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Introduction

Black holes, regions of spacetime which appear black since not even light can escape, are perhaps the most striking prediction of Einstein's General Relativity. Since their discovery, black holes have inspired physicists to raise many deep questions about the fundamental laws of nature.

It is by now widely believed that black holes, or at least objects which can be approximately described as classical black holes, actually exist in our universe, at the center of galaxies or resulting from stellar collapse, and are thus not merely theoretical constructs.

The study of black holes is still an extremely active topic of research, and is central to the ongoing quest to find a quantum theory of gravity. For one thing, black holes typically encompass spacetime singularities, which signal the breakdown of the classical theory. It is believed that such singularities will be resolved in a full theory of quantum gravity.

Other clues toward the construction of a microscopic theory of gravity come from black hole thermodynamics. Classical black holes obey the four laws of black hole mechanics [1], which are formally analogous to the laws of thermodynamics. If one wishes to take the analogy seriously, then thermodynamic quantities should be assigned to a black hole, in particular a temperature T and an entropy S_{BH} [2] proportional to the area of the horizon. This thermodynamic picture was corroborated by the famous calculation by Hawking [3], who showed that in the semiclassical regime a black hole emits thermal radiation corresponding to the temperature T . The statistical interpretation of thermodynamics then leads to a drastic change in the understanding of what a black hole is. While in general relativity a black hole is a simple object, completely characterized by a small number of parameters, the fact that it possesses an entropy suggests the existence of a microscopic description in terms of a large number of degenerate states, all with the same values of the macroscopic parameters. Any viable theory of quantum gravity should then be able to reproduce, from the counting of microstates, the macroscopically determined entropy.

One of the leading candidates for a theory of quantum gravity is string theory, or more precisely its supersymmetric incarnation, superstring theory. According to superstring theory all the elementary particles and forces of nature, including gravity, can be described as vibrations of one-dimensional objects called strings, propagating in a

10-dimensional spacetime. The low energy limit of superstring theory is a field theory including gravity and enjoying a local symmetry relating bosonic and fermionic fields to each other, i.e. a local supersymmetry, which is called supergravity.

To recover the universe we experience, which is 4-dimensional and possesses no unbroken supersymmetry, from these 10-dimensional supersymmetric theories, it is generally believed that six of the ten dimensions are small and compact, giving rise to an effective 4-dimensional theory, and that our universe is described by a supersymmetry-breaking vacuum of this effective theory.

While supersymmetric vacua cannot describe our universe, they have played and continue to play a key role in the development of string theory. In particular, the first successful computation of black hole entropy from microstate counting by Strominger and Vafa [4] made use of the unbroken supersymmetry of the considered system. Supersymmetry allows, owing to non-renormalization theorems, to extrapolate the result at weak string coupling, where the system can be described in terms of strings and branes, to strong coupling, where a description in terms of a black hole is valid. Following this work similar tests have been performed successfully for a large class of extremal and near-extremal black holes, strengthening string theory's position as a theory of quantum gravity.

Black hole spacetimes, both with and without unbroken supersymmetries, which asymptote to anti-de Sitter space are also interesting in the context of the conjectured AdS/CFT correspondence. The correspondence, originally postulated by Maldacena in 1997 [5], asserts that string theory in anti-de Sitter space in d dimensions is equivalent to a conformal quantum field theory without gravity living on the $d-1$ conformal boundary of AdS. In a weaker form, the low energy limit of string theory, supergravity, is dual to a strongly coupled CFT. Asymptotically AdS black hole solutions can be used in this framework to approximate systems of physical interest described by field theories in the strongly coupled regime, such as the quark-gluon plasma and various condensed matter systems.

To find supersymmetric solutions of a supergravity theory one has to solve first order differential equations called Killing spinor equations. This is typically easier than trying to solve directly the equations of motion, which include the second order Einstein equations. In this sense, supersymmetry can be regarded as a solution generating technique. It turns out that it is possible to extend this method to non-supersymmetric theories. This means that, at least for certain theories, it is possible to find a set of first order equations, which in this case are not related to an underlying supersymmetry of the theory, whose solutions are also solutions of all the equations of motion. This approach is what is known as fake supergravity [6]. Fake supersymmetric solutions can exhibit properties that are generally absent from genuine supersymmetric solutions, such as a positive cosmological constant and explicit time-dependence.

Many open problems in black hole physics, for instance whether the cosmic censor-

ship conjecture really holds, or what happens when black holes collide, or again how accretion influences the growth and thermodynamics of black holes, are dynamical in nature. To study such problems it would therefore be desirable to have dynamical black hole solutions at our disposal. Unfortunately, not many such solutions are available in the literature. Fake supergravity provides a way to obtain new dynamical black hole solutions.

The goal of this thesis is to obtain novel black hole solutions, both supersymmetric and non-supersymmetric, with potential application to the issues outlined above. The thesis is organized as follows. In chapter 1 we briefly introduce the extended $N = 2$ supergravity in four dimensions, writing the bosonic action, both for the ungauged and the gauged theory, the equations of motion and the supersymmetry variations for vanishing fermionic fields. In chapter 2 we review the classification of the timelike supersymmetric solutions of $N = 2, d = 4$ gauged supergravity coupled to matter supermultiplets originally published in [7]. We also present a simple supersymmetric black hole solution we found using this classification, which is, to the best of our knowledge, the first supersymmetric solution to gauged supergravity with nontrivial hyperscalars. Chapter 3 reviews the classification [8] of the fake supersymmetric solutions to fake $N = 2, d = 4$ gauged supergravity coupled to vector multiplets, a theory obtained from the corresponding genuine supergravity by analytic continuation. Making use of these results, in chapter 4 we obtain some fake supersymmetric solutions, representing multi-centered black holes in a cosmological Friedmann-Lemaître-Robertson-Walker background, with and without rotation, and with flat or curved spatial sections. We also study in some detail the physical properties of the non-rotating single-centered solution. In chapter 5 we present a different multi-centered solution in a FLRW background, which is not obtained from either genuine or fake supersymmetry, but rather as a generalization of the previously known charged McVittie spacetime [9, 10]. As a particular subcase, this solution describes multiple black holes in a background that is locally anti-de Sitter. We also discuss some physical properties of the single-centered asymptotically AdS case and generalize the solution to arbitrary dimension. Appendix A contains the conventions we use throughout the thesis, while appendix B is a review of various geometric structures used in the main text, in particular to define $N = 2, d = 4$ supergravity. Finally, in appendix C we give a succinct account of the formalism for dynamical black holes proposed by Hayward [11, 12, 13, 14], giving in particular a generalized definition of black holes, and generalized laws of black hole dynamics.

$N = 2, d = 4$ gauged supergravity

A field theory with local supersymmetry is necessarily invariant also under local space-time translations, i.e. under spacetime diffeomorphisms. This means that it must include gravity, which is why these theories are called supergravities. The first supergravity action was constructed in 1976 by Freedman, Ferrara and van Nieuwenhuizen [15], and more general theories were discovered afterwards, in higher dimensions, coupled with matter supermultiplets, and with more than one supersymmetry generator.

Supergravity was initially considered as a good candidate for a quantum theory of gravity, since, given the good high-energy behaviour at low perturbative order, there was hope that it could be ultraviolet finite. This however turned out not to be the case, and supergravity is by now considered to be just the low energy effective limit of a more fundamental theory, namely superstring theory or M-theory.

Supergravities symmetric under N supersymmetry generators, with $N > 1$, are known as *extended* supergravities. It is possible to use some or all of the vector fields of a supergravity to gauge a Yang-Mills subgroup of the internal symmetries of the theory; in this case the theory is called a *gauged* supergravity.

In what follows we will focus on extended $N = 2$ gauged supergravity in four dimensions. This class of theories is interesting on its own, e.g. for studying black hole solutions, since it has enough symmetry to be manageable while still allowing for interesting scalar manifold geometries and matter couplings. Moreover it has applications in string theory, as it emerges naturally from compactifications of 10-dimensional superstring theory and 11-dimensional M-theory.

In this chapter we briefly review $N = 2, d = 4$ gauged supergravity coupled to matter. In section 1.1 we introduce the field content and the ungauged action for the bosonic sector of the theory, in section 1.2 we discuss the internal symmetries and the gauging, and in section 1.3 we give the expressions for the bosonic equations of motion and the supersymmetry variations for vanishing fermions.

The discussion here is far from being exhaustive. For a more complete treatment we refer e.g. to [16] or [17].

1.1 $N = 2, d = 4$ supergravity

The field content of the theory includes of course the gravity multiplet, which consists of the graviton e_μ^a , two *gravitinos* $\psi_{I\mu}$ ($I = 1, 2$) and a vector field A_μ^0 , called *graviphoton*. The gravity multiplet can in general be coupled to matter multiplets, specifically to a number n_V of *vector multiplets* and a number n_H of *hypermultiplets*.

Each of the n_V vector multiplets, labeled by an index $i = 1, \dots, n_V$, is composed of one complex scalar Z^i , two *gaugini* λ^{Ii} , and one vector field A_μ^i , while each hypermultiplet contains four real scalars (the *hyperscalars*) and two *hyperini*. The hyperscalars and hyperini of all the hypermultiplets in the theory are collectively denoted as respectively q^u ($u = 1, \dots, 4n_H$) and ζ_α ($\alpha = 1, \dots, 2n_H$). The $\bar{n} \equiv n_V + 1$ vector fields of the theory, A_μ^0, A_μ^i , are also referred to as the array A_μ^Λ ($\Lambda = 0, \dots, n_V$).

	Gravity			Vector			Hyper	
Fields	e_μ^a	$\psi_{I\mu}$	A_μ^0	A_μ^i	λ^{Ii}	Z^i	ζ_α	q^u
Spin	2	3/2	1	1	1/2	0	1/2	0

Table 1.1: Field content of $N = 2, d = 4$ supergravity.

We describe every fermionic field of the theory, meaning the gravitinos, gauginos and hyperini, as Weyl spinors.

In the ungauged theory, the self-coupling of both the complex scalars Z^i and the hyperscalars q^u is described by non-linear sigma models. The complex scalars parametrize a target manifold \mathcal{M}_V of complex dimension n_V which turns out to be a special Kähler manifold (see Appendix B.4). The special Kähler geometry, through the period matrix $\mathcal{N}_{\Lambda\Sigma}$ defined by the relations (B.29), determines also the couplings of the n_V scalars Z^i with the \bar{n} vector fields A^Λ . The hyperscalars, on the other hand, parametrize a quaternionic Kähler manifold \mathcal{M}_H (described in Appendix B.6) of real dimension $4n_H$. In the ungauged theory, besides the self-coupling determined by the sigma model, the hyperscalars are only coupled minimally to gravity and do not couple directly to the vector multiplet fields.

In this thesis we are interested in bosonic solutions, that is, solutions on which all the fermionic fields vanish. We will therefore concentrate on the bosonic sector of the theory. The bosonic sector of the action for the ungauged $N = 2, d = 4$ supergravity is

$$\begin{aligned}
 S = \int d^4x \sqrt{|g|} \big[& R + 2 \mathcal{G}_{i\bar{j}} \partial_\mu Z^i \partial^\mu \bar{Z}^{\bar{j}} + 2 H_{uv} \partial_\mu q^u \partial^\mu q^v \\
 & + 2 I_{\Lambda\Sigma} F^{\Lambda\mu\nu} F^\Sigma_{\mu\nu} - 2 R_{\Lambda\Sigma} F^{\Lambda\mu\nu} \star F^\Sigma_{\mu\nu} \big] , \quad (1.1)
 \end{aligned}$$

where $\mathcal{G}_{i\bar{j}}(Z, \bar{Z})$ and $H_{uv}(q)$ are the metrics respectively on \mathcal{M}_V and \mathcal{M}_H , the field

strengths are defined as usual, $F^\Lambda = dA^\Lambda$, $\star F$ is the Hodge dual of F , and we have introduced the following abbreviations for the real and imaginary parts of the period matrix:

$$I_{\Lambda\Sigma} \equiv \Im(\mathcal{N}_{\Lambda\Sigma}), \quad R_{\Lambda\Sigma} \equiv \Re(\mathcal{N}_{\Lambda\Sigma}). \quad (1.2)$$

1.2 Symmetries and gauging

From the expression (1.1) it is clear that all the isometries of the quaternionic Kähler metric H_{uv} are symmetries of the action. To preserve the supersymmetry, the isometries must also preserve the quaternionic Kähler structure, which means that they must be generated by quaternionic Killing vectors (see Appendix B.7).

The isometries of the Kähler metric $\mathcal{G}_{i\bar{j}}$, on the other hand, should preserve the special Kähler structure, meaning that they should not only be generated by holomorphic Killing vectors, but also that they must be embedded in the symplectic group $Sp(\bar{n}, \mathbb{R})$ as explained in Appendix B.5. This however is still not sufficient to guarantee that these isometries are symmetries of the action (1.1). The isometries of $\mathcal{G}_{i\bar{j}}$ which are symmetries of the action are only those which can be embedded in the subgroup of $Sp(\bar{n}, \mathbb{R})$ generated by $2\bar{n} \times 2\bar{n}$ matrices

$$\mathcal{S} = \begin{pmatrix} a & b \\ c & -a^T \end{pmatrix} \quad (1.3)$$

with $b = c = 0^1$. If the theory admits a prepotential $\mathcal{F}(\chi)$ these are exactly the symplectic transformations leaving $\mathcal{F} = \frac{1}{2}\chi^\Lambda \mathcal{F}_\Lambda$ invariant.

Since the theory contains $\bar{n} = n_V + 1$ vector fields, we can choose to gauge a symmetry subgroup of the product scalar manifold $\mathcal{M}_V \otimes \mathcal{M}_H$ with dimension up to \bar{n} . This is generated by holomorphic Killing vectors $k_\Lambda(Z) = k_\Lambda{}^i(Z)\partial_i + k_\Lambda{}^{\bar{i}}(\bar{Z})\partial_{\bar{i}}$ and quaternionic Killing vectors $k_\Lambda(q) = k_\Lambda{}^u(q)\partial_u$ satisfying the same Lie algebra

$$[k_\Lambda, k_\Sigma] = -f_{\Lambda\Sigma}{}^\Gamma k_\Gamma, \quad [k_\Lambda, k_\Sigma] = -f_{\Lambda\Sigma}{}^\Gamma k_\Gamma. \quad (1.4)$$

The matrices \mathcal{S}_Λ associated with k_Λ should also provide a representation of the same Lie algebra, $[\mathcal{S}_\Lambda, \mathcal{S}_\Sigma] = -f_{\Lambda\Sigma}{}^\Gamma \mathcal{S}_\Gamma$, which means

$$\mathcal{S}_\Lambda = \begin{pmatrix} a_\Lambda & 0 \\ 0 & -(a_\Lambda)^T \end{pmatrix} \quad \text{with} \quad (a_\Lambda)^\Gamma{}_\Sigma = f_{\Lambda\Sigma}{}^\Gamma. \quad (1.5)$$

The constraint (B.52) can then be written, for the section \mathcal{V} , as

$$f_{\Lambda\Sigma}{}^\Omega \mathcal{L}^\Sigma \mathcal{M}_\Omega = 0, \quad (1.6)$$

¹It would be possible to modify the action by adding a Chern-Simons term in such a way that transformations with $c \neq 0$ would also be symmetries of the action. We will not however consider this possibility.

and using equation (B.51) it implies the conditions

$$\mathcal{L}^\Lambda \mathcal{P}_\Lambda = 0, \quad \mathcal{L}^\Lambda k_\Lambda{}^i = 0, \quad \bar{\mathcal{L}}^\Lambda k_\Lambda{}^i = -i \bar{f}^\Lambda{}^i \mathcal{P}_\Lambda. \quad (1.7)$$

After the gauging procedure, the action (1.1) is modified to

$$S = \int d^4x \sqrt{|g|} \left[R + 2 \mathcal{G}_{i\bar{j}} \mathfrak{D}_\mu Z^i \mathfrak{D}^\mu \bar{Z}^{\bar{j}} + 2 H_{uv} \mathfrak{D}_\mu q^u \mathfrak{D}^\mu q^v \right. \\ \left. + 2 I_{\Lambda\Sigma} F^{\Lambda\mu\nu} F^\Sigma{}_{\mu\nu} - 2 R_{\Lambda\Sigma} F^{\Lambda\mu\nu} \star F^\Sigma{}_{\mu\nu} - V(Z, \bar{Z}, q) \right], \quad (1.8)$$

where now the field strengths are

$$F^\Lambda = dA^\Lambda + \frac{1}{2} g f_{\Sigma\Gamma}{}^\Lambda A^\Sigma \wedge A^\Gamma, \quad (1.9)$$

where g is the gauge coupling constant, and the covariant derivatives acting on the scalars are

$$\mathfrak{D}_\mu Z^i = \partial_\mu Z^i + g A_\mu^\Lambda k_\Lambda{}^i(Z), \quad \mathfrak{D}_\mu q^u = \partial_\mu q^u + g A_\mu^\Lambda k_\Lambda{}^u(q). \quad (1.10)$$

The gauged theory includes a scalar potential,

$$V(Z, \bar{Z}, q) =$$

$$\frac{g^2}{2} \left[\bar{\mathcal{L}}^\Lambda \mathcal{L}^\Sigma (4 H_{uv} k_\Lambda{}^u k_\Sigma{}^v - 3 P_\Lambda{}^x P_\Sigma{}^x) + \mathcal{G}^{i\bar{j}} f^\Lambda{}_i \bar{f}^\Sigma{}_{\bar{j}} P_\Lambda{}^x P_\Sigma{}^x - \frac{1}{2} I^{\Lambda\Sigma} \mathcal{P}_\Lambda \mathcal{P}_\Sigma \right], \quad (1.11)$$

where $I^{\Lambda\Sigma} I_{\Sigma\Gamma} \equiv \delta^\Lambda{}_\Gamma$ and the other quantities, including the moment maps $\mathcal{P}_\Lambda(Z, \bar{Z})$ and $P_\Lambda{}^x(q)$, are defined in Appendix B. Using the identity

$$\mathcal{G}^{i\bar{j}} f^\Lambda{}_i \bar{f}^\Sigma{}_{\bar{j}} = -\frac{1}{2} I^{\Lambda\Sigma} - \bar{\mathcal{L}}^\Lambda \mathcal{L}^\Sigma \quad (1.12)$$

the potential can be rewritten in the form

$$V(Z, \bar{Z}, q) = g^2 \left[2 \bar{\mathcal{L}}^\Lambda \mathcal{L}^\Sigma (H_{uv} k_\Lambda{}^u k_\Sigma{}^v - P_\Lambda{}^x P_\Sigma{}^x) - \frac{1}{4} I^{\Lambda\Sigma} (\mathcal{P}_\Lambda \mathcal{P}_\Sigma + P_\Lambda{}^x P_\Sigma{}^x) \right]. \quad (1.13)$$

The gauging procedure introduces new couplings in the theory. In particular the hyperscalars are now directly coupled to the gauge fields and to the vector multiplets complex scalars through the covariant derivatives and the scalar potential.

We will later restrict to abelian gaugings. If the gauge subgroup is abelian, the structure constants $f_{\Lambda\Sigma}{}^\Gamma$ are zero. Therefore the matrices \mathcal{S}_Λ in (1.5) vanish, and so do also the moment maps \mathcal{P}_Λ (B.51), and the holomorphic Killing vectors k_Λ (B.41). The covariant derivatives acting on the complex scalars Z^i reduce then to ordinary derivatives. This is a consequence of the fact that the sections \mathcal{L}^Λ (or equivalently χ^Λ) transform in the adjoint representation of the gauge group, which is trivial in the abelian case.

It should also be noted that even in the absence of hypermultiplets, $n_H = 0$, the triholomorphic moment maps $P_\Lambda{}^x$ can be non-vanishing. In this case they are constants known as Fayet-Iliopoulos terms, and correspond to the gauging of a subgroup of the $SU(2) \times U(1)$ R -symmetry group.

1.3 Equations of motion and supersymmetry variations

Following [7] we write the bosonic equations of motion obtained from the action (1.8) as

$$\begin{aligned}\mathcal{E}_a{}^\mu &\equiv -\frac{1}{2\sqrt{|g|}} \frac{\delta S}{\delta e^a{}_\mu} = 0, & \mathcal{E}_i &\equiv -\frac{1}{2\sqrt{|g|}} \frac{\delta S}{\delta Z^i} = 0, \\ \mathcal{E}_\Lambda{}^\mu &\equiv \frac{1}{8\sqrt{|g|}} \frac{\delta S}{\delta A^\Lambda{}_\mu} = 0, & \mathcal{E}^u &\equiv -\frac{1}{4\sqrt{|g|}} H^{uv} \frac{\delta S}{\delta q^v} = 0,\end{aligned}\tag{1.14}$$

and the Bianchi identities for the vector field strengths as

$$\mathcal{B}^{\Lambda\mu} \equiv \mathfrak{D}_\nu \star F^{\Lambda\nu\mu},\tag{1.15}$$

with

$$\begin{aligned}\mathcal{E}_{\mu\nu} &= G_{\mu\nu} + 8I_{\Lambda\Sigma} F^{\Lambda+}{}_\mu{}^\rho F^{\Sigma-}{}_{\nu\rho} + 2\mathcal{G}_{i\bar{j}} [\mathfrak{D}_\mu Z^i \mathfrak{D}_\nu \bar{Z}^{\bar{j}} - \frac{1}{2} g_{\mu\nu} \mathfrak{D}_\rho Z^i \mathfrak{D}^\rho \bar{Z}^{\bar{j}}] \\ &\quad + 2H_{uv} [\mathfrak{D}_\mu q^u \mathfrak{D}_\nu q^v - \frac{1}{2} g_{\mu\nu} \mathfrak{D}_\rho q^u \mathfrak{D}^\rho q^v] + \frac{1}{2} g_{\mu\nu} V(Z, \bar{Z}, q),\end{aligned}\tag{1.16}$$

$$\mathcal{E}_\Lambda{}^\mu = \mathfrak{D}_\nu \star F_\Lambda{}^{\nu\mu} + \frac{1}{4} g(k_{\Lambda\bar{i}} \mathfrak{D}^\mu \bar{Z}^{\bar{i}} + \bar{k}_{\Lambda i} \mathfrak{D}^\mu Z^i) + \frac{1}{2} g k_{\Lambda u} \mathfrak{D}^\mu q^u,\tag{1.17}$$

$$\mathcal{E}^i = \mathfrak{D}^2 Z^i + \partial^i F_\Lambda{}^{\mu\nu} \star F^\Lambda{}_{\mu\nu} + \frac{1}{2} \partial^i V(Z, \bar{Z}, q),\tag{1.18}$$

$$\mathcal{E}^u = \mathfrak{D}^2 q^u + \frac{1}{4} \partial^u V(Z, \bar{Z}, q),\tag{1.19}$$

where the dual field strengths F_Λ are defined by

$$F_{\Lambda\mu\nu} \equiv -\frac{1}{4\sqrt{|g|}} \frac{\delta S}{\delta \star F^\Lambda{}_{\mu\nu}} = 2\Re(\bar{\mathcal{N}}_{\Lambda\Sigma} F^{\Sigma+}{}_{\mu\nu}) = R_{\Lambda\Sigma} F^\Sigma{}_{\mu\nu} + I_{\Lambda\Sigma} \star F^\Sigma{}_{\mu\nu}.\tag{1.20}$$

The supersymmetry transformation rules of the bosons are the same as in the un-

gauged case,

$$\delta_\epsilon e^a{}_\mu = -\frac{i}{4}\bar{\psi}_{I\mu}\gamma^a\epsilon^I + \text{c.c.}, \quad (1.21)$$

$$\delta_\epsilon A^\Lambda{}_\mu = \frac{1}{4}\bar{\mathcal{L}}^\Lambda\varepsilon^{IJ}\bar{\psi}_{I\mu}\epsilon_J + \frac{i}{8}f^\Lambda{}_{i\varepsilon IJ}\bar{\lambda}^{Ii}\gamma_\mu\epsilon^J + \text{c.c.}, \quad (1.22)$$

$$\delta_\epsilon Z^i = \frac{1}{4}\bar{\lambda}^{Ii}\epsilon_I, \quad (1.23)$$

$$\delta_\epsilon q^u = \frac{1}{4}\mathbf{U}_{\alpha I}{}^u\bar{\zeta}^\alpha\epsilon^I + \text{c.c.}, \quad (1.24)$$

while for vanishing fermionic fields, the rules for the fermions are

$$\delta_\epsilon\psi_{I\mu} = \mathfrak{D}_\mu\epsilon_I + \left[T^+{}_{\mu\nu}\varepsilon_{IJ} - \frac{1}{2}S^x\eta_{\mu\nu}\varepsilon_{IK}(\sigma^x)^K{}_J\right]\gamma^\nu\epsilon^J, \quad (1.25)$$

$$\delta_\epsilon\lambda^{Ii} = i\mathfrak{D}Z^i\epsilon^I + \left[\left(\mathcal{G}^{i+} + W^i\right)\varepsilon^{IJ} + \frac{i}{2}W^{ix}(\sigma^x)^I{}_K\varepsilon^{KJ}\right]\epsilon_J, \quad (1.26)$$

$$\delta_\epsilon\zeta_\alpha = i\mathbf{U}_{\alpha I}{}^u\mathfrak{D}q^u\epsilon^I + N_\alpha{}^I\epsilon_I, \quad (1.27)$$

where the covariant derivative on spinors is given by

$$\mathfrak{D}_\mu\epsilon_I = \left[\nabla_\mu + \frac{i}{2}(\mathcal{Q}_\mu + gA^\Lambda{}_\mu\mathcal{P}_\Lambda)\right]\epsilon_I + \frac{i}{2}(A^x{}_u\partial_\mu q^u + gA^\Lambda{}_\mu\mathbf{P}_\Lambda{}^x)\sigma^x{}_I{}^J\epsilon_J, \quad (1.28)$$

and \mathcal{Q}_μ is the pullback to spacetime of the Kähler connection

$$\mathcal{Q} = -\frac{i}{2}(\partial_i\mathcal{K}dZ^i - \partial_{\bar{i}}\mathcal{K}d\bar{Z}^{\bar{i}}). \quad (1.29)$$

The quantities S^x , W^i , W^{ix} and $N_\alpha{}^I$ appearing in these rules are called *fermion shifts* and are defined as

$$S^x = \frac{1}{2}g\mathcal{L}^\Lambda\mathbf{P}_\Lambda{}^x, \quad (1.30)$$

$$W^i = \frac{1}{2}g\bar{\mathcal{L}}^\Lambda k_\Lambda{}^i = -\frac{i}{2}g\mathcal{G}^{i\bar{j}}\bar{f}^\Lambda{}_{\bar{j}}\mathcal{P}_\Lambda, \quad (1.31)$$

$$W^{ix} = g\mathcal{G}^{i\bar{j}}\bar{f}^\Lambda{}_{\bar{j}}\mathbf{P}_\Lambda{}^x, \quad (1.32)$$

$$N_\alpha{}^I = g\mathbf{U}_{\alpha I}{}^u\bar{\mathcal{L}}^\Lambda k_\Lambda{}^u, \quad (1.33)$$

while $T_{\mu\nu}$ and $G^i{}_{\mu\nu}$ are respectively the graviphoton and matter vector field strengths,

defined by

$$T_{\mu\nu} \equiv 2i\mathcal{L}^\Sigma I_{\Sigma\Lambda} F^\Lambda_{\mu\nu}, \quad (1.34)$$

$$G^i_{\mu\nu} \equiv -\mathcal{G}^{i\bar{j}} \bar{f}^\Sigma_{\bar{j}} I_{\Sigma\Lambda} F^\Lambda_{\mu\nu}, \quad (1.35)$$

or equivalently, combining the vector field strengths F^Λ and their duals F_Λ into a symplectic vector $\mathcal{F} = (F^\Lambda, F_\Lambda)^T$, by

$$T^+ = \langle \mathcal{V} | \mathcal{F} \rangle \quad \text{and} \quad G^{i+} = \frac{i}{2} \mathcal{G}^{i\bar{j}} \langle \mathcal{D}_{\bar{j}} \bar{\mathcal{V}} | \mathcal{F} \rangle. \quad (1.36)$$

Supersymmetric solutions

The first efforts towards a systematic characterization of supersymmetric solutions to supergravity theories dates back to 1982, when Gibbons and Hull [18] obtained a partial classification for pure $N = 2, d = 4$ ungauged supergravity. Since then many more results have been obtained, and in recent years powerful techniques have been developed [19, 20, 21], allowing further advances in the field.

Restricting our attention to $N = 2, d = 4$ supergravity, the most complete classification to date is the one given by Meessen and Ortín in [7], where they characterize all the bosonic timelike supersymmetric solutions to the theory coupled both to vector multiplets and to hypermultiplets, with non-abelian gauging of the isometries of the scalar manifold, using the bilinear method of [20]. Previous, less complete classifications include those in [22, 23, 24, 25, 26, 27].

In this chapter we review the results of [7], and subsequently use them to obtain a new supersymmetric black hole solution, which we will present in a paper currently under preparation [28]. This is, to the best of our knowledge, the first analytic¹ supersymmetric black hole solution to gauged supergravity having nontrivial hyperscalars, with the exception of the solutions that can be obtained with the method outlined in [30], in which however the hyperscalars are required to be covariantly constant.

The chapter is organized as follows. In section 2.1 we first introduce the Killing spinor identities for a generic supergravity theory following the treatment in [31]. Then we apply the formalism to $N = 2, d = 4$ supergravity, obtaining the minimal set of equations of motion that must be imposed on a supersymmetric configuration to ensure that all the equations of motion are satisfied. In section 2.2, starting from the Killing spinor equations, we obtain the equations characterizing supersymmetric field configurations. In section 2.3 we impose the residual equations of motion, and we summarize the form of the fields of a supersymmetric solution and the equations they have to satisfy. Finally in section 2.4 we apply these results to a simple theory with one vector multiplet and one hypermultiplet, and obtain our black hole solution.

¹Numerical supersymmetric black hole solutions in AdS_4 with nontrivial hypermultiplets were obtained in [29]

2.1 Killing Spinor Identities

Consider a generic supergravity theory with action S . The local supersymmetry of the theory means that there exists some supersymmetric transformation, with parameter $\epsilon(x)$, acting on all the bosonic fields ϕ^b and fermionic fields ϕ^f of the theory such that

$$\delta_\epsilon S \equiv \sum_b \frac{\delta S}{\delta \phi^b} \delta_\epsilon \phi^b + \sum_f \frac{\delta S}{\delta \phi^f} \delta_\epsilon \phi^f = 0. \quad (2.1)$$

If we now vary this identity over the fermions, and subsequently set the fermionic fields to zero, we obtain

$$\begin{aligned} \left. \frac{\delta(\delta_\epsilon S)}{\delta \phi^{f2}} \right|_{\phi^f=0} &= \\ \left[\sum_b \left(\frac{\delta S}{\delta \phi^b} \frac{\delta(\delta_\epsilon \phi^b)}{\delta \phi^{f2}} + \frac{\delta^2 S}{\delta \phi^{f2} \delta \phi^b} \delta_\epsilon \phi^b \right) + \sum_{f_1} \left(\frac{\delta S}{\delta \phi^{f_1}} \frac{\delta(\delta_\epsilon \phi^{f_1})}{\delta \phi^{f2}} + \frac{\delta^2 S}{\delta \phi^{f2} \delta \phi^{f_1}} \delta_\epsilon \phi^{f_1} \right) \right]_{\phi^f=0} &= \\ \left[\sum_b \frac{\delta S}{\delta \phi^b} \frac{\delta(\delta_\epsilon \phi^b)}{\delta \phi^{f2}} + \sum_{f_1} \frac{\delta^2 S}{\delta \phi^{f2} \delta \phi^{f_1}} \delta_\epsilon \phi^{f_1} \right]_{\phi^f=0} &= 0, \quad (2.2) \end{aligned}$$

where the last equality follows because the bosonic quantities $\frac{\delta S}{\delta \phi^b}$ and $\delta_\epsilon \phi^f$ are of second order in fermionic fields, and consequently their fermionic variation vanishes for vanishing fermionic fields. Equation (2.2) follows purely from the supersymmetry of the action and is true for every bosonic field configuration.

If we restrict eq. (2.2) to bosonic supersymmetric field configurations, i.e. bosonic field configurations satisfying

$$\delta_{\epsilon_K} \phi^f \big|_{\phi^f=0} = 0, \quad (2.3)$$

for some supersymmetry parameter $\epsilon_K(x)$ (a *Killing spinor*), then we are left with

$$\left. \frac{\delta(\delta_{\epsilon_K} S)}{\delta \phi^f} \right|_{\phi^f=0} = \sum_b \frac{\delta S}{\delta \phi^b} \frac{\delta(\delta_{\epsilon_K} \phi^b)}{\delta \phi^f} \bigg|_{\phi^f=0} = 0. \quad (2.4)$$

These are known as *Killing Spinor Identities* [31], and relate the bosonic equations of motion $\frac{\delta S}{\delta \phi^b}$ to each other through the variation with respect to the fermionic fields of the supersymmetry variation $\delta_{\epsilon_K} \phi^b$ of the bosonic fields. This means that in general one needs to impose only a subset of the equations of motion on a bosonic supersymmetric configuration to ensure that all the equations of motion are satisfied.

We now want to apply this formalism to $N = 2, d = 4$ supergravity. Using the supersymmetry transformation rules of the bosonic fields eqs. (1.21–1.24) and the definitions

of the equations of motion (1.14) inside eq. (2.4), the Killing spinor identities become

$$\mathcal{E}_a{}^\mu \gamma^a \epsilon^I - 4i \varepsilon^{IJ} \mathcal{L}^\Lambda \mathcal{E}_\Lambda{}^\mu \epsilon_J = 0, \quad (2.5)$$

$$\mathcal{E}^i \epsilon^I - 2i \varepsilon^{IJ} \bar{f}^{i\Lambda} \mathcal{E}_\Lambda \epsilon_J = 0, \quad (2.6)$$

$$\mathcal{E}^u \mathcal{U}^{\alpha I}{}_{u \epsilon_I} = 0. \quad (2.7)$$

By performing duality rotations on the above identities, it is possible to obtain a formally electric-magnetic duality-covariant version of them. These are:

$$\mathcal{E}_a{}^\mu \gamma^a \epsilon_I - 4i \langle \mathcal{E}^\mu | \mathcal{V} \rangle \varepsilon_{IJ} \epsilon^J = 0, \quad (2.8)$$

$$\mathcal{E}^i \epsilon^I + 2i \langle \mathcal{E} | \bar{\mathcal{U}}^i \rangle \varepsilon^{IJ} \epsilon_J = 0, \quad (2.9)$$

$$\mathcal{E}^u \mathcal{U}^{\alpha I}{}_{u \epsilon_I} = 0, \quad (2.10)$$

where \mathcal{E}^μ is a symplectic vector containing both the Maxwell equations and Bianchi identities:

$$\mathcal{E}^\mu \equiv \begin{pmatrix} \mathcal{B}^{\Lambda \mu} \\ \mathcal{E}_\Lambda{}^\mu \end{pmatrix}. \quad (2.11)$$

The vector bilinear $V^a \equiv i \bar{\epsilon}^I \gamma^a \epsilon_I$ constructed out of Killing vectors can be either a null or a timelike vector, dividing the supersymmetric field configurations in two classes. We will consider only the timelike case. In this case we can use an orthonormal frame whose time component e^0 is given by $V/|V|$. Acting on the identities (2.8–2.10) on the left with gamma matrices and conjugate spinors, then, we obtain

$$\mathcal{E}^{0m} = \mathcal{E}^{mn} = 0, \quad (2.12)$$

$$\langle \mathcal{V}/X | \mathcal{E}^0 \rangle = \frac{1}{4} |X|^{-1} \mathcal{E}^{00}, \quad (2.13)$$

$$\langle \mathcal{V}/X | \mathcal{E}^m \rangle = 0, \quad (2.14)$$

$$\langle \bar{\mathcal{U}}_{\bar{i}} | \mathcal{E}^0 \rangle = \frac{1}{2} e^{-i\alpha} \mathcal{E}_{\bar{i}}, \quad (2.15)$$

$$\langle \bar{\mathcal{U}}_{\bar{i}} | \mathcal{E}^m \rangle = 0, \quad (2.16)$$

$$\mathcal{E}^u = 0, \quad (2.17)$$

where $X \equiv e^{i\alpha}|X| \equiv \frac{1}{2}\varepsilon^{IJ}\bar{\epsilon}_I\epsilon_J$ is the scalar bilinear.

Using the special geometric completeness relation (B.34), these identities imply that every timelike supersymmetric configuration automatically satisfies all the equations of motion except $\mathcal{E}^{00} = 0$, $\mathcal{E}^i = 0$ and the symplectic vector of equations $\mathcal{E}^0 = 0$. They also imply that to guarantee that all the equations of motion are satisfied, it suffices to impose that the time components of the Maxwell equations and Bianchi identities both vanish, $\mathcal{E}^0 = 0$.

2.2 Supersymmetric configurations

Our goal is to obtain supersymmetric bosonic solutions of the equations of motion (1.16–1.19) derived from the action (1.8). We will first look for supersymmetric field configurations, and later impose the remaining equations of motion, namely the 0-components of the Maxwell equations and Bianchi identities, as explained in section 2.1.

We will for the moment consider the field strengths F^Λ and the vector potentials A^Λ as independent fields; they will become related once we impose the Bianchi identities. Supersymmetric field configurations are those for which the supersymmetry variations of the fermionic fields (1.25–1.27) vanish. More precisely, they are field configurations for which the equations $\delta_\epsilon\psi_{I\mu} = \delta_\epsilon\lambda^{Ii} = \delta_\epsilon\zeta_\alpha = 0$, which are first order differential equations for the supersymmetry parameters, admit at least one solution ϵ_I . The equations are known as *Killing Spinor Equations* and their solutions as *Killing spinors*.

The Killing spinor equations imply other equations for the bilinears

$$X = \frac{1}{2}\varepsilon^{IJ}\bar{\epsilon}_I\epsilon_J, \quad V_a = i\bar{\epsilon}^I\gamma_a\epsilon_I, \quad V^x{}_a = i\sigma^x{}_I{}^J\bar{\epsilon}^I\gamma_a\epsilon_J, \quad \Phi^x{}_{ab} = i\sigma^x{}_I{}^J\bar{\epsilon}_I\gamma_{ab}\epsilon_J \quad (2.18)$$

constructed out of Killing spinors. These can be obtained by acting on the left with gamma matrices and conjugate spinors on the Killing spinor equations.

From the gravitino supersymmetry transformation rule eq. (1.25) the independent equations we get in this way are

$$\mathfrak{D}_\mu X = iV^\nu T^+{}_{\nu\mu} + \frac{i}{\sqrt{2}}S^x V^x{}_\mu, \quad (2.19)$$

$$\nabla_{(\mu}V_{\nu)} = 0, \quad (2.20)$$

$$dV = 4iX\bar{T}^- - \sqrt{2}\bar{S}^x\Phi^x + \text{c.c.}, \quad (2.21)$$

$$\mathfrak{D}_{(\mu}V^x{}_{\nu)} = \bar{T}^-{}_{(\mu|\rho}\Phi^x{}_{|\nu)}{}^\rho + \frac{i}{\sqrt{2}}X\bar{S}^x g_{\mu\nu} + \text{c.c.}, \quad (2.22)$$

$$\mathfrak{D}V^x = -i\epsilon^{xyz}\bar{S}^y\Phi^z + \text{c.c.}, \quad (2.23)$$

where V , V^x and Φ^x are the differential forms associated with the corresponding bilinears, and the $SU(2)$ -covariant derivative is given by

$$\mathfrak{D}V^x = dV^x + \epsilon^{xyz} A^y \wedge V^z. \quad (2.24)$$

From the rule for the gauginos, eq. (1.26), we get the equation

$$\begin{aligned} i\bar{X}\varepsilon^{KI}\mathfrak{D}^\mu Z^i + i\Phi^{KI\mu\nu}\mathfrak{D}_\nu Z^i - 4i\varepsilon^{IJ}G^{i+\mu}{}_\nu V^K{}_{J\mu}{}^\nu \\ - iW^i\varepsilon^{IJ}V^K{}_{J\mu} - iW^i{}_{IJ}V^K{}_{J\mu} = 0, \end{aligned} \quad (2.25)$$

while the rule for the hyperinos eq. (1.27), which using (B.61) and the completeness relation for Pauli matrices (A.31), can be rewritten as

$$\mathfrak{D}q^u\epsilon^I - iK^x{}^u{}_v\sigma^x{}_J{}^I\mathfrak{D}q^v\epsilon^J - ig\varepsilon^{IJ}\tilde{\mathcal{L}}^\Lambda k_\Lambda{}^u\epsilon_J + \frac{1}{2}g\tilde{\mathcal{L}}^\Lambda\mathfrak{D}^u P_\Lambda{}^x\sigma^x{}_{IJ}\epsilon_J = 0, \quad (2.26)$$

gives the equation:

$$V^I{}_{K\mu}\mathfrak{D}^\mu q^u - iK^x{}^u{}_v\sigma^x{}_J{}^I V^J{}_{K\mu}\mathfrak{D}^\mu q^v + gX\delta^I{}_K\tilde{\mathcal{L}}^\Lambda k_\Lambda{}^u + \frac{i}{2}gX\tilde{\mathcal{L}}^\Lambda\mathfrak{D}^u P_\Lambda{}^x\sigma^x{}_{IK} = 0. \quad (2.27)$$

Equation (2.20) tells us that V^μ , which as said before we are taking to be timelike, is a Killing vector, as usual in supergravity, while the V^x are not Killing because of eq. (2.22).

