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From Oscillations to Normal Modes: an educational path for the upper secondary school

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Contents

Abstract

The present thesis work starts from the assumption that harmonic oscillations and normal modes are key physical concepts. They are fundamental in quantum physics, in electromagnetism (especially in treating coupled oscillating circuits and electromagnetic waves), in acoustics and in mechanical systems. The conceptual and practical importance of normal modes emerges also clearly from the fact that every small and sufficiently smooth oscillation of a complex system is given by a linear superposition of its normal modes. The notion of normal modes is thus a powerful conceptual organizer. Nevertheless, in teaching practice (at least in Italy), only short time is devoted to harmonic motion, rarely coupled oscillators are treated and, in secondary school text-books, normal modes are usually not even present. The purpose of this thesis work is to develop an effective path on scillations for the upper secondary school that leads to the normal modes of oscillations. To do this, an educational reconstruction of the concept of harmonic motion has been necessary as the harmonic motion is a fundamental prerequisite for the understanding of normal modes. The introduction of normal modes is, for upper secondary school students, complicated by the complexity of the mathematics involved. In our path we propose to overcome the mathematical difficulties through an experimental approach and the use of different tools such as video and picture analysis, also in slow motion, data logging and data analysis techniques and applet simulations, with the goal of being as simple as possible from the mathematical point of view but without losing the advantages that mathematics (even at simple level) can provide. In this perspective, a multiple representation approach has been used. The path on oscillations that we present here is the result of a Design Based Research on normal modes with Italian upper secondary school students. The complete path has been proposed to three classes of 11th grade students during curricular lessons. A version of the sequence has been proposed also to other three classes (one of grade 11th and two of grade 12th) during afternoon extra-curricular lessons, and a version with university-level formalism has also been proposed to a group of undergraduate students in mathematics during the third year course “Preparation of Didactical Experiments”. A reduced version of the path has also been proposed to a number of classes of 12th grade students within the one-shot lessons on oscillations (afternoon extra-curricular activities) in the framework of PLS (Piano Lauree Scientifiche) activities. The one-shot lessons have been attended, over time, by about six hundred students.

The all path is based on a number of activities in which we start from a real experiment or a video or else an applet simulation to introduce and discuss a limited topic. The general purpose is to identify, among the oscillations, those that give rise

to a peculiar kind of motion, the harmonic motion, and determine the conditions under which such motion can be obtained. A number of significant situations of harmonic and anharmonic motions are investigated and criteria to establishing the harmonicity/anharmonicity of the oscillation are discussed. An important tool for the analysis of the data is then introduced: the Fast Fourier Transform. The FFT is introduced as a tool and not discussed through mathematics. Then the concept on resonance is introduced in a phenomenological way through experiments and exploring related videos in the Internet videos database. The next step is the introduction of the coupling between two oscillators and the discovery of particular motion configurations: the Normal Modes of Oscillation. We then extend the experiments to three, four, five....many coupled oscillators until we arrive to the continuous case; first in one dimension with the string and then in two dimensions with the Cladni plates and we study the normal modes of such complex systems.

The structure of the thesis is as follows: an introduction to the motivations, a description of the state of the art and the formulation of the research questions. Then a brief description of the methodological framework mainly based on the Design Based Research approach and the Model of the Educational Reconstruction. The reconstruction of the Harmonic motion at university level follows; the translation of such a reconstruction into upper secondary school level is developed in chapter five. In chapter four normal modes for a system of two, three, N coupled oscillators are treated with the proper formalism; also in this case the translation into upper secondary school level is developed in chapter five. Then the very core of the thesis follows , namely the developing of the path, as briefly described above. The next chapter reports a path developed with undergraduate students as an implementation of the study of oscillations. It is the study of the modes of oscillation in the interesting case of a parametric oscillator were there is a non-linear coupling between modes of oscillation. In the last section, the main results of the experimentation of the path with threes classes of 11th grade students are briefly presented. These results are based on questionnaires (pre-test and post-test), discussions and interviews.

1. Introduction

1.1. State of the art

In the Italian school, students face the topic "*oscillations*" between the 11th and 12th grade, that is between the third and the fourth year of upper secondary school, as an introduction to the wider topic of waves. Generally, in teaching practice, only a short time is devoted to harmonic motion, coupled oscillators are rarely treated and almost never normal modes of oscillation are presented. Often harmonic motion, in school textbooks, is treated as a complement of kinematics [Caforio & Ferilli, 2006, Bergamaschini et al., 2007, Fazio & Montano, 2000, Amaldi, 2011, Amaldi, 2005, Papucci, 2008]. Moreover harmonic and coupled oscillations are rarely supported by experiments in lab activities. Not only in the Italian school, but also in the literature it is difficult to find out teaching paths on normal modes for secondary school with a detailed analysis of disciplinary knots and learning problems. Nonetheless harmonic oscillations and normal modes of oscillations have a great importance for the understanding of many fundamental topics such as acoustics and optics and, moreover, they are fundamental for studying modern physics.

1.2. Overview

Normal modes are peculiar ways of oscillation of complex systems such that when a system oscillates in one of its normal modes each part of the system moves of harmonic motion at the same frequency of the other parts and with a fixed phase relation [Barbieri & Giliberti, 2012, Crowford, 1968, Fitzpatrick, 2013]. The conceptual and practical importance of normal modes clearly emerges from the fact that every oscillation of a system is given by a linear superposition of its normal modes [Barbieri & Giliberti, 2012, Smith, 2010]. Moreover normal modes are important conceptual organizers; in fact they allow the description of almost all oscillating systems (and related phenomena) of physical interest in many physical contexts from a unifying point of view [Barbieri & Giliberti, 2012]. They are fundamental in quantum physics [Giliberti, 2007, Smith, 2010], in electromagnetism (especially in the comprehension of coupled oscillating circuits and electromagnetic waves), in acoustics, in mechanical systems and, in addition, they give the possibility of introducing the Fourier Transform in a simple but meaningful way

[Fitzpatrick, 2013, Smith, 2010][Fitzpatrick, 2013, Smith, 2010], at least within a phenomenological approach. From a pedagogical point of view normal modes may therefore give students a deeper (and sometimes also faster) learning about the many faces of oscillations in different contexts. One of the reasons of the fact that normal modes are commonly not treated in upper secondary school is the complexity of the mathematics involved. In fact this makes the treating of normal modes best suited for university courses rather than for those of upper secondary school. Even if at university level many scientific papers can be found in the literature about normal modes, nonetheless a clear educational path about them and their didactical implications for secondary school with a detailed analysis of disciplinary knots and learning problems is, to the best of our knowledge, still missing. The Physics Education Research Group of the University of Milano has been studying for years ways and paths for a meaningful introduction of modern physics in secondary school. For this purpose an educational reconstruction [Duit et al., 2005][Duit et al., 2005]of many physics topics has become a necessity. In this context the possibility of introduction of normal modes in secondary school has become a priority. The work on oscillations we are going to describe in the following is part of this general research.

1.3. The research questions

As said above, the introduction of the study of oscillations with normal modes is, for upper secondary school students, complicated by the complexity of the mathematics involved. In the path that we have developed and that we are testing, we propose to overcome the mathematical difficulties through an experimental approach and the use of different tools such as video and picture analysis [tra,], also in slow motion, data logging and data analysis techniques [Log,] and applet simulations [Martinez et al., 2010, Fis, , Falstad, 2014] , with the goal of being as simple as possible from the mathematical point of view but without losing the advantages that mathematics (even at simple level) can provide. The harmonic motion is a fundamental prerequisite for the understanding of normal modes [Barbieri & Giliberti, 2012]. Therefore it is important to construct a definition of harmonic motion that is both effective for learning and easy to handle in different, also non-standard, situations. In a dynamic perspective, we think that the definition of harmonic motion as the motion of a body subjected to a force that is the linearization of a restoring force, suits the task. Data logging techniques and video analysis are used to overcome the necessity of a full analytical treatment of normal modes. Moreover, we think that the analysis of the diagrams obtained with the data logging can help students to face the well-known difficulties [McDermott et al., 1987] in the representation and interpretation of graphs. In the data analysis techniques of oscillating complex systems, the FFT (Fast Fourier Transform) tool plays a fundamental role. We are investigating whether the use of FFT tool is effective even without its mathematical treatment as this is suitable only for students of undergraduate courses. The main research questions can be summarized as follows:

- Is the dynamical choice of defining the harmonic motion through the linearization of the restoring force more effective than the kinematical definition through the projection of a uniform circular motion on a diameter (as it is usually done in text-books in Italy)?
- Are we able to build an effective calculus-less path on normal modes for high school students?
- Can the use of data-logging techniques, applied to real experiments, help students to overcome some important difficulties in the representation and interpretation of graphs?
- Can the use of the FFT tool, even without a specific mathematical treatment, help students to reach a deeper understanding of oscillatory phenomena?

1.4. Methods

The path has been developed on the basis of an educational reconstruction of the content [Kattmann et al., 1995] namely oscillations and normal modes. In this context the researcher designs and creates learning environments, experimental devices and teaching/learning sequences that s(he) experiments, evaluates, revises and develops within authentic educational settings [Duit et al., 2005]. Following the main steps of the Educational Reconstruction Model [Kattmann et al., 1995] we: 1) started with the analysis of the contents: from the analysis of publications, university textbooks, high school textbooks and websites to the analysis of the key points and the conceptual nodes; 2) developed the educational path; 3) experimented the path with students; 4) performed an empirical investigation of the learning process. The path has been experimented, revised and implemented three times with 11th and 12th grade upper secondary students. The first goal has been the translation of the focus from a kinematic point of view to a dynamic one in the discussion of simple harmonic motion with the possibility of the recognition at a glance of the harmonic or non-harmonic oscillations even without a careful mathematical analysis of the forces involved. After this, the overall goal has been the introduction of normal modes of oscillation for upper secondary students. In the reconstruction of the contents an extensive use of multiple representations [Simon, 1977] has been made. The whole experimentation has been made in the perspective of a Designed Based Research (DBR) oriented to the production of Teaching/Learning Sequences (TLS).

The DBR recognizes the deep complexity of the teaching/learning process due to the many variables involved. These variables are related to the social context; the project itself to be realized; the teachers who have to implement the project. The irreducibility of these variables makes very interesting the analysis of the relations between the context and the results. Therefore an accurate analysis of the results becomes very useful. Therefore we wanted to design a research that, as suggested by the Design Based Research Collective[Edr, 2003, DBR, 2003] is characterised by these key points: 1) the goal is the design of the learning context and the development of learning theories; 2) the development and the implementation of the project take place in continuous cycles of design, implementation and redesign; 3) the research takes into account how the project actually works in real context: it has to document the successes and failures in order to refine the understanding of issues related to learning. Therefore, in our experimentations we used written tests, interviews and audio recordings in 3 classes of upper secondary school students in curricular hours with a guided Inquiry Based Science Education (IBSE) approach and in the 60 hours university open inquiry course “Preparazione di Esperienze Didattiche” for undergraduate students in Mathematics. A reduced path has been tested also with 600 secondary school students attending the extracurricular PLS [PLS,] lab on oscillations.

2. The methodological framework

2.1. Design and research methodology

2.1.1. The Design-Based Research approach

The Design-Based Research (DBR) methodological approach was born and has developed in recent years as a response to the difficulties that the experimental-type research had to address the problems and to acknowledge the insistent demands coming from the real educational contexts [Ligorio & Cacciamani, 2013]. This approach starts from pioneering studies of Ann Brown [Brown, 1992] and Alan Collins [Collins, 1992] who were pioneers in performing design experiments in real classes of students [Collins et al., 2004]. The DBR approach has been taken up by a group of North American researchers who have given themselves the name of “The Design-Based Researche Collective” [DBR, 2003]. The DBR has been defined as a systematic yet flexible methodology aimed to improve the educational practicies. It is based on a number of cycles that include: **design**, **implementation**, **analysis** and **re-design**. The main purpose is to guide the researcher towards the definition of principles and theories which are sensitive to the context in which these innovations are tested[Wang & Hannafin, 2005]. There is a clear difference with respect to the experimental approach. In the experimental approach the researchers formulate the hypothesys on the basis of previous observations or theories, then they design the experiment and, according to the results, they can validate (or rebut) the hypotesis. In the DBR approach, instead, the researchers analyze the practical problems, then they develop the solutions on the basis of the design and they implement such solutions through many cycles. At the end of each cycle, the researchers improve the principles of the project and find new solutions to introduct in the further cycle [Ligorio & Cacciamani, 2013].

According to the *Collective* directions, the DBR is characterized by the following features:

- The DBR has two main goals: to design learning environments and to develop theories and “proto-theories” on learning processes. The two goals are closely linked.
- The development and the implementation of the project, as well as the research which arises from the control of its quality, take place through cycles of design, implementation, analysis and re-design.

- The research based on projects must lead to shareable theories. They must help comunicate to other researchers the relevant informations both on the design and the educational implications.
- The research must account for how the project works in real environments. It should register successes and failures and should also put into evidence the many interactions which can make us understand the learning problems.
- The DBR makes use of mixed methods to maximize the credibility and the adaptability of the project. The methods and the techniques can be very different: questionnaires, case studies, interviews etc. The different kind of data, qualitative and quantitative, are mixed together in the analysis of the implementation of the project.

All these features summarize the outlines of the DBR and the assumptions on which it is based, together with some operative indications. First of all there is the awareness that the teaching/learning process is a very complex process. Such a complexity is due to the number of variables interacting. The variables refer to the social context (the students and the context they belong to), to the didactical project and to the teachers and researchers who carry out the project. Because of the irreducibility of these variables and their interactions, the role of context/environment on the results of the educational interventions, must be considered. For this reason it is necessary to distinguish the results that are context-dependent from the general ones. It is also clear the necessity of implementing the analysis methods as to take into account the interactions between variables. That's why the DBR gives priority to internal-type assesment criteria instead of external-type ones (for istance the ones based on the comparison with a control group) [Battaglia, 2011].

The DBR provides many benefits that are of interest for the educational research [DBR, 2003]. In fact it is a very versatile methodology that allows to design and investigate many different innovations, from the curricula, to the didactical activities to the introduction of artifacts such as the new technologies to support learning, and many others. Moreover, the DBR is a methodology where the research fits the context and is modulated, in a flexible way, on the needs of the class. Another important point is the fact that with DBR the researcher and the teachers work together to innovations in teaching/learning practice in real educational environments. DBR is widely used for the construction of learning environments based on the use of information technologiy and comunication technology. It gives many important indications about the links between theory and in-classroom didactical practice, regarding such tecnologies [Squire, 2005].

There are many interesting examples of use of DBR in real classes. For istance the *River City* project [Clarke et al., 2006] in which a multimedia virtual environment has been contructed for teaching science and in particular biology and ecology. Another interesting example of use of the DBR approach is the italian project *CROSS (Comunità di ricerca online per lo studio delle scienze)* born within the wider project *SeT* (Scienza e Tecnologia) and implemented in six cycles in a network of italian schools [Cacciamani, 2008].

It appears clear that the DBR approach is an importand research method. It is a research method “for the school” rather than “on the school” as it is able to support the continuous innovation within the educational contexts [Ligorio & Cacciamani, 2013].

2.1.2. The Model of the Educational Reconstruction

A major concearn of Science Education Research is to improve instructional practices for schools at all levels. About this, an intensive international debate has developed since the early 90s in scientific literacy [Duit et al., 2012]. Such a debate has been powered by different international monitoring studies. For istance: TIMSS (Trends in International Mathematics and Science Study) since 1995 [TIM, 2014] and PISA (Programme for International Student Assesment) since 2000 [PIS, 2014].

Among the various strands of science educational research, in the middle 1990s, the Model of Educational Reconstruction (MER) has been proposed by a group of German researchers [Kattmann et al., 1995, Duit et al., 2005, Duit, 2007]. The Model of Educational Reconstruction represents the merger of two lines of research in science education: the more pedagogical oriented research which has a European footprint and the more empirical research which is quite American-style. The first one is more oriented to the improvement of the teaching practice while the second is more oriented towards the specific learning outcomes of the organization of the contents and to the curriculum [Jenkins, 2001]. The European current has highlighted the need to rethink the scientific contents to be rebuilt in educational perspective [Fensham, 2001]. In this context the model of ’"Educational reconstruction" combines the hermeneutical tradition on scientific content with a constructivist approach to teaching / learning. A key concearn of the Model is that you have to give equal attention to science subject matter as well as to students learning needs and capabilities. There are three intimately linked components of the Model of Educational Reconstruction:

1. *Analysis of the content structure*[1] which is made of two processes: *clarification of subject matter* and the *analysis of educational significance.* Clarification of subject matter can take into account content analyses of leading textbooks, key pubblications on the topic and its historical development. Also student’s pre-instructional conceptions not in accordance with the science concepts to be learned, should be taken into account [Driver & Erickson, 1983].
2. *Research on teaching and learning* comprise empirical studies on various features of the learning setting. It is the investigation into student and teacher perspectives regarding the choosen subject. This includes pre-instructional conceptions, affective variables like interests, self-concepts, attitudes and skills.

[1] *Content* denotes science subject matter, *structure* is related to the significance of internal structure of the content

3. *Development and evaluation of instruction* concerns the design of instructional materials, learning activities and teaching and learning sequences. The design of the learning environment is the very heart of this point. It has to be structured by the specific needs and learning capabilities of the students. Various empirical methods are used to evaluate the activities, like interviews with students and teachers, questionnaires on the development of students' cognitive and affective variables and analyses of the video registrations of instructional practice.

