite dimensional if and only if $\ker\alpha/\mathrm{W}(k)pe_0$ is finitely generated. This is the case if and only if $\ker\alpha$ is finitely generated, which holds if and only if M itself is finitely generated.

(*iv*) Every $\beta \in \mathrm{Lift}_\alpha(R)$ is uniquely determined by images $\beta(e_0) \in R_1^\times$ and $\beta(e_1), \ldots, \beta(e_m) \in \mathfrak{m}_R$. Using the first part of this lemma and keeping in mind that $\{\beta \in \mathrm{Lift}_\alpha(R) \mid \beta(e_0) = 1\}$ is non-empty if and only if $a_0 = 0$ in R, we easily obtain the claim reasoning similarly as in the proof of Lemma 3.9.(*iv*). □

3.5 Conclusions

Theorem 3.11. *Given $M \in \mathrm{Ob}(\mathcal{M})$ and $\alpha \in \mathrm{Hom}_{\mathcal{M}}(M, k)$, define G_M and $\rho_\alpha : G_M \to \mathrm{GL}_n(k)$ as in Definition 3.2.*

(*i*) *If M is not finitely generated over $\mathrm{W}(k)$ then ρ_α has no versal deformation ring.*

(*ii*) *Suppose $\alpha \equiv 0$. If M is finitely generated, there exist $a_1, \ldots, a_m \in \mathrm{W}(k)$ such that $M \cong \bigoplus_{i=1}^m \mathrm{W}(k)/(a_i)$. Assuming $p > 3$ we obtain that*

$$R_v := \mathrm{W}(k)[[X_1, \ldots, X_m]]/(a_1X_1, \ldots, a_mX_m)$$

is the versal deformation ring of ρ_α; however, if $M \neq \{0\}$ then R_v is not the universal deformation ring.

(iii) Suppose $\alpha \not\equiv 0$. If M is finitely generated, there exist $a_0, \ldots, a_m \in \mathrm{W}(k)$ and submodules M_1, $M_2 \leqslant M$ such that $M = M_1 \oplus M_2$, $M_1 \cong \mathrm{W}(k)/(a_0)$, $M_2 \cong \bigoplus_{i=1}^{m} \mathrm{W}(k)/(a_i)$, $M_1 \nsubseteq \ker\alpha$ and $M_2 \subseteq \ker\alpha$. Assuming $k \neq \mathbb{F}_2, \mathbb{F}_3$ we obtain that

$$R_u := \mathrm{W}(k)[[X_1, \ldots, X_m]]/(a_0, a_1X_1, \ldots, a_mX_m)$$

is the universal deformation ring of ρ_α.

Proof. Combine Proposition 3.5 with Lemmas 3.9 and 3.10 (using also Lemma 3.7, part *(iii)* and *(iv)*, for the statements regarding the structure of M in case it is finitely generated). □

Example 3.12. When $p > 3$, $a \in pW(k)$, $M := \mathrm{W}(k)/(a)$ and α is trivial, the above construction yields $R_v = \mathrm{W}(k)[[X]]/(aX)$ as a versal (but not universal) deformation ring.

Example 3.13. For $M := k^{r+1}$ ($r \in \mathbb{N}$), $\alpha : M \to k$ projection on the first coordinate, $k \neq \mathbb{F}_2, \mathbb{F}_3$, the above construction yields $R_u = k[[X_1, \ldots, X_r]]$ as the universal deformation ring.

We actually have the following more general corollary.

Corollary 3.14. *If $k \neq \mathbb{F}_2, \mathbb{F}_3$ then for arbitrary positive integers k_0, k_1, …, k_m the ring*

$$\mathrm{W}(k)[[X_1, \ldots, X_m]]/(p^{k_0}, p^{k_1}X_1, \ldots, p^{k_m}X_m)$$

can be obtained as a universal deformation ring of a finite group representation.

Remark 3.15. The condition $\mathrm{Ad}(\rho_\alpha)^{G_M} = kI_n$ is satisfied if and only if α is non-trivial. It is interesting to observe explicitly how this property (or its lack) influences the existence of the universal deformation ring in Theorem 3.11.

It should be also underlined that in this chapter we have obtained representability results explicitly constructing versal or universal deformation rings and not relying on Schlessinger criteria giving their existence a priori.

Chapter 4

Towards determining the universal deformation rings

This chapter has an auxiliary character. We present here a collection of various technical results that can be used to gain information about the universal deformation ring of a given group representation. Our considerations originate from answers to several questions that arose in the course of preparation of Chapter 5.

The main idea of the first part of this chapter is to identify and look closer at deformations with particularly interesting properties. We present now sample results in this direction, which we will apply in the next chapter (for the proof, see Remark 4.11, Example 4.18 and Lemma 4.19). Here and below G is a profinite group and $\bar{\rho} : G \to \mathrm{GL}_n(k)$ is a continuous representation.

Proposition 4.1. *Suppose $\bar{\rho}$ has a universal deformation ring $R_{\bar{\rho}}$, let $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ and $\rho \in \mathrm{Lift}_{\bar{\rho}}(R)$ be given and assume that for some $i, j \in [n]$ we have $I_n + Re_{ij} \subseteq \operatorname{im} \rho$. Then:*

(i) R is a quotient of $R_{\bar{\rho}}$.

(ii) $R_{\bar{\rho}} \cong R$ holds if and only if $[\rho]$ is a universal deformation of $\bar{\rho}$.

(iii) The natural transformation $h_R \to \mathrm{Def}_{\bar{\rho}}$ that associates, given $S \in \mathrm{Ob}(\hat{\mathcal{C}})$, the map $f \in h_R(S)$ with the deformation $[f \circ \rho] \in \mathrm{Def}_{\bar{\rho}}(S)$ is injective.

In the second part we choose a different approach and try to relate universal deformation rings of different representations. For example, we obtain the following result, which is a shortened version of Proposition 4.40.

Proposition 4.2. *Suppose that $N \lhd G$ is a closed normal subgroup such that $\mathrm{Ad}(\bar{\rho})^N = kI_n$ and $\bar{\rho}|_N$ has a universal deformation ring R. Assume moreover that there exists a universal lift of $\bar{\rho}|_N$ that may be extended to a lift $G \to \mathrm{GL}_n(R)$ of $\bar{\rho}$.*

If $\mathrm{CHom}(G/N, \mathbb{Z}/p\mathbb{Z})$ is finite then $R[[(G/N)^{ab,p}]]$ is a universal deformation ring of $\bar{\rho}$. Otherwise $\mathrm{Def}_{\bar{\rho}}$ is not representable over $\hat{\mathcal{C}}$.

Many of our ideas culminate in Proposition 4.29 which links both approaches and we obtain the following simple corollary (see Example 4.31).

Proposition 4.3. *Let $R_{\bar{\rho}}$, R and ρ be as in Proposition 4.1. Then there exists a universal deformation ring R_k of the representation $\mathrm{im}\,\bar{\rho} \hookrightarrow \mathrm{GL}_n(k)$. Moreover, the fiber product $R \times_k R_k$ is a quotient of $R_{\bar{\rho}}$ and a necessary condition for $R_{\bar{\rho}} \cong R$ is $R_k \cong k$.*

The above presented results comprise everything we will need in Chapter 5. On the other hand, rather than proving them in an ad hoc manner, we prefer to develop a more systematic approach.

Doing this we naturally obtain several further results. Among others, in Corollary 4.13 we resolve the problem with two definitions of a versal deformation ring, mentioned in section 2.3.1. Corollary 4.27 is not essential in the next chapter, but gives a good motivation for our considerations, as described in section 5.1. We present also some results for which we do not have an immediate application, but which we believe make our exposition more complete. Finally, let us note that despite being primarily interested in universal deformation rings, for greater generality we try to formulate our results, whenever possible, in terms of versal rings.

4.1 Properties of lifts

Let G be a profinite group and $\bar{\rho} : G \to \mathrm{GL}_n(k)$ a continuous representation for which a versal deformation ring $R_{\bar{\rho}} \in \mathrm{Ob}(\hat{\mathcal{C}})$ exists. Suppose $R_{\bar{\rho}}$ is not known, but a ring $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ and a lift $\rho \in \mathrm{Lift}_{\bar{\rho}}(R)$ are given. What conclusions about $R_{\bar{\rho}}$ can one draw from this fact and which properties of ρ are worth studying? These and related questions will occupy us in the current section.

Even though we are interested only in deformation functors of group representations, we find it convenient to analyze our problems first in an abstract category theoretic setting. In what follows F will denote an arbitrary set valued functor $F : \hat{\mathcal{C}} \to \text{Sets}$.

4.1.1 Properties of natural transformations $h_R \to F$

Notation 4.4. *For $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ and $\xi \in F(R)$ we will denote by ξ^* the natural transformation $h_R \to F$ defined as*

$$\forall S \in \mathrm{Ob}(\hat{\mathcal{C}}) : \quad h_R(S) \ni f \mapsto F(f)(\xi) \in F(S).$$

By Yoneda's lemma, every natural transformation $\Theta : h_R \to F$ is of the above form. More specifically, $\Theta = \Theta(\mathrm{id}_R)^*$.

Remark 4.5. Our notation is motivated by the following observation. For $T \in \mathrm{Ob}(\hat{\mathcal{C}})$, $F = h_T$, $\xi \in h_T(R)$ the obtained natural transformation $h_R \to h_T$ coincides with the pull-back transformation, standardly denoted by ξ^*.

We are mainly interested in functors that are representable or at least have a versal hull. Consequently, we are mostly interested in the case when ξ^* is a natural isomorphism or at least is étale. However, it is convenient to introduce also the following weaker properties.

Definition 4.6. Given $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ and a natural transformation $\Phi : h_R \to F$, we define the following potential properties of Φ:

(**S**) Φ is surjective on every object of $\hat{\mathcal{C}}$.

(**I**) Φ is injective on $k[\varepsilon]$.

(**SI**) Φ satisfies (**S**) and (**I**).

If, working with the above properties, we will want to emphasize the choice of the functor, we will add a suitable subscript to their names and, for example, write: property ($\mathbf{S}_\mathrm{F}$), property ($\mathbf{I}_\mathrm{G}$) etc.

It is clear that if Φ is a natural isomorphism, then Φ satisfies (**SI**). The same implication holds true if Φ is étale, provided that $\#F(k) = 1$ (Remark 1.44).

We will say that $\xi \in F(R)$ has one of the above defined properties if and only if ξ^* does. Moreover, we will say that R satisfies one of them if and only if there exists $\xi \in F(R)$ satisfying it.

Definition 4.7. We will denote by $F\hat{\mathcal{C}}$ the so called *category of elements* of F. The objects of $F\hat{\mathcal{C}}$ are pairs (R, ξ_R) of $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ and $\xi_R \in F(R)$; the morphisms are defined as

$$\mathrm{Hom}_{F\hat{\mathcal{C}}}\left((R, \xi_R),\, (S, \xi_S)\right) := \{f \in \mathrm{Hom}_{\hat{\mathcal{C}}}(R, S) \text{ such that } F(f)(\xi_R) = \xi_S\},$$

for every pair of objects $(R, \xi_R), (S, \xi_S)$.

We will say that $(R, \xi) \in \mathrm{Ob}(F\hat{\mathcal{C}})$ has one of the properties (**S**), (**I**), (**SI**) if and only if ξ does. Similarly, we will say that (R, ξ) represents F (is a versal hull of F) if and only if ξ^* is a natural isomorphism (is étale).

4.1.2 Relations between properties of different transformations

Before stating the next lemma, we recall that in category theory a split monomorphism (or: a section) is a morphism that is a right inverse of some other morphism. Similarly, a split epimorphism (or: a retraction) is a left inverse of some morphism.

Lemma 4.8. *For a $\hat{\mathcal{C}}$-morphism $\varphi : R \to S$ let $\varphi^* : h_S \to h_R$ denote the pull-back transformation. Then:*

(i) φ^* is injective if and only if $\varphi^*_{k[\varepsilon]}$ is injective and if and only if φ is surjective.

(ii) φ^* is surjective if and only if φ is a split monomorphism in $\hat{\mathcal{C}}$.

Proof. For part (i) see the proof of [Maz1, Lemma, p. 279], part (ii) follows easily from definitions. □

Lemma 4.9. *Let $f : (R, \xi_R) \to (S, \xi_S)$ be a morphism of $F\hat{\mathcal{C}}$ and denote by $f^* : h_S \to h_R$ the pull-back transformation. Then $\xi_S^* = \xi_R^* \circ f^*$ and we have the following implications:*

- *($\boldsymbol{S}$) for $\xi_S \Rightarrow$ ($\boldsymbol{S}$) for ξ_R.*
- *($\boldsymbol{I}$) for $\xi_S \Rightarrow f$ surjective.*
- *($\boldsymbol{S}$) for ξ_R and f is a split monomorphism $\Rightarrow$ ($\boldsymbol{S}$) for ξ_S.*
- *($\boldsymbol{I}$) for ξ_R and f surjective $\Rightarrow$ ($\boldsymbol{I}$) for ξ_S.*

$$\begin{tikzcd} h_S \arrow[rr, "f^*"] \arrow[dr, "\xi_S^*"'] & & h_R \arrow[dl, "\xi_R^*"] \\ & F & \end{tikzcd}$$

Proof. For every $T \in \mathrm{Ob}(\hat{\mathcal{C}})$ and $g \in \mathrm{Hom}_{\hat{\mathcal{C}}}(S,T)$ we have

$$\xi_S^*(g) = F(g)(\xi_S) = F(g \circ f)(\xi_R) = \xi_R^*(f^*(g)),$$

which shows that $\xi_S^* = \xi_R^* \circ f^*$. All the claims follow easily from this observation and Lemma 4.8. $\square$

Proposition 4.10. *If there exists $(R_F, \xi_F) \in \mathrm{Ob}(F\hat{\mathcal{C}})$ satisfying ($\boldsymbol{SI}$) then for every $(R, \xi) \in \mathrm{Ob}(F\hat{\mathcal{C}})$ we have:*

(i) *(R,ξ) satisfies ($\boldsymbol{S}$) if and only if a surjection $(R,\xi) \to (R_F,\xi_F)$ exists. If this is the case then every map $(R,\xi) \to (R_F,\xi_F)$ is a surjection.*

(ii) *(R,ξ) satisfies ($\boldsymbol{I}$) if and only if a surjection $(R_F,\xi_F) \to (R,\xi)$ exists. If this is the case then every map $(R_F,\xi_F) \to (R,\xi)$ is a surjection.*

(iii) *(R,ξ) satisfies ($\boldsymbol{SI}$) if and only if an isomorphism $(R_F,\xi_F) \to (R,\xi)$ exists. If this is the case then every map $(R_F,\xi_F) \to (R,\xi)$ is an isomorphism.*

Sketch of the proof. We use the implications listed in Lemma 4.9 in the following way:

(i) The "if" part follows from the first implication, the "only if" part and the additional statement follow from the second implication.

(ii) Similarly, we use the fourth implication for the "if" part and the second implication for the "only if" part as well as the additional statement.

(*iii*) The "if" part is obvious. For the other statements we use part (*i*), part (*ii*) and Lemma 1.10. □

Remark 4.11. In the rest of this chapter we will concentrate on property (**I**) rather than property (**S**). This is motivated by the following arguments:

(1) Testing property (**I**) is much easier than testing property (**S**) and can be performed using some universal criteria. On contrary, arguments proving property (**S**) usually depend heavily on the particular case considered. This will be very well seen in Chapter 5.

(2) We find very useful the following observations following from Proposition 4.10.(*ii*). Let F be such that $\#F(k) = 1$ and F has a versal hull. Suppose we are given $(R, \xi) \in F\hat{\mathcal{C}}$. If we can show that ξ has property (**I**), we obtain that R is a quotient of the versal hull of F, even without having determined the versal hull explicitly. Moreover, in this case we also conclude using Lemma 1.10 that R is a versal hull of F if and only if (R, ξ) is a versal hull (note that in general such implication is of course not true).

4.1.3 An application: two definitions of a versal deformation ring

The following result fills the small gap mentioned in section 2.3.1.

Proposition 4.12. *Suppose* $F : \hat{\mathcal{C}} \to \mathrm{Sets}$ *satisfies properties* (***H0***)-(***H2***) *of Theorem 1.47. If* $(R, \xi) \in \mathrm{Ob}(F\hat{\mathcal{C}})$ *has property* (***SI***), *then it is a versal hull of* F.

Proof. If $(R, \xi) \in \mathrm{Ob}(F\hat{\mathcal{C}})$ has property (**SI**) then $t_F \cong t_R$ is finite and a versal hull (R_v, ξ_v) of F exists by Theorem 1.47. Since (R_v, ξ_v) has property (**SI**) by Remark 1.44, Proposition 4.10.(*iii*) shows that there exists an isomorphism $(R_v, \xi_v) \to (R, \xi)$. This proves the claim. □

Corollary 4.13. *Let* F *be a deformation functor of some group representation. As an immediate consequence of Proposition 4.12 and Proposition 2.14, we obtain that Definition 2.19 and Definition 2.22 of a versal deformation ring are equivalent.*

Note that, as the below example shows, property (**SI**) does not imply in general that a transformation is smooth.

Example 4.14. Given $R \in \mathrm{Ob}(\hat{\mathcal{C}})$, set $F(R) := \mathfrak{m}_R/(\mathfrak{m}_R^2 \cap \mathrm{Ann}\, p)$. For $f \in \mathrm{Hom}_{\hat{\mathcal{C}}}(R,S)$ let $F(f) : F(R) \to F(S)$ be the induced map. We obtain a continuous functor $F : \hat{\mathcal{C}} \to \mathrm{Sets}$, for which $\#F(k) = 1$.

Denote the functor $R \mapsto \mathfrak{m}_R$ by G. Assigning to $x \in G(R)$ its equivalence class in $F(R)$, we obtain a natural transformation $\Theta : G \to F$ that is surjective and bijective on $k[\varepsilon]$. To see that it is not smooth, consider $A := \mathrm{W}(k)$, $B := \mathrm{W}(k)/(p^3)$. One easily checks that the map $G(A) \to G(B) \times_{F(B)} F(A)$ is not surjective. For instance: $(0, \Theta_A(p^2))$ does not lie in its image. Using the natural isomorphism $h_{\mathrm{W}(k)[[X]]} \cong G$, we obtain a non-smooth natural transformation $h_{\mathrm{W}(k)[[X]]} \to F$ that satisfies property $(\mathbf{SI}_{\mathrm{F}})$.

4.1.4 Property $(\mathbf{I})$ for deformation functors

Let G be a profinite group and $\bar{\rho} : G \to \mathrm{GL}_n(k)$ a continuous representation. We specialize our general considerations to the case of $F = \mathrm{Def}_{\bar{\rho}}$ and, motivated by Remark 4.11, focus on property $(\mathbf{I})$. We want to determine the deformations of $\bar{\rho}$ having this property. The aim of this section is to present an easy sufficient condition serving this purpose.

A criterion for property $(\mathbf{I})$

Notation 4.15. *Given a subset $H \subseteq \ker \bar{\rho}$, let us denote by $\mathfrak{c}_\rho^H$ the ideal*

$$\mathfrak{c}_\rho^H := \left(\left(\rho(g) - I_n\right)(i,j) \mid g \in H,\ i,j \in [n] \right) \lhd R$$

generated by all entries of all matrices of the form $\rho(g) - I_n$, $g \in H$.

Lemma 4.16. *Let $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ and $\rho \in \mathrm{Lift}_{\bar{\rho}}(R)$ be given. Suppose that $f \in h_R(k[\varepsilon])$ lies in the kernel of the map $[\rho]^*_{k[\varepsilon]} : h_R(k[\varepsilon]) \to \mathrm{Def}_{\bar{\rho}}(k[\varepsilon])$. Then $\mathfrak{c}_\rho^{\ker \bar{\rho}} \subseteq \ker f$.*

Proof. Choose an arbitrary $g \in \ker \bar{\rho}$ and suppose $\rho(g) = I_n + A$ for some $A \in M(\mathfrak{m}_R)$. Since $[f \circ \rho] = [\bar{\rho}]$, there exists $K \in I_n + \varepsilon M_n(k)$ such that $K f \rho K^{-1} = \bar{\rho}$. In particular, evaluating both sides at g we obtain $K(I_n + fA)K^{-1} = I_n$. Both K and $I_n + fA$ belong to $I_n + \varepsilon M_n(k)$ and since this is an abelian group, we conclude that $fA = 0$, i.e., $fA(i,j) = 0$ for every $i,j \in [n]$. □

Proposition 4.17. *If ρ is such that $(\mathfrak{c}_\rho^{\ker \bar{\rho}}, p) = \mathfrak{m}_R$ then $[\rho]$ has $(\boldsymbol{I})$.*

Proof. It is clear that $p \in \ker f$ for every $f \in h_R(k[\varepsilon])$. This observation and the assumption combined with Lemma 4.16 show that every $f \in \ker[\rho]^*_{k[\varepsilon]}$ coincides with the canonical reduction modulo $\mathfrak{m}_R$. Hence, $\ker[\rho]^*_{k[\varepsilon]}$ is trivial, $[\rho]^*_{k[\varepsilon]}$ is injective and $[\rho]$ has $(\mathbf{I})$. □

Example 4.18. If for some $i, j \in [n]$ we have $I_n + Re_{ij} \subseteq \operatorname{im}\rho$ then $\mathfrak{e}_\rho^{\ker\bar{\rho}} = \mathfrak{m}_R$. Consequently, $[\rho]$ has property $(\mathbf{I})$.

Actually, the condition considered in Example 4.18 has even stronger consequences.