Equation (2.19) can be rewritten in the form

$$V^\nu T^+{}_{\nu\mu} = -i\mathfrak{D}_\mu X - \frac{1}{\sqrt{2}}S^x V^x{}_\mu, \quad (2.28)$$

and its consistency requires

$$V^\mu\mathfrak{D}_\mu X = 0. \quad (2.29)$$

The antisymmetric part of equation (2.25) gives

$$V^\nu G^{i+}{}_{\nu\mu} = \frac{1}{2}\bar{X}\mathfrak{D}_\mu Z^i + \frac{1}{4}W^i V_\mu - \frac{i}{4\sqrt{2}}W^{ix}V^x{}_\mu, \quad (2.30)$$

which implies

$$V^\mu\mathfrak{D}_\mu Z^i + 2XW^i = 0. \quad (2.31)$$

The special geometry completeness relation (B.34) implies

$$F^\Lambda{}^+ = i\tilde{\mathcal{L}}^\Lambda T^+ + 2f^\Lambda{}_i G^{i+}. \quad (2.32)$$

Substituting equations (2.28) and (2.30) in (2.32), and using (1.7) and (1.12) we obtain

$$V^\nu F^\Lambda{}^+{}_{\nu\mu} = \tilde{\mathcal{L}}^\Lambda\mathfrak{D}_\mu X + \bar{X}\mathfrak{D}_\mu \mathcal{L}^\Lambda + \frac{i}{8}gI^{\Lambda\Sigma}(\mathcal{P}_\Sigma V_\mu + \sqrt{2}P_\Sigma{}^x V^x{}_\mu), \quad (2.33)$$

which through (A.17) allows us to obtain for the field strengths F^Λ the expression

$$F^\Lambda = -\frac{1}{2}\mathfrak{D}[\mathcal{R}^\Lambda V] - \frac{1}{2}\star\left\{V\wedge\left[\mathfrak{D}I^\Lambda + \sqrt{2}g\left(\mathcal{R}^\Lambda\mathcal{R}^\Sigma P_\Sigma{}^x - \frac{1}{8|X|^2}I^{\Lambda\Sigma}P_\Sigma{}^x\right)V^x\right]\right\}, \quad (2.34)$$

in terms of the zero Kähler weight sections

$$\mathcal{R} \equiv \Re \left(\frac{\mathcal{V}}{X} \right), \quad \mathcal{I} \equiv \Im \left(\frac{\mathcal{V}}{X} \right). \quad (2.35)$$

The trace of equation (2.27) is

$$V^\mu \mathfrak{D}_\mu q^u - i\sqrt{2} K^{x\ u}{}_v V^{x\ \mu} \mathfrak{D}_\mu q^v + 2gX \bar{\mathcal{L}}^\Lambda k_\Lambda{}^u = 0, \quad (2.36)$$

and its real and imaginary parts are

$$V^\mu \mathfrak{D}_\mu q^u + 2g|X|^2 \mathcal{R}^\Lambda k_\Lambda{}^u = 0, \quad (2.37)$$

$$K^{x\ u}{}_v V^{x\ \mu} \mathfrak{D}_\mu q^v + \sqrt{2}g|X|^2 \mathcal{I}^\Lambda k_\Lambda{}^u = 0. \quad (2.38)$$

To make further progress we introduce a time coordinate t associated to the timelike Killing vector V by

$$V^\mu \partial_\mu \equiv \sqrt{2} \partial_t. \quad (2.39)$$

It is then always possible to make the gauge choice

$$V^\mu A^\Lambda{}_\mu = \sqrt{2} A^\Lambda{}_t = -2|X|^2 \mathcal{R}^\Lambda. \quad (2.40)$$

In this gauge, because of (1.7), equations (2.29), (2.31) and (2.37) reduce to the requirement of time-independence for all the scalar fields and the bilinear X ,

$$\partial_t Z^i = \partial_t X = \partial_t q^u = 0, \quad (2.41)$$

which of course implies also the time-independence of the \mathcal{R} and \mathcal{I} sections.

The definition (2.39) and the Fierz identity $V^2 = 4|X|^2$ imply that the 1-form V must take the form

$$V = 2\sqrt{2}|X|^2(dt + \omega), \quad (2.42)$$

where ω is a spatial 1-form, time-independent since V is Killing, which by definition must satisfy

$$d\omega = \frac{1}{2\sqrt{2}} d \left(\frac{V}{|X|^2} \right). \quad (2.43)$$

This last expression, using equations (2.19) and (2.21), becomes

$$d\omega = -\frac{i}{2\sqrt{2}} \star \left[\left(X \mathfrak{D} \bar{X} - \bar{X} \mathfrak{D} X + ig\sqrt{2}|X|^2 \mathcal{R}^\Lambda P_\Lambda{}^x V^x \right) \wedge \frac{V}{|X|^4} \right]. \quad (2.44)$$

From the Fierz identities we know that the V^x are mutually orthogonal and that they are also orthogonal to V . Furthermore they imply

$$V^\mu V_\mu = 4|X|^2 \quad \text{and} \quad V^{x\ \mu} V^x{}_\mu = -2|X|^2. \quad (2.45)$$

This means that the metric can be written as

$$ds^2 = \frac{1}{4|X|^2} V \otimes V - \frac{1}{2|X|^2} \delta_{xy} V^x \otimes V^y, \quad (2.46)$$

and that the V^x are a Dreibein for a 3-dimensional Euclidean metric:

$$\delta_{xy} V^x \otimes V^y \equiv h_{mn} dx^m dx^n, \quad (2.47)$$

where we introduced the remaining 3 spatial coordinates x^m ($m = 1, 2, 3$). The 4-dimensional metric takes the coordinate-form

$$ds^2 = 2|X|^2 (dt + \omega)^2 - \frac{1}{2|X|^2} h_{mn} dx^m dx^n. \quad (2.48)$$

In what follows all objects with flat (x, y, \dots) or curved (m, n, \dots) 3-dimensional indices will refer to the above Dreibein and the corresponding 3-dimensional metric. The position of the flat x, y, \dots indices is irrelevant, since they are raised and lowered with δ_{xy} .

Using these conventions, eq. (2.44) takes the 3-dimensional form

$$(d\omega)_{xy} = 2\varepsilon_{xyz} \left\{ \langle \mathcal{I} | \tilde{\mathcal{D}}_z \mathcal{I} \rangle - \frac{g}{2\sqrt{2}|X|^2} \mathcal{R}^\Lambda \mathcal{P}_\Lambda^z \right\}, \quad (2.49)$$

where $\tilde{\mathcal{D}}$ is the covariant derivative with respect to the effective 3-dimensional gauge connection

$$\tilde{A}^\Lambda_m \equiv A^\Lambda_m - \omega_m A^\Lambda_t = A^\Lambda_m + \sqrt{2}|X|^2 \mathcal{R}^\Lambda \omega_m. \quad (2.50)$$

More information on the spatial 3-dimensional metric comes from equation (2.23). Its purely spatial part of equation takes the form

$$dV^x + \epsilon^{xyz} \tilde{A}^y \wedge V^z + T^x = 0, \quad (2.51)$$

with

$$\tilde{A}^x_m \equiv A^x_m - g \tilde{A}^\Lambda_m \mathcal{P}_\Lambda^x, \quad (2.52)$$

$$T^x = \frac{g}{\sqrt{2}} \mathcal{I}^\Lambda \mathcal{P}_\Lambda^y V^y \wedge V^x. \quad (2.53)$$

Equation (2.51) can be interpreted as Maurer-Cartan's first structure equation for the Dreibein V^x , with spin connection

$$\varpi_{xyz}(V) = -\varepsilon_{yzw} \tilde{A}^w_x + \sqrt{2}g \mathcal{I}^\Lambda \mathcal{P}_\Lambda^{[y} \delta^{z]x}. \quad (2.54)$$

To summarize, we have shown that, in the timelike case, a bosonic supersymmetric field configuration, with the gauge choice (2.40), necessarily satisfies equations (2.48), (2.49), (2.51), (2.41), (2.34) and (2.38). In particular, at this stage no constraint is imposed

on \mathcal{R} and \mathcal{I} , and consequently on the complex scalars Z^i , other than t -independence. These necessary conditions are also sufficient to have supersymmetry, since as proven in [7] for any such configuration there is always a Killing spinor, taking the form

$$\epsilon_I = X^{1/2} \eta_I, \quad (2.55)$$

where η_I is a constant spinor satisfying the constraints

$$\eta^I + i\gamma^0 \varepsilon^{IJ} \eta_J = 0 \quad \text{and} \quad \eta_I + \gamma^{0(x)} \sigma^{(x)J}{}_I \eta_J = 0 \quad (\text{no sum over } x). \quad (2.56)$$

Each of the four compatible constraints in (2.56) is able to project out half of the components of η_I . However only three of the constraints are independent, so that of the eight real components one always survives. The configurations are then at least $\frac{1}{8}$ -BPS.

2.3 Timelike supersymmetric solutions

As argued in section 2.1, if a supersymmetric configuration satisfies the time components of the Maxwell equations and of the Bianchi identities, then it solves all the equations of motion of the theory.

The time component of the Hodge dual of the Bianchi identities is just the Bianchi identity of the effective 3-dimensional field strength \tilde{F}^Λ , which has the following 3-dimensional expression:

$$\tilde{F}^\Lambda{}_{xy} \equiv -\frac{1}{\sqrt{2}} \varepsilon_{xyz} \{ \tilde{\mathcal{D}}_z \mathcal{I}^\Lambda + g \mathcal{B}^\Lambda{}_z \}, \quad (2.57)$$

where

$$\mathcal{B}^\Lambda{}_z \equiv \sqrt{2} \left[\mathcal{R}^\Lambda \mathcal{R}^\Sigma + \frac{1}{8|X|^2} I^{\Lambda\Sigma} \right] \mathcal{P}_\Sigma{}^z. \quad (2.58)$$

The integrability equation of (2.57) takes the form of a generalized gauge covariant Laplace equation for the \mathcal{I}^Λ ,

$$\tilde{\mathcal{D}}^2 \mathcal{I}^\Lambda + g \tilde{\mathcal{D}}_x \mathcal{B}^\Lambda{}_x = 0, \quad (2.59)$$

where the covariant derivatives include both the gauge connection and the spin connection for the 3-dimensional base space with metric h_{mn} .

The time component of the Maxwell equations takes instead the form of a sort of Bianchi identity for the dual field strengths F_Λ , which can be written as

$$\begin{aligned} -\frac{1}{\sqrt{2}} \varepsilon_{xyz} \tilde{\mathcal{D}}_x \tilde{F}_{\Lambda yz} &= \frac{1}{\sqrt{2}} g \langle \mathcal{I} | \tilde{\mathcal{D}}_x \mathcal{I} \rangle \mathcal{P}_\Lambda^x + \frac{1}{2} g^2 f_{\Lambda(\Omega}{}^\Gamma f_{\Delta)\Gamma}{}^\Sigma \mathcal{I}^\Omega \mathcal{I}^\Delta \mathcal{I}_\Sigma \\ &+ \frac{g^2}{4|X|^2} \mathcal{R}^\Sigma [\mathbf{k}_{\Lambda u} \mathbf{k}_{\Sigma}{}^u - \mathcal{P}_\Lambda{}^x \mathcal{P}_\Sigma{}^x], \end{aligned} \quad (2.60)$$

where \tilde{F}_Λ is defined by

$$\tilde{F}_{\Lambda xy} \equiv -\frac{1}{\sqrt{2}}\varepsilon_{xyz}\{\tilde{\mathfrak{D}}_z\mathcal{I}_\Lambda + g\mathcal{B}_{\Lambda z}\}, \quad (2.61)$$

with

$$\mathcal{B}_{\Lambda x} \equiv \sqrt{2}\left[\mathcal{R}_\Lambda\mathcal{R}^\Sigma + \frac{1}{8|X|^2}R_{\Lambda\Gamma}I^{\Gamma\Sigma}\right]\mathcal{P}_\Sigma^x. \quad (2.62)$$

To summarize, a timelike supersymmetric solution of $N = 2$, $d = 4$ gauged supergravity as defined by the action (1.8) is given by a metric $g_{\mu\nu}$, $\bar{n} = n_v + 1$ vector fields A_μ^Λ , n_V complex scalar fields Z^i and $4n_H$ real hyperscalars q^u such that the metric and vector fields take the form

$$ds^2 = 2|X|^2(d\tau + \omega)^2 - \frac{1}{2|X|^2}h_{mn}dy^m dy^n, \quad (2.63)$$

$$A^\Lambda = -\frac{1}{2}\mathcal{R}^\Lambda V + \tilde{A}_m^\Lambda dy^m, \quad (2.64)$$

where the 3-dimensional metric h_{mn} must admit a Dreibein V^x satisfying the structure equation

$$dV^x + \epsilon^{xyz}\left(A^y - g\tilde{A}^\Lambda\mathcal{P}_\Lambda^y\right) \wedge V^z + \frac{g}{\sqrt{2}}\mathcal{I}^\Lambda\mathcal{P}_\Lambda^y V^y \wedge V^x = 0, \quad (2.65)$$

$|X|^2$ can be determined from \mathcal{R} and \mathcal{I} ,

$$\frac{1}{2|X|^2} = \langle\mathcal{R}|\mathcal{I}\rangle, \quad (2.66)$$

the 1-form V is given by

$$V = 2\sqrt{2}|X|^2(d\tau + \omega), \quad (2.67)$$

and the spatial 1-form ω satisfies

$$(d\omega)_{xy} = 2\varepsilon_{xyz}\left\{\langle\mathcal{I}|\tilde{\mathfrak{D}}_z\mathcal{I}\rangle - \frac{g}{2\sqrt{2}|X|^2}\mathcal{R}^\Lambda\mathcal{P}_\Lambda^z\right\}, \quad (2.68)$$

The complex scalars Z^i , the sections \mathcal{R} and \mathcal{I} , the 1-form ω , the function X and the hyperscalars q^u are all time-independent. The complex scalars are determined, in a way that depends on the chosen parametrization of the special Kähler manifold, from the sections \mathcal{R} and \mathcal{I} .

The effective 3-dimensional gauge connection \tilde{A}^Λ must satisfy

$$(\tilde{\mathfrak{D}}\tilde{A}^\Lambda)_{xy} = \tilde{F}_{xy}^\Lambda = -\frac{1}{\sqrt{2}}\varepsilon_{xyz}\{\tilde{\mathfrak{D}}_z\mathcal{I}^\Lambda + g\mathcal{B}_{\Lambda z}\}, \quad (2.69)$$

from which follows the integrability condition (2.59). A similar condition for the \mathcal{I}_Λ 's is given by (2.60), which can be rewritten as

$$\begin{aligned} \tilde{\mathfrak{D}}^2 \mathcal{I}_\Lambda + g \tilde{\mathfrak{D}}_x \mathcal{B}_{\Lambda x} &= \frac{g}{\sqrt{2}} \langle \mathcal{I} | \tilde{\mathfrak{D}}_x \mathcal{I} \rangle P_\Lambda^x + \frac{g^2}{2} f_{\Lambda(\Omega}{}^\Gamma f_{\Delta)\Gamma}{}^\Sigma \mathcal{I}^\Omega \mathcal{I}^\Delta \mathcal{I}_\Sigma \\ &+ \frac{g^2}{4|X|^2} \mathcal{R}^\Sigma [k_{\Lambda u} k_{\Sigma}{}^u - P_\Lambda^x P_{\Sigma}{}^x] . \end{aligned} \quad (2.70)$$

Finally, the hyperscalars must satisfy

$$K^{x u}{}_v V^{x \mu} \mathfrak{D}_\mu q^v + \sqrt{2} g |X|^2 \mathcal{I}^\Lambda k_{\Lambda}{}^u = 0 . \quad (2.71)$$

For a given special geometric model the sections \mathcal{R} can always, at least in principle, be determined in terms of the sections \mathcal{I} , by solving the so-called *stabilisation equations*. This means that to obtain a supersymmetric solution one needs to solve the above equations for \mathcal{I}^Λ , \mathcal{I}_Λ , ω , V^x and q^u .

In what follows we will restrict ourselves to abelian gauging, in which case some of the above equations simplify. Namely, equation (2.49) becomes

$$(d\omega)_{xy} = 2 \varepsilon_{xyz} \left\{ \langle \mathcal{I} | \partial_z \mathcal{I} \rangle - \frac{g}{2\sqrt{2}|X|^2} \mathcal{R}^\Lambda P_\Lambda^z \right\} , \quad (2.72)$$

the expression for \tilde{F}^Λ

$$(d\tilde{A}^\Lambda)_{xy} = \tilde{F}^\Lambda{}_{xy} = -\frac{1}{\sqrt{2}} \varepsilon_{xyz} \{ \partial_z \mathcal{I}^\Lambda + g \mathcal{B}^\Lambda{}_z \} , \quad (2.73)$$

and equations (2.59) and (2.60) simplify to

$$\tilde{\nabla}^2 \mathcal{I}^\Lambda + g \tilde{\nabla}_x \mathcal{B}^\Lambda{}_x = 0 \quad (2.74)$$

$$\tilde{\nabla}^2 \mathcal{I}_\Lambda + g \tilde{\nabla}_x \mathcal{B}_{\Lambda x} = \frac{g}{\sqrt{2}} \langle \mathcal{I} | \partial_x \mathcal{I} \rangle P_\Lambda^x + \frac{g^2}{4|X|^2} \mathcal{R}^\Sigma [k_{\Lambda u} k_{\Sigma}{}^u - P_\Lambda^x P_{\Sigma}{}^x] , \quad (2.75)$$

where $\tilde{\nabla}_m$ is the covariant derivative associated with the 3-dimensional metric h_{mn} .

2.4 A black hole solution

We now turn to the task of obtaining an explicit solution with non-trivial hyperscalars. To do so, we consider a simple theory with just one vector multiplet and one hypermultiplet, $n_V = n_H = 1$.

More specifically, let the hypermultiplet be the *universal hypermultiplet* [32]. This is called universal since it arises as a subsector in every $N = 2$ Calabi-Yau compactification of M-theory or Type II string theory, and it parameterizes the quaternionic space

$SU(2,1)/U(2)$. The metric on this space can be written in terms of the hyperscalars (ϕ, a, ξ^0, ξ_0) as:

$$H_{uv}dq^u dq^v = d\phi^2 + \frac{1}{4}e^{4\phi} \left(da - \frac{1}{2}\langle \xi | d\xi \rangle \right)^2 + \frac{1}{4}e^{2\phi} [(d\xi^0)^2 + (d\xi_0)^2], \quad (2.76)$$

and the corresponding $SU(2)$ connection has components

$$A^1 = e^\phi d\xi_0, \quad A^2 = e^\phi d\xi^0, \quad A^3 = \frac{e^{2\phi}}{2} \left(da - \frac{1}{2}\langle \xi | d\xi \rangle \right). \quad (2.77)$$

As of the vector multiplet, we choose a special geometric model specified by the prepotential

$$\mathcal{F}(\chi) = -i\chi^0\chi^1, \quad (2.78)$$

with the parametrization $\chi^0 = 1, \chi^1 = Z$. Then it is easy to obtain from (B.21) the Kähler potential $\mathcal{K} = -\log[4\Re(Z)]$ and the scalar metric

$$\mathcal{G} = \partial_Z \partial_{\bar{Z}} \mathcal{K} = \frac{1}{4\Re(Z)^2}, \quad (2.79)$$

while the period matrix $\mathcal{N}_{\Lambda\Sigma}$, giving the scalar-vector couplings, is calculated from eq. (B.31) to be

$$\mathcal{N} = -i \begin{pmatrix} Z & 0 \\ 0 & \frac{1}{Z} \end{pmatrix}. \quad (2.80)$$

Using the definition (2.35), the dependence of the \mathcal{R} section on the \mathcal{I} section for this special geometric model is readily seen to be

$$\mathcal{R}^0 = -\mathcal{I}_1 \quad \mathcal{R}^1 = -\mathcal{I}_0 \quad \mathcal{R}_0 = \mathcal{I}^1 \quad \mathcal{R}_1 = \mathcal{I}^0, \quad (2.81)$$

so that the complex scalar is given by

$$Z = \frac{\mathcal{R}^1 + i\mathcal{I}^1}{\mathcal{R}^0 + i\mathcal{I}^0} = \frac{\mathcal{I}_0 - i\mathcal{I}^1}{\mathcal{I}_1 - i\mathcal{I}^0}, \quad (2.82)$$

and

$$\frac{1}{2|X|^2} = \langle \mathcal{R} | \mathcal{I} \rangle = 2(\mathcal{I}^0\mathcal{I}^1 + \mathcal{I}_0\mathcal{I}_1). \quad (2.83)$$

Since the theory includes two vector fields, we can choose to gauge up to two isometries of the metric H_{uv} . We choose to gauge the (commuting) isometries generated by the Killing vectors

$$k_\Lambda = \tilde{k}_\Lambda \partial_a + \delta_\Lambda^0 c (\xi_0 \partial_{\xi^0} - \xi^0 \partial_{\xi_0}) \quad (2.84)$$

where \tilde{k}_Λ and c are constants, meaning that we are gauging the \mathbb{R} group of the translations along a with the combination $A^\Lambda \tilde{k}_\Lambda$, and the $U(1)$ group of rotations in the ξ^0 - ξ_0

plane with the field A^0 . The triholomorphic moment maps associated with the Killing vectors (2.84) can be obtained from (B.66), and are

$$\begin{aligned} P_\Lambda^1 &= -\delta_\Lambda^0 c \xi^0 e^\phi, & P_\Lambda^2 &= \delta_\Lambda^0 c \xi_0 e^\phi, \\ P_\Lambda^3 &= \delta_\Lambda^0 c \left[1 - \frac{1}{4} e^{2\phi} ((\xi^0)^2 + (\xi_0)^2) \right] + \frac{1}{2} \tilde{k}_\Lambda e^{2\phi}. \end{aligned} \quad (2.85)$$

With these choices the scalar potential (1.13) reads

$$\begin{aligned} V &= \frac{g^2}{2} \left\{ \frac{1}{Z + \bar{Z}} \left[\frac{e^{4\phi}}{4} \left[\tilde{k}_0 - \frac{c}{2} ((\xi^0)^2 + (\xi_0)^2) \right]^2 - c^2 - \tilde{k}_0 c e^{2\phi} \right] \right. \\ &\quad \left. + \frac{Z\bar{Z}}{Z + \bar{Z}} \frac{e^{4\phi}}{4} \tilde{k}_1^2 - \tilde{k}_1 c e^{2\phi} \right\}. \end{aligned} \quad (2.86)$$

For simplicity we will look for solutions with $\mathcal{R}^0 = \mathcal{R}^1 = \mathcal{I}_0 = \mathcal{I}_1 = 0$, which implies from (2.82) that the scalar Z is real and from (2.64) that the gauge fields are in a purely magnetic configuration. From eq. (2.72) follows that ω is a closed 1-form, and can be reabsorbed with a redefinition of the coordinate t , leading to static solutions. This choice also implies that eq. (2.75) is satisfied trivially.

We will also take the hyperscalar a to be constant and $\xi^0 = \xi_0 = 0$, so that the moment maps (2.85) become

$$P_\Lambda^1 = P_\Lambda^2 = 0, \quad P_\Lambda^3 = \delta_\Lambda^0 c + \frac{1}{2} \tilde{k}_\Lambda e^{2\phi}. \quad (2.87)$$

Eq. (2.65) implies then $dV^3 = 0$, making it possible to define a coordinate r such that locally

$$V^3 = dr. \quad (2.88)$$

We will impose radial symmetry on the solution by requiring the scalar fields Z , ϕ and the sections \mathcal{I}^Λ to depend only on r .

The ϕ , ξ^0 and ξ_0 components of equation (2.38) reduce then to the constraint

$$A_x^\Lambda \tilde{k}_\Lambda = 0, \quad \implies \quad \tilde{A}^\Lambda \tilde{k}_\Lambda = 0 \quad (2.89)$$

while the a component becomes

$$\phi' = \frac{g}{2\sqrt{2}} e^{2\phi} \mathcal{I}^\Lambda \tilde{k}_\Lambda, \quad (2.90)$$

where the prime stands for a derivative with respect to r .

If we now introduce the remaining coordinates θ and ϕ by choosing

$$V^1 = e^{W(r)} d\theta \quad \text{and} \quad V^2 = e^{W(r)} f(\theta) d\phi, \quad (2.91)$$

where at this stage f is an arbitrary function of θ , the remaining components of eq. (2.65) are satisfied provided that the following conditions are met

$$W'(r) = -\frac{g}{\sqrt{2}} \mathbf{P}_\Lambda^3 \mathcal{I}^\Lambda = -\frac{g}{\sqrt{2}} \left(c \mathcal{I}^0 + \frac{e^{2\phi}}{2} \mathcal{I}^\Lambda \tilde{k}_\Lambda \right), \quad (2.92)$$

$$\tilde{A}^0 = -\frac{f'(\theta)}{gc} d\varphi. \quad (2.93)$$

From (2.93) and the constraint (2.89) we also have

$$\tilde{A}^1 = \frac{\tilde{k}_0}{\tilde{k}_1} \frac{f'(\theta)}{gc} d\varphi. \quad (2.94)$$

Finally, eq. (2.73) gives the following two equations

$$\left[\left(\mathcal{I}^\Lambda \tilde{k}_\Lambda \right)' - \frac{g}{\sqrt{2}} (\mathcal{I}^\Lambda)^2 \tilde{k}_\Lambda \mathbf{P}_\Lambda^3 \right] e^{2W(r)} = (-1)^\Lambda \frac{\sqrt{2} \tilde{k}_0}{gc} \frac{f''(\theta)}{f(\theta)}, \quad (\text{no sum over } \Lambda), \quad (2.95)$$

while eq. (2.74) is automatically satisfied since we obtained \tilde{F}^Λ as the exterior derivative of the effective connection \tilde{A}^Λ .

Equation (2.90) allows us to use the chain rule to trade the coordinate r for ϕ in eq. (2.95), which summing over Λ becomes

$$\frac{1}{2} \partial_\phi \left[\left(\mathcal{I}^\Lambda \tilde{k}_\Lambda \right)^2 \right] - \left(\mathcal{I}^\Lambda \tilde{k}_\Lambda \right)^2 + 2 \mathcal{I}^0 \tilde{k}_0 \left(\mathcal{I}^1 \tilde{k}_1 - \mathcal{I}^0 c e^{-2\phi} \right) = 0. \quad (2.96)$$

If we impose the condition

$$\mathcal{I}^1 \tilde{k}_1 = \mathcal{I}^0 c e^{-2\phi} \quad (2.97)$$

this equation is solved by

$$\mathcal{I}^0 = \frac{\alpha e^\phi}{\tilde{k}_0 + c e^{-2\phi}}, \quad \mathcal{I}^1 = \frac{c}{\tilde{k}_1} \frac{\alpha e^{-\phi}}{\tilde{k}_0 + c e^{-2\phi}}, \quad (2.98)$$

where α is an integration constant. Substituting these expressions back in (2.95) for $\Lambda = 0$ or $\Lambda = 1$, we obtain an expression for the function $W(r)$,

$$e^{2W(r)} = \left[\frac{2}{\alpha gc} \left(\tilde{k}_0 + c e^{-2\phi} \right) e^{-\phi} \right]^2 \frac{f''(\theta)}{f(\theta)}. \quad (2.99)$$

The expression (2.99) is also a solution of equation (2.92), which is non-trivial, proving the constraint eq. (2.97) to be consistent with all the equations. From (2.99) we also conclude that $f''(\theta)/f(\theta)$ should be a positive constant, therefore $f(\theta)$ in general takes the form

$$f(\theta) = \gamma \sinh(\delta\theta + \rho), \quad (2.100)$$

where γ , δ and ρ are constants. We can now go back to the coordinate r by solving equation (2.90) to obtain the dependence of ϕ on r , obtaining

$$\phi = -\frac{1}{3} \log \left(-\frac{3\alpha g}{2\sqrt{2}} r + \beta \right), \quad (2.101)$$

where β is yet another integration constant.

All the integration constants can be reabsorbed with the coordinate change

$$(t, r, \theta, \varphi) \longrightarrow \left(\frac{4\sqrt{2}\alpha}{g\tilde{k}_1 c} t, -\frac{2\sqrt{2}}{3\alpha g} (r^3 - \beta), \frac{\theta - \rho}{\delta}, \frac{\varphi}{\delta\gamma} \right), \quad (2.102)$$

allowing us to write the complete solution as

$$ds^2 = \frac{16r^2}{g^2\tilde{k}_1 c} \left[\left(1 + \frac{\tilde{k}_0}{c} \frac{1}{r^2} \right)^2 r^2 dt^2 - \left(1 + \frac{\tilde{k}_0}{c} \frac{1}{r^2} \right)^{-2} \frac{dr^2}{r^2} - \frac{1}{2} (d\theta^2 + \sinh^2 \theta d\varphi^2) \right], \quad (2.103)$$

$$A^0 = -\frac{\cosh \theta}{gc} d\varphi, \quad A^1 = \frac{\tilde{k}_0}{\tilde{k}_1} \frac{\cosh \theta}{gc} d\varphi, \quad (2.104)$$

$$\phi = -\log r, \quad Z = \frac{c}{\tilde{k}_1} r^2. \quad (2.105)$$

We start the analysis of the solution by noting that it has no free parameters, since all the constants appearing in (2.103-2.105) are completely determined by the choice of gauging. Observe also that in order to maintain the correct signature and to have $Z > 0$, which is required to have a real Kähler potential, we have to impose $\tilde{k}_1 c > 0$.

The metric (2.103) is singular in $r = 0$ and, if $\tilde{k}_0 c < 0$, also in $r = \sqrt{-k_0/c}$. The singularity in $r = r_S \equiv 0$ is a true curvature singularity, while the one in $r = r_H \equiv \sqrt{-k_0/c}$ is not and corresponds instead to a Killing horizon, always covering the curvature singularity.

With the metric written in the form (2.103), it is immediate to see that in the asymptotic limit $r \rightarrow +\infty$ it reduces to

$$ds^2 = \frac{16r^2}{g^2\tilde{k}_1 c} \left[r^2 dt^2 - \frac{dr^2}{r^2} - \frac{1}{2} (d\theta^2 + \sinh^2 \theta d\varphi^2) \right], \quad (2.106)$$

which is manifestly conformally equivalent to $\text{AdS}_2 \times \text{H}^2$. Note that even if the scalar fields diverge in this limit, this does not constitute a problem since the boundary terms resulting from the integration by parts of the action vanish.

In the near horizon limit, $r \rightarrow r_H$, after the coordinate change $t \rightarrow t/4$, the metric takes the form

$$ds^2 = -\frac{4}{g^2 c^2} \frac{k_0}{\tilde{k}_1} \left[r^2 dt^2 - \frac{dr^2}{r^2} - 2 (d\theta^2 + \sinh^2 \theta d\varphi^2) \right], \quad (2.107)$$

which exhibits the geometry of $\text{AdS}_2 \times \text{H}^2$, while the scalar fields take the values

$$\phi = -\frac{1}{2} \log \left(-\frac{k_0}{c} \right), \quad Z = -\frac{k_0}{k_1}. \quad (2.108)$$

The magnetic charges are given by

$$P^\Lambda = \frac{1}{4\pi} \int F^\Lambda = p^\Lambda V, \quad V = \int \sinh \theta d\theta \wedge d\varphi, \quad (2.109)$$

yielding for the magnetic charge densities

$$p^0 = -\frac{1}{4\pi g c} \quad p^1 = \frac{k_0}{k_1} \frac{1}{4\pi g c}. \quad (2.110)$$

The Bekenstein-Hawking entropy density can then be written as

$$s = \frac{S}{V} = -\frac{k_0}{k_1} \frac{2}{g^2 c^2} = 32\pi^2 p^0 p^1. \quad (2.111)$$

Fake supergravity

Supersymmetric solutions to a supergravity theory are obtained by looking for field configurations for which the Killing spinor equations admit a solution. The general form of the Killing spinor equation arising from the vanishing of the gravitino supersymmetry variation is given by

$$(\nabla_\mu + M_\mu) \epsilon = 0, \quad (3.1)$$

where ϵ is a spinorial function, ∇_μ is the general covariant derivative on spinors and M_μ are matrix-valued functions.

Finding supersymmetric solutions is in general simpler than trying to solve directly the equations of motion of the theory, since the first order Killing spinor equations are usually easier to solve than the second order Einstein equations. It is then natural to wonder whether this procedure can be generalized to find broader classes of solutions. In other words, one would like to know if there are first order equations of the form (3.1) for which the field configurations admitting a nonzero solution ϵ are solutions of the equations of motion of some field theory with gravity, not necessarily supersymmetric.

This kind of approach is known as *fake supergravity* [6], since the equations (3.1), while having a form similar to the Killing spinor equations, are not related to an underlying supersymmetry of the theory. Fake supergravity allows to find solutions of theories that are only loosely related to supergravity, but also non-supersymmetric solutions of genuine supergravities.

In this chapter we review the classification of fake supersymmetric solutions presented by Meessen and Palomo–Lozano in [8]¹. The considered theory is obtained by analytic continuation from genuine $N = 2, d = 4$ gauged supergravity coupled to vector supermultiplets (see chapter 1).

In section 3.1 we set up the theory, write the fake Killing spinor equations and their integrability conditions, which relate the equations of motion, greatly reducing the number of independent equations of motion for a fake supersymmetric configuration. In section 3.2 we proceed to characterize the fake supersymmetric solutions by deducing first

¹Related work in 4 and 5 dimensions was published in [33, 34, 35].

order equations for the fields from the fake Killing spinor equations and imposing the residual equations of motion. We conclude by summarizing the form of the fields in a fake supersymmetric configuration and the equations they must satisfy.

3.1 Fake $N = 2, d = 4$ supergravity

We start from genuine $N = 2, d = 4$ gauged supergravity as presented in chapter 1, but without hypermultiplets, $n_H = 0$, and with one (combination) of the $\bar{n} = n_V + 1$ vector fields gauging a $U(1)$ subgroup of the $SU(2)$ factor of the full R -symmetry group $SU(2) \times U(1)$ through the Fayet-Iliopoulos mechanism. This corresponds to having a constant triholomorphic moment map.

Then we perform a Wick rotation on the Fayet-Iliopoulos term, or in other words we take the constant triholomorphic moment map P_Λ^x to be imaginary,

$$P_\Lambda^x \longrightarrow iC_\Lambda \delta_2^x, \quad (3.2)$$

where C_Λ is a constant and we have taken $P_\Lambda^1 = P_\Lambda^3 = 0$ without loss of generality.

The effect of this Wick rotation is that instead of gauging a $U(1)$ group, we are gauging an \mathbb{R} -symmetry through the effective connection $C_\Lambda A^\Lambda$. The presence of a Fayet-Iliopoulos term is compatible with the gauging of a non-Abelian subgroup of isometries of the scalar manifold with structure constants $f_{\Lambda\Sigma}{}^\Gamma$, provided that the constraint

$$f_{\Lambda\Sigma}{}^\Gamma C_\Gamma = 0, \quad (3.3)$$

which is a consequence of the equivariance condition (B.72), is satisfied. Since one vector field is used to gauge this \mathbb{R} -symmetry, this isometry subgroup can have at most dimension n_V .

As long as we restrict to the bosonic sector without hyperscalars, the action after the Wick rotation is real and still describes a valid theory of gravity. We rewrite it here for convenience:

$$S = \int d^4x \sqrt{|g|} \left[R + 2\mathcal{G}_{i\bar{j}} \mathfrak{D}_\mu Z^i \mathfrak{D}^\mu \bar{Z}^{\bar{j}} + 2I_{\Lambda\Sigma} F^{\Lambda\mu\nu} F^\Sigma_{\mu\nu} - 2R_{\Lambda\Sigma} F^{\Lambda\mu\nu} \star F^\Sigma_{\mu\nu} - V \right] \quad (3.4)$$

The action (3.4) has exactly the same form as in the case of genuine supergravity (1.8) without hypermultiplets, the only difference being in the scalar potential that from the expression (1.13) is now modified to

$$V(Z, \bar{Z}) = g^2 \left[2|C_\Lambda \mathcal{L}^\Lambda|^2 + \frac{1}{4} I^{\Lambda\Sigma} (C_\Lambda C_\Sigma - \mathcal{P}_\Lambda \mathcal{P}_\Sigma) \right], \quad (3.5)$$

allowing in particular to have a positive cosmological constant.

The idea behind fake supergravity is to find first order differential equations for which the existence of a solution implies that the equations of motion of the theory, or at least a subset of them, are satisfied. This is analogous to what happens in genuine supergravity, in which the Killing spinor equations, through the Killing spinor identities, reduce the number of equations of motion that one has to actually impose on a field configuration to obtain a solution.

Since we are considering a theory that is obtained by a slight modification of a genuine supergravity, it makes sense to try to obtain such *fake Killing spinor equations* by modifying in some way the true Killing spinor equations of the theory from which we started. As mentioned, we are introducing an \mathbb{R} -connection which together with the existent Kähler $U(1)$ -symmetry due to the vector coupling means that we should modify the covariant derivative on spinors (1.28) as

$$\mathbb{D}_\mu \epsilon_I = \nabla_\mu \epsilon_I + \frac{i}{2} \mathcal{Q}_\mu \epsilon_I + \frac{ig}{2} A_\mu^\Lambda [\mathcal{P}_\Lambda + iC_\Lambda] \epsilon_I. \quad (3.6)$$

In this chapter we will denote respectively with \mathbb{D} and with \mathfrak{D} derivatives with or without the \mathbb{R} -connection.

We then alter equations (1.25-1.26) to obtain the following fake Killing spinor equations:

$$\mathbb{D}_\mu \epsilon_I + \left[T^+_{\mu\nu} \varepsilon_{IJ} + \frac{ig}{4} C_\Lambda \mathcal{L}^\Lambda \eta_{\mu\nu} \varepsilon_{IJ} \right] \gamma^\nu \epsilon^J = 0, \quad (3.7)$$

$$i\mathfrak{D} Z^i \epsilon^I + \left[G^{i+} + W^i \right] \varepsilon^{IJ} \epsilon_J = 0, \quad (3.8)$$

where the object

$$W^i \equiv -\frac{ig}{2} \bar{f}^{i\Lambda} [\mathcal{P}_\Lambda + iC_\Lambda] \quad (3.9)$$

now combines the contributions of both the fermion shifts W^i and W^{ix} defined in (1.31) and (1.32), and the field strengths T^+ and G^{i+} are still as defined in (1.34) and (1.35),

$$T^+ \equiv 2i\mathcal{L}^\Sigma I_{\Sigma\Lambda} F^{\Lambda+}, \quad (3.10)$$

$$G^{i+} \equiv -\mathcal{G}^{i\bar{j}} \bar{f}^{\Sigma} I_{\Sigma\Lambda} F^{\Lambda+}. \quad (3.11)$$

Note that the fake Killing spinor equations (3.7-3.9) are not simply obtained from the supersymmetry rules (1.25-1.26) with the substitution (3.2), since this would not lead to the correct equations of motion², but rather with

$$P_\Lambda^x \sigma^x I_J \longrightarrow -iC_\Lambda \delta^I_J. \quad (3.12)$$

Since the above fake Killing spinor equations do not come from an underlying supersymmetry of the theory, we cannot obtain Killing spinor identities with the procedure

²We thank P. Meessen for clarifications on this point.

explained in section 2.1. In this case the relations between the different equations of motion will instead be obtained from the integrability conditions for equations (3.7-3.9). These can easily be calculated and give rise respectively to

$$\mathcal{B}_{\mu\nu}\gamma^\nu\epsilon_I = -2i\mathcal{L}^\Lambda\left[\mathcal{B}_\Lambda - \mathcal{N}_{\Lambda\Sigma}\mathcal{B}^\Sigma\right]\varepsilon_{IJ}\gamma_\mu\epsilon^J, \quad (3.13)$$

and

$$\mathcal{B}^i\epsilon_I = -2i\bar{f}^{i\Lambda}\left[\mathcal{B}_\Lambda - \mathcal{N}_{\Lambda\Sigma}\mathcal{B}^\Sigma\right]\varepsilon_{IJ}\epsilon^J, \quad (3.14)$$

with the Bianchi identity

$$\star\mathcal{B}^\Lambda = \mathfrak{D}F^\Lambda = 0 \quad (3.15)$$

and the equations of motion defined by

$$\mathcal{B}_{\mu\nu} = R_{\mu\nu} + 2\mathcal{G}_{ij}\mathfrak{D}_{(\mu}Z^i\mathfrak{D}_{\nu)}\bar{Z}^{\bar{j}} + 4I_{\Lambda\Sigma}\left[F_{\mu\rho}^\Lambda F_{\nu}^{\Sigma\rho} - \frac{1}{4}\eta_{\mu\nu}F_{\rho\sigma}^\Lambda F^{\Sigma\rho\sigma}\right] - \frac{1}{2}g_{\mu\nu}V, \quad (3.16)$$

$$\star\mathcal{B}_\Lambda = \mathfrak{D}F_\Lambda - \frac{g}{4}\left(k_{\Lambda\bar{i}}\star\mathfrak{D}\bar{Z}^{\bar{i}} + \bar{k}_{\Lambda i}\star\mathfrak{D}Z^i\right), \quad (3.17)$$

$$\mathcal{B}^i = \square Z^i - i\partial^i\bar{\mathcal{N}}_{\Lambda\Sigma}F_{\rho\sigma}^{\Lambda+}F^{\Sigma+ \rho\sigma} + i\partial^i\mathcal{N}_{\Lambda\Sigma}F_{\rho\sigma}^{\Lambda-}F^{\Sigma- \rho\sigma} + \frac{1}{2}\partial^iV, \quad (3.18)$$

with the dual field strengths

$$F_\Lambda \equiv \mathcal{N}_{\Lambda\Sigma}F^{\Sigma-} + \bar{\mathcal{N}}_{\Lambda\Sigma}F^{\Sigma+}. \quad (3.19)$$

As was the case for supersymmetric field configurations in supergravity, the independent number of equations of motion one has to impose in order to ensure that a configuration for which equations (3.7–3.8) admit a solution is a solution to the full set of equations of motion is greatly reduced. If the squared norm of the vector bilinear $V_\mu = i\bar{\epsilon}^I\gamma_\mu\epsilon_I$ is positive, meaning that V is timelike, we only need to solve the time components of the Bianchi identity, and of the Maxwell equations,

$$\iota_V\star\mathcal{B}^\Lambda = 0, \quad \iota_V\star\mathcal{B}_\Lambda = 0. \quad (3.20)$$

The authors of [8] studied also the null case, when V has vanishing norm, but we will only consider the timelike case.