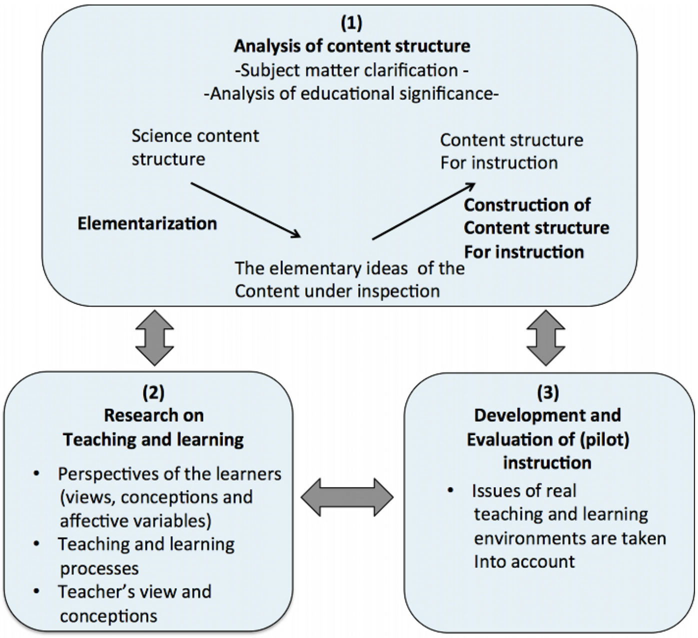


Figure 2.1.: The Model of Educational Reconstruction

As said above the three components of the Model are intimately linked. At the initial steps of the design, the science content structure has to be transformed into a content structure for instruction. The science content structure may not be directly transferred into the content structure for instruction as the two are substantially different. This is made possible by a process of *elementarization* of the content followed by the *construction of the content structure for instruction*. During these processes the content has to be semplified (to be accessible for students) and transferred into contexts that make sense to learners. The phase of the construction of the structure for the instruction starts from students' pre-knowledge. In this perspective learning is meant as the construction of knowledge that students perform their own on the grounds of the already existing knowledge. So the conceptions and the believes that students already have are not seen as ostacles of learning but as starting points [Driver & Easly, 1978]. In this sense the Model of Educational Reconstruction is part of a constructivist epistemological framework [Phillips, 2000, Duit & Treagust, 1998, Duit & Treagust, 2003, Widodo, 2004]. The second point is that the science knowledge itself is a human construction [Abd-El-Khalick & Lederman, 2000]: there is no true content structure of a particular content area but a consensus of a particular science comunity. Every presentation of this consensus, that we can find also in the textbooks, is an idiosyncratic reconstruction of the explicit or implicit aims of the authors [Kattmann et al., 1995]. Consequently the science content for instruction has to be built by the designer of the curriculum on the basis of the aims of the teaching of that particular content. In short, the science content stucture has to be reconstructed from a educational perspective: this is the essence of the term "educational reconstruction".

Many teachers think that the content structure for instruction should be simpler than the science content structure to become accessible to students. Accordingly, they proceed to a "reduction" of the contents. This is not what the Educational Reconstruction Model suggests. On the contrary, according to the Model, the content structure for instruction has to be more complex of the science content structure if we want to meet the learning needs of the students. In fact, it is necessary to include the abstract science knowledge into various contexts if we want to reach out both the learning potentialities and difficulties of students.

The instructional design, according to the *MER* has to take into account also the empirical studies on learning processes and on students' interests. In this phase we have to consider also the role of teaching methods, experiments and other didactic supports [Duit, 2006]. It appears clear that the experimentation and validation of the didactical materials and the activities for the instruction are intimatley linked with their design. Accordingly, in the Educational Reconstruction Model, the developing of the materials for the instruction and the research activities are interconnected.

In agreement with the MER, this design-based research has been developed in four phases: the design of the content structure for instruction, the construction of the teaching/learning sequence and of the instructional materials, the experimentation of the sequence and the analysis of the risults.

2.2. Methods and instruments for data analysis

In this section are briefly described the instruments and methods we used to asses the level of comprehension of the contents and the effectiveness of the educational path that we have designed, developed and experimented.

2.2.1. Analysis of wrtitten texts

The written test in form of a pre-test and a post-test, (where the post-test is equal to the pre-test) is a powerful instrument to investigate the thinking, the preconception and the misconception of students. It also helps to investigate the evolution that students' thinking undergo as a result of the teaching/learning sequence. The analysis of the tests are a good starting point for the final interviews.

2.2.2. Interviews

The use of the interview in research marks a move away from seeing human subjects as simply manipulable and data as somehow external to individuals, and toward regarding knowledge as generated between humans, often through conversation [Kvale, 1996a]. The term itself *"inter-view"* denotes an interchange of views between two or more people on the same topic. This interaction is responsible for knowledge production and generating data [Kvale, 1996b]. In this perspective the interview is not either subjective or objective but intersubjective [Laing, 1967].

The interview is a flexible tool for data collection as it allows the use of multi-sensory channels: verbal, non-verbal, spoken and heard [Cohen et al., 2007]. It allows you to keep under control the carrying out of the activities. The interview type we chose is a semi structured interview with guide approach where the topics and issues are decided in advance but the interviewer let the answers of students guide the ongoing of the interview. Generally the interviews were performed on focus-groups of three or four students (possibly the same group of students that worked together in the experimental activities).

2.2.3. Audio and video recordings

The instructional activities in classroom have been video-recorded and reviewed several times with the aim of identifying critical episodes, expecially during group discussion. The analysis of recordings is very helpful to isolate particular moments in which it is evident that the use of didactical instruments, peer discussion and negotiation with the teacher/researcher facilitate the development of some skills (e. g. the skills of observing, describing, interpreting and acting). Reviewing the recordings has helped to follow the dynamics of the negoziations of meaning. This clearly emerges from the discussion between students, both in small working groups, both in the entire class. In particular, it has been possible to characterize the development of precise concepts and the changes in the observation strategies and interpretation models, that take place as a result of the negotiations.

The final interviews have been audio recorded and the recordings analysed together with the material (writing, sketches etc.) produced by students during the interview itself.

The analysis of recordings is particularly useful because allows to us to grasp details that inevitably escape during classroom activities. Moreover, they can be viewed as many times as it is necessary an do not only provide precise evidence of what students said, but they allow us to grasp, or understanding aspects of insecurity that come through body language.

3. The Harmonic Motion

3.1. Introduction

The concept of harmonic motion is fundamental for the comprehension of the world we live in. Harmonic motion is present almost everywhere, even if we usually do not think of this fact. For istance, if we consider any solid object around us, we know that any atom within it has a well defined position (it is solid!). However, if we could magnify the object as to be able to distinguish its single atoms, we' ll see each of those atoms vibrating relative to this assigned position[Smith, 2010]. The hotter the object the more violent the vibration. This is true for every atom in every solid object in the Universe. If we consider one single vibrating atom, its vibration can have a pattern similar to the one showed in Fig. 3.1. It could seem a rather complicated pattern but it isn't really if we consider that it can be obtained by the summation of three simple sinusoids, the ones showed in Fig. 3.2. Each of these sinusoids describes the harmonic motion associated with a "*normal mode*" of the solid that contains the atoms. The pattern in Fig. 3.1 is so complex just because the solid has many "*degrees of freedom*" and thereby many normal modes as will be described in the next chapter; every atom in the solid object can move in three dimensions and it is affected by the motion of the sorrounding atoms.

Despite its importance, harmonic motion isn't always treated properly in the italian upper secondary school. Often, harmonic motion is presented as a complement to the kinematics, after studying rectilinear uniform and accelerated motion and circular motion. In this perspective harmonic motion is simply defined as an oscillatory motion whose amplitude obeys a sinusoid-like law such as: $x(t) = A\sin(\omega t + \phi)$. Alternatively, simple harmonic motion is defined as the motion described by the projection of a uniform circular motion along a diameter. Usually no more than a couple of examples are given to students or met in lab activities, namely the simple pendulum and the mass-spring oscillator.

Both the above kinematic definitions are not very useful from an operative point of view. In fact, it is difficult for students recognize in a real situation if an oscillating system performs harmonic motion or not, using such definitions. A reconstruction of the content, in our experience, is necessary.

We believe that a dynamic definition is much more fruitful in that sense. This is true not only for upper secondary school students but also for university level students, at least in our experience.

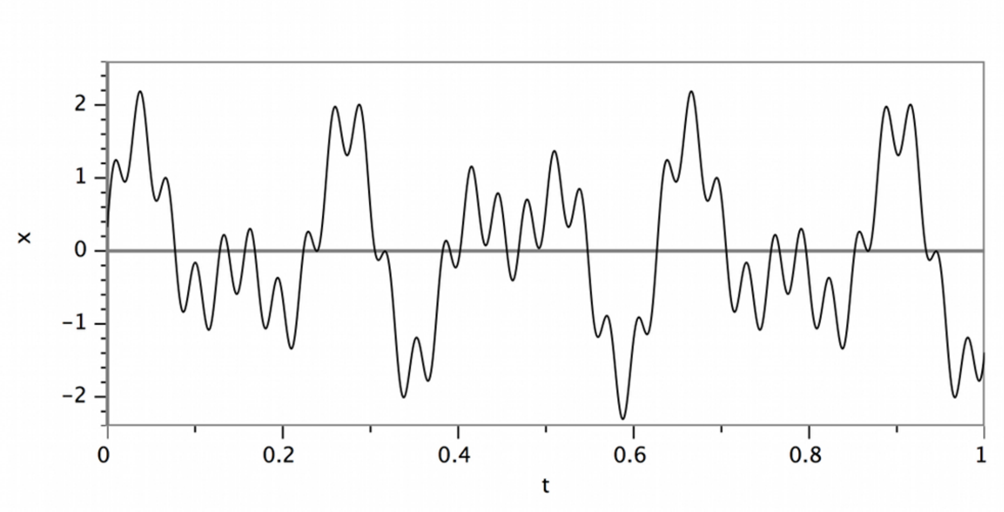


Figure 3.1.: The complex pattern of motion for an atom in a solid object in arbitrary units

In fact our lab experience with graduate students in mathematics showed that the comprehension of the link between mathematics and physics in the study of oscillations is far from clear. This fact prevents to grasp the importance of the harmonic motion as a conceptual organizer that should emerge from the choice/recognition of particular deep similarities/diversities among different types of periodic motions. The kinematic definition of harmonic motion is not enough to understand the physics implied; the dynamical definition, which stems from the analysis of the potential energy, is often uneffective.

From these considerations the necessity arises of a fruitful definition of harmonic motion which establishes some effective criteria for the recognition of the harmonicity/anharmonicity in real contexts.

In the following we propose an approach to harmonic motion that, starting from a dynamic definition, sets a criterion that allows to realize at a glance the anharmonicity/harmonicity of an oscillation and to understand the link with the mathematical aspects of the problem [Giliberti et al., 2014]. The goal is to make students able to recognize if a motion is harmonic or not even without knowing the exact expression of the acting forces, but simply by watching the oscillations and sometimes by listening to the sound generated by the oscillations themselves. We also discuss the role of the constant and of the linear damping in relation to the concepts of anharmonicity/harmonicity.

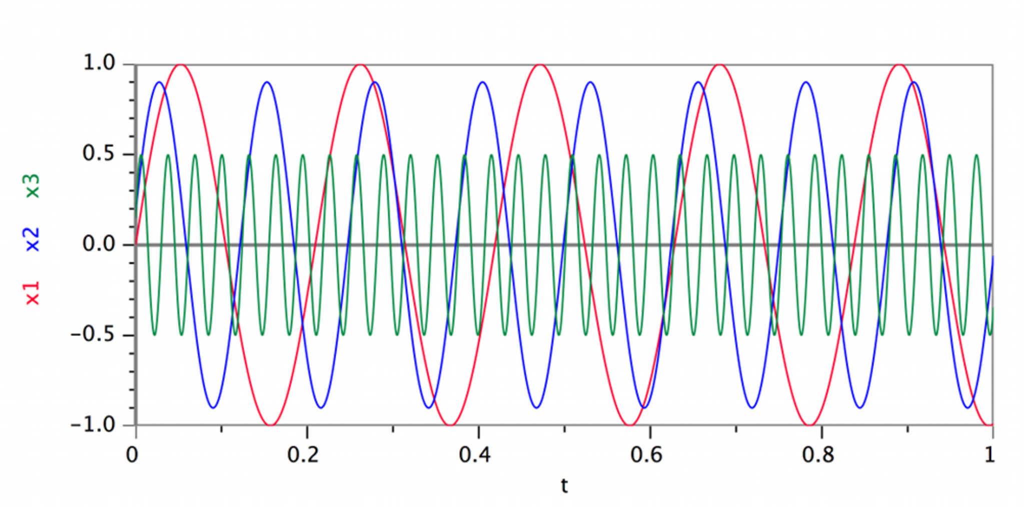


Figure 3.2.: Three sinusoidal components of the motion for an atom in a solid object in arbitrary units

3.2. Definition of Harmonic motion

Let us consider a one degree of freedom system subject to a restoring force, that is a force that gives rise to a motion with a stable equilibrium point. Let us call ξ the curvilinear coordinate measuring the oriented distance along the trajectory described by the moving body, with the zero corresponding to the equilibrium position. In this case, at least in a neighbourhood of $\xi = 0$, the graph of the ξ-component of the restoring force, F_ξ, *vs* ξ lies in the second and in the fourth quadrant (solid line in figure Fig. 3.3). Moreover, if the restoring force is sufficiently regular, that is the function $F_\xi(\xi)$ is continuous and differentiable in the origin, it can be approximated by its tangent line in that point, provided the amplitude of the oscillation is small enough (dashed line in figure Fig. 3.3).

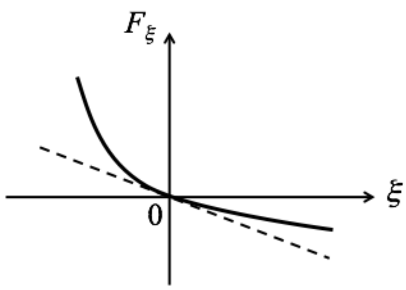


Figure 3.3.: A restoring force and its linearization in the origin.

As a consequence, a body subject to a sufficiently regular restoring force with the first derivative different from zero in the equilibrium point, for small amplitude oscillations, will obey the equation of motion:

$$F_\xi = -k\xi, \qquad (k > 0). \tag{3.1}$$

Denoting with m the mass of the body, we immediately get:

$$\ddot{\xi} + \omega_0^2 \xi = 0, \tag{3.2}$$

where

$$\omega_0 \equiv \sqrt{\frac{k}{m}}. \tag{3.3}$$

We define harmonic motion, the motion of a point mass satisfying equation (3.1) or, equivalently, equation (3.2). This is an intrinsic definition, in the sense that, once a reference frame is fixed, it refers only to the trajectory and to the resultant acting force, which are intrinsic characteristics of the motion. In many cases, however, it is useful to perform an invertible regular transformation:

$$\xi = \xi\left(q\right) \tag{3.4}$$

from the coordinate ξ to a new coordinate q (in general an angle or a position on a linear axis) that is more suitable to discuss lab experiments or to make a mathematical model of the system. Since equation (3.2) is a linear equation, if we require that the transformation (3.4) preserves the description of the motion, or equivalently preserves the form of equation (3.2), the only allowed tranformations will be the linear ones. Nonetheless, since in most situations we are mainly interested in the small oscillations around the equilibrium point, there is no need that the whole tranformation (3.4) is linear. It is enough that it can be linearized in the origin; namely, that in the neighbourhood of $q = 0$, $\xi(q)$ it can be approximated to the first order in q by:

$$\xi\left(q\right) = \frac{d\xi}{dq}(0) \cdot q; \quad \frac{d\xi}{dq}(0) \neq 0. \tag{3.5}$$

Equation (3.2) can be easily integrated to obtain the general solution [Fitzpatrick, 2013, Mencuccini & Silvestrini, 2013, Feynmann et al., 1964]:

$$\xi\left(t\right) = A\cos\left(\omega_0 t + \varphi_0\right), \tag{3.6}$$

where the amplitude A and the initial phase φ_0 are two integration constants that are completely independent on the value of the (angular) frequency ω_0 which is solely determined by dynamic conditions. The stated independence of A from ω_0 is generally referred to as isochronism.

We would like to stress that the harmonic motion defined above is not necessarily rectilinear, as ξ is a curvilinear coordinate (for instance, the ends of a torsional pendulum describe an arc of circumference performing harmonic oscillations over a wide range of angles). It is also important to emphasize that F_ξ must not be confused with the intensity of the total acting force, but it is only its component along the direction of motion. This is a conceptual aspect for which particular care is needed in describing motions on curved trajectories. In fact, in these cases, the resultant force is different from zero even in the equilibrium position, because the contribution of the centripetal component has to be considered; while, on the contrary F_ξ is, indeed, null.

In conclusion, the path towards the previous definition leads us to formulate a four-point criterion saying that the small oscillations of a one degree of freedom system are harmonic if $\xi = 0$:

(a) is a stable equilibrium point;

and in $\xi = 0$:

(b) the function $F_\xi(\xi)$ is continuous;

(c) the function $F_\xi(\xi)$ is differentiable;

(d) $\frac{dF_\xi}{d\xi} \neq 0$.

Obviously, condition (c) implies condition (b). Nevertheless we believe that, from a didactical point of view, keeping these conditions separate allows a clearer comprehension of the physics involved.

3.3. Harmonic or not?

The prototype of harmonic motion is the *mass-spring oscillator* that has been discussed in many papers and textbooks [Fitzpatrick, 2013, Mencuccini & Silvestrini, 2013, Feynmann et al., 1964, Arnold, 1979, Barbieri & Giliberti, 2012, Bergomi & et al, 1997]. An analysis of the motion of a one degree of freedom vertical mass-spring oscillator can be done via sonar detection and the “Logger Pro” software [**?**]. In figure Fig. 3.4, the acceleration *vs* position is shown. It will not be discussed here, but it will be used in the following as gold standard in analyzing other oscillations.

Let us now discuss how the mathematical conditions (a) to (d) of our four-point criterion are linked to real experiments and how the anharmonicity/harmonicity of small amplitude oscillations can be in many cases decided “at sight”, even without explicitly knowing the equation of motion. Listening to the sound produced by the

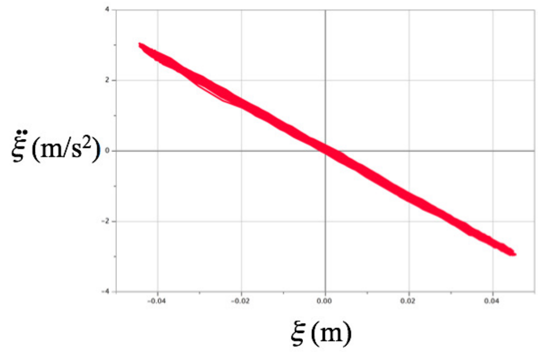


Figure 3.4.: Vertical mass-spring oscillator. Measured acceleration *vs* position: the graph represents a typical restoring force. The data have been collected and processed by a “Logger-Pro” sonar system.

oscillations can sometimes discriminate between anharmonicity and harmonicity. Let us consider some examples.