Lemma 4.19. *If for some $i, j \in [n]$ we have $I_n + Re_{ij} \subseteq \operatorname{im}\rho$ then $[\rho]^*$ is injective.*

Proof. Let $S \in \mathrm{Ob}(\hat{\mathcal{C}})$ be given and consider $f, g \in h_R(S)$ with $[f\circ\rho] = [g\circ\rho]$. By definition, there exists $X \in I + M_n(\mathfrak{m}_S)$ such that $f \circ \rho = X(g \circ \rho)X^{-1}$. In particular, for all $r \in R$ we have $I_n + f(r)e_{ij} = X(I_n + g(r)e_{ij})X^{-1}$ or, equivalently, $f(r)e_{ij} = g(r)Xe_{ij}X^{-1}$. Substituting $r = 1$ we obtain $e_{ij} = Xe_{ij}X^{-1}$, so we conclude that $f(r)e_{ij} = g(r)e_{ij}$ holds for all $r \in R$. Hence, $f = g$. □

Property (I) and fiber products

Proposition 4.17 can be seen as a special case of the following more general result, which in turn is a preparatory step for Proposition 4.29. Let $\pi_1 : R_1 \to S$, $\pi_2 : R_2 \to S$ be surjective $\hat{\mathcal{C}}$-morphisms and denote by $R_1 \times_S R_2$ the corresponding fiber product (cf. Lemma 1.25). Given deformations $[\rho_i] \in \mathrm{Def}_{\bar{\rho}}(R_i)$ with the same reduction to S, it is easy to see that there exists $[\rho] \in \mathrm{Def}_{\bar{\rho}}(R_1 \times_S R_2)$ such that $[\rho_i] = [p_i \circ \rho]$ for $i = 1, 2$. One can wonder whether it is possible to conclude from properties of $[\rho_i]$ that $[\rho]$ has property $(\mathbf{I})$.

$$\begin{tikzcd}
G \arrow[rrr, dotted, "\rho"] \arrow[drr, dashed, "\rho_1"'] \arrow[drrrr, dashed, "\rho_2"] & & & \mathrm{GL}_n(R_1 \times_S R_2) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
& & \mathrm{GL}_n(R_1) \arrow[dr, "\pi_1"'] & & \mathrm{GL}_n(R_2) \arrow[dl, "\pi_2"] \\
& & & \mathrm{GL}_n(S) &
\end{tikzcd}$$

Proposition 4.20. *Define* $R := R_1 \times_S R_2$ *and let* $\rho \in \mathrm{Lift}_{\bar\rho}(R)$ *be a lift projecting to* $\rho_1 \in \mathrm{Lift}_{\bar\rho}(R_1)$ *and* $\rho_2 \in \mathrm{Lift}_{\bar\rho}(R_2)$. *Suppose* $[\rho_2]$ *has property* (***I***) *and* $\ker \pi_1 \subseteq (\mathfrak{e}_{\rho_1}^{\ker \rho_2}, \operatorname{char} R_2)$ *holds. Then* $[\rho]$ *has property* (***I***).

Proof. Let $f \in \ker \rho^*_{k[\varepsilon]}$. Observe that $\mathfrak{e}_{\rho_1}^{\ker \rho_2} \times \{0\} \subseteq \mathfrak{e}_{\rho}^{\ker \bar\rho}$, so $\mathfrak{e}_{\rho_1}^{\ker \rho_2} \times \{0\} \subseteq \ker f$, by Lemma 4.16. Moreover, since $p = (p, p) \in \ker f$, we also have that $(\operatorname{char} R_2, 0) \in \ker f$. The assumption implies now that $\ker p_2 = \ker \pi_1 \times \{0\} \subseteq \ker f$. Hence, f factors via R_2 and using the assumption that $[\rho_2]$ has property (**I**), it follows that f is trivial. We conclude that $[\rho]$ has property (**I**). $\square$

Remark 4.21. In the special case $R_2 = S = k$, $\rho_2 = \bar\rho$ we obtain Proposition 4.17.

4.2 Relations between different universal deformation rings

As mentioned in the introduction, our aim for the second part of this chapter is to investigate relations between deformation rings of different representations. This is done in Proposition 4.25, devoted to representations of quotient groups and Propositions 4.34 and 4.40, devoted to representations of subgroups. The most elaborate Proposition 4.29 links the approach of this section with the approach of the first part of the chapter.

4.2.1 General setup

Let G_1, G_2 be profinite groups and $\alpha : G_1 \to G_2$ be a continuous group homomorphism. Consider a continuous representation $\bar\rho_2 : G_2 \to \mathrm{GL}_n(k)$ and let $\bar\rho_1 := \bar\rho_2 \circ \alpha$. Then $\bar\rho_1 : G_1 \to \mathrm{GL}_n(k)$ is continuous and if both $\bar\rho_1$ and $\bar\rho_2$ have versal deformation rings, we want to study how they are related. As in the preceding section, we will first analyze the problem in a more general setting, basing on the observation that α induces a natural transformation $\Theta_\alpha : \mathrm{Def}_{\bar\rho_2} \to \mathrm{Def}_{\bar\rho_1}$, defined as follows:

$$\forall S \in \mathrm{Ob}(\hat{\mathcal{C}}) : \ \mathrm{Def}_{\bar\rho_2}(S) \ni [\rho] \to [\rho \circ \alpha] \in \mathrm{Def}_{\bar\rho_1}(S).$$

$$
\begin{tikzcd}
 & & \mathrm{GL}_n(R) \arrow[d] \\
G_1 \arrow[r, "\alpha"] \arrow[urr, dotted, "\rho \circ \alpha"] \arrow[rr, bend right, "\bar{\rho}_1"'] & G_2 \arrow[r, "\bar{\rho}_2"] \arrow[ur, "\rho"] & \mathrm{GL}_n(k)
\end{tikzcd}
$$

4.2.2 Relations between different functors

Let F, G be covariant functors $\hat{\mathcal{C}} \to \mathrm{Sets}$ such that there exist $(R_F, \xi_F) \in \mathrm{Ob}(F\hat{\mathcal{C}})$ and $(R_G, \xi_G) \in \mathrm{Ob}(G\hat{\mathcal{C}})$ with properties $(\mathbf{SI}_{\mathrm{F}})$ and $(\mathbf{SI}_{\mathrm{G}})$, respectively. Suppose moreover that we are given a natural transformation $\Theta : F \to G$. We want to check how the existence of Θ and its properties relate R_F and R_G.

Our assumptions lead to the following setup. We obtain the diagram

$$
\begin{tikzcd}
h_{R_F} \arrow[d, "\xi_F^*"] & h_{R_G} \arrow[d, "\xi_G^*"] \\
F \arrow[r, "\Theta"] & G
\end{tikzcd}
$$

which we want to complete by adding an arrow between h_{R_F} and h_{R_G} (corresponding, clearly, to a $\hat{\mathcal{C}}$-morphism between R_F and R_G). In the next lemma we discuss the two possibilities for doing this.

Lemma 4.22. *Let $\varphi : R_G \to R_F$ and $\psi : R_F \to R_G$ be $\hat{\mathcal{C}}$-morphisms.*

(i) The first diagram below commutes if and only if $G(\varphi)(\xi_G) = \Theta(\xi_F)$.

(ii) The second diagram below commutes if and only if $\Theta(F(\psi)(\xi_F)) = \xi_G$.

(iii) If φ and ψ satisfy conditions listed in parts (i) and (ii) then $\psi\varphi \in \mathrm{Aut}_{\hat{\mathcal{C}}}(R_G)$. If ξ_G^* is moreover a natural isomorphism, even $\psi\varphi = \mathrm{id}_{R_G}$ holds true.

$$\begin{tikzcd} h_{R_F} \arrow[r, dashed, "\varphi^*"] \arrow[d, "\xi_F^*"] & h_{R_G} \arrow[d, "\xi_G^*"] \\ F \arrow[r, "\Theta"] & G \end{tikzcd} \qquad \begin{tikzcd} h_{R_F} \arrow[d, "\xi_F^*"] & h_{R_G} \arrow[l, dashed, "\psi^*"'] \arrow[d, "\xi_G^*"] \\ F \arrow[r, "\Theta"] & G \end{tikzcd}$$

Proof. (*i*), (*ii*) In both cases we apply Yoneda's lemma, due to which it is sufficient to check where the identity maps are taken by different compositions of maps. In the first case we obtain $(\Theta \circ \xi_F^*)(\mathrm{id}_F) = \Theta(\xi_F)$ and $(\xi_G^* \circ \varphi^*)(\mathrm{id}_F) = G(\varphi)(\xi_G)$. In the second case: $\xi_G^*(\mathrm{id}_G) = \xi_G$ and $(\Theta \circ \xi_F^*) \circ \psi^*(\mathrm{id}_G) = \Theta(F(\psi)(\xi_F))$. In each case the diagram commutes if and only if the two presented expressions are equal.

(*iii*) If $G(\varphi)(\xi_G) = \Theta(\xi_F)$ and $\Theta(F(\psi)(\xi_F)) = \xi_G$ hold, then using the definition of a natural transformation we obtain:

$$G(\psi\varphi)(\xi_G) = G(\psi)(\Theta\xi_F) = \Theta(F(\psi)(\xi_F)) = \xi_G.$$

Hence, $\xi_G^*(\psi\varphi) = \xi_G^*(\mathrm{id}_R)$. If ξ_G^* is a natural isomorphism then $\psi\varphi = \mathrm{id}_{R_G}$ and we are done. In general, property $(\mathbf{SI}_{\mathrm{G}})$ of ξ_G and Proposition 4.10.(*iii*) imply that $\psi\varphi$ is an automorphism of R_G. $\square$

Observe that property $(\mathbf{S}_{\mathrm{G}})$ of ξ_G implies that there exists at least one $G\hat{\mathcal{C}}$-morphism $\varphi : (R_G, \xi_G) \to (R_F, \Theta(\xi_F))$, i.e., φ satisfying the first condition given in Lemma 4.22.

Proposition 4.23. *Let a $G\hat{\mathcal{C}}$-morphism $\varphi : (R_G, \xi_G) \to (R_F, \Theta(\xi_F))$ be given. Then:*

(i) φ is surjective if and only if $\Theta_{k[\varepsilon]}$ is injective.

(ii) The following conditions are equivalent:

(a) φ is a split monomorphism,

(b) Θ is surjective,

(c) $\xi_G \in \operatorname{im} \Theta_{R_G}$,

(d) there exists $\psi : R_F \to R_G$ satisfying the condition given in Lemma 4.22.(ii).

Proof. (*i*) By Lemma 4.8, φ is surjective if and only if $\varphi^*_{k[\varepsilon]}$ is injective. Since $(\xi^*_G)_{k[\varepsilon]}$ and $(\xi^*_F)_{k[\varepsilon]}$ are isomorphisms and the first diagram in Lemma 4.22 commutes, $\varphi^*_{k[\varepsilon]}$ is injective if and only if $\Theta_{k[\varepsilon]}$ has this property.

(*ii*) (*a*) ⇒ (*b*): By Lemma 4.8, φ is a split monomorphism if and only if φ^* is surjective. If this is the case, then Θ is surjective as well, given that ξ^*_G is surjective and the first diagram in Lemma 4.22 commutes.

(*b*) ⇒ (*c*): This implication is obvious.

(*c*) ⇒ (*d*): By assumption, there exists $\xi \in F(R_G)$ such that $\Theta(\xi) = \xi_G$. By property $(\mathbf{S}_\mathrm{F})$ of ξ_F we obtain a map $\psi : R_F \to R_G$ such that $\xi = F(\psi)(\xi_F)$ and this ψ satisfies the equivalent conditions of Lemma 4.22.(*ii*).

(*d*) ⇒ (*a*): Follows from Lemma 4.22.(*iii*). □

Representability and existence of versal hulls

Note that in the setup considered above, in some cases representability or existence of a versal hull of one of the functors implies the analogous property of the other functor. For example, we have the following result.

Lemma 4.24. *Assume that F and G are continuous functors satisfying properties ($\boldsymbol{H0}$)-($\boldsymbol{H2}$) of Theorem 1.47.*

(*i*) *Suppose $\Theta_{k[\varepsilon]}$ is surjective. If F has a versal hull then G has a versal hull as well.*

(*ii*) *Suppose $\Theta_{k[\varepsilon]}$ is injective. If G has a versal hull (is representable) then F has a versal hull (is representable) as well.*

Proof. (*i*) By Theorem 1.47, F has a versal hull if and only if F satisfies (**H3**), i.e., if and only if $\dim_k t_F < \infty$. Surjectivity of $\Theta_{k[\varepsilon]} : t_F \to t_G$ implies that $\dim_k t_G < \infty$. Hence, G satisfies (**H3**) and has a versal hull as well.

(*ii*) The proof in the case of versal hulls is analogous to the previous one. If (R_G, ξ_G) represents G, we obtain that there exists a versal hull (R_F, ξ_F) of F. In order to prove that it represents F, assume that for some $S \in \mathrm{Ob}(\hat{\mathcal{C}})$ and $f, g \in h_{R_F}(S)$ we have $\xi^*_F(f) = \xi^*_F(g)$. By

Proposition 4.23, the map $\varphi : (R_G, \xi_G) \to (R_F, \Theta(\xi_F))$ is surjective. We obtain

$$\xi_G^*(f \circ \varphi) = G(f \circ \varphi)(\xi_G) = G(f)(\Theta(\xi_F)) = \Theta(F(f)(\xi_F)) = \Theta(\xi_F^*(f))$$

and, similarly, $\xi_G^*(g \circ \varphi) = \Theta(\xi_F^*(g))$. Hence, $\xi_G^*(f \circ \varphi) = \xi_G^*(f \circ \varphi)$ and we conclude using injectivity of ξ_G^* that $f \circ \varphi = g \circ \varphi$. By surjectivity of φ, this implies $f = g$. □

Note that in the first case G need not be representable even if F is. For example: consider the surjective transformation $\mathrm{Lift}_{\bar{\rho}} \to \mathrm{Def}_{\bar{\rho}}$ corresponding to an arbitrary group representation $\bar{\rho}$ having a versal deformation ring that is not universal.

4.2.3 Representations of quotient groups

We return to the general setup described in section 4.2.1 and specialize the above results to the case $F = \mathrm{Def}_{\bar{\rho}_2}$, $G = \mathrm{Def}_{\bar{\rho}_1}$ and $\Theta = \Theta_\alpha$. We will analyze first the special case of α surjective.

Proposition 4.25. *If $\alpha : G_1 \to G_2$ is surjective then:*

(i) The natural transformation $\Theta_\alpha : \mathrm{Def}_{\bar{\rho}_2} \to \mathrm{Def}_{\bar{\rho}_1}$ is injective.

(ii) If $\bar{\rho}_1$ has a versal deformation ring $R_{\bar{\rho}_1}$ then $\bar{\rho}_2$ has a versal deformation ring $R_{\bar{\rho}_2}$, which is a quotient of $R_{\bar{\rho}_1}$. If $R_{\bar{\rho}_1}$ is a universal deformation ring of $\bar{\rho}_1$, then $R_{\bar{\rho}_2}$ is a universal deformation ring of $\bar{\rho}_2$.

Proof. (*i*) If $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ and $[\rho], [\phi] \in \mathrm{Def}_{\bar{\rho}_2}(R)$ are such that $[\rho \circ \alpha] = [\phi \circ \alpha]$ then by definition there exists $K \in I_n + M_n(\mathfrak{m}_R)$ for which $\rho \circ \alpha = K(\phi \circ \alpha)K^{-1}$. Since α is surjective, we conclude that $\rho = K\phi K^{-1}$ and, hence, $[\rho] = [\phi]$.

(*ii*) The claim follows from the first part of the lemma and Propositions 4.23, 4.24. □

Example 4.26. Let G be a profinite group and $\bar{\rho} : G \to \mathrm{GL}_n(k)$ a continuous representation. Suppose $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ and $\rho \in \mathrm{Lift}_{\bar{\rho}}(R)$ are given. Define $\tilde{G} := \operatorname{im} \rho$, let ι be the inclusion $\tilde{G} \hookrightarrow \mathrm{GL}_n(R)$ and set $\tilde{\rho} := \pi_{\mathfrak{m}_R}\iota$. If $\bar{\rho}$ has a versal (universal) deformation ring $R_{\bar{\rho}}$ then:

$$\begin{tikzcd}
 & & \mathrm{GL}_n(R) \arrow[d, "\pi_{\mathfrak{m}_R}"] \\
G \arrow[r, two heads] \arrow[urr, "\rho"] & \tilde{G} \arrow[r, "\tilde{\rho}"] \arrow[ur, hook, "\iota"] & \mathrm{GL}_n(k)
\end{tikzcd}$$

(*i*) The representation $\tilde{\rho}$ has a versal (universal) deformation ring $R_{\tilde{\rho}}$ which is a quotient of $R_{\bar{\rho}}$.

(*ii*) If $R \cong R_{\bar{\rho}}$ and $[\rho]$ is a universal deformation of $\bar{\rho}$ then $R_{\tilde{\rho}} \cong R_{\bar{\rho}}$.

The first statement follows directly from Proposition 4.25. In the particular case considered in the second statement, $\mathrm{Def}_{\tilde{\rho}} \to \mathrm{Def}_{\bar{\rho}}$ is also surjective by Proposition 4.23.(*ii*).

The result presented in the second part of Example 4.26 has the following interesting consequence for studying the inverse problem. We will comment on it more in Section 5.1.

Corollary 4.27. *If $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ is a universal deformation ring of some representation, then it is also a universal deformation ring of some representation for which the universal lift is injective.*

We shall not use the following fact, but we mention that it is actually easy to describe the relation between $R_{\bar{\rho}_1}$ and $R_{\bar{\rho}_2}$ in Proposition 4.25 precisely.

Lemma 4.28. *In the setting of Proposition 4.25, suppose $[\rho_v]$ is a versal deformation of $\bar{\rho}_1$ to $R_{\bar{\rho}_1}$. Then $R_{\bar{\rho}_2} \cong R_{\bar{\rho}_1}/\mathfrak{e}_{\rho_v}^{\ker\alpha}$.*

Proof. We have an isomorphism of topological groups $G_1/\ker\alpha \cong \operatorname{im}\alpha = G_2$ (note that this statement is not true for arbitrary topological groups; we use here the fact that G_1 and G_2 are profinite, hence compact), so lifts of $\bar{\rho}_2$ can be identified with lifts of $\bar{\rho}_1$ mapping $\ker\alpha$ trivially. Hence, given $S \in \mathrm{Ob}(\hat{\mathcal{C}})$, deformations of $\bar{\rho}_2$ to S can be identified with deformations $[f \circ \rho_v]$ obtained for all $f \in \mathrm{Hom}_{\hat{\mathcal{C}}}(R_{\bar{\rho}_1}, S)$ for which $(f \circ \rho_v)|_{\ker\alpha}$ is trivial. The last condition is equivalent to $\mathfrak{e}_{\rho_v}^{\ker\alpha} \subseteq \ker f$ and it is easy to conclude that $R_{\bar{\rho}_1}/\mathfrak{e}_{\rho_v}^{\ker\alpha}$ is a versal deformation ring of $\bar{\rho}_2$. □

Quotient groups, fiber product and property (I)

We refine now the result presented in Example 4.26 and link it with the approach of the first part of this chapter. A concrete motivation for our considerations comes from Chapter 5, see Remark 5.20.

Proposition 4.29. *Let G be a profinite group and $\bar{\rho} : G \to \mathrm{GL}_n(k)$ be a continuous representation having a versal deformation ring $R_{\bar{\rho}}$. Moreover, assume that $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ and $\xi \in \mathrm{Def}_{\bar{\rho}}(R)$ with property ($\boldsymbol{I}$) are given and choose a lift $\rho \in \xi$.*

For every proper ideal $I \lhd R$ we define $G_I := \mathrm{im}(\pi_I \rho)$ and let $\bar{\rho}_I : G_I \to \mathrm{GL}_n(k)$ be the composition of the inclusion $\iota_I : G_I \hookrightarrow \mathrm{GL}_n(R/I)$ with the reduction $\pi_{\mathfrak{m}_{R/I}}$. We also define $\alpha_I : G \to G_I$ as the unique map for which $\pi_I \circ \rho = \iota_I \circ \alpha_I$.

$$\begin{tikzcd}
\mathrm{GL}_n(R_{I,\varphi}) \arrow[rr, dotted] \arrow[dr, dotted] & & \mathrm{GL}_n(R_{\bar{\rho}_I}) \arrow[dr, dotted, "\varphi"] & \\
& \mathrm{GL}_n(R) \arrow[rr, two heads, "\pi_I"] & & \mathrm{GL}_n(R/I) \arrow[d, "\pi_{\mathfrak{m}_{R/I}}"] \\
G \arrow[uu, dotted, "\tilde{\rho}"] \arrow[ur, "\rho"] \arrow[rr, two heads, "\alpha_I"] & & G_I \arrow[uu, dotted, "\rho_I"] \arrow[ur, hook, "\iota_I"] \arrow[r, "\bar{\rho}_I"] & \mathrm{GL}_n(k)
\end{tikzcd}$$

Under these assumptions, for every proper ideal $I \lhd R$:

(i) There exists a versal deformation $(R_{\bar{\rho}_I}, \xi_I)$ of $\bar{\rho}_I$.

(ii) For every morphism $\varphi : (R_{\bar{\rho}_I}, \xi_I) \to (R/I, [\iota_I])$, the fiber product $R_{I,\varphi} := R \times_{R/I} R_{\bar{\rho}_I}$ of π_I and φ is a $\hat{\mathcal{C}}$-ring.

(iii) Given ξ_I and φ as above, there exists a versal lift $\rho_I \in \xi_I$ such that $\varphi \circ \rho_I = \iota_I$. For this ρ_I we obtain a lift $\tilde{\rho} := (\rho, \rho_I \circ \alpha_I) : G \to \mathrm{GL}_n(R_{I,\varphi})$ of $\bar{\rho}$.

(iv) If $\mathfrak{e}_\rho^{\ker(\pi_I \circ \rho)} = I$ then $[\tilde{\rho}]$ has property ($\boldsymbol{I}$). In particular: $R_{I,\varphi}$ is a quotient of $R_{\bar{\rho}}$.

Proof. (*i*) The claim follows from Proposition 4.25, cf. Example 4.26.