3.2 Fake supersymmetric solutions

We now proceed as we did in the supersymmetric case. By acting on the left with conjugate spinors and gamma matrices on the fake Killing spinor equations (3.7-3.8) we obtain equations for the bilinears.

From equation (3.7) we get, in terms of the real symplectic sections of Kähler weight zero \mathcal{R} and \mathcal{I} defined in (2.35),

$$\mathbb{D}X = \frac{g}{4} C_\Lambda \mathcal{L}^\Lambda V + i \iota_V T^+, \quad (3.21)$$

$$\mathbb{D}_\mu V_\nu = g|X|^2 C_\Lambda \mathcal{R}^\Lambda g_{\mu\nu} + 4\mathfrak{Im}(\bar{X} T_{\mu\nu}^+) , \quad (3.22)$$

$$\mathbb{D}V^x = \frac{g}{2} C_\Lambda \mathcal{R}^\Lambda V \wedge V^x + \frac{g}{2} C_\Lambda \mathcal{I}^\Lambda \star [V \wedge V^x] , \quad (3.23)$$

while equation (3.8) leads to

$$2\bar{X} \mathfrak{D}Z^i = 4 \iota_V G^{i+} - W^i V , \quad (3.24)$$

The bilinear V in the supersymmetric case was a Killing vector, but this is not the case here, as one can see from equation (3.22). We are still free however to introduce a time coordinate τ from V by choosing an adapted coordinate system through

$$V^\mu \partial_\mu = \sqrt{2} \partial_\tau , \quad (3.25)$$

but now the components of the metric will depend explicitly on τ .

From equation (3.23) we can calculate

$$\mathcal{L}_V V^x = \iota_V dV^x + d(\iota_V V^x) = g C_\Lambda \iota_V A^\Lambda V^x + 2g|X|^2 C_\Lambda \mathcal{R}^\Lambda V^x , \quad (3.26)$$

which, if we make the same gauge choice as in the supersymmetric case,

$$\iota_V A^\Lambda = -2|X|^2 \mathcal{R}^\Lambda , \quad (3.27)$$

reduces to

$$\mathcal{L}_V V^x = 0 , \quad (3.28)$$

telling us that the 1-forms V^x are τ -independent.

From the contraction of equation (3.21) with V we get

$$\frac{1}{\bar{X}} \mathfrak{D}_V \frac{1}{X} = -g \langle \mathcal{R} | C \rangle + ig \langle \mathcal{I} | C \rangle , \quad (3.29)$$

where we introduced the symplectic vector

$$C \equiv \begin{pmatrix} 0 \\ C_\Lambda \end{pmatrix} . \quad (3.30)$$

Using the identity

$$\frac{1}{\bar{X}} \mathfrak{D} \frac{1}{X} = 2 (\langle \mathcal{R} | \mathfrak{D} \mathcal{I} \rangle - i \langle \mathcal{I} | \mathfrak{D} \mathcal{I} \rangle) , \quad (3.31)$$

that can be easily proven using eqs. (B.28) and (B.32), the real and imaginary parts of eq. (3.29) can be written as

$$\langle \mathcal{R} | \nabla_V \mathcal{I} + \frac{g}{2} C \rangle = 0 \quad (3.32)$$

$$\langle \mathcal{I} | \nabla_V \mathcal{I} + \frac{g}{2} C \rangle = 0, \quad (3.33)$$

where we used the gauge choice (3.27) and the second constraint in (1.7).

Contracting instead equation (3.24) with V leads to

$$\mathfrak{D}_V Z^i = -2 X W^i, \quad (3.34)$$

which upon using again the gauge-fixing (3.27) and the constraints (1.7) takes the form

$$\nabla_V Z^i = -g X \bar{f}^{\Lambda i} C_\Lambda. \quad (3.35)$$

Using then the first special geometry identity in (B.32) we can rewrite the above equation to

$$\langle \nabla_V \mathcal{I} + g C | \bar{\mathcal{U}}_{\bar{j}} \rangle = i \langle \nabla_V \mathcal{R} | \bar{\mathcal{U}}_{\bar{j}} \rangle, \quad (3.36)$$

which can be manipulated by using the special geometry properties and again eq. (3.34) to give

$$\langle \bar{\mathcal{U}}_{\bar{j}} | \nabla_V \mathcal{I} + \frac{g}{2} C \rangle = 0. \quad (3.37)$$

Equations (3.32), (3.33) and (3.37), together with the special geometric completeness relation (B.34), then imply

$$\nabla_V \mathcal{I} = -\frac{g}{2} C, \quad (3.38)$$

which means that only the lower components \mathcal{I}_Λ of the sections \mathcal{I} are τ -dependent, and that this dependence is linear.

As in the supersymmetric case, the Fierz identities imply that the 1-form associated with the vector bilinear V must take the form

$$V = 2\sqrt{2}|X|^2 (d\tau + \omega), \quad (3.39)$$

but now the 1-form ω can be τ -dependent. The Fierz identities also imply that the V^x are mutually orthogonal and orthogonal to V , allowing us to introduce the remaining coordinates and write the metric in the form

$$ds^2 = 2|X|^2 (d\tau + \omega)^2 - \frac{1}{2|X|^2} h_{mn} dy^m dy^n, \quad (3.40)$$

where the three dimensional Riemannian metric h_{mn} has the 1-forms V^x as a Dreibein,

$$h_{mn} = V_m^x V_n^x. \quad (3.41)$$

If we define

$$\tilde{A}^\Lambda \equiv \tilde{A}_m^\Lambda dy^m \equiv A^\Lambda + \frac{1}{2} \mathcal{R}^\Lambda V, \quad (3.42)$$

equation (3.23) in the chosen coordinate system reads

$$dV^x = g C_\Lambda \tilde{A}^\Lambda \wedge V^x + \frac{g}{2\sqrt{2}} C_\Lambda \mathcal{I}^\Lambda \varepsilon^{xyz} V^y \wedge V^z. \quad (3.43)$$

Equation (3.43) tells us that the base three dimensional space with metric h_{mn} must be a Gauduchon-Tod space (see Appendix B.8), with the conformal transformations sending one element of the conformal class into another corresponding to the residual gauge freedom

$$C_\Lambda \tilde{A}^\Lambda \rightarrow C_\Lambda \tilde{A}^\Lambda + dw(y), \quad V^x \rightarrow e^w V^x. \quad (3.44)$$

From the definition of ω eq. (3.39) and the antisymmetric part of equation (3.22) follows

$$d\omega + g C_\Lambda \tilde{A}^\Lambda \wedge (d\tau + \omega) = \sqrt{2} \star [V \wedge \langle \mathcal{I} | \mathfrak{D} \mathcal{I} \rangle], \quad (3.45)$$

where we made use of the identity (A.17) to write explicitly $T^+_{\mu\nu}$ starting from the expression for $\iota_V T^+$ that can be obtained from eq. (3.21). The time dependence of ω can be determined by contracting equation (3.45) with V . In this way we get

$$\mathcal{L}_V \omega = g\sqrt{2} C_\Lambda \tilde{A}^\Lambda \longrightarrow \omega = g C_\Lambda \tilde{A}^\Lambda \tau + \tilde{\omega}, \quad (3.46)$$

where $\tilde{\omega} = \tilde{\omega}_m dy^m$ is τ -independent. Substituting the above result into equation (3.45) and keeping only the τ -independent part, we arrive to

$$\tilde{\mathbb{D}} \tilde{\omega} = \varepsilon^{xyz} \langle \tilde{\mathcal{I}} | \tilde{\mathfrak{D}}_x \tilde{\mathcal{I}} - \tilde{\omega}_x \partial_\tau \mathcal{I} \rangle V^y \wedge V^z, \quad (3.47)$$

where we defined

$$\tilde{\mathcal{I}} \equiv \mathcal{I}|_{\tau=0} \quad (3.48)$$

and the derivatives associated with the effective three dimensional connection \tilde{A} by

$$\tilde{\mathbb{D}} \tilde{\omega} = d\tilde{\omega} + g C_\Lambda \tilde{A}^\Lambda \wedge \tilde{\omega}, \quad (3.49)$$

$$\tilde{\mathfrak{D}}_m \mathcal{I} = \partial_m \mathcal{I} + g \tilde{A}_m^\Lambda \mathcal{S}_\Lambda \mathcal{I}. \quad (3.50)$$

The field strength F^Λ can be deduced as in the supersymmetric case from the expression (2.32), using equations (3.21) and (3.24) to obtain $\iota_V T^+$ and $\iota_V G^{i+}$, and then the identity (A.17). The result reads

$$\begin{aligned} F^\Lambda &= -\frac{1}{2} \mathfrak{D} (\mathcal{R}^\Lambda V) - \frac{1}{2} \star [V \wedge \mathbb{D} \mathcal{I}^\Lambda] \\ &= -\frac{1}{2} \mathfrak{D} (\mathcal{R}^\Lambda V) - \frac{1}{2\sqrt{2}} \varepsilon^{xyz} \tilde{\mathbb{D}}_x \mathcal{I}^\Lambda V^y \wedge V^z. \end{aligned} \quad (3.51)$$

At this point we proceed to impose the remaining equations. The time component of the Bianchi identity translates to

$$(\tilde{\mathfrak{D}}\tilde{A}^\Lambda)_{xy} = \tilde{F}_{xy}^\Lambda = -\frac{1}{\sqrt{2}}\varepsilon^{xyz}\tilde{\mathbb{D}}_z\mathcal{I}^\Lambda, \quad (3.52)$$

which due to eq. (3.38) is manifestly τ -independent, implying that the potentials \tilde{A}^Λ are also τ -independent. The integrability condition for eq. (3.52) is a generalized Laplace equation for the \mathcal{I}^Λ 's.

A similar condition for the \mathcal{I}_Λ 's comes from the imposition of the time component of the Maxwell equations, and after some manipulations it reads

$$\tilde{\mathbb{D}}_x^2\tilde{\mathcal{I}}_\Lambda - \left(\tilde{\mathbb{D}}_x\tilde{\omega}_x\right)\partial_\tau\mathcal{I}_\Lambda = \frac{g^2}{2}f_{\Lambda(\Omega}{}^\Gamma f_{\Delta)\Gamma}{}^\Sigma\mathcal{I}^\Omega\mathcal{I}^\Delta\tilde{\mathcal{I}}_\Sigma - \frac{g^2}{2}f_{\Lambda\Omega}{}^\Sigma\mathcal{I}^\Omega\tilde{\mathcal{I}}_\Sigma C_\Gamma\mathcal{I}^\Gamma. \quad (3.53)$$

To summarize the results of this section, a fake supersymmetric solution to the theory defined by the action (3.4) is given by a metric and $n_V + 1$ vector fields of the form

$$ds^2 = 2|X|^2(d\tau + \omega)^2 - \frac{1}{2|X|^2}h_{mn}dy^m dy^n, \quad (3.54)$$

$$A^\Lambda = -\frac{1}{2}\mathcal{R}^\Lambda V + \tilde{A}_m^\Lambda dy^m, \quad (3.55)$$

and n_V complex scalars Z^i whose dependence on the sections \mathcal{V} , or equivalently on \mathcal{R} and \mathcal{I} , in general depends on the chosen parametrization of the special Kähler manifold. In what follows however we will always take the scalars to be given by

$$Z^i = \frac{\mathcal{L}^i}{\mathcal{L}^0} = \frac{\mathcal{R}^i + i\mathcal{I}^i}{\mathcal{R}^0 + i\mathcal{I}^0}. \quad (3.56)$$

The 1-form V is given by the expression (3.39), $\omega = \omega_m dy^m$ is a 1-form which in general depends on τ , and h is the metric on a three-dimensional Gauduchon-Tod base space. In particular there must exist a dreibein V^x for h satisfying

$$dV^x = g C_\Lambda \tilde{A}^\Lambda \wedge V^x + \frac{g}{2\sqrt{2}} C_\Lambda \mathcal{I}^\Lambda \varepsilon^{xyz} V^y \wedge V^z. \quad (3.57)$$

Furthermore the following equations must hold:

$$\omega = g C_\Lambda \tilde{A}^\Lambda \tau + \tilde{\omega}, \quad (3.58)$$

$$\tilde{F}_{xy}^\Lambda = -\frac{1}{\sqrt{2}} \varepsilon^{xyz} \tilde{\mathbb{D}}_z \mathcal{I}^\Lambda, \quad (3.59)$$

$$\partial_\tau \mathcal{I}^\Lambda = 0, \quad \partial_\tau \mathcal{I}_\Lambda = -\frac{g}{2\sqrt{2}} C_\Lambda, \quad (3.60)$$

$$\tilde{\mathbb{D}}_x^2 \tilde{\mathcal{I}}_\Lambda - \left(\tilde{\mathbb{D}}_x \tilde{\omega}_x \right) \partial_\tau \mathcal{I}_\Lambda = \frac{g^2}{2} f_{\Lambda(\Omega}{}^\Gamma f_{\Delta)\Gamma}{}^\Sigma \mathcal{I}^\Omega \mathcal{I}^\Delta \tilde{\mathcal{I}}_\Sigma - \frac{g^2}{2} f_{\Lambda\Omega}{}^\Sigma \mathcal{I}^\Omega \tilde{\mathcal{I}}_\Sigma C_\Gamma \mathcal{I}^\Gamma, \quad (3.61)$$

$$\tilde{\mathbb{D}} \tilde{\omega} = \varepsilon^{xyz} \langle \tilde{\mathcal{I}} | \tilde{\mathfrak{D}}_x \tilde{\mathcal{I}} - \tilde{\omega}_x \partial_\tau \mathcal{I} \rangle V^y \wedge V^z, \quad (3.62)$$

with

$$\tilde{F}^\Lambda \equiv \tilde{\mathfrak{D}} \tilde{A}^\Lambda, \quad \tilde{\omega} \equiv \omega|_{\tau=0}, \quad \tilde{\mathcal{I}} \equiv \mathcal{I}|_{\tau=0}, \quad (3.63)$$

$$\tilde{\mathbb{D}}_m \mathcal{I} \equiv \partial_m \mathcal{I} + g C_\Lambda \tilde{A}_m^\Lambda \mathcal{I} + g \tilde{A}_m^\Lambda \mathcal{S}_\Lambda \mathcal{I}, \quad \tilde{\mathbb{D}}_x \mathcal{I} \equiv W_x^m \tilde{\mathbb{D}}_m \mathcal{I}. \quad (3.64)$$

Black holes in an expanding universe from fake supergravity

Not much is known on dynamical processes involving black holes, since only a few time-dependent black hole solutions have been constructed so far. The first and perhaps most famous one is the McVittie spacetime [9], whose interpretation as a black hole in a FLRW universe has been the subject of some controversy in the literature [36, 37, 38]. Another example, which however violates the energy conditions, was constructed by Sultana and Dyer [39] using conformal techniques.

Kastor and Traschen (KT) [40] obtained a solution describing an arbitrary number of black holes in a de Sitter universe, each carrying an electric charge equal to the mass. This leads to a no-force condition, such that the whole system is just comoving with the cosmological expansion. This solution allowed an analytical discussion of black hole collisions and of the issue of whether such processes lead to a violation of the cosmic censorship conjecture [40, 41]. The KT solution is a time-dependent generalization of the Majumdar–Papapetrou (MP) spacetime [42, 43], which describes maximally charged Reissner–Nordström black holes in static equilibrium in an asymptotically flat space. The MP solution is supersymmetric, and in this case the no-force condition allowing to take arbitrary superpositions of black holes despite the high non-linearity of Einstein’s equations can be traced back to linear differential equations arising as a consequence of the existence of a Killing spinor.

Supersymmetry however is only compatible with a negative or vanishing cosmological constant, thus no true Killing spinor can exist in a theory with positive cosmological constant. Despite this, it was shown in [44] that the KT solution admits a *fake* Killing spinor, leading to an explanation of the black hole superposition similar to that for the supersymmetric MP solution. The fake Killing spinor equations are obtained in this case from the Killing spinor equations of pure $N = 2$ gauged supergravity, simply taking the gauge coupling constant to be imaginary.

Maeda, Ohta and Uzawa (MOU) obtained [45], from the compactification of higher dimensional intersecting brane solutions, four- and five-dimensional spacetimes, further studied in [46], describing black holes in a FLRW universe filled with stiff matter. In

[47] Gibbons and Maeda presented a class of spacetimes interpolating between the KT and the four-dimensional MOU black holes as solutions to a theory with a Liouville-type scalar potential, later generalized to arbitrary dimension and further analyzed in [48].

As of time-dependent rotating black hole solutions, only a few examples are known. A spinning generalization of the KT solution in a string-inspired theory was given by Shiromizu in [49], while five-dimensional multi-centered rotating charged de Sitter black holes were constructed in [50, 51], and a rotating generalization of the five-dimensional MOU solution was obtained in [52] by solving fake Killing spinor equations.

In this chapter we use the classification of fake supersymmetric solutions [8] reviewed in chapter 3 to construct explicitly some time-dependent solutions, describing multi-centered black holes in a cosmological background, which were originally presented in our papers [53, 54]. All these solutions are obtained considering theories with only one vector multiplet, $n_V = 1$, and with Abelian gauging, i.e. with the Fayet-Iliopoulos gauging of the \mathbb{R} -symmetry and no additional gauging of the isometries of the special Kähler manifold. In section 4.1 we consider a rather generic special geometric model, but with the further restriction that the complex field Z takes on real values on the solution, which will lead to non-rotating spacetimes, which turn out to be generalizations of the Gibbons–Maeda black holes [47].

We then proceed to analyze in some detail the physical properties of the solutions in the single-centered case, making use of the formalism introduced in appendix C.

In section 4.2 we consider a different truncation of the same model, and obtain again the spacetime of [47] but with one of the gauge fields dualized to an electric, rather than magnetic, configuration.

In section 4.3 we consider in a general way some possible choices for the Gauduchon-Tod structure of the base space, without referring to a specific special geometric model and without restrictions on the values of the scalars. We give explicit expressions for the section \mathcal{I} and the rotation 1-form ω suitable to describe black holes, both single- and multi-centered.

Finally in sections 4.4 and 4.5 we apply the results of section 4.3 to two specific special geometric models. Since we are no longer requiring the scalar field to be real we are able to obtain multi-centered solutions with rotation and NUT-charge, in a cosmological background with flat or curved spatial sections. In particular for the first choice of prepotential we are able to write the solutions with flat spatial slices in terms of two complex harmonic functions, in a form similar to the Israel–Wilson–Perjés class of metrics [55, 56], of which they are time-dependent generalizations.

4.1 The $\mathcal{F}(\chi) = -\frac{i}{4}(\chi^0)^n(\chi^1)^{2-n}$ model

Given this prepotential with $n \neq 0, 2$, from (B.21) we can derive the Kähler potential

$$e^{-\kappa} = \frac{n}{4}Z^{2-n} + \frac{2-n}{4}\bar{Z}Z^{1-n} + \text{c.c.}, \quad (4.1)$$

where we took $|\chi^0| = 1$.

If we consider the truncation $\Im m(Z) = 0$, the Kähler metric becomes

$$\mathcal{G} = \partial_Z \partial_{\bar{Z}} \mathcal{K}|_{\Im m(Z)=0} = \frac{n(2-n)}{4} \Re e(Z)^{-2} = \frac{n_0 n_1}{16} e^{-2\phi}, \quad (4.2)$$

where we defined

$$n_0 \equiv 2n, \quad n_1 \equiv 2(2-n) = 4 - n_0, \quad \phi \equiv \log \Re e(Z). \quad (4.3)$$

From equation (B.31) we obtain then

$$\mathcal{N} = -\frac{i}{8} \begin{pmatrix} n_0 e^{\frac{n_1}{2}\phi} & 0 \\ 0 & n_1 e^{-\frac{n_0}{2}\phi} \end{pmatrix}, \quad (4.4)$$

and for the scalar potential (3.5) we get

$$V = \frac{1}{2} \left[\frac{n_0(n_0-1)}{t_0^2} e^{-\frac{n_1}{2}\phi} + 2 \frac{n_0 n_1}{t_0 t_1} e^{\frac{n_0-n_1}{4}\phi} + \frac{n_1(n_1-1)}{t_1^2} e^{\frac{n_0}{2}\phi} \right], \quad (4.5)$$

with the definition

$$t_\Lambda \equiv -\frac{n_\Lambda}{2gC_\Lambda}. \quad (4.6)$$

If one wishes to have a non-zero potential in the particular cases $n_0 = 1$ and $n_0 = 3$ one has to require respectively $C_1 \neq 0$ and $C_0 \neq 0$.

Plugging these expressions into (3.4) leads to the bosonic Lagrangian

$$\begin{aligned} e^{-1}\mathcal{L} = & R + \frac{n_0 n_1}{8} \partial_\mu \phi \partial^\mu \phi - \frac{n_0}{4} e^{\frac{n_1}{2}\phi} F_{\mu\nu}^0 F^{0\mu\nu} - \frac{n_1}{4} e^{-\frac{n_0}{2}\phi} F_{\mu\nu}^1 F^{1\mu\nu} \\ & - \frac{1}{2} \left[\frac{n_0(n_0-1)}{t_0^2} e^{-\frac{n_1}{2}\phi} + 2 \frac{n_0 n_1}{t_0 t_1} e^{\frac{n_0-n_1}{4}\phi} + \frac{n_1(n_1-1)}{t_1^2} e^{\frac{n_0}{2}\phi} \right]. \end{aligned} \quad (4.7)$$

We see that in order to avoid ghost fields in the Lagrangian one has to impose $0 < n_0 < 4$, corresponding to $0 < n < 2$ in the prepotential. One can check that for $t_1 \rightarrow \infty$ (i.e., $C_1 = 0$), (4.7) reduces to the Lagrangian used in [47], if we identify $n_0 = n_T$, $n_1 = n_S$.

Rather than solving the stabilisation equations and express the \mathcal{R} section in terms of \mathcal{I} , in the present case it is more convenient to express all the components of \mathcal{R} and \mathcal{I} in

terms of \mathcal{I}^0 , \mathcal{I}_1 and the scala field ϕ . The definition (2.35), together with the constraint $\Im(Z) = 0$, leads to

$$\begin{aligned}\mathcal{I}^1 &= e^\phi \mathcal{I}^0, & \mathcal{I}_0 &= \frac{n_0}{n_1} e^\phi \mathcal{I}_1, \\ \mathcal{R}^0 &= -\frac{8}{n_1} e^{\frac{n_0-n_1}{4}\phi} \mathcal{I}_1, & \mathcal{R}^1 &= -\frac{8}{n_1} e^{\frac{n_0}{2}\phi} \mathcal{I}_1, \\ \mathcal{R}_0 &= \frac{n_0}{8} e^{\frac{n_1}{2}\phi} \mathcal{I}^0, & \mathcal{R}_1 &= \frac{n_1}{8} e^{\frac{n_1-n_0}{4}\phi} \mathcal{I}^0,\end{aligned}\tag{4.8}$$

as well as

$$\frac{1}{2|X|^2} = \langle \mathcal{R} | \mathcal{I} \rangle = \frac{1}{2} e^{\frac{n_1}{2}\phi} (\mathcal{I}^0)^2 + \frac{32}{n_1^2} e^{\frac{n_0}{2}\phi} (\mathcal{I}_1)^2.\tag{4.9}$$

Notice that since both \mathcal{I}^0 and \mathcal{I}^1 must be independent of τ , either $\mathcal{I}^0 = 0$ or ϕ is also independent of τ . In this second case using (3.60) we see that $C_0 = 0 \Leftrightarrow C_1 = 0$, so that if we require a non vanishing scalar potential we must impose $C_0, C_1 \neq 0$; we also find that $e^\phi = \frac{n_1}{n_0} \frac{C_0}{C_1} = \frac{t_1}{t_0}$.

4.1.1 Construction of the solution

The simplest solution of eq. (3.43) is the flat three-dimensional space, with

$$V_m^x = \delta_m^x, \quad C_\Lambda \tilde{A}^\Lambda = C_\Lambda \mathcal{I}^\Lambda = 0.\tag{4.10}$$

With this choice for the base space we don't need to distinguish between x, y, z, \dots and lower m, n, p, \dots indices.

If we require a nonvanishing scalar potential $V \neq 0$, then $C_\Lambda \mathcal{I}^\Lambda = 0$ together with (4.8) implies either

$$\mathcal{I}^0 = 0 \quad \Rightarrow \quad \mathcal{I}^1 = \mathcal{R}_0 = \mathcal{R}_1 = 0,\tag{4.11}$$

or a constant ϕ with

$$e^\phi = -\frac{C_0}{C_1},\tag{4.12}$$

and $C_0, C_1 \neq 0$; but if ϕ is constant we should also have $e^\phi = \frac{n_1}{n_0} \frac{C_0}{C_1}$, so this choice is clearly inconsistent. The only consistent possibility is then $\mathcal{I}^0 = 0$. Using equation (3.59) this immediately implies

$$\tilde{F}^0 = \tilde{F}^1 = 0.\tag{4.13}$$

Because of (4.11) and $C_\Lambda \tilde{A}^\Lambda = 0$, eq. (3.62) implies $d\tilde{\omega} = 0$, and thus locally

$$\tilde{\omega} = df,\tag{4.14}$$

where f is a generic function of the spatial coordinates.

Equation (3.61), since we are in the Abelian case and the structure constants vanish, then becomes

$$\begin{cases} \partial_p \partial_p (\tilde{\mathcal{I}}_0 + \frac{gC_0}{2\sqrt{2}} f) = 0, \\ \partial_p \partial_p (\tilde{\mathcal{I}}_1 + \frac{gC_1}{2\sqrt{2}} f) = 0, \end{cases} \implies \begin{cases} \partial_p \partial_p (e^{\tilde{\phi}} \tilde{\mathcal{I}}_1 - \frac{n_1}{4\sqrt{2}} \frac{f}{t_0}) = 0, \\ \partial_p \partial_p (\tilde{\mathcal{I}}_1 - \frac{n_1}{4\sqrt{2}} \frac{f}{t_1}) = 0, \end{cases} \quad (4.15)$$

with $\tilde{\phi} \equiv \phi|_{\tau=0}$. This can be solved by introducing two generic harmonic functions of the spatial coordinates $\mathcal{H}_0, \mathcal{H}_1$ as

$$\tilde{\mathcal{I}}_1 = \frac{n_1}{4\sqrt{2}} (f/t_1 + \mathcal{H}_1), \quad e^{\tilde{\phi}} = \frac{f/t_0 + \mathcal{H}_0}{f/t_1 + \mathcal{H}_1}. \quad (4.16)$$

At this point, using (3.60) and $\mathcal{I}_0 = e^{\phi} \mathcal{I}_1$ we obtain

$$\begin{aligned} \mathcal{I}_1 &= \frac{n_1}{4\sqrt{2}} \left(\frac{\tau + f}{t_1} + \mathcal{H}_1 \right), \quad \mathcal{I}_0 = \frac{n_0}{4\sqrt{2}} \left(\frac{\tau + f}{t_0} + \mathcal{H}_0 \right), \\ e^{\phi} &= \frac{(\tau + f)/t_0 + \mathcal{H}_0}{(\tau + f)/t_1 + \mathcal{H}_1}, \end{aligned} \quad (4.17)$$

and from (4.9) one gets

$$\frac{1}{2|X|^2} = \left(\frac{\tau + f}{t_0} + \mathcal{H}_0 \right)^{\frac{n_0}{2}} \left(\frac{\tau + f}{t_1} + \mathcal{H}_1 \right)^{\frac{n_1}{2}}. \quad (4.18)$$

We have now all the elements needed to write down the complete solution in terms of the two generic harmonic functions \mathcal{H}_0 and \mathcal{H}_1 . Since f appears everywhere as a shift in the time coordinate τ we can set it equal to zero with the coordinate change $t = \tau + f$ to obtain

$$ds^2 = \mathcal{U}^{-2} dt^2 - \mathcal{U}^2 d\vec{y}^2, \quad (4.19)$$

$$A^\Lambda = \left(\frac{t}{t_\Lambda} + \mathcal{H}_\Lambda \right)^{-1} dt, \quad \phi = \ln \left(\frac{t/t_0 + \mathcal{H}_0}{t/t_1 + \mathcal{H}_1} \right),$$

with

$$\mathcal{U} \equiv \left(\frac{t}{t_0} + \mathcal{H}_0 \right)^{\frac{n_0}{4}} \left(\frac{t}{t_1} + \mathcal{H}_1 \right)^{\frac{n_1}{4}}. \quad (4.20)$$

Here one clearly recognizes the substitution principle originally put forward by Behrndt and Cvetič in [57], which amounts to adding a linear time dependence to the harmonic functions in a supersymmetric solution of $N = 2, d = 4$ supergravity.

4.1.2 Physical discussion

As a first remark if we set $C_1 = 0$, corresponding to $t_1 \rightarrow \infty$, and make the choice of harmonic functions

$$\mathcal{H}_0 = \sum_{i=1}^N \frac{Q_0^{(i)}}{|\vec{y} - \vec{y}_i|}, \quad \mathcal{H}_1 = 1 + \sum_{i=1}^N \frac{Q_1^{(i)}}{|\vec{y} - \vec{y}_i|}, \quad (4.21)$$

we recover precisely the solution presented in [47]. The same is true if we set $C_0 = 0$, change the sign of the scalar field and exchange everywhere 0 and 1 indices. This solution represents a system of multiple maximally charged black holes in a universe expanding with arbitrary equation of state $P = w\rho$, with $w = \frac{8-5n_0}{3n_0}$ so that $-1 \leq w \leq 1$ for $1 \leq n_0 \leq 4$ (for $n_0 < 1$ the scalar potential is unbounded from below). Note that one can have $w < -1$ by allowing $n_0 < 0$ or $n_0 > 4$, but then of course the action (4.7) contains ghosts. In this case, we would have black holes embedded in an expanding universe filled with phantom energy. In the limit $n_0 = 4$ one obtains the Kastor-Traschen solution [40], describing multiple black holes in a de Sitter background, while for $n_0 = 0$ the scalar potential is zero and the solution is the Majumdar-Papapetrou spacetime, describing multiple extremal Reissner-Nordström black holes in an asymptotically flat background. Notice that we can also recover the Kastor-Traschen solution keeping both t_0 and t_1 finite and taking $t_0\mathcal{H}_0 = t_1\mathcal{H}_1$.

Retaining both C_0 and C_1 the scalar potential has critical points; the derivative of the scalar potential can be written as

$$V'[\phi] = \frac{n_0 n_1}{4t_1^2} e^{-\frac{n_1}{4}\phi} \left[\frac{t_1}{t_0} - e^\phi \right] \left[\frac{t_1}{t_0} (1 - n_0) + (1 - n_1) e^\phi \right]. \quad (4.22)$$

We can see that if we take $t_0 t_1 > 0$ there is, for every value of $0 < n_0 < 4$, a minimum in $e^\phi = \frac{t_1}{t_0}$, $V_{\min} = 6 t_1^{-n_1/2} t_0^{-n_0/2}$. For $0 < n_0 < 1$ or $3 < n_0 < 4$ there is also a maximum in $e^\phi = -\frac{1-n_0}{1-n_1} \frac{t_1}{t_0}$, $V_{\max} = 2 \left(\frac{n_0-1}{1-n_1} \right)^{\frac{n_0-n_1}{4}} t_1^{-n_1/2} t_0^{-n_0/2}$; however for these values of n_0 the potential is not bounded from below. For $1 < n_0 < 3$ the potential is bounded and the minimum is global.

If on the other hand we take $t_0 t_1 < 0$, there is only a negative minimum in $e^\phi = -\frac{1-n_0}{1-n_1} \frac{t_1}{t_0}$ if $1 < n_0 < 3$, $V_{\min} = -2 \left(\frac{1-n_0}{1-n_1} \right)^{\frac{n_0-n_1}{4}} |t_1|^{-n_1/2} |t_0|^{-n_0/2}$, while there are no critical points for $0 < n_0 < 1$ or $3 < n_0 < 4$.

For $t_0 t_1 > 0$ and assuming that the harmonic functions have a well-defined limit for $|\vec{y}| \rightarrow \infty$, one can study the asymptotic behaviour of the metric; swapping the coordinate t for \tilde{t} defined by

$$\frac{d\tilde{t}}{dt} = \left(\frac{t}{t_0} + k_0 \right)^{-\frac{n_0}{4}} \left(\frac{t}{t_1} + k_1 \right)^{-\frac{n_1}{4}}, \quad k_i \equiv \lim_{|\vec{y}| \rightarrow \infty} \mathcal{H}_i, \quad (4.23)$$

the metric asymptotically assumes a Friedmann-Lemaître-Robertson-Walker form,

$$ds^2 = d\tilde{t}^2 - a^2(\tilde{t}) d\vec{y}^2, \quad a(\tilde{t}) = \frac{dt}{d\tilde{t}}. \quad (4.24)$$

The explicit form of $a(\tilde{t})$ is complicated; however it is possible to obtain the time dependence of the density and pressure,

$$\rho(\tilde{t}) = \frac{3}{128\pi} \frac{n_0^2}{t_0^2} \frac{1}{R(\tilde{t})^{\frac{n_0}{2}}} \left(R(\tilde{t}) + \frac{n_1}{n_0} \frac{t_0}{t_1} \right)^2, \quad (4.25)$$

$$P(\tilde{t}) = -\frac{5}{128\pi} \frac{n_0^2}{t_0^2} \frac{1}{R(\tilde{t})^{\frac{n_0}{2}}} \left[\left(R(\tilde{t}) + \frac{n_1}{n_0} \frac{t_0}{t_1} \right)^2 - \frac{8}{5n_0} \left(R^2(\tilde{t}) + \frac{n_1}{n_0} \frac{t_0^2}{t_1^2} \right) \right], \quad (4.26)$$

where

$$R(\tilde{t}) \equiv \frac{t(\tilde{t})/t_1 + k_1}{t(\tilde{t})/t_0 + k_0}, \quad (4.27)$$

so that

$$\frac{P(\tilde{t})}{\rho(\tilde{t})} = w(\tilde{t}) = -\frac{5}{3} \left[1 - \frac{8}{5n_0} \frac{R^2(\tilde{t}) + \frac{n_1}{n_0} \frac{t_0^2}{t_1^2}}{\left(R(\tilde{t}) + \frac{n_1}{n_0} \frac{t_0}{t_1} \right)^2} \right], \quad (4.28)$$

that gives the correct value of [47] in the limits $t_0 \rightarrow \infty$ or $t_1 \rightarrow \infty$.

If both t_0 and t_1 are finite, w is time-independent only if $t_0 k_0 = t_1 k_1$, which is equivalent to consider $k_0 = k_1 = 0$, since we are free to set $k_0 = 0$ without loss of generality by shifting t . In this case $a(\tilde{t}) = e^{\tilde{t}/\tilde{t}_0}$, with $\tilde{t}_0 = t_0^{n_0/4} t_1^{n_1/4}$, $w = -1$ and the spacetime is asymptotically de Sitter independently of the value of n_0 , while the scalar field tends to the critical value $e^\phi = t_1/t_0$.

Note that in the case $t_0 t_1 > 0$, the solution (4.19) tends to de Sitter for $|\vec{y}| \rightarrow \infty$ and arbitrary k_Λ either for $t \rightarrow \infty$ or $t \rightarrow -\infty$ (for positive or negative t_Λ respectively).

Since we are interested in black hole systems, we consider harmonic functions of the form

$$H_\Lambda(t, \vec{y}) \equiv \frac{t}{t_\Lambda} + \mathcal{H}_\Lambda = \frac{t}{t_\Lambda} + k_\Lambda + \sum_{i=1}^N \frac{Q_\Lambda^{(i)}}{|\vec{y} - \vec{y}_i|}, \quad (4.29)$$

and take $k_0 = 0$ since it can be eliminated by shifting t . Notice that while we could take some of the charges to be zero, this would lead to a divergent scalar field in the limit $|\vec{y} - \vec{y}_i| \rightarrow 0$.

The scalar curvature of (4.19) reads

$$\begin{aligned} R = & \frac{3}{8} \frac{n_1(3n_1 - 4)t_0^2 H_0^2 + 6n_0 n_1 t_0 t_1 H_0 H_1 + n_0(3n_0 - 4)t_1^2 H_1^2}{t_0^2 t_1^2 H_0^{\frac{n_1}{2}} H_1^{\frac{n_0}{2}}} \\ & + \frac{n_0 n_1}{8 H_0^{\frac{n_0}{2}} H_1^{\frac{n_1}{2}}} \left(\partial_p \ln \frac{H_0}{H_1} \right)^2, \end{aligned} \quad (4.30)$$

which is singular for $H_0 = 0$ or $H_1 = 0$.

We can also consider the limit $|\vec{y} - \vec{y}_i| \equiv r_i \rightarrow 0$ for some i ; then the time dependence drops out and the metric reduces to $\text{AdS}_2 \times S^2$,

$$ds_{r_i \rightarrow 0}^2 = \frac{r_i^2}{l_i^2} dt^2 - \frac{l_i^2}{r_i^2} dr_i^2 - l_i^2 d\Omega_2^2, \quad (4.31)$$

with $l_i \equiv (Q_0^{(i)})^{n_0/4} (Q_1^{(i)})^{n_1/4}$. As we shall see later, (4.31) does actually not describe the geometry near the event horizon of our time-dependent solution.

We turn now to study in more detail the system with a single black hole. Since in this case there is spherical symmetry, we will work in spherical coordinates,

$$H_0(t, r) = \frac{t}{t_0} + \frac{Q_0}{r}, \quad H_1(t, r) = \frac{t}{t_1} + k_1 + \frac{Q_1}{r}. \quad (4.32)$$

If $Q_0, Q_1 \neq 0$ we will assume in the following, without loss of generality, $|Q_1 t_1| \geq |Q_0 t_0|$.

Since $r = 0$ is not a curvature singularity unless one of the charges is zero, the space-time can be extended to $r < 0$. The singularities are represented in the r - t plane by two hyperbolae having the asymptotes $r = 0$ and respectively $t = 0$ or $t = -k_1 t_1$; if $k_1 \neq 0$ they intersect unless $Q_0 t_0 = Q_1 t_1$. To ensure the regularity of the solution we must require $H_0 H_1 > 0$; this corresponds to the area external to the singularities in the r - t plane for $t_0 t_1 > 0$ or to the area between them if $t_0 t_1 < 0$ (see figure 4.1).