3.3.1. Bouncing disk

The bouncing disk consists of a disk moving back and forth between two elastic edges of an air table. Are these oscillations harmonic? No, they are not, because the system has not a single stable equilibrium point, but an infinite set of neutral equilibrium positions; a fact which implies that we cannot even introduce the notion of small oscillations. In this case, the first point of our criterion is not satisfied. A video of the motion has been analyzed by the software “Tracker” [tra,]. The results are shown in Fig. 3.5.

While in the case of the mass-spring oscillator the acceleration *vs* position diagram

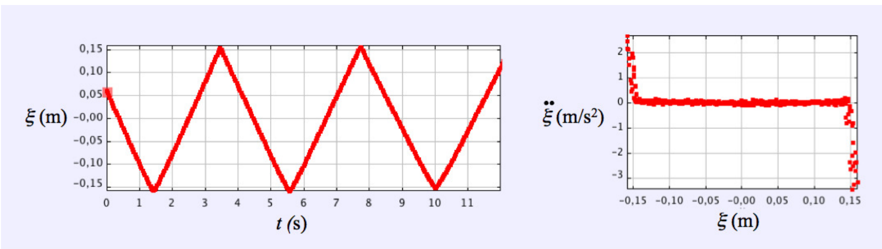


Figure 3.5.: Bouncing disk. (a) Measured position *vs* time; (b) Measured acceleration *vs* position. The data have been processed by the “Tracker” video analysis software.

is a straight line lying in the second and fourth quadrant passing through the origin of the axes, in the case of the bouncing disk such a diagram is completely flat, except at the edges where the force is impulsive (figure 3(b)). In fact the position *vs* time graph shows that we are in presence of a uniform rectilinear motion between the turning points (figure 3(a)). The anharmonicity of the motion can be either stated with standard methods such as the analysis of the motion waveform obtained by data-logging techniques. In Fig. 3.6 it is reported the FFT (Fast Fpourier Transform graph obtained by the analysis of the motion waveform. The many harmonics clearly prove the anharmonicity of the motion.

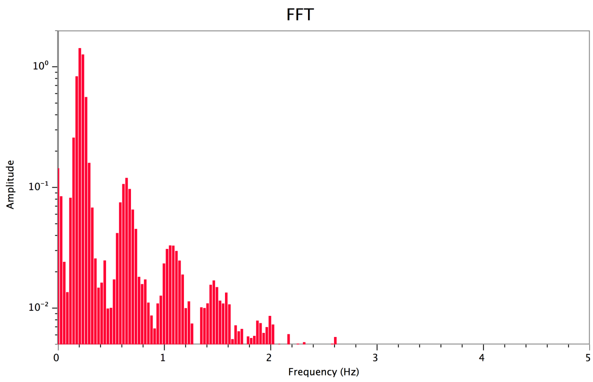


Figure 3.6.: The FFT graph for the Bouncing Disk

3.3.2. Galileo oscillator

Let us consider a V-shaped track, as in figure Fig. 3.7, with a ball rolling over it, a device that is sometimes referred to as the Galileo oscillator [?]. The function $F_\xi(\xi)$ is not continuous in $\xi = 0$, therefore the second point of our criterion is not fulfilled and the motion cannot be harmonic.

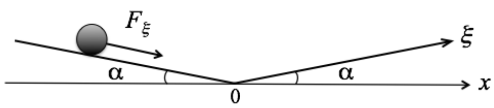


Figure 3.7.: The Galileo oscillator. The ξ-coordinate along the trajectory and the x-axis along which the motion is detected.

In fact, neglecting both the friction and the rolling energy, which indeed are always present in a real experiment, the force acting on an object of mass m along this kind of trajectory is given by:

$$mg \sin\alpha;\ \xi < 0 \qquad -mg \sin\alpha;\ \xi > 0. \tag{3.7}$$

If the angle α is sufficiently small so that the ball can really oscillate back and forth between the two parts of the track, we can make a video analysis of the motion. The results are shown in figure Fig. 3.8. The coordinate $q = x$ is used instead of ξ; the transformation (3.4) can be written as:

$$\xi(x) = \frac{x}{\cos\alpha}, \tag{3.8}$$

that is clearly a linear transformation and, as already said, preserves the harmonicity/anharmonicity of the motion.

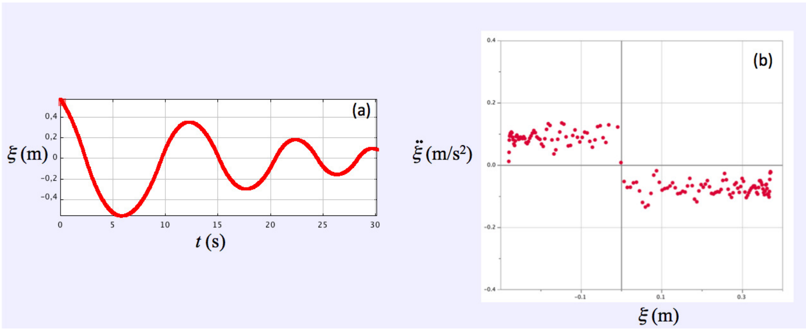


Figure 3.8.: The Galileo oscillator. (a) Position *vs* time; (b) Acceleration *vs* position. The data have been processed by the "Tracker" video analysis software; they show two different values of the acceleration, as can be inferred from equation (3.7).

The motion is much more similar to that of a freely falling bouncing ball than to that of a mass-spring oscillator and it cannot be approximated by a harmonic motion, however small the amplitude of oscillation. Of course also in this case, the anharmonicity of the motion can be stated with standard methods such as the analysis of the motion waveform obtained by data-logging techniques. In Fig. 3.9 it is reported the FFT graph obtained by the analysis of the motion waveform. The spread of frequencies and the presence of harmonics clearly proves the anharmonicity of the motion.

3.3.3. The interrupted pendulum

Let us analyze a simple pendulum of length l_1 that is interrupted in its motion by a peg between the point of suspension and the equilibrium point, so that its length abruptly changes to l_2 (see figure Fig. 3.10(a)) [**?**]. The equation of motion is:

$$\ddot{\xi} + g\sin\frac{\xi}{l_1} = 0;\ \xi \leq 0 \qquad \ddot{\xi} + g\sin\frac{\xi}{l_2} = 0;\ \xi > 0, \tag{3.9}$$

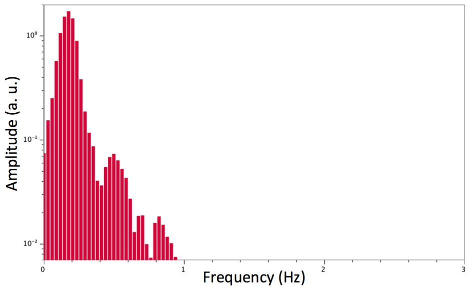


Figure 3.9.: The FFT graph obtained by the analysis of the motion waveform of Fig. 3.8

which shows that $\xi = 0$ is a corner point of the function $\ddot{\xi}(\xi)$ (figure Fig. 3.10(b)). Therefore the function $F_\xi(\xi)$ is not differentiable in $\xi = 0$, the third point of our criterion does not hold, and the motion of the pendulum cannot be approximated by a harmonic motion, not even in the small oscillation limit.

For what concerns the period T_{12} of the small oscillations of this asymmetric pendulum, we obviously have:

$$T_{12} = \frac{T_1}{2} + \frac{T_2}{2}, \tag{3.10}$$

where T_1 and T_2 are the oscillation periods of the two pendulums of length l_1 and l_2, respectively. Consequently, the interrupted pendulum is isochronous for small angles of oscillation without being harmonic. The FFT of the motion waveform (figure Fig. 3.10(c)) shows clearly, even for small oscillations, the first harmonic of frequency $\nu_1 = {}^1\!/_{T_{12}}$ and some of its multiples, as one expects for a non-sinusoidal periodic motion of period T_{12}, at variance to what one could naively expect, i.e. the presence of the two frequencies ${}^1\!/_{T_1}$ and ${}^1\!/_{T_2}$. We stress that the request of differentiability of the function $F_\xi(\xi)$ in the origin, that at first might seem a mere mathematical question, corresponds to detectable physical effects.

We want also to observe that another case in which the function $F_\xi(\xi)$ is not differentiable in $\xi = 0$ is when it has a vertical tangent line in that point. Physically, we can think of a very tough spring that we are not able to stretch with our lab equipment, so that the motion of one of its end cannot even occur.

3.3.4. The x^4-track

Here we want to analyze the case when only the condition (d) of our four-point criterion is not satisfied, that is $\frac{d}{d\xi}F_\xi(0)$ is zero. If the force and its first derivative are zero in the origin, also the second derivative $\frac{d^2}{d\xi^2}F_\xi(0)$ must be zero, otherwise $\xi = 0$ would not be a stable equilibrium point. Thus, the first term different from zero in the MacLaurin expansion of F_ξ must be at least proportional to ξ^3. We

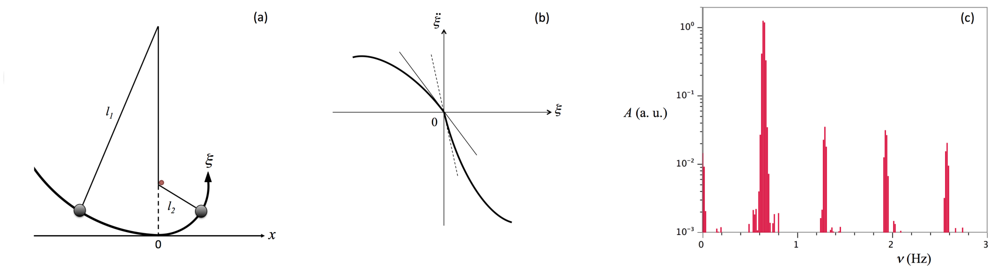


Figure 3.10.: The interrupted pendulum. (a) The apparatus: the bob is shown at two different times; (b) Acceleration *vs* position; (c) FFT of the motion waveform. The data for the FFT have been collected and processed by a "Logger-Pro" system.

want to show that, in this condition, the motion is not only anharmonic, but also not-isochronous.

Let $U(\xi)$ be the potential energy of the ball of mass m. Energy conservation can be written as:

$$\frac{1}{2}m\dot{\xi}^2 + U(\xi) = E, \tag{3.11}$$

where E is the total, constant energy of the system. Let A_1 and A_2 be the extremes of oscillation (solution of $U(\xi) = E$) then the period of oscillation T is twice the time spent by the body for going from A_1 to A_2. That is:

$$T = 2\int_{A_1}^{A_2} \frac{d\xi}{\sqrt{\frac{2}{m}\left[E - U(\xi)\right]}}. \tag{3.12}$$

In the case of our interest, $U(\xi) = c\xi^4$, where c is a positive constant. Therefore, putting $A_1 \equiv -A$ and $A_2 \equiv A$; with $cA^4 = E$, we obtain:

$$T = 2\sqrt{\frac{m}{2}}\int_{-A}^{A} \frac{d\xi}{\sqrt{cA^4 - c\xi^4}} = 4\sqrt{\frac{m}{2c}}\int_0^A \frac{d\xi}{\sqrt{A^4 - \xi^4}}. \tag{3.13}$$

Making the substitution $\xi \equiv Ax$, T is given by:

$$T = 4\sqrt{\frac{m}{2c}}\frac{1}{A}\int_0^1 \frac{dx}{\sqrt{1 - x^4}} = 4\sqrt{\frac{m}{2c}}\alpha\frac{1}{A}, \tag{3.14}$$

where the constant

$$\alpha \equiv \int_0^1 \frac{dx}{\sqrt{1-x^4}} \sim 1.31 \tag{3.15}$$

has been numerically calculated. The period of oscillation depends on the amplitude as A^{-1} and, therefore, it is not isochronous however small the amplitude A. From equation (3.14), we observe that when $A \to 0$, $T \to \infty$. To better understand this behaviour, we note that the linearized equation (3.2) reads:

$$\ddot{\xi} = 0, \tag{3.16}$$

a fact that makes a small neighbourhood of $\xi = 0$ similar to a region of neutral equilibrium.

In order to have a mechanical example of this kind of motion one can consider a ball moving on a x^4-shaped track. One observes that about the equilibrium point the track is nearly flat and realizes that, in order to have only small amplitude oscillations, the ball has to be substantially at rest with "long" oscillation periods. On the contrary, to get not too long periods of oscillation, one is forced to consider sufficiently great amplitudes. In this case, by just listening to the sound produced by the sliding ball, one can infer that the motion is manifestly anharmonic.

Similar considerations can be done, more in general, when the first term in the MacLaurin expansion of $F_\xi(\xi)$ is proportional to a generic ξ^{2n-1} with $n > 1$; the dependence of the period T on the amplitude A would be $T \propto A^{-(n-1)}$.

3.4. Damped oscillations

In additions to the conditions previously discussed, a useful way to recognize anharmonicity is to find an amplitude dependence of the period of oscillation. Since real motions are always damped, we have to exclude that this dependence comes from damping, instead of being due to an intrinsic anharmonicity.

In the interesting case of sliding friction, it can be demonstrated that a constant friction force does not affect the frequency of a simple harmonic motion, but only the amplitude of oscillation which is decreased each cycle [Barrat & Strobel, 1981, Onorato et al.,]. Therefore sliding friction cannot produce an amplitude dependence of the oscillation period.

In the case of viscous friction, the restoring force no longer depends only on position, but also on velocity and the situation is a little bit more complicated. In the simplest case, for small velocities, the damping can be assumed to be linear with the velocity and the equation of motion is, therefore:

$$\ddot{\xi} + \omega_0^2 \xi + 2\gamma\dot{\xi} = 0, \tag{3.17}$$

where γ is the damping coefficient and ω_0 is the zero-friction angular frequency. In the case of neither critical ($\gamma = \omega_0$) nor over-damped ($\gamma > \omega_0$) motion (that we do not consider here), the solution of equation (3.17) is:

$$\xi\left(t\right) = Ae^{-\gamma t}\cos\left(\omega t + \varphi_0\right); \qquad \omega \equiv \sqrt{\omega_0^2 - \gamma^2}; \qquad \gamma < \omega_0, \tag{3.18}$$

$$\omega \equiv \omega_0 \sqrt{1 - \left(\frac{\gamma}{\omega_0}\right)^2} \tag{3.19}$$

where A and φ_0 are integration constants and ω is the actual angular frequency of the damped harmonic motion. To be more precise, since the motion is not strictly speaking periodic, instead of the period we should refer to the notion of pseudo-frequency (and of pseudo-period).

It is important to deeply understand the physical meaning of the constants appearing in equation (3.18). Let us consider the problem of writing down the explicit solution of equation (3.17) for an oscillator that starts from rest at the initial position $\xi(0) = \xi_0$. For that, we have to solve che Cauchy problem:

$$\ddot{\xi} + \omega_0^2 \xi + 2\gamma\dot{\xi} = 0; \quad \xi(0) = \xi_0, \quad \dot{\xi}(0) = 0. \tag{3.20}$$

With simple calculations, we obtain the constants A and φ_0:

$$\frac{A}{\xi_0} = \left[\frac{1}{1 - \left(\frac{\gamma}{\omega_0}\right)^2}\right]^{\frac{1}{2}}; \qquad \varphi_0 = -\arctan\frac{\frac{\gamma}{\omega_0}}{\left[1 - \left(\frac{\gamma}{\omega_0}\right)^2\right]^{\frac{1}{2}}}, \tag{3.21}$$

where the relation on the left shows that A is always greater than ξ_0. We can observe that, although A depends on ξ_0 and on the ratio γ/ω_0, the pseudo-frequency does not depend on ξ_0 and, therefore, a pseudo-period dependence on the amplitude cannot come from linear damping. Furthermore, the fact that φ_0 is negative implies that the elapsed time T_Q between the start ($t = 0$) and the first passage through the equilibrium position is more than a quarter of a period, namely:

$$T_Q = \frac{T}{4} - \frac{T\varphi_0}{2\pi}, \tag{3.22}$$

while the time interval between the equilibrium position and the first maximum is less than $T/4$; the period of a complete oscillation is $T = {}^{2\pi}/_{\omega}$. From equations (3.18) and (3.21) we obtain the fractional increase:

$$I_Q = \frac{T_Q - \frac{T}{4}}{\frac{T}{4}} = \frac{2}{\pi} \arctan \frac{{}^{\gamma}/_{\omega_0}}{\sqrt{1 - \left(\frac{\gamma}{\omega_0}\right)^2}}, \tag{3.23}$$

that can be seen as an estimate of the "anharmonicity degree" of the motion. The moving towards the equilibrium position and the moving away from the equilibrium position are not symmetric, because in the former case the friction force is antiparallel to the restoring force while, in the latter, it is parallel.

From equation (3.18) it is straightforward to obtain:

$$\ddot{\xi} = -\left(\omega^2 - \gamma^2\right)\xi + 2\gamma\omega A e^{-\gamma t} \sin\left(\omega t + \varphi_0\right), \tag{3.24}$$

which shows that the acceleration is the sum of two contributions: a harmonic-like term, plus a temporal damped sinusoidal term. The first term in equation (3.24) could be surprising, because the coefficient of ξ is not ω^2, as one could at first expect. The fact is that the pseudo-frequency must not be confused with the frequency. While frequency is a property of a periodic motion that, for a harmonic oscillator, is fixed by the slope of the acceleration *vs* position graph, the pseudo-frequency is the rhythm given by the oscillator which performs a complete oscillation around the origin.

The analysis of the graphs of the acceleration *vs* position allows us to understand the differences between the harmonic and the linearly damped harmonic motions.