(*ii*) By Lemma 1.25, it is sufficient to prove that φ is surjective. This in turn follows from Proposition 4.10.(*ii*), once we check that $[\iota_I]$ has property (**I**). If $f, g \in \mathrm{Hom}_{\hat{\mathcal{C}}}(R/I, k[\varepsilon])$ are such that $[f \circ \iota_I] = [g \circ \iota_I]$ then also $[f \circ \pi_I \circ \rho] = [g \circ \pi_I \circ \rho]$. Hence, by property (**I**) of $[\rho]$, we obtain $f \circ \pi_I = g \circ \pi_I$ and, consequently, $f = g$. This proves that $[\iota_I]$ indeed has property (**I**).

(*iii*) Choose an arbitrary lift $\tilde{\rho}_v \in \xi_I$. Since $\varphi \circ \xi_I = [\iota_I]$, for some $X \in I_n + M_n(\mathfrak{m}_{R/I})$ we have $X(\varphi \circ \tilde{\rho}_v)X^{-1} = \iota_I$. By surjectivity of φ, there exists $Y \in M_n(R_{\bar{\rho}_I})$ such that $\varphi(Y) = X$ and it is sufficient to set $\rho_I := Y\tilde{\rho}_v Y^{-1}$. The second statement follows easily, since $\pi_I \circ \rho = \iota_I \circ \alpha_I = \varphi \circ \rho_I \circ \alpha_I$.

(*iv*) We apply Proposition 4.20, with $R_1 = R$, $\rho_1 = \rho$ and $R_2 = R_{\bar{\rho}_I}$, $\rho_2 = \rho_I \circ \alpha_I$. Firstly, we check that $[\rho_I \circ \alpha_I]$ has property (**I**). Indeed, if $f, g \in \mathrm{Hom}_{\hat{\mathcal{C}}}(R_{\bar{\rho}_I}, k[\varepsilon])$ are such that $[f \circ \rho_I \circ \alpha_I] = [g \circ \rho_I \circ \alpha_I]$ then, by surjectivity of α_I, we also have $[f \circ \rho_I] = [g \circ \rho_I]$. Since $[\rho_I]$ is a versal deformation of $\bar{\rho}_I$, we conclude that $f = g$. Secondly, since ι_I is injective, ρ_I is injective as well and hence $\ker(\rho_I \circ \alpha_I) = \ker \alpha_I = \ker(\iota_I \circ \alpha_I) = \ker(\pi_I \circ \rho)$. By assumption,

$$\mathfrak{e}_\rho^{\ker(\rho_I \circ \alpha)} = \mathfrak{e}_\rho^{\ker(\pi_I \circ \rho)} = I = \ker \pi_I.$$

We see that assumptions of Proposition 4.20 are satisfied, so $[\tilde{\rho}]$ has property (**I**). The fact that $R_{I,\varphi}$ is a quotient of $R_{\bar{\rho}}$ follows from Proposition 4.10.(*ii*). □

Corollary 4.30. *Let us use the notation of Proposition 4.29. If for some proper ideal $I \lhd R$ condition $\mathfrak{e}_\rho^{\ker(\pi_I \circ \rho)} = I$ holds then $R_{\bar{\rho}_I} \cong R/I$ is a necessary condition for $R_{\bar{\rho}} \cong R$.*

Proof. By Proposition 4.29, if $R_{\bar{\rho}} \cong R$ then there exists a surjection $\varphi : R_{\bar{\rho}_I} \to R/I$ such that the fiber product $R \times_{R/I} R_{\bar{\rho}_I}$ of π_I and φ is a quotient of R. On the other hand, R clearly is a quotient of $R \times_{R/I} R_{\bar{\rho}_I}$. By noetherianity (cf. Lemma 1.10), it must be $R \times_{R/I} R_{\bar{\rho}_I} \cong R$, which holds if and only if φ is an isomorphism $R_{\bar{\rho}_I} \cong R/I$. □

Example 4.31. Let G be a profinite group, $\bar{\rho} : G \to \mathrm{GL}_n(k)$ a continuous group representation having a versal deformation ring $R_{\bar{\rho}}$. If $R \in \mathrm{Ob}(\hat{\mathcal{C}})$, $\rho \in \mathrm{Lift}_{\bar{\rho}}(R)$ are such that for some $i, j \in [n]$ we have $I_n + Re_{ij} \subseteq \operatorname{im} \rho$ then ρ has property $(\mathbf{I})$ by Example 4.18 and $\mathfrak{c}_{\rho}^{\ker \pi_I \rho} = I$ holds for every proper ideal $I \lhd R$; in particular: also for $I = \mathfrak{m}_R$. Denoting by R_k the versal deformation ring of $\operatorname{im} \bar{\rho} \hookrightarrow \mathrm{GL}_n(k)$, we conclude from Proposition 4.29 that $R \times_k R_k$ is a quotient of $R_{\bar{\rho}}$. Hence, a necessary condition for $R_{\bar{\rho}} \cong R$ is that $R_k \cong k$.

Remark 4.32. In what follows we use the notation of Proposition 4.29.

(1) For $I = (0)$ all resulting rings $R_{(0),\varphi}$ are isomorphic to $R_{\tilde{\rho}}$ considered in Example 4.26.

(2) In Proposition 4.29 we introduce definitions which depend on many choices we make. We want to note that in case $\mathrm{Def}_{\bar{\rho}}$ is representable, the resulting ring $R \times_{R/I} R_{\bar{\rho}_I}$ is (up to isomorphism) determined only by the deformation class $\xi \in \mathrm{Def}_{\bar{\rho}}(R)$ and ideal I.

    Firstly, G_I and $\bar{\rho}_I$ depend on the choice of $\rho \in \xi$, but the corresponding deformation functors $\mathrm{Def}_{\bar{\rho}_I}$ are pairwise isomorphic and can be identified. If $\mathrm{Def}_{\bar{\rho}}$ is representable, Proposition 4.25 shows that $\mathrm{Def}_{\bar{\rho}_I}$ is representable as well. Due to Proposition 4.10.(*iii*), the universal deformation $(R_{\bar{\rho}_I}, \xi_v)$ is uniquely defined up to a $\hat{\mathcal{C}}$-isomorphism. Finally, since the deformation ξ_v is universal, the map φ is uniquely determined (note also that it depends only on ξ and not on the particular representative $\rho \in \xi$). We conclude that in this case we can simply talk about the uniquely determined (up to isomorphism) ring $R_I := R \times_{R/I} R_{\bar{\rho}_I}$, corresponding to ξ and I.

(3) If condition $\mathfrak{c}_{\rho}^{\ker(\pi_I \circ \rho)} = I$ holds for more than one ideal I, one may be interested in comparing the resulting rings $R_{I,\varphi}$. We have the following result:

    If $\mathfrak{c}_{\rho}^{\ker(\pi_I \circ \rho)} = I$ and $J \subseteq I$ then for every $\varphi : (R_{\bar{\rho}_I}, \xi_I) \to (R/I, [\iota_I])$ there exists $\psi : (R_{\bar{\rho}_J}, \xi_J) \to (R/J, [\iota_J])$ such that $R_{I,\varphi}$ is a quotient of $R_{J,\psi}$.

    Sketch of the proof. Since ρ has property $(\mathbf{I})$, so does ι_J (this has been shown in Proposition 4.29.(*ii*)) and one checks that

$$\mathfrak{e}_{\iota_J}^{\ker(\pi_{I/J}\circ\iota_J)} = \mathfrak{e}_{\rho}^{\ker(\pi_I\circ\rho)}/J = I/J.$$

Hence, we can apply Proposition 4.29.(*iv*) to $R := R/J$, $I := I/J$, $\bar{\rho} := \bar{\rho}_J$ and $\rho := \iota_J$ and for every $\varphi : (R_{\bar{\rho}_I}, \xi_I) \to (R/I, [\iota_I])$ we obtain a surjective $\hat{\mathcal{C}}$-homomorphism $(\psi, f) : R_{\bar{\rho}_J} \twoheadrightarrow R/J \times_{R/I} R_{\bar{\rho}_I}$ onto the fiber product of $\pi_{I/J}$ and φ. We easily conclude that

$$R \times_{R/J} \left(R/J \times_{R/I} R_{\bar{\rho}_I}\right) \cong R \times_{R/I} R_{\bar{\rho}_I} = R_{I,\varphi}$$

is an epimorphic image of $R_{J,\psi} = R \times_{R/J} R_{\bar{\rho}_J}$. □

Assume $\mathrm{Def}_{\bar{\rho}}$ is represented by $R_{\bar{\rho}}$ and define $X_\rho := \{I \lhd R, I \neq R \mid \mathfrak{e}_\rho^{\ker(\pi_I\circ\rho)} = I\}$. Using the convention introduced in part (2) we obtain the family $Y_\rho := \{R_I \mid I \in X_\rho\}$. If we order the proper ideals of R by inclusion and (isomorphism classes of) $\hat{\mathcal{C}}$-rings by the relation of being a quotient (i.e., $R \leqslant S \Leftrightarrow R$ is a quotient of S), the main results of this subsection may be summarized as follows:

The map $X_\rho \ni I \mapsto R_I \in Y_\rho$ is order-reversing. The family Y_ρ contains the greatest element, which is $R_{\tilde{\rho}}$ defined in Example 4.26, has an upper bound $R_{\bar{\rho}}$ and a lower bound R. In particular, a necessary condition for $R_{\bar{\rho}} \cong R$ is that $Y_{\bar{\rho}} = \{R\}$.

4.2.4 Representations of subgroups

Suppose that, in the setting of section 4.2.1, α is injective. We may identify G_1 with the closed subgroup $H := \operatorname{im}\alpha$ of $G := G_2$ and α with the inclusion $H \hookrightarrow G$ (note that, similarly as in the proof of Lemma 4.28, we use here compactness of profinite groups). We will denote $\bar{\rho}_2$ simply by $\bar{\rho}$. The representation $\bar{\rho}_1$ in this special case is just the restriction $\bar{\rho}|_H$. Moreover, we will suppose that $\bar{\rho}$ and $\bar{\rho}|_H$ have versal deformation rings R_G and R_H, respectively.

Remark 4.33. In contrast to the case of representations of quotient groups (see Proposition 4.25), there is no unique pattern for the relation between R_G and R_H. To observe this, consider the extreme case in which H is the trivial subgroup. Proposition 2.34 implies that $R_H \cong \mathrm{W}(k)$, regardless of $\bar{\rho}$.

As we will show in the next chapter, R_G can be an arbitrary $\hat{\mathcal{C}}$-ring. Hence, depending on $\bar{\rho}$, a map $R_H \to R_G$ may be injective, as well as surjective, or not have any of these properties at all.

Keeping the above observation in mind, we present some criteria only for selected properties of maps $R_H \to R_G$.

Proposition 4.34. *Suppose $[\rho_G] \in \mathrm{Def}_{\bar{\rho}}(R_G)$, $[\rho_H] \in \mathrm{Def}_{\bar{\rho}|_H}(R_H)$ are versal deformations and consider a $\mathrm{Def}_{\bar{\rho}|_H}$ $\hat{\mathcal{C}}$-morphism $\varphi : (R_H, [\rho_H]) \to (R_G, [(\rho_G)|_H])$.*

(i) φ is surjective if and only if the restriction map $H^1(G, \mathrm{Ad}(\bar{\rho})) \to H^1(H, \mathrm{Ad}(\bar{\rho}))$ is injective.

(ii) φ is a split monomorphism if and only if ρ_H may be extended to a lift of $\bar{\rho}$, i.e., if and only if $\rho_H = \rho|_H$ for some $\rho \in \mathrm{Lift}_{\bar{\rho}}(R_H)$.

Proof. Both claims follow easily from Proposition 4.23. In the first case we also make use of the cohomological interpretation of the tangent spaces $\mathrm{Def}_{\bar{\rho}}(k[\varepsilon])$ and $\mathrm{Def}_{\bar{\rho}|_H}(k[\varepsilon])$ as $H^1(G, \mathrm{Ad}(\bar{\rho}))$ and $H^1(H, \mathrm{Ad}(\bar{\rho}))$, respectively (cf. Lemma 2.17). It is easy to check that the map $\mathrm{Def}_{\bar{\rho}}(k[\varepsilon]) \ni [\rho] \to [\rho|_H] \in \mathrm{Def}_{\bar{\rho}|_H}(k[\varepsilon])$ corresponds in this interpretation with the restriction map of cohomology groups. □

The following corollary is an extended version of [Ra, Lemma 3.1.4] in which the same result was stated, but only for finite groups and universal deformation rings. As it is customary (cf. [Se2, §1.3]), when working with profinite groups the subgroup index $[G : H]$ should be understood as the supernatural number $lcm\{[G : U] \mid H \subseteq U, U \leqslant G \text{ open}\}$. In particular: the notion of being "prime to p" appearing in the next corollary is well-defined also when the index of H in G is not a finite number.

Corollary 4.35. *If $H \leqslant G$ is a closed subgroup of index coprime to p then R_G is a quotient of R_H.*

Proof. We have the following general result: if H is a closed subgroup of a profinite group G, of index coprime to p, then for every $q \geqslant 0$ and every discrete G-module A the restriction map $H^q(G, A) \to H^q(H, A)$ is injective on the p-primary component of $H^q(G, A)$ ([Se2, §2, Proposition 9 and following Corollary]). Hence, the claim follows from Lemma 4.34.(i). □

Note that R_G may be a quotient of R_H also when $[G : H]$ is divisible by p. This is the case, for example, when H is trivial and $\bar{\rho}$ has k as its universal deformation ring.

Corollary 4.36. *Suppose that the injection $H \hookrightarrow G$ is split. Then $\varphi : R_H \to R_G$ is a split monomorphism.*

Proof. If $\iota : H \hookrightarrow G$ is split and $\pi : G \twoheadrightarrow H$ is such that $\pi \circ \iota = \mathrm{id}_H$ then every lift ρ of $\bar{\rho}|_H$ extends to a lift $\rho \circ \pi$ of G. The claim follows from Lemma 4.34.(ii). ☐

Example 4.37. The representation considered in [BCdS, Theorem 4.1] is of the above type. The authors construct there groups K, G, $\Gamma = K \rtimes G$ and a representation of Γ with universal deformation ring $\mathbb{Z}_p[[t]]/(p^n t, t^2)$. The universal deformation ring of its restriction to G is $\mathbb{Z}_p$, hence: indeed a retract of $\mathbb{Z}_p[[t]]/(p^n t, t^2)$.

It is worth noting that it may be the case that there exist surjections or split monomorphisms $\varphi : R_H \to R_G$, but that none of them is of the type considered in Lemma 4.34, see example below. In particular: injectivity of the restriction map $H^1(G, \mathrm{Ad}(\bar{\rho})) \to H^1(H, \mathrm{Ad}(\bar{\rho}))$ is only a sufficient, but not a necessary condition for R_G being a quotient of R_H.

Example 4.38. Consider $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ and $\phi \in \mathrm{End}_{\hat{\mathcal{C}}}(R)$ that is injective, but neither surjective, nor split (for instance, $R := \mathrm{W}(k)[[X]]$, $\phi : R \xrightarrow{X \mapsto X^2} R$) and define $G := \mathrm{SL}_4(R)$, $H := \phi(G)$.

In the next chapter we prove that the injection $\iota : G \to \mathrm{GL}_4(R)$ is a universal lift for the representation $\bar{\rho} := \pi_{\mathfrak{m}_R} \circ \iota : G \to \mathrm{GL}_n(k)$. Denote by $\psi : H \to G$ the inverse of the isomorphism $\phi : G \to H$. Then $[\iota \circ \psi]$ is a universal lift for $\bar{\rho}|_H$ and $R_H \cong R_G \cong R$, but the corresponding map $\varphi : (R_H, [\iota \circ \psi]) \to (R_G, [\iota])$ coincides with ϕ. Due to Proposition 4.10.(iii), the maps corresponding to other choices of universal deformations are obtained composing ϕ with automorphisms of R_G and R_H. Hence: none of them defines an isomorphism between R_H and R_G.

4.2.5 Representations of normal subgroups

We present a special case of Lemma 4.34.(ii), in which the relation between the universal deformation rings can be precisely described.

Lemma 4.39. *If* $\mathrm{Ad}(\bar\rho)^G = kI_n$ *and* ρ *is a lift of* $\bar\rho$ *to* $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ *then* $\mathrm{Ad}(\rho)^G = RI_n$.

Sketch of the proof, cf. [Go, Lemma 3.8]. Let $X \in \mathrm{Ad}(\rho)^G$. It suffices to show inductively that for every $l \geqslant 1$ matrix X is scalar modulo $\mathfrak{m}_R^l$. For $l = 1$ this is assumed in the lemma statement. In the inductive step we have $X = \lambda I_n + Y$, where $\lambda \in R$ and $Y \in M_n(\mathfrak{m}_R^{l-1})$. Let W be the k-vector space $\mathfrak{m}_R^{l-1}/\mathfrak{m}_R^l$ and d its dimension. Then $M_n(W) \cong \mathrm{Ad}(\bar\rho)^d$ as kG-modules and $M_n(W)^G \cong (\mathrm{Ad}(\bar\rho)^d)^G = (kI_n)^d$. Clearly $Y \in \mathrm{Ad}(\rho)^G$, so its image modulo $\mathfrak{m}_R^l$ lies in $M_n(W)^G$. It follows that Y, hence also X, is a scalar modulo $\mathfrak{m}_R^l$. □

Proposition 4.40. *Let* G *be a profinite group and* $\bar\rho : G \to \mathrm{GL}_n(k)$ *a continuous representation. Suppose that* $N \lhd G$ *is a closed normal subgroup such that* $\mathrm{Ad}(\bar\rho)^N = kI_n$ *and* $\bar\rho|_N$ *has a universal deformation ring* R. *Assume moreover that there exists a universal lift of* $\bar\rho|_N$ *that may be extended to a lift* $\phi : G \to \mathrm{GL}_n(R)$ *of* $\bar\rho$.

(i) *For every* $S \in \mathrm{Ob}(\hat{\mathcal{C}})$ *and* $\xi \in \mathrm{Def}_{\bar\rho}(S)$ *there exist unique* $f \in h_R(S)$ *and* $\lambda \in \mathrm{CHom}(G, S_1^\times)$ *with* $N \subseteq \ker\lambda$ *such that* $\xi = [\lambda \cdot (f \circ \phi)]$.

(ii) *If* $\mathrm{CHom}(G/N, \mathbb{Z}/p\mathbb{Z})$ *is finite then* $R[[(G/N)^{ab,p}]]$ *is a universal deformation ring of* $\bar\rho$. *Otherwise* $\mathrm{Def}_{\bar\rho}$ *is not representable over* $\hat{\mathcal{C}}$.

Proof. (i) Consider $S \in \mathrm{Ob}(\hat{\mathcal{C}})$ and $\xi = [\rho] \in \mathrm{Def}_{\bar\rho}(S)$. Restricting to N we obtain the unique $f \in h_R(S)$ such that $[\rho|_N] = [(f \circ \phi)|_N]$. Replacing ρ by a strictly equivalent lift (if necessary) we will assume that $\rho|_N = (f \circ \phi)|_N$. In what follows, for brevity we denote $f \circ \phi$ by ϕ_f.

Let $g \in G$ and $n \in N$ be given. Since N is normal, we have $\phi_f(gng^{-1}) = \rho(gng^{-1}) = \rho(g)\phi_f(n)\rho(g)^{-1}$. Consequently, $\phi_f(g)^{-1}\rho(g)$ commutes with all $\phi_f(n)$, $n \in N$, and is therefore a scalar matrix by Lemma 4.39. Let $\lambda : G \to S_1^\times$ be such that $\rho(g) = \lambda(g)\phi_f(g)$. Then λ is a continuous group homomorphism factoring via G/N. Conversely, given $f \in h_R(S)$ and $\lambda \in \mathrm{CHom}(G, S_1^\times)$ factoring via G/N, the map $\phi_{f,\lambda} : g \mapsto \lambda(g)\phi_f(g)$ is a lift of $\bar\rho$. Moreover, if $[\phi_{f,\lambda}] = [\phi_{f',\lambda'}]$ then $\phi_{f,\lambda} = X\phi_{f',\lambda'}X^{-1}$ for some $X \in \mathrm{GL}_n(R)$ and restricting to N we see that X is scalar by Lemma 4.39; consequently: $\phi_{f,\lambda} = \phi_{f',\lambda'}$, i.e., $f = f'$, $\lambda = \lambda'$.

(ii) Consider the trivial one-dimensional representation $\bar\rho_1 : G/N \to k^\times$. It is easy to conclude from the first part of the lemma that we have a natural isomorphism $\mathrm{Def}_{\bar\rho} \cong \mathrm{Def}_{\bar\rho_1} \times \mathrm{Def}_{\bar\rho|_N}$. The claim follows from Lemma 1.28, Proposition 2.30 and the observation that $R \widehat{\otimes}_{\mathrm{W}(k)} \mathrm{W}(k)[[(G/N)^{ab,p}]] \cong R[[(G/N)^{ab,p}]]$. $\square$

Remark 4.41. (1) Working not in $\hat{\mathcal{C}}$, but in a bigger category, we could simply state that $\mathrm{Def}_{\bar\rho}$ is represented by $R[[(G/N)^{ab,p}]]$ (cf. our remarks in section 2.3.3).

(2) The condition $\mathrm{Ad}(\bar\rho)^N = kI_n$ in Proposition 4.40 is crucial. For instance, consider G and $\bar\rho$ as in Example 2.35 and let $N \lhd G$ be the trivial subgroup. By Proposition 2.34, the universal deformation ring of both $\bar\rho|_N$ and $\bar\rho$ is $\mathrm{W}(k)$, whereas $\mathrm{W}(k)[[(G/N)^{ab,p}]] = \mathrm{W}(k)[C_2]$.