The present spacetime satisfies the weak energy condition; to see this, compute the energy-momentum tensor components T_{ab} for an observer with orthonormal frame

$$e^0 = \mathcal{U}^{-1} dt, \quad e^1 = \mathcal{U} dr, \quad e^2 = \mathcal{U} r d\theta, \quad e^3 = \mathcal{U} r \sin \theta d\varphi.$$

One obtains

$$\begin{aligned} \rho^\phi &= \mathcal{F}_1 + \mathcal{F}_2 + \mathcal{F}_3, & P_r^\phi &= \mathcal{F}_1 + \mathcal{F}_2 - \mathcal{F}_3, & P_\Omega^\phi &= \mathcal{F}_1 - \mathcal{F}_2 - \mathcal{F}_3, \\ \rho^{\text{em}} &= -P_r^{\text{em}} = P_\Omega^{\text{em}} = \mathcal{F}_4, & T_{01}^\phi &= -\mathcal{F}_5, \end{aligned} \quad (4.33)$$

where $\rho = T_{00}$, $P_r = T_{11}$, $P_\Omega = T_{22} = T_{33}$, the other off-diagonal components are zero, and

$$\mathcal{F}_1 = \mathcal{U}^2 \frac{n_0 n_1}{16} \left(\frac{1}{t_0 H_0} - \frac{1}{t_1 H_1} \right)^2, \quad \mathcal{F}_2 = \mathcal{U}^{-2} \frac{n_0 n_1}{16 r^4} \left(\frac{Q_0}{H_0} - \frac{Q_1}{H_1} \right)^2, \quad (4.34)$$

$$\mathcal{F}_3 = \frac{\mathcal{U}^2}{4} \left[\left(\frac{n_0}{t_0 H_0} - \frac{n_1}{t_1 H_1} \right)^2 - \frac{n_0}{t_0^2 H_0^2} - \frac{n_1}{t_1^2 H_1^2} \right], \quad (4.35)$$

$$\mathcal{F}_4 = \frac{\mathcal{U}^{-2}}{4 r^4} \left(\frac{n_0 Q_0^2}{H_0^2} + \frac{n_1 Q_1^2}{H_1^2} \right), \quad \mathcal{F}_5 = \frac{n_0 n_1}{8 r^2} \left(\frac{1}{t_0 H_0} - \frac{1}{t_1 H_1} \right) \left(\frac{Q_0}{H_0} - \frac{Q_1}{H_1} \right). \quad (4.36)$$

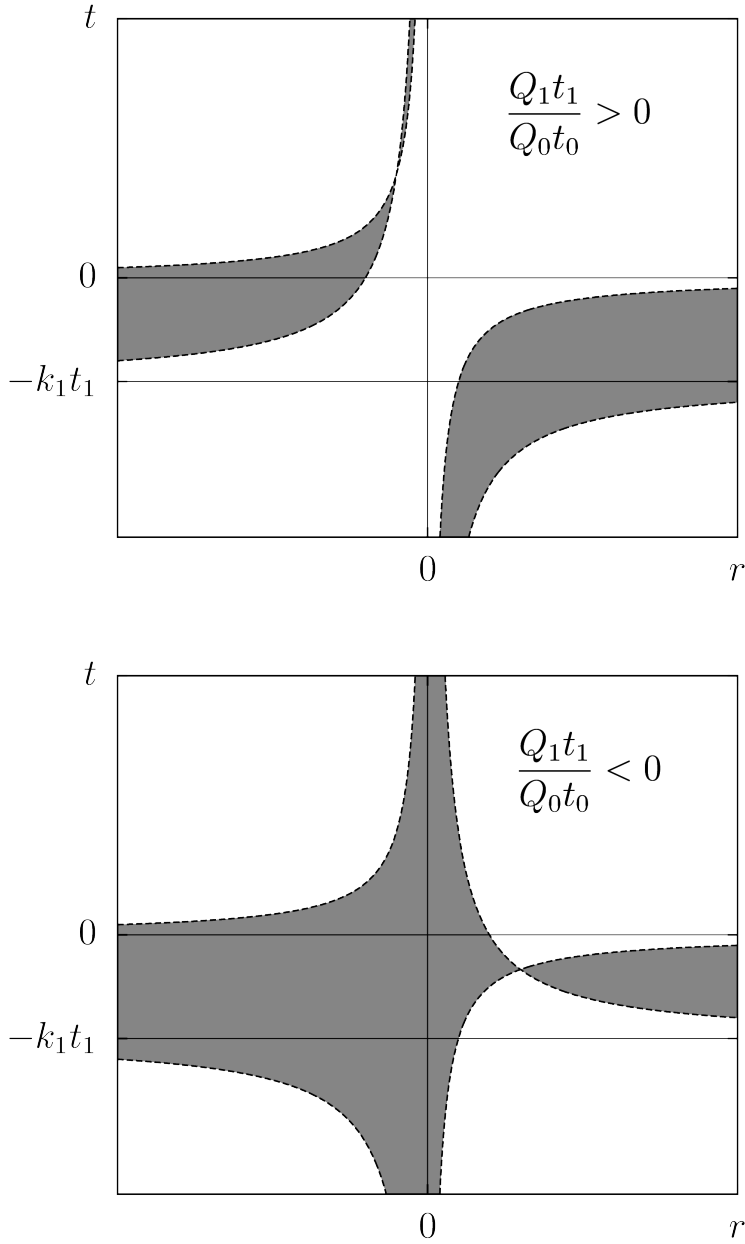


Figure 4.1: Allowed coordinate ranges in the $r-t$ plane. The dashed curves denote the curvature singularities, the allowed range is the white area for $t_0t_1 > 0$ or the grey area for $t_0t_1 < 0$. We assume here k_1t_1 and Q_0t_0 positive; the other cases can be obtained by reflection or rotation.

Since $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_4$ and $\mathcal{F}_1 + \mathcal{F}_3 = \mathcal{U}^2 \frac{3}{16} \left(\frac{n_0}{t_0 H_0} + \frac{n_1}{t_1 H_1} \right)^2$ are positive definite, the energy densities ρ^ϕ and ρ^{em} are positive. Notice also that $\mathcal{F}_4 - \mathcal{F}_2 = \mathcal{U}^{-2} \left(\frac{n_0 Q_0}{H_0} + \frac{n_1 Q_1}{H_1} \right)^2 / (16r^4)$ is positive definite and that $\mathcal{F}_5^2 = 4\mathcal{F}_1\mathcal{F}_2$.

T_b^a can always be diagonalized by changing to a different orthonormal basis. Its eigenvalues are

$$\hat{\rho} = \frac{1}{2} \left(\rho - P_r + \sqrt{(\rho + P_r)^2 - 4T_{01}^2} \right), \quad (4.37)$$

$$-\hat{P}_r = -\frac{1}{2} \left(\rho - P_r + \sqrt{(\rho + P_r)^2 - 4T_{01}^2} \right), \quad (4.38)$$

$$-\hat{P}_\Omega = -P_\Omega. \quad (4.39)$$

In terms of these the weak energy condition can be stated as

$$\hat{\rho} \geq 0, \quad \hat{\rho} + \hat{P}_r \geq 0, \quad \hat{\rho} + \hat{P}_\Omega \geq 0. \quad (4.40)$$

We have

$$\hat{\rho} = \mathcal{F}_3 + \mathcal{F}_4 + |\mathcal{F}_1 - \mathcal{F}_2| \geq (\mathcal{F}_1 + \mathcal{F}_3) + (\mathcal{F}_4 - \mathcal{F}_2) \geq 0, \quad (4.41)$$

$$\hat{\rho} + \hat{P}_r = 2|\mathcal{F}_1 - \mathcal{F}_2| \geq 0, \quad (4.42)$$

$$\hat{\rho} + \hat{P}_\Omega = \mathcal{F}_1 + \mathcal{F}_4 + (\mathcal{F}_4 - \mathcal{F}_2) + |\mathcal{F}_1 - \mathcal{F}_2| \geq 0, \quad (4.43)$$

and thus (4.40) holds. Whether the strong and dominant energy conditions are satisfied depends on the values of the parameters; it has been shown in particular that the Gibbons-Maeda solution ($t_1 \rightarrow \infty$) satisfies the strong energy condition if and only if the asymptotic cosmological background does [48], and that the Maeda-Ohta-Uzawa solution ($t_1 \rightarrow \infty, n_0 = 1$) satisfies the dominant energy condition [46].

The spherical symmetry allows us to covariantly define the circumference radius $R = |r|\mathcal{U} = |r|H_0^{n_0/4}H_1^{n_1/4}$; it is immediate to see that this radius vanishes on the singularities. In a spherically symmetric spacetime it is also possible to compute the Misner-Sharp quasilocal energy [58], that can be interpreted as the energy inside a closed surface of radius R ,

$$m = 4\pi R (1 + \nabla_\mu R \nabla^\mu R), \quad (4.44)$$

where

$$\nabla_\mu R \nabla^\mu R = -\frac{1}{16} \left[\left(\frac{tn_0}{t_0 H_0} + \frac{(t + k_1 t_1)n_1}{t_1 H_1} \right)^2 - r^2 H_0^{n_0} H_1^{n_1} \left(\frac{n_0}{t_0 H_0} + \frac{n_1}{t_1 H_1} \right)^2 \right]. \quad (4.45)$$

Following [46, 48] we can look for trapping horizons [11]. Introducing the Newman-Penrose null tetrads

$$\begin{aligned} l &= \frac{1}{\sqrt{2}} (\mathcal{U}^{-1} dt - \mathcal{U} dr) , \\ n &= \frac{1}{\sqrt{2}} (\mathcal{U}^{-1} dt + \mathcal{U} dr) , \\ m &= \mathcal{U} \frac{r}{\sqrt{2}} (d\theta + i \sin \theta d\varphi) , \end{aligned} \quad (4.46)$$

and the complex conjugate \bar{m} , satisfying $l^\mu n_\mu = 1 = -m^\mu \bar{m}_\mu$, the expansions of the outgoing and ingoing radial null geodesics are defined by

$$\theta_+ \equiv -2m^{(\mu} \bar{m}^{\nu)} \nabla_\mu l_\nu , \quad \theta_- \equiv -2m^{(\mu} \bar{m}^{\nu)} \nabla_\mu n_\nu , \quad (4.47)$$

which evaluated explicitly are

$$\theta_\pm = \frac{1}{2\sqrt{2}r\mathcal{U}} \left[r\mathcal{U}^2 \left(\frac{n_0}{t_0 H_0} + \frac{n_1}{t_1 H_1} \right) \pm \left(\frac{tn_0}{t_0 H_0} + \frac{(t+k_1 t_1)n_1}{t_1 H_1} \right) \right]. \quad (4.48)$$

While θ_\pm are not covariant quantities, their product is; comparing (4.45) and (4.48) it is straightforward to conclude that

$$\theta_+ \theta_- = \frac{2}{R^2} \nabla_\mu R \nabla^\mu R. \quad (4.49)$$

A metric sphere is said to be *trapped* or *untrapped* if $\theta_+ \theta_- > 0$ or $\theta_+ \theta_- < 0$ respectively, and to be *marginal* if $\theta_+ \theta_- = 0$. A trapping horizon is the closure of a hypersurface foliated by marginal surfaces, which means that it occurs when $\theta_+ \theta_- = 0$, or equivalently when $\nabla_\mu R$ becomes null.

It is possible to geometrically define on trapping horizons a local surface gravity k_l and the associated Hawking temperature $T_l = \frac{k_l}{2\pi}$ [13, 14],

$$\begin{aligned} k_l \equiv -\frac{1}{2} \tilde{\nabla}_\mu \tilde{\nabla}^\mu R|_{\text{TH}} = \\ -\frac{1}{8R} \left\{ \left(\frac{tn_0 t_1 H_1 + (t+k_1 t_1)n_1 t_0 H_0}{n_0 t_1 H_1 + n_1 t_0 H_0} \right)^2 \left[\left(\frac{n_0}{t_0 H_0} + \frac{n_1}{t_1 H_1} \right)^2 - \left(\frac{n_0}{t_0^2 H_0^2} + \frac{n_1}{t_1^2 H_1^2} \right) \right] \right. \\ \left. + \left(\frac{t^2 n_0}{t_0^2 H_0^2} + \frac{(t+k_1 t_1)^2 n_1}{t_1^2 H_1^2} \right) - 2 \left(\frac{tn_0}{t_0 H_0} + \frac{(t+k_1 t_1)n_1}{t_1 H_1} \right) \right\}, \end{aligned} \quad (4.50)$$

where $\tilde{\nabla}$ is the covariant derivative associated with the two dimensional metric normal to the spheres of symmetry. This surface gravity satisfies on the trapping horizons an identity similar to the usual relation for stationary black holes,

$$K^\mu \nabla_{[\nu} K_{\mu]} = k_l K_\nu , \quad (4.51)$$

where in place of a Killing vector we have the Kodama vector $K \equiv g^{-1}(*dR)$, with $*$ evaluated with respect to the normal metric. It should be noted however that an observer whose worldline is an integral curve of K does not measure the temperature T_l near the trapping horizons; the observed temperature is, to first order, $T = T_l C^{-1/2}$, with redshift factor $C = \nabla_\mu R \nabla^\mu R$.

Now if we take $k_1 = 0$ or equivalently consider the limit $r \rightarrow 0, t \rightarrow \infty$ with rt kept finite, (4.45) vanishes for $t^2 = r^2 H_0^{n_0} H_1^{n_1}$, i.e.,

$$t^2 r^2 = \left(\frac{tr}{t_0} + Q_0 \right)^{n_0} \left(\frac{tr}{t_1} + Q_1 \right)^{n_1}, \quad (4.52)$$

or $n_1 t_0 H_0 + n_0 t_1 H_1 = 0$ if $t_0 t_1 < 0$. However the latter solution doesn't correspond to a change of sign in $\theta_+ \theta_-$, so it doesn't identify a trapping horizon. Notice that the solutions of (4.52) have constant circumference radius R , and since the gradient of R becomes null there, the trapped horizons are null surfaces in the limit $r \rightarrow 0, t \rightarrow \infty$ with rt fixed. In this limit the geometric surface gravity (4.50) simplifies to

$$k_l = \frac{1}{8R} \left(\frac{tn_0}{t_0 H_0} + \frac{tn_1}{t_1 H_1} \right) \left(2 - \frac{tn_0}{t_0 H_0} - \frac{tn_1}{t_1 H_1} \right). \quad (4.53)$$

The identification of event horizons is a nontrivial task for dynamical black holes, since it requires the knowledge of the entire causal structure of the spacetime. Nevertheless, we can argue as in [46], and use the fact that the event horizon has to cover the trapped surfaces provided the outside region of a black hole behaves sufficiently well [59]. Since the spacetime (4.19) is indeed well-behaved for positive r (as long as we are outside the forbidden regions in fig. 4.1), and the trapping horizons contain null surfaces (4.52) in the limit $r \rightarrow 0, t \rightarrow \infty$, we shall examine in the following if these null surfaces are possible candidates for the black hole event horizon. As we said, the limit $r \rightarrow 0, t \rightarrow \infty$ with rt kept finite is equivalent to taking $k_1 = 0$. In this case, the metric is invariant under the transformation $t \rightarrow \alpha t, r \rightarrow r/\alpha$, and thus admits the Killing vector $\xi = t\partial_t - r\partial_r$, which is hypersurface orthogonal. Introducing the coordinates

$$T = \pm \log |t| + \int^{\mathcal{R}} \frac{g^2(\mathcal{R})}{\mathcal{R}f(\mathcal{R})} d\mathcal{R}, \quad \mathcal{R} = \frac{rt}{Q_0 t_0}, \quad (4.54)$$

$$f(\mathcal{R}) \equiv (Q_0 t_0)^2 \mathcal{R}^2 - g^2(\mathcal{R}), \quad g(\mathcal{R}) \equiv Q_0^2 (\mathcal{R} + 1)^{\frac{n_0}{2}} \left(\frac{t_0}{t_1} \mathcal{R} + \frac{Q_1}{Q_0} \right)^{\frac{n_1}{2}}, \quad (4.55)$$

such that $\xi = \partial_T$, the metric can be written in static form as

$$ds^2 = \frac{f(\mathcal{R})}{g(\mathcal{R})} dT^2 - (Q_0 t_0)^2 \frac{g(\mathcal{R})}{f(\mathcal{R})} d\mathcal{R}^2 - g(\mathcal{R}) d\Omega^2. \quad (4.56)$$

From (4.56) it is clear that there are Killing horizons where $f(\mathcal{R}) = 0$, that is, in (r, t) coordinates, $t^2 = r^2 H_0^{n_0} H_1^{n_1}$; thus the Killing horizons coincide with the trapping horizons (4.52). As the near-horizon geometry (4.56) enjoys the unexpected symmetry under

translations of the time coordinate T (which is not a symmetry of the original spacetime (4.19)), our solution (4.19) provides (like the ones in [45, 46]) a realization of asymptotic symmetry enhancement at the horizon of a dynamical black hole. The fact that the horizon does not grow, i.e., the ambient matter does not accrete onto the black hole, was conjectured in [52] to be related to fake supersymmetry.

Since the spacetime (4.56) is static, we can calculate the surface gravity on the horizons which is given by

$$k^2 = -\frac{1}{2}\nabla_\mu\xi_\nu\nabla^\mu\xi^\nu = \frac{1}{4}\left(\frac{n_0\mathcal{R}}{\mathcal{R}+1} + \frac{n_1\mathcal{R}}{\mathcal{R} + \frac{Q_1t_1}{Q_0t_0}} - 2\right)^2, \quad (4.57)$$

that depends only on \mathcal{R} (or equivalently on rt) and where \mathcal{R} is one root of $f(\mathcal{R}) = 0$. Note that, contrary to the asymptotically flat case, there is no preferred normalization for the Killing vector ξ here, and that the surface gravity is sensitive to this norm. Notice also that in general (4.57) is nonvanishing. A temperature different from zero would be in contradiction with supersymmetry, but not with fake supersymmetry: Following the explanation in [60], consider a black hole with temperature T . A spinor in the Euclidean section must then be antiperiodic under translation of the Euclidean time through a period $\beta = 1/T$. Supersymmetry implies the existence of a spinor field solving the Killing spinor equation, and this spinor must be periodic to give a regular solution. Both requirements are compatible only if the period is infinite, or equivalently when the temperature vanishes. Now, in fake supergravity, there are no fermions whose variation under a putative fake supersymmetry transformation is associated to the fake Killing spinor equation. The latter is just an auxiliary construction, which implies (under certain conditions) the second order field equations. Thus, the above contradiction for nonzero temperature does not arise.

Rewriting (4.53) in static coordinates,

$$k_l = \frac{1}{8\sqrt{|Q_0t_0\mathcal{R}|}}\left(\frac{n_0\mathcal{R}}{\mathcal{R}-1} + \frac{n_1\mathcal{R}}{\mathcal{R} - \frac{Q_1t_1}{Q_0t_0}}\right)\left(2 - \frac{n_0\mathcal{R}}{\mathcal{R}-1} - \frac{n_1\mathcal{R}}{\mathcal{R} - \frac{Q_1t_1}{Q_0t_0}}\right), \quad (4.58)$$

we see that it agrees with (4.57) up to a normalization factor constant over each Killing horizon. This is the same factor that ties the Kodama vector K to the Killing vector ξ on the horizons,

$$K|_{\text{KH}} = \pm \frac{1}{4\sqrt{|Q_0t_0\mathcal{R}|}}\left(\frac{n_0\mathcal{R}}{\mathcal{R}-1} + \frac{n_1\mathcal{R}}{\mathcal{R} - \frac{Q_1t_1}{Q_0t_0}}\right)\xi. \quad (4.59)$$

The horizon condition can be rewritten as

$$|\mathcal{R}| = a|\mathcal{R}+1|^{\frac{n_0}{2}}|\mathcal{R}+b|^{\frac{n_1}{2}}, \quad a \equiv \left|\frac{Q_0}{t_0}\right|\left|\frac{t_0}{t_1}\right|^{\frac{n_1}{2}}, \quad b \equiv \frac{Q_1t_1}{Q_0t_0}. \quad (4.60)$$

If $t_0t_1 > 0$ the accessible regions of spacetime are those with $\mathcal{R} > \max(-1, -b)$ and $\mathcal{R} < \min(-1, -b)$.

We see that for $b > 1$ there are always exactly two horizons for negative \mathcal{R} , one for $\mathcal{R} < -b$ and one for $-1 < \mathcal{R} < 0$; however only one of these is accessible since they are located in disconnected regions of the spacetime. For $a \geq 1/4$ one has $\mathcal{R} \leq a(\mathcal{R} + 1)^2 < a(\mathcal{R} + 1)^{n_0/2}(\mathcal{R} + b)^{n_1/2}$ for every positive \mathcal{R} and consequently there are no other horizons. On the other hand if $a \leq 1/(4b)$ there is an interval for which $\mathcal{R} \geq a(\mathcal{R} + b)^2 > a(\mathcal{R} + 1)^{n_0/2}(\mathcal{R} + b)^{n_1/2}$ and there are thus two distinct horizons for positive \mathcal{R} . For intermediate values of a there can be zero or two, possibly coincident, horizons for positive \mathcal{R} depending on the value of the parameters. $b = 1$ corresponds to the single-centered Kastor-Traschen solution, or Reissner-Nordström-de Sitter with mass equal to the charge, the extremal case corresponding to $a = 1/4$. We can then identify the three horizons in the $\mathcal{R} > -1$ region as respectively inner and outer black hole horizons and cosmological horizon.

For $b \leq -1$ there is always one horizon in the region $\mathcal{R} > -b$ and at least one, at most three horizons for $\mathcal{R} < -1$. In this case $\mathcal{R} = 0$ is not accessible.

For $b = 0$, corresponding to a black hole charged under only one of the gauge fields, there is a solution in $\mathcal{R} = 0$ which is not a horizon since it is coincident with a singularity; depending on the value of the parameters there can be zero, one or two horizons for $\mathcal{R} > 0$. In the region $\mathcal{R} < -1$ there is always a single horizon.

If $t_0 t_1 < 0$ the accessible region is given by the values of \mathcal{R} between -1 and $-b$. For $b > 1$ there can be zero, one or two, possibly coincident, horizons; for $b < -1$ there are always two horizons, one with negative and one with positive \mathcal{R} . For $b = 0$ there is again a solution in $\mathcal{R} = 0$ coincident with a singularity; depending on the value of the parameters there can be zero, one or two additional solutions corresponding to horizons.

With the choice of coordinates we made, the radial null geodesic equations simplify to

$$\ddot{T} + 2\Gamma^T_{T\mathcal{R}}\dot{T}\dot{\mathcal{R}} = 0, \quad \ddot{\mathcal{R}} = 0, \quad (4.61)$$

which means that \mathcal{R} is an affine parameter for the radial null geodesics and consequently all horizons and singularities are reached within a finite value of the affine parameter. From the null condition $d\mathcal{R} = \pm f(\mathcal{R})dT/(Q_0 t_0 g(\mathcal{R}))$ we obtain the expressions for the radial null geodesics in the near-horizon and near-singularity limits,

$$\begin{aligned} \mathcal{R} \sim \mathcal{R}_{\text{hor}} \quad &: \quad T = \pm \frac{1}{2k} \log |\mathcal{R} - \mathcal{R}_{\text{hor}}| + c_1, \\ \mathcal{R} \sim -1 \quad &: \quad T = \pm 2 \frac{Q_0}{t_0} \left(\frac{Q_1}{Q_0} - \frac{t_0}{t_1} \right)^{\frac{n_1}{2}} \frac{(\mathcal{R}+1)^{1+\frac{n_0}{2}}}{2+n_0} + c_2, \\ \mathcal{R} \sim -\frac{Q_1 t_1}{Q_0 t_0} \quad &: \quad T = \pm 2 \frac{Q_1}{t_1} \left(\frac{Q_0}{Q_1} - \frac{t_1}{t_0} \right)^{\frac{n_0}{2}} \frac{\left(\frac{Q_0 t_0}{Q_1 t_1} \mathcal{R} + 1 \right)^{1+\frac{n_1}{2}}}{2+n_1} + c_3, \end{aligned} \quad (4.62)$$

where c_i are constants and k is the surface gravity (4.57).

4.2 Alternative truncation

In section 4.1 we considered the truncation $\Im m(Z) = 0$; we could instead have taken $\Re e(Z) = 0$, but this choice is not consistent for every value of n with the prepotential we had there. Here we consider a slightly modified prepotential,

$$\mathcal{F}(\chi) = \frac{i^{n-1}}{4(1-n)} (\chi^0)^n (\chi^1)^{2-n}, \quad (4.63)$$

with $n \neq 0, 1, 2$, that leads to consistent results with the truncation $\Re e(Z) = 0$ (but not with $\Im m(Z) = 0$). The model (4.63) is of course related to the one of section 4.1 by a complex rescaling of the χ^Λ , and thus the truncations considered here and in the preceding section are actually two different truncations of the same model.

From (B.21), taking $|\chi^0| = 1$ we obtain the Kähler potential

$$e^{-\mathcal{K}} = \frac{i^n}{4(1-n)} Z^{1-n} [nZ + (2-n)\bar{Z}] + \text{c.c.}, \quad (4.64)$$

and, imposing $\Re e(Z) = 0$, the Kähler metric

$$\mathcal{G} = \partial_Z \partial_{\bar{Z}} \mathcal{K}|_{\Re e(Z)=0} = -\frac{n(2-n)}{4} \Im m(Z)^{-2} = \frac{n_0 n_1}{(n_1 - n_0)^2} e^{\frac{n_0 - n_1}{2} \phi}, \quad (4.65)$$

with $n_0 \equiv -\frac{2n}{1-n}$, $n_1 \equiv \frac{2(2-n)}{1-n} = 4 - n_0$, $\phi \equiv \frac{4}{n_1 - n_0} \log \Im m(Z)$.

From equation (B.31) one obtains the vectors' kinetic matrix

$$\mathcal{N} = -\frac{i}{8} \begin{pmatrix} n_0 e^{\frac{n_1}{2} \phi} & 0 \\ 0 & n_1 e^{\frac{n_0}{2} \phi} \end{pmatrix}, \quad (4.66)$$

while (3.5) leads to the scalar potential

$$V = \frac{1}{2} \left[\frac{n_0(n_0 - 1)}{t_0^2} e^{-\frac{n_1}{2} \phi} + \frac{n_1(n_1 - 1)}{t_1^2} e^{-\frac{n_0}{2} \phi} \right], \quad (4.67)$$

where we defined as before $t_\Lambda \equiv -\frac{n_\Lambda}{2gC_\Lambda}$.

Substituting in eq. (3.4) we have

$$\begin{aligned} e^{-1} \mathcal{L} = & R + \frac{n_0 n_1}{8} \partial_\mu \phi \partial^\mu \phi - \frac{n_0}{4} e^{\frac{n_1}{2} \phi} F_{\mu\nu}^0 F^{0\mu\nu} - \frac{n_1}{4} e^{\frac{n_0}{2} \phi} F_{\mu\nu}^1 F^{1\mu\nu} \\ & - \frac{1}{2} \left[\frac{n_0(n_0 - 1)}{t_0^2} e^{-\frac{n_1}{2} \phi} + \frac{n_1(n_1 - 1)}{t_1^2} e^{-\frac{n_0}{2} \phi} \right], \end{aligned} \quad (4.68)$$

which differs from the Lagrangian obtained in the previous section only by a sign in front of n_0 in the exponents and the absence of the cross term in the potential. To avoid ghost fields in the action we must restrict n_0 and n_1 to positive values, which corresponds to have in the prepotential either $n < 0$ or $n > 2$.

We can again, starting from the definitions (2.35) and imposing this time the constraint $\Re(Z) = 0$, write all the components of the sections \mathcal{R} and \mathcal{I} in terms of two components, namely \mathcal{I}^0 and \mathcal{I}^1 , and of the scalar field ϕ , obtaining

$$\begin{aligned}\mathcal{I}_0 &= -\frac{n_0}{8}e^\phi\mathcal{I}^1, & \mathcal{I}_1 &= \frac{n_1}{8}e^\phi\mathcal{I}^0, \\ \mathcal{R}^0 &= e^{\frac{n_0-n_1}{4}\phi}\mathcal{I}^1, & \mathcal{R}^1 &= -e^{\frac{n_1-n_0}{4}\phi}\mathcal{I}^0, \\ \mathcal{R}_0 &= \frac{n_0}{8}e^{\frac{n_1}{2}\phi}\mathcal{I}^0, & \mathcal{R}_1 &= \frac{n_1}{8}e^{\frac{n_0}{2}\phi}\mathcal{I}^1,\end{aligned}\tag{4.69}$$

as well as

$$\frac{1}{2|X|^2} = \langle \mathcal{R} | \mathcal{I} \rangle = \frac{1}{2} \left[e^{\frac{n_1}{2}\phi} (\mathcal{I}^0)^2 + e^{\frac{n_0}{2}\phi} (\mathcal{I}^1)^2 \right]. \tag{4.70}$$

From (4.69) and (3.60) we see that, since we exclude the case $C_0 = C_1 = 0, \mathcal{I}^1 = 0$ is equivalent to $C_0 = 0, \mathcal{I}^0 = 0$ is equivalent to $C_1 = 0$, and $C_1\mathcal{I}^1 = -\frac{n_1}{n_0}C_0\mathcal{I}^0$.

4.2.1 Construction of the solution

As before we take

$$V_m^x = \delta_m^x, \quad C_\Lambda \tilde{A}^\Lambda = C_\Lambda \mathcal{I}^\Lambda = 0. \tag{4.71}$$

Since $C_1\mathcal{I}^1 = -\frac{n_1}{n_0}C_0\mathcal{I}^0$, $C_\Lambda \mathcal{I}^\Lambda = 0$ with $n_0 \neq 2^1$ implies $C_0\mathcal{I}^0 = 0$. One has thus either $C_0 = \mathcal{I}^1 = 0$ or $C_1 = \mathcal{I}^0 = 0$. We will consider just the first case since the second can be obtained simply by exchanging 0 and 1 indices. We have thus

$$C_0 = 0, \quad \mathcal{I}^1 = \mathcal{I}_0 = \mathcal{R}^0 = \mathcal{R}_1 = 0, \tag{4.72}$$

and from $C_\Lambda \tilde{A}^\Lambda = 0$, taking into account that $C_1 \neq 0$,

$$\tilde{A}^1 = 0. \tag{4.73}$$

Eq. (3.59) yields

$$\tilde{F}_{mn}^0 = -\frac{1}{\sqrt{2}}\varepsilon_{mnp}\partial_p\mathcal{I}^0, \tag{4.74}$$

and from the Bianchi identity $d\tilde{F}^0 = 0$ we obtain

$$\partial_p\partial_p\mathcal{I}^0 = 0 \quad \Rightarrow \quad \mathcal{I}^0 = \sqrt{2}\mathcal{H}_0, \tag{4.75}$$

where \mathcal{H}_0 is a generic harmonic function of the spatial coordinates.

Using (4.72) and $C_\Lambda \tilde{A}^\Lambda = 0$, from eq. (3.62) we conclude as before $\tilde{\omega} = df$, where f is a generic function of the spatial coordinates. (3.61) implies then

$$\partial_p\partial_p\left(\tilde{\mathcal{I}}_1 + \frac{gC_1}{2\sqrt{2}}f\right) = 0 \quad \Rightarrow \quad \tilde{\mathcal{I}}_1 = \frac{n_1}{4\sqrt{2}}(f/t_1 + \mathcal{H}_1), \tag{4.76}$$

¹For $n_0 = 2$ ($n \rightarrow \pm\infty$) we could take both $C_0, C_1 \neq 0$ (equivalently, $\mathcal{I}^0, \mathcal{I}^1 \neq 0$); however this would lead to exactly the same solution we obtain here, with just a field redefinition.

where \mathcal{H}_1 is another harmonic function of the spatial coordinates. Using (3.60) and (4.69) one gets

$$\mathcal{I}_1 = \frac{n_1}{4\sqrt{2}}((\tau + f)/t_1 + \mathcal{H}_1), \quad e^\phi = \frac{(f + \tau)/t_1 + \mathcal{H}_1}{\mathcal{H}_0}, \quad (4.77)$$

and from (4.70) one computes

$$\frac{1}{2|X|^2} = \left(\frac{\tau + f}{t_1} + \mathcal{H}_1 \right)^{\frac{n_1}{2}} \mathcal{H}_0^{\frac{n_0}{2}}. \quad (4.78)$$

Eliminating f by introducing the new time coordinate $t = \tau + f$, the solution can be written as

$$ds^2 = \mathcal{U}^{-2} dt^2 - \mathcal{U}^2 d\vec{y}^2,$$

$$F^0 = -\frac{1}{2}\varepsilon_{mnp}\partial_p\mathcal{H}_0 dy^m \wedge dy^n, \quad A^1 = \left(\frac{t}{t_1} + \mathcal{H}_1 \right)^{-1} dt, \quad (4.79)$$

$$\phi = \ln \left(\frac{t/t_1 + \mathcal{H}_1}{\mathcal{H}_0} \right),$$

with

$$\mathcal{U} \equiv \left(\frac{t}{t_1} + \mathcal{H}_1 \right)^{\frac{n_1}{4}} \mathcal{H}_0^{\frac{n_0}{4}}. \quad (4.80)$$

This is, with the right choice for \mathcal{H}_0 and \mathcal{H}_1 , the spacetime found in [47] and discussed further in [48]; however in this case, instead of having two gauge fields in an electric configuration, one of them is magnetic due to the different sign in the exponent of its scalar coupling. In other words, one of the field strengths in the Gibbons-Maeda solution is dualized here.

4.3 Choice of base space

4.3.1 Flat space

The simplest solution of eq. (3.43) is three-dimensional flat space, with

$$V_m^x = \delta_m^x, \quad C_\Lambda \tilde{A}^\Lambda = C_\Lambda \mathcal{I}^\Lambda = 0. \quad (4.81)$$

With this choice for the base space we don't need to distinguish between $x, y, z \dots$ and lower m, n, p, \dots indices.

If $C_0 = C_1 = 0$, $C_\Lambda \mathcal{I}^\Lambda = 0$ is automatically satisfied and the section \mathcal{I} is time-independent. Using equation (3.59) and the Bianchi identity $d\tilde{F}^\Lambda = 0$ it can be seen that the \mathcal{I}^Λ must be harmonic,

$$\mathcal{I}^0 \equiv \sqrt{2}H^0, \quad \mathcal{I}^1 \equiv \sqrt{2}H^1. \quad (4.82)$$

Moreover, (3.61) implies that the \mathcal{I}_Λ are harmonic as well,

$$\mathcal{I}_0 \equiv \frac{H_0}{2\sqrt{2}}, \quad \mathcal{I}_1 \equiv \frac{H_1}{2\sqrt{2}}. \quad (4.83)$$

Equation (3.62) becomes

$$d\tilde{\omega} = \star_3 (H_0 dH^0 + H_1 dH^1 - H^0 dH_0 - H^1 dH_1). \quad (4.84)$$

If at least one of the C_Λ is nonzero, e.g. $C_1 \neq 0$, $C_\Lambda \mathcal{I}^\Lambda = 0$ implies $\mathcal{I}^1 = -\frac{C_0}{C_1} \mathcal{I}^0$. Then, (3.59) and the Bianchi identity $d\tilde{F}^0 = 0$ yield

$$\mathcal{I}^0 = \sqrt{2} H_{\text{im}}, \quad \mathcal{I}^1 = -\sqrt{2} \frac{C_0}{C_1} H_{\text{im}}, \quad (4.85)$$

where H_{im} is a time-independent harmonic function².

(3.61) together with (3.60) implies that the time-independent combination $\mathcal{I}_0 - \frac{C_0}{C_1} \mathcal{I}_1$ is harmonic. It proves convenient to express this defining

$$\mathcal{I}_0 \equiv \frac{C_0}{C_1} \left(\mathcal{I}_1 - \frac{1}{2\sqrt{2}} H_1 \right) + \frac{1}{2\sqrt{2}} H_0, \quad (4.86)$$

with H_0, H_1 harmonic functions independent of τ . Since there are no further constraints on $\tilde{\mathcal{I}}_1$, the \mathcal{I}_Λ can be written as

$$\mathcal{I}_1 = \frac{1}{2\sqrt{2}} \left(\frac{\tau}{t_1} + f \right), \quad \mathcal{I}_0 = \frac{1}{2\sqrt{2}} \left[\frac{\tau}{t_0} + H_0 + \frac{t_1}{t_0} (f - H_1) \right], \quad (4.87)$$

where $t_\Lambda \equiv -(gC_\Lambda)^{-1}$ and f is a generic function of the spatial coordinates.

Equation (3.62) becomes

$$d\tilde{\omega} = \star_3 \left[\left(H_0 - \frac{t_1}{t_0} H_1 \right) dH_{\text{im}} - H_{\text{im}} d \left(H_0 - \frac{t_1}{t_0} H_1 \right) \right], \quad (4.88)$$

and from (3.61) one gets

$$\partial_p \tilde{\omega}_p = t_1 \partial_p \partial_p f. \quad (4.89)$$

It is always possible to set f to zero with a shift in the time coordinate, $\tau = t - t_1 f + t_1 H_1$, and replacing $\tilde{\omega}$ by $\hat{\omega} = \tilde{\omega} - t_1 df + t_1 dH_1$, such that

$$\mathcal{I}_1 = \frac{1}{2\sqrt{2}} \left(\frac{t}{t_1} + H_1 \right), \quad \mathcal{I}_0 = \frac{1}{2\sqrt{2}} \left(\frac{t}{t_0} + H_0 \right),$$

$$d\hat{\omega} = \star_3 \left[\left(H_0 - \frac{t_1}{t_0} H_1 \right) dH_{\text{im}} - H_{\text{im}} d \left(H_0 - \frac{t_1}{t_0} H_1 \right) \right], \quad \partial_p \hat{\omega}_p = 0, \quad (4.90)$$

$$d\tau + \tilde{\omega} = dt + \hat{\omega}.$$

²Since H_{im} is related to the imaginary part of the scalar field Z , the label ‘im’ stands for ‘imaginary’.

An explicit choice for the harmonic functions, best expressed in Boyer-Lindquist coordinates (r, θ, ϕ) with $x + iy = \sqrt{r^2 + a^2} \sin \theta e^{i\varphi}$ and $z = r \cos \theta$, is

$$H = k + q \Re(V) + Q \Im(V), \quad (4.91)$$

with

$$V = \frac{1}{r - ia \cos \theta}. \quad (4.92)$$

If all the harmonics have this form, (4.90) is solved by

$$\hat{\omega} = \frac{1}{\Sigma} \left[-\frac{1}{2} a \sin^2 \theta \left(2 \widehat{kQ} r + \widehat{qQ} \right) + \widehat{kq} (r^2 + a^2) \cos \theta \right] d\varphi, \quad (4.93)$$

where

$$\Sigma = r^2 + a^2 \cos^2 \theta, \quad \widehat{xy} = \tilde{x} y_{\text{im}} - x_{\text{im}} \tilde{y}, \quad \tilde{x} = x_0 - \frac{t_1}{t_0} x_1. \quad (4.94)$$

This choice is also suitable to be generalized to the multi-centered case. To this end, define

$$V(\vec{x}, a) = \frac{1}{\sqrt{x^2 + y^2 + (z - ia)^2}}, \quad (4.95)$$

and consider harmonic functions of the form

$$H = k + \sum_I (q_I \Re(V_I) + Q_I \Im(V_I)), \quad (4.96)$$

with $V_I \equiv V(\vec{x} - \vec{x}_I, a_I)$, where \vec{x}_I is an arbitrary point in \mathbb{R}^3 and the parameter a_I in general depends on I . As long as the charges are taken to satisfy $q_{\text{im}I} = \alpha \tilde{q}_I$, $Q_{\text{im}I} = \alpha \tilde{Q}_I$ for every I , with α independent of I , (4.90) reduces to

$$d\hat{\omega} = (\alpha \tilde{k} - k_{\text{im}}) \star_3 d\tilde{H}, \quad (4.97)$$

where $\tilde{H} = H_0 - t_1 H_1 / t_0$. $\hat{\omega}$ is thus given by a sum over I of terms of the form (4.93), with $\widehat{qQ} = 0$. More explicitly, (4.93) with these charge constraints can be written in Cartesian coordinates and generalized to

$$\hat{\omega} = -2(\alpha \tilde{k} - k_{\text{im}}) \sum_I \left[\frac{\tilde{Q}_I \Re(V_I)}{|\vec{x} - \vec{x}_I|^2 + a_I^2 + 1/|V_I|^2} - \frac{\tilde{q}_I \Im(V_I)}{|\vec{x} - \vec{x}_I|^2 + a_I^2 - 1/|V_I|^2} \right] \cdot a_I [(x - x_I) dy - (y - y_I) dx]. \quad (4.98)$$

4.3.2 Three-sphere

Since Gauduchon-Tod spaces are actually conformal classes, it would be possible to take any conformally flat three-dimensional manifold as a base space simply by applying a conformal transformation to the quantities in 4.3.1 with appropriate conformal weights,

leading to a nonzero $C_\Lambda \tilde{A}^\Lambda$. This would however result in the same four-dimensional solutions expressed in different coordinates.