Let us first analyze the case $\gamma T{=}1$, when the amplitude of oscillation is reduced by a factor ${}^{1}/_{e}$ in a period, so that some complete oscillations are still easily visible, though clearly damped. In this case, we have:

$$\frac{\gamma}{\omega_0} = \left(4\pi^2 + 1\right)^{-\frac{1}{2}} \sim 0.16, \tag{3.25}$$

and $I_Q \sim 0.1$ In figure Fig. 3.11(a), a plot of ξ *vs* t shows four complete oscillations. Panel (b) of figure Fig. 3.11 displays $\ddot{\xi}$ *vs* ξ and gives us a picture of the general structure of the acceleration *vs* displacement for the so called damped *harmonic* motions. When the body passes through the equilibrium position, the harmonic force (see first term on the right side of equation (3.24) is indeed zero, but the viscous drag opposes to the direction of motion, so that the curve $F_\xi(\xi)$ (that is no more a single valued function of ξ) does not touch the origin, except when the motion stops). In figure Fig. 3.11(b) the pseudo-period is the time taken to go from point A to point B or, equivalently, from the extremes C and D. It is this interval of time that remains constant during the oscillation.

It is interesting to analyze how the physical behaviour of the damped oscillator changes for smaller or higher values of γ/ω_0.

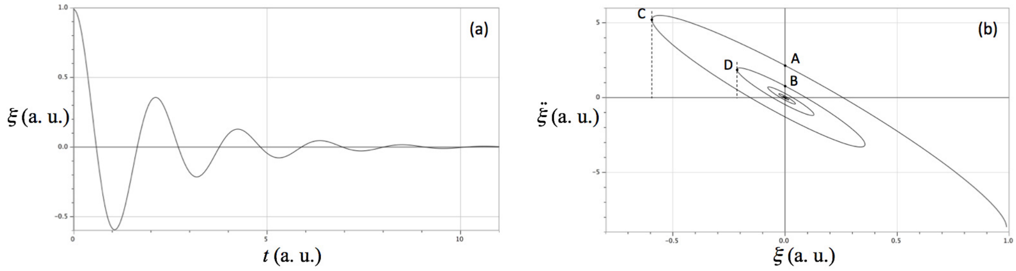


Figure 3.11.: The damped harmonic oscillator. (a) Simulated amplitude decay with a ratio $\gamma/\omega_0 \sim 0.16$; (b) Simulated acceleration *vs* position with the same ratio $\gamma/\omega_0 \sim 0.16$ as in panel (a).

For $\gamma/\omega_0 \ll 1$

$$I_Q \sim \frac{2}{\pi}\frac{\gamma}{\omega_0} \ll 1, \quad \text{and} \quad T_Q \sim T/4. \tag{3.26}$$

Moreover,

$$\omega \sim \omega_0 \left[1 - \frac{1}{2}\left(\frac{\gamma}{\omega_0}\right)^2\right], \tag{3.27}$$

a quantity that is difficult to distinguish from ω_0 in most simple lab experiments. In figure Fig. 3.12 a real damped mass-spring oscillator is analyzed. In this case, $\gamma/\omega_0 \sim 0.0045$ and $I_Q \sim 0.003$. While the damping has a clear and visible effect on the amplitude (see figure Fig. 3.12(a)), since $\omega_0 \sim 7.73\,\mathrm{s}^{-1}$, from equation (3.27) it follows that, in order to observe the increase of T_Q with respect to $T/4$, we should measure time with an accuracy of 10^{-4} s that exceeds most of the standard didactical lab devices. The acceleration *vs* position plot has the same structure of that of figure Fig. 3.11(b), but is completely squeezed so that, to resolve the coils of the spiral, one should have great precision tools.

From equation (3.24) we can infer that for $\gamma > \omega$ the first term on the right side changes sign, therefore the harmonic term of the force is substituted by a moving away one. If $\gamma = \omega$, then $\gamma/\omega_0 = 1/\sqrt{2}$, $I_Q = 1/2$ and the harmonic term in equation (3.24) is zero.

The graphs of this motion are sketched in figure Fig. 3.13, where the anharmonicity is clearly visible in panel (b). In fact, the more the curve lies in the first and third

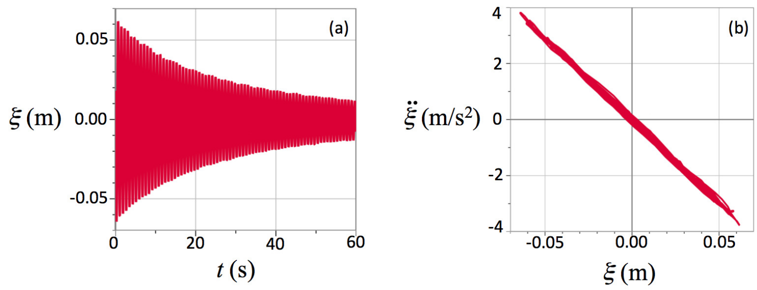


Figure 3.12.: The damped harmonic oscillator (with a CD-rom stuck to the mass). (a) Measured amplitude decay with a ratio $\gamma/\omega_0 \sim 0.0045$; (b) Measured acceleration *vs* position with the same ratio $\gamma/\omega_0 \sim 0.0045$ as in panel (a). The data have been collected and processed by a "Logger-Pro" sonar system.

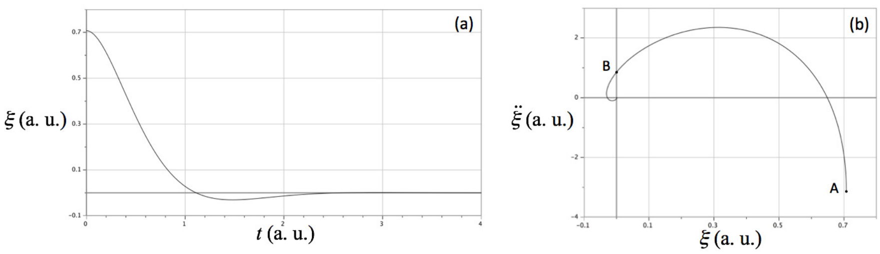


Figure 3.13.: The damped harmonic oscillator. (a) Simulated amplitude decay with a ratio $\gamma/\omega_0 = 1/\sqrt{2}$; (b) Simulated acceleration *vs* position with the same ratio $\gamma/\omega_0 = 1/\sqrt{2}$ as in panel (a).

quadrant, where $\ddot{\xi}$ has the same sign of ξ , the more the motion is anharmonic because the acting force has the opposite sign of a harmonic force. The body is accelerated towards the equilibrium position (**A** to **B**) in an initial brief interval of time and then it is slowed down.

If $\gamma/\omega_0 \sim 1$ then:

$$I_Q \sim 1 \quad \text{and} \quad T_Q \sim T/2, \tag{3.28}$$

and the motion is practically just a coming back to the equilibrium position.

In conclusion, we stress the importance of analyzing the behaviour of the force (acceleration) as a function of position to discriminate how much a damped motion is different from an ideal harmonic one. The physics of the problem is regulated by the ratio γ/ω_0. If γ/ω_0 is generally less then 0.16 (figure Fig. 3.11(b)) we can roughly say that the oscillation is substantially harmonic. On the contrary, when γ/ω_0 is greater then $\frac{1}{\sqrt{2}}$ (figure Fig. 3.13(b)), a clear anharmonicity appears.

3.5. Some didactical considerations

From the previous analysis, it appears that it is often extremely simple to infer the anharmonicity/harmonicity of the small oscillations, even without knowing the exact dependence of the force on the displacement.

First of all, in first approximation, linear damping does not substantially alter the harmonicity, provided some complete oscillations are clearly visible. Keeping in mind the four-point criterion of section 3.2, just looking at the motion and, if needed, also listening to the sound produced by the oscillations, we can label the oscillations as harmonic or not. If we have just a look at the motion of a bouncing disk, we immediately see that it has not a stable equilibrium position; by simply watching the form of the track of a Galileo oscillator, we understand that in the equilibrium point there is a problem of continuity of the force. Therefore we immediately conclude that these motions are not harmonic. If one is not yet convinced, just by listening to the changing of the ticking rhythm, which reflects a dependence of the frequency on the amplitude, s(he) will finally be persuaded. Looking at the trajectory of the bob of an interrupted pendulum, an abrupt curvature change as well as a sudden variation of the half-period are evident. The case of the x^4-track is particularly intriguing, since viewing could not be enough, hearing could be necessary to realize the great dependence of the frequency on the amplitude.

From a laboratory point of view, a little warning is also worth mentioning, in fact a restoring force of the form

$$F_{\xi}(\xi) = -k_1\xi - k_3\xi^3; \qquad k_1 > 0; \quad k_3 > 0 \tag{3.29}$$

clearly satisfies the four-point criterion; nevertheless if $k_3 \gg k_1$, anharmonic oscillations will appear as soon as the amplitude of oscillation is not small enough. This example shows that the observed anharmonicity/harmonicity of the small oscillations depends, in general, both on the experimental set up and on the measurement devices.

In a didactical path it should be very instructive to ask students to try to understand at a glance the anharmonicy/harmonicity of two seesaws: a bar put on a cylindrical shaped pivot (figure Fig. 3.14(a)) and a bar put on a square pivot (figure Fig. 3.14(b)) [Pecori & Torzo, 2001] . It should be clear that, for small oscillations, the former seesaw performs harmonic motion, while the latter does not (the eye sees an edge and the ear hears an increasing frequency ticking).

For completeness, in Fig. 3.15 are reported the FFT graphs obtained by the analysis of the motion via sonar detector for a seesaw on a round pivot (a) and on a flat pivot (b). It is evident that in the case of the round pivot, the frequency obtained is neat and with a precise value as expected for the harmonic motion. On the other hand, in the case of the flat pivot, the frequency is widely spread around a middle value and it is detectable the presence of higher harmonics to confirm the anharmonicity of the motion.

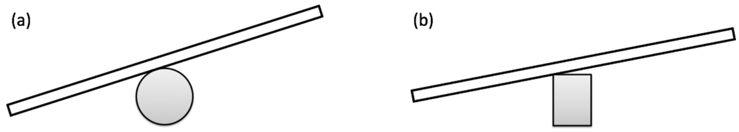


Figure 3.14.: (a) The seesaw with a cylindrical shaped pivot; (b) The seesaw with a square pivot.

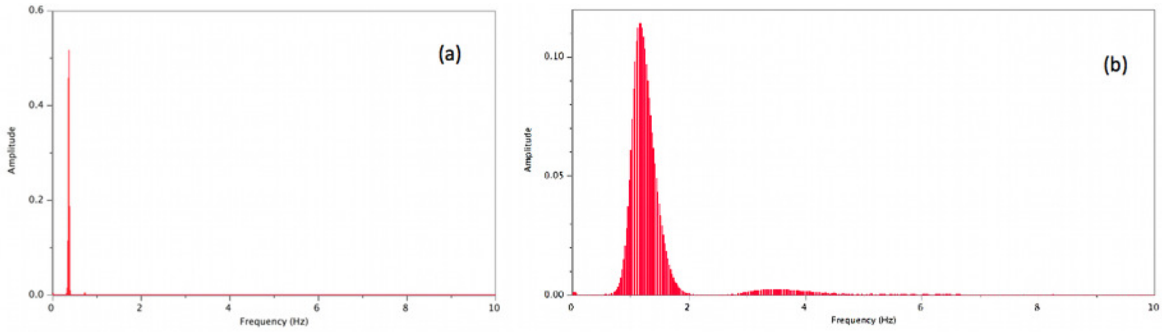


Figure 3.15.: (a) The FFT for the seesaw on a round pivot and (b) on a flat pivot.

As we have already said in subsection 3.3, while harmonicity implies isochronism, the viceversa is not true. Let us consider a body moving on a cycloidal track under the effect of gravity Fig. 3.16.

Neglecting friction, the motion is harmonic along the cycloid, whatever the ampli-

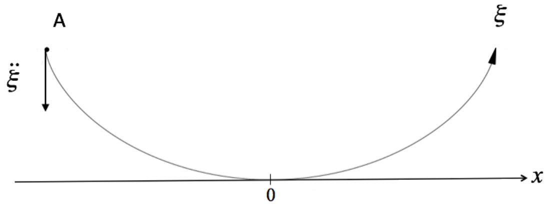


Figure 3.16.: The cycloidal track.

tude. The equation of motion is [Onorato et al., 2013]:

$$\ddot{\xi} + \frac{g}{4R}\xi = 0, \tag{3.30}$$

where R is the radius of the circle generating the cycloid and g the gravity acceleration. As a consequence, the oscillations are all isochronous. It is, therefore, straightforward to understand that even the x-component of the motion is isochronous for every amplitude. But it is harmonic only for small oscillations, while with increasing amplitude the anharmonicity clearly appears. In fact, the invertible transformation from ξ to x is not linear and, therefore, it does not preserve the form of equation (3.30). To better understand this fact, we can consider an oscillation starting from the point A in Fig. 3.16: the x-component of the acceleration is zero in that point, at variance to what a harmonic motion requires. For continuity reasons, the motion will therefore be anharmonic even in the intermediate amplitude region.

4. The Normal Modes of Oscillation

4.1. Overview

Every single oscillator has its own unique natural frequency[1] of oscillation that is determined by the nature of such oscillator. For instance, the natural frequency for the mass-spring depends on the mass of the bob and the elastic constant of the spring and it is given by $\omega = \sqrt{k/m}$. We want to see what happens to the oscillation when two, three, N mass-spring are connected between springs as to generalize to the continuous systems. We will inspire to the lectures given by Professor David Morin in his undergraduate courses in the Physics Department at Harvard University [Morin,].

4.2. Two masses

4.2.1. The motion equations

The systems of two coupled mass-springs consists of two mass-spring oscillators connected by a thirs spring as in Fig. 4.1 . For semplicity we will consider all the springs with the same elastic constant k.

Figure 4.1.: The system of two coupled mass-spring oscillators

Let x_1 and x_2 measure the displacements of the left and right masses from their respective equilibrium positions. We can assume that all of the springs are unstretched at equilibrium. The middle spring is stretched (or compressed) by $x_2 - x_1$ so the $F = ma$ equations for the two masses are

$$\begin{aligned} m\ddot{x}_1 &= -kx_1 - k\left(x_1 - x_2\right), \\ m\ddot{x} &= -kx_2 - k\left(x_2 - x_1\right) \end{aligned} \tag{4.1}$$

[1] we here call frequency the angular frequency ω that is proportional to the frequency f. More properly $\omega = 2\pi f$.

These two $F = ma$ equations are "coupled", in the sense that both x_1 and x_2 appear in both equations. How do we go about solving for $x_1(t)$ and $x_2(t)$? There are (at least) two ways to solve this system of equations. The first is substantially the one used in section 8.3.4.2 that is quick but it works only with simple and symmetric sistems. The second that we use here, is more complex but it works almost with any system. Our strategy will be to look for simple kinds of motions where both masses move with the same frequency. We will then build up the most general solution from these simple motions. For all we know, such motions might not even exist, but we have nothing to lose by trying to findnd them. We will find that they do in fact exist. You might want to try to guess now what they are for our two-mass system, but it isn't necessary to know what they look like before undertaking this method. Let's guess solutions of the form $x_1(t) = A_1 \exp(i\omega t)$ and $x_2(t) = A_2 \exp(i\omega t)$. Substituting these equations into the 4.1 and canceling the factor $\exp(i\omega t)$ we obtain

$$\begin{aligned} m\omega^2 A_1 &= -kA_1 - k(A_1 - A_2), \\ m\omega^2 A_2 &= -kA_2 - k(A_2 - A_1). \end{aligned} \tag{4.2}$$

that can be written as a matrix

$$\begin{pmatrix} -m\omega^2 + 2k & k \\ -k & -m\omega^2 + 2k \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{4.3}$$

We can multiply both sides of this equation by the inverse of the matrix. This leads to (A1, A2) = (0, 0). This is obviously a solution, but we're looking for a nontrivial solution that actually contains some motion. The only way is that the inverse of the matrix doesn't exist. So if the determinant is zero, then the inverse doesn't exist. This is therefore what we want. Setting the determinant equal to zero gives the quartic equation,

$$det \begin{pmatrix} -m\omega^2 + 2k & k \\ -k & -m\omega^2 + 2k \end{pmatrix} = 0 \qquad \Longrightarrow \omega^2 = \frac{k}{m} \quad or \quad \omega^2 = 3\frac{k}{m}. \tag{4.4}$$

The four solutions thus are: $\omega = \pm\sqrt{k/m}$ and $\omega = \pm\sqrt{3k/m}$. For the case where $\omega^2 = k/m$, we can plug this value of ω^2 back into 4.3 to obtain

$$k \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{4.5}$$

Both rows of this equation yield the same result , namely $A_1 = A_2$. So (A_1, A_2) is proportional to the vector (1, 1). For the case $\omega = \sqrt{3k/m}$, 4.3 gives

$$k \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{4.6}$$

Both rows now give $A_1 = -A_2$. So (A_1, A_2) is proportional to the vector (1, -1). If we call $\omega_s = \sqrt{k/m}$ and $\omega_f = \sqrt{3k/m}$, the general solution is the sum of the four solutions we have found. In vector notation, $x_1(t)$ and $x_2(t)$ are given by

$$\begin{aligned} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} &= C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \exp(i\omega_s t) + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \exp(-i\omega_s t) \\ &+ C_3 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \exp(i\omega_f t) + C_4 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \exp(-i\omega_f t) \end{aligned} \tag{4.7}$$

The positions $x_1(t)$ and $x_2(t)$ must be real for all t. This yields that standard result that $C_1 = C_2^* \equiv (A_s/2)\, e^{i\phi_s}$ and $C_3 = C_4^* \equiv (A_f/2)\, e^{i\phi_f}$. We have included the factors of 1/2 in these definitions so that we won't have a bunch of factors of 1/2 in our final answer. The imaginary parts in 4.7 cancel, and we obtain

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = A_s \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_s t + \phi_s) + A_f \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\omega_f t + \phi_f) \tag{4.8}$$

Therefore

$$\begin{aligned} x_1(t) &= A_s \cos(\omega_s t + \phi_s) + A_f \cos(\omega_f t + \phi_f)\,, \\ x_2(t) &= A_s \cos(\omega_s t + \phi_s) - A_f \cos(\omega_f t + \phi_f)\,. \end{aligned} \tag{4.9}$$

4.2.2. Normal modes and normal coordinates

Normal Modes Let us now see what we have found solving for $x_1(t)$ and $x_2(t)$. If in 4.9 $A_f = 0$, then

$$x_1(t) = x_2(t) = A_s \cos(\omega_s t + \phi_s) \tag{4.10}$$

So both masses move in exactly the same manner. Both to the right, then both to the left, and so on. This means that the middle spring is never stretched, as if it did't be there. It is as if we have two copies of a simple spring-mass system. This is consistent with the fact that ω_s equals the standard expression $\sqrt{k/m}$, independent of k. This nice motion, where both masses move with the same frequency, is called a *normal mode*. To specify what a normal mode looks like, you have to give the frequency and also the relative amplitudes. So this mode has frequency $\sqrt{k/m}$, and the amplitudes are equal. If, on the other hand, $A_s = 0$ in 4.9, then we have

$$x_1(t) = -x_2(t) = A_f \cos(\omega_f t + \phi_f) \tag{4.11}$$

Now the masses move oppositely. Both outward, then both inward, and so on. The frequency is now $\omega_f = \sqrt{3k/m}$. This second frequency is larger than the previous because now the central spring is stretched or compressed, so it adds to the restoring force. This motion is the second normal mode. It has frequency $\sqrt{3k/m}$, and the amplitudes are equal and opposite. The 4.9 tells us that any arbitrary motion of the system can be thought of as a linear combination of these two normal modes.