Chapter 5

The special linear group and a solution to the inverse problem

This chapter contains the most important results of the thesis. Namely, we show that every $\hat{\mathcal{C}}$-ring R can be realized as the universal deformation ring of a continuous linear representation of a profinite group. The example we use for this goal is the special linear group $G := \mathrm{SL}_n(R)$ together with the natural representation (induced by the reduction $R \twoheadrightarrow k$) in $\mathrm{GL}_n(k)$, with the assumption $n \geqslant 4$. This is the main result of the chapter. We moreover discuss similar representations for $n = 2, 3$ and the results of our considerations may be summarized as follows:

Theorem 5.1. *Let R be a complete noetherian local ring with a finite residue field k, $n \geqslant 2$ and consider the natural representation $\bar{\rho}$ of* $\mathrm{SL}_n(R)$ *in* $\mathrm{GL}_n(k)$*. Then R is the universal deformation ring of $\bar{\rho}$ if and only if* $(n, k) \notin \{(2, \mathbb{F}_2), (2, \mathbb{F}_3), (2, \mathbb{F}_5), (3, \mathbb{F}_2)\}$.

We conclude the chapter generalizing our considerations to the closed subgroups G of $\mathrm{GL}_n(R)$ that contain $\mathrm{SL}_n(R)$. We consider analogous representations $\bar{\rho}$ of G (coming from the reduction $R \twoheadrightarrow k$) and discuss the problem whether, given G, the corresponding $\bar{\rho}$ has R as its universal deformation ring. Our results show, in particular, that this is not the case for $G = \mathrm{GL}_n(R)$, unless $R = k$ and $(n, k) \notin \{(2, \mathbb{F}_2), (2, \mathbb{F}_3), (3, \mathbb{F}_2)\}$. On the other hand, for $G = \{A \in \mathrm{GL}_n(R) \mid (\det A)^{\#k-1} = 1\}$, we obtain R as the universal deformation ring of the corresponding $\bar{\rho}$ if and only if $(n, k) \notin \{(2, \mathbb{F}_2), (2, \mathbb{F}_3), (3, \mathbb{F}_2)\}$. We also show that, in contrast to the case

$G = \mathrm{SL}_n(R)$, for some choices of G and R a universal deformation ring of the corresponding $\bar{\rho}$ even does not exist.

This chapter is a modified version of author's preprint [Dor]. Compared to the preprint, there are several changes in notation, order of exposition and even some of the proofs, but the mathematical content remains almost the same. The only improvement worth mentioning is presented in Remark 5.20, to which we would like to draw reader's attention. It is a corollary of results discussed in the preceding chapter.

Remark. Similar results have been obtained independently by Eardley and Manoharmayum in their preprint [EM], published at almost the same time as the first version of [Dor]. However, the methods of both papers are different. The reader is encouraged to get familiar also with the approach of Eardley and Manoharmayum, which is based on cohomology computations. We present a more elementary and self-contained approach treating also some cases ($n = 2$; $n = 4$, $k = \mathbb{F}_2$; the general linear group) that [EM] does not cover.

5.1 Motivation

We begin describing and motivating a general framework in which we will be working in this chapter.

One often studies some naturally occurring group representations in order to understand better the structure of a given group. In our case, since we focus on the inverse problem, we are free to choose groups and representations the way it is convenient for us. Corollary 4.27 shows that it is sufficient to restrict to representations with an injective universal lift. Such a lift gives us a way of identifying the represented group with a subgroup of a general linear group and we naturally arrive at the following setup.

Let $R \in \mathrm{Ob}(\hat{\mathcal{C}})$, $n \in \mathbb{N}$ be given and suppose G is a closed subgroup of $\mathrm{GL}_n(R)$. Then it is a profinite group and the inclusion $\iota_G : G \hookrightarrow \mathrm{GL}_n(R)$ is a continuous representation of G, lifting the residual representation $\bar{\rho}_G := \pi_{\mathfrak{m}_R}\iota_G$.

$$\begin{tikzcd} & \mathrm{GL}_n(R) \arrow[d, "\pi_{\mathfrak{m}_R}"] \\ G \arrow[ur, hook, "\iota_G"] \arrow[r, "\bar{\rho}_G"] & \mathrm{GL}_n(k) \end{tikzcd}$$

We are interested in finding a group G such that ι_G is a universal lift of $\bar{\rho}_G$. The first and most obvious candidate to consider would be the group $\mathrm{GL}_n(R)$ itself. Note that Rainone has studied the deformations of the identity map $\mathrm{GL}_2(\mathbb{F}_p) \to \mathrm{GL}_2(\mathbb{F}_p)$ in [Ra] and obtained $\mathbb{F}_p$ as the universal deformation ring for all $p > 3$. However, one quickly notices that in general the condition described in Proposition 2.33 may not be satisfied (see Example 5.23), so $\mathrm{Def}_{\bar{\rho}_G}$ may even not be representable over $\hat{\mathcal{C}}$. And even if it is, then not necessarily by R, as we will show at the end of this chapter.

The described problems with $G = \mathrm{GL}_n(R)$ are some of the reasons why we turn our attention to the group $G = \mathrm{SL}_n(R)$. A big advantage of this choice is that the special linear group has a nice set of generators satisfying many interesting properties (described in the next section), which will play a key role in our considerations.

5.2 Structure of the special linear group

In this section R denotes a commutative ring and n an integer. Moreover, we assume $n \geqslant 2$.

Notation 5.2. *Let $a, b \in [n]$, $a \neq b$. We introduce the following notation for some of the elements of $\mathrm{GL}_n(R)$.*

- $t^r_{ab} := I_n + re_{ab}$, *for $r \in R$.*
- $d(r_1, \dots, r_n)$, *where $r_i \in R^\times$, is the diagonal matrix with consecutive diagonal entries $r_1, \dots, r_n$.*
- $d^r_{ab} := d(r_1, \dots, r_n)$, *where $r \in R^\times$ and $r_a := r$, $r_b := r^{-1}$, $r_i := 1$ for $i \neq a, b$.*
- $\sigma^r_{ab} := I_n - e_{aa} - e_{bb} + re_{ab} - r^{-1}e_{ba}$, *for $r \in R^\times$.*

This notation suppresses n and R, but that should not cause any problems as n will usually be fixed and R easily deducible from the element r used.

Lemma 5.3. *The following relations hold for $a, b, c, d \in [n]$, $a \neq b$:*

(R1)	*If* $r, s \in R$	*then*	$t_{ab}^{r}\, t_{ab}^{s} = t_{ab}^{r+s}$,
(R2)	*If* $r, s \in R$ *and* $c \neq a, b$	*then*	$[t_{ab}^{r}, t_{bc}^{s}] = t_{ac}^{rs}$,
(R3)	*If* $r, s \in R$ *and* $\{a, c\} \cap \{b, d\} = \varnothing$	*then*	$[t_{ab}^{r}, t_{cd}^{s}] = 1$
(R4)	*If* $r \in R$, $D = d(\lambda_1, \ldots, \lambda_n), \lambda_i \in R^\times$	*then*	$D t_{ab}^{r} D^{-1} = t_{ab}^{\frac{\lambda_a}{\lambda_b} r}$
(R5)	*If* $u \in R^\times$	*then*	$\sigma_{ab}^{u} = t_{ab}^{u}\, t_{ba}^{-\frac{1}{u}}\, t_{ab}^{u}$,
(R6)	*If* $u \in R^\times$	*then*	$d_{ab}^{u} = \sigma_{ab}^{u} \sigma_{ab}^{-1}$,
(R7)	*If* $u \in R^\times$, $r, s \in R$ *and* $u = 1 + rs$	*then*	$d_{ab}^{u} = t_{ab}^{r}\, t_{ba}^{s}\, t_{ab}^{\frac{-r}{u}}\, t_{ba}^{-su}$.

Sketch of the proof. These identities follow directly from definitions and straightforward computations in which one uses the fact that $e_{ab} e_{cd} = \delta_{bc} e_{ad}$ (δ_{bc} being the Kronecker delta symbol). □

Lemma 5.4. *Given an ideal $\mathfrak{a} \trianglelefteq R$ let $U_{\mathfrak{a}} := \mathrm{SL}_n(R) \cap (I_n + M_n(\mathfrak{a}))$ and $V_{\mathfrak{a}} := \langle t_{ab}^{r} \in \mathrm{GL}_n(R) \mid r \in \mathfrak{a} \rangle$. If R is local then $U_{\mathfrak{a}^2} \leqslant V_{\mathfrak{a}} \leqslant U_{\mathfrak{a}}$. In particular, $U_R = \mathrm{SL}_n(R)$ is generated by all the elements of the form t_{ab}^{r}.*

Sketch of the proof. The inclusion $V_{\mathfrak{a}} \subseteq U_{\mathfrak{a}}$ is obvious. For the other inclusion, let $M \in U_{\mathfrak{a}^2}$. Observe that multiplying M by t_{ab}^{r} amounts to adding a multiple of one of its rows or columns to some other. We claim that performing such operations on M we may obtain a diagonal matrix lying in $U_{\mathfrak{a}^2}$. If $\mathfrak{a}$ is contained in the maximal ideal $\mathfrak{m}$ of the ring R then all the diagonal entries of M are invertible and we may simply cancel all other entries proceeding row by row. In case $\mathfrak{a} = R$ every row contains an invertible element (since $\det M \notin \mathfrak{m}$), so each diagonal entry either is invertible or becomes such after one of the described operations. We proceed as follows: make $M(n, n)$ invertible, cancel all other entries in the n-th row and column, repeat the procedure recursively on the leading $(n-1) \times (n-1)$ submatrix.

Every diagonal matrix in $U_{\mathfrak{a}^2}$ may be decomposed as a finite product of matrices of the form d_{ab}^{r}, $r \in (1 + \mathfrak{a}^2) \cap R^\times$. To finish the proof, we show that each of them is generated by some elements of the form t_{ab}^{r}, $r \in \mathfrak{a}$. If $\mathfrak{a} = R$, the claim follows from relations (R5) and (R6) described in Lemma 5.3. If $\mathfrak{a} \subseteq \mathfrak{m}$ we use relation (R7) together with the observation that every element of $1+\mathfrak{a}^2$ is a finite product of elements of the form $1+rs$, where r, $s \in \mathfrak{a}$. $\square$

Lemma 5.5. *Assume R is local with residue field k.*

(i) Using the notation of Lemma 5.4 we have that for every proper finitely generated ideal $\mathfrak{a} \lhd R$ there exists $r \in \mathbb{N}$ such that the commutator subgroup $U'_{\mathfrak{a}}$ contains $U_{\mathfrak{a}^r}$.

(ii) If either $n \geqslant 3$ or $k \neq \mathbb{F}_2, \mathbb{F}_3$ and $n = 2$ then $\mathrm{SL}_n(R)' = \mathrm{SL}_n(R)$.

Sketch of the proof. (i) Suppose first that $n \geqslant 3$. Then relation (R2) from Lemma 5.3 implies that $V_{\mathfrak{a}^2} \subseteq V'_{\mathfrak{a}}$ and hence, using Lemma 5.4, we obtain $U_{\mathfrak{a}^4} \subseteq U'_{\mathfrak{a}}$.

If $n = 2$ and a proper ideal $\mathfrak{a} \lhd R$ is given, define $\mathfrak{b} := \langle x^2 - 2x \mid x \in \mathfrak{a} \rangle \lhd R$. Due to relation (R4) from Lemma 5.3, for every x, $y \in \mathfrak{a}$ we have $\left(\begin{smallmatrix} 1 & (x^2-2x)y \\ 0 & 1 \end{smallmatrix}\right) = [d_{12}^{(1-x)}, t_{12}^{y}] \in U'_{\mathfrak{a}}$ and, analogously, $\left(\begin{smallmatrix} 1 & 0 \\ (x^2-2x)y & 1 \end{smallmatrix}\right) \in U'_{\mathfrak{a}}$. Hence, $V_{\mathfrak{b}\mathfrak{a}} \subseteq U'_{\mathfrak{a}}$. Observe that $\{x^3 \mid x \in \mathfrak{a}\} \subseteq \mathfrak{b}$. Indeed, for every $x \in \mathfrak{a}$ we have $x^3 \equiv 2x^2 \equiv 4x \pmod{\mathfrak{b}}$ and $4x = (x^2 + 2x) - (x^2 - 2x) \in \mathfrak{b}$. If $\mathfrak{a}$ is finitely generated and $l \in \mathbb{N}$ is the cardinality of some finite set of its generators then it is easy to observe that $\mathfrak{a}^{2l+1} \subseteq \langle x^3 \mid x \in \mathfrak{a} \rangle$. Hence, $\mathfrak{a}^{2l+2} \subseteq \mathfrak{b}\mathfrak{a}$ and we conclude (using also Lemma 5.4) that $U_{\mathfrak{a}^{4l+4}} \subseteq V_{\mathfrak{a}^{2l+2}} \subseteq V_{\mathfrak{b}\mathfrak{a}} \subseteq U'_{\mathfrak{a}}$.

(ii) It is sufficient to show that generators of $\mathrm{SL}_n(R)$ lie in $\mathrm{SL}_n(R)'$. To this end use Lemma 5.4 and suitable relations from Lemma 5.3: (R2) in case $n \geqslant 3$ or (R4) in case $n = 2$, $k \neq \mathbb{F}_2, \mathbb{F}_3$. $\square$

Lemma 5.6. *If $M \in M_n(R)$ commutes with all $t_{ab}^1 \in \mathrm{GL}_n(R)$ then M is a scalar matrix.*

Proof. The claim follows from the observation that $t_{ab}^1 M = M t_{ab}^1$ is equivalent to $e_{ab}M = Me_{ab}$, which holds if and only if $M(a,a) = M(b,b)$ and $\forall\ x \neq a, y \neq b :\ M(x,a) = M(y,b) = 0$. $\square$

5.3 The special linear group and deformations

Let us fix a finite field k and work in the resulting category $\hat{\mathcal{C}}$. The following assumption will be made for the whole of this section.

Assumption 5.7. *Let $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ and $n \geqslant 2$ be given, define $G := \mathrm{SL}_n(R)$ and consider the representation $\bar{\rho} : G \to \mathrm{GL}_n(k)$ induced by the reduction $R \twoheadrightarrow k$. We will denote by ι the inclusion $G = \mathrm{SL}_n(R) \hookrightarrow \mathrm{GL}_n(R)$ and by $\mathfrak{J}$ the set $\{(a, b) \in [n] \times [n] \mid a \neq b\}$.*

As a closed subgroup of $\mathrm{GL}_n(R)$, the group G is profinite and $\bar{\rho}$ is continuous. We are interested in the following question: is $\mathrm{Def}_{\bar{\rho}}$ represented by R?

5.3.1 General observations

Recall that, according to Notation 4.4, we will denote by $[\iota]^* : h_R \to \mathrm{Def}_{\bar{\rho}}$ the natural transformation that, given $S \in \mathrm{Ob}(\hat{\mathcal{C}})$, associates with $f \in \mathrm{Hom}_{\hat{\mathcal{C}}}(R, S)$ the deformation $[f \circ \iota] \in \mathrm{Def}_{\bar{\rho}}(S)$.

Lemma 5.8. *(i) There exists a universal deformation ring of $\bar{\rho}$.*

(ii) G satisfies the p-finiteness condition (Φ_p).

(iii) The ring R is the universal deformation ring of $\bar{\rho}$ if and only if ι is a universal lift of $\bar{\rho}$.

(iv) The map $[\iota]^* : h_R \to \mathrm{Def}_{\bar{\rho}}$ is injective.

Proof. (*i*) By Proposition 2.23, it is sufficient to show that $\mathrm{Def}_{\bar{\rho}}$ satisfies properties (**H3**) and (**H4**). The latter follows from Lemma 5.6 and part (*iv*) of Proposition 2.23. For the former, we use part (*iii*) of the same proposition and Lemma 5.5. More precisely, we have $\ker \bar{\rho} = U_{\mathfrak{m}_R}$ and want to check that $\mathrm{CHom}(U_{\mathfrak{m}_R}, \mathbb{Z}/p\mathbb{Z})$ is finite. This holds true since $U'_{\mathfrak{m}_R}$ is open in G (and hence the abelianization $U^{ab}_{\mathfrak{m}_R}$ is finite) by Lemma 5.5.

(*ii*) Similarly as above, for every $r \in \mathbb{N}_+$ the group $\mathrm{CHom}(U_{\mathfrak{m}_R^r}, \mathbb{Z}/p\mathbb{Z})$ is finite, due to the finiteness of $U^{ab}_{\mathfrak{m}_R^r}$. Since $\{U_{\mathfrak{m}_R^r} \mid r \in \mathbb{N}_+\}$ forms a basis of open neighbourhoods of G, the claim follows.

(*iii*), (*iv*) Follow from Proposition 4.1. □

We would like to point out that the first claim of Lemma 5.8 has only a motivating character. According to the last two claims of the lemma, in order to conclude that R is a universal deformation ring of $\bar{\rho}$, it is enough to prove that $[\iota]^*$ is surjective. As the reader will observe, our arguments in the next sections will not use the existence of a universal deformation ring of $\bar{\rho}$, but will rather reprove it.

Preparing for the main argument, we present the following auxiliary result.

Lemma 5.9. *Let $S \in \mathrm{Ob}(\hat{\mathcal{C}})$. If $\xi \in \mathrm{Def}_{\bar{\rho}}(S)$ then $\xi \in \mathrm{im}\,[\iota]^*_S$ holds if and only if there exists a lift $\rho \in \xi$ satisfying the following condition:*

$$(\diamondsuit) \qquad \forall (a,b) \in \mathfrak{J}\,, r \in R \;\; \exists\; c^r_{ab} \in S: \quad \rho(t^r_{ab}) = t^{c^r_{ab}}_{ab}.$$

Proof. Every lift of $\bar{\rho}$ of the form $\mathrm{GL}_n(f) \circ \iota$, $f \in \mathrm{Hom}_{\hat{\mathcal{C}}}(R,S)$ obviously satisfies $(\diamondsuit)$. Conversely, consider ρ satisfying $(\diamondsuit)$ and suppose first that $\mathbf{n} \geqslant \mathbf{3}$. Conjugating with the diagonal matrix $d(1, c^1_{12}, \ldots, c^1_{1n}) \in I_n + M_n(\mathfrak{m}_S)$ we obtain a lift $\tilde{\rho}$ strictly equivalent to ρ. It satisfies $(\diamondsuit)$ as well and in addition $\forall j \in [n]\backslash\{1\}: \;\tilde{\rho}(t^1_{1j}) = t^1_{1j}$, due to Lemma 5.3, (R4). We may thus suppose without loss of generality that $c^1_{1j} = 1$ for all $j \in [n]\backslash\{1\}$. Lemma 5.3, (R2) implies then that $\forall (j,k) \in \mathfrak{J}$, $j,k \neq 1 : c^1_{jk} = c^1_{1k}/c^1_{1j} = 1$. Furthermore, for every $j \in [n]\backslash\{1\}$ there exists $k \in [n]\backslash\{1,j\}$, so $c^1_{j1} = c^1_{1k}/c^1_{1j} = 1$ as well. We conclude that $\forall (a,b) \in \mathfrak{J}: \; c^1_{ab} = 1$.

Due to Lemma 5.3, (R1) and (R2), the following relations are satisfied for all $r, s \in R$ and pairwise distinct $a, b, c \in [n]$:

$$\begin{cases} c^{r+s}_{ab} & = \quad c^r_{ab} + c^s_{ab} \\ c^{rs}_{ac} & = \quad c^r_{ab}\, c^s_{bc} \end{cases}$$

Substituting in the second relation firstly $r = 1$, then $s = 1$, we see that the value c^r_{ab} with a fixed $r \in R$ does not depend neither on a, nor on b. Denote this common value by $\varphi(r)$. We have obtained a function $\varphi : R \to S$, which is additive by the first relation and multiplicative by the second one, satisfies $\varphi(1) = 1$ and for which $\varphi(r)$ and r have the same image in k, i.e., $\varphi \in h_R(S)$. Since G is generated by the elements of the form t^r_{ab} (Lemma 5.4) we conclude that $\rho = \mathrm{GL}_n(\varphi) \circ \iota$ and so $[\rho] = [\iota]^*_S(\varphi) \in \mathrm{im}\,[\iota]^*_S$.

Suppose now $\mathbf{n} = \mathbf{2}$. We may similarly assume that $c^1_{12} = 1$. Define $\varphi, g : R \to S$ by $\varphi(r) := c^r_{12}$ and $g(r) := c^r_{21}$. We claim that $g = \varphi$ and $\varphi \in \mathrm{Hom}_{\hat{\mathcal{C}}}(R,S)$, which clearly implies that $[\rho] = [\iota]^*_S(\varphi) \in \mathrm{im}\,[\iota]^*_S$. Since φ is additive by the relation (R1) of Lemma 5.3, $\varphi(1) = 1$ and for all $r \in R$ the images of $\varphi(r)$ and r in k coincide, we only need to check that φ is multiplicative. Furthermore, it is sufficient to check multiplicativity only on $R^\times$, because of additivity of φ and the fact that every non-invertible $r \in R$ is a sum of two invertible elements (e.g. $r = (r-1)+1$). Similarly, it is sufficient to check that $g(r) = \varphi(r)$, for $r \in R^\times$.