On the other hand there is a different Gauduchon-Tod structure that can be defined on the same conformal class, giving nonequivalent four-dimensional solutions. Start from a 3-sphere, with metric in the form

$$ds_3^2 = \frac{1}{4} \left[d\theta^2 + \sin^2 \theta d\varphi^2 + (d\psi + \cos \theta d\varphi)^2 \right], \quad (4.99)$$

and choose the dreibein

$$\begin{aligned} V^1 &= \frac{1}{2} (\sin \psi d\theta - \sin \theta \cos \psi d\varphi), \\ V^2 &= \frac{1}{2} (\cos \psi d\theta + \sin \theta \sin \psi d\varphi), \\ V^3 &= \frac{1}{2} (d\psi + \cos \theta d\varphi), \end{aligned} \quad (4.100)$$

that obeys

$$dV^x = -\varepsilon^{xyz} V^y \wedge V^z. \quad (4.101)$$

Thus, equation (3.43) is satisfied with

$$C_\Lambda \tilde{A}^\Lambda = 0, \quad C_\Lambda \mathcal{I}^\Lambda = -\frac{2\sqrt{2}}{g}. \quad (4.102)$$

A useful consequence of (4.101) is that with this frame choice we have for the associated spin connection $\omega_y^x{}_z - \omega_z^x{}_y = 2\varepsilon^{xyz}$, where $\omega_x^y{}_z \equiv V_x^\mu \omega_\mu^y{}_z$, as can easily be seen from Maurer-Cartan's first structure equation. This in particular implies that for a scalar function f on the sphere

$$\partial_x \partial_x f = \nabla_m \nabla^m f, \quad (4.103)$$

where ∇ is the Levi-Civita connection associated with the metric (4.99), and

$$[\partial_x, \partial_y] = 2\varepsilon^{xyz} \partial_z. \quad (4.104)$$

From (4.102) it is clear that the ungauged theory, $C_0 = C_1 = 0$, is incompatible with this GT-structure, hence at least one of the C_Λ must be nonzero. If $C_1 \neq 0$, (4.102) gives $\mathcal{I}^1 = 2\sqrt{2}t_1 - \frac{t_1}{t_0}\mathcal{I}^0$, where the t_Λ were defined in 4.3.1. The Bianchi identity $d\tilde{F}^0 = 0$, using (4.101), immediately implies $\varepsilon^{xyz} \partial_x \tilde{F}_{yz}^0 = 0$. Plugging in the expression for \tilde{F}_{xy}^0 given by (3.59) and using (4.103) one concludes that \mathcal{I}^0 must be harmonic on the sphere,

$$\mathcal{I}^0 = \sqrt{2}H_{\text{im}}, \quad \mathcal{I}^1 = \sqrt{2} \left(2t_1 - \frac{t_1}{t_0}H_{\text{im}} \right). \quad (4.105)$$

Equations (3.60) and (3.61) again imply that the combination $\mathcal{I}_0 - \frac{t_1}{t_0}\mathcal{I}_1$ is harmonic on the base space,

$$\mathcal{I}_0 = \frac{t_1}{t_0} \left(\mathcal{I}_1 - \frac{1}{2\sqrt{2}} H_1 \right) + \frac{1}{2\sqrt{2}} H_0, \quad (4.106)$$

while no additional constraint is imposed on $\tilde{\mathcal{I}}_1$, so one has

$$\mathcal{I}_1 = \frac{1}{2\sqrt{2}} \left(\frac{\tau}{t_1} + f \right), \quad \mathcal{I}_0 = \frac{1}{2\sqrt{2}} \left(\frac{\tau}{t_0} + H_0 + \frac{t_1}{t_0} (f - H_1) \right), \quad (4.107)$$

where a generic function f on S^3 was introduced. (3.62) becomes

$$d\tilde{\omega} = \star_3 \left[\left(H_0 - \frac{t_1}{t_0} H_1 \right) dH_{\text{im}} - H_{\text{im}} d \left(H_0 - \frac{t_1}{t_0} H_1 \right) - 2t_1 df + 2\tilde{\omega} \right], \quad (4.108)$$

with $\partial_x \tilde{\omega}_x = t_1 \partial_x \partial_x f$ due to (3.61). Setting as before $f = 0$ by taking $\tau = t - t_1 f + t_1 H_1$ and $\tilde{\omega} = \hat{\omega} + t_1 df - t_1 dH_1$, one gets

$$\mathcal{I}_0 = \frac{1}{2\sqrt{2}} \left(\frac{t}{t_0} + H_0 \right), \quad \mathcal{I}_1 = \frac{1}{2\sqrt{2}} \left(\frac{t}{t_1} + H_1 \right), \quad (4.109)$$

and $\hat{\omega}$ satisfies

$$d\hat{\omega} = \star_3 \left[\tilde{H} dH_{\text{im}} - H_{\text{im}} d\tilde{H} - 2t_1 dH_1 + 2\hat{\omega} \right], \quad \partial_x \hat{\omega}_x = \nabla^m \hat{\omega}_m = 0, \quad (4.110)$$

with $\tilde{H} \equiv H_0 - \frac{t_1}{t_0} H_1$. If the harmonics are chosen such as to satisfy $dH_{\text{im}} \wedge d\tilde{H} = 0$, the simplest solution to these equations is $\hat{\omega} = \frac{1}{2} H_{\text{im}} d\tilde{H} - \frac{1}{2} \tilde{H} dH_{\text{im}} + t_1 dH_1$, with $d\hat{\omega} = 0$, and all other solutions can be obtained by adding arbitrary solutions of $d\omega - 2\star_3 \omega = 0$, which implies $\nabla^m \omega_m = 0$; these are clearly independent of the choice of harmonic functions.

To make an explicit choice for $\hat{\omega}$ and the harmonics it is convenient to work with the usual hyperspherical coordinates,

$$ds_{S^3}^2 = d\Psi^2 + \sin^2 \Psi (d\Theta^2 + \sin^2 \Theta d\Phi^2). \quad (4.111)$$

In these coordinates the simplest nontrivial choice of harmonic function on S^3 is

$$H = k + q \frac{\cos \Psi}{\sin \Psi}, \quad (4.112)$$

which is singular in the points $\Psi = 0, \pi$. In a neighbourhood of the singularities the metric on S^3 is well approximated by the flat metric in spherical coordinates with Ψ playing the role of a radial coordinate, and $H \sim k + \frac{q}{\Psi}$. If all the harmonics are chosen to be of the form (4.112), the minimal $\hat{\omega}$ becomes

$$\hat{\omega} = \frac{1}{2} \frac{\tilde{k} q_i - k_i \tilde{q} - 2q_1 t_1}{\sin^2 \Psi} d\Psi, \quad (4.113)$$

which is the differential of a harmonic function and as such can be set to zero by a shift in the time coordinate and a redefinition of the harmonics H_0 and H_1 . This is equivalent to take $\hat{\omega} = 0$ from the beginning by imposing the constraint

$$\tilde{k} q_i - k_i \tilde{q} - 2q_1 t_1 = 0. \quad (4.114)$$

The equation $d\omega = 2 \star_3 \omega$, together with (4.101) and (4.104), implies $\partial_x \partial_x \omega_y = -8\omega_y$, which means that the components of ω with respect to the dreibein V^x are spherical harmonics on S^3 with eigenvalue $1 - n^2 = -8$. Using the well-known expressions for these spherical harmonics and rewriting the one-forms V^x in the coordinates (4.111) it is possible to obtain the most general solution for ω which is regular on the three-sphere. The metric (4.99) is obtained by considering S^3 embedded in \mathbb{C}^2 , $|z_1|^2 + |z_2|^2 = 1$, and taking the parametrization

$$z_1 = \cos \frac{\theta}{2} e^{\frac{i}{2}(\varphi+\psi)}, \quad z_2 = \sin \frac{\theta}{2} e^{\frac{i}{2}(\varphi-\psi)}. \quad (4.115)$$

Comparing this with the usual parametrization for S^3 in \mathbb{R}^4 one obtains in the coordinates (4.111) the expressions

$$\begin{aligned} V^1 &= -\sin \Theta \sin \Phi d\Psi + \sin \Psi (\sin \Psi \cos \Phi - \cos \Psi \cos \Theta \sin \Phi) d\Theta \\ &\quad - \sin \Psi \sin \Theta (\cos \Psi \cos \Phi + \sin \Psi \cos \Theta \sin \Phi) d\Phi, \\ V^2 &= \sin \Theta \cos \Phi d\Psi + \sin \Psi (\sin \Psi \sin \Phi + \cos \Psi \cos \Theta \cos \Phi) d\Theta \\ &\quad - \sin \Psi \sin \Theta (\cos \Psi \sin \Phi - \sin \Psi \cos \Theta \cos \Phi) d\Phi, \\ V^3 &= \cos \Theta d\Psi - \sin \Psi \cos \Psi \sin \Theta d\Theta - \sin^2 \Psi \sin^2 \Theta d\Phi, \end{aligned} \quad (4.116)$$

and the most general regular ω is

$$\begin{aligned} \omega &= (a \cos \Phi - b \sin \Phi) (\sin \Theta d\Psi + \sin \Psi \cos \Psi \cos \Theta d\Theta - \sin^2 \Psi \sin \Theta \cos \Theta d\Phi) \\ &\quad - \sin \Psi (a \sin \Phi + b \cos \Phi) (\sin \Psi d\Theta + \cos \Psi \sin \Theta d\Phi) \\ &\quad - c (\cos \Theta d\Psi - \sin \Psi \cos \Psi \sin \Theta d\Theta + \sin^2 \Psi \sin^2 \Theta d\Phi), \end{aligned} \quad (4.117)$$

where a , b , and c are constants.

It is also possible to construct multicentered solutions by taking sums of harmonic functions with singularities in arbitrary points on the 3-sphere. Given the standard embedding of S^3 in \mathbb{R}^4 , the harmonic function $\frac{\cos \Psi}{\sin \Psi}$ can be written as

$$h = \frac{x_1}{\sqrt{1 - x_1^2}}, \quad (4.118)$$

and the analogous harmonic function with singularities in any couple of antipodal points can be simply obtained by a rotation in \mathbb{R}^4 sending the point $(1, 0, 0, 0)$, corresponding

to $\psi = 0$, in one of the new points. However in this case one has in general $d\hat{\omega} \neq 0$, and in order to reinstate $d\hat{\omega} = 0$ while keeping the possibility of having an arbitrary number of black holes in arbitrary positions and with independent charges one has to impose $q_{\text{im}} = \alpha \tilde{q}$ for each of them, where α is a proportionality constant.

4.3.3 Berger sphere

A more general Gauduchon-Tod space can be defined starting from the Berger sphere [61], which is a squashed S^3 with metric

$$ds_3^2 = d\theta^2 + \sin^2 \theta d\varphi^2 + \cos^2 \mu (d\psi + \cos \theta d\varphi)^2. \quad (4.119)$$

Given the well-known expressions for the left-invariant 1-forms

$$\sigma_1^L = \sin \psi d\theta - \sin \theta \cos \psi d\varphi, \quad \sigma_2^L = \cos \psi d\theta + \sin \theta \sin \psi d\varphi, \quad \sigma_3^L = d\psi + \cos \theta d\varphi,$$

and for the right-invariant 1-forms

$$\sigma_1^R = \sin \varphi d\theta - \sin \theta \cos \varphi d\psi, \quad \sigma_2^R = \cos \varphi d\theta + \sin \theta \sin \varphi d\psi, \quad \sigma_3^R = d\varphi + \cos \theta d\psi,$$

one can define the dreibein [62]

$$\begin{aligned} V^1 &= \cos \mu \sigma_1^R \pm \sin \mu (\cos \theta \sigma_2^R - \sin \theta \sin \varphi \sigma_3^R), \\ V^2 &= \cos \mu \sigma_2^R \mp \sin \mu (\cos \theta \sigma_1^R + \sin \theta \cos \varphi \sigma_3^R), \\ V^3 &= \cos \mu \sigma_3^R \pm \sin \mu \sin \theta (\sin \varphi \sigma_1^R + \cos \varphi \sigma_2^R), \end{aligned} \quad (4.120)$$

that satisfies

$$dV^x = \pm \sin \mu \cos \mu \sigma_3^L \wedge V^x - \frac{\cos \mu}{2} \varepsilon^{xyz} V^y \wedge V^z, \quad (4.121)$$

so that equation (3.43) is satisfied with

$$C_\Lambda \tilde{A}^\Lambda = \pm \frac{\sin \mu \cos \mu}{g} \sigma_3^L, \quad C_\Lambda \mathcal{I}^\Lambda = -\frac{\sqrt{2}}{g} \cos \mu. \quad (4.122)$$

Using Maurer-Cartan's first structure equation it is possible to see that for a scalar function on the Berger sphere

$$\partial_x \partial_x f \pm 2 \sin \mu \cos \mu (\sigma_3^L)_x \partial_x f = \nabla_m \nabla^m f. \quad (4.123)$$

Again at least one of the C_Λ must be nonzero. If we assume $C_1 \neq 0$, (4.122) yields $\mathcal{I}^1 = \sqrt{2} t_1 \cos \mu - \frac{t_1}{t_0} \mathcal{I}^0$, where the t_Λ are defined as before.

The Bianchi identity $d\tilde{F}^\Lambda = 0$, using (4.121), implies

$$\varepsilon^{xyz} (\partial_x \pm 2 \sin \mu \cos \mu \sigma_3^L_x) \tilde{F}_{yz}^\Lambda = 0.$$

Substituting the expression for F_{xy}^Λ given by (3.59) and using (4.123) one gets for $K_{\text{im}} \equiv \frac{1}{\sqrt{2}}\mathcal{I}^0$:

$$\nabla_m \left[\nabla^m \pm \sin \mu \cos \mu \sigma_3^L \right] K_{\text{im}} = \left[\nabla^m \pm \sin \mu \cos \mu \sigma_3^L \right] \nabla_m K_{\text{im}} = 0. \quad (4.124)$$

Eqns. (3.60) and (3.61) imply that the combination $\tilde{K} \equiv 2\sqrt{2}(\mathcal{I}_0 - \frac{t_1}{t_0}\mathcal{I}_1)$ satisfies

$$(\nabla_m \nabla^m - \sin^2 \mu) \tilde{K} = 0, \quad (4.125)$$

while no additional constraint is imposed on $\tilde{\mathcal{I}}_1$, so one has

$$\mathcal{I}_1 = \frac{1}{2\sqrt{2}} \left(\frac{\tau}{t_1} + f \right), \quad \mathcal{I}_0 = \frac{1}{2\sqrt{2}} \left(\frac{\tau}{t_0} + \tilde{K} + \frac{t_1}{t_0} f \right), \quad (4.126)$$

where a generic function $f(\theta, \varphi, \psi)$ was introduced. (3.62) becomes

$$d\tilde{\omega} \pm \sin \mu \cos \mu \sigma_3^L \wedge \tilde{\omega} = \star_3 \left[\tilde{K} dK_{\text{im}} - K_{\text{im}} d\tilde{K} - t_1 \cos \mu df + \cos \mu \tilde{\omega} \right], \quad (4.127)$$

and from (3.61) we get

$$\nabla_m \tilde{\omega}^m \mp \sin \mu \cos \mu \sigma_3^L \tilde{\omega}^m = t_1 (\nabla_m \nabla^m - \sin^2 \mu) f. \quad (4.128)$$

It is possible to set $f = 0$ by taking $\tau = t - t_1 f + t_1 K_1$ and $\tilde{\omega} = \hat{\omega} + t_1 d(f - K_1) \pm \sin \mu \cos \mu \sigma_3^L t_1 (f - K_1)$, where $K_1(\theta, \varphi, \psi)$ satisfies (4.125). In this way

$$\mathcal{I}_0 = \frac{1}{2\sqrt{2}} \left(\frac{t}{t_0} + K_0 \right), \quad \mathcal{I}_1 = \frac{1}{2\sqrt{2}} \left(\frac{t}{t_1} + K_1 \right), \quad (4.129)$$

with $K_0 \equiv \tilde{K} + \frac{t_1}{t_0} K_1$, and $\hat{\omega}$ satisfies

$$d\hat{\omega} \pm \sin \mu \cos \mu \sigma_3^L \wedge \hat{\omega} = \star_3 \left[\tilde{K} dK_{\text{im}} - K_{\text{im}} d\tilde{K} - t_1 \cos \mu dK_1 + \cos \mu \hat{\omega} \right],$$

$$\nabla^m \hat{\omega}_m \mp \sin \mu \cos \mu \sigma_3^L \hat{\omega}^m = 0. \quad (4.130)$$

There is no obvious way of finding solutions to the eqns. (4.124) and (4.125) that in the limit $\mu \rightarrow 0$ reduce to harmonic functions of the form given in 4.3.2, which is what one would expect for black hole solutions. It is however possible to consider simple solutions given by the trivial choices

$$K_0 = K_1 = 0, \quad K_{\text{im}} = k_{\text{im}}, \quad \hat{\omega} = 0, \quad (4.131)$$

with k_{im} constant.

4.4 The $\mathcal{F}(\chi) = -\frac{i}{4}\chi^0\chi^1$ model

Given this prepotential, from (B.21) we can derive the Kähler potential

$$e^{-\mathcal{K}} = \Re \mathfrak{e}(Z), \quad (4.132)$$

where we fixed $|\chi^0| = 1$. The Kähler metric is then

$$\mathcal{G} = \partial_Z \partial_{\bar{Z}} \mathcal{K} = \frac{1}{4} \Re(Z)^{-2}. \quad (4.133)$$

From equation (B.31) one obtains

$$\mathcal{N} = -\frac{i}{4} \begin{pmatrix} Z & 0 \\ 0 & \frac{1}{\bar{Z}} \end{pmatrix}, \quad (4.134)$$

and for the scalar potential (3.5) one gets

$$V = g^2 \left[\frac{C_0^2}{\Re(Z)} + 4C_0 C_1 + \frac{C_1^2}{\Re(1/Z)} \right]. \quad (4.135)$$

(2.35) leads to

$$\mathcal{R}^0 = -4\mathcal{I}_1, \quad \mathcal{R}^1 = -4\mathcal{I}_0, \quad \mathcal{R}_0 = \frac{1}{4}\mathcal{I}^1, \quad \mathcal{R}_1 = \frac{1}{4}\mathcal{I}^0, \quad (4.136)$$

as well as

$$\frac{1}{2|X|^2} = \langle \mathcal{R} | \mathcal{I} \rangle = \frac{1}{2} \mathcal{I}^0 \mathcal{I}^1 + 8 \mathcal{I}_0 \mathcal{I}_1. \quad (4.137)$$

4.4.1 Flat base space

Using the results of section 4.3.1, one gets in the ungauged case from (4.137)

$$\frac{1}{2|X|^2} = H^0 H^1 + H_0 H_1, \quad (4.138)$$

and the solution takes the well-known form [63]

$$ds^2 = 2|X|^2 (d\tau + \tilde{\omega})^2 - \frac{1}{2|X|^2} d\tilde{y}^2, \quad Z = \frac{H_0 - iH^1}{H_1 - iH^0}, \quad (4.139)$$

$$F^0 = d(2|X|^2 H_1 (d\tau + \tilde{\omega})) - \star_3 dH^0, \quad F^1 = d(2|X|^2 H_0 (d\tau + \tilde{\omega})) - \star_3 dH^1, \quad (4.140)$$

with $\tilde{\omega}$ satisfying (4.84). In the gauged case the solution can be written as

$$ds^2 = 2|X|^2 (dt + \hat{\omega})^2 - \frac{1}{2|X|^2} d\tilde{y}^2, \quad Z = \frac{t/t_0 + H_0 + it_1/t_0 H_{\text{im}}}{t/t_1 + H_1 - iH_{\text{im}}}, \quad (4.141)$$

$$F^0 = d \left[2|X|^2 \left(\frac{t}{t_1} + H_1 \right) (dt + \hat{\omega}) \right] - \star_3 dH_{\text{im}},$$

$$F^1 = d \left[2|X|^2 \left(\frac{t}{t_0} + H_0 \right) (dt + \hat{\omega}) \right] + \frac{t_1}{t_0} \star_3 dH_{\text{im}},$$

where

$$\frac{1}{2|X|^2} = \left(\frac{t}{t_0} + H_0\right) \left(\frac{t}{t_1} + H_1\right) - \frac{t_1}{t_0} H_{\text{im}}^2 \quad (4.142)$$

and $\hat{\omega} \equiv \tilde{\omega} - t_1 df + t_1 dH_1$ satisfies eq. (4.90).

Both solutions can also be rewritten in terms of two complex harmonic functions \mathcal{H}_Λ as follows:

$$ds^2 = \frac{1}{\Re(\mathcal{H}_0 \bar{\mathcal{H}}_1)} (dt + \omega)^2 - \Re(\mathcal{H}_0 \bar{\mathcal{H}}_1) d\vec{y}^2, \quad Z = \frac{\mathcal{H}_0}{\mathcal{H}_1}, \quad (4.143)$$

$$F^0 = d \left[\frac{\Re(\mathcal{H}_1)}{\Re(\mathcal{H}_0 \bar{\mathcal{H}}_1)} (dt + \omega) \right] + \star_3 d\Im(\mathcal{H}_1),$$

$$F^1 = d \left[\frac{\Re(\mathcal{H}_0)}{\Re(\mathcal{H}_0 \bar{\mathcal{H}}_1)} (dt + \omega) \right] + \star_3 d\Im(\mathcal{H}_0),$$

where ω is time-independent and satisfies

$$d\omega = \star_3 \Im(\mathcal{H}_0 d\bar{\mathcal{H}}_1 + \mathcal{H}_1 d\bar{\mathcal{H}}_0). \quad (4.144)$$

In the ungauged case, the only additional constraint on the complex harmonics is that they are independent of time. In terms of the harmonics defined above they are given by

$$\mathcal{H}_0 = H_0 - iH^1, \quad \mathcal{H}_1 = H_1 - iH^0. \quad (4.145)$$

In the gauged case the time dependence of the harmonics is completely determined by $\partial_t \mathcal{H}_\Lambda = 1/t_\Lambda^3$. In addition they must satisfy $\Im(\mathcal{H}_0) = -\frac{t_1}{t_0} \Im(\mathcal{H}_1)$, and thus

$$\mathcal{H}_0 = \frac{t}{t_0} + H_0 + i\frac{t_1}{t_0} H_{\text{im}}, \quad \mathcal{H}_1 = \frac{t}{t_1} + H_1 - iH_{\text{im}}. \quad (4.146)$$

In this case there is also the additional constraint $\partial_p \omega_p = 0$.

Notice that (4.143) reduces to the Israel-Wilson-Perjés [56, 55] solution for $\mathcal{H}_0 = \mathcal{H}_1$. This means in particular that we can recover the Kerr-Newman solution with mass equal to the charge by taking

$$\mathcal{H}_0 = \mathcal{H}_1 = 1 + qV \equiv 1 + \frac{q}{r - ia \cos \theta}, \quad \omega = \frac{qa \sin^2 \theta (2r + q)}{r^2 + a^2 \cos^2 \theta} d\varphi, \quad (4.147)$$

expressed in Boyer-Lindquist coordinates.

³Here one recognizes the substitution principle originally put forward by Behrndt and Cvetič in [57], which amounts to adding a linear time dependence to the harmonic functions in a supersymmetric black hole of $N = 2, d = 4$ supergravity.

This construction suggests the more general form (4.91) for the harmonics, with ω given by (4.93). With these choices the gauged solution explicitly reads

$$ds^2 = \frac{\Sigma^2}{\Delta} dt^2 + \frac{\Sigma}{\Delta} \left[-a \sin^2 \theta \left(2 \widehat{kQ} r + \widehat{qQ} \right) + 2 \widehat{kq} (r^2 + a^2) \cos \theta \right] dt d\varphi$$

$$- \frac{\Delta}{\Sigma(r^2 + a^2)} dr^2 - \frac{\Delta}{\Sigma} d\theta^2 \quad (4.148)$$

$$+ \left[\frac{1}{4\Delta} \left[-a \sin^2 \theta \left(2 \widehat{kQ} r + \widehat{qQ} \right) + 2 \widehat{kq} (r^2 + a^2) \cos \theta \right]^2 - \frac{\Delta}{\Sigma^2} (r^2 + a^2) \sin^2 \theta \right] d\varphi^2,$$

$$A^0 = \frac{\Sigma}{\Delta} (\Sigma(t/t_1 + k_1) + q_1 r + Q_1 a \cos \theta) dt$$

$$- \frac{1}{2} \left[\frac{\Sigma}{\Delta} (\Sigma(t/t_1 + k_1) + q_1 r + Q_1 a \cos \theta) \left(2 \widehat{kQ} r + \widehat{qQ} \right) - 2 Q_i r \right] \frac{a \sin^2 \theta}{\Sigma} d\varphi$$

$$+ \left[\frac{\Sigma}{\Delta} (\Sigma(t/t_1 + k_1) + q_1 r + Q_1 a \cos \theta) \widehat{kq} - q_i \right] \frac{(r^2 + a^2) \cos \theta}{\Sigma} d\varphi, \quad (4.149)$$

$$A^1 = \frac{\Sigma}{\Delta} (\Sigma(t/t_0 + k_0) + q_0 r + Q_0 a \cos \theta) dt$$

$$- \frac{1}{2} \left[\frac{\Sigma}{\Delta} (\Sigma(t/t_0 + k_0) + q_0 r + Q_0 a \cos \theta) \left(2 \widehat{kQ} r + \widehat{qQ} \right) + 2 \frac{t_1}{t_0} Q_i r \right] \frac{a \sin^2 \theta}{\Sigma} d\varphi$$

$$+ \left[\frac{\Sigma}{\Delta} (\Sigma(t/t_0 + k_0) + q_0 r + Q_0 a \cos \theta) \widehat{kq} + \frac{t_1}{t_0} q_i \right] \frac{(r^2 + a^2) \cos \theta}{\Sigma} d\varphi, \quad (4.150)$$

$$Z = \frac{\Sigma(t/t_0 + k_0) + q_0 r + Q_0 a \cos \theta + it_1/t_0 (\Sigma k_i + q_i r + Q_i a \cos \theta)}{\Sigma(t/t_1 + k_1) + q_1 r + Q_1 a \cos \theta - i(\Sigma k_i + q_i r + Q_i a \cos \theta)}, \quad (4.151)$$

where

$$\Delta = \left[\Sigma \left(\frac{t}{t_0} + k_0 \right) + q_0 r + Q_0 a \cos \theta \right] \left[\Sigma \left(\frac{t}{t_1} + k_1 \right) + q_1 r + Q_1 a \cos \theta \right]$$

$$- \frac{t_1}{t_0} [\Sigma k_i + q_i r + Q_i a \cos \theta]^2, \quad (4.152)$$

$$\Sigma = r^2 + a^2 \cos^2 \theta, \quad \widehat{xy} = \tilde{x} y_i - x_i \tilde{y}, \quad \tilde{x} = x_0 - \frac{t_1}{t_0} x_1. \quad (4.153)$$

It can be seen from these expressions that the constant \widehat{kq} in ω represents essentially a NUT charge.

4.4.2 Spherical base space

Using the results of 4.3.2, the complete solution can be written in terms of harmonic functions H_{im}, H_0, H_1 on S^3 and a time-independent one-form $\hat{\omega}$ as

$$\begin{aligned} ds^2 &= 2|X|^2(dt + \hat{\omega})^2 - \frac{1}{2|X|^2}ds_{S^3}^2, \\ F^0 &= d\left[2|X|^2\left(\frac{t}{t_1} + H_1\right)(dt + \hat{\omega})\right] - \star_3 dH_{\text{im}}, \\ F^1 &= d\left[2|X|^2\left(\frac{t}{t_0} + H_0\right)(dt + \hat{\omega})\right] + \frac{t_1}{t_0}\star_3 dH_{\text{im}}, \\ Z &= \frac{t/t_0 + H_0 - i2t_1 + it_1H_{\text{im}}/t_0}{t/t_1 + H_1 - iH_{\text{im}}}, \end{aligned} \quad (4.154)$$

where

$$\frac{1}{2|X|^2} = \left(\frac{t}{t_0} + H_0\right)\left(\frac{t}{t_1} + H_1\right) + H_{\text{im}}\left(2t_1 - \frac{t_1}{t_0}H_{\text{im}}\right), \quad (4.155)$$

and $\hat{\omega}$ satisfies (4.110). In particular the harmonics can be taken to be of the form (4.112), with $\hat{\omega}$ as in 4.3.2. The curvature scalars $R, R_{\mu\nu}R^{\mu\nu}$ and $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ are singular for $\frac{1}{2|X|^2} = 0$, but not in the points $\psi = 0, \pi$ unless $q_0q_1 = \frac{t_1}{t_0}q_i^2$.

Note finally that the scalar field (4.154) assumes the constant value $Z = t_1/t_0$ (where the potential (4.135) has an extremum⁴) if $t_0H_0 = t_1H_1$ and $H_i = t_0$. In this case, $\tilde{H} = 0$ and $\hat{\omega} = t_1dH_1$. If we take $\omega = 0$ and define a new time coordinate τ by $t + t_1H_1 = t_0t_1 \sinh \tau$, the metric becomes

$$ds^2 = t_0t_1 \left[d\tau^2 - \cosh^2 \tau ds_{S^3}^2 \right], \quad (4.156)$$

and the gauge field strengths F^Λ vanish, so that the solution is dS_4 . For $\omega \neq 0$, one gets a deformation of dS_4 with nonzero F^Λ . This is what happens also in the “asymptotic” limit $\Psi \sim \pi/2$ of the solution with the explicit choice (4.112) and with $t_0k_0 = t_1k_1, k_i = t_0$.

⁴We assume $t_1/t_0 > 0$.

4.4.3 Berger sphere

For this base space, the results of section 4.3.3 imply that the complete solution can be written in the form

$$\begin{aligned}
 ds^2 &= 2|X|^2 (dt \pm \sin \mu \cos \mu \sigma_3^L t + \hat{\omega})^2 - \frac{1}{2|X|^2} ds_3^2, \\
 F^0 &= d \left[2|X|^2 \left(\frac{t}{t_1} + K_1 \right) (dt \pm \sin \mu \cos \mu \sigma_3^L t + \hat{\omega}) \right] \\
 &\quad - \star_3 [dK_{\text{im}} \pm \sin \mu \cos \mu \sigma_3^L K_{\text{im}}], \\
 F^1 &= d \left[2|X|^2 \left(\frac{t}{t_0} + K_0 \right) (dt \pm \sin \mu \cos \mu \sigma_3^L t + \hat{\omega}) \right] \\
 &\quad + \frac{t_1}{t_0} \star_3 [dK_{\text{im}} \pm \sin \mu \cos \mu \sigma_3^L (K_{\text{im}} - t_0 \cos \mu)], \\
 Z &= \frac{t/t_0 + K_0 - it_1 \cos \mu + it_1 K_{\text{im}}/t_0}{t/t_1 + K_1 - iK_{\text{im}}}, \tag{4.157}
 \end{aligned}$$

where

$$\frac{1}{2|X|^2} = \left(\frac{t}{t_0} + K_0 \right) \left(\frac{t}{t_1} + K_1 \right) + K_{\text{im}} \left(t_1 \cos \mu - \frac{t_1}{t_0} K_{\text{im}} \right), \tag{4.158}$$

the functions K_0 and K_1 satisfy (4.125), K_{im} satisfies (4.124), and the time-independent one-form $\hat{\omega}$ is a solution of (4.130).

With the trivial choices (4.131) the solution reduces to

$$\begin{aligned}
 ds^2 &= \frac{t_0 t_1}{t^2 + \alpha_0 \alpha_1} (dt \pm \sin \mu \cos \mu \sigma_3^L t)^2 - \frac{t^2 + \alpha_0 \alpha_1}{t_0 t_1} ds_3^2, \\
 A^\Lambda &= \pm t_\Lambda \sin \mu \left(\frac{t^2 \cos \mu}{t^2 + \alpha_0 \alpha_1} - \frac{\alpha_\Lambda}{t_0 t_1} \right) \sigma_3^L, \quad Z = \frac{t_1}{t_0} \frac{t - i\alpha_1}{t - i\alpha_0}, \tag{4.159}
 \end{aligned}$$

with

$$\alpha_0 = t_1 k_{\text{im}}, \quad \alpha_1 = t_0 t_1 \cos \mu - \alpha_0 = t_0 t_1 \cos \mu - t_1 k_{\text{im}}. \tag{4.160}$$

Imposing $\alpha_0 = \alpha_1$, the scalar becomes constant and one obtains a solution of Einstein-Maxwell-de Sitter theory already found by Meessen [64]. This can be seen as a deformation of dS_4 , which is recovered when $\mu = 0$.

4.5 The $\mathcal{F}(\chi) = -\frac{1}{8} \frac{(\chi^1)^3}{\chi^0}$ model

Using (B.21) this prepotential leads to the Kähler potential

$$e^{-\mathcal{K}} = \Im(Z)^3, \quad (4.161)$$

where we took $|\chi^0| = 1$, and to the Kähler metric

$$\mathcal{G} = \partial_Z \partial_{\bar{Z}} \mathcal{K} = \frac{3}{4} \Im(Z)^{-2}. \quad (4.162)$$

The vectors' kinetic matrix is, according to equation (B.31),

$$\mathcal{N} = \frac{1}{4} \begin{pmatrix} -Z \Re(Z)^2 - \frac{i}{2} |Z|^2 \Im(Z) & \frac{3}{2} Z \Re(Z) \\ \frac{3}{2} Z \Re(Z) & -3Z + i \frac{3}{2} \Im(Z) \end{pmatrix}, \quad (4.163)$$

and from (3.5) one gets the scalar potential

$$V = \frac{4}{3} g^2 \frac{C_1^2}{\Im(Z)}. \quad (4.164)$$

It is worth noting that with the choice $C_1 = 0$ the potential vanishes, and the fake supersymmetric solutions constructed here are also solutions to the equations of motion of the corresponding ungauged supergravity.

Requiring $\Re(Z)$, $\Im(Z) \neq 0$ and $\langle \mathcal{R} | \mathcal{I} \rangle > 0$ the stabilization equations give

$$\begin{aligned} \mathcal{R}^0 &= \frac{1}{2\mathcal{S}} \left[(\mathcal{I}^1)^3 + 4\mathcal{I}^0 \mathcal{I}_1 \mathcal{I}^1 + 4\mathcal{I}_0 (\mathcal{I}^0)^2 \right], \\ \mathcal{R}^1 &= -\frac{2}{9\mathcal{S}} \left[16\mathcal{I}^0 (\mathcal{I}_1)^2 + 3\mathcal{I}_1 (\mathcal{I}^1)^2 - 9\mathcal{I}_0 \mathcal{I}^0 \mathcal{I}^1 \right], \\ \mathcal{R}_0 &= \frac{2}{27\mathcal{S}} \left[16(\mathcal{I}_1)^3 - 27(\mathcal{I}_0)^2 \mathcal{I}^0 - 27\mathcal{I}_0 \mathcal{I}_1 \mathcal{I}^1 \right], \\ \mathcal{R}_1 &= \frac{1}{6\mathcal{S}} \left[4(\mathcal{I}_1)^2 \mathcal{I}^1 - 12\mathcal{I}_0 \mathcal{I}^0 \mathcal{I}_1 - 9\mathcal{I}_0 (\mathcal{I}^1)^2 \right], \end{aligned} \quad (4.165)$$

with

$$\mathcal{S} \equiv \sqrt{-4(\mathcal{I}_0 \mathcal{I}^0)^2 + \frac{4}{3} (\mathcal{I}_1 \mathcal{I}^1)^2 + \frac{128}{27} \mathcal{I}^0 (\mathcal{I}_1)^3 - 2\mathcal{I}_0 (\mathcal{I}^1)^3 - 8\mathcal{I}_0 \mathcal{I}^0 \mathcal{I}_1 \mathcal{I}^1}, \quad (4.166)$$

and

$$\frac{1}{2|X|^2} = \langle \mathcal{R} | \mathcal{I} \rangle = \mathcal{S}. \quad (4.167)$$

4.5.1 Flat base space

Using again the results of 4.3.1 the solution in the gauged case can be written in terms of harmonic functions H_0 , H_1 and H_{im} and a time-independent one-form ω as

$$ds^2 = \mathcal{S}^{-1}(dt + \omega)^2 - \mathcal{S}d\vec{y}^2, \quad Z = \frac{\mathcal{T}^1 + it_1\mathcal{S}/t_0}{\mathcal{T}^0 - i\mathcal{S}}, \quad (4.168)$$

$$F^0 = d\left(\frac{H_{\text{im}}\mathcal{T}^0}{\mathcal{S}^2}(dt + \omega)\right) - \star_3 dH_{\text{im}}, \quad F^1 = d\left(\frac{H_{\text{im}}\mathcal{T}^1}{\mathcal{S}^2}(dt + \omega)\right) + \frac{t_1}{t_0} \star_3 dH_{\text{im}},$$

with

$$\mathcal{S} = \sqrt{H_{\text{im}}\mathcal{H}_0 \left[\mathcal{T}^0 + \left(\frac{t_1}{t_0}\right)^3 H_{\text{im}}^2 \right] + H_{\text{im}}\mathcal{H}_1 \left[\mathcal{T}^1 - \frac{4}{27}\mathcal{H}_1^2 \right]},$$

$$\mathcal{T}^0 = H_{\text{im}} \left[\left(\frac{t_1}{t_0}\right)^3 H_{\text{im}} - \mathcal{H}_0 + \frac{t_1}{t_0}\mathcal{H}_1 \right],$$

$$\mathcal{T}^1 = \frac{4}{9}\mathcal{H}_1^2 + \frac{1}{3}\left(\frac{t_1}{t_0}\right)^2 H_{\text{im}}\mathcal{H}_1 + \frac{t_1}{t_0}H_{\text{im}}\mathcal{H}_0,$$

$$\mathcal{H}_0 = \frac{t}{t_0} + H_0, \quad \mathcal{H}_1 = \frac{t}{t_1} + H_1, \quad (4.169)$$

while ω solves eq. (4.90).