Normal coordinates By adding and subtracting the expressions for $x_1(t)$ and $x_2(t)$ in 4.9, we see that for any arbitrary motion of the system, the quantity $x_1(t) + x_2(t)$ oscillates with frequency ω_s, and the quantity $x_1(t) - x_2(t)$ oscillates with frequency ω_f . These combinations of the coordinates are known as the normal coordinates of the system. The $x_1(t) + x_2(t)$ normal coordinate is associated with the normal mode (1, 1), in fact they both have frequency ω_s. Equivalently, any contribution from the other mode, where $x_1(t) = -x_2(t)$ will vanish in the sum $x_1(t) + x_2(t)$. Basically, the sum $x_1(t) + x_2(t)$ picks out the part of the motion with frequency ω_s and discards the part with frequency ω_f . Similarly, the $x_1(t) - x_2(t)$ normal coordinate is associated with the normal mode (1, -1), because they both have frequency ω_f . Equivalently, any contribution from the other mode (where $x_1(t) = x_2(t)$) will vanish in the difference $x_1(t) - x_2(t)$.

4.3. Three masses

As an intermediate step to the general case of N masses connected by springs, let's consider at the case of three masses, as shown in Fig. 4.2. We'll just deal with

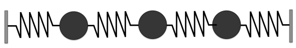


Figure 4.2.: Three coupled mass-spring system

undriven and undamped motion here, and we'll choose all the masses equal we'll also assume that all the spring constants are equal, lest the math get intractable. If x_1, x_2 and x_3 are the displacements of the three masses from their equilibrium positions, then the three $F = ma$ equations are

$$\begin{aligned} m\ddot{x}_1 &= -kx_1 - k(x_1 - x_2) \\ m\ddot{x}_2 &= -k(x_2 - x_1) - k(x_1 - x_3) \\ m\ddot{x}_3 &= -k(x_3 - x_2) - kx_3 \end{aligned} \tag{4.12}$$

We will use the determinant method and guess a solution of the form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \exp(i\omega t) . \tag{4.13}$$

Substituting in 4.12 and cancelling the $\exp(i\omega t)$ factor, we obtain

$$\begin{pmatrix} -\omega^2+2\omega_0^2 & -\omega_0^2 & 0 \\ -\omega_0^2 & -\omega^2+2\omega_0^2 & -\omega_0^2 \\ 0 & -\omega_0^2 & -\omega^2+2\omega_0^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{4.14}$$

where $\omega_0^2 = k/m$. A nonzero solution for (A_1, A_2, A_3) exists only if the determinant of this matrix is zero. Setting it equal to zero gives

$$\begin{gathered} \left(-\omega^2+2\omega_0^2\right)\left(\left(-\omega^2+2\omega_0^2\right)^2-2\omega_0^4\right)+\omega_0^2\left(-\omega_0^2\left(-\omega^2+2\omega_0^2\right)\right)=0 \\ \Longrightarrow \left(-\omega^2+2\omega_0^2\right)\left(\omega^4-4\omega_0^2\omega^2+2\omega_0^4\right)=0. \end{gathered} \tag{4.15}$$

This is a 6th-order equations. Anyway it is just cubic in ω^2 and, as $(-\omega^2+2\omega_0^2)$ is a factor, at the end we have a quadratic equation in ω^2. Using the quadratic formula, the roots to 4.15 are

$$\omega^2 = 2\omega_0^2 \qquad \textit{and} \qquad \omega^2 = \left(2 \pm \sqrt{2}\right)\omega_0^2 \tag{4.16}$$

If weplug these values back into 4.14 to find the relations among A_1, A_2, A_3 we have the three normal modes

$$\begin{aligned} \omega = \pm\sqrt{2}\omega_0 \quad &\Longrightarrow \quad \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \propto \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \\ \omega = \pm\sqrt{2+\sqrt{2}}\omega_0 \quad &\Longrightarrow \quad \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \propto \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} \\ \omega = \pm\sqrt{2-\sqrt{2}}\omega_0 \quad &\Longrightarrow \quad \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \propto \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \end{aligned} \tag{4.17}$$

The most general solution is obtained by taking an arbitrary linear combination of the six solutions corresponding to the six possible values of ω:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \exp\left(i\sqrt{2}\omega_0 t\right) + C_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \exp\left(-i\sqrt{2}\omega_0 t\right) + \ldots \tag{4.18}$$

However, the x's must be real, so C_2 must be the complex conjugate of C_1. Likewise for the two C's corresponding to the $\left(1, -\sqrt{2}, 1\right)$ mode, and also for the two C's corresponding to the $\left(1, \sqrt{2}, 1\right)$ mode. Following the procedure that transformed

4.7 into 4.8, we see that the most general solution can be written as

$$\begin{aligned}\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} &= A_m \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cos\left(\sqrt{2}\omega_0 t + \phi_m\right) \\ &+ A_f \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} \cos\left(\sqrt{2+\sqrt{2}}\omega_0 t + \phi_f\right) \\ &+ A_s \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \cos\left(\sqrt{2-\sqrt{2}}\omega_0 t + \phi_s\right)\end{aligned} \tag{4.19}$$

where the subscriptions "m", "f", and "s" stand for middle, fast, and slow. The six unknowns, A_m, A_f, A_s, ϕ_m, ϕ_f and ϕ_s are determined by the six initial conditions (three positions and three velocities). If A_m is the only nonzero coefficient, then the motion is purely in the middle mode. Likewise for the cases where only A_f or only A_s is nonzero.

4.4. N masses

Let's now consider the general case of N masses between two fied walls. The masses are all equal to m, and the spring constants are all equal to k. The method we'll use below will actually work even if we don't have walls at the ends, that is, even if the masses extend infinitely in both directions. Let the displacements of the masses relative to their equilibrium positions be x_1, x_2, ..., x_N. If the displacements of the walls are called x_0 and x_{N+1}, then the boundary conditions that we'll eventually apply are $x_0 = x_{N+1} = 0$.

The force on the nth mass is

$$F_n = -k(x_n - x_{n-1}) - k(x_n - x_{n+1}) = kx_{n-1} - 2kx_n + kx_{n+1} \tag{4.20}$$

So at the end we have a collection of $F = ma$ equations that look like

$$m\ddot{x} = kx_{n-1} - 2kx_n + kx_{n+1} \tag{4.21}$$

These can be callected in the matrix equation

$$m\frac{d^2}{dt^2}\begin{pmatrix} \vdots \\ x_{n-1} \\ x_n \\ x_{n+1} \\ \vdots \end{pmatrix} = \begin{pmatrix} & & & \vdots & & & & \\ \cdots & k & -2k & k & & & & \\ & & k & -2k & k & & & \\ & & & k & -2k & k & \cdots & \\ & & & & \vdots & & & \end{pmatrix}\begin{pmatrix} \vdots \\ x_{n-1} \\ x_n \\ x_{n+1} \\ \vdots \end{pmatrix}. \tag{4.22}$$

We can guess a solution of the form

$$\begin{pmatrix} \vdots \\ x_{n-1} \\ x_n \\ x_{n+1} \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ A_{n-1} \\ A_n \\ A_{n+1} \\ \vdots \end{pmatrix} \exp(i\omega t). \tag{4.23}$$

Setting the resulting determinant equal to zero for large N, it would be completely intractable to solve for the ω's by using the determinant method. Instead of the determinant method, we'll look at each of the $F = ma$ equations individually. Let us consider the nth equation. Substituting $x_n = A_n \exp(i\omega t)$ in the 4.21 and cancelling the $\exp(i\omega t)$ factor, we obtain

$$\begin{aligned} -\omega^2 A_n = \omega_0^2 \left(A_{n-1} - 2A_n + A_{n+1}\right) \\ \implies \quad \tfrac{A_{n-1}+A_{n+1}}{A_n} = \tfrac{2\omega_0^2-\omega^2}{\omega_0^2} \end{aligned} , \tag{4.24}$$

where $\omega_0 = \sqrt{k/m}$. This equation must hold for all values of n from 1 to N, so we have N equations of this form. For a given mode with a given frequency ω, the quantity $\left(2\omega_0^2 - \omega^2\right)/\omega_0^2$ on the righthand side is a constant, independent of n. So the ratio $(A_{n-1} + A_{n+1})/A_n$ on the lefthand side must also be independent of n. The problem therefore reduces to finding the general form of a string of A's that has the ratio $(A_{n-1} + A_{n+1})/A_n$ being independent of n. If we know three adjacent A's, then this ratio is determined, so we can recursively find the A's for all other n . Or equivalently, if we know two adjacent A's and also ω, so that the value of $\left(2\omega_0^2 - \omega^2\right)/\omega_0^2$ is known (we're assuming that ω_0 is given), then all the other A's can be determined. The following claim tells us what form the A's take. It is this claim that allows us to avoid using the determinant method.

Claim: If $\omega \leq 2\omega_0$, then any set of A_n's satisfying the system of N equations in 4.24 can be written as

$$A_n = B\cos n\theta + C\sin n\theta, \tag{4.25}$$

for certain values of B, C, and θ. (The fact that there are three parameters here is consistent with the fact that three A's, or two A's and ω, determine the whole set.).

Proof: We'll start by defning

$$\cos\theta \equiv \frac{A_{n-1} + A_{n+1}}{2A_n}. \tag{4.26}$$

The righthand side is independent of n, so θ is well defined. If we're looking at a given normal mode with frequency ω, then in view of 4.24, an equivalent defnition of θ is

$$2\cos\theta \equiv \frac{2\omega_0^2 - \omega^2}{\omega_0^2}. \tag{4.27}$$

These defnitions are permitted only if they yield a value of $\cos\theta$ that satisfies $|\cos\theta| \leq 1$. This condition is equivalent to the condition that ω must satisfy $-2\omega_0 \leq \omega \leq 2\omega_0$. We'll just deal with positive ω here (negative ω yields the same results, because only its square enters into the problem), but we must remember to also include the $e^{i\omega t}$ solution in the end. So this is where the $\omega \leq 2\omega_0$ condition in the claim comes from.

We can prove that with walls at the ends, θ (and hence ω) can take on only a certain set of discrete values. If there are no walls, that is, if the system extends infinitely in both directions, then θ (and hence ω) can take on a continuous set of values. The N equations represented in 4.24 tell us that if we know two of the A's, and if we also have a value of ω, then we can use the equations to successively determine all the other A's. Let's say that we know what A_0 and A_1 are (For instance, if there are walls, $A_0 = 0$). The rest of the An's can be determined as follows. We define B by

$$A_0 \equiv B\cos(0\cdot\theta) + C\sin(0\cdot\theta) \quad \Longrightarrow \quad A_0 \equiv B. \tag{4.28}$$

(So $B = 0$ if there are walls). Now we define C by

$$A_1 \equiv B\cos(1\cdot\theta) + C\sin(1\cdot\theta) \quad \Longrightarrow \quad A_1 \equiv B\cos\theta + C\sin\theta. \tag{4.29}$$

For any A_0 and A_1, these two equations uniquely determine B and C (θ was already determined by ω). By construction of these definitions, the proposed $A_n = B\cos n\theta + C\sin n\theta$ relation holds for n = 0 and n = 1. We will now show inductively that it holds for all n.

If we solve for A_{n+1} in 4.27 and use the inductive hypothesis that the $A_n = B\cos n\theta + C\sin n\theta$ result holds for $n-1$ and n, we have

$$\begin{aligned}
A_{n+1} &= (2\cos\theta)\,A_n - A_{n-1} \\
&= 2\cos\theta\,(B\cos n\theta + C\sin n\theta) - (B\cos(n-1)\,\theta + C\sin(n-1)\,\theta) \\
&= B\,(2\sin n\theta\cos\theta - (\cos n\theta\cos\theta + \sin n\theta\sin\theta)) \\
&\qquad + C\,(2\sin n\theta\cos\theta - (\sin n\theta\cos\theta - \cos n\theta\sin\theta)) \\
&= B\,(\cos n\theta\cos\theta - \sin n\theta\sin\theta) + C\,(\sin n\theta\cos\theta + \cos n\theta\sin\theta) \\
&= B\cos(n+1)\,\theta + C\sin(n+1)\,\theta,
\end{aligned} \tag{4.30}$$

which is the desired expression for the case of $n+1$. (Note that this works independently for the B and C terms.) Therefore, since the $A_n = B\cos n\theta + C\sin n\theta$ result holds for $n = 0$ and $n = 1$, and since the inductive step is valid, the result therefore holds for all n. We could have instead solved for A_{n-1} in 4.30 and demonstrated that the inductive step works in the negative direction too. Therefore, starting with two arbitrary masses anywhere in the line, the $A_n = B\cos n\theta + C\sin n\theta$ result holds even for an infinite number of masses extending in both directions.

This claim tells us that we have found a solution of the form,

$$x_n(t) = A_n e^{i\omega t} = (B\cos n\theta + C\sin n\theta)\, e^{i\omega t}. \tag{4.31}$$

As for convention ω is positiuve, the solution $e^{-i\omega t}$ works as well. So another solution is

$$x_n(t) = A_n e^{-i\omega t} = (D\cos n\theta + E\sin n\theta)\, e^{-i\omega t}. \tag{4.32}$$

The coefficients in this solution need not be the same as those in the $e^{i\omega t}$ solution. Since the $F = ma$ equations in 4.21 are all linear, the sum of two solutions is again a solution. So the most general solution (for a given value of ω) is the sum of the above two solutions (each of which is itself a linear combination of two solutions). As usual, we now invoke the fact that the positions must be real. This implies that the above two solutions must be complex conjugates of each other. And since this must be true for all values of n, we see that B and D must be complex conjugates, and likewise for C and E. Let's define $B = D* \equiv (F/2)\, e^{i\beta}$ and $C = E* \equiv (G/2)\, e^{i\gamma}$. There is no reason why B, C, D, and E (or equivalently the A's in 4.23) have to be real. The sum of the two solutions then becomes

$$x_n(t) = C_1\cos n\theta\cos\omega t + C_2\cos n\theta\sin\omega t + C_3\sin n\theta\cos\omega t + C_4\sin n\theta\sin\omega t, \tag{4.33}$$

where θ is determined by ω via the 4.27, which it is possible to write in the form

$$\theta \equiv \cos^{-1}\left(\frac{2\omega_0^2 - \omega^2}{2\omega_0^2}\right). \tag{4.34}$$

The constants C_1, C_2, C_3, C_4 in 4.33 are related to the constants F, G, β, γ in 4.32 in the usual way ($C_1 = F\cos\beta$, etc.).

4.33 is the most general form of the positions for the mode that has frequency ω. This set of the $x_n(t)$ functions (N of them) satisfies the $F = ma$ equations in 4.21 (N of them) for any values of C_1, C_2, C_3, C_4. These four constants are determined by four initial values, for example, $x_0(0)$, $\dot{x}_0(0)$, $x_1(0)$ and $\dot{x}_1(0)$. Of course, if $n = 0$ corresponds to a fixed wall, then the first two of these are zero.

5. The Path on Oscillation for Upper Secondary School Students

5.1. Introduction

The path on oscillations that we present here is the result of a Design Based Research on normal modes with Italian upper secondary school students. The complete path has been proposed to three classes of 11th grade students during curricular lessons. A version of the sequence has been proposed also to other three classes (one of grade 11th and two of grade 12th) during afternoon extra-curricular lessons, and a version with university-level formalism as also been proposed to a group of undergraduated students in mathematics during the third year course *"Preparation of didactical experiments"*. A reduced version of the path has also been proposed to a number of classes of 12th grade students within the one-shot lessons on oscillations (afternoon extracurricular activities) in the framework of PLS[PLS,] activities. The one-shot lessons have been attended, in time, by about six hundred students.

This works origins from the necessity of introducing a complete path on oscillations that in the Italian upper secondary school is still in part missing. The path starts from the oscillation of single simple oscillators, considers two coupled oscillators and presents the normal modes of oscillation up to more complex systems of many coupled oscillators until you get to the continuous case, for instance, oscillating strings and membranes. Coupled oscillations and normal modes are considered complicated topics, in part because of the mathematics involved. The aim is that to introduce such topics to 11th and 12th grade students avoiding the use of too complex mathematics and calculus but without losing rigor and completeness.

5.2. The data-logging techniques

The path on oscillation is entirely based on an experimental approach. We chose to collect data with students with two different data-logging and data analysis systems: the commercial *Vernier Logger Pro* software, with sonar motion detector [Log,] and the freeware video analysis software *Tracker* [tra,]. The two data acquisition systems are not exclusive but complementary. In fact, both have such characteristics as to make one more suitable than the other, depending on the experiment and the measurement to be made (see Fig. 5.1).

Logger Pro	Tracker
• Real time measurements • High sensitivity • Handles different probes (sound, light, motion, force etc.) • Tracks also fast objects	• Free software • Tracks also in two dimensions • Virtually no limit to the number of targets • Technology students friendly (smartphone)

Figure 5.1.: Comparison between the two data-logging techniques: *Vernier Logger Pro* and *Tracker*

5.2.1. The Logger Pro System

The Vernier data-logging system consists of the Vernier motion detector2 connected to the Vernier A/D converter interface "LabPro". The interface is connected to a PC and the system is driven by its dedicated software, namely "Logger Pro". Up to two digital sensors can be connected to LabPro (Fig. 5.2, Fig. 5.3). All needed data analysis can be performed online with Logger Pro software.