Let $r \in R^\times$, $a := \varphi(r)$, $b := g(-r^{-1})$ and $\sigma_r := \left(\begin{smallmatrix} 0 & r \\ -r^{-1} & 0 \end{smallmatrix}\right)$. Applying relation (R5) of Lemma 5.3 twice, we get $t_{12}^r\, t_{21}^{-1/r}\, t_{12}^r = \sigma_r = t_{21}^{-1/r} t_{12}^r t_{21}^{-1/r}$, hence:

$$\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ b & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) = \rho(\sigma_r) = \left(\begin{smallmatrix} 1 & 0 \\ b & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ b & 1 \end{smallmatrix}\right)$$
$$\begin{pmatrix} 1+ab & 2a+a^2b \\ b & 1+ab \end{pmatrix} = \rho(\sigma_r) = \begin{pmatrix} 1+ab & a \\ 2b+ab^2 & 1+ab \end{pmatrix}$$

It follows that $a + a^2 b = 0$, so $ab = -1$, since a is invertible. This means $\varphi(r)g(r^{-1}) = 1$ and $\rho(\sigma_r) = \begin{pmatrix} 0 & \varphi(r) \\ -\varphi(r)^{-1} & 0 \end{pmatrix}$. Relation (R6) implies now $\rho\left(\left(\begin{smallmatrix} r & \\ & r^{-1} \end{smallmatrix}\right)\right) = \rho(\sigma_r)\rho(\sigma_1)^{-1} = \begin{pmatrix} \varphi(r) & \\ & \varphi(r)^{-1} \end{pmatrix}$. As $R^\times \ni r \mapsto \left(\begin{smallmatrix} r & \\ & r^{-1} \end{smallmatrix}\right)$ is a group homomorphism, we conclude that φ is multiplicative on $R^\times$. In particular, $\varphi(r)g(r^{-1}) = 1$ implies that $g(r^{-1}) = \varphi(r^{-1})$, so $g = \varphi$ on $R^\times$. This finishes the proof. $\square$

5.3.2 Main result

Theorem 5.10. *If $n \geqslant 4$ then ι is universal.*

Proof. Let $S \in \mathrm{Ob}(\hat{\mathcal{C}})$. By Lemma 5.8, we only need to show that $[\iota]_S^*$ is surjective, i.e., that every lift of $\bar{\rho}$ to S is strictly equivalent to $\rho_f := \mathrm{GL}_n(f)|_G$ for some $f \in \mathrm{Hom}_{\hat{\mathcal{C}}}(R, S)$. Moreover, we may restrict to the case $S \in \mathrm{Ob}(\mathcal{C})$, since all rings in $\hat{\mathcal{C}}$ are inverse limits of artinian rings.

For $S \in \mathrm{Ob}(\mathcal{C})$ let $n(S)$ be the smallest $j \in \mathbb{N}$ such that $\mathfrak{m}_S^j = 0$. We proceed by induction on $n(S)$. For $n(S) = 1$, i.e., $S = k$, the statement is obvious. For the inductive step consider S with $n(S) \geqslant 2$, a lift $\rho : G \to \mathrm{GL}_n(S)$ of $\bar{\rho}$ and set $l := n(S) - 1$. By the inductive hypothesis, we may suppose (considering a strictly equivalent lift if necessary) that ρ reduced to $S/\mathfrak{m}_S^l$ is induced by a morphism $g : R \to S/\mathfrak{m}_S^l$. For every $r \in R$ choose $p_r \in S$ such that $p_r \equiv g(r) \mod \mathfrak{m}_S^l$; for $r = 1$ we choose $p_1 = 1$. This way

$$\forall\ (a,b) \in \mathfrak{J}, r \in R : \quad \rho(t_{ab}^r) = t_{ab}^{p_r} + M_{ab}^r \quad , \quad \text{for some } M_{ab}^r \in M_{n\times n}(\mathfrak{m}_S^l).$$

We will analyze the structure of the matrices M_{ab}^r proving a series of claims. In the calculations we use the fact that $J := M_{n\times n}(\mathfrak{m}_S^l)$ is a two-sided ideal of $M_n(S)$ such that $\mathfrak{m}_S J = J^2 = 0$.

Claim 1. *If $a, b, c, d \in [n]$ are such that $\{a, c\} \cap \{b, d\} = \varnothing$ then for all $r, s \in R$:*

$$p_s(M^r_{ab}e_{cd} - e_{cd}M^r_{ab}) = p_r(M^s_{cd}e_{ab} - e_{ab}M^s_{cd})$$

Proof. Since t^r_{ab} and t^s_{cd} commute (Lemma 5.3, (R3)), so do their lifts. Denoting $t := t^{p_r}_{ab}$, $M := M^r_{ab}$, $z := t^{p_s}_{cd}$, $N := M^s_{cd}$ we obtain:

$$\begin{aligned}
(t+M)\ (z+N) &= (z+N)\ (t+M) \\
tz\ +\ tN\ +\ Mz &= zt\ +\ zM\ +\ Nt \\
Mz - zM &= Nt - tN \\
M(I_n + p_s e_{cd}) - (I_n + p_s e_{cd})M &= N(I_n + p_r e_{ab}) - (I_n + p_r e_{ab})N \\
p_s(M^r_{ab}e_{cd} - e_{cd}M^r_{ab}) &= p_r(M^s_{cd}e_{ab} - e_{ab}M^s_{cd})
\end{aligned}$$ □

Claim 2. $\forall (a, b) \in \mathfrak{J}, r \in R: \quad M^r_{ab}(i, j) = 0$ *when $i \neq a$, $j \neq b$ and $i \neq j$.*

Proof. If we fix $r \in R$ and $(a, b) \in \mathfrak{J}$ then given $j \in [n]\backslash\{b\}$ we may choose $d \in [n]\backslash\{a, b, j\}$ (here we use the assumption $n \geqslant 4$). Such a, b, j, d satisfy the assumptions of Claim 1, so we obtain $M^r_{ab}e_{jd} - e_{jd}M^r_{ab} = p_r(M^1_{jd}e_{ab} - e_{ab}M^1_{jd})$. If $i \in [n]\backslash\{a, j\}$ then a comparison of the (i, d)-entries of both sides of the relation shows that $M^r_{ab}(i, j) = 0$. □

Claim 3. $\forall (a, b) \in \mathfrak{J}, r \in R: \quad \operatorname{tr} M^r_{ab} = 0.$

Proof. Lemma 5.5 implies that $\det \rho(t^r_{ab}) = 1$, while $\det \rho(t^r_{ab}) = \prod_{i=1}^n (1 + M^r_{ab}(i, i)) = 1 + \operatorname{tr} M^r_{ab}$ by Claim 2. □

Claim 4. $\forall (a, b) \in \mathfrak{J}, r \in R: \quad M^r_{ab}(i, i) = 0$ *for $i \in [n]\backslash\{a, b\}$.*

Proof. Consider $r \in R$ and $(a, b) \in \mathfrak{J}$. If $c \in [n]\backslash\{a, b\}$ then since $n \geqslant 4$ we may choose $d \in [n]\backslash\{a, b, c\}$. Let $U_c := \{A \in \mathrm{GL}_n(S) \mid \forall x \in [n]\backslash\{c\} : A(x, c), A(c, x) \in \mathfrak{m}^l_S\}$. It is easy to see that U_c is a group and $\chi : U_c \to S^\times$, $A \mapsto A(c, c)$ a group homomorphism (due to the fact that $(\mathfrak{m}^l_S)^2 = 0$). Moreover, $\rho(t^r_{ab}), \rho(t^1_{ad}), \rho(t^r_{db}) \in U_c$ and since $[t^1_{ad}, t^r_{db}] = t^r_{ab}$ by Lemma 5.3, (R2), we have that $\chi(\rho(t^r_{ab})) = [\chi(\rho(t^1_{ad}))\ ,\ \chi(\rho(t^r_{db}))] = 1$. We conclude that $M^r_{ab}(c, c) = 0$. □

Claim 5. $\forall (a, b) \in \mathfrak{J}, r \in R: \quad M^r_{ab}(a, a) = -M^r_{ab}(b, b).$

Proof. This is an immediate consequence of Claim 3 and Claim 4. □

Claim 6. *If $a, b, c, d \in [n]$ are such that $\{a, c\} \cap \{b, d\} = \varnothing$ and $(a, b) \neq (c, d)$ then for all $r, s \in R$:*

$$\begin{cases} p_s M_{ab}^r(a, c) & = -p_r M_{cd}^s(b, d), \\ p_s M_{ab}^r(d, b) & = -p_r M_{cd}^s(c, a). \end{cases}$$

Proof. Thanks to Claim 2 and Claim 4 the formula of Claim 1 reduces to

$$p_s\Big(M_{ab}^r(a, c)e_{ad} \; - \; e_{cb}M_{ab}^r(d, b)\Big) \; = \; p_r\Big(M_{cd}^s(c, a)e_{cb} \; - \; e_{ad}M_{cd}^s(b, d)\Big),$$

If $(a, b) \neq (c, d)$ then the coefficients at e_{ad} (resp. e_{cb}) on both sides must be equal. □

Claim 7. *There exists $X \in M_n(\mathfrak{m}_S^l)$ such that $\forall (a, b) \in \mathfrak{J}, r \in R \; \exists c_{ab}^r \in \mathfrak{m}_S^l$:*

$$M_{ab}^r = p_r(e_{ab}X - Xe_{ab}) + c_{ab}^r e_{ab}.$$

Proof. Let $(a, b) \in \mathfrak{J}$ and $c, d \in [n] \backslash \{a, b\}$. The quadruple (a, b, a, d) satisfies the assumptions of Claim 6, so $M_{ab}^1(a, a) = -M_{ad}^1(b, d)$ (the first relation). Combining with Claim 5 we obtain $M_{ab}^1(b, b) = M_{ad}^1(b, d)$, so $M_{ay}^1(b, y)$ is independent of the choice of $y \in [n] \backslash \{a\}$. We will denote this common value by $Y(b, a)$. Analogously, using the quadruple (a, b, c, b) and the second relation of Claim 6 we prove that the value of $M_{xb}^1(x, a)$, with x ranging over $[n] \backslash \{b\}$, is constant. We will denote it $X(b, a)$.

Setting $X(a, a) := Y(a, a) := 0$ for all $a \in [n]$ we obtain well defined matrices $X, Y \in M_n(\mathfrak{m}_S^l)$. Since $M_{ab}^1(a, a) = -M_{ab}^1(b, b)$ by Claim 5, we have $X(b, a) = -Y(b, a)$ for all $(a, b) \in \mathfrak{J}$, hence $X = -Y$.

Consider $(a, b) \in \mathfrak{J}$ and $c \in [n] \backslash \{b\}$. Then it is possible to find $d \in [n]$ such that a, b, c, d satisfy the assumptions of Claim 6 (if $a = c$ choose any $d \in [n] \backslash \{a, b\}$, if $a \neq c$ let $d := b$; note that this argument relies only on the fact that $n \geqslant 3$). The first relation gives $\forall r \in R : M_{ab}^r(a, c) = -p_r M_{cd}^1(b, d) = p_r X(b, c)$. Similarly, $\forall d \in [n] \backslash \{a\}, r \in R : M_{ab}^r(d, b) = -p_r X(d, a)$. We conclude that $M_{ab}^r = p_r(e_{ab}X - Xe_{ab}) + M_{ab}^r(a, b)e_{ab}$. □

Let X be as in the last claim and consider the representation $\tilde{\rho} := (I_n + X)\rho(I_n + X)^{-1}$. It follows that $\tilde{\rho}(t_{ab}^r) = t_{ab}^{p_r} + M_{ab}^r + Xt_{ab}^{p_r} - t_{ab}^{p_r}X = t_{ab}^{\phi_{ab}(r)}$, where $\phi_{ab}(r) := p_r + c_{ab}^r$. The lift $\tilde{\rho}$, strictly equivalent to ρ, satisfies thus $(\diamondsuit)$ and Lemma 5.9 finishes the proof of the theorem. □

Corollary 5.11. *Every $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ can be obtained as a universal deformation ring of a continuous representation of a profinite group satisfying the condition (Φ_p) .*

5.4 Lower dimensions

In this section we continue working under Assumption 5.7 and discuss the possibility of extending Theorem 5.10 to the cases $n = 2$ and $n = 3$.

5.4.1 Case $n = 3$

Theorem 5.12. *Suppose $n = 3$, $k \neq \mathbb{F}_2$. Then ι is universal.*

Proof. A closer look at the proof of Theorem 5.10 shows that assuming Claim 2 and Claim 4 the rest of the argument would hold also for $n \geqslant 3$. We provide thus a different argument for both of the claims in case $n = 3$ and $k \neq \mathbb{F}_2$ (this second assumption is actually needed only for proving Claim 4). In what follows, we assume $[n] = \{a, b, c\}$.

A proof of Claim 2: Considering $(a, b) \in \mathfrak{J}$, $r \in R$ we need to show that $M^r_{ab}(i, j) = 0$ for $(i, j) \in \{(b, a), (c, a), (b, c)\}$. We see that the fact that t^r_{ab} and t^1_{ac} commute implies $M^r_{ab}(i, a) = 0$ for $i \neq a$, just as in the case $n \geqslant 4$. Similarly, the fact that t^r_{ab} and t^1_{cb} commute implies $M^r_{ab}(b, j) = 0$ for $j \neq b$.

A proof of Claim 4: Let $(a, b) \in \mathfrak{J}$, $r \in R$ and define U_c, χ just as in the case $n \geqslant 4$. We need to show $M^r_{ab}(c, c) = 0$. Making use of the assumption $k \neq \mathbb{F}_2$ we choose $\lambda \in R$ such that $\lambda \not\equiv 0, 1 \mod \mathfrak{m}_R$ and consider the elements $d := d^{\lambda}_{ac}$, $t := t^{\frac{r}{\lambda-1}}_{ab}$. According to Lemma 5.3, relation (R4), we have $[d, t] = t^{\frac{\lambda r}{\lambda-1}}_{ab} t^{\frac{-r}{\lambda-1}}_{ab} = t^r_{ab}$. Since $\rho(d), \rho(t), \rho(t^r_{ab}) \in U_c$, evaluating χ at $\rho(t^r_{ab})$ we conclude that $M^r_{ab}(c, c) = 0$, just as in the case $n \geqslant 4$. □

As the following lemma shows, the case $k = \mathbb{F}_2$ must really be excluded in Theorem 5.12.

Proposition 5.13. *Assume $n = 3$ and $k = \mathbb{F}_2$.*

(i) There exists a lift ρ_0 of $\bar{\rho}$ to $\mathbb{Z}_2$.

(ii) There is no $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ for which ι is universal.

Proof. (i) Since $\operatorname{im}\bar{\rho} \subseteq \mathrm{SL}_3(\mathbb{F}_2)$, it is enough to prove the claim for $R = k$. There exists an irreducible 3-dimensional representation of $\mathrm{SL}_3(\mathbb{F}_2)$ over the ring $\mathbb{Z}[\omega]$, $\omega = \frac{-1+\sqrt{7}}{2}$, defined in [ATL] by

$$A := \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & \omega & -1-\omega \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \; B := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

In order to check that a representation of $\mathrm{SL}_3(\mathbb{F}_2) = \langle A, B \rangle$ may be defined this way recall that $\mathrm{SL}_3(\mathbb{F}_2)$ is known to be isomorphic to $\mathrm{PSL}_2(\mathbb{F}_7)$, which has an abstract presentation $\langle S, T \mid S^7 = T^2 = (ST)^3 = (S^4T)^4 = 1 \rangle$ due to Sunday ([Su]). One checks directly that the defining relations are satisfied both by $T := A$, $S := BA$ and their proposed images. We obtain ρ_0 by sending ω to the root of $X^2 + X + 2$ that lies in $1 + 2\mathbb{Z}_2$.

(ii) The first part of the proposition implies the claim in case $R = k$. In the general case, the claim follows from this observation and Proposition 4.3. □

5.4.2 Case $n = 2$

Theorem 5.14. *Suppose $n = 2$ and $k \neq \mathbb{F}_2$, $\mathbb{F}_3$, $\mathbb{F}_5$. Then ι is universal.*

Proof. Let $S \in \mathrm{Ob}(\hat{\mathcal{C}})$ and $\xi \in \mathrm{Def}_{\bar{\rho}}(S)$ be given. According to Lemma 5.8 we only need to show that $\xi \in \operatorname{im}[\iota]_S^*$.

Define $H := \{\left(\begin{smallmatrix} u & \\ & u^{-1} \end{smallmatrix}\right) \mid u \in \mu_R\}$, $M := R$ and let $\alpha : M \to k$ be the reduction modulo $\mathfrak{m}_R$. Due to the assumption $k \neq \mathbb{F}_2$, $\mathbb{F}_3$, $\mathbb{F}_5$ there exists $\alpha \in \mu_R$ such that $\alpha^4 \neq 1$. Using the notation of Remark 3.6 we obtain $\left(\begin{smallmatrix} \alpha & \\ & \alpha^{-1} \end{smallmatrix}\right) \in H \cap \chi^{-1}(X_H)$. Therefore, restricting ρ to the subgroup $G_{M,H}$ of $\mathrm{SL}_2(R)$, we conclude from Remark 3.6 that there exist $\rho \in \xi$ and $f : R \to S$ such that $\rho|_H = \mathrm{id}_H$ and $\rho(t_{12}^r) = t_{12}^{f(r)}$ for every $r \in R$. Considering the representation $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \rho \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ we similarly obtain from Remark 3.6 that there exists $g : R \to S$ such that $\rho(t_{21}^r) = t_{21}^{g(r)}$ for every $r \in R$. We see that lift ρ satisfies condition $(\diamondsuit)$ of Lemma 5.9, hence $\xi = [\rho] \in \operatorname{im}[\iota]_S^*$. □

Proposition 5.15. *Assume $n = 2$ and $k \in \{\mathbb{F}_2, \mathbb{F}_3, \mathbb{F}_5\}$.*

(i) There exists a lift ρ_0 of $\bar{\rho}$ to, respectively, $\mathbb{Z}_2$, $\mathbb{Z}_3$ or $\mathbb{Z}_5[\sqrt{5}]$.

(ii) There is no $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ for which ι is universal.

Proof. (i) It is enough to prove the claim for $R = k$. One easily checks that $\mathrm{SL}_2(\mathbb{F}_2) = \langle\tau\rangle \rtimes \langle\varepsilon\rangle$, where $\tau := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, $\varepsilon := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and that $\tau \mapsto \left(\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}\right)$, $\varepsilon \mapsto \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ defines a lift of $\mathrm{SL}_2(\mathbb{F}_2)$ to $\mathbb{Z}_2$. In case $p \in \{3, 5\}$ it is known ([Cox, §7.6]) that $\mathrm{SL}_2(\mathbb{F}_p)$ has presentation

$$\langle A, B, C \mid A^p = B^3 = C^2 = ABC\rangle = \langle A, C \mid A^p = (A^{-1}C)^3 = C^2 \rangle,$$

realized for example by the following choice of generators: $A := \left(\begin{smallmatrix} -1 & 0 \\ -1 & -1 \end{smallmatrix}\right)$, $C := \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Using this fact and defining $t \in \mathbb{Z}_3$ by $t^2 = -2$, $t \equiv 2 (\mathrm{mod}\ 3)$ it is easy to check that

$$\left(\begin{smallmatrix} -1 & 0 \\ -1 & -1 \end{smallmatrix}\right) \mapsto \frac{1}{2}\left(\begin{smallmatrix} 1 & t+1 \\ t-1 & 1 \end{smallmatrix}\right) \quad , \quad \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$$

extends to a lift of $\mathrm{SL}_2(\mathbb{F}_3)$ to $\mathbb{Z}_3$. Similarly, defining $i, \varphi \in \mathbb{Z}_5[\sqrt{5}]$ by $i^2 = -1$, $i \equiv 2 (\mathrm{mod}\ 5)$ and $\varphi := \frac{1+\sqrt{5}}{2}$ we have that

$$\left(\begin{smallmatrix} -1 & 0 \\ -1 & -1 \end{smallmatrix}\right) \mapsto \frac{1}{2}\begin{pmatrix} \varphi & i(\varphi-1)+1 \\ i(\varphi-1)-1 & \varphi \end{pmatrix} \quad , \quad \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$$

extends to a lift of $\mathrm{SL}_2(\mathbb{F}_5)$ to $\mathrm{GL}_2(\mathbb{Z}_5[\sqrt{5}])$.

(ii) The claim follows from (i) and Proposition 4.3, similarly as in Proposition 5.13. □

Corollary 5.16. *Combining Theorems 5.10, 5.12 and 5.14 with Propositions 5.13 and 5.15 we obtain Theorem 5.1 stated in the introduction to this chapter.*

5.5 Special cases

It would be interesting to know what are the universal deformation rings of $\bar{\rho}$ in the cases not treated by Theorem 5.12 and Theorem 5.14. We present a complete answer in case $R = k$.

Proposition 5.17. *The universal deformation ring of $\bar{\rho}$ for $n = 3$ and $R = \mathbb{F}_2$ is $\mathbb{Z}_2$.*

Proof. It is sufficient to check that the tangent space to $\mathrm{Def}_{\bar{\rho}}$ is zero dimensional. Indeed, this implies that a versal deformation ring $R_{\bar{\rho}}$ of $\mathrm{Def}_{\bar{\rho}}$ exists (Proposition 2.23) and is a quotient of $\mathbb{Z}_2$ (Proposition 1.46). By

Proposition 5.13, it can not be a proper quotient of $\mathbb{Z}_2$, so it will follow that $R_{\bar\rho} = \mathbb{Z}_2$. This last condition implies also that $R_{\bar\rho}$ is a universal deformation ring of $\bar\rho$.

We have to check that every deformation of $\bar\rho$ to $S = \mathbb{F}_2[\varepsilon]$ is induced by a $\hat{\mathcal{C}}$-morphism $R \to S$. This can be done modifying the argument used in the inductive step of the proof of Theorem 5.10, in the special case $S = \mathbb{F}_2[\varepsilon]$, $R = \mathbb{F}_2$. In Theorem 5.10 we have assumed that $n \geqslant 4$, but, as mentioned in the proof of Theorem 5.12, this condition is crucial only for proving Claims 2 and 4; the rest of the argument uses only a weaker assumption $n \geqslant 3$. Moreover, in Theorem 5.12 we have presented an alternative argument for Claim 2 that holds in case $n = 3$. Therefore, the only difficulty lies in finding a different argument for Claim 4, which asserts that given $(a, b) \in \mathfrak{J}$ and $r \in R$, we have $M^r_{ab}(i, i) = 0$ for all $i \in [n] \backslash \{a, b\}$. In our case, since $n = 3$, there is only one such i for given a and b. Moreover, since $R = \mathbb{F}_2$, the only non-trivial case is $r = 1$.