In the case $C_0 = 0$ ($t_0 \rightarrow \infty$) and with the convenient redefinitions $H_1 \rightarrow 3/2 H_1$, $\tilde{t}_1 = 3/2 t_1$ the solution simplifies to

$$ds^2 = \mathcal{S}^{-1}(dt + \omega)^2 - \mathcal{S}d\vec{y}^2, \quad Z = -\frac{\mathcal{H}_1^2}{H_0 H_{\text{im}} + i\mathcal{S}}, \quad (4.170)$$

$$F^0 = -d\left(\frac{H_0 H_{\text{im}}^2}{\mathcal{S}^2}(dt + \omega)\right) - \star_3 dH_{\text{im}}, \quad F^1 = d\left(\frac{H_{\text{im}}\mathcal{H}_1^2}{\mathcal{S}^2}(dt + \omega)\right),$$

where

$$\mathcal{S} = \sqrt{H_{\text{im}}\mathcal{H}_1^3 - H_{\text{im}}^2 H_0^2}. \quad (4.171)$$

With the choice (4.91) and (4.93), this can be explicitly written as

$$ds^2 = \frac{\Sigma^2}{\Delta} dt^2 + \frac{\Sigma}{\Delta} \left[-a \sin^2 \theta \left(2 \widehat{kQ} r + \widehat{qQ} \right) + 2 \widehat{kq} (r^2 + a^2) \cos \theta \right] dt d\varphi$$

$$- \frac{\Delta}{\Sigma(r^2 + a^2)} dr^2 - \frac{\Delta}{\Sigma} d\theta^2 \quad (4.172)$$

$$+ \left[\frac{1}{4\Delta} \left[-a \sin^2 \theta \left(2 \widehat{kQ} r + \widehat{qQ} \right) + 2 \widehat{kq} (r^2 + a^2) \cos \theta \right]^2 - \frac{\Delta}{\Sigma^2} (r^2 + a^2) \sin^2 \theta \right] d\varphi^2,$$

$$A^0 = - \frac{\Sigma}{\Delta^2} (\Sigma k_0 + q_0 r + Q_0 a \cos \theta) (\Sigma k_{\text{im}} + q_{\text{im}} r + Q_{\text{im}} a \cos \theta)^2 dt$$

$$+ \left[\frac{\Sigma}{2\Delta^2} (\Sigma k_0 + q_0 r + Q_0 a \cos \theta) (\Sigma k_{\text{im}} + q_{\text{im}} r + Q_{\text{im}} a \cos \theta)^2 \left(2 \widehat{kQ} r + \widehat{qQ} \right) + Q_{\text{im}} r \right] \cdot$$

$$\cdot \frac{a \sin^2 \theta}{\Sigma} d\varphi - \left[\frac{\Sigma}{\Delta^2} (\Sigma k_0 + q_0 r + Q_0 a \cos \theta) (\Sigma k_{\text{im}} + q_{\text{im}} r + Q_{\text{im}} a \cos \theta)^2 \widehat{kq} + q_{\text{im}} \right] \cdot$$

$$\cdot \frac{(r^2 + a^2) \cos \theta}{\Sigma} d\varphi, \quad (4.173)$$

$$A^1 = \frac{\Sigma}{\Delta^2} (\Sigma k_{\text{im}} + q_{\text{im}} r + Q_{\text{im}} a \cos \theta) \left[\Sigma(t/\tilde{t}_1 + k_1) + q_1 r + Q_1 a \cos \theta \right]^2$$

$$\cdot \left[dt - \frac{1}{2\Sigma} \left[\left(2 \widehat{kQ} r + \widehat{qQ} \right) a \sin^2 \theta + \widehat{kq} (r^2 + a^2) \cos \theta \right] d\varphi \right], \quad (4.174)$$

$$Z = - \frac{\left[\Sigma(t/\tilde{t}_1 + k_1) + q_1 r + Q_1 a \cos \theta \right]^2}{(\Sigma k_{\text{im}} + q_{\text{im}} r + Q_{\text{im}} a \cos \theta) (\Sigma k_0 + q_0 r + Q_0 a \cos \theta) + i\Delta}, \quad (4.175)$$

where

$$\Delta = \left\{ \left[\Sigma(t/\tilde{t}_1 + k_1) + q_1 r + Q_1 a \cos \theta \right]^3 [\Sigma k_{\text{im}} + q_{\text{im}} r + Q_{\text{im}} a \cos \theta] \right.$$

$$\left. - [\Sigma k_0 + q_0 r + Q_0 a \cos \theta]^2 [\Sigma k_{\text{im}} + q_{\text{im}} r + Q_{\text{im}} a \cos \theta]^2 \right\}^{\frac{1}{2}}$$

$$\Sigma = r^2 + a^2 \cos^2 \theta, \quad \widehat{xy} = x_0 y_{\text{im}} - x_{\text{im}} y_0. \quad (4.176)$$

In the case of flat gauging, $C_1 = 0$ (which is inequivalent to $C_0 = 0$ for this model), the results of 4.3.1 are still valid provided one exchanges 0 and 1 indices everywhere. Redefining $H_1 \rightarrow 3 H_1$, the solution simplifies to

$$ds^2 = \mathcal{S}^{-1}(dt + \omega)^2 - \mathcal{S}d\vec{y}^2, \quad Z = -\frac{H_1 - i\mathcal{U}}{H_{\text{im}}}, \quad (4.177)$$

$$F^0 = -d\left(\frac{H_{\text{im}}}{\mathcal{U}^2}(dt + \omega)\right), \quad F^1 = d\left(\frac{H_1}{\mathcal{U}^2}(dt + \omega)\right) - \star_3 dH_{\text{im}},$$

with

$$\mathcal{S} = H_{\text{im}}\mathcal{U} = H_{\text{im}}\sqrt{3H_1^2 - 2\mathcal{H}_0 H_{\text{im}}}. \quad (4.178)$$

The metric with the same harmonic functions and ω as before can again be written in the form (4.172), but where now

$$\begin{aligned} \widehat{xy} &= 3(x_1 y_{\text{im}} - x_{\text{im}} y_1), \quad \Delta = [\Sigma k_{\text{im}} + q_{\text{im}} r + Q_{\text{im}} a \cos \theta] \tilde{\Delta}, \\ \tilde{\Delta} &= \left\{ 3 [\Sigma k_1 + q_1 r + Q_1 a \cos \theta]^2 - [\Sigma(t/t_0 + k_0) + q_0 r + Q_0 a \cos \theta] \cdot \right. \\ &\quad \left. \cdot [\Sigma k_{\text{im}} + q_{\text{im}} r + Q_{\text{im}} a \cos \theta] \right\}^{\frac{1}{2}}, \end{aligned} \quad (4.179)$$

while the other fields read

$$\begin{aligned} A^0 &= -\frac{\Sigma}{\tilde{\Delta}^2} (\Sigma k_{\text{im}} + q_{\text{im}} r + Q_{\text{im}} a \cos \theta) \cdot \\ &\quad \cdot \left\{ dt + \frac{1}{2\Sigma} \left[(2\widehat{kQ}r + \widehat{qQ}) a \sin^2 \theta - 2\widehat{kq}(r^2 + a^2) \cos \theta \right] d\varphi \right\}, \end{aligned} \quad (4.180)$$

$$\begin{aligned} A^1 &= \frac{\Sigma}{\tilde{\Delta}^2} (\Sigma k_1 + q_1 r + Q_1 a \cos \theta) dt \\ &\quad - \frac{1}{2} \left[\frac{\Sigma}{\tilde{\Delta}^2} (\Sigma k_1 + q_1 r + Q_1 a \cos \theta) (2\widehat{kQ}r + \widehat{qQ}) - 2Q_{\text{im}} r \right] \frac{a \sin^2 \theta}{\Sigma} d\varphi \\ &\quad + \left[\frac{\Sigma}{\tilde{\Delta}^2} (\Sigma k_1 + q_1 r + Q_1 a \cos \theta) \widehat{kq} - q_{\text{im}} \right] \frac{(r^2 + a^2) \cos \theta}{\Sigma} d\varphi, \end{aligned} \quad (4.181)$$

$$Z = -\frac{\Sigma k_1 + q_1 r + Q_1 a \cos \theta - i\tilde{\Delta}}{\Sigma k_{\text{im}} + q_{\text{im}} r + Q_{\text{im}} a \cos \theta}. \quad (4.182)$$

4.5.2 Spherical base space

Using the results of 4.3.2, the complete solution can be written as

$$ds^2 = \mathcal{S}^{-1}(dt + \hat{\omega})^2 - \mathcal{S} ds_{\mathbb{S}^3}^2, \quad Z = -\frac{\mathcal{T}^1 - i\mathcal{S}\tilde{H}_{\text{im}}}{\mathcal{T}^0 + i\mathcal{S}H_{\text{im}}}, \quad (4.183)$$

$$F^0 = -d\left[\frac{\mathcal{T}^0}{\mathcal{S}^2}(dt + \hat{\omega})\right] - \star_3 dH_{\text{im}}, \quad F^1 = d\left[\frac{\mathcal{T}^1}{\mathcal{S}^2}(dt + \hat{\omega})\right] + \frac{t_1}{t_0} \star_3 dH_{\text{im}},$$

where

$$\mathcal{S} = \sqrt{-\mathcal{H}_0(\mathcal{T}^0 + \tilde{H}_{\text{im}}^3) + \mathcal{H}_1\left(\mathcal{T}^1 - \frac{4}{27}H_{\text{im}}\mathcal{H}_1^2\right)},$$

$$\mathcal{T}^0 = \tilde{H}_{\text{im}}^3 + H_{\text{im}}\tilde{H}_{\text{im}}\mathcal{H}_1 + H_{\text{im}}^2\mathcal{H}_0, \quad \mathcal{T}^1 = \frac{4}{9}H_{\text{im}}\mathcal{H}_1^2 + \frac{1}{3}\tilde{H}_{\text{im}}^2\mathcal{H}_1 - H_{\text{im}}\tilde{H}_{\text{im}}\mathcal{H}_0,$$

$$\mathcal{H}_\Lambda = \frac{t}{t_\Lambda} + H_\Lambda, \quad \tilde{H}_{\text{im}} = 2t_1 - \frac{t_1}{t_0}H_{\text{im}}, \quad (4.184)$$

and $\hat{\omega}$ satisfies (4.110). An explicit solution can be obtained with harmonics of the form (4.112), obeying the constraint (4.114), and $\hat{\omega}$ given by (4.117).

4.5.3 Berger sphere

Making use of the results of 4.3.3 the complete solution can be written as

$$ds^2 = \mathcal{S}^{-1}(dt \pm \sin \mu \cos \mu \sigma_3^L t + \hat{\omega})^2 - \mathcal{S} ds_3^2, \quad Z = -\frac{\mathcal{T}^1 - i\mathcal{S}\tilde{K}_{\text{im}}}{\mathcal{T}^0 + i\mathcal{S}K_{\text{im}}}, \quad (4.185)$$

$$F^0 = -d\left[\frac{\mathcal{T}^0}{\mathcal{S}^2}(dt \pm \sin \mu \cos \mu \sigma_3^L t + \hat{\omega})\right] - \star_3[dK_{\text{im}} \pm \sin \mu \cos \mu \sigma_3^L K_{\text{im}}],$$

$$F^1 = d\left[\frac{\mathcal{T}^1}{\mathcal{S}^2}(dt \pm \sin \mu \cos \mu \sigma_3^L t + \hat{\omega})\right] - \star_3[d\tilde{K}_{\text{im}} \pm \sin \mu \cos \mu \sigma_3^L \tilde{K}_{\text{im}}],$$

where

$$\mathcal{S} = \sqrt{-\mathcal{K}_0 \left(\mathcal{T}^0 + \tilde{K}_{\text{im}}^3 \right) + \mathcal{K}_1 \left(\mathcal{T}^1 - \frac{4}{27} K_{\text{im}} \mathcal{K}_1^2 \right)},$$

$$\mathcal{T}^0 = \tilde{K}_{\text{im}}^3 + K_{\text{im}} \tilde{K}_{\text{im}} \mathcal{K}_1 + K_{\text{im}}^2 \mathcal{K}_0, \quad \mathcal{T}^1 = \frac{4}{9} K_{\text{im}} \mathcal{K}_1^2 + \frac{1}{3} \tilde{K}_{\text{im}}^2 \mathcal{K}_1 - K_{\text{im}} \tilde{K}_{\text{im}} \mathcal{K}_0,$$

$$\mathcal{K}_\Lambda = \frac{t}{t_\Lambda} + K_\Lambda, \quad \tilde{K}_{\text{im}} = t_1 \cos \mu - \frac{t_1}{t_0} K_{\text{im}}. \quad (4.186)$$

Here the functions K_0 and K_1 satisfy eq. (4.125), K_{im} obeys (4.124), and the one-form $\hat{\omega}$ is a time-independent solution of (4.130).

Multi-centered black holes with a negative cosmological constant

Composite objects formed by elementary constituents with mass-to-charge ratio equal to one have been studied for a long time in general relativity. In the Newtonian theory of gravity it is clearly possible to obtain a system of point charges in static equilibrium by fine-tuning the charge suitably with the particle mass in such a way as to balance the gravitational and electrostatic forces. On the other hand, the non linearity of the equations of motion imply that in general relativity the existence of a similar system is not guaranteed.

The first indications that such a general relativistic analogue actually exists dates back to 1917, when Weyl obtained his famous two-body static axially symmetric electrovacuum solution [65], later generalized independently by Majumdar [42] and Papapetrou [43], who removed the requirement of axial symmetry. A stationary generalization of the MP solution was constructed by Israel and Wilson [55] and Perjés [56]. All these multi-black-hole geometries are supersymmetric and thus admit supercovariantly constant spinors [18, 22]. As a consequence, they satisfy rather simple first-order equations, which explains why one can build arbitrary superpositions of the elementary constituents, in spite of the nonlinear nature of the Einstein-Maxwell equations.

Nevertheless, supersymmetry does not seem to be necessary for the existence of these bound states, since by now we know many examples of multicentered black holes that are not BPS, see e.g. [66] or chapter 4.

The study of composite systems like “black hole molecules” has played a crucial role in several recent developments of supergravity and string theory, especially in attempts to understand the quantum structure of black holes. Moreover, they are of interest in the field of holography, in particular for applications of the gauge/gravity correspondence to condensed matter phenomena (see e.g. [67] for a review). In this context, it was established recently in [68] that stable and metastable stationary bound states in four-dimensional anti-de Sitter space exist, and it was argued that their holographic duals represent structural glasses. The glassy feature of these black hole bound states is related

to their rugged free energy landscape, which in turn is a consequence of the fact that the constituents can have a wide range of different possible charges.

In this chapter we present the work published in [69], showing how to construct multicenter solutions in any FLRW spacetime and for arbitrary dimension. These geometries are sourced by a $U(1)$ gauge field and by a charged perfect fluid, and their construction doesn't rely on genuine or fake supersymmetry. Generically, the black holes that we construct are determined by a function satisfying the conformal Laplace equation on the spatial slices of the FLRW background universe. This can be seen as a generalization of the characterization in terms of harmonic functions taking place for supersymmetric or fake supersymmetric multicentered black holes.

Since anti-de Sitter space can be written in a FLRW form with hyperbolic spatial slices and trigonometric scale factor, our recipe allows, as a particular subcase, to obtain multicenter solutions in AdS. Like the underlying FLRW universe, these are highly dynamical, and thus different from the bound states of [68]. Unfortunately, the big bang/big crunch coordinate singularities that appear when one writes AdS in FLRW coordinates become true curvature singularities once such a dynamical black hole is present. We show that this implies that actually only one point of the conformal boundary of AdS survives. This makes it questionable if our solutions admit an AdS/CFT interpretation in the usual sense.

The chapter is organized as follows. In section 5.1 we show how to construct multicenter solutions in an arbitrary FLRW universe, starting from the charged generalization of the McVittie spacetime [9] originally presented in a little-known paper by Shah and Vaidya [10]. In section 5.2, we discuss some physical properties of the single-centered solution in AdS, both for mass-to-charge ratio generic and equal to one. In particular, we determine the curvature singularities and trapping horizons, compute the surface gravity of the latter, and show that the generalized first law of black hole dynamics proposed by Hayward [13] holds. Finally in section 5.3, the higher-dimensional case is considered.

5.1 Multi-centered maximally charged McVittie solutions

In [10], Shah and Vaidya presented a charged generalization of the McVittie solution [9], with metric

$$ds^2 = \frac{\left[1 - (M^2 - Q^2) \frac{1+kr^2}{4a^2 r^2}\right]^2}{\left[1 + M \frac{\sqrt{1+kr^2}}{ar} + (M^2 - Q^2) \frac{1+kr^2}{4a^2 r^2}\right]^2} dt^2 - 4a^2 \left[1 + M \frac{\sqrt{1+kr^2}}{ar} + (M^2 - Q^2) \frac{1+kr^2}{4a^2 r^2}\right]^2 \frac{dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2}{(1+kr^2)^2}, \quad (5.1)$$

and U(1) field strength given by

$$F = \frac{Q}{ar^2} \frac{1}{\sqrt{1+kr^2}} \frac{\left[1 - (M^2 - Q^2) \frac{1+kr^2}{4a^2r^2}\right]}{\left[1 + M \frac{\sqrt{1+kr^2}}{ar} + (M^2 - Q^2) \frac{1+kr^2}{4a^2r^2}\right]^2} dr \wedge dt. \quad (5.2)$$

(5.1) and (5.2) satisfy the Einstein-Maxwell equations

$$G_{\mu\nu} = 8\pi T_{\mu\nu}, \quad \nabla_\nu F^{\mu\nu} = 4\pi J^\mu, \quad (5.3)$$

$$T_{\mu\nu} = \frac{1}{4\pi} \left[-F_{\mu\rho} F_\nu{}^\rho + \frac{1}{4} g_{\mu\nu} F_{\rho\lambda} F^{\rho\lambda} \right] + \rho u_\mu u_\nu + p(u_\mu u_\nu - g_{\mu\nu}), \quad J^\mu = \sigma u^\mu, \quad (5.4)$$

where the pressure, energy density, charge density and four-velocity of the charged perfect fluid source read respectively

$$8\pi p = -2 \left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} \right) \frac{\left[1 + M \frac{\sqrt{1+kr^2}}{ar} + (M^2 - Q^2) \frac{1+kr^2}{4a^2r^2}\right]}{\left[1 - (M^2 - Q^2) \frac{1+kr^2}{4a^2r^2}\right]} - 3 \frac{\dot{a}^2}{a^2} - k \left\{ a^2 \left[1 + M \frac{\sqrt{1+kr^2}}{ar} + (M^2 - Q^2) \frac{1+kr^2}{4a^2r^2}\right]^2 \left[1 - (M^2 - Q^2) \frac{1+kr^2}{4a^2r^2}\right] \right\}^{-1}, \quad (5.5)$$

$$8\pi\rho = 3 \frac{\dot{a}^2}{a^2} + \frac{3k}{2a^2} \left[1 + M \frac{\sqrt{1+kr^2}}{ar} + (M^2 - Q^2) \frac{1+kr^2}{4a^2r^2}\right]^{-3} \left[2 + M \frac{\sqrt{1+kr^2}}{ar}\right], \quad (5.6)$$

$$4\pi\sigma = -\frac{3kQ}{4a^3} \frac{\sqrt{1+kr^2}}{r} \left[1 + M \frac{\sqrt{1+kr^2}}{ar} + (M^2 - Q^2) \frac{1+kr^2}{4a^2r^2}\right]^{-3}, \quad (5.7)$$

$$u = \frac{1 - (M^2 - Q^2) \frac{1+kr^2}{4a^2r^2}}{1 + M \frac{\sqrt{1+kr^2}}{ar} + (M^2 - Q^2) \frac{1+kr^2}{4a^2r^2}} dt. \quad (5.8)$$

Moreover, $k = 0, \pm 1$ determines the geometry of the spatial slices. From (5.7) it is clear that the cosmic fluid is required to be charged if the spatial geometry of the underlying FLRW universe is curved.

In the maximally charged case $M = |Q|$ (obtained in [70]), after the coordinate change $r = \frac{1}{\sqrt{k}} \tan \frac{\sqrt{k}\psi}{2}$, (5.1) and (5.2) boil down to

$$ds^2 = \frac{1}{\left[1 + M \frac{\sqrt{k}}{a \sin(\sqrt{k}\psi/2)}\right]^2} dt^2 - a^2 \left[1 + M \frac{\sqrt{k}}{a \sin(\sqrt{k}\psi/2)}\right]^2 \left[d\psi^2 + \frac{\sin^2(\sqrt{k}\psi)}{k} (d\theta^2 + \sin^2\theta d\phi^2) \right], \quad (5.9)$$

$$F = \frac{Mk \cos(\sqrt{k} \psi/2)}{2a \sin^2(\sqrt{k} \psi/2)} \frac{d\psi \wedge dt}{\left[1 + M \frac{\sqrt{k}}{a \sin(\sqrt{k} \psi/2)}\right]^2} = d \left[\left(1 + M \frac{\sqrt{k}}{a \sin(\sqrt{k} \psi/2)}\right)^{-1} dt \right], \quad (5.10)$$

while the pressure, energy- and current density become

$$8\pi p = -3 \frac{\dot{a}^2}{a^2} - \frac{k}{a^2} \left[1 + M \frac{\sqrt{k}}{a \sin(\sqrt{k} \psi/2)}\right]^{-2} \left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right) \left[1 + M \frac{\sqrt{k}}{a \sin(\sqrt{k} \psi/2)}\right], \quad (5.11)$$

$$8\pi\rho = 3 \frac{\dot{a}^2}{a^2} + \frac{3}{2} \frac{k}{a^2} \left[1 + M \frac{\sqrt{k}}{a \sin(\sqrt{k} \psi/2)}\right]^{-3} \left[2 + M \frac{\sqrt{k}}{a \sin(\sqrt{k} \psi/2)}\right], \quad (5.12)$$

$$4\pi J = -\frac{3}{4} \frac{kM}{a^3} \frac{\sqrt{k}}{\sin(\sqrt{k} \psi/2)} \left[1 + M \frac{\sqrt{k}}{a \sin(\sqrt{k} \psi/2)}\right]^{-4} dt. \quad (5.13)$$

This solution appears to be characterized by the function

$$H = \frac{M\sqrt{k}}{\sin(\sqrt{k} \psi/2)}, \quad (5.14)$$

which happens to satisfy the conformal Laplace equation on E^3 , S^3 or H^3 ,

$$\nabla^2 H = \frac{1}{8} R H, \quad (5.15)$$

where $R = 6k$ is the corresponding scalar curvature. It is straightforward to verify that one can take any function \mathcal{H} solving (5.15), and the resulting fields still satisfy the Einstein-Maxwell equations (5.3). This allows to generalize (5.9) to a multi-centered solution by choosing \mathcal{H} to be a linear combination of terms obtained by acting on H with the isometries of the three-dimensional base space metric. Alternatively, one can use the conformal invariance of (5.15), which implies

$$\tilde{\nabla}^2 \tilde{H} = \frac{1}{8} \tilde{R} \tilde{H}, \quad (5.16)$$

where $\tilde{\nabla}^2$ and \tilde{R} denote the Laplacian and scalar curvature of the conformally related metric $\tilde{g}_{ij} = \Omega^2 g_{ij}$ respectively, and $\tilde{H} = \Omega^{-1/2} H$. Now let g_{ij} be the flat metric, $g_{ij} dx^i dx^j = d\vec{x}^2$, and

$$\tilde{g}_{ij} dx^i dx^j = \frac{4d\vec{x}^2}{[1 + k\vec{x}^2]^2}.$$

Starting from the usual one-center solution for a flat base, $H = \sqrt{2}M/|\vec{x}|$, one gets

$$\tilde{H} = \frac{M}{|\vec{x}|} \sqrt{1 + k\vec{x}^2} = \frac{M\sqrt{k}}{\sin(\sqrt{k}\psi/2)},$$

which is the function appearing in (5.9). Taking instead

$$\mathcal{H} = \sum_{I=1}^N \frac{Q_I}{|\vec{x} - \vec{x}_I|}$$

leads to

$$\tilde{\mathcal{H}} = \frac{1}{\sqrt{2}} [1 + k\vec{x}^2]^{1/2} \sum_{I=1}^N \frac{Q_I}{|\vec{x} - \vec{x}_I|}. \quad (5.17)$$

It would be interesting to understand whether there is a deeper reason for the appearance of this conformal structure.

Notice that the existence of this multi-centered generalization of (5.9) is also suggested by considering a charged probe particle in the geometry (5.9), whose equation of motion is

$$\nabla_v p^\mu = -q F^\mu{}_\nu v^\nu. \quad (5.18)$$

If the particle is BPS, $m = q$, and we take $v = v^t \partial_t$ for its four-velocity, it is easy to show that the attractive gravitational force encoded in the Christoffel connection exactly cancels the repulsive Lorentz force, such that the particle can stay at rest at fixed ψ, θ, ϕ .

5.2 Singularities and horizons in the single-centered asymptotically AdS case

In this section, we shall discuss some physical properties of the single-centered (non necessarily maximally charged) solution in AdS, which does not coincide with the well-known Reissner-Nordström-AdS black hole, but is highly dynamical.

Let us choose $k = -1$ and $a(t) = l \sin(t/l)$, with $l > 0$ and $0 < t/l < \pi$. Then, far from the black hole ($\psi \rightarrow \infty$ or $r \rightarrow 1$), the energy density and pressure approach the values given by a negative cosmological constant $\Lambda = -3/l^2$, while the charge density (5.7) goes to zero. In this limit, the metric (5.1) tends to AdS in FLRW coordinates, i.e.,

$$ds^2 \rightarrow dt^2 - l^2 \sin^2 \frac{t}{l} (d\psi^2 + \sinh^2 \psi d\Omega^2). \quad (5.19)$$

The FLRW form is related to global coordinates τ, \hat{r} by

$$\hat{r} = l \sin \frac{t}{l} \sinh \psi, \quad \cos \frac{t}{l} = \left(1 + \frac{\hat{r}^2}{l^2}\right)^{1/2} \cos \frac{\tau}{l}, \quad (5.20)$$

which casts (5.19) into

$$ds^2 = \left(1 + \frac{\hat{r}^2}{l^2}\right) d\tau^2 - \left(1 + \frac{\hat{r}^2}{l^2}\right)^{-1} d\hat{r}^2 - \hat{r}^2 d\Omega^2. \quad (5.21)$$

(5.19) has a lightlike big bang/big crunch singularity in $t = 0$ and $t = l\pi$ respectively, that are of course artefacts of the coordinate system t, ψ . In fact, by introducing τ, \hat{r} , one

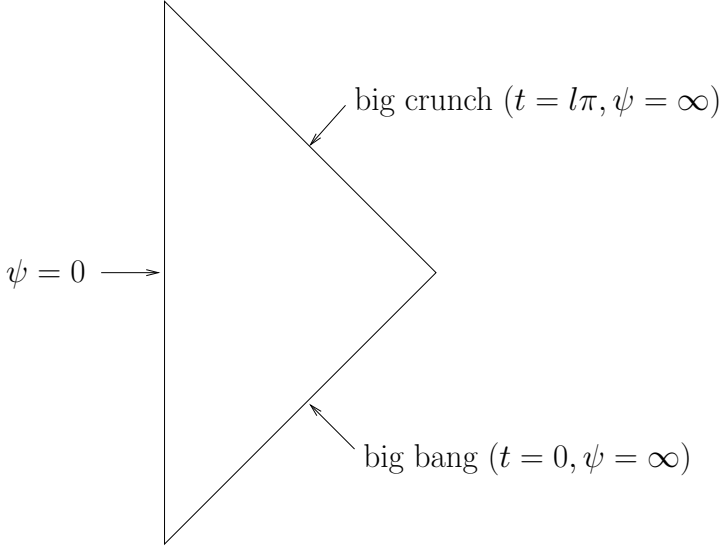


Figure 5.1: Carter-Penrose diagram for AdS in FLRW coordinates.

extends the spacetime beyond these singularities. The causal structure of AdS in FLRW coordinates is visualized in the Carter-Penrose diagram fig. 5.1.

Notice also that, due to $\cos^2(t/l) \leq 1$, the last eq. of (5.20) implies $\tau/l \rightarrow \pi/2$ for $\hat{r} \rightarrow \infty$, so that actually only the point $\tau = l\pi/2$ (which is of course a two-sphere) of the conformal boundary of AdS is visible in FLRW coordinates.

Rewriting the metric (5.1) for brevity as

$$ds^2 = \frac{g^2}{f^2} dt^2 - a^2 f^2 \left(d\psi^2 + \frac{\sin^2(\sqrt{k}\psi)}{k} d\Omega^2 \right),$$

with

$$f = 1 + \frac{\sqrt{k}M}{a \sin(\sqrt{k}\psi/2)} + k \frac{M^2 - Q^2}{4a^2 \sin^2(\sqrt{k}\psi/2)}, \quad g = 1 - k \frac{M^2 - Q^2}{4a^2 \sin^2(\sqrt{k}\psi/2)}, \quad (5.22)$$

the scalar curvature is

$$R = -12 \frac{\dot{a}^2}{a^2} - 6 \frac{f}{g} \left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} \right) - \frac{3}{2} k \frac{f(g+2) + g^2}{a^2 f^3 g}. \quad (5.23)$$

The spacetime with $k = -1$ has thus curvature singularities in $a(t) = 0$, $\sinh(\psi/2) = \pm \frac{\sqrt{M^2 - Q^2}}{2a}$ and $\sinh(\psi/2) = \frac{\pm Q - M}{2a}$; however the only singularity that is connected with the asymptotic region $\psi \rightarrow +\infty$ is the hypersurface $\sinh(\psi/2) = \frac{\sqrt{M^2 - Q^2}}{2a}$. In the maximally charged case, $M = |Q|$, this singular hypersurface becomes the union of the hypersurfaces $t = 0$, $t = l\pi$ and $\psi = 0$.

To determine if the present spacetime describes a black hole, one can look for trapping horizons [11]. Introducing the Newman-Penrose null tetrads

$$l = \frac{1}{\sqrt{2}} \left(\frac{g}{f} dt - a f d\psi \right), \quad n = \frac{1}{\sqrt{2}} \left(\frac{g}{f} dt + a f d\psi \right),$$

$$m = \frac{a f \sinh \psi}{\sqrt{2}} (d\theta + i \sin \theta d\varphi), \quad (5.24)$$

and the complex conjugate \bar{m} , the expansions of the outgoing and ingoing radial null geodesics are respectively

$$\theta_+ \equiv -2m^{(\mu} \bar{m}^{\nu)} \nabla_\mu l_\nu, \quad \theta_- \equiv -2m^{(\mu} \bar{m}^{\nu)} \nabla_\mu n_\nu, \quad (5.25)$$

and once evaluated read

$$\theta_\pm = \frac{\sqrt{2}}{a} \left[\dot{a} \pm \frac{g + \sinh^2(\psi/2)(f + g)}{\sinh \psi f^2} \right]. \quad (5.26)$$

Marginal surfaces are defined as spacelike 2-surfaces on which $\theta_+ = 0$ ($\theta_- = 0$), and trapping horizons are defined as the closure of 3-surfaces foliated by marginal surfaces such that $\theta_- \neq 0$ and $\mathcal{L}_- \theta_+ \neq 0$ ($\theta_+ \neq 0$ and $\mathcal{L}_+ \theta_- \neq 0$) on the 3-surface, where \mathcal{L}_\pm is the Lie derivative along the outgoing or ingoing radial null geodesics. From eq. (5.26) it is clear that if $t \neq l\pi/2$ the two expansions can't both vanish at the same time, while in $t = l\pi/2$ they only vanish behind or on the singularity, since outside of the singularity both f and g are positive, so that no horizon can exist in any case for $t = l\pi/2$. Furthermore $\mathcal{L}_- \theta_+$ and $\mathcal{L}_+ \theta_-$ are negative in the whole considered region; as a consequence the only condition necessary to locate the trapping horizons is the vanishing of θ_+ or θ_- .

For $M \neq |Q|$ there are always solutions to $\theta_\pm = 0$ that lie on the singularity; this means that the horizons intersect the singularity and there is a time interval around $t = l\pi/2$ for which they are not defined. On the other hand, if $M = |Q|$ the horizons are defined for every $t \neq l\pi/2$, while for $t = l\pi/2$ they tend to coincide on the singularity $\psi = 0$. For $\psi \rightarrow +\infty$, $\theta_\pm = 0$ implies $\dot{a} \rightarrow \pm 1$ which means that the horizons tend to the axes $t = 0$ and $t = l\pi$.

There are always two trapping horizons: One for $t > l\pi/2$ where $\theta_+ = 0$ and $\theta_- = 2\sqrt{2}\frac{\dot{a}}{a} < 0$, and the other for $t < l\pi/2$ where $\theta_- = 0$ and $\theta_+ = 2\sqrt{2}\frac{\dot{a}}{a} > 0$. Since $\mathcal{L}_- \theta_+$ and $\mathcal{L}_+ \theta_-$ are negative these are respectively an outer future trapping horizon, which can be interpreted as the horizon of a black hole, and an outer past trapping horizon, which can be interpreted as the horizon of a white hole.

In figures 5.2 and 5.3 we display, respectively in the cosmological (FLRW) coordinates (t, ψ) and in the global coordinates (τ, \hat{r}) as defined in (5.20), the curvature singularity, the trapping horizons and the radial null geodesics intersecting in a point with $t = l\pi/2$ or $\tau = l\pi/2$, for arbitrarily chosen parameters; the plots are obtained by numerical methods.

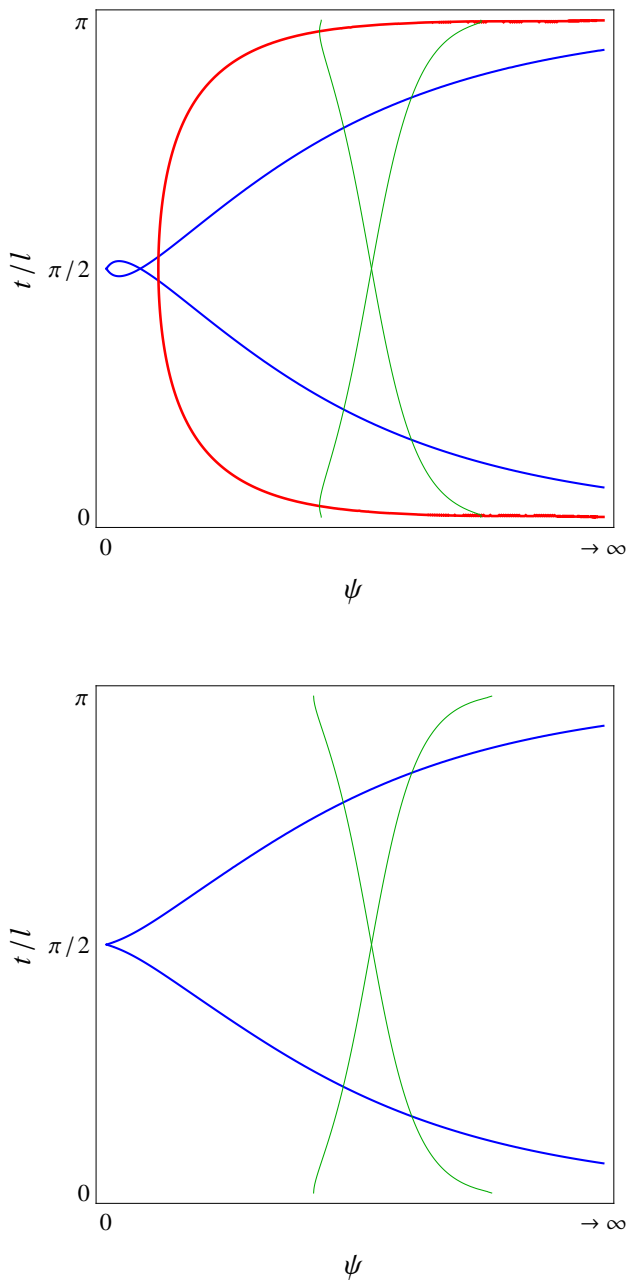


Figure 5.2: Plots of curvature singularity (red), trapping horizons (blue) and one pair of radial null geodesics (green) crossing in $t = l\pi/2$, in FLRW coordinates (t, ψ) for $M \neq |Q|$ (left) and $M = |Q|$ (right). For $M = |Q|$ the curvature singularities coincide with the axes $\psi = 0$, $t = 0$ and $t = l\pi$.

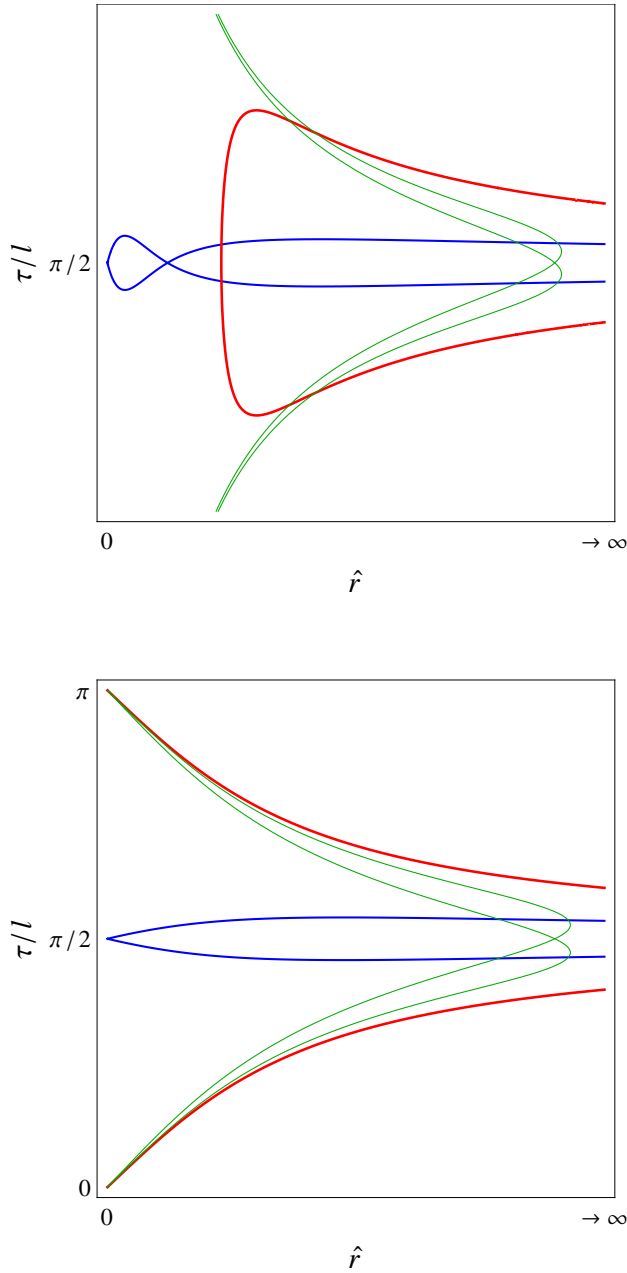


Figure 5.3: Plots of curvature singularity (red), trapping horizons (blue) and one pair of radial null geodesics (green) crossing in $\tau = l\pi/2$, in the coordinate system (τ, \hat{r}) for $M \neq |Q|$ (left) and $M = |Q|$ (right). The plot for $M \neq |Q|$ is zoomed in on the vertical axis to show its relevant features. For $M = |Q|$, the axis $\hat{r} = 0$ belongs to the curvature singularity.

The radial null geodesics satisfy

$$\frac{dt}{d\psi} = \pm \frac{af^2}{g}. \quad (5.27)$$

For the case $M = |Q|$ this means that the singularity at $\psi = 0$ is never reached, since for finite a the derivative tends to infinity. On the other hand the t -component of the geodesic equation for radial null or timelike geodesics in $a \sim 0$ and finite ψ , using $|\dot{\psi}| \leq |\dot{t}|/(af^2)$ reads

$$\ddot{t} + \frac{\cos(t/l)}{l \sin(t/l)} \dot{t}^2 \sim 0, \quad (5.28)$$

where a dot indicates a derivative with respect to the affine parameter. The solution, $t \sim \pm l \cos^{-1}(c_1 \lambda + c_2)$ shows that the singularities in $t = 0, l\pi$ are always reached for finite values of the affine parameter.

Taking advantage of the spherical symmetry, it is possible to define in a simple, geometrical way the surface gravity k_l on the trapping horizons [13] and the associated local Hawking temperature $T = \frac{k_l}{2\pi}$ [14], according to

$$k_l = -\frac{1}{2} \tilde{\nabla}_\mu \tilde{\nabla}^\mu \mathcal{R} \Big|_{\theta_\pm=0} = -\frac{\mathcal{R}}{2} \left[\frac{f}{g} \left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} \right) + 2 \frac{\dot{a}^2}{a^2} \right. \\ \left. + \frac{1}{2} \frac{1}{a^2 f^3 \sinh^2(\psi/2)} \left(\frac{3}{2} g - \cosh^2(\psi/2) \right) \pm \frac{\dot{a}}{a^2 f} \frac{\cosh(\psi/2)}{\sinh(\psi/2)} \left(\frac{1}{2} + \frac{1}{g} - \frac{g}{f} \right) \right], \quad (5.29)$$

where $\mathcal{R} = af \sinh \psi$ is the areal radius, $\tilde{\nabla}$ is the covariant derivative operator associated with the two-dimensional metric normal to the spheres of symmetry, and the vanishing of expression (5.26) was used. k_l is in general not zero even in the maximally charged case, and is positive on the horizons, as is expected for outer trapping horizons. It is straightforward to verify that the generalized first law of black hole dynamics proposed by Hayward in [13],

$$E' = \frac{k_l A'}{8\pi} + \frac{1}{2} \mathcal{T} V', \quad (5.30)$$

holds on the trapping horizons. Here a prime represents a derivative along a vector field tangent to the trapping horizon, $A = 4\pi \mathcal{R}^2$ is the area of the spheres of symmetry, $V = \frac{4}{3}\pi \mathcal{R}^3$ is the areal volume, \mathcal{T} is the trace of the total energy-momentum tensor T with respect to the two-dimensional normal metric, and E is the Misner-Sharp energy, defined as

$$E = \frac{1}{2} \mathcal{R} (1 + \nabla_\mu \mathcal{R} \nabla^\mu \mathcal{R}). \quad (5.31)$$

Notice that $\nabla_\mu \mathcal{R} \nabla^\mu \mathcal{R} = \theta_+ \theta_- \mathcal{R}^2/2$ is identically zero on the trapping horizons, implying $E' = \frac{1}{2} \mathcal{R}'$.

5.3 Higher-dimensional generalization

It is possible to construct higher-dimensional generalizations of the multi-centered solutions found in section 5.1. To this aim, inspired by previous results [71, 72], we use the ansatz

$$ds^2 = \frac{g^2}{f^2} dt^2 - a^2 f^{\frac{2}{D-2}} ds_D^2, \quad (5.32)$$

$$F = \sqrt{\frac{D-1}{2(D-2)}} \frac{g}{f} \left[\left(1 - \frac{g}{f}\right)^2 + 4 \frac{g}{f} \left(1 - \frac{1}{g}\right) \right]^{1/2} \frac{dH}{H} \wedge dt,$$

with

$$f = 1 + M \frac{H}{a^{D-2}} + \frac{M^2 - Q^2}{4} \frac{H^2}{a^{2(D-2)}}, \quad g = 1 - \frac{M^2 - Q^2}{4} \frac{H^2}{a^{2(D-2)}}, \quad (5.33)$$

where $a(t)$ is a function of time, $H(\vec{x})$ is a function of the spatial coordinates, D and $ds_D^2 \equiv h_{ij} dx^i dx^j$ are respectively the dimension and the metric of the spatial slices. Notice that the square bracket in the expression of F is equal to $QH/(a^{D-2}f)$ and is just a way to express the charge Q in terms of the functions f and g .