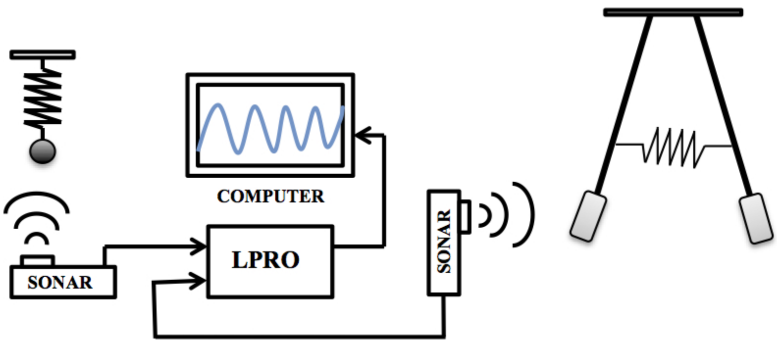


Figure 5.2.: Scheme of the Vernier data-logging System

The sonar motion detector emits short bursts of ultrasonic sound waves from the gold foil of the transducer. These waves fill a cone-shaped area about 15° to 20° off the axis of the centerline of the beam. The range of detection spans between 0.150 - 6.000 meters with a resolution of 0.001 meters. The interface resolution is 12 - bit

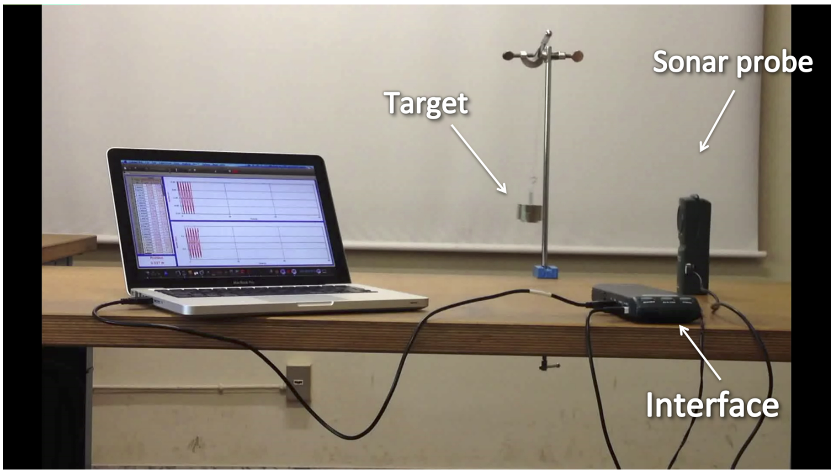


Figure 5.3.: The apparatus

and maximum sampling rate is 50,000 samples per second [det, 2014]. In our plots the uncertainty bars are not shown because they have the same size of dots.

This system has some characteristics that make it very suitable to track the motion also of fast objects, provided the motion takes place along a rectilinear trajectory. In fact the sonar detector can only detect objects that move in front of it within an angle of about 20° off the axis of the centerline of the beam. The Motion Detector is capable of measuring objects as close as 0.15 m and as far away as 6 m with a resolution of less than 1 mm. Of course the maximum distance for tracking the motion depends on the reflectivity of ultrasonic waves of the object. For instance, metallic bodies are easily detectable at greater distance than plastic or wooden bodies. An important characteristic is that this system performs real time measurements. The duration of the measure is virtually indefinite since it depends only on the memory available on your computer. The high sample rate available (up to 50,000 samples per second) enables to track the motion of fast moving objects with a good resolution. Moreover, the system manages a number of different probes in addition to the sonar: sound and light sensor, force sensor, pressure sensor and many others; this makes it useful for many didactical lab activities, not only for the study of oscillations.

The Logger Pro software has many interesting tools for managing and analysing the collected data, in particular it can easily provide the FFT graph of the waveform of the motion.

5.2.2. The Tracker Video Analysis Software

The Tracker Video Analysis Software is a freeware software developed at Cabrillo University (Aptos, California) [Cab, 2014] by Douglas Brown [Dou, 2014] and it is designed to be used in physics education.

Tracker is an image and video analysis package and modelling tool that is built upon the Open Source Physics Java code library. Features include object tracking with position, velocity and acceleration overlays and graphs, special effect filters, multiple reference frames, calibration points and line profiles for analysis of spectra and interference patterns. It is designed to be used in introductory college physics labs and lectures but it is suitable for physics courses and laboratories at upper secondary school level, as well [Ope, 2014]. Tracker can handle any video. in particular the videos of experiments, taken by smartphones. This turns a smartphone into an important and versatile laboratory instruments available almost everywhere and always. The clear advantage is the fact that the software is freeware, and so no cost for the school, and nowadays almost every student owns a smartphone. Unlike the sonar detector, Tracker can track the motion of an object also in two dimensions so it can be used to study the motions in a plane such as circular motion, parabolic motion and many others. Moreover, there is virtually no limit to the number of targets to track and some important modelling can be performed: for instance the study of the motion of the center of mass. Another advantage is the fact that the students like very much the use of a technology very friendly to them such as their smartphones and in our experience they result very enthusiastic in making the experiments. There are a couple of main disadvantages. The first is that Tracker does not allow a real time measurement and plotting of the results. It needs that the video be uploaded on the computer and elaborated. This process, depending on the quality of the video and on its length, can last a long time. The second important limit for the use of Tracker is the fact that it is based on the recognition of the position of the target object in subsequent video frames. Thus, if the motion is too fast, the image of the object in each frame can be not well defined and difficult to be tracked. As the refresh time of smartphones is usually of 25-30 frames per second, it is difficult to track the videos that are too fast. Also the Tracker software has many tools for the analysis of the collected data, such as the FFT tool. In Fig. 5.4 it is shown the typical graphic interface of the Tracker software.

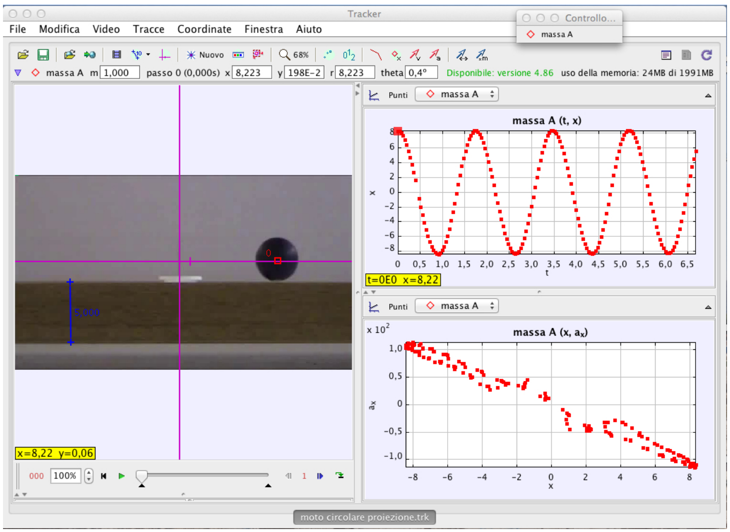


Figure 5.4.: A typical example of the graphic interface of Tracker software. In this case a target ball fixed on a rotating disk is being tracked as to study the projection of a circular motion along one axis. The video has been recorded by an Iphone 4s device.

5.3. The Path

The path on oscillations is based on a number of activities in which we start from a real experiment or a video or else an applet simulation to introduce and discuss a limited topic [Stellato et al., 2014b, Stellato et al., 2014a]. The general purpose is to identify, among the oscillations, those that give rise to a peculiar kind of motion: the harmonic motion, and determine the conditions under which such motion can be obtained. A number of significative situations of harmonic and anharmonic motions are investigated and criteria to estabilishing the armonicity/anharmonicity of the oscillation are discussed. An important tool for the analysis of the data is then introduced: the Fast Fourier Trasform. The FFT is introduced as a tool and not discussed through mathematics. Then the concept on resonance is introduced in a phenomenological way through experiments and exploring related videos in the Internet videos database [YOU, 2014]. The next step is the introduction of the coupling between two oscillators and the discovery of particular motion configurations: the *Normal Modes of Oscillation*. We then extend the experiments to three, four, five....many coupled oscillator until we arrive to the continuous case; first in one dimension with the string and then in two dimensions with the Cladni plates and study the normal modes of such complex systems.

Here the main activities follow:

5.3.1. The restoring forces acting on oscillators

The path on oscillation is oriented to the introduction of normal modes in the Italian upper secondary school. Thus a deep comprehension of the harmonic motion is needed. We think that a fruitful definition of harmonic motion is the dynamic one, even if it is not usually present in the textbooks and in the courses for upper secondary students, in Italy. For this reason it is necessary that the students understand the role of forces on oscillations, in fact a conceptual knot for harmonic motion is the concept of elastic (linearized) restoring force [Giliberti et al., 2014].

The activity starts with a brief brainstorming on what is an oscillation and what is a periodic phenomenon. In this phase the teacher/researcher just guides the discussion without giving definitions. The ideas must come from students, mainly in peer to peer discussion, and definitions must eventually come as the result of negotiation between students. Here the teacher/researcher has the role of guide/facilitator only.

A particular attention should be deserved to the consideration that, in nature, almost none oscillation is really periodic because of friction. Anyway we can consider periodic those motions for which damping is small enough and for a short time of observation, provided the shapes of the motion repeats at regular time intervals.

After the brainstorming phase, some real oscillating objects are shown to students (if requested, they are repeated and students can try themselves): a *vertically bouncing ball*, a *simple pendulum*, a *disk bouncing between two elastic edges on an air table*, a *vertical mass on a spring*, a *seesaw on flat pivot*, a *seesaw on a round pivot*, *a ball on a semi-circular track and a Galileo-pendulum*. Students are asked to group these experiments by some common characteristics they can identify. If it doesn't emerge from the peer discussion, the teacher/researchers guides the students to take into account, in grouping the oscillations, the existence of a stable equilibrium point. Then the analysis of the total force acting along the trajectory is performed. At this point the students are asked to group again the oscillations taking into account the characteristic of the forces. So two groups of oscillations can be obtained: oscillations which have a single stable equilibrium position, subject to a restoring force, and all the others. In a first time, the students try to produce the graphs of the force as a function of the position, then similar graphs (acceleration as a function of position) are produced via the two data-logging systems described above. Of course a brief training on the use of the data-logging systems is required.

5.3.1.1. The vertical bouncing ball

The vertical bouncing ball is simply an elastic ball (it can be either made of rubber, steel or glass) that is let free to fall from a height and bouncing on the floor as in Fig. 5.5(a). The experiment can be performed with zero initial velocity or with an initial vertical velocity. Let us consider a complete "oscillation", that is a complete bounce, in which the ball starts falling from a height, impacts with the floor, bounces and goes back toward the start position. Of course at each bounce the maximum height decreases because of the energy loss in the impact with the ground and the friction with the surrounding air. Once falling, the ball is subject only to gravitational force, if we neglect the small friction due to the air. So during the fall the total force acting on the ball is directed vertically towards the ground; the same is true after the impact, when the ball is no more in contact with the ground and goes back up to the starting position. Things are different during the impact with the floor: in the very short time of impact, the ball deforms and an impulsive elastic force superimpose to the gravitational force with an opposite sign. If we choose a reference system oriented upwards with the origin centered in the impact point [1] the overall force acting on the ball is of the form of the one of Fig. 5.5(b). Note that the force is intended acting in the center of mass of the ball and, even if the ball is squeezed during the impact, its thickness never equals zero. So the graph of the force lies in the first and in the fourth quadrant and never intersects the axis of the force (so it does not pass through the origin of the reference system).

Trying to draw the graph of the force vs the position (acceleration vs position) with Loggerpro system or with Tracker video analysis is quite difficult in this case. In fact in the first case, the sonar sensor needs to be in front of the falling ball during all the measurements and, in the second case, the velocity of the ball is too high to obtain well-enough defined phtograms with smartphones, to be tracked with Tracker. It is possible to track the experiment using cameras with high photogram-refresh rate; this is not our case.

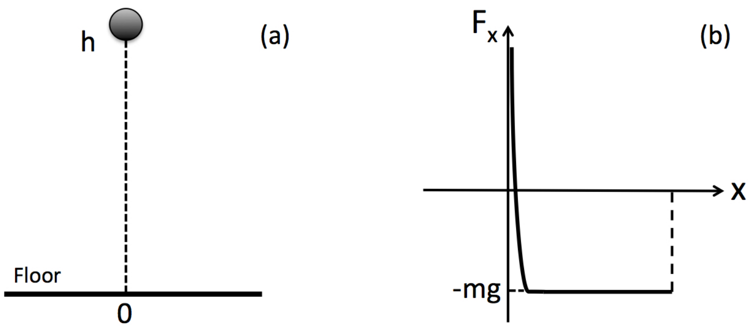


Figure 5.5.: (a) scheme of a ball bouncing on the floor; (b) diagram of the force versus position for the bouncing ball

[1] the impact point is not properly a stable equilibrium point, in fact if we suppose to dig a hole in the floor, the ball would continue to fall downwards

5.3.1.2. The simple pendulum

The simple pendulum is the classic pendulum that 11th and 12th grade students already know (Fig. 5.6). In our experiment we chose to use a bifilar pendulum because the bifilar version guarantees stability of the plan of the oscillation: this is important if we want to track the motion by sonar detector. Also in this case we neglect the friction with air.

The rest position for the pendulum is a stable equilibrium position: if the pendulum is displaced rightwards, it tends to move left to return to the equilibrium position; vice versa, if the pendulum is displaced leftwards, it moves rightwards to return to the equilibrium position. The only force responsible for the motion is the weight (gravitational force), in particular its component along the trajectory. A detailed analysis of the forces with students is important, in fact, while almost all students (in our experimentation classes) were able to identify and draw the components of the gravitational force, none of them could predict the presence of the centripetal force due to the fact that the bob of the pendulum moves with velocity along an arc of circumference.

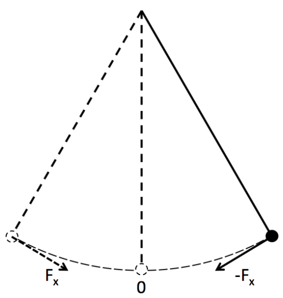


Figure 5.6.: Scheme of the simple pendulum. It is schetched the total force acting along the trajectory

At this point we ask the students to draw the force vector in different points of the trajectory as to make them aware that in each point the force acting along the trajectory is directed towards the equilibrium position. If we choose the equilibrium point as the origin of the system of curvilinear coordinate describing the trajectory, we can state that, in each point, the force results opposite to the displacement. In this way we can define the general concept of *restoring force*. So, in a system with a stable equilibrium position, a restoring force is a force that is always directed towards the equilibrium point (the origin of the reference system) and opposite to the displacement; namely, a force that tends to return the system to its equilibrium position. If we try to draw such a force as a function of the position, without too many details, it must pass through the origin of the reference system and lies in the second and fourth quadrant as depicted in Fig. 5.7. The accurate graph of the force vs position can be discussed with students and their results can be compared with the graph obtained (for acceleration vs position) with the one of the data-logging techniques. For instance, if we process the data obtained by the sonar detector, we obtain a graph like the one in Fig. 5.8.

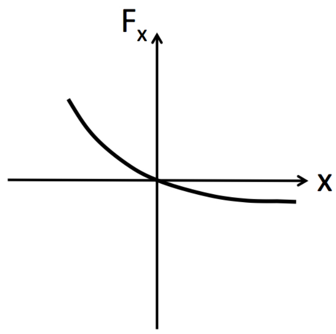


Figure 5.7.: The restoring force, a general graph

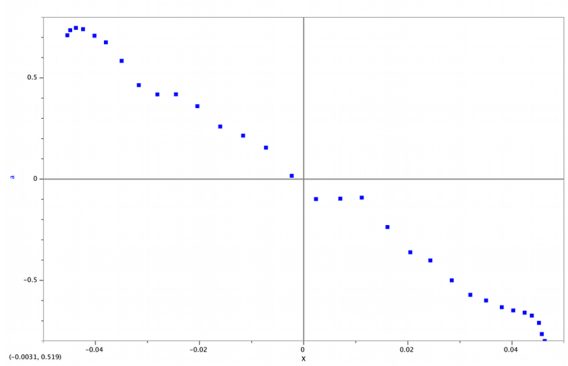


Figure 5.8.: Diagram of the acceleration vs displacement for a simple pendulum. The graph refers to a single oscillation and the data have been collected via sonar detector and processed with the LoggerPro System.

5.3.1.3. The bouncing disk

The bouncing disk consists of a disk moving on an air table[2], so to reduce friction, and bouncing between two opposite elastic edges (see Fig. 5.9). If we give the disc

[2]The air table is simply a horizontal plane, with many uniformly distributed holes through which an air compressor blows air with constant flow. The disk placed on the surface of the table is like floating on a thin layer of air; in this way the sliding friction with the floor is practically zero.

an initial velocity, it will bounce back and forth between elastic edges. It is clear that in this case there isn't a single equilibrium position: almost all the infinite points between the two opposite edges are equilibrium points, although not stable equilibrium points. We want, as in previous experiments, to plot the total force acting along the trajectory as a function of the position. It comes quite natural to place the origin of the reference system in the midpoint of the trajectory, but any other point is fine. The total force acting along the trajectory is zero in each position between the edges. In fact the only force is the gravitational one that results perpendicular to the plane of motion (and therefore to the trajectory) and it is balanced by the impact force of the air from the compressor. As in the case of the bouncing ball, things change at the edges, during the collision of the disk with the elastic bands. During the impact of the disk with the rubber bands an impulsive elastic force arises and reverses the direction of the speed of the disk. The only effect of the impact with the edges is the reversing of the velocity (of course in the assumption of perfectly elastic impact). The graph of the force acting on the disk along its straight trajectory is like that of Fig. 5.10. So in this case the graph of the force vs displacement, interests all the quadrants. In fact, the section where it is zero belongs to the first and fourth quadrant as well when x is positive and it belongs to the second and third quadrant as well when x is negative.