Suppose $[n] = \{a, b, c\}$ and set $t_{ab} := t^1_{ab}$, $M := M^1_{ab}$. We need to check that $M(c, c) = 0$. Note that t_{ab} is of order 2 and so is its lift $\rho(t_{ab}) = t_{ab} + M$. Using the fact that $\operatorname{char} S = 2$, we compute:

$$I_n = (t_{ab} + M)^2 = I_n + t_{ab}M + Mt_{ab} = I_n + e_{ab}M + Me_{ab}.$$

In particular, comparison of (a, b)-entries yields: $M(a, a) + M(b, b) = 0$. Since $M(a, a) + M(b, b) + M(c, c) = \operatorname{tr} M = 0$ by Claim 3, we conclude that $M(c, c) = 0$. □

Lemma 5.18. *Let polynomials $f_n \in \mathbb{Z}[X]$ be defined recursively by $f_0 = 0$, $f_1 = 1$, $f_{n+1} = Xf_n - f_{n-1}$. Consider a commutative ring R and a matrix $M \in M_2(R)$ such that $\det M = 1$ and at least one of its off-diagonal entries is not a zero divisor. If $n = 2k+1$ is an odd positive integer then $M^n = -I_n$ holds if and only if $t := \operatorname{tr} M$ is a root of the polynomial $f_{k+1} - f_k$.*

Proof. By Cayley-Hamilton, $M^2 = tM - 1$ and it is easy to check that $\forall n \geqslant 1 : M^n = f_n(t)M - f_{n-1}(t)I_n$. It follows that $M^n = -I_n$ if and only if $f_n(t) = 0$ and $f_{n-1}(t) = 1$. If $I_n = (f_n, f_{n-1} - 1)$ is the ideal of $\mathbb{Z}[X]$ generated by f_n and $f_{n-1} - 1$, then one easily proves by induction on l that $\forall l \in \{0, \ldots, n-1\} : I_n = (f_{n-l} - f_l, f_{n-1-l} - f_{l+1})$. In particular, for $l = k$ we obtain $I_n = (f_{k+1} - f_k)$. □

Proposition 5.19. *Assume $n = 2$ and $k \in \{\mathbb{F}_2, \mathbb{F}_3, \mathbb{F}_5\}$. The universal deformation rings of $\bar{\rho}$ for $R = k$ are, respectively: $\mathbb{Z}_2$, $\mathbb{Z}_3[X]/(X^3 - 1)$ and $\mathbb{Z}_5[\sqrt{5}]$.*

Proof. For $G = \mathrm{SL}_2(\mathbb{F}_2)$ we observe that the $\mathbb{F}_2G$-module $V_{\bar{\rho}}$ is projective. Indeed, for a field k of characteristic p and a finite group G with p-Sylow subgroup S, a kG-module V is projective if and only if V is projective as a kS-module ([Alp, p. 66, Corollary 3]). In this case $S \cong \langle \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \rangle \cong C_2$ (cyclic group of order 2) and $V_{\bar{\rho}|_S} \cong \mathbb{F}_2[C_2]$ is even a free $\mathbb{F}_2S$-module. The claim follows now from Proposition 2.34.

The case $G = \mathrm{SL}_2(\mathbb{F}_3)$ can be approached via Proposition 4.2. It follows from the discussion in [Cox, §7.2, §7.6] that $G \cong G' \rtimes C_3$, where $G' = \langle \left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 2 & 0 \end{smallmatrix}\right) \rangle$ is isomorphic to the quaternion group of order 8. Since G' has order coprime to $p = 3$, it follows from Proposition 2.34 that $\mathbb{Z}_3$ is the universal deformation ring for $\bar{\rho}|_{G'}$. Proposition 5.15 shows that there exists a universal lift of $\bar{\rho}|_{G'}$ that may be extended to G. Moreover, it is easy to check that $\mathrm{Ad}(\bar{\rho})^{G'} = kI_n$. Thus, given that $G/G' \cong C_3$, the universal deformation ring of $\bar{\rho}$ is $\mathbb{Z}_3[C_3] \cong \mathbb{Z}_3[X]/(X^3 - 1)$.

In case $R = \mathbb{F}_5$ we will simply check that the lift described in Proposition 5.15 is universal. Consider $S \in \mathrm{Ob}(\hat{\mathcal{C}})$, $\xi \in \mathrm{Def}_{\bar{\rho}}(S)$ and let $A := \left(\begin{smallmatrix} -1 & 0 \\ -1 & -1 \end{smallmatrix}\right)$, $C := \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in \mathrm{SL}_2(R)$; we moreover identify C with $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in \mathrm{SL}_2(S)$. Since $H := \langle C \rangle$ is of order 4, it follows from Proposition 2.34 that there is precisely one deformation of $\bar{\rho}|_H$ to S. Hence, ξ has a representative $\rho \in \xi$ satisfying $\rho(C) = C$. We claim that there is precisely one $\rho \in \xi$ satisfying this condition and such that the diagonal entries of $\rho(A)$ are equal. Indeed, if $\rho(A) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $X \in I_n + M_n(\mathfrak{m}_S)$ is a matrix commuting with $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ then there exist $u, v \in S$ such that $X = \left(\begin{smallmatrix} u & v \\ -v & u \end{smallmatrix}\right)$ and writing $t := v/u \in \mathfrak{m}_S$ we obtain

$$\left(\begin{smallmatrix} u & v \\ -v & u \end{smallmatrix}\right) \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \left(\begin{smallmatrix} u & v \\ -v & u \end{smallmatrix}\right)^{-1} = \frac{1}{1+t^2} \left(\begin{smallmatrix} a+ct+bt+dt^2 & b+dt-at-ct^2 \\ c-at+dt-bt^2 & d-bt-ct+at^2 \end{smallmatrix}\right).$$

The equation $a + ct + bt + dt^2 = d - bt - ct + at^2$ is equivalent to $t^2(d - a) + 2t(b + c) + (a - d) = 0$ and has precisely one solution $t \in \mathfrak{m}_S$, due to Hensel's lemma.

Since A and C generate G, a lift ρ is uniquely determined by $\rho(A)$ and $\rho(C)$. Note that $\det \rho(A) = \det \rho(C) = 1$ due to Lemma 5.5. Using all the above observations and the presentation of $\mathrm{SL}_2(\mathbb{F}_5)$ introduced in the proof of Proposition 5.15, we see that deformations of $\bar{\rho}$ to S correspond bijectively with matrices $M = \left(\begin{smallmatrix} a & b \\ c & a \end{smallmatrix}\right) \in M_n(S)$ such that $\left(\begin{smallmatrix} a & b \\ c & a \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} -1 & 0 \\ -1 & -1 \end{smallmatrix}\right) \bmod \mathfrak{m}_S$, $\det M = 1$ and $M^5 = \left(M^{-1}C\right)^3 = -I_n$.

By Lemma 5.18, the last condition is equivalent to $\operatorname{tr} M = 2a$ being a root of $f_3 - f_2 = X^2 - X - 1$ and $\operatorname{tr}(M^{-1}C) = b - c$ being a root of $f_2 - f_1 = X - 1$. If $(2a)^2 = 2a + 1$ then solving the quadratic equation $b(b-1) + 1 - a^2 = 1 - \det M = 0$ we obtain that $b = \frac{1-i(1-2a)}{2}$ with $i^2 = -1$. We conclude that the full set of conditions imposed on a, b, c is as follows: $a = \frac{\varphi}{2}$, $b = \frac{1-i(1-\varphi)}{2}$, $c = b - 1$, where $\varphi^2 = \varphi + 1$, $\varphi \equiv -2 \bmod \mathfrak{m}_S$ and $i^2 = -1$, $i \equiv 2 \bmod \mathfrak{m}_S$. It follows that every deformation of $\bar{\rho}$ to S is induced by a morphism $\mathbb{Z}_5[\sqrt{5}] \to S$ applied to the universal lift defined in the proof of Proposition 5.15. □

Remark 5.20. In view of Proposition 4.3, the results of this chapter provide a valuable information about the exceptional universal deformation rings in general. More precisely, we have that the universal deformation ring of $\bar{\rho}$ has, depending on the case, $R \times_{\mathbb{F}_2} \mathbb{Z}_2$, $R \times_{\mathbb{F}_3} \mathbb{Z}_3[X]/(X^3 - 1)$ or $R \times_{\mathbb{F}_5} \mathbb{Z}_5[\sqrt{5}]$ as its quotient.

Remark 5.21. The above results obtained for $n = 2$ seem to be not entirely new. For example, Rainone in [Ra] has considered the case $k = \mathbb{F}_2$ and Mazur mentions the case $k = \mathbb{F}_5$ in [Maz2, §1.9] though without giving a proof. Also Bleher and Chinburg obtained analogous results for an algebraically closed field in [BC3]. However, there does not seem to be an easy and complete treatment of all the cases in the literature.

Remark 5.22. It is worth noting that even though in case $k = \mathbb{F}_2$ we have obtained the same universal deformation ring for $n = 2$ and $n = 3$, the kG-module $V_{\bar{\rho}}$ is not projective when $n = 3$. Indeed, it is known ([Alp, p. 33, Corollary 7]) that if a kG-module V is projective then the order of the p-Sylow subgroup S of G divides $\dim_k V$. Here $|S| = 8$ and $\dim_k V_{\bar{\rho}} = 3$.

5.6 The general linear group

Concluding this chapter, we turn back to the more general picture sketched in Section 5.1. Let $R \in \operatorname{Ob}(\hat{\mathcal{C}})$ be given. We will consider the family

$$\mathfrak{F} := \{G \leqslant \mathrm{GL}_n(R) \mid G \text{ closed and } \mathrm{SL}_n(R) \leqslant G\}$$

and the corresponding representations $\bar{\rho}_G$, defined as in Section 5.1. In particular, we want to analyze what results would be obtained in the preceding sections if we considered the general linear group instead of the special linear one.

All elements of $\mathfrak{F}$ are clearly profinite groups. Note that the determinant map gives a bijective correspondence between $\mathfrak{F}$ and closed subgroups of $R^\times$. In particular, every $G \in \mathfrak{F}$ is a normal subgroup of $\mathrm{GL}_n(R)$. As mentioned in Section 5.1, not every $G \in \mathfrak{F}$ satisfies the necessary condition presented in Proposition 2.33.

Example 5.23. Let $R := \mathbb{F}_p[[X]]$. One may check that $R_1^\times \cong \mathbb{Z}_p^{\mathbb{N}}$. Using the determinant map and isomorphism $R^\times \cong \mu_R \oplus R_1^\times$ we obtain that $\mathrm{CHom}(\mathrm{GL}_n(R), \mathbb{Z}/p\mathbb{Z})$ is infinite.

Consequently, $\mathrm{Def}_{\bar{\rho}_G}$ need not be representable over $\hat{\mathcal{C}}$. If it is, we will denote an object representing it by $R_u(G)$.

Proposition 5.24. *Let $G, H \in \mathfrak{F}$ be such that $H \subseteq G$ and $\mathrm{Def}_{\bar{\rho}_H}$ is represented by R. If $\mathrm{CHom}(G/H, \mathbb{Z}/p\mathbb{Z})$ is finite then $\mathrm{Def}_{\bar{\rho}_G}$ is represented by $R[[(G/H)^p]]$. Otherwise it is not representable over $\hat{\mathcal{C}}$.*

Proof. It is an immediate consequence of Proposition 4.2 and Lemma 2.26. □

We conclude that Rainone's results about $\mathrm{GL}_n(k)$, mentioned in Section 5.1, generalize much better and in a more natural way to the group $\mu L_n(R)$ defined below than to the group $\mathrm{GL}_n(R)$:

Corollary 5.25. *Suppose $(n, k) \notin \{(2, \mathbb{F}_2), (2, \mathbb{F}_3), (2, \mathbb{F}_5), (3, \mathbb{F}_2)\}$ and let $\bar{\rho} := \bar{\rho}_{\mathrm{GL}_n(k)}$.*

(i) Either $R_u(\mathrm{GL}_n(R)) \cong R[[R_1^\times]]$ or $\mathrm{Def}_{\bar{\rho}}$ is not representable over $\hat{\mathcal{C}}$. In particular, R represents $\mathrm{Def}_{\bar{\rho}}$ if and only if $R = k$.

(ii) Let $\mu L_n(R) := \{A \in \mathrm{GL}_n(R) \mid \det A \in \mu_R\}$. The set $\mathfrak{G}$ of all $G \in \mathfrak{F}$ for which $\mathrm{Def}_{\bar{\rho}_G}$ is represented by R coincides with the set $\{G \in \mathfrak{F} \mid G \leqslant \mu L_n(R)\}$.

Proof. (i) By Theorems 5.10, 5.12 and 5.14 we have $R_u(\mathrm{SL}_n(R)) = R$, so we may apply Proposition 5.24 with $G = \mathrm{GL}_n(R)$ and $H = \mathrm{SL}_n(R)$. Since

$G/H \cong R^\times \cong \mu_R \oplus R_1^\times$, μ_R is of finite order coprime to p and $R_1^\times$ is a pro-p group, we have that $(R^\times)^p \cong R_1^\times$. Hence, the first claim follows.

(*ii*) A similar reasoning as in part (*i*) shows that elements of $\mathfrak{G}$ correspond (via the determinant map) with these closed subgroups of $R^\times \cong \mu_R \oplus R_1^\times$ that have a trivial pro-p completion. Since $R_1^\times$ is a pro-p group and μ_R finite of order coprime to p, every closed subgroup of $\mu_R \oplus R_1^\times$ is a product $A \oplus B$ of closed subgroups $A \leqslant \mu_R$ and $B \leqslant R_1^\times$. Moreover, $(A \oplus B)^p \cong B$, so the elements of $\mathfrak{G}$ correspond with subgroups of μ_R. $\square$

Remark 5.26. For $(n,k) \in \{(2,\mathbb{F}_2),(2,\mathbb{F}_3),(3,\mathbb{F}_2)\}$ there exists a lift of $\mathrm{GL}_n(k)$ to $\mathbb{Z}_p$. Indeed, in case $k = \mathbb{F}_2$ we have $\mathrm{GL}_n(k) = \mathrm{SL}_n(k)$, so we already know it; for $n = 2$, $k = \mathbb{F}_3$ see [Ra] (it is also not difficult to check it directly, knowing that $\mathrm{SL}_2(\mathbb{F}_3)$ lifts to $\mathbb{Z}_3$). This fact and a reasoning as in Proposition 5.13 show that in these cases R does not represent $\mathrm{Def}_{\bar\rho_G}$ for any $G \in \mathfrak{F}$.

If $n = 2$, $k = \mathbb{F}_5$ then $R_u(\mathrm{SL}_n(R)) \not\cong R$, but $R_u(\mu L_n(R)) \cong R$. This can be proved using the proof of Theorem 5.14 with only a small modification. Namely, instead of $H = \{\left(\begin{smallmatrix} u & \\ & u^{-1}\end{smallmatrix}\right) \mid u \in \mu_R\}$ we consider $H := \{\left(\begin{smallmatrix} u & \\ & 1\end{smallmatrix}\right) \mid u \in \mu_R\}$ and instead of $\alpha \in k^\times$ satisfying the condition $\alpha^4 \neq 1$, we choose α such that $\alpha^2 \neq 1$. Then $\left(\begin{smallmatrix} \alpha & \\ & 1\end{smallmatrix}\right) \in H \cap \chi^{-1}(X_H)$ and a combination of Remark 3.6, Lemma 5.9 and Lemma 5.8 proves the claim. As a corollary, we conclude, using Proposition 5.24, that the first part of Corollary 5.25 holds also in the case $n = 2$, $k = \mathbb{F}_5$.

Chapter 6

The inverse problem restricted to finite groups

Let k be a finite field and let us consider the category $\hat{\mathcal{C}}$. As we already know, every $\hat{\mathcal{C}}$-ring can be realized as a universal deformation ring of a continuous representation of a profinite group. This leads to a new question: does the same result hold if we restrict to representations of *finite* groups?

We show that, unlike in the general case, there exist some rings which do not occur as universal deformation rings in the restricted setting. The main result is Theorem 6.30 and its reformulation in Theorem 6.31 in which we present a non-trivial necessary condition for characteristic zero universal deformation rings of finite group representations.

6.1 Initial remarks and a cardinality argument

Definition 6.1. By $\mathfrak{U}$ we will denote the class of all $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ for which there exists a finite group G and a representation $\bar{\rho} : G \to \mathrm{GL}_n(k)$ such that R is the universal deformation ring of $\bar{\rho}$.

Formally speaking, in this chapter we are interested in determining the class $\mathfrak{U}$. To begin with, our earlier considerations lead to the following conclusion.

Observation 6.2. *Every finite ring* $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ *belongs to* $\mathfrak{U}$.

Proof. If $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ is finite then for every natural n the group $\mathrm{SL}_n(R)$ is finite and by results of Chapter 5 we have $R \in \mathfrak{U}$. $\square$

Moreover, note that we have already encountered examples of infinite rings in $\mathfrak{U}$:

- $\mathrm{W}(k)$, obtained for representations of groups of order coprime to p (see Proposition 2.34);
- Rings of the form

$$\mathrm{W}(k)[X_1, \ldots, X_m]/(X_1^{p^{k_i}} - 1, \ldots, X_m^{p^{k_m}} - 1), \quad m, k_1, \ldots, k_m \in \mathbb{N},$$

  obtained for one-dimensional representations of finite groups (see Proposition 2.30);
- $\mathbb{Z}_5[\sqrt{5}]$, obtained as the exceptional universal deformation ring in Chapter 5;
- In case $k \neq \mathbb{F}_2, \mathbb{F}_3$, rings of the form

$$\mathrm{W}(k)[[X_1, \ldots, X_m]]/(p^{k_0}, p^{k_1}X_1, \ldots, p^{k_m}X_m), \quad k_0, k_1, \ldots, k_m \in \mathbb{N}_+,$$

  obtained in Corollary 3.14.

Finally, let us observe that a cardinality argument shows that contrary to the general case, there exist rings which can not be obtained as universal deformation rings in the new setting.

Proposition 6.3. *The class* $\mathrm{Ob}(\hat{\mathcal{C}})$ *is uncountable.*

Proof. For every $\alpha \in \mathrm{W}(k)$ denote by R_α the following $\hat{\mathcal{C}}$-ring:

$$R_\alpha := \mathrm{W}(k)[X, Y]/\big(\,(X, Y)^5 + (\,X^4,\ Y^4 - X^2Y^2 - \alpha X^3 Y)\,\big)$$

We will prove that $R_\alpha \cong_{\hat{\mathcal{C}}} R_\beta$ if and only if $\alpha = \pm\beta$. Since $\mathrm{W}(k)$ is uncountable, this will imply that we can choose an uncountable family of pairwise non-isomorphic rings of the form R_α.

Given $\alpha \in \mathrm{W}(k)$ we will denote by x, y the images of X, Y in R_α and by I_α the ideal that they generate. Note that $I_\alpha = \operatorname{nil} R_\alpha$.

If $F \in \mathrm{Hom}_{\hat{\mathcal{C}}}(R_\alpha, R_\beta)$ is an isomorphism then $F(\mathrm{nil}\, R_\alpha) = F(\mathrm{nil}\, R_\beta)$, so it must be $F(x), F(y) \in I_\beta$. Suppose $a, b, c, d \in \mathrm{W}(k)$ are such that

$$\begin{aligned} F(x) &\equiv ax + by \pmod{I_\beta^2} \\ F(y) &\equiv cx + dy \pmod{I_\beta^2}. \end{aligned}$$

Then $F(x^4) \equiv (ax+by)^4 \pmod{I_\beta^5}$, but since $I_\beta^5 = 0$, we simply obtain $(ax+by)^4 = 0$. Let us expand the left hand side:

$$\begin{aligned} (ax+by)^4 &= 4(ax)^3 by + 6(ax)^2(by)^2 + 4ax(by)^3 + (by)^4 \\ &= (4a^3b + b^4\beta)x^3y + (6a^2b^2 + b^4)x^2y^2 + 4ab^3xy^3 \end{aligned}$$

It must be thus $(4a^3b + b^4\beta) = (6a^2b^2 + b^4) = ab^3 = 0$, which is possible if and only if $b = 0$. Note that this implies that $d \neq 0$, otherwise we would have $y \notin F(R_\alpha)$.

Similarly, from $F(y^4) = F(x^2y^2 + \alpha x^3 y)$ we obtain

$$(cx+dy)^4 = (ax)^2(cx+dy)^2 + \alpha(ax)^3(cx+dy).$$

By analogy to the earlier case, the left hand side of this equation equals $(cx+dy)^4 = (4c^3d + d^4\beta)x^3y + (6c^2d^2 + d^4)x^2y^2 + 4(cd^3)xy^3$ and the right right hand side can be expanded as

$$\begin{aligned} (ax)^2(cx+dy)^2 + \alpha(ax)^3(cx+dy) &= a^2x^2(2cd\,xy + d^2y^2) + (\alpha a^3 d)x^3y \\ &= (2a^2cd + \alpha a^3 d)x^3y + (a^2d^2)x^2y^2. \end{aligned}$$

Comparing both sides we conclude that

$$(4c^3d + d^4\beta - 2a^2cd - \alpha a^3 d) = (6c^2d^2 + d^4 - a^2d^2) = cd^3 = 0.$$

Since $d \neq 0$, these equations reduce to $c = 0$, $d^2 = a^2$ and $d\beta = \alpha a$. Consequently, $\beta = \frac{a}{d}\alpha = \pm\alpha$. On the other hand, it is clear that sending $x \mapsto -x$, $y \mapsto y$ we define an isomorphism between R_α and $R_{-\alpha}$. This finishes the proof. $\square$

Corollary 6.4. *The classes $\mathfrak{U}$ and $\mathrm{Ob}(\hat{\mathcal{C}})$ do not coincide.*

Proof. There exist only countably many finite groups G and each of them has only finitely many representations $\bar{\rho} : G \to \mathrm{GL}_n(k)$ over the finite field k. Consequently, $\mathfrak{U}$ is at most countable, whereas $\mathrm{Ob}(\hat{\mathcal{C}})$ is uncountable by Proposition 6.3. $\square$

We see that the interesting part of the modified inverse problem consists in determining which infinite rings belong to $\mathfrak{U}$ and which do not. Since the argument of Corollary 6.4 is not constructive, a first step towards solving this problem is to provide concrete examples of $\hat{\mathcal{C}}$-rings not in $\mathfrak{U}$. This will occupy us in the rest of this chapter.