The nonvanishing components of the Einstein tensor for (5.32) are given by

$$\begin{aligned} G_{tt} &= \frac{D(D-1)}{2} \frac{\dot{a}^2}{a^2} \frac{g^2}{f^2} + \frac{\hat{R}}{2a^2} \frac{g^2}{f^{\frac{D-1}{D-2}}} - \frac{D-1}{D-2} \left(1 - \frac{g}{f}\right) \frac{g^2}{f^{\frac{D-1}{D-2}}} \frac{\hat{\nabla}^2 H}{a^2 H} \\ &+ \left\{ \frac{2}{D-2} \frac{1}{f} \left(1 - \frac{1}{g}\right) [g(D-1) + D-3] + \frac{1}{2} \frac{D-1}{D-2} \left(1 - \frac{g}{f}\right)^2 \right\} \frac{g^2}{f^{\frac{D-1}{D-2}}} \frac{\partial_l H h^{lm} \partial_m H}{a^2 H^2}, \\ G_{ij} &= (1-D) h_{ij} a^2 f^{\frac{D}{D-2}} \left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} \right) - \frac{D(D-1)}{2} \dot{a}^2 f^{\frac{2}{D-2}} h_{ij} \\ &+ \hat{R}_{ij} - \frac{1}{2} \hat{R} h_{ij} + 2 \left(1 - \frac{1}{g}\right) \left(\frac{\hat{\nabla}^2 H}{H} h_{ij} - \frac{\hat{\nabla}_i \hat{\nabla}_j H}{H} \right) \\ &+ \left\{ \frac{D-1}{2} \left(1 - \frac{g}{f}\right)^2 - 2 \left(1 - \frac{1}{g}\right) \left[1 - (D-1) \frac{g}{f}\right] \right\} \frac{1}{D-2} \frac{\partial_l H h^{lm} \partial_m H}{H^2} h_{ij} \\ &- \left\{ (D-1) \left(1 - \frac{g}{f}\right)^2 - 2 \left(1 - \frac{1}{g}\right) \left[D + 2(1-D) \frac{g}{f} \right] \right\} \frac{1}{D-2} \frac{\partial_i H \partial_j H}{H^2}, \quad (5.34) \end{aligned}$$

where $\hat{\nabla}$, \hat{R}_{ij} and \hat{R} represent respectively the covariant derivative, Ricci tensor and scalar curvature of the spatial metric h_{ij} . From the expression for F one obtains for the electromagnetic energy-momentum tensor

$$8\pi T_{tt}^{\text{em}} = \frac{1}{2} \frac{D-1}{D-2} \left[\left(1 - \frac{g}{f}\right)^2 + 4 \frac{g}{f} \left(1 - \frac{1}{g}\right) \right] \frac{g^2}{f^{2\frac{D-1}{D-2}}} \frac{\partial_l H h^{lm} \partial_m H}{a^2 H^2}, \quad (5.35)$$

$$8\pi T_{ij}^{\text{em}} = -\frac{D-1}{D-2} \left[\left(1 - \frac{g}{f}\right)^2 + 4 \frac{g}{f} \left(1 - \frac{1}{g}\right) \right] \frac{1}{H^2} \left(\partial_i H \partial_j H - \frac{1}{2} h_{ij} \partial_l H h^{lm} \partial_m H \right).$$

The requirement to have a perfect fluid as matter source translates into the condition $G_{ij} - 8\pi T_{ij}^{\text{em}} \propto h_{ij}$. This implies that $\hat{R}_{ij} \propto h_{ij}$, that is, the spatial slices must be Einstein manifolds, and that the function H must satisfy the condition

$$-\frac{\hat{\nabla}_i \hat{\nabla}_j H}{H} + \frac{D}{D-2} \frac{\partial_i H \partial_j H}{H^2} \propto h_{ij}. \quad (5.36)$$

Notice that (5.36) is conformally invariant on *Einstein manifolds*, in the sense that under a conformal transformation that maps h_{ij} to $\tilde{h}_{ij} = e^{2\omega} h_{ij}$, assuming that H transforms as $\tilde{H} = e^{\frac{2-D}{2}\omega} H$, one has

$$-\frac{\tilde{\nabla}_i \tilde{\nabla}_j \tilde{H}}{\tilde{H}} + \frac{D}{D-2} \frac{\partial_i \tilde{H} \partial_j \tilde{H}}{\tilde{H}^2} = -\frac{\hat{\nabla}_i \hat{\nabla}_j H}{H} + \frac{D}{D-2} \frac{\partial_i H \partial_j H}{H^2} + \frac{\hat{R}_{ij} - \tilde{\hat{R}}_{ij}}{2}. \quad (5.37)$$

For a metric, U(1) gauge field, fluid velocity and current density of the form

$$ds^2 = V(t, x^i) dt^2 - g_{ij} dx^i dx^j, \quad A = \phi dt, \quad u = \sqrt{V} dt, \quad J = \rho_e dt, \quad (5.38)$$

(which is precisely what we have here), the conservation laws $\nabla_\mu T^{\mu\nu} = 0$ imply

$$\partial_t p + \frac{p+\rho}{2} g^{ij} \partial_t g_{ij} = 0, \quad \partial_i p + \frac{p+\rho}{2V} \partial_i V - \frac{\rho_e}{\sqrt{V}} \partial_i \phi = 0. \quad (5.39)$$

These equations carry information on how the pressure gradients balance the equilibrium of the system. In particular, the second one shows that the spatial gradient of the pressure cancels the gravitational and electromagnetic forces. Note that, due to the explicit time-dependence, there is one additional equation w.r.t. (17) of [73].

Let us now turn to (5.36). In the particular case of a conformally flat spatial metric, $h_{ij} = e^{2\omega} \delta_{ij}$, for it to be Einstein it must also be of constant curvature, and one can always take $e^{-\omega} = 1 + \frac{k}{4} r^2$, with $r^2 \equiv \sum x^i x^i$. Then we have $H = \left(1 + \frac{k}{4} r^2\right)^{\frac{D-2}{2}} H_0$, with H_0 satisfying (5.36) on flat space, i.e.,

$$H_0 = (\alpha r^2 + \beta^i x^i + \gamma)^{\frac{2-D}{2}}, \quad (5.40)$$

where α can always be set to 1 by rescaling the parameters M and Q . In this case the energy density and pressure of the fluid are given by

$$\begin{aligned}
 8\pi\rho &= \frac{f^2}{g^2} (G_{tt} - 8\pi T_{tt}^{\text{em}}) = \frac{D(D-1)}{2} \frac{\dot{a}^2}{a^2} + \frac{kD(D-1)}{2a^2} \frac{1}{f^{\frac{2}{D-2}}} \\
 &\quad - \frac{D-1}{D-2} \left(1 - \frac{g}{f}\right) \frac{1}{f^{\frac{2}{D-2}}} \frac{\hat{\nabla}^2 H}{a^2 H} + 2 \frac{D-3}{D-2} \left(1 - \frac{1}{g}\right) \frac{1}{f^{\frac{D}{D-2}}} \frac{\partial_l H h^{lm} \partial_m H}{a^2 H^2}, \\
 8\pi p &= \frac{h^{ij}}{Da^2 f^{\frac{2}{D-2}}} (G_{ij} - 8\pi T_{ij}^{\text{em}}) = -\frac{k}{2a^2} \frac{(D-1)(D-2)}{f^{\frac{2}{D-2}}} + (1-D) \frac{f}{g} \left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right) \\
 &\quad - \frac{D(D-1)}{2} \frac{\dot{a}^2}{a^2} + \frac{2}{a^2 f^{\frac{2}{D-2}}} \left(1 - \frac{1}{g}\right) \frac{D-1}{D} \frac{\hat{\nabla}^2 H}{H}, \tag{5.41}
 \end{aligned}$$

while the current density reads

$$4\pi J = -\sqrt{\frac{D-1}{2(D-2)}} \left[\left(1 - \frac{g}{f}\right)^2 + 4 \frac{g}{f} \left(1 - \frac{1}{g}\right) \right]^{1/2} \frac{g}{a^2 f^{\frac{D}{D-2}}} \frac{\hat{\nabla}^2 H}{H} dt. \tag{5.42}$$

In the maximally charged case, $|Q| = M$, the ansatz (5.32) reduces to

$$ds^2 = \frac{1}{f^2} dt^2 - a^2 f^{\frac{2}{D-2}} ds_D^2, \quad F = \sqrt{\frac{D-1}{2(D-2)}} \frac{1}{f} \left(1 - \frac{1}{f}\right) \frac{dH}{H} \wedge dt,$$

with

$$f = 1 + M \frac{H}{a^{D-2}}, \tag{5.43}$$

and the Einstein tensor boils down to

$$\begin{aligned}
 G_{tt} &= \frac{D(D-1)}{2} \frac{\dot{a}^2}{a^2} \frac{1}{f^2} + \frac{\hat{R}}{2a^2} \frac{1}{f^{\frac{D-1}{D-2}}} - \frac{D-1}{D-2} \left(1 - \frac{1}{f}\right) \frac{1}{f^{\frac{D-1}{D-2}}} \frac{\hat{\nabla}^2 H}{a^2 H} \\
 &\quad + \frac{1}{2} \frac{D-1}{D-2} \left(1 - \frac{1}{f}\right)^2 \frac{1}{f^{\frac{D-1}{D-2}}} \frac{\partial_l H h^{lm} \partial_m H}{a^2 H^2}, \\
 G_{ij} &= (1-D) h_{ij} a^2 f^{\frac{D}{D-2}} \left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right) - \frac{D(D-1)}{2} \dot{a}^2 f^{\frac{2}{D-2}} h_{ij} \\
 &\quad + \hat{R}_{ij} - \frac{1}{2} \hat{R} h_{ij} + \frac{D-1}{D-2} \left(1 - \frac{1}{f}\right)^2 \left[\frac{1}{2} \frac{\partial_l H h^{lm} \partial_m H}{H^2} h_{ij} - \frac{\partial_i H \partial_j H}{H^2} \right]. \tag{5.44}
 \end{aligned}$$

Finally, the electromagnetic energy-momentum tensor becomes

$$8\pi T_{tt}^{\text{em}} = \frac{1}{2} \frac{D-1}{D-2} \left(1 - \frac{1}{f}\right)^2 \frac{1}{f^{\frac{D-1}{D-2}}} \frac{\partial_l H h^{lm} \partial_m H}{a^2 H^2},$$

$$8\pi T_{ij}^{\text{em}} = -\frac{D-1}{D-2} \left(1 - \frac{1}{f}\right)^2 \frac{1}{H^2} \left(\partial_i H \partial_j H - \frac{1}{2} h_{ij} \partial_l H h^{lm} \partial_m H \right). \quad (5.45)$$

In this case the condition to have a perfect fluid source, $G_{ij} - 8\pi T_{ij}^{\text{em}} \propto h_{ij}$, simply reduces to the requirement that the spatial slices are Einstein manifolds, $\hat{R}_{ij} \propto h_{ij}$, while H can now be any function of the spatial coordinates, i.e., (5.36) does not need to hold anymore. The maximally charged solution is thus less constrained. This is of course also true in the four-dimensional case, with suitable forms for the density, pressure and current of the fluid. For a spatial metric of constant curvature the energy density, pressure and current density of the fluid are respectively given by

$$8\pi\rho = \frac{D(D-1)}{2} \frac{\dot{a}^2}{a^2} + \frac{kD(D-1)}{2a^2} \frac{1}{f^{\frac{2}{D-2}}} - \frac{D-1}{D-2} \left(1 - \frac{1}{f}\right) \frac{1}{f^{\frac{2}{D-2}}} \frac{\hat{\nabla}^2 H}{a^2 H},$$

$$8\pi p = -\frac{k}{2a^2} \frac{(D-1)(D-2)}{f^{\frac{2}{D-2}}} + (1-D)f \left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} \right) - \frac{D(D-1)}{2} \frac{\dot{a}^2}{a^2},$$

$$4\pi J = -\sqrt{\frac{D-1}{2(D-2)}} \left(1 - \frac{1}{f}\right) \frac{1}{a^2 f^{\frac{D}{D-2}}} \frac{\hat{\nabla}^2 H}{H} dt. \quad (5.46)$$

Given that H can be an arbitrary function in the extremal case, what was the reason for the appearance of the conformal Laplace equation in section 5.1? To answer this question, let us go back to the nonextremal solution, and consider the case where h_{ij} is the metric on a space of constant curvature. As we already said, one has then (setting $\alpha = 1$)

$$H = \left(1 + \frac{k}{4} r^2\right)^{\frac{D-2}{2}} H_0, \quad H_0 = (r^2 + \beta^i x^i + \gamma)^{\frac{2-D}{2}}. \quad (5.47)$$

If the parameters in (5.47) satisfy the constraint $\gamma = \beta^i \beta^i / 4$, H_0 can be rewritten as

$$H_0 = \frac{1}{|\vec{x} - \vec{x}_0|^{D-2}}, \quad (x_0^i \equiv -\beta^i / 2), \quad (5.48)$$

which is harmonic on D -dimensional flat space. In this case, H in (5.47) satisfies the conformal Laplace equation

$$\hat{\nabla}^2 H = \frac{D-2}{4(D-1)} \hat{R} H. \quad (5.49)$$

(5.49) results thus from extrapolating the nonextremal case (where (5.36) must hold) to the maximally charged situation, under the additional assumption that h_{ij} has constant curvature.

Conclusions

In this thesis we obtained new black hole solutions, with and without supersymmetry, with particular emphasis on multi-centered black holes in cosmological Friedmann–Lemaître–Robertson–Walker backgrounds. These are highly dynamical spacetimes, and we hope that they can be used to shed some light on some long-standing problems in black hole physics which are dynamical in nature, such as what happens when black holes collide and whether the cosmic censorship hypothesis holds.

In chapter 2, using the characterization of supersymmetric solutions to $N = 2$, $d = 4$ gauged supergravity coupled to matter by Meessen and Ortín [7], we were able to obtain what is, as far as we know, the first analytical supersymmetric black hole solution with a nontrivial hyperscalar field. This solution however has a rather peculiar asymptotic behaviour, and the hyperscalar is not charged under the two $U(1)$ gauge fields of the theory. Our actual goal was to find analytic supersymmetric black holes with charged hyperscalars in anti-de Sitter space, possibly generalizing the AdS black hole solutions of [74]. This is a difficult task, since the inclusion of hypermultiplets makes the BPS equations much more complicated. Nevertheless, we still hope to be able to obtain some solution of this kind, perhaps by considering a different gauging or special geometric model.

In chapter 4 we used the classification of fake supersymmetric solutions [8] to obtain multi-centered solutions in a cosmological background. Fake supergravity allows to apply the powerful techniques used to classify supersymmetric solutions of supergravities also to non-supersymmetric theories. In our case, it enabled us to find solutions with and without rotation, and with both flat and curved spatial geometries. The linearity of the fake BPS equations makes it possible to superimpose an arbitrary number of black holes, obtaining multi-centered spacetimes with metrics characterized by harmonic functions on the three-dimensional base space. We first obtained spacetimes generalizing the black holes previously obtained by different methods in [47], which in turn were a generalization of the KT [40] and MOU [45] solutions, and analyzed some physical properties of the single-centered case. Then we added rotation and NUT-charge to

a subclass of these solutions, and also found multi-centered solutions in a FLRW background whose spatial sections are 3-spheres.

Some possible extensions of this work include a more detailed study of the physical properties of our solutions, in particular in the rotating case. It would also be worthwhile to try to extend the analytic studies of nonrotating black hole collisions in de Sitter space performed in [40, 41] to the more general solutions considered here, and see how the results depend on the rotation, the cosmological scale factor different from dS , and the spatial curvature of the underlying FLRW cosmology.

In chapter 5 we obtained a multi-centered generalization of the charged McVittie spacetime [9, 10], sourced by a $U(1)$ gauge field and a charged perfect fluid, in arbitrary dimension. While there is no obvious relation of these solutions to genuine or fake supergravity, the metric is given in terms of a function satisfying the conformal Laplace equation on the spatial slices of the FLRW background, generalizing the usual construction in terms of harmonic functions encountered in multi-centered (fake) supersymmetric black holes. Since the background of these solutions can be any FLRW spacetime, as a particular subcase we obtained multi-centered black holes in a background that is locally anti-de Sitter space in cosmological coordinates. For the single-centered asymptotically AdS case we discussed some physical properties, and verified the validity of a generalized first law of black hole mechanics.

It would be interesting to investigate whether it is possible to mimic the perfect fluid with one or more scalar fields, and to embed our solutions in some simple model of matter-coupled genuine or fake supergravity. Since the charge density σ of the cosmic fluid is nonvanishing for $k \neq 0$, these scalars would have to be charged under a $U(1)$ gauge field.

Notation and conventions

The conventions used in this thesis are the same as those of [8, 7], which in turn are based upon the ones in [75]. We report them here for convenience.

A.1 Tensors

We use Greek letters μ, ν, ρ, \dots as *curved* tensor indices in a coordinate basis and Latin letters a, b, c, \dots as *flat* tensor indices in a tetrad basis.

We symmetrize () and antisymmetrize [] with weight one (i.e. dividing by $n!$).

We use mostly minus signature $(+ - - -)$ for the metric of a four dimensional space-time. η is the Minkowski metric and a general metric is denoted by g .

Flat and curved indices are related by tetrads e_a^μ and their inverses e^a_μ , satisfying

$$e_a^\mu e_b^\nu g_{\mu\nu} = \eta_{ab}, \quad e^a_\mu e^b_\nu \eta_{ab} = g_{\mu\nu}. \quad (\text{A.1})$$

∇ is the total general and Lorentz covariant derivative, whose action on tensors and spinors (ψ) is given by

$$\begin{aligned} \nabla_\mu \xi^\nu &= \partial_\mu \xi^\nu + \Gamma_{\mu\rho}^\nu \xi^\rho, \\ \nabla_\mu \xi^a &= \partial_\mu \xi^a + \omega_{\mu b}^a \xi^b, \\ \nabla_\mu \psi &= \partial_\mu \psi - \frac{1}{4} \omega_\mu^{ab} \gamma_{ab} \psi, \end{aligned} \quad (\text{A.2})$$

where γ_{ab} is the antisymmetric product of two gamma matrices (see next section), $\omega_{\mu b}^a$ is the spin connection and $\Gamma_{\mu\rho}^\nu$ is the affine connection. The respective curvatures are defined through the Ricci identities

$$\begin{aligned} [\nabla_\mu, \nabla_\nu] \xi^\rho &= R_{\mu\nu\sigma}^\rho(\Gamma) \xi^\sigma + T_{\mu\nu}^\sigma \nabla_\sigma \xi^\rho, \\ [\nabla_\mu, \nabla_\nu] \xi^a &= R_{\mu\nu b}^a(\omega) \xi^b, \\ [\nabla_\mu, \nabla_\nu] \psi &= -\frac{1}{4} R_{\mu\nu}^{ab}(\omega) \gamma_{ab} \psi. \end{aligned} \quad (\text{A.3})$$

and given in terms of the connections by

$$R_{\mu\nu\rho}{}^{\sigma}(\Gamma) = 2\partial_{[\mu}\Gamma_{\nu]\rho}{}^{\sigma} + 2\Gamma_{[\mu|\lambda}{}^{\sigma}\Gamma_{\nu]\rho}{}^{\lambda}, \quad (\text{A.4})$$

$$R_{\mu\nu a}{}^b(\omega) = 2\partial_{[\mu}\omega_{\nu]a}{}^b - 2\omega_{[\mu|a}{}^c\omega_{|\nu]c}{}^b.$$

These two connections are related by the tetrad postulate

$$\nabla_{\mu}e_a{}^{\mu} = 0, \quad (\text{A.5})$$

as

$$\omega_{\mu a}{}^b = \Gamma_{\mu a}{}^b + e_a{}^{\nu}\partial_{\mu}e_{\nu}{}^b, \quad (\text{A.6})$$

which implies that the curvatures are, in turn, related by

$$R_{\mu\nu\rho}{}^{\sigma}(\Gamma) = e_{\rho}{}^a e^{\sigma}{}_b R_{\mu\nu a}{}^b(\omega). \quad (\text{A.7})$$

Finally, the requirements of metric compatibility and vanishing torsion fully determine the connections to be of the form

$$\Gamma_{\mu\nu}{}^{\rho} = \frac{1}{2}g^{\rho\sigma}\{\partial_{\mu}g_{\nu\sigma} + \partial_{\nu}g_{\mu\sigma} - \partial_{\sigma}g_{\mu\nu}\}, \quad (\text{A.8})$$

$$\omega_{abc} = -\Omega_{abc} + \Omega_{bca} - \Omega_{cab}, \quad \Omega_{ab}{}^c = e_a{}^{\mu}e_b{}^{\nu}\partial_{[\mu}e_{\nu]}^c.$$

The 4-dimensional fully antisymmetric Levi-Civita symbol is defined by

$$\epsilon^{0123} = +1, \quad \Rightarrow \quad \epsilon_{0123} = -1. \quad (\text{A.9})$$

We define the Hodge dual of a completely antisymmetric tensor of rank k , $F_{(k)}$ by

$$\star F_{(k)}{}^{\mu_1\cdots\mu_{(d-k)}} = \frac{1}{k!\sqrt{|g|}}\epsilon^{\mu_1\cdots\mu_{(d-k)}\mu_{(d-k+1)}\cdots\mu_d}F_{(k)\mu_{(d-k+1)}\cdots\mu_d}. \quad (\text{A.10})$$

Differential forms of rank k are normalized as follows:

$$F_{(k)} \equiv \frac{1}{k!}F_{(k)\mu_1\cdots\mu_k}dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_k}. \quad (\text{A.11})$$

The exterior and interior derivatives act on differential forms as:

$$dF_{(k)} = \frac{1}{k!}\partial_{\nu}F_{(k)\mu_1\cdots\mu_k}dx^{\nu} \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_k}, \quad (\text{A.12})$$

$$\iota_V F_{(k)} = \frac{1}{(k-1)!}V^{\nu}F_{(k)\nu\mu_2\cdots\mu_k}dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_k}. \quad (\text{A.13})$$

For any 4-dimensional 2-form, we define the self-dual and anti-self-dual forms as

$$F^{\pm} \equiv \frac{1}{2}(F \pm i \star F), \quad \Rightarrow \quad \pm i \star F^{\pm} = F^{\pm}. \quad (\text{A.14})$$

For any two 2-forms F and G , we have then

$$F^\pm \cdot G^\mp = 0, \quad F^\pm_{[\mu}{}^\rho \cdot G^\mp_{\nu]\rho} = 0. \quad (\text{A.15})$$

Given any 2-form $F = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu$ and a non-null 1-form $\hat{V} = V_\mu dx^\mu$, we can express F in the form

$$F = V^{-2}[E \wedge \hat{V} - \star(B \wedge \hat{V})], \quad E_\mu \equiv F_{\mu\nu}V^\nu, \quad B_\mu \equiv \star F_{\mu\nu}V^\nu. \quad (\text{A.16})$$

For the complex combinations F^\pm we have

$$F^\pm = V^{-2}[C^\pm \wedge \hat{V} \pm i \star(C^\pm \wedge \hat{V})], \quad C_\mu^\pm \equiv F_{\mu\nu}^\pm V^\nu. \quad (\text{A.17})$$

A.2 Gamma matrices and spinors

We work with a purely imaginary representation

$$\gamma^{a*} = -\gamma^a, \quad (\text{A.18})$$

and our convention for their anticommutator is

$$\{\gamma^a, \gamma^b\} = +2\eta^{ab}. \quad (\text{A.19})$$

Thus,

$$\gamma^0 \gamma^a \gamma^0 = \gamma^{a\dagger} = \gamma^{a-1} = \gamma_a. \quad (\text{A.20})$$

The chirality matrix is defined by

$$\gamma_5 \equiv -i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{4!}\epsilon_{abcd}\gamma^a\gamma^b\gamma^c\gamma^d, \quad (\text{A.21})$$

and satisfies

$$\gamma_5^\dagger = -\gamma_5^* = \gamma_5, \quad (\gamma_5)^2 = 1. \quad (\text{A.22})$$

With this chirality matrix, we have the identity

$$\gamma^{a_1 \dots a_n} = \frac{(-1)^{[n/2]}i}{(4-n)!}\epsilon^{a_1 \dots a_n b_1 \dots b_{4-n}}\gamma_{b_1 \dots b_{4-n}}\gamma_5. \quad (\text{A.23})$$

Our convention for Dirac conjugation is

$$\bar{\psi} = i\psi^\dagger\gamma_0. \quad (\text{A.24})$$

Using the identity Eq. (A.23) the general $d = 4$ Fierz identity for *commuting* spinors takes the form

$$\begin{aligned}
 (\bar{\lambda} M \chi)(\bar{\psi} N \varphi) &= \frac{1}{4}(\bar{\lambda} M N \varphi)(\bar{\psi} \chi) + \frac{1}{4}(\bar{\lambda} M \gamma^a N \varphi)(\bar{\psi} \gamma_a \chi) - \frac{1}{8}(\bar{\lambda} M \gamma^{ab} N \varphi)(\bar{\psi} \gamma_{ab} \chi) \\
 &\quad - \frac{1}{4}(\bar{\lambda} M \gamma^a \gamma_5 N \varphi)(\bar{\psi} \gamma_a \gamma_5 \chi) + \frac{1}{4}(\bar{\lambda} M \gamma_5 N \varphi)(\bar{\psi} \gamma_5 \chi).
 \end{aligned} \tag{A.25}$$

We use 4-component chiral spinors whose chirality is related to the position of the $SU(2)$ index:

$$\gamma_5 \chi_I = +\chi_I, \quad \gamma_5 \psi_{\mu I} = -\psi_{\mu I}, \quad \gamma_5 \epsilon_I = -\epsilon_I. \tag{A.26}$$

Both chirality and position of the $SU(2)$ index are reversed under complex conjugation:

$$\gamma_5 \chi_I^* \equiv \gamma_5 \chi^I = -\chi^I, \quad \gamma_5 \psi_{\mu I}^* \equiv \gamma_5 \psi_{\mu}^I = +\psi_{\mu}^I, \quad \gamma_5 \epsilon_I^* \equiv \gamma_5 \epsilon^I = +\epsilon^I. \tag{A.27}$$

We take this fact into account when Dirac-conjugating chiral spinors:

$$\bar{\chi}^I \equiv i(\chi_I)^\dagger \gamma_0, \quad \bar{\chi}^I \gamma_5 = -\bar{\chi}^I, \quad \text{etc.} \tag{A.28}$$

A.3 Pauli matrices

The Hermitean, unitary, traceless, 2×2 Pauli matrices σ^x ($x = 1, 2, 3$) are

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A.29}$$

and they satisfy the following properties:

$$(\sigma^x)^I{}_J (\sigma^y)^J{}_K = \delta^{xy} \delta^I{}_K + i \varepsilon^{xyz} (\sigma^z)^I{}_K, \tag{A.30}$$

$$(\sigma^x)^K{}_I (\sigma^x)^L{}_J = 2 \delta^K{}_J \delta^L{}_I - \delta^K{}_I \delta^L{}_J \tag{A.31}$$

$$(\sigma^{[x]})^I{}_J (\sigma^{[y]})^K{}_L = -\frac{i}{2} \varepsilon^{xyz} [\delta^I{}_L (\sigma^z)^K{}_J - (\sigma^z)^I{}_L \delta^K{}_J]. \tag{A.32}$$

We can also define σ -matrices with only upper or lower indices as

$$(\sigma^x)^{IJ} \equiv (\sigma^x)^I{}_K \varepsilon^{KJ}, \quad (\sigma^x)_{IJ} \equiv \varepsilon_{IK} (\sigma^x)^K{}_J, \tag{A.33}$$

which are symmetric matrices

$$(\sigma^x)^{[IJ]} = (\sigma^x)_{[IJ]} = 0, \tag{A.34}$$

and satisfy the property

$$i(\sigma^x)_{IJ} = [i(\sigma^x)^{IJ}]^* . \quad (\text{A.35})$$

It is possible to use the σ -matrices to switch between $SU(2)$ indices and three dimensional vector indices using the convention

$$A^I{}_J \equiv \frac{i}{2} A^x (\sigma^x)^I{}_J . \quad (\text{A.36})$$

A.4 Bilinears and Fierz identities

Starting from (fake) Killing spinors it is possible to construct the following bilinears. The scalar bilinear is defined by

$$X \equiv \frac{1}{2} \varepsilon^{IJ} \bar{\epsilon}_I \epsilon_J , \quad (\text{A.37})$$

and can be inverted to give

$$\bar{\epsilon}_I \epsilon_J = \varepsilon_{IJ} X . \quad (\text{A.38})$$

The vector bilinears are defined by the decomposition

$$V_a{}^I{}_J \equiv i \bar{\epsilon}^I \gamma_a \epsilon_J = \frac{1}{2} V_a \delta^I{}_J + \frac{1}{\sqrt{2}} V_a^x (\sigma^x)^I{}_J , \quad (\text{A.39})$$

whose inversion gives

$$V_a = V_a^I{}_I \quad \text{and} \quad V_a^x = \frac{1}{\sqrt{2}} (\sigma^x)^J{}_I V_a^I{}_J . \quad (\text{A.40})$$

Finally there are three imaginary self-dual 2-forms defined by

$$\Phi_{IJ\ ab} \equiv \bar{\epsilon}_I \gamma_{ab} \epsilon_J = \frac{i}{2} (\sigma^x)_{IJ} \Phi_{ab}^x \longrightarrow \Phi^x = i (\sigma^x)^{IJ} \Phi_{IJ} . \quad (\text{A.41})$$

The Fierz identities imply

$$\eta_{ab} = \frac{1}{4|X|^2} [V_a V_b - 2 V_a^x V_b^x] , \quad (\text{A.42})$$

and

$$\iota_V V^x = 0 , \quad g(V, V) = 4|X|^2 , \quad g(V^x, V^y) = -2|X|^2 \delta^{xy} . \quad (\text{A.43})$$

The bilinear 2-form Φ can also be obtained from the vector bilinears V and V^x as

$$\Phi^x = \frac{i}{\sqrt{2}X} [V^x \wedge V + i \star (V^x \wedge V)] . \quad (\text{A.44})$$

In this appendix we present the definitions and review some properties of the geometries mentioned in the main text. The discussion here is mostly based on references [16, 76, 17], the appendices of [7] and, for section B.8, appendix C of [35].

In the literature there are different definitions of special Kähler manifold, not all of them equivalent. Our definition is based on the definition 2 of [76], which does not rely on the existence of a prepotential.

For a more in depth analysis of the topics discussed here, and for the omitted proofs of some statements, we refer the reader to the sources mentioned above and references therein.

B.1 Complex manifolds

A topological space \mathcal{M} is a complex manifold of complex dimension n if there is an open cover $\{U_I\}$ of \mathcal{M} such that on each U_I there is a homeomorphism $\psi_I : U_I \rightarrow V_I \subset \mathbb{C}^n$, and on intersections $U_I \cap U_J$ the transition maps $\psi_I \circ \psi_J^{-1}$ are holomorphic.

From a local point of view, an n -dimensional complex manifold can be viewed as a $2n$ -dimensional real manifold parametrized by n complex coordinates Z^i . We can start from the $2n$ real coordinates $x^1, \dots, x^n, x^{n+1}, \dots, x^{2n}$ and define

$$Z^i = x^i + ix^{i+n}, \quad i = 1, \dots, n \tag{B.1}$$

$$\bar{Z}^{\bar{i}} = x^i - ix^{i+n} = \overline{Z^i}.$$

We can also write the $2n$ complex coordinates as Z^a , where the index a runs first through the holomorphic coordinates i and then through the anti-holomorphic coordinates \bar{i} . The map (B.1) between the coordinates x^i and the coordinates Z^a can then be considered as a standard coordinate transformation, and the usual relations from differential geometry remain valid. While a generic coordinate transformation $Z'^i = f^i(Z, \bar{Z})$ would not preserve the splitting between holomorphic and anti-holomorphic coordinates, holomorphic transformations of the form $Z'^i = f^i(Z)$ do.

A different characterization of a complex manifold can be given in terms of a complex structure. On every real manifold of even dimension it is possible to locally define an *almost complex structure*, which is a tensor $J_a{}^b(x)$ mapping the tangent space in itself, with the property $J_a{}^c J_c{}^b = -\delta_a{}^b$. If this structure can be extended globally to define a smooth tensor field, then the manifold is an *almost complex manifold*. Every complex manifold has a global almost complex structure, which can be given in holomorphic coordinates by

$$J = \begin{pmatrix} i\delta_i{}^{\bar{j}} & 0 \\ 0 & -i\delta_{\bar{i}}{}^j \end{pmatrix}, \quad (\text{B.2})$$

but the converse is in general not true. There is a theorem stating that an almost complex manifold is a complex manifold if and only if the Nijenhuis tensor vanishes:

$$N_{ab}{}^c \equiv J_a{}^d (\partial_d J_b{}^c - \partial_b J_d{}^c) - J_b{}^d (\partial_d J_a{}^c - \partial_a J_d{}^c) = 0. \quad (\text{B.3})$$

B.2 Kähler manifolds

If a Riemannian metric is defined on a complex manifold, the line element takes the general form

$$ds^2 = \mathcal{G}_{ab} dZ^a dZ^b = 2\mathcal{G}_{i\bar{j}} dZ^i d\bar{Z}^{\bar{j}} + \mathcal{G}_{ij} dZ^i dZ^j + \mathcal{G}_{\bar{i}\bar{j}} d\bar{Z}^{\bar{i}} d\bar{Z}^{\bar{j}}. \quad (\text{B.4})$$

The metric is said to be *Hermitian* if there are choices of coordinates for which $\mathcal{G}_{ij} = \mathcal{G}_{\bar{i}\bar{j}} = 0$, or equivalently the line element takes the form

$$ds^2 = \mathcal{G}_{ab} dZ^a dZ^b = 2\mathcal{G}_{i\bar{j}} dZ^i d\bar{Z}^{\bar{j}}. \quad (\text{B.5})$$

These coordinate systems are then said to be adapted to the Hermitian structure. Given a Hermitian metric, we can define the fundamental 2-form:

$$\mathcal{J} = i\mathcal{G}_{i\bar{j}} dZ^i \wedge d\bar{Z}^{\bar{j}}. \quad (\text{B.6})$$

A *Kähler manifold* is a complex manifold with Hermitian metric such that its fundamental form, in this case also called Kähler form, is closed, $d\mathcal{J} = 0$. This is equivalent to the condition

$$\partial_k \mathcal{G}_{i\bar{j}} - \partial_i \mathcal{G}_{k\bar{j}} = 0, \quad (\text{B.7})$$

which implies the existence in every coordinate patch of a real function $\mathcal{K}(Z, \bar{Z})$, the *Kähler potential*, such that the metric is locally given by

$$\mathcal{G}_{i\bar{j}} = \partial_i \partial_{\bar{j}} \mathcal{K}. \quad (\text{B.8})$$

The Kähler potential is not uniquely defined, since a Kähler transformation

$$\mathcal{K}(Z, \bar{Z}) \rightarrow \mathcal{K}'(Z, \bar{Z}) = \mathcal{K}(Z, \bar{Z}) + f(Z) + \bar{f}(\bar{Z}) \quad (\text{B.9})$$

leaves the metric invariant. In general there is no globally defined Kähler potential, but potentials on overlapping coordinate patches differ by a Kähler transformation.

A Kähler manifold can equivalently be defined as a real even dimensional manifold with an almost complex structure J and Hermitian metric \mathcal{G} such that J is covariantly constant with respect to the Levi-Civita connection:

$$\nabla_a J_b{}^c = 0. \quad (\text{B.10})$$

The condition for the metric to be Hermitian can be restated as the requirement that it is invariant under the action of the almost complex structure, $J\mathcal{G}J^T = \mathcal{G}$. The condition (B.10) is sufficient to ensure that the Kähler form, which in real coordinates can be written as

$$\mathcal{J} = \frac{1}{2} J_{ab} dx^a \wedge dx^b, \quad J_{ab} = J_a{}^c \mathcal{G}_{cb}, \quad (\text{B.11})$$

is closed and that the Nijenhuis tensor (B.3) vanishes.

It is worth noting that the Levi-Civita connection for a Kähler metric is greatly simplified. The only non-vanishing components are those of the form

$$\Gamma_{jk}^i = \mathcal{G}^{i\bar{l}} \partial_j \mathcal{G}_{k\bar{l}}, \quad \Gamma_{\bar{j}\bar{k}}^{\bar{i}} = \mathcal{G}^{l\bar{i}} \partial_{\bar{j}} \mathcal{G}_{l\bar{k}}. \quad (\text{B.12})$$

B.3 Kähler-Hodge manifolds

Consider a holomorphic line bundle, that is a rank one complex vector bundle with holomorphic transition functions, over a Kähler manifold $L \rightarrow \mathcal{M}$. If we take a Hermitian fibre metric $h(Z, \bar{Z})$ on L , there is a unique connection compatible with both the holomorphic and the Hermitian structures. The curvature form of this compatible connection is given by

$$\mathcal{F} = \partial_i \partial_{\bar{j}} \log(h) dZ^i \wedge d\bar{Z}^{\bar{j}}. \quad (\text{B.13})$$

Since \mathcal{F} is a closed 2-form, the equivalence class $c_1(L) \equiv \frac{i}{2\pi} [\mathcal{F}]$ defined from \mathcal{F} up to addition of exact differential forms is a cohomology. This cohomology class is what is known as the first (and only) Chern class of the line bundle, and it is independent of the choice of h . The fundamental form \mathcal{J} of a Kähler manifold is also a closed differential form, and like the curvature form it defines a cohomology class $\frac{1}{2\pi} [\mathcal{J}]$, called Kähler class. Integrating a representative of any cohomology class on a closed surface one obtains a number which clearly does not depend on the choice of the representative itself, and is thus a characteristic of the class.

A Kähler manifold \mathcal{M} is a *Kähler-Hodge manifold* if and only if there exists a holomorphic line bundle $L \rightarrow \mathcal{M}$ such that the first Chern class of the bundle equals the Kähler class:

$$c_1(L) = \frac{1}{2\pi} [\mathcal{J}]. \quad (\text{B.14})$$

The first Chern class of a line bundle is always an element of the integral cohomology group, which means that the integral over a closed surface of a member of $c_1(L)$ is an

integer. This, together with (B.14), implies that the Kähler class is of integral cohomology too. Conversely, it can be shown that on a Kähler manifold with Kähler class of integral cohomology it is always possible to define a holomorphic line bundle whose first Chern class equals the Kähler class.

In supergravity there are in general complex scalar fields Z^i parametrizing a Kähler target manifold \mathcal{M} , and Kähler transformations of the form (B.9) are symmetries of the action. Under this symmetry the fermions of the theory transform as

$$\psi \rightarrow e^{-\frac{1}{4}(f(Z) - \bar{f}(\bar{Z}))} \psi, \quad (\text{B.15})$$

defining in a natural way a $U(1)$ -bundle over \mathcal{M} .

The Kähler potentials in different coordinate patches on intersections $U_I \cap U_J$ satisfy

$$\mathcal{K}_I - \mathcal{K}_J = f_{IJ}(Z) + \bar{f}_{IJ}(\bar{Z}), \quad \text{with} \quad f_{IJ} = -f_{JI}, \quad (\text{B.16})$$

and for the $U(1)$ -bundle to be well defined on triple intersections $U_I \cap U_J \cap U_K$ a *cocycle condition* must hold:

$$e^{-\frac{1}{4}(f_{IJ}(Z) - \bar{f}_{IJ}(\bar{Z}))} e^{-\frac{1}{4}(f_{JK}(Z) - \bar{f}_{JK}(\bar{Z}))} e^{-\frac{1}{4}(f_{KI}(Z) - \bar{f}_{KI}(\bar{Z}))} = 1 \quad (\text{B.17})$$

$$\implies \quad \Im(f_{IJ} + f_{JK} + f_{KI}) = 4\pi c_{IJK},$$

where c_{IJK} is a real constant. At the same time the identity $(\mathcal{K}_I - \mathcal{K}_J) + (\mathcal{K}_J - \mathcal{K}_K) + (\mathcal{K}_K - \mathcal{K}_I) = 0$ implies from (B.16)

$$\Re(f_{IJ} + f_{JK} + f_{KI}) = 0. \quad (\text{B.18})$$

From equations (B.17) and (B.18) follows that we can define a holomorphic line bundle on \mathcal{M} , with sections given by holomorphic functions $\chi(Z)$ and transition functions $e^{f_{IJ}(Z)}$. This bundle is well defined since it too satisfies the cocycle condition on triple intersections,

$$e^{f_{IJ}(Z)} e^{f_{JK}(Z)} e^{f_{KI}(Z)} = 1. \quad (\text{B.19})$$

The exponential of the Kähler potential $e^{\mathcal{K}(Z, \bar{Z})}$ can be seen as a Hermitian metric on this bundle, and the curvature of the unique compatible connection is

$$\mathcal{F} = \partial_i \partial_{\bar{j}} \mathcal{K} dZ^i \wedge d\bar{Z}^{\bar{j}} = \mathcal{G}_{i\bar{j}} dZ^i \wedge d\bar{Z}^{\bar{j}} = -i \mathcal{J} \implies c_1(L) = \frac{1}{2\pi} [\mathcal{J}]. \quad (\text{B.20})$$

In supergravity then the existence and regularity of fermion fields transforming under Kähler transformations as in (B.15) implies that the target space of the complex scalars is a Kähler-Hodge manifold.