Usually the students have some difficulties to plot the correct graph in this case. They tend to associate the force to velocity so imagine that the effect of the impact with the edges is just the reversing of such acting force. The use of data-logging techniques is very useful to convince students that no forces act on the disk (along the trajectory) far from the edges. This can be done in two ways: the first is the measure ov the velocity that results constant between the edges (Fig. 5.11(a)), hence there is no acceleration and therefore, no force; the second is the direct plotting of the graph of the acceleration vs the position (Fig. 5.11(b)). This experiments requires the use of Tracker because it is difficult to target with a sonar the motion of an object which can have two motion components [Stellato et al., 2014a]. As shown in Fig. 5.9 , it is necessary to mark the tracked object by a well contrasted target.

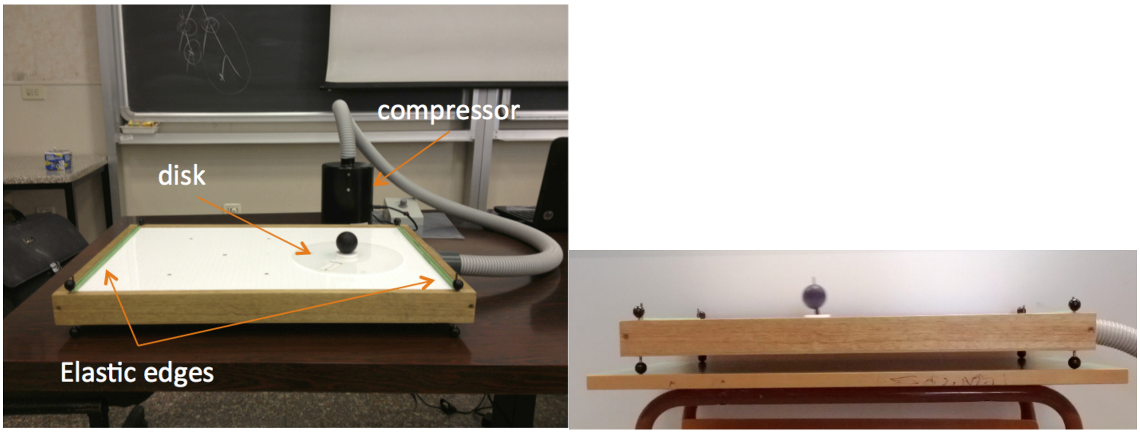


Figure 5.9.: The bouncing disk on an air table. The black bob fixed at the center of the disk is the target body to be tracked with Tracker video-analysis software; the color has been chosen so as to provide the maximum possible contrast.

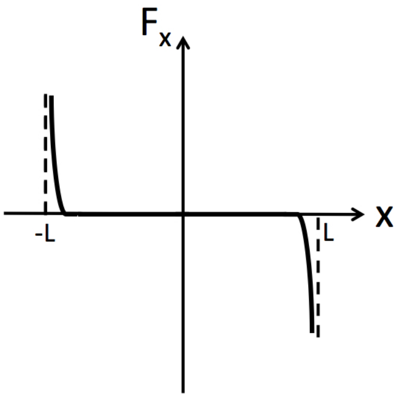


Figure 5.10.: diagram of the force versus position for the bouncing disk

5.3.1.4. The vertical mass on a spring

The vertical mass-spring oscillator consists of a mass appended at the bottom of a vertical spring (see Fig. 5.12). The vertical configuration avoids the problem of the friction with surfaces. The mass, given the elastic constant of the spring, is chosen so as to have a stable vertical oscillation. In fact, a parametric resonance between the vertical spring mode and the transverse pendulum mode is estabilished when the spring oscillation frequency doubles that of the pendulum [Olsson, 1976, Cayton, 1977]. Also in this experiment, we clarify with students that we are neglecting the effect of the friction with the air (and spring internal friction).

In this configuration, the system settles itself in a stable equilibrium position where

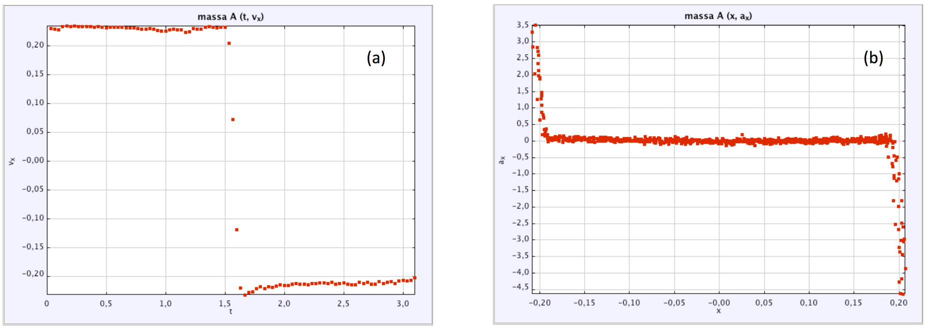


Figure 5.11.: The bouncing disk: (a) diagram of the velocity vs time before and after an impact with an elastic edge: the effect of the impact is just the reversing of velocity; (b) diagram of the acceleration vs displacement. The graphs have been obtained processing by Tracker video-analysis software the video of the experiment. The video has been recorded by an Iphone 4s device.

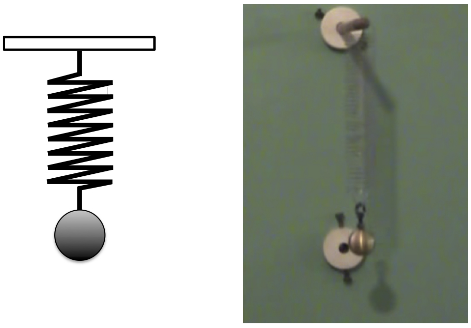


Figure 5.12.: The vertical spring-mass system

the gravitational force, due to the appended mass and the mass of the spring, is balanced by the elastic force due to a proper elongation of the spring. If we pull the mass downwards, an additional contribute due to elastic force, arising from the further stretching of the spring, recalls the mass towards the equilibrium position. The same thing happens if we push the mass upwards. In both cases, the elastic force is directed in the direction opposite to the displacement, in accordance with Hooke's Law. The 11th and 12th grade students already know well the Hook's Law and are able to make an accurate graph of the force along the trajectory versus the displacement, which results to be linear (see Fig. 5.13). So also in this case, if we choose the equilibrium position as the origin of the reference system, the graph of the force passes through the origin and it lies in the second and fourth quadrant. Moreover, this graph is a straight line: it not only represents a restoring force, but a linear (or elastic) restoring force.

In this experiment it is particularly simple to plot the graph of the force (acceler-

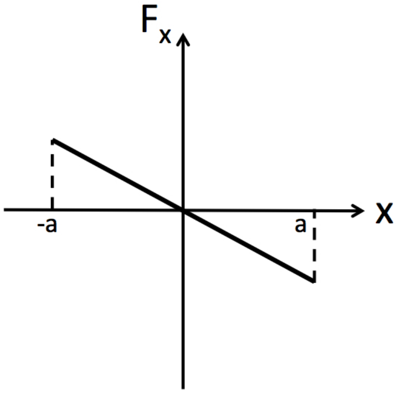


Figure 5.13.: The graph of force vs displacement for the vertical spring-mass system. The x cohordinate in this case indicates the vertical axis, oriented upwards

ation) by collecting and processing data with a data-logging system, because the trajectory is rectilinear. Both the Logger Pro and the Tracker techniques fit the goal. In Fig. 5.14 it is reported the graph obtained by sonar detector and Logger Pro. Note that in this case the amplitude of the oscillation is not very small (nearly 15 cm), nonetheless the graph of the acceleration vs position is nearly a perfect straight line passing through the origin of the reference system.

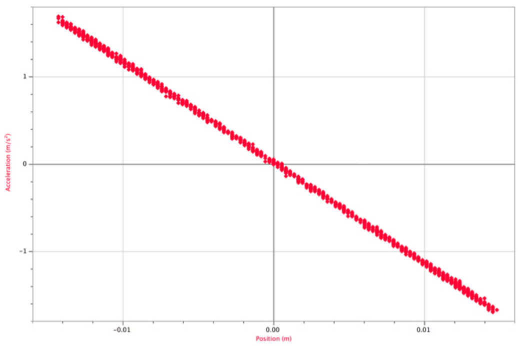


Figure 5.14.: The vertical spring-mass: graph of the acceleration versus position. The data have been collected by sonar detector and processed by Logger Pro.

5.3.1.5. The seesaw on a flat pivot

The seesaw on a flat pivot is simply an homogeneous bar oscillating around a flat pivot as in Fig. 5.15. In this experiment we ask the students to draw the graph of the force vs position of a point (the dot in Fig. 5.15) at the end of the bar. This is a very complicated object to describe for 11th and 12th grade students. In fact a complete description of the experiment would require the knowledge of dynamics of the rigid body. During the discussion the teacher can help students with a simplified representation of the experiment. Namely, when the bar is lifted, as in Fig. 5.16, the contact point between the seesaw and the pivot is such that the right part of the bar, with respect to the contact point, is longer than the left one. The difference in length equals the width of the pivot. As the only force responsible for the oscillation is the weight of the seesaw, we can assume that equal parts of the bar, in opposite side with respect of the contact point, balance each other. So the motion is substantially due only to the part of the bar that remains unbalanced (the gray segment in Fig. 5.16 labeled by the letter “d” corresponding to the width of the pivot). The fall of the seesaw is therefore caused by the effect of the weight of this unbalanced piece of bar and it is a vertical force directed downwards. When the seesaw reaches the horizontal position, the contact point is the opposite side of the pivot and things just invert. In this case, the unbalanced piece of bar is at the opposite side of the seesaw. This results in the fact that now, the apparent force we can see acting in our dot is vertical, directed upwards. The system has a stable equilibrium position: it is the natural configuration with the bar resting horizontally on the pivot. If we chose the stable equilibrium position, as the origin of the reference system, we can obtain a graph for the force vs position like the one in Fig. 5.17. So also in this case, if we choose the equilibrium position as the origin of the reference system, the graph of the force passes through the origin and it lies in the second and fourth quadrant. Moreover, if we choose small-amplitude oscillations, this graph is a straight line: it not only represents a restoring force, but a linear (or elastic) restoring force as in

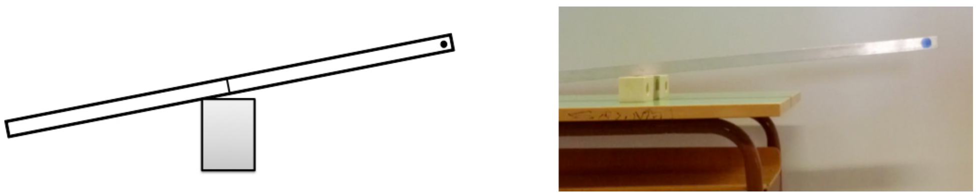


Figure 5.15.: The seesaw on a flat pivot: the scheme (left) and the picture of the real experiment (right). The dot at the very end of the bar is the target to be tracked with Tracker software.

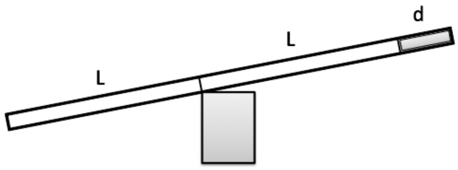


Figure 5.16.: Qualitative explanation of the effect of the unbalanced part of a seesaw on a flat pivot

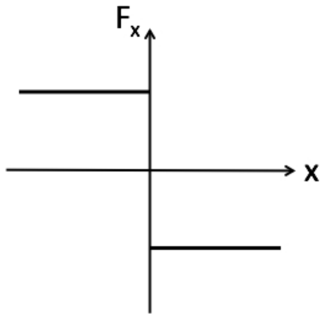


Figure 5.17.: The seesaw on a flat pivot: the qualitative diagram of the force vs position

So in this case the force in the diagram force vs displacement, lies almost in the second and fourth quadrant but it is not a regular graph as it has a discontinuity in the origin. In this experiment, a data logging is possible both with Logger Pro both with Tracker. A typical graph obtained is the one in Fig. 5.18 where in the left is represented the acceleration versus time and in the right, the acceleration versus position. Here the action of the teacher is needed to explain the discrepancy between the graphs of Fig. 5.18 and the expected graph of Fig. 5.17. In fact the graphs of Fig. 5.18 seem to be quite regular and one could think that they do not present a discontinuity in the origin, because of the presence of some few dots in the neighborhood of the origin. These dots are due to the fact that both the sonar detector and the video analysis do not really measure the acceleration: they measure the positions as a function of time. The acceleration is obtained via the second derivative of the position with a continuous step numerical method.

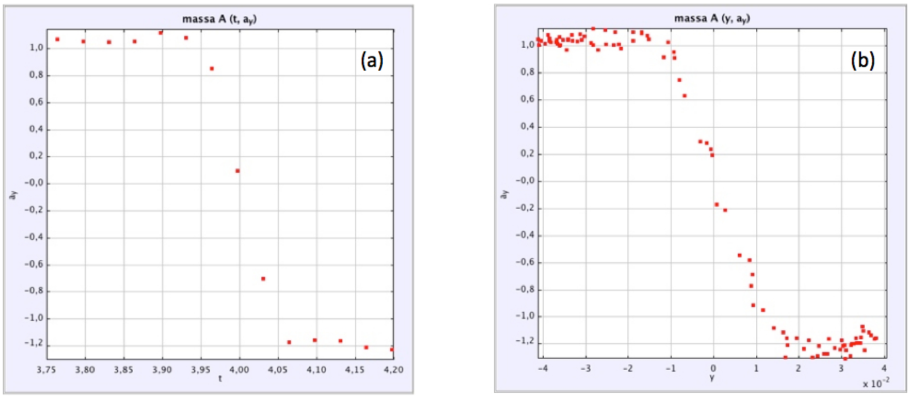


Figure 5.18.: The seesaw on a flat pivot: (a) acceleration vs time and (b) acceleration vs position. In this case the graphs have been obtained by traking the video of the experiment by Tracker software. The video has been recorded by an iphone 4s device.

5.3.1.6. The seesaw on a round pivot

This is the same experiment as the previous one with the only (but significant) difference that the pivot is no more a flat surface but it is a perfectly circular surface. The situation is represented in Fig. 5.19. Also in this experiment we ask the students to draw the graph of the force vs position of a point at the end of the bar. Also in this case we choose to avoid the description of the experiment through the dynamics of the rigid body. During the discussion the teacher can help students with the same simplified representation of the experiment as in the previous case. Namely, when the bar is lifted, the contact point between the seesaw and the pivot is such that the right part of the bar, with respect to the contact point, is longer than the left one. As the only force responsible for the oscillation is the weight of the seesaw, we can assume that equal parts of the bar, in opposite side with respect to the contact point, balance each other. So the motion is substantially due only to the part of the bar that remains unbalanced . The fall of the seesaw is therefore caused by the effect of the weight of this unbalanced piece of bar and it is a vertical force directed downwards. The main difference with the seesaw on a flat pivot consists in the fact that in this case there are no more only two tilting points but infinite. Thus the unbalanced part of the seesaw changes in a continuous way. If the oscillation is small enough, the unbalanced part of the rod results proportional to the inclination angle and therefore to the displacement. In such a way, the force responsible for the motion (that is the weight of the unbalanced part of the seesaw) results proportional to the displacement. If we choose the position of the end of the rod when the seesaw is perfectly horizontal as the origin of the reference system and we draw the force as a function of the displacement, we expect to have a straight line passing through the origin and lying in the second and fourth quadrant as in Fig. 5.13.

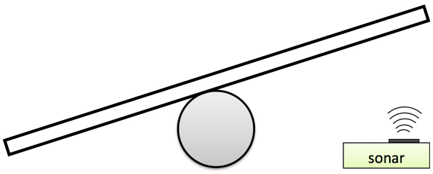


Figure 5.19.: The seesaw on a round pivot: scheme of the experiment. The sonar detector is placed under one end of the seesaw.

A tracking of the experiment can confirm the rude model described above and convince the students. In this case both the Tracker video analysis and the sonar detection and subsequent analysis by Loggerpro software are suitable. In fact the oscillations in this experiments are slower than in the case of the seesaw on a flat pivot because the force responsible for the motion, as described above, varies from zero to its maximum value very smoothly. It is interesting letting the students perform the data-logging and analysis twice, with both data-logging techniques and discuss the effects of errors on measures, in the two cases. In Fig. 5.20 it is reported the graph of the force as a function of the vertical position of one end of the seesaw obtained collecting data by a sonar and processing them by the Loggerpro software. Despite the many errors due to the experimental conditions and the relevant effect of friction, the graph can confirm our rude model in the case of small oscillations.

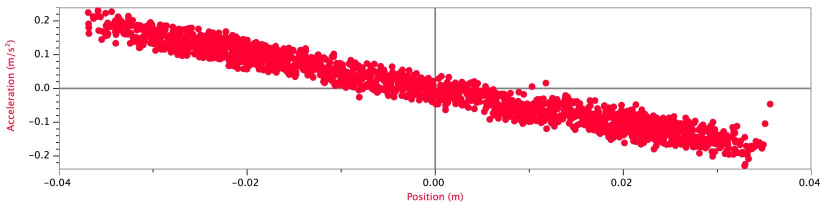
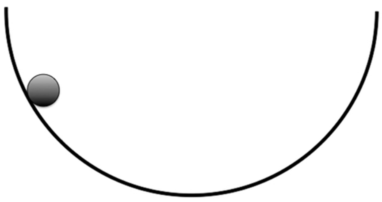


Figure 5.20.: The seesaw on a round pivot: the graph of the acceleration as a function of the vertical displacement of the end of the seesaw. The graph has been obtained processing by Loggerpro software the data collected by the sonar detector.

At the end, regarding to the acting force, the seesaw on a round pivot experiment is much more similar to the pendulum and the mass-spring than to the seesaw on a flat pivot.