6.2 A motivating example

Before developing a general approach, let us present a motivating example. For the rest of this subsection $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ will be a discrete valuation ring with a uniformizing element π and field of fractions K. The group G will be finite and we will consider a residual representation $\bar{\rho} : G \to \mathrm{GL}_n(k)$ about which we additionally assume that $\mathrm{Ad}(\bar{\rho})^G = kI_n$.

Proposition 6.5. *Lifts ρ_1, $\rho_2 \in \mathrm{Lift}_{\bar{\rho}}(R)$ are strictly equivalent if and only if they are equivalent as representations over K.*

Proof. The "only if" part is obvious. Conversely, suppose there exists $A \in \mathrm{GL}_n(K)$ such that $\rho_1 = A\rho_2A^{-1}$. Since R is a discrete valuation ring, there exist $B \in M_n(R)\backslash\pi M_n(R)$ and $s \in \mathbb{Z}$ such that $A = \pi^s B$. Clearly $\rho_1 B = B\rho_2$. Reducing to k we obtain $\bar{\rho}\bar{B} = \bar{B}\bar{\rho}$ and the assumption $\mathrm{Ad}(\bar{\rho})^G = kI_n$ implies that the image $\bar{B}$ of B is a (non-zero) scalar matrix. Therefore, there exist $u \in \mu_R$ and $B_0 \in I_n + M_n(\mathfrak{m}_R)$ such that $B = uB_0$. Since $\rho_1 = B_0\,\rho_2\,B_0^{-1}$, it follows that ρ_1 and ρ_2 are strictly equivalent. □

Corollary 6.6. *If* $\mathrm{char}\, R = 0$*, then* $\mathrm{Def}_{\bar{\rho}}(R)$ *is finite.*

Proof. The field K has characteristic zero, so the group ring KG is semisimple by Maschke's theorem. Therefore, up to equivalence, there exist only finitely many n-dimensional representations of G over K and the claim follows from Lemma 6.5. □

Corollary 6.7. *If* $\mathrm{char}\, R = 0$*, then* $R[[X]]$ *is not a universal deformation ring of* $\bar{\rho}$*.*

Proof. Combine the preceding corollary with the fact that $\mathrm{Hom}_{\hat{\mathcal{C}}}(R[[X]], R)$ is infinite. □

Our aim for the next section is to generalize these observations. We will prove that the claim of Corollary 6.6 holds even without the assumption $\mathrm{Ad}(\bar{\rho})^G = kI_n$. This will imply that $R[[X]] \notin \mathfrak{U}$. Moreover, we will develop an approach that will allow us to bound the number of deformations to a wider class of rings of characteristic zero, not only discrete valuation rings. This in turn will allow us to identify explicitly a class of $\hat{\mathcal{C}}$-rings not in $\mathfrak{U}$.

Remark 6.8. Proposition 6.5 applies also to discrete valuation rings of characteristic p. However, if $\operatorname{char} K = \operatorname{char} R = p$, there may exist infinitely many non-equivalent representations of G over K and Corollary 6.6 may not hold. Consequently, in this case one can not draw a similar conclusion that $R[[X]] \notin \mathfrak{U}$, cf. Example 3.13.

6.3 Finiteness bounds

Definition 6.9. We will denote by $\mathfrak{F}$ the class of all rings $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ such that for every finite group G and representation $\bar{\rho} : G \to \mathrm{GL}_n(k)$ the set $\mathrm{Def}_{\bar{\rho}}(R)$ is finite.

It is clear that every finite $\hat{\mathcal{C}}$-ring is in $\mathfrak{F}$, but what we are really interested in, is identifying some large subclass of infinite rings belonging to $\mathfrak{F}$. This will be done using the following key lemma, inspired by [Mar, Theorem 2].

Lemma 6.10. *Let G be a finite group, $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ be a ring in which $|G|$ is not a zero-divisor and define $J := |G| \cdot \mathfrak{m}_R \lhd R$. Then representations $\rho_1, \rho_2 : G \to \mathrm{GL}_n(R)$ are strictly equivalent if and only if their reductions $\pi_J\rho_1$ and $\pi_J\rho_2$ to R/J are strictly equivalent.*

Proof. The "only if" part of the lemma is obvious. For the "if" part note that $\pi_J\rho_1$ and $\pi_J\rho_2$ are strictly equivalent if and only if there exists $A \in I_n + M_n(\mathfrak{m}_R)$ such that

$$\forall g \in G: \quad \rho_1(g)A \equiv A\rho_2(g) \mod M_n(J).$$

If this is the case then $B := \sum_{g\in G} \rho_1(g)\, A\, \rho_2(g)^{-1}$ satisfies $B \equiv |G| \cdot A \mod M_n(J)$. Using the definition of J and the assumption that $|G|$ is not a zero divisor in R, we define $B_0 := \frac{1}{|G|} \cdot B \in M_n(R)$ and observe that

$B_0 \in I_n + M_n(\mathfrak{m}_R)$. Moreover,

$$\rho_1(h)\,B_0\,\rho_2(h)^{-1} = \frac{1}{|G|}\sum_{g\in G}\rho_1(hg)\,A\,\rho_2(hg)^{-1} = \frac{1}{|G|}\sum_{g\in G}\rho_1(g)\,A\,\rho_2(g)^{-1} = B_0,$$

so ρ_1 and ρ_2 are strictly equivalent. □

Remark 6.11. In the setting of the above lemma let us write $|G| = p^r s$, $r \geqslant 0$, $p \nmid s$. Since s is invertible in R, we have that $|G|$ is a zero-divisor if and only if p^r is. Moreover, $|G|\mathfrak{m}_R = p^r\mathfrak{m}_R$.

In particular, if $p \nmid |G|$ then the assumption that $|G|$ is not a zero-divisor in R is satisfied for every $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ and the above lemma implies that there is at most one deformation to every $R \in \mathrm{Ob}(\hat{\mathcal{C}})$. And actually there is exactly one deformation to every $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ – see Proposition 2.34.

Lemma 6.12. *Consider $R \in \mathrm{Ob}(\hat{\mathcal{C}})$, $r \in \mathbb{Z}_{\geqslant 1}$. The following conditions are equivalent:*

(i) p^r is not a zero-divisor in R and $R/p^r\mathfrak{m}_R$ is finite.

(ii) p is not a zero-divisor in R and $\dim R = 1$.

(iii) R is a finitely generated $\mathrm{W}(k)$*-module with trivial p-torsion part.*

Proof. It is clear that p^r is a zero-divisor if and only if p is a zero-divisor. Since k is finite we have that a ring $S \in \mathrm{Ob}(\hat{\mathcal{C}})$ is finite if and only if it is artinian, i.e., if and only if $\dim S = 0$. The fact that $p^r\mathfrak{m}_R \subseteq pR \subseteq \mathrm{rad}(p^r\mathfrak{m}_R)$ implies $\dim R/p^r\mathfrak{m}_R = \dim R/pR$. Assume that p is not a zero-divisor. Then $\dim R/pR = \dim R - 1$, so $\dim R/p^r\mathfrak{m}_R = 0$ if and only if $\dim R = 1$. This proves the equivalence of the first two statements. It is also clear that *(iii)* implies *(ii)*. The converse statement follows from Theorem 1.16 (see also Remark 1.18). □

Definition 6.13. We will denote by $\mathfrak{W}$ the subclass of all rings $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ satisfying the equivalent conditions of Lemma 6.12.

Corollary 6.14. *If $R \in \mathfrak{W}$ then $R \in \mathfrak{F}$.*

Proof. By the first property of Lemma 6.12 the ring $R/n\mathfrak{m}_R$ is finite for every $n \geqslant 1$ (see also the discussion in Remark 6.11). Since all finite $\hat{\mathcal{C}}$-rings are in $\mathfrak{F}$, the claim follows easily from Lemma 6.10. □

The result obtained in Corollary 6.14 is fully satisfactory for our applications in the next sections, but we note that it can be further extended.

Definition 6.15. Given an abelian group A and a prime number p we will denote by $T_{p^\infty}(A)$ its p-torsion subgroup, i.e., $T_{p^\infty}(A) = \bigcup_{r=1}^{\infty}\{a \in A \mid p^r a = 0\}$.

Observation 6.16. *If R is a ring then $T_{p^\infty}(R)$ is an ideal. Suppose that* $\operatorname{char} R = 0$ *and let $\tilde{R} := R/T_{p^\infty}(R)$. Then* $\operatorname{char}\tilde{R} = 0$ *and p is not a zero-divisor in $\tilde{R}$.*

Lemma 6.17. *Let $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ be of characteristic zero and finitely generated as a* $\mathrm{W}(k)$*-module. Then $R \in \mathfrak{F}$.*

Proof. Let G be a finite group and $\bar{\rho} : G \to \mathrm{GL}_n(k)$ its representation. The ring $\tilde{R} := R/T_{p^\infty}(R)$ is in $\mathfrak{W}$ (Observation 6.16 implies easily that $\tilde{R}$ satisfies the condition (*iii*) of Lemma 6.12), so by Corollary 6.14 the set $\mathrm{Def}_{\bar{\rho}}(\tilde{R})$ is finite. Noetherianity of R implies that there exists $r \geqslant 1$ such that $T_{p^\infty}(R) = \operatorname{Ann} p^r$. Hence $T_{p^\infty}(R)$ is a finite $\mathrm{W}(k)/(p^r)$-module and, consequently, a finite set. It follows that the fibers of the map $\mathrm{Lift}_{\bar{\rho}}(R) \to \mathrm{Lift}_{\bar{\rho}}(\tilde{R})$ induced by the reduction modulo $T_{p^\infty}(R)$ are finite and so $\mathrm{Def}_{\bar{\rho}}(R)$ is finite as well. □

6.4 Properties of $\mathfrak{W}$-rings

Notation 6.18. *In the rest of this chapter we reserve the letter K to denote the field of fractions of the ring* $\mathrm{W}(k)$.

Lemma 6.19. *Consider $R \in \mathfrak{W}$ and its localization $R' := R[\frac{1}{p}]$ away from p.*

(i) The natural map $R \to R'$ is injective and R is a domain (is reduced) if and only if R' is a domain (is reduced).

(ii) R' is naturally isomorphic to $R \otimes_{\mathrm{W}(k)} K$.

(iii) R' is an integral extension of K.

(iv) *R' is a domain if and only if it is a finite field extension of K. If this is the case then R' is the field of fractions of R.*

(v) *R' is reduced if and only if it is a finite étale K-algebra (i.e., a finite product of finite field extensions of K).*

Proof. (*i*) This is an easy consequence of the fact that p is not a zero-divisor in R.

(*ii*) Note that $K = \mathrm{W}(k)[\frac{1}{p}]$ and use the identification $B_S \cong B \otimes_A A_S$, valid for any A-algebra B and multiplicative subset $S \subseteq A$.

(*iii*) The extension $\mathrm{W}(k) \subseteq R$ is integral. Localizing away from p we obtain that $K = \mathrm{W}(k)[\frac{1}{p}] \subseteq R[\frac{1}{p}] = R'$ is integral as well.

(*iv*) The first claim follows from part (*iii*) and general properties of integral extensions (see for example [At, Proposition 5.7]). The second one is clear, since the fraction fields of domains R and $R[\frac{1}{p}]$ coincide.

(*v*) In general, part (*iii*) implies that R' is artinian, so by the structure theorem ([At, Theorem 8.7]), there exist artinian local rings $A_1, A_2, \ldots A_s$ such that $R \cong A_1 \oplus A_2 \oplus \ldots \oplus A_s$. Note that these rings are necessarily integral extensions of K.

The ring R' is reduced if and only if all A_i's have this property. To finish the proof, observe that a local artinian ring A is reduced if and only if it is a domain and use again [At, Proposition 5.7]. □

6.5 Excluding rings from being in $\mathfrak{U}$

Keeping in mind Corollary 6.14 and the general idea outlined in section 6.2, we turn our attention to the following problem: for which $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ does there exist $S \in \mathfrak{W}$ such that $\mathrm{Hom}_{\hat{\mathcal{C}}}(R, S)$ is infinite?

Observation 6.20. *If S is a ring in which p is not a zero-divisor then every ring homomorphism $R \to S$ factors via $R/T_{p^\infty}(R)$.*

The above observation implies that it is enough to solve the problem for $\hat{\mathcal{C}}$-rings R in which $T_{p^\infty}(R) = 0$, i.e., in which p is not a zero-divisor. Observe that such rings are of characteristic zero, hence infinite and of Krull dimension greater than zero.

In what follows we assume that $T_{p^\infty}(R) = 0$ and set $d := \dim R$. By Theorem 1.16 there exists a subring $R_0 \subseteq R$, isomorphic to the ring $\mathrm{W}(k)[[X_1, \ldots, X_{d-1}]]$, over which R is a finite module.

We divide further discussion into two cases, depending on whether $d \geqslant 2$ or $d = 1$.

6.5.1 Case $\dim R \geqslant 2$

The results of this subsection have been obtained developing a rough idea of Jakub Byszewski. I would like to express him my gratitude for suggesting this approach.

Lemma 6.21. *There exist (up to isomorphism) exactly countably many integral domains in $\mathfrak{W}$.*

Proof. Since for every $n \in \mathbb{N}$ we have the integral domain $\mathrm{W}(k)[\sqrt[n]{p}] \in \mathfrak{W}$, the interesting part of the proof consists in showing that $\mathfrak{W}$ contains at most countably many integral domains.

It is a classical fact, following from Krasner's lemma, that for every $n \in \mathbb{N}$ the field $\mathbb{Q}_p$ has (up to isomorphism) only finitely many extensions of degree n, see [St, Theorem 4.8]. As a consequence, there exist only countably many finite field extensions of K. Therefore, in view of Lemma 6.19.(i) and (iv), it is sufficient to prove that every family of pairwise non-isomorphic domains $S \in \mathfrak{W}$ with the same field of fractions L is at most countable.

Let L be a finite field extension of K and consider $S \subseteq L$ such that $S \in \mathfrak{W}$ and L is the field of fractions of S. Since S is integrally dependent on $\mathrm{W}(k)$, it is contained in the integral closure of $\mathrm{W}(k)$ in L, i.e., in the ring of integers $\mathcal{O}_L$ of L. On the other hand, $\mathcal{O}_L$ is a finitely generated $\mathrm{W}(k)$-module and by Lemma 6.19.(iv) we have $L = S[\frac{1}{p}]$, so there exists $r \in \mathbb{N}$ such that $p^r\mathcal{O}_L \subseteq S$. To finish the proof, it is sufficient to show that for every $r \in \mathbb{N}$ there exist only finitely many $\mathrm{W}(k)$-modules M such that $p^r\mathcal{O}_L \subseteq M \subseteq \mathcal{O}_L$. But this is obvious: such modules correspond bijectively with $\mathrm{W}(k)$-submodules of $\mathcal{O}_L/p^r\mathcal{O}_L$, which is a finite set (indeed: it is a finitely generated $\mathrm{W}(k)/(p^r)$-module). $\square$

Theorem 6.22. *If $d \geqslant 2$ then there exists an integral domain $S \in \mathfrak{W}$ such that the set $\mathrm{Hom}_{\hat{\mathcal{C}}}(R, S)$ is infinite.*

Proof. Due to the assumption $d \geqslant 2$, there exist uncountably many prime ideals $\mathfrak{p}$ of R_0 with the property $R_0/\mathfrak{p} \cong \mathrm{W}(k)$ (recall that by definition $R_0 \cong \mathrm{W}(k)[[X_1, \ldots, X_{d-1}]]$). If $\mathfrak{p}$ is one of them then there exists a prime ideal $\mathfrak{q}$ of R such that $\mathfrak{q} \cap R_0 = \mathfrak{p}$ ([At, Theorem 5.10]). The domain $R/\mathfrak{q}$ is an integral extension of $S/\mathfrak{p} \cong \mathrm{W}(k)$, hence belongs to $\mathfrak{W}$. We obtain thus uncountably many surjections $R \to R/\mathfrak{q}$ from R to some integral domain in $\mathfrak{W}$. Lemma 6.21 and infinite pigeonhole principle imply that for some integral domain $S \in \mathfrak{W}$ the set $\mathrm{Hom}_{\hat{\mathcal{C}}}(R, S)$ is infinite (even uncountable). $\square$

Remark 6.23. It is tempting to "refine" the above theorem by changing "integral domain" to "discrete valuation ring", using the following argument:

If a domain $S \in \mathfrak{W}$ has the field of fractions L then $S \subseteq \mathcal{O}_L$. Let us thus compose the considered morphisms $R \to S$ with inclusions $S \hookrightarrow \mathcal{O}_L$, in order to obtain infinitely many morphisms $R \to \mathcal{O}_L$.

However, $\mathcal{O}_L$ is not necessarily a $\hat{\mathcal{C}}$-ring. Even though it is complete, local and noetherian, its residue field may be strictly larger than k. See also the example below.

Example 6.24. Suppose p is a prime number satisfying $p \equiv 3 \pmod 4$, i.e., such that -1 is not a quadratic residue in $\mathbb{F}_p$ and let $R := \mathbb{Z}_p[[X, Y]]/(X^2 + Y^2)$.

Consider the integral domain $S := \mathbb{Z}_p[T]/(T^2 + p^2) \in \mathfrak{W}$. For every $a \in \mathbb{Z}_p$ we have a $\hat{\mathcal{C}}$-morphism defined by $X \mapsto aT$, $Y \mapsto ap$, so $\mathrm{Hom}_{\hat{\mathcal{C}}}(R, S)$ is infinite.

On the other hand, if $S \in \mathfrak{W}$ is a discrete valuation ring then the only $a, b \in S$ for which $a^2 + b^2 = 0$ are $a = b = 0$. Hence, $\mathrm{Hom}_{\hat{\mathcal{C}}}(R, S)$ is a one-element set.

6.5.2 Case $\dim R = 1$

If $\dim R = 1$ and $T_{p^\infty}(R) = 0$ then R itself belongs to $\mathfrak{W}$. It can be shown that $\mathrm{Spec}\, R$ is finite and so the approach of the preceding subsection can not be applied in this case. Yet there may still exist some $S \in \mathfrak{W}$ for which the set $\mathrm{Hom}(R, S)$ is infinite.

Example 6.25. Let $R := \mathrm{W}(k)[\varepsilon]$. The only $\hat{\mathcal{C}}$-morphism $R \to S$ such that S is a characteristic zero integral domain is the reduction $R \xrightarrow{\varepsilon \mapsto 0} \mathrm{W}(k)$, but for $S := R$ there exist infinitely many morphisms $R \to S$. Indeed, for every $C \in \mathrm{W}(k)$ we can define a $\hat{\mathcal{C}}$-homomorphism

$$\mathrm{W}(k)[\varepsilon] \ni x + \varepsilon y \longmapsto x + C\,\varepsilon y \in \mathrm{W}(k)[\varepsilon].$$

Consequently, $R \notin \mathfrak{U}$.

Note that all infinitely many $\hat{\mathcal{C}}$-homomorphisms $R \to S$ constructed in the above example reduce to the same morphism modulo $I := \varepsilon S$. Moreover, the ideal $I \lhd S$ has the property $I^2 = 0$. In the rest of this subsection we will construct pairs (R, S) with similar properties.

In what follows, we will need the notions of derivations and Kähler differentials. Their definitions and basic properties can be found for example in [Ei, §16].

Proposition 6.26. *Let A be a ring and $f : R \to S$ be a homomorphism of A-algebras. Suppose there exists an ideal $I \lhd S$ with property $I^2 = 0$ and let $g : R \to S$ be an additive map such that $f \equiv g \pmod{I}$. Then g is an A-algebra homomorphism if and only if the map $(f - g) : R \to I$ is an A-derivation.*

Proof. Apply [Ei, Proposition 16.11]. □

Notation 6.27. *Given a ring A and an A-algebra R we will denote by $\Omega_{R/A}$ the R-module of relative Kähler differentials of R.*

Lemma 6.28. *If $R, S \in \mathfrak{W}$ and S is reduced then $\mathrm{Hom}_{\hat{\mathcal{C}}}(R, S)$ is finite.*

Proof. By Lemma 6.19.(i) it is sufficient to show that there exist only finitely many $\mathrm{W}(k)$-algebra homomorphisms $R \to S[\frac{1}{p}] =: S'$. Let $x_1, \ldots, x_m \in R$ be such that $R = \mathrm{W}(k)[x_1, \ldots, x_m]$. For every $i \in \{1, \ldots, m\}$ there exists a monic polynomial $F_i \in \mathrm{W}(k)[X]$ of which x_i is a root. If S is reduced then S' is a finite product of fields by Lemma 6.19.(v). Hence, each of these polynomials has only a finite number of roots in S', so there exist only finitely many values in S' to which x_i, $i \in \{1, \ldots, m\}$, can be mapped. This proves the claim. □

Theorem 6.29. *Consider $R \in \mathfrak{W}$ and define $R' := R[\frac{1}{p}]$, $\Omega := \Omega_{R/\mathrm{W}(k)}$, $\Omega' := \Omega_{R'/K}$. The following conditions are equivalent:*

(i) There exists $S \in \mathfrak{W}$ such that $\mathrm{Hom}_{\hat{\mathcal{C}}}(R, S)$ is infinite.

(ii) $\Omega \neq T_{p^\infty}(\Omega)$.