B.4 Special Kähler manifolds

Consider an n -dimensional Kähler-Hodge manifold \mathcal{M} , and let $L \rightarrow \mathcal{M}$ be the holomorphic line bundle with first Chern class equal to the Kähler class of \mathcal{M} . \mathcal{M} is a *special Kähler manifold* if there exist on \mathcal{M} :

1. A holomorphic flat vector bundle $\mathcal{H} \rightarrow \mathcal{M}$ with fibre dimension $2(n+1)$, and whose structure group is the symplectic group $Sp(2(n+1), \mathbb{R})$.
2. A holomorphic section $\Omega(Z)$ of $L \otimes \mathcal{H}$ satisfying the constraints

$$\mathcal{K} = -\log [i\langle \Omega | \bar{\Omega} \rangle] \quad \text{and} \quad \langle \Omega | \mathfrak{D}_i \Omega \rangle = \langle \Omega | \partial_i \Omega \rangle = 0, \quad (\text{B.21})$$

where the Kähler covariant derivative of $\Omega(Z)$ is given by $\mathfrak{D}_i \Omega = \partial_i \Omega + (\partial_i \mathcal{K}) \Omega$.

The section $\Omega(Z)$ is locally represented by symplectic vectors of the form

$$\Omega(Z) = \begin{pmatrix} \chi^\Lambda(Z) \\ \mathcal{F}_\Lambda(Z) \end{pmatrix} \quad \Lambda = 0, \dots, n, \quad (\text{B.22})$$

and on coordinate patch intersections $U_I \cap U_J$ the transition maps act on $\Omega(Z)$ as

$$\Omega(Z) \longrightarrow e^{f_{IJ}(z)} M_{IJ} \Omega(Z) \quad (\text{B.23})$$

where the maps $f_{IJ} : U_I \cap U_J \rightarrow \mathbb{C}$ are holomorphic, the M_{IJ} are constant matrices in $Sp(2(n+1), \mathbb{R})$, and on triple intersections a cocycle condition holds:

$$e^{f_{IJ}f_{JK}f_{KI}} = 1, \quad M_{IJ}M_{JK}M_{KI} = \mathbb{1}. \quad (\text{B.24})$$

If we assume that the components \mathcal{F}_Λ depend on Z^i only through $\chi^\Lambda(Z)$, the second equation in (B.21) becomes

$$\partial_i \chi^\Lambda [2\mathcal{F}_\Lambda - \partial_\Lambda (\chi^\Sigma \mathcal{F}_\Sigma)] = 0, \quad (\text{B.25})$$

and is satisfied if there is a holomorphic homogeneous function of second degree $\mathcal{F}(\chi)$, called a *prepotential*, such that $\mathcal{F}_\Lambda = \partial_\Lambda \mathcal{F}$. It is possible to prove that if the $(n+1) \times (n+1)$ matrix $(\chi^\Lambda, \mathfrak{D}_i \chi^\Lambda)$ is invertible a prepotential always exists. In general a prepotential may not exist, but it's always possible, for a given special Kähler manifold, to change to a frame in which there is a prepotential by acting on $\Omega(Z)$ with a symplectic transformation [76].

A section $\mathcal{V}(Z)$ of a different bundle, which is only covariantly holomorphic, is often introduced

$$\mathcal{V}(Z) = \begin{pmatrix} \mathcal{L}^\Lambda(Z) \\ \mathcal{M}_\Lambda(Z) \end{pmatrix} = e^{\mathcal{K}(Z, \bar{Z})/2} \Omega(Z), \quad \mathfrak{D}_{\bar{i}} \mathcal{V} = \partial_{\bar{i}} \mathcal{V} - \frac{1}{2} (\partial_{\bar{i}} \mathcal{K}) \mathcal{V} = 0, \quad (\text{B.26})$$

together with the objects

$$\mathcal{U}_i \equiv \mathfrak{D}_i \mathcal{V} = \partial_i \mathcal{V} + \frac{1}{2} (\partial_i \mathcal{K}) \mathcal{V} = \begin{pmatrix} f_i^\Lambda \\ h_{\Lambda i} \end{pmatrix}. \quad (\text{B.27})$$

In terms of \mathcal{V} and \mathcal{U}_i the constraints (B.21) become

$$\langle \bar{\mathcal{V}} | \mathcal{V} \rangle = i \quad \text{and} \quad \langle \mathcal{V} | \mathfrak{D}_i \mathcal{V} \rangle = \langle \mathcal{V} | \mathcal{U}_i \rangle = 0. \quad (\text{B.28})$$

In extended supergravities, the coupling of the complex scalars to the vector fields is given in terms of a $(n+1) \times (n+1)$ *period matrix* \mathcal{N} . This matrix is entirely determined from special geometric data, and is defined through the relations

$$\mathcal{M}_\Lambda = \mathcal{N}_{\Lambda\Sigma} \mathcal{L}^\Sigma, \quad h_{\Lambda i} = \bar{\mathcal{N}}_{\Lambda\Sigma} f_i^\Sigma. \quad (\text{B.29})$$

The $(n+1) \times (n+1)$ matrix $(\mathcal{L}^\Sigma, \bar{f}_i^\Sigma)$ is always invertible [76], so \mathcal{N} is given by the matrix product

$$\mathcal{N}_{\Lambda\Sigma} = (\mathcal{M}_\Lambda, \bar{h}_{\Lambda\bar{i}}) (\mathcal{L}^\Sigma, \bar{f}_i^\Sigma)^{-1}, \quad (\text{B.30})$$

which when a prepotential $\mathcal{F}(\chi)$ exists becomes

$$\mathcal{N}_{\Lambda\Sigma} = \bar{\mathcal{F}}_{\Lambda\Sigma} + 2i \frac{(N_{\Lambda\Gamma} \chi^\Gamma) (N_{\Sigma\Delta} \chi^\Delta)}{\chi^\Omega N_{\Omega\Psi} \chi^\Psi}, \quad (\text{B.31})$$

with $N_{\Lambda\Sigma} \equiv \Im(\mathcal{F}_{\Lambda\Sigma}) = \Im(\partial_\Lambda \partial_\Sigma \mathcal{F})$, showing explicitly that \mathcal{N} is symmetric. In the general case the symmetry of \mathcal{N} follows from the vanishing of the matrix product $(\mathcal{L}^\Sigma, \bar{f}_i^\Sigma)^T (\bar{\mathcal{N}} - \bar{\mathcal{N}}^T)_{\Lambda\Sigma} (\mathcal{L}^\Sigma, \bar{f}_i^\Sigma)$, due to the second constraint in (B.28), together with the invertibility of $(\mathcal{L}^\Sigma, \bar{f}_i^\Sigma)$.

The definitions (B.26) and (B.28) imply the relations

$$\langle \mathcal{U}_i | \bar{\mathcal{U}}_{\bar{i}} \rangle = i\mathcal{G}_{i\bar{i}}, \quad \langle \mathcal{U}_i | \bar{\mathcal{V}} \rangle = 0, \quad (\text{B.32})$$

while the definition of the period matrix $\mathcal{N}_{\Lambda\Sigma}$ (B.29) together with its symmetry leads to

$$\langle \mathcal{U}_i | \mathcal{U}_j \rangle = 0. \quad (\text{B.33})$$

The identities (B.28), (B.32) and (B.33) tell us that the $n+1$ sections \mathcal{V} and \mathcal{U}_i , and their complex conjugates, are linearly independent. This allows us to write the useful completeness relation, for a generic symplectic section \mathcal{A} ,

$$\mathcal{A} = i\langle \mathcal{A} | \bar{\mathcal{V}} \rangle \mathcal{V} - i\langle \mathcal{A} | \mathcal{V} \rangle \bar{\mathcal{V}} + i\langle \mathcal{A} | \mathcal{U}_i \rangle \mathcal{G}^{i\bar{i}} \bar{\mathcal{U}}_{\bar{i}} - i\langle \mathcal{A} | \bar{\mathcal{U}}_{\bar{i}} \rangle \mathcal{G}^{i\bar{i}} \mathcal{U}_i. \quad (\text{B.34})$$

B.5 Symmetries of special Kähler manifolds

For a generic Riemannian or pseudo-Riemannian manifold, the isometries are the transformations that preserve the metric \mathcal{G} . This means that they are generated by a vector field k satisfying the Killing equation

$$\mathcal{L}_k \mathcal{G}_{ab} = \nabla_a k_b + \nabla_b k_a = 0. \quad (\text{B.35})$$

When the manifold is Kähler, one also wants to preserve the complex structure

$$\mathcal{L}_k J_a{}^b = \nabla_a k^c J_c{}^b - \nabla_c k^b J_a{}^c = 0. \quad (\text{B.36})$$

Since the complex structure in holomorphic coordinates has the form (B.2), when the indices a and b are both holomorphic or anti-holomorphic this equation is automatically satisfied, while for mixed indices it translates to the conditions

$$\partial_{\bar{i}} k^j = 0, \quad \partial_i k^{\bar{j}} = 0 \quad \implies \quad k = k^i(Z) \partial_i + k^{\bar{i}}(\bar{Z}) \partial_{\bar{i}}, \quad (\text{B.37})$$

which is the reason why a vector field k satisfying both eq. (B.35) and eq. (B.36) is called *holomorphic Killing vector*. If k is holomorphic, the completely holomorphic or anti-holomorphic components of eq. (B.35) are automatically satisfied for a Kähler metric, while the components with mixed indices become

$$0 = \nabla_i k_{\bar{j}} + \nabla_{\bar{j}} k_i = \partial_i k_{\bar{j}} + \partial_{\bar{j}} k_i, \quad (\text{B.38})$$

which is equivalent to require that the Kähler form is preserved:

$$0 = \mathcal{L}_k \mathcal{J} = (\iota_k d + d\iota_k) \mathcal{J} = d\iota_k \mathcal{J} = -id (k_i dZ^i - k_{\bar{i}} d\bar{Z}^{\bar{i}}) \quad (\text{B.39})$$

where we made use of $d\mathcal{J} = 0$ and ι_k is the interior derivative with respect to k . Since $\iota_k \mathcal{J}$ is a closed form, by Poincaré's lemma there exists a (real) function $\mathcal{P}(Z, \bar{Z})$, called a *moment (or momentum) map*, such that locally

$$\iota_k \mathcal{J} = -d\mathcal{P}. \quad (\text{B.40})$$

This means that a holomorphic Killing vector, if the function $\mathcal{P}(Z, \bar{Z})$ is known, is given by

$$k^i(Z) = i\mathcal{G}^{i\bar{j}} \partial_{\bar{j}} \mathcal{P}(Z, \bar{Z}), \quad (\text{B.41})$$

which is the reason why \mathcal{P} is often called Killing prepotential. Equation (B.41) can be inverted to give

$$\begin{aligned} \mathcal{P}(Z, \bar{Z}) &= -i [k^i \partial_i \mathcal{K}(Z, \bar{Z}) - \lambda(Z)] = i [k^{\bar{i}} \partial_{\bar{i}} \mathcal{K}(Z, \bar{Z}) - \bar{\lambda}(\bar{Z})] \\ &= -\frac{i}{2} [k^i \partial_i \mathcal{K}(Z, \bar{Z}) - k^{\bar{i}} \partial_{\bar{i}} \mathcal{K}(Z, \bar{Z})] - \frac{i}{2} (\lambda(Z) - \bar{\lambda}(\bar{Z})), \end{aligned} \quad (\text{B.42})$$

where $\lambda(Z)$ is a generic holomorphic function. This expression is completely general since eq. (B.40) defines \mathcal{P} up to a constant, which can be absorbed in the function $\lambda(Z)$. Eq. (B.42) implies also

$$0 = i (k^i \partial_i \mathcal{K} + k^{\bar{i}} \partial_{\bar{i}} \mathcal{K}) - i (\lambda + \bar{\lambda}) = i (\mathcal{L}_k \mathcal{K} - \lambda - \bar{\lambda}), \quad (\text{B.43})$$

meaning that in general, under an isometry generated by k , the Kähler potential changes by a Kähler transformation, determined by the holomorphic function $\lambda(Z)$ appearing in the expression of the moment map \mathcal{P} .

The Killing vectors of a manifold generate a Lie algebra. In particular for holomorphic Killing vectors the Lie bracket relation is

$$\begin{aligned} [k_\Lambda, k_\Sigma] &= \left(k_\Lambda^i \partial_i k_\Sigma^{\bar{j}} - k_\Sigma^i \partial_i k_\Lambda^{\bar{j}} \right) \partial_{\bar{j}} + \left(k_\Lambda^{\bar{i}} \partial_{\bar{i}} k_\Sigma^j - k_\Sigma^{\bar{i}} \partial_{\bar{i}} k_\Lambda^j \right) \partial_j \\ &= -f_{\Lambda\Sigma}^\Gamma \left(k_\Gamma^i \partial_i + k_\Gamma^{\bar{i}} \partial_{\bar{i}} \right), \end{aligned} \quad (\text{B.44})$$

showing that the holomorphic and anti-holomorphic components of the Killing vectors satisfy the algebra separately. The Lie derivative must satisfy the condition $[\mathcal{L}_\Lambda, \mathcal{L}_\Sigma] = \mathcal{L}_{[k_\Lambda, k_\Sigma]}$, which when applied to eq. (B.43) gives

$$\mathcal{L}_\Lambda \lambda_\Sigma - \mathcal{L}_\Sigma \lambda_\Lambda = -f_{\Lambda\Sigma}^\Gamma \lambda_\Gamma. \quad (\text{B.45})$$

This in turn implies that the moment maps transform in the adjoint representation of the symmetry group,

$$\mathcal{L}_\Lambda \mathcal{P}_\Sigma = \left(k_\Lambda^i \partial_i + k_\Lambda^{\bar{i}} \partial_{\bar{i}} \right) \mathcal{P}_\Sigma = i \left(k_\Lambda^i \mathcal{G}_{i\bar{j}} k_\Sigma^{\bar{j}} - k_\Sigma^i \mathcal{G}_{i\bar{j}} k_\Lambda^{\bar{j}} \right) = -f_{\Lambda\Sigma}^\Gamma \mathcal{P}_\Gamma, \quad (\text{B.46})$$

where \mathcal{L}_Λ is the Lie derivative with respect to k_Λ . Equation (B.46) is called *equivariance condition*.

Up to now we have considered a generic Kähler manifold. For a special Kähler manifold, we want to consider isometries that also preserve the special structure. This amounts to require that the isometries are embedded in the symplectic group, or more precisely that the sections $\Omega(Z)$ transform as follows:

$$\mathcal{L}_\Lambda \Omega = \mathcal{S}_\Lambda \Omega - \lambda_\Lambda(Z) \Omega, \quad (\text{B.47})$$

where the functions $\lambda_\Lambda(Z)$ generate Kähler transformations and the real, constant, $2(n+1) \times 2(n+1)$ matrices

$$\mathcal{S}_\Lambda = \begin{pmatrix} a & b \\ c & -a^T \end{pmatrix}, \quad \text{with} \quad b = b^T \quad \text{and} \quad c = c^T, \quad (\text{B.48})$$

generate transformations in $Sp(n+1)$. Note that the matrices \mathcal{S}_Λ must provide a representation of the Lie algebra of the symmetry group, $[\mathcal{S}_\Lambda, \mathcal{S}_\Sigma] = -f_{\Lambda\Sigma}^\Gamma \mathcal{S}_\Gamma$. Since Ω is holomorphic, its Lie derivative is given by

$$\mathcal{L}_\Lambda \Omega = k_\Lambda^i \partial_i \Omega = k_\Lambda^i \mathcal{D}_i \Omega - k_\Lambda^i \partial_i \mathcal{K} \Omega = k_\Lambda^i \mathcal{D}_i \Omega - i \mathcal{P}_\Lambda \Omega - \lambda_\Lambda \Omega, \quad (\text{B.49})$$

where we made use of (B.42). Comparing the expression (B.49) with the one in (B.47), and taking the symplectic product with $\bar{\Omega}$, one obtains

$$k_\Lambda^i \langle \bar{\Omega} | \mathcal{D}_i \Omega \rangle - i \mathcal{P}_\Lambda \langle \bar{\Omega} | \Omega \rangle = \langle \bar{\Omega} | \mathcal{S}_\Lambda \Omega \rangle. \quad (\text{B.50})$$

The first constraint in (B.21) implies that $\langle \bar{\Omega} | \mathcal{D}_i \Omega \rangle = 0$, and we are left with the simple relation

$$\mathcal{P}_\Lambda = e^{\mathcal{K}} \langle \bar{\Omega} | \mathcal{S}_\Lambda \Omega \rangle. \quad (\text{B.51})$$

If instead we take the symplectic product with Ω we get the constraint

$$\langle \Omega | \mathcal{S}_\Lambda \Omega \rangle = 0. \quad (\text{B.52})$$

B.6 Quaternionic-Kähler manifolds

A *quaternionic-Kähler manifold* is a $4n$ -dimensional Riemannian manifold (with $n > 1$) whose holonomy group is a subgroup of $USp(2n) \times SU(2)$.

An equivalent, but perhaps more transparent, characterization of such manifolds is the following: a quaternionic-Kähler manifold is a $4n$ -dimensional Riemannian manifold admitting a locally defined triplet \vec{K}_u^v of almost complex structures satisfying the quaternion relation

$$K^1 K^2 = K^3, \quad (\text{B.53})$$

and whose Levi-Civita connection preserves \vec{K} up to a rotation,

$$\nabla_w \vec{K}_u^v + \vec{A}_w \times \vec{K}_u^v = 0, \quad (\text{B.54})$$

where $\vec{A} \equiv \vec{A}_u(q) dq^u$ is a triplet of 1-forms on the manifold.

For $n > 1$ it is possible to prove that a quaternionic-Kähler manifold is necessarily Einstein and that the $SU(2)$ curvature is proportional to the complex structures:

$$R_{uv} = \frac{1}{4n} H_{uv} R \quad (\text{B.55})$$

$$F^x \equiv dA^x + \frac{1}{2} \varepsilon^{xyz} A^y \wedge A^z = \varkappa K^x, \quad \varkappa \equiv \frac{R}{4n(n+2)}, \quad (\text{B.56})$$

where H_{uv} is the metric tensor of the manifold and $K^x \equiv \frac{1}{2} K_u^x{}^z H_{zv} dq^u \wedge dq^v$ are the 2-forms associated with the almost complex structures. For $n = 1$ we will consider equations (B.55) and (B.56) part of the definition of quaternionic-Kähler manifold. In supergravity the constant \varkappa in (B.56) must be negative, in particular we take $\varkappa = -2$, which implies that the quaternionic-Kähler manifolds relevant for supergravity have negative scalar curvature.

It should be noted that, despite the name, quaternionic-Kähler manifolds are not in general Kähler manifolds. In fact in many cases they don't even admit a globally defined almost complex structure (see e.g. [77]). If however $\varkappa = 0$, which means that the $SU(2)$ curvature is zero, the manifold is *hyper-Kähler*, and in particular it is Ricci flat and Kähler, having a triplet of covariantly constant complex structures. Hyper-Kähler manifolds do not appear in supergravity, but they are relevant for rigid supersymmetry.

In supergravity it is customary to introduce frame fields $U_{\alpha I}^u(q)$ connecting the scalar fields q^u , that are coordinates on a quaternionic-Kähler manifold, to the fermions ζ^α . The index α runs from 1 to $2n$, while $I = 1, 2$, so that the $U_{\alpha I}^u(q)$ can be seen as $4n \times 4n$

invertible matrices when (αI) is considered as a single index. The inverse is written $U^{\alpha I}_u(q)$:

$$U^{\alpha I}_u U^v_{\alpha I} = \delta^v_u, \quad U^{\alpha I}_u U^u_{\beta J} = \delta^I_J \delta^\alpha_\beta, \quad (\text{B.57})$$

and there is a reality condition

$$U_{\alpha I u} \equiv \left(U^{\alpha I}_u \right)^* = \varepsilon_{IJ} \mathbb{C}_{\alpha\beta} U^{\beta J}_u, \quad U^{u\alpha I} \equiv (U^u_{\alpha I})^* = \varepsilon^{IJ} \mathbb{C}^{\alpha\beta} U^u_{\beta J}, \quad (\text{B.58})$$

where $\mathbb{C}_{\alpha\beta}$ is a non-degenerate tensor satisfying

$$\mathbb{C}_{\alpha\beta} = -\mathbb{C}_{\beta\alpha}, \quad \mathbb{C}_{\alpha\beta} \mathbb{C}^{\beta\gamma} = -\delta^\gamma_\alpha, \quad \mathbb{C}^{\alpha\beta} = (\mathbb{C}_{\alpha\beta})^*. \quad (\text{B.59})$$

The metric is given by

$$H_{uv} = U^{\alpha I}_u \varepsilon_{IJ} \mathbb{C}_{\alpha\beta} U^{\beta J}_v, \quad (\text{B.60})$$

so that $\varepsilon_{IJ} \mathbb{C}_{\alpha\beta}$ can be interpreted as a metric in tangent space.

From the above definitions follows that it is possible to define a triplet of almost complex structures starting from the frame fields,

$$K^x_u{}^v = -i U^{\alpha I}_u U^v_{\alpha J} \sigma^x_I{}^J, \quad (\text{B.61})$$

where the σ^x are the Pauli matrices. The frame fields are covariantly constant using a connection on every index,

$$\nabla_v U^{\alpha I}_u \equiv \partial_v U^{\alpha I}_u + A_{vJ}{}^I U^{\alpha J}_u + \Delta_{v\beta}{}^\alpha U^{\beta I}_u - \Gamma_{vu}{}^w U^{\alpha I}_w = 0, \quad (\text{B.62})$$

where $A_{uI}{}^J \equiv \frac{i}{2} \vec{A}_u \cdot \vec{\sigma}_I{}^J$ and $\Delta_{u\alpha}{}^\beta$ is the $USp(2n)$ connection. It is then easily verified that the almost complex structures in (B.61) satisfy (B.54).

B.7 Symmetries of quaternionic-Kähler manifolds

To preserve the quaternionic-Kähler structure we must require that the isometries preserve the complex structures K^x up to a rotation. This means that the “quaternionic” Killing vectors k generating the isometries satisfy the equations

$$\mathcal{L}_k H_{uv} = \nabla_u k_v + \nabla_v k_u = 0 \quad (\text{B.63})$$

$$\mathcal{L}_k K^x_u{}^v = k^w \nabla_w K^x_u{}^v + \nabla_u k^w K^x_w{}^v - \nabla_w k^v K^x_u{}^w = -\varepsilon^{xyz} W^y K^z_u{}^v, \quad (\text{B.64})$$

where we introduced a compensator field W^x . Using (B.54), equation (B.64) can be rewritten as

$$\nabla_u k^w K^x_w{}^v - \nabla_w k^v K^x_u{}^w = \varepsilon^{xyz} P^y K^z_u{}^v, \quad \text{with } P^x \equiv k^u A_u{}^x - W^x. \quad (\text{B.65})$$

The objects P^x are called *triholomorphic moment maps*, and they are in many ways analogous to the moment maps introduced in section B.5.

Contracting equation (B.65) with $K_v^l{}^u$, and using the quaternion algebra relation $K_u^x{}^w K_w^y{}^v = -\delta^{xy} \delta_u^v + \varepsilon^{xyz} K_u^z{}^v$, we obtain the expression

$$P^x = \frac{1}{2n} K_u^x{}^v \nabla_v k^u, \quad (\text{B.66})$$

which gives the moment maps P^x in terms of the Killing vector k .

Equation (B.64) is equivalent to the analogous equation for the 2-forms K^x ,

$$\mathcal{L}_k K^x = (\iota_k d + d\iota_k) K^x = -\varepsilon^{xyz} W^y K^z, \quad (\text{B.67})$$

which, since from eq. (B.54) we have $dK^x = -\varepsilon^{xyz} A^y \wedge K^z$, can be rewritten as

$$D\iota_k K^x \equiv d\iota_k K^x + \varepsilon^{xyz} A^y \wedge \iota_k K^x = \varepsilon^{xyz} P^y K^z. \quad (\text{B.68})$$

Taking the exterior derivative of (B.68), after some straightforward calculations and making use of the proportionality relation (B.56) between F^x and K^x , we arrive to

$$\varepsilon^{xyz} \omega^y \wedge K^z = 0 \quad \text{with} \quad \omega^x \equiv dP^x + \varepsilon^{xyz} A^y P^z + \varkappa \iota_k K^x. \quad (\text{B.69})$$

Equation (B.69) in components is $\varepsilon^{xyz} \omega_{[u}^y K_{v]w}^z = 0$, and contracting with K^{uv} it becomes $K_u^x{}^v \omega_v^x = 0$. Since it is always possible to rotate to a frame in which two of the three ω^x are zero, and since $K_u^x{}^v$ are invertible matrices, this implies $\omega^x = 0$, or

$$D_u P^x \equiv \partial_u P^x + \varepsilon^{xyz} A_u^y P^z = \varkappa K_{uv}^x k^v. \quad (\text{B.70})$$

Equation (B.70) is the analogous for quaternionic-Kähler manifolds of eq. (B.41), and can be considered as the defining equation of the triholomorphic moment maps.

We assume that the Killing vectors satisfy the Lie algebra $[k_\Lambda, k_\Sigma] = -f_{\Lambda\Sigma}^\Gamma k_\Gamma$. The Lie derivative should then satisfy the condition $[\mathcal{L}_\Lambda, \mathcal{L}_\Sigma] = \mathcal{L}_{[k_\Lambda, k_\Sigma]}$. Applying this constraint to equation (B.64) we obtain for the compensator fields the relation

$$2\mathcal{L}_{[\Lambda} W_{\Sigma]}^x + \varepsilon^{xyz} W_\Lambda^y W_\Sigma^z = -f_{\Lambda\Sigma}^\Gamma W_\Gamma^x, \quad (\text{B.71})$$

which from eq. (B.70) implies for the moment maps the equivariance condition

$$\varepsilon^{xyz} P_\Lambda^y P_\Sigma^z - \varkappa k_\Lambda^u K_{uv}^x k_\Sigma^v = f_{\Lambda\Sigma}^\Gamma P_\Gamma^x, \quad (\text{B.72})$$

analogous to the equivariance condition (B.46) for Kähler manifolds.

B.8 Gauduchon-Tod spaces

A *Weyl manifold* is an n -dimensional manifold \mathcal{M} together with a conformal class $[g]$ of metrics on \mathcal{M} , and a torsionless connection D which preserves the conformal class, i.e.

$$D_\rho g_{\mu\nu} = 2\theta_\rho g_{\mu\nu}, \quad (\text{B.73})$$

where g is any representative of $[g]$ and θ is a 1-form on \mathcal{M} . From the above definition follows that we can express the connection as

$$D_\mu \xi_\nu = \nabla_\mu \xi_\nu + \gamma_{\mu\nu}{}^\rho \xi_\rho \quad \text{with} \quad \gamma_{\mu\nu}{}^\rho = \gamma_{\nu\mu}{}^\rho = \delta_\mu^\rho \theta_\nu + \delta_\nu^\rho \theta_\mu - g_{\mu\nu} \theta^\rho, \quad (\text{B.74})$$

where ∇ is the Levi-Civita connection for the chosen $g \in [g]$. The curvature tensor of this connection and the associated Ricci and scalar curvature are defined by

$$[D_\mu, D_\nu] \xi_\rho = -W_{\mu\nu\rho}{}^\sigma \xi_\sigma, \quad W_{\mu\nu} \equiv W_{\mu\rho\nu}{}^\rho, \quad W \equiv W_\mu{}^\mu. \quad (\text{B.75})$$

The Ricci tensor is not symmetric and its antisymmetric and symmetric parts are given respectively by

$$W_{[\mu\nu]} = -\frac{n}{2} F_{\mu\nu}, \quad \text{where} \quad F \equiv d\theta \quad (\text{B.76})$$

and

$$W_{(\mu\nu)} = R_{\mu\nu} - (n-2)\nabla_{(\mu}\theta_{\nu)} - (n-2)\theta_\mu\theta_\nu - g_{\mu\nu}[\nabla_\rho\theta^\rho - (n-2)\theta_\rho\theta^\rho]. \quad (\text{B.77})$$

Equation (B.73) implies that, under a conformal transformation $g \rightarrow e^{2w}g$ sending one element of $[g]$ to another, the 1-form θ transforms as $\theta \rightarrow \theta + dw$. This means that θ can be seen as a gauge field gauging a \mathbb{R} -symmetry. $W_{\mu\nu\rho}{}^\sigma$ and $W_{\mu\nu}$ are conformally invariant, while the scalar curvature transforms as $W \rightarrow e^{-2w}W$.

In analogy with the definition of Einstein manifolds, a Weyl manifold is said to be *Einstein-Weyl* if the curvatures satisfy the conformally invariant condition

$$W_{(\mu\nu)} = \frac{1}{n} g_{\mu\nu} W. \quad (\text{B.78})$$

A smooth manifold with three complex structures, that is three globally defined almost complex structures whose Nijenhuis tensor vanishes, satisfying the quaternion algebra, is called an *hypercomplex manifold*. If in addition a Riemannian metric is defined on the manifold which is Hermitian with respect to each of the three complex structures, $g(J^x V, J^x W) = g(V, W)$, then the manifold is called *hyper-Hermitian*. In [61], Gauduchon and Tod studied the structure of four dimensional hyper-Hermitian Riemannian spaces admitting a tri-holomorphic Killing vector, i.e. a Killing vector compatible with the three complex structures on the hyper-Hermitian space. A result of that study is that the three dimensional base-space, that is the space of the orbits of the tri-holomorphic Killing vector with the induced metric, is determined by an orthonormal frame, E^x , a 1-form θ and a real function κ that must satisfy

$$dE^x = \theta \wedge E^x - \kappa \star E^x, \quad (\text{B.79})$$

where \star is the Hodge operator associated to the metric constructed out of the frame fields E^x . Equation (B.79) is equivalent to the statement that the space is a three dimensional

Einstein-Weyl manifold with the additional conditions:

$$W = -\frac{3}{2}\kappa^2, \quad (\text{B.80})$$

$$\star d\theta = d\kappa + \kappa\theta. \quad (\text{B.81})$$

A three dimensional Einstein-Weyl manifold that satisfies (B.80)-(B.81), or equivalently a three dimensional Riemannian manifold satisfying (B.79), is called a *Gauduchon-Tod space*.

Dynamical black holes

Black holes are conventionally defined by the existence of an event horizon. A rigorous definition of event horizon however relies on the global assumption of asymptotic flatness (see e.g. [78]). Furthermore the global nature of event horizons means that they cannot be located by a physical observer, since this would require the knowledge of the entire causal structure of the spacetime.

In this appendix we present an alternative definition of black hole, proposed by S. Hayward and based on the quasi-local concept of *trapping horizons* [11], which does not require the spacetime to be asymptotically flat and can be used also for dynamical spacetimes. Following [13] we will also give, in the spherically symmetric case, a generalized first law of black hole dynamics.

C.1 Trapping horizons

The intuitive idea behind the definition of trapping horizons is that inside a black hole both the ingoing and the outgoing light rays converge, so that all signals are confined in a shrinking region.

Start from the definition of marginal surface. Assuming that the spacetime is time-orientable, let θ_+ and θ_- be the expansions of the future-pointing null geodesics orthogonal to a spatial two-surface S . Then S is a *marginal surface* if one of the expansions, which we take to be θ_+ , vanishes on S :

$$\theta_+|_S = 0. \quad (\text{C.1})$$

A *trapping horizon* is the closure, \bar{H} , of a three-surface H foliated by marginal surfaces on which

$$\theta_-|_H \neq 0 \quad \text{and} \quad \mathcal{L}_-\theta_+|_H \neq 0, \quad (\text{C.2})$$

where \mathcal{L}_- is the Lie derivative along the direction of the null geodesics with expansion θ_- . Marginal surfaces and trapping horizons are called

$$\left\{ \begin{array}{ll} \text{outer} & \text{if } \mathcal{L}_-\theta_+|_H < 0 \\ \text{inner} & \text{if } \mathcal{L}_-\theta_+|_H > 0 \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{ll} \text{future} & \text{if } \theta_-|_H < 0 \\ \text{past} & \text{if } \theta_-|_H > 0 \end{array} \right. . \quad (\text{C.3})$$

The future outer trapping horizon provides a quasi-local definition of black hole, which does not rely on the global structure of spacetime or on the assumption of asymptotic flatness. This definition captures the idea that the ingoing light rays should converge, $\theta_-|_H < 0$, while the outgoing light rays should diverge outside the black hole and converge inside, so that they are parallel on the horizon, $\theta_+|_H = 0$ with $\mathcal{L}_-\theta_+|_H < 0$. In the same way the past outer trapping horizon defines a white hole, while inner trapping horizons are associated with cosmological horizons.

C.2 Spherical symmetry

In a spherically symmetric spacetime, the area A of the spheres of symmetry is a geometrical invariant. This enables us to define in a coordinate-independent way the *areal radius* $\mathcal{R} \equiv \sqrt{\frac{A}{4\pi}}$.

The line element can always be locally written in double null form,

$$ds^2 = 2g_{+-}d\xi^+d\xi^- - \mathcal{R}^2d\Omega^2, \quad (\text{C.4})$$

where the null vectors ∂_{ξ^\pm} are taken to be future directed and \mathcal{R} is a function of ξ_\pm , and the null expansions are given simply by

$$\theta_\pm = \frac{\partial_\pm A}{A} = 2\frac{\partial_\pm \mathcal{R}}{\mathcal{R}}. \quad (\text{C.5})$$

Marginal surfaces are then spheres on which $g^{-1}d\mathcal{R}$ is null,

$$\partial_\mu \mathcal{R} \partial^\mu \mathcal{R} = 0, \quad (\text{C.6})$$

and trapping horizons are the closure of three dimensional hypersurfaces foliated by marginal spheres. It also follows that marginal spheres and trapping horizons are outer if $\nabla^2 \mathcal{R} < 0$, inner if $\nabla^2 \mathcal{R} > 0$ and degenerate if $\nabla^2 \mathcal{R} = 0$; and that they are future if $g^{-1}d\mathcal{R}$ is future directed, and past if $g^{-1}d\mathcal{R}$ is past directed.

Spherical symmetry also allows to define the quasi-local Misner-Sharp energy [58],

$$E \equiv \frac{\mathcal{R}}{2} (1 + \partial_\mu \mathcal{R} \partial^\mu \mathcal{R}), \quad (\text{C.7})$$

which can be interpreted as the active gravitational energy and enjoys many desirable properties, in particular reducing to the correct expression in a number of physically interesting limits [12].

In a dynamical spacetime there is no timelike Killing vector. If the spacetime is spherically symmetric, however, it is still possible to identify a preferred time coordinate. This dynamic time is determined by an analogue of a timelike Killing vector, the *Kodama vector* [79] defined as

$$K \equiv g^{-1} (*_2 d\mathcal{R}) \quad (\text{C.8})$$

where $*_2$ denotes the Hodge dual with respect to the two dimensional metric normal to the spheres of symmetry. From the definition follows that

$$K^\mu \partial_\mu \mathcal{R} = 0 \quad \text{and} \quad K^\mu K_\mu = \partial_\mu \mathcal{R} \partial^\mu \mathcal{R} = 1 - \frac{2E}{\mathcal{R}}, \quad (\text{C.9})$$

which means that K is spatial on trapped spheres, temporal on untrapped spheres, and null on marginal spheres. In particular, a trapping horizon is a hypersurface on which K is null, and can be seen as a generalization of a Killing horizon, which is defined as a hypersurface on which a Killing vector is null. Note however that unlike a Killing horizon, a trapping horizon is in general not null.

In the stationary, spherically symmetric case the Kodama and Killing vectors agree if K and $g^{-1}d\mathcal{R}$ commute.

C.3 Dynamical surface gravity and local Hawking temperature

A dynamical surface gravity can be defined in a spherically symmetric spacetime as

$$k_l \equiv -\frac{1}{2} \tilde{\nabla}^2 \mathcal{R} \quad (\text{C.10})$$

where $\tilde{\nabla}$ is the derivative operator associated with the normal two dimensional metric. From the definition it is immediate to conclude that k_l is positive, negative or vanishes respectively on outer, inner and degenerate trapping horizons.

From the Einstein equations follows that

$$k_l = \frac{E}{\mathcal{R}^2} - 2\pi \mathcal{T} \mathcal{R}, \quad (\text{C.11})$$

where \mathcal{T} is the trace of the total energy-momentum tensor $T_{\mu\nu}$ with respect to the normal metric. This expression shows that when $\mathcal{T} = 0$, k_l has the form of the Newtonian gravitational acceleration, with E substituting the Newtonian mass.

The Kodama vector satisfies everywhere the relation

$$K^\mu \nabla_{[\nu} K_{\mu]} = k_l \partial_\nu \mathcal{R}, \quad (\text{C.12})$$

which on trapping horizons can be written in the form

$$K^\mu \nabla_{[\nu} K_{\mu]}|_{\text{TH}} = \pm k_l K_\nu, \quad (\text{C.13})$$

analogous to the relation in the stationary case defining the surface gravity from the timelike Killing vector. As a consequence, when the Kodama and Killing vectors agree, e.g. in the Reissner-Nordström case, k_l reduces to the usual surface gravity.

In [14] the authors used the Hamilton-Jacobi tunneling method to show that the dynamical surface gravity (C.10) can be identified, near a trapping horizon, with a temperature

$$T \simeq \frac{k_l}{2\pi}, \quad (\text{C.14})$$

analogously to the identification of the surface gravity with the Hawking temperature in the stationary case. T must be positive, so this local temperature is only defined near outer trapping horizons, on which $k_l > 0$. It was also argued that the temperature measured to leading order near the horizon by an observer, whose worldline is an integral curve of the Kodama vector, is not T , but instead

$$\hat{T} \simeq \frac{T}{\sqrt{C}}, \quad \text{with} \quad C \equiv 1 - \frac{2E}{\mathcal{R}}. \quad (\text{C.15})$$

The presence of the redshift factor C means that the temperature diverges on the trapping horizon, where $\mathcal{R} = 2E$. More precisely, T is the (finite) limit on the horizon of the redshift-renormalized temperature:

$$\hat{T}\sqrt{C} \xrightarrow{\text{TH}} T. \quad (\text{C.16})$$

C.4 Generalized first law of black hole dynamics

The definition of the Misner-Sharp Energy (C.7) together with the Einstein equations implies

$$\partial_\mu E = A\psi_\mu + \frac{1}{2}\mathcal{T}\partial_\mu V \quad (\text{C.17})$$

where $V \equiv \frac{4}{3}\pi\mathcal{R}^3$ is the areal volume and

$$\psi_\mu \equiv T_{\mu\nu}\partial^\nu\mathcal{R} + \frac{1}{2}\mathcal{T}\partial_\mu\mathcal{R}. \quad (\text{C.18})$$

In [13] it is argued that eq. (C.17) effectively expresses conservation of energy, with the two terms on the right representing respectively an energy supply term and a work term.

Projecting eq. (C.17), evaluated on a trapping horizon, along a vector z tangent to the horizon, we obtain the generalized *first law of black hole dynamics*:

$$E' = \frac{k_l A'}{8\pi} + \frac{1}{2}\mathcal{T}V', \quad (\text{C.19})$$

where the prime stands for a derivative along z . If an electromagnetic field is present, excluding magnetic monopoles it is constrained by spherical symmetry to be in a purely electric configuration, and (C.19) can be rewritten as

$$E' = \frac{k_l A' + \mathcal{E}^2 V'}{8\pi} + \frac{1}{2}\mathcal{T}_{\text{other}} V', \quad (\text{C.20})$$

where \mathcal{E} is the magnitude of the electric field. The contribution of the electric field to the work term agrees with the standard expression for electric work in special relativity,

$$W = \frac{1}{8\pi} \int_\Sigma \mathcal{E}^2 dV, \quad (\text{C.21})$$

substantiating the interpretation as work term.

In the stationary case, without gravitational sources besides the electromagnetic field, the only spherically symmetric solution is the Reissner-Nordström solution. If again we exclude magnetic monopoles this solution is characterized by two constants, the electric charge e and the asymptotic energy m , and for $m > |e|$ it represents a black hole with Killing horizon at $\mathcal{R} = m + \sqrt{m^2 - e^2}$ and surface gravity $k = \frac{1}{\mathcal{R}^2} \sqrt{m^2 - e^2}$. For this black hole the first law is usually given as

$$dm = \frac{k dA}{8\pi} + \frac{e de}{\mathcal{R}}, \quad (\text{C.22})$$

where the differentials are taken in state space (m, e) . The Misner-Sharp energy for the Reissner-Nordström black hole is

$$E = m - \frac{e^2}{2\mathcal{R}}, \quad (\text{C.23})$$

so that eq. (C.22) can be rewritten as

$$dE = \frac{k dA + \mathcal{E}^2 dV}{8\pi}, \quad (\text{C.24})$$

which has a form similar to eq. (C.20), showing that eq. (C.19) can be considered a dynamical generalization of the usual first law. The main differences are that the generalized law is evaluated on trapping horizons, and that the state space differentials are replaced by derivatives along the horizon.

In [11] Hayward also introduced a generalized second law of black hole dynamics. For a spherically symmetric spacetime the generalized *second law of black hole dynamics* states that if the null energy condition holds on a trapping horizon, then the area of the spheres of symmetry A , or equivalently the energy E , is nondecreasing along the horizon if the horizon is future outer or past inner, or nonincreasing if the horizon is past outer or future inner.

The analogue of the zeroth law is trivial in spherical symmetry, since it simply states that k_l is constant on the marginal spheres foliating a trapping horizon.

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