5.3.1.7. The ball on a semi-circular track

This experiment simply consists of a steel ball rolling on a semi-circular track as shown in the scheme and photograph of Fig. 5.21. The dynamics of the real experiment is too complicated for 11th and 12th grade students due to the fact that the ball, while sliding along the track, rotates and the rigid body rotational dynamics should be taken into consideration. We will neglect the rotational component of the problem without loosing too much descriptive accuracy. The track has to be well fixed to the floor so as to ensure that the only possible motion is the motion of the ball. Positioning the ball at different points on the track and leaving it move, the students can see that the system has a stable equilibrium position: the point at the bottom of the track. Moreover, the students can be guided to realize that the motion of the ball is governed by a restoring force. Such a force is mainly the component of the weight along tangent to the track. The analysis of the video of the oscillation via tracker software can provide the instant acceleration vector as shown in Fig. 5.22. So also in this case, if we choose the equilibrium position as the origin of the reference system, the graph of the force passes through the origin and it lies in the second and fourth quadrant. Moreover, for small-amplitude oscillations, this graph is a straight line: it not only represents a restoring force, but a linear (or elastic) restoring force as in Fig. 5.13.



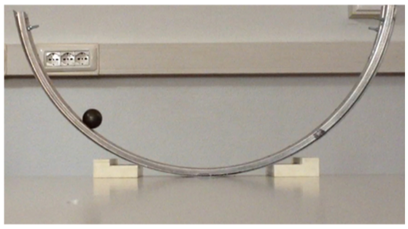


Figure 5.21.: The ball on a semicircular guide: scheme and photograph

In this experiment it is easy to track the motion of the ball via the Tracker software, provided that the background is well contrasted with respect to the target ball [3]. In Fig. 5.23 it is reported the graph of the acceleration vs position obtained by the analysis of the video in the case of small oscillations. The graph is almost a straight line passing through the origin of the reference system and lying in the second and third quadrant. The students should realize that such a graph for the acceleration (and thus the force) as a function of the position is very similar to the one of the pendulum and the vertical mass on a spring.

[3] In this case the use of the sonar detector is not useful because of the shape of the trajectory.

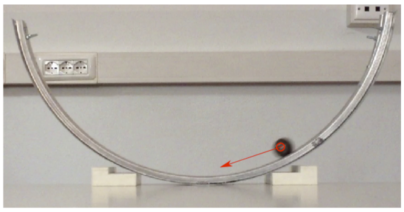


Figure 5.22.: The acceleration acting on a ball on a semicircular guide. The red vector representing istant acceleration was obtained by the Tracker software.

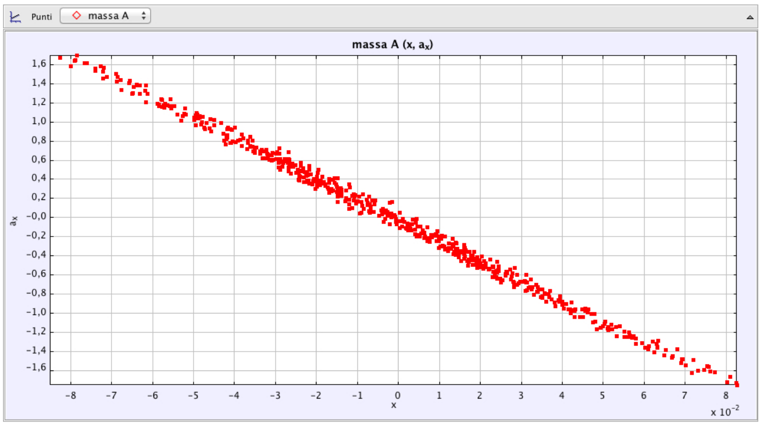


Figure 5.23.: The semi-circular track: graph of the acceleration as a function of the position in the case of small-amplitude oscillations. The video has been taken by an Iphone 4s device.

5.3.1.8. The Galileo oscillator

The Galileo oscillator simply it is a V-shaped track [Torzo, 2014] which consists of two inclined planes joined to their base so as to form a V, with a steel ball free to roll over the track as shown in the scheme of Fig. 5.24.

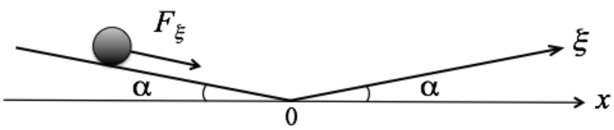


Figure 5.24.: The scheme of the Galileo Oscillator. In figure it is also represented the weight component responsible for the motion of the ball.

Also in this case, as in the case of the semi-circular track, the rotation of the ball (and friction) should be taken into account for a complete treatment of the experiment. Of course this is not suitable for 11th and 12th degree students. So we will neglect the rotational component of the problem without loosing too much descriptive accuracy. For simplicity we chose to have a symmetric V, that is the two planes have the same angle of inclination. The students can easily check that the system has a stable equilibrium position, namely the bottom of the V (indicated by the label “0” in Fig. 5.24). In fact, no matter how you displace the ball along the track, its motion always comes to a stop in the lower point of the V. If we try to analyse the forces acting on the ball along the track, we can realize that, if we neglect the friction, the only force acting on the ball is the unbalanced component of the weight, directed along the track (see Fig. 5.24). This component of the weight is constant in value in each point of the track. It simply changes in sign passing from left to right with respect to the equilibrium position and vice-versa. It is always points to the equilibrium position so we can say it is a restoring force. The diagram of such a force as a function of position is the same we can plot in the case of the seesaw on a flat pivot (see Fig. 5.17). In fact, if we choose the position coordinate ξ along the trajectory as in Fig. 5.24, we can plot the graph of the total acting force as in

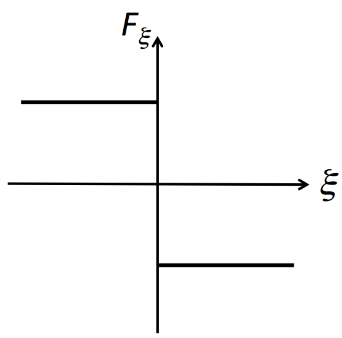


Figure 5.25.: The Galileo oscillator: diagram of the total acting force as a function of the position

If we want to perform a data analysis of this experiment, the tracking of the video is more suitable than the sonar technique because of the peculiar shape of the trajectory. To perform a good tracking we need a well contrasted background with respect to the moving ball. We need also that the velocity of the ball be small enough as to have well defined images to track in each photogram of the video. We reach this condition simply having small angles of inclination for the two planes (that is: a “very open” V). In Fig. 5.26 a photogram from the video for the real experiment performed with students.

In Fig. 5.27 it is reported the graph of the acceleration (which is similar to the one of the force, except for the scale factor introduced by the mass value) vs the

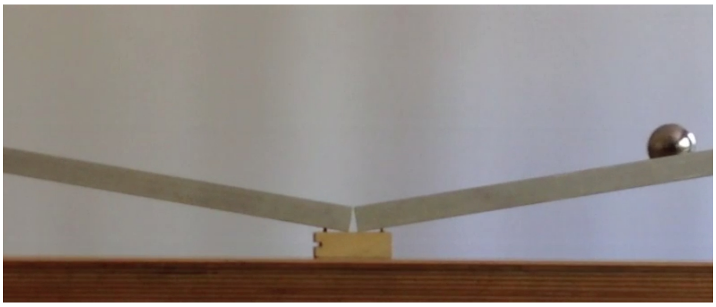


Figure 5.26.: The real Galileo Oscillator experiment: the picture is a photogram extracted from the video. The angle of inclination is about 10°.

displacement. The data have been obtained by tracking the video of the experiment. It is necessary to discuss the results with the students to show that, within the many "noises" and experimental errors in the measurement, the graph obtained is compatible with the expected diagram of Fig. 5.25. So students can point out that the force acting on the Galileo oscillator is similar to the one acting on the seesaw on a flat pivot. In fact it is limited to the second and third quadrant where it is constant and has the same value, but opposite sign depending on the quadrant. In other words the graph presents a discontinuity in the origin.

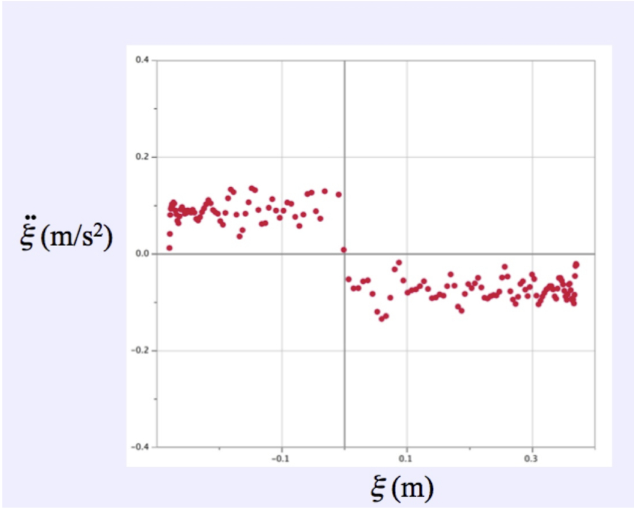


Figure 5.27.: The Galileo oscillator: the graph of the acceleration as a function of the position. The two points next to the origin result from the calculation algorithm used by the Tracker software. In fact only position are really measured; velocity and acceleration are obtained respectively by a derivative and second derivative of the position with respect to the time. In this operation, in the vicinity of the discontinuity, some approximations are introduced due to the fact that the position as a function of time is made of a discrete set of values. The video has been recorded by an Iphone 4s device.

5.3.2. The harmonic oscillation

At this point, the students are able to recognize that some of the experiments performed have some peculiar common characteristics. In fact, both the *simple pendulum*, the *vertical mass on a spring*, the *seesaw on a round pivot* and the *ball on a semi-circular track,* have a stable equilibrium position and are driven by a restoring force. Once we choose the equilibrium position as the origin of the reference system, when we draw such a force as a function of the position, it passes through the origin of the reference system and lies in the second and fourth quadrant. Moreover, these forces are "regular" in the neighborhood of the origin[4] and here their graph can be approximated to the tangent line, provided the amplitude of oscillation is small enough. So the central point is that any body subject to a sufficiently regular restoring force, for small amplitude of oscillation, is governed by forces of the kind $F = -kx$. The teacher has to stress that the oscillation driven by such linearizable forces have a great importance in almost every branch of Physics, they are called *harmonic oscillation*. Nevertheless this definition can lead students to think that

[4]This means that the force is differentiable in the origin. With 11th and 12th grade students which have not been introduced to calculus yet, we simply state that the graph of the force is regular in the origin, that is smooth (no discontinuities, no corner point etc.)

essentially all the restoring forces give rise to harmonic motion if the oscillation is sufficiently small. This is not always true. It is necessary to introduce a criterion for identifying quickly the harmonicity or anharmonicity of the motion in real cases. Namely, according to the four-point criterion [Giliberti et al., 2014] of section 3.2 we can state that the small oscillations of a one degree of freedom system are harmonic if $x = 0$:

(a) is a stable equilibrium point; and in $x = 0$:

(b) the function $F_x(x)$ is continuous;

(c) the function $F_x(x)$ admits tangent line in $x = 0$;

(d) The tangent line in $x = 0$ is not horizontal.

Obviously, condition (c) implies condition (b). Nevertheless we believe that, from a didactical point of view, keeping these conditions separate allows a clearer comprehension of the physics involved.

Condition (d) is too sophisticated for 11th and 12th grade students to be discussed by mathematics. It can be seen with the experiment of the x^4-track reported in section 3.3.4. In this case around the equilibrium point (the middle point of the track) the track is nearly flat thus it is similar to a region of neutral equilibrium so the ball can be substantially considered at rest with very long oscillation period.

5.3.2.1. The motion law for the harmonic motion

Once the definition of harmonic motion, as the one ruled by the dynamical law $F = -kx$, has been given, one has to face the problem of finding a way to integrate the differential equation $a = -k'x$ to obtain x as a function of t (with students that have no calculus background). Our strategy is to use the projection on a diameter of a point-mass moving in circular motion. In fact, in this way it is easy to observe that the projection of the acceleration is given by $a = -k'x$ and that the projected velocity and position have a sinusoidal dependence on time. Most of Italian text-books define harmonic motion just as the projection of a circular motion over a diameter in a cinematic perspective. We, on the contrary, have chosen a very different dynamical approach and use circular motion only as a device to integrate a differential equation. Moreover this is quite simple to obtain tracking the motion of a target dot on a rotating disk. According to this dynamic approach we can start comparing the general expression for the force: $F = ma$ with the linearized restoring force: $F = -kx$ to obtain the expression:

$$a = -\frac{k}{m}x \tag{5.1}$$

So we are looking for an expression of the position $x(t)$, as a function of time, that is proportional to the acceleration through the proportionality constant k/m and opposite to the acceleration. This means that the graph of $x(t)$ has the same shape of the graph of $a(t)$ but it is flipped with respect to the time axis. Let us consider a point that is moving along a circumference with a fixed angular velocity ω. Let us now consider the projection of such a point along a diameter. If P is the position of the point at a certain time and Q is its projection, let's say along the x axis, it is straightforward to see that:

$$x = r \cos \vartheta \tag{5.2}$$

where r is the radius of the circumference and ϑ is the angular position of the point, as depicted in Fig. 5.28. As for the uniform circular motion the angle ϑ is linear with respect to the time following the expression:

$$\vartheta(t) = \omega t + \phi \tag{5.3}$$

then the expression 5.2 becomes:

$$x(t) = r \cos(\omega t + \phi) \tag{5.4}$$

Now we have to obtain the expression for the acceleration. If we consider the triangles of Fig. 5.29, because of their similitude we can wright:

$$a \, : \, r = a_x \, : \, x \tag{5.5}$$

from which we can obtain:

$$a_x = (\frac{a}{r})\, x \tag{5.6}$$

As in the uniform circular motion the acceleration a is centripetal acceleration, that is $a = \omega^2 r$, the 5.6 becomes:

$$a_x = \omega^2 x \tag{5.7}$$

From the Fig. 5.28 we can see that a_x and x have always opposite direction. So, taking into account also the directions of the vectors, we can wright:

$$a_x = -\omega^2 x \tag{5.8}$$

Comparing the 5.7 with the 5.1 which is the expression for the harmonic motion, we can establish the relationship between the angular velocity ω and the constant k:

$$\omega^2 = \frac{k}{m} \tag{5.9}$$

We can obtain the expression for the velocity in a similar way. In fact, as we can

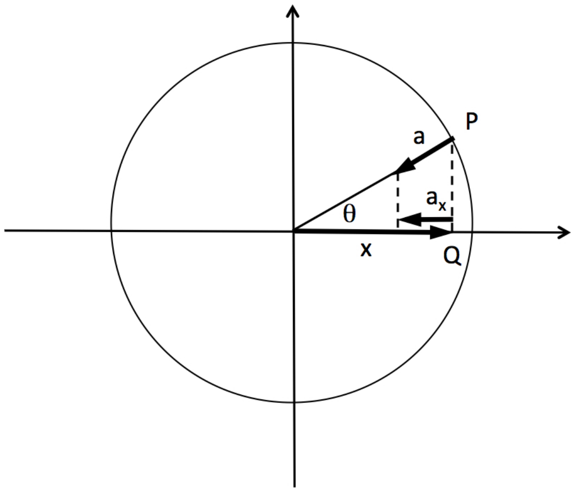


Figure 5.28.: The motion of a point along a circumference: projection of the position and of the acceleration along the x axis.

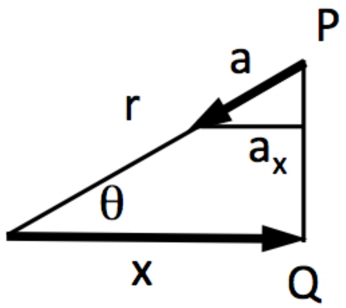


Figure 5.29.: The similitude between the triangles

see in Fig. 5.30, the projection of the velocity along the x axis is:

$$v_x(t) = v\ \sin\vartheta = v\ \sin(\omega t + \phi) \tag{5.10}$$

and, considering that $v = \omega r$,

$$v_x(t) = \omega r\ \sin(\omega t + \phi) \tag{5.11}$$

and taking into account also the directions of the vectors:

$$v_x(t) = -\omega r\ \sin(\omega t + \phi) \tag{5.12}$$

A simple tracking of a dot performing a uniform circular motion can convince students that $x(t) = r\cos(\omega t + \phi)$ indeed represents the solution of the equation:

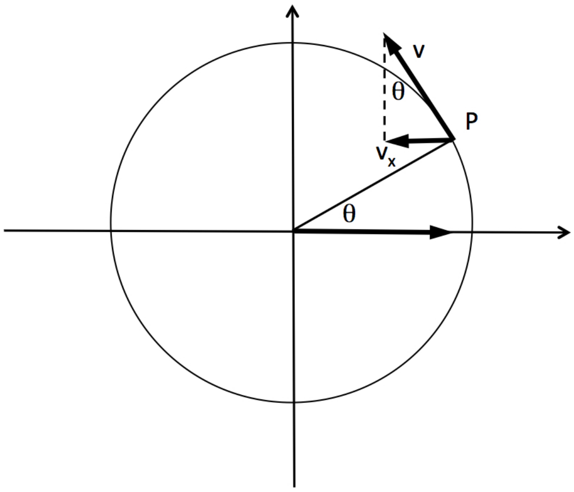


Figure 5.30.: The motion of a point along a circumference: projection of the velocity along the x axis.

$a(t) = -\omega^2 x(t)$, where ω^2 is a constant depending on the particular oscillator we are considering (k/m for the mass on a spring, g/l for the pendulum and so on...). In Fig. 5.31 it is reported the screenshot of a tracking for a point on a disk rotating with uniform angular velocity. All the quantities are projected along the x axis. In Fig. 5.32 are represented the projection along the x axis for position, acceleration and velocity as a function of time. For students it is easy to see that at each time the position x is opposite to the acceleration a_x, according to equation 5.1. Moreover they can see the graph for the velocity is shifted by $\pi/2$ with respect to the graph for the position, according to the fact that if the motion law for the position is sinus-like, the law for the velocity is cosinus-like and vice versa.

At this point it is important to stress with students that in 5.4 r (the amplitude of the oscillation) and ω (the angular frequency) are independent one from each other and are the only two parameters that characterize the motion. This is a key point as it states that in a harmonic motion, the frequency does not depend on the amplitude. Such a point is very easy to verify by data-logging techniques. For instance, the teacher can let students try and analyze the harmonic motion of a vertical mass-spring in the case of different initial displacements of the mass, so they can measure the same period (thus the same frequency) regardless the different displacements. In short, the data-logging techniques allow students to state the harmonicity/anharmonicity of