(iii) Ω' *is not trivial.*

(iv) R *is not reduced.*

Proof. $(i) \Rightarrow (ii)$: Let $N := \operatorname{nil} S$ and observe that by noetherianity there exists $m \in \mathbb{N}$ such that $N^m = 0$. We have a finite chain of morphisms

$$S = S/N^m \twoheadrightarrow S/N^{m-1} \twoheadrightarrow \ldots \twoheadrightarrow S/N^2 \twoheadrightarrow S/N.$$

By assumption, $\#\operatorname{Hom}_{\hat{\mathcal{C}}}(R, S/N^m) = \infty$ and $\#\operatorname{Hom}_{\hat{\mathcal{C}}}(R, S/N) < \infty$ by Lemma 6.28. Hence, there exists $r \in \{1, 2, \ldots, m-1\}$ such that

$$\#\operatorname{Hom}_{\hat{\mathcal{C}}}(R, S/N^{r+1}) = \infty \quad \text{and} \quad \#\operatorname{Hom}_{\hat{\mathcal{C}}}(R, S/N^r) < \infty.$$

Let $\tilde{S} := S/N^{r+1}$, $I := N^r/N^{r+1}$. Then I is a non-zero ideal of $\tilde{S}$ such that $I^2 = 0$. By infinite pigeonhole principle, there exists an infinite family of $\hat{\mathcal{C}}$-morphisms $R \to \tilde{S}$ with the same reduction modulo I. By Proposition 6.26, it corresponds to an infinite family of derivations $R \to I$, so $\operatorname{Hom}_R(\Omega, I)$ is infinite by the definition of Ω. Observe that both Ω and I are finitely generated $W(k)$-modules: Ω because R is a finitely generated $W(k)$-module and I because it is finitely generated over $\tilde{S}$, which itself is finitely generated over $W(k)$. Both Ω and I are therefore infinite – otherwise $\operatorname{Hom}_{W(k)}(\Omega, I)$ would be finite, which would yield a contradiction. This means that they both must be distinct from their p-torsion subgroups, in particular $\Omega \neq T_{p^\infty}(\Omega)$.

$(ii) \Rightarrow (i)$: Define $M := \Omega/T_{p^\infty}(\Omega)$ and let us adopt the convention of writing $[\omega]$ for the corresponding class of $\omega \in \Omega$ in M. Note that Ω is a finitely generated R-module (because R is a finitely generated W(k)-algebra) and hence, so are M and $S := R \oplus M$.

We introduce the ring structure on S using the scalar multiplication and setting $xy = 0$ for every $x, y \in M$. The obtained ring is clearly local, complete and noetherian (here it is important that M is a finitely generated R-module) and has the same residue field as R. Moreover, $T_{p^\infty}(S)$ is trivial and since $R \subseteq S$ is an integral extension, we also have $\dim S = \dim R = 1$. Therefore, $S \in \mathfrak{W}$. Furthermore, for every $C \in \mathrm{W}(k)$ the map $R \ni x \mapsto x + C \cdot [dx] \in S$ is a well-defined $\hat{\mathcal{C}}$-morphism. Since M is a (by assumption non-trivial) free W(k)-module, all these morphisms are pairwise distinct. We conclude that S satisfies all the required properties.

$(ii) \Leftrightarrow (iii)$: This part follows from $\Omega' = \Omega_{R'/K} \cong \Omega_{R/\mathrm{W}(k)} \otimes_{\mathrm{W}(k)} K \cong \Omega[\frac{1}{p}]$ (note that formation of differentials commutes with base change: [Ei, Proposition 16.4]) and an easy observation that $\Omega[\frac{1}{p}] = 0$ if and only if $\Omega = T_{p^\infty}(\Omega)$.

$(iii) \Leftrightarrow (iv)$: R' is a finitely generated K-algebra, so $\Omega' = 0$ if and only if R' is a finite direct product of fields, each finite and separable over K ([Ei, Corollary 16.16]). Note that $\operatorname{char} K = 0$, so every field extension of K is separable and combine this result with Lemma 6.19.(v). □

6.6 Conclusions and comments

The following result is an immediate consequence of Corollary 6.14, Theorem 6.22 and Theorem 6.29.

Theorem 6.30. *Let $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ be of characteristic zero and in $\mathfrak{U}$. Then $R/T_{p^\infty}(R)$ is reduced and of Krull dimension* 1.

Proof. If R is a universal deformation ring of a representation $\bar{\rho}$ of a finite group G and $S \in \mathfrak{W}$, then $\mathrm{Def}_{\bar{\rho}}(S)$ is finite by Corollary 6.14. Hence, so is $\mathrm{Hom}_{\hat{\mathcal{C}}}(R, S)$. Theorem 6.22 implies thus that $\dim R/T_{p^\infty}(R) = 1$ and Theorem 6.29 implies that $R/T_{p^\infty}(R)$ is reduced. □

Alternatively, using Lemma 6.12 and Lemma 6.19 we can phrase this theorem as follows:

Theorem 6.31. *If $R \in \mathfrak{U}$ has characteristic zero then $R \otimes_{\mathrm{W}(k)} K$ is a finite étale K-algebra.*

Note that the infinite rings belonging to $\mathfrak{U}$ that were mentioned in Section 6.1 indeed satisfy the conditions of the above theorems.

Remark 6.32. Let us return to the argument proving Corollary 6.4 and analyze the construction described in the proof of Proposition 6.3. The above theorems imply that if $\operatorname{char} R = 0$, then actually none of the constructed uncountably many rings belongs to $\mathfrak{U}$. However, Proposition 6.3 still provides some extra information. Namely, it implies that there are uncountably many rings that can not be obtained even as a *versal* deformation ring of a finite group representation. This is something that we can not conclude in any way using the approach that we have just developed.

Remark 6.33. Note that contrary to the case $R = \mathrm{W}(k)[X]/(X^2)$, every ring of the form $\mathrm{W}(k)[X]/(X^2, p^r X)$, $r \in \mathbb{N}_{\geqslant 1}$, belongs to $\mathfrak{U}$, as was shown by Bleher, Chinburg and de Smit in [BCdS]. The above theorem explains where does the main difference lie between these two similar cases.

Observe also how easy it is to arrive at an unsolved case: the author is not aware of any result concerning the problem whether, given $r \in \mathbb{N}_{\geqslant 1}$, the ring $\mathrm{W}(k)[X]/(X^2 - p^r X)$ belongs to $\mathfrak{U}$ or not.

6.6.1 Quotients of universal deformation rings

Definition 6.34. Let us denote by $\mathfrak{Q}$ the subclass of all $\hat{\mathcal{C}}$-rings of the form R/I, where $R \in \mathfrak{U}$ and I is its proper ideal.

Remark 6.35. Observe that Theorem 6.30 holds true also if we replace "in $\mathfrak{U}$", by "in $\mathfrak{Q}$". Indeed, if $S \in \mathfrak{W}$ and $R \in \mathfrak{Q}$ is a quotient of $R' \in \mathfrak{U}$, then finiteness of the set $\mathrm{Hom}_{\hat{\mathcal{C}}}(R', S)$ implies finiteness of the set $\mathrm{Hom}_{\hat{\mathcal{C}}}(R, S)$. The proof of Theorem 6.30 is thus valid also in case $R \in \mathfrak{Q}$.

One could hope that this strengthening would allow to obtain new results about the class $\mathfrak{U}$ itself. That is, a priori, it could happen that Theorem 6.30 does not exclude R from being in $\mathfrak{U}$, but some of its quotients is excluded from being in $\mathfrak{Q}$ by Remark 6.35. However, this is not the case, which can be easily seen using Theorem 6.31 and the following easy observation.

Observation 6.36. *A quotient of a finite étale K-algebra is itself finite étale.*

Thus, the following problem remains open:

Question 6.37. Obviously $\mathfrak{U} \subseteq \mathfrak{Q}$. But is $\mathfrak{Q}$ strictly larger than $\mathfrak{U}$?

6.6.2 Open questions related to extending the main result

Trying to extend the results of Theorem 6.30 one is lead to following questions:

Question 6.38. Let $S \in \mathrm{Ob}(\hat{\mathcal{C}})$ be a one-dimensional, reduced ring in which p is not a zero-divisor (in particular: of characteristic zero).

(*i*) Does $S \in \mathfrak{Q}$ hold?

(*ii*) If $S \in \mathfrak{Q}$, which rings R such that $R/T_{p^\infty}(R) \cong S$ are in $\mathfrak{U}$?

It would be interesting to solve these questions at least in some special cases, for example: for $S = \mathrm{W}(k)$ (only the second question), for integral domains in general (more restrictively: for discrete valuation rings), for rings with one-dimensional tangent space. Thus, as a challenge for an interested reader and starting point for new research we formulate the following concrete problem:

Question 6.39. Which of the following rings are in $\mathfrak{U}$ (are in $\mathfrak{Q}$)?

- $\mathrm{W}(k)[\sqrt[r]{p}\,]$
- $\mathrm{W}(k)[X]/(X^2 - p^r X\,)$
- $\mathrm{W}(k)[[X]]/(p^r X)$

Here r is an integer, $r > 1$ in the first case and $r \geqslant 1$ in the other cases.

The author is only aware of the fact that $\mathbb{Z}_5[\sqrt{5}]$ and $\mathbb{Z}_p[[X]]/(pX)$ for $p = 3$ ($p \geqslant 3$) is in $\mathfrak{U}$ (is in $\mathfrak{Q}$). See also Example 3.12.

6.6.3 Other remarks

It is worth noting that Lemma 6.10, on which we based the argument of this section, can be applied also in a slightly different way. Not only in order to find some rings that are not in $\mathfrak{U}$, but also in order to give a lower bound on the size of a group whose representation can realize R as a universal deformation ring. More precisely:

Lemma 6.40. *Let $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ be given and suppose there exist $S \in \mathrm{Ob}(\hat{\mathcal{C}})$, $r \in \mathbb{N}$ and $f_1, f_2 \in \mathrm{Hom}_{\hat{\mathcal{C}}}(R, S)$ such that $T_{p^\infty}(S) = 0$, $f_1 \neq f_2$, and $f_1 \equiv f_2 \pmod{p^r \mathfrak{m}_S}$. If R is a universal deformation ring of a representation $\bar{\rho}$ of a finite group G, then $p^{r+1} \mid \#G$.*

Proof. Let p^l be the largest power of p dividing $\#G$. By definition of a universal deformation ring, morphisms f_1 and f_2 induce two different deformations of $\bar{\rho}$ to S, so f_1 and f_2 are different modulo $p^l \mathfrak{m}_S$ by Lemma 6.10. Using the assumption we conclude that $l > r$ and the claim follows. ☐

Example 6.41. Let $r \geqslant 1$ be an integer and suppose $R := \mathrm{W}(k)[X]/(X^2 - p^r X)$ is a universal deformation ring of a representation of a finite group G. Since $T_{p^\infty}(R) = 0$ and we have $f_1 : R \xrightarrow{X \mapsto 0} R$, $f_2 : R \xrightarrow{X \mapsto p^r} R$ with the same reduction modulo $p^{r-1}\mathfrak{m}_R$, Lemma 6.40 implies that $p^r \mid \#G$.

6.6.4 Positive characteristic rings in $\mathfrak{U}$

It is clear that the approach of the preceding section can not be directly generalized to rings of positive characteristic, because it relied heavily on properties of rings in which p is not a zero-divisor. Therefore, new techniques must be developed to handle this case.

We only want to observe that also the results in the positive characteristic case will differ from the ones obtained in this chapter. For instance, in contrast to the characteristic zero case we have the following result.

Proposition 6.42. *If $k \neq \mathbb{F}_2, \mathbb{F}_3$ then for every $d \in \mathbb{N}$, the class $\mathfrak{U}$ contains a characteristic p domain of Krull dimension d.*

Proof. The claim follows easily from Example 3.13. □

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Summary

In this thesis we study representations of profinite groups over some particular type of local rings. More specifically, suppose a finite field k is given. Then the rings that are complete, local, noetherian and whose residue field is isomorphic to k form a category, which we denote by $\hat{\mathcal{C}}$. Given a profinite group G and its continuous finite dimensional representation $\bar{\rho}$ over k, we are interested in describing all the possibilities of lifting $\bar{\rho}$ to some object of $\hat{\mathcal{C}}$.

For each problem of the above described type, an associated *deformation functor* from $\hat{\mathcal{C}}$ to the category of sets can be defined. If such a functor is representable (in the sense of category theory) then the object representing it is called *the universal deformation ring* of the given representation. The following *inverse problem* is central in the thesis: which rings do occur as universal deformation rings in the introduced setting?

The main results of the thesis go in two directions. Firstly, we completely answer the stated question in its general form. Secondly, we introduce its modification and begin a systematic study of the analogous problem restricted to representations of finite groups.

Our main contribution consists in providing a complete solution to the inverse problem. We show that in fact every $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ can be obtained as a universal deformation ring. The example we use for this goal is the special linear group $G := \mathrm{SL}_n(R)$, together with the natural representation (induced by the reduction $R \twoheadrightarrow k$) in $\mathrm{GL}_n(k)$. Interestingly, in order to obtain R as a universal deformation ring it is important that $n \geqslant 4$, because the lower dimensional cases admit some puzzling exceptions, requiring a careful analysis (also carried out in the thesis). We also discuss deformations of analogous representations of closed subgroups of $\mathrm{GL}_n(R)$ containing $\mathrm{SL}_n(R)$.

As mentioned, we are moreover interested in determining the rings that can be obtained as universal deformation rings of representations of *finite* groups. The methods outlined above allow us to conclude merely that every finite ring belonging to $\hat{\mathcal{C}}$ can be obtained this way. However, infinite rings having this property exist as well. The second most important contribution of the thesis is thus the following criterion: *Denote by* $\mathrm{W}(k)$ *the ring of Witt vectors over* k*, by* K *its field of fractions and suppose* $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ *can be obtained as a universal deformation ring of some finite group representation. If* R *has characteristic zero, then* $R \otimes_{\mathrm{W}(k)} K$ *is a finite étale* K*-algebra.*

Samenvatting

In dit proefschrift bestuderen wij representaties van pro-eindige groepen over bepaalde lokale ringen. Specifieker, zij k een eindig lichaam. De ringen die compleet, lokaal en noethers zijn, en die een restklassenlichaam isomorf met k hebben, vormen een categorie, die we aanduiden met $\hat{\mathcal{C}}$. Gegeven een pro-eindige groep G en haar continue eindig-dimensionale representatie $\bar{\rho}$ over k, zijn wij geïnteresseerd in het beschrijven van alle mogelijkheden van het liften van $\bar{\rho}$ tot een object van $\hat{\mathcal{C}}$. Voor elk probleem van het hierboven beschreven soort, kan men een bijbehorende *deformatiefunctor* van $\hat{\mathcal{C}}$ naar de categorie van verzamelingen definiëren. Als deze functor representeerbaar is (in de zin van categorietheorie), wordt het representerende object de *universele deformatiering* van de gegeven representatie genoemd. Het volgende *inverse probleem* staat centraal in het proefschrift: welke ringen ontstaan als universele deformatieringen in de geïntroduceerde opzet?

De belangrijkste resultaten van het proefschrift gaan in twee richtingen. Ten eerste geven we een volledige antwoord op de gestelde vraag in zijn algemene vorm. Ten tweede introduceren wij een modificatie en beginnen een systematische studie van het analoge probleem, beperkt tot representaties van eindige groepen.

Onze belangrijkste bijdrage is een complete oplossing voor het inverse probleem. We laten zien dat elke $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ kan worden verkregen als een universele deformatiering. Het voorbeeld dat wij voor dit doel gebruiken is de speciale lineaire groep $G := \mathrm{SL}_n(R)$, tezamen met de natuurlijke representatie (geïnduceerd door de reductie $R \twoheadrightarrow k$) in $\mathrm{GL}_n(k)$. Interessant genoeg, om R als de universele deformatiering te krijgen, is het belangrijk dat $n \geqslant 4$; de lagerdimensionale gevallen laten uitzonderingen toe (die in dit proefschrift ook worden bestudeerd). We bespreken bovendien deformaties van analoge representaties van gesloten ondergroepen van $\mathrm{GL}_n(R)$, die $\mathrm{SL}_n(R)$ bevatten.

Zoals gezegd, zijn wij bovendien geïnteresseerd in het bepalen van de ringen die kunnen worden verkregen als universele deformatieringen van representaties van *eindige* groepen. De hierboven beschreven methoden laten ons slechts concluderen dat elke eindige ring die tot $\hat{\mathcal{C}}$ behoort, kan worden verkregen op deze manier. Echter, oneindige ringen met deze eigenschap bestaan ook. De tweede belangrijkste bijdrage van het proefschrift is het volgende criterium: *Zij* $\mathrm{W}(k)$ *de ring van Witt vectoren over* k*,* K *zijn quotiëntenlichaam en neem aan dat* $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ *als een universele deformatiering van een eindige groepsrepresentatie verkregen kan worden. Als* R *karakteristiek nul heeft, is* $R \otimes_{\mathrm{W}(k)} K$ *een eindige étale* K*-algebra.*

Sommario

In questa tesi studiamo le rappresentazioni di gruppi profiniti su un particolare tipo di anelli locali. Per essere più precisi, supponiamo sia dato un campo finito k. Gli anelli che sono completi, locali, noetheriani e con campo residuo isomorfo a k costituiscono una categoria, che indichiamo con $\hat{\mathcal{C}}$. Dato un gruppo profinito G e una rappresentazione continua e finito dimensionale $\overline{\rho}$ a valori in k, siamo interessati a descrivere tutti i possibili sollevamenti di $\overline{\rho}$ ad una rappresentazione su un oggetto di $\hat{\mathcal{C}}$.

Per ciascun problema del tipo sopra descritto può essere definito un *funtore di deformazione* associato che va da $\hat{\mathcal{C}}$ alla categoria degli insiemi. Se tale funtore è rappresentabile (nel senso della teoria di categorie) allora l'oggetto rappresentante è chiamato *l'anello universale di deformazione* della rappresentazione data. Il seguente *problema inverso* è l'oggetto di studio centrale nella tesi: quali anelli si realizzano come anelli universali di deformazione?

I principali risultati della tesi vanno in due direzioni. In primo luogo, abbiamo dato una risposta completa alla domanda sopra indicata nella sua forma generale. Secondariamente, introduciamo una variante per gruppi finiti e iniziamo uno studio sistematico del problema analogo limitato a tale caso.

Il nostro contributo principale consiste nel fornire una soluzione completa per il problema inverso. Abbiamo dimostrato che ogni $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ può essere ottenuto come un anello universale di deformazione. L'esempio che usiamo per questo obiettivo è il gruppo lineare speciale $G := \mathrm{SL}_n(R)$, insieme alla rappresentazione naturale (indotta dalla riduzione $R \twoheadrightarrow k$) in $\mathrm{GL}_n(k)$. Per ottenere R come l'anello universale di deformazione è importante che $n \geqslant 4$, in quanto i casi di dimensione inferiore ammettono eccezioni (di cui effettuiamo un'attenta analisi). Discutiamo anche le de-

formazioni delle analoghe rappresentazioni di sottogruppi chiusi di $\mathrm{GL}_n(R)$ che contengono $\mathrm{SL}_n(R)$.

Come detto, siamo inoltre interessati a determinare gli anelli che possono essere ottenuti come anelli universali di deformazione per rappresentazioni di gruppi *finiti*. I metodi sopra descritti ci permettono di concludere solo che ogni anello finito appartenente a $\hat{\mathcal{C}}$ può essere ottenuto in questo modo. Tuttavia, esistono anelli infiniti con la stessa proprietà. Il secondo contributo importante della tesi è quindi il seguente criterio: *Indichiamo con* $\mathrm{W}(k)$ *l'anello dei vettori di Witt su* k*, con* K *il suo campo di frazioni e supponiamo che* $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ *possa essere ottenuto come l'anello universale di deformazione di una rappresentazione di gruppo finito. Se* R *ha la caratteristica zero, allora* $R \otimes_{\mathrm{W}(k)} K$ *è una* K*-algebra finita étale.*

Acknowledgments

I think I was very lucky with my research topic. Working on it was an opportunity to deepen my knowledge of several branches of mathematics, there were many potential research directions to explore and it was possible to obtain a significant progress in a reasonable time. All these factors make me think it was a perfect topic for a PhD student and I would like to thank a lot Bart de Smit for proposing it to me. I also thank him and Fabrizio Andreatta for all their comments and advices that helped me choose right directions to follow as well as formulate and understand better my own ideas.

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I am greatly indebted to the mentioned people for their contributions to Chapter 6. The main question studied there comes from Hendrik and some of his deeper ideas are reflected in a simplified manner in my "cardinality argument". Jakub had a very nice idea which I have used to deal with one of the cases in Section 6.5, whereas Fabrizio considerably simplified my proofs in the other case.

At some point a preprint containing results partially overlapping with those of Chapter 5 has been published by some other authors. Overlooking this fact could have been very painful, so I am very grateful to Jakub, who informed me about the mentioned work. I also appreciate a lot the help I have received from Bas Edixhoven when dealing with this situation.

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Curriculum Vitae

Krzysztof Dorobisz was born on 1$^{\text{st}}$ January 1988 in Kraków, Poland, where he also attended the August Witkowski 5$^{\text{th}}$ High School. During this period he participated in several mathematical competitions, including the International Mathematical Olympiad, where he won a bronze medal in 2006. In the same year he started a 5-year Master's degree programme in mathematics at the Jagiellonian University in his hometown.

During the last year of his studies Krzysztof participated in a special double degree programme (Short Track Master's) offered jointly by the Jagiellonian University and Vrije Universiteit in Amsterdam; in the summer 2011 he successfully graduated from both universities.

Thanks to the stay in Amsterdam Krzysztof came into contact with Leiden University and learned about the Algant-DOC program. Having obtained a Ph.D. fellowship, he started his work under the supervision of dr. Bart de Smit and dr. Fabrizio Andreatta in October 2011. This thesis summarizes the research done during the past three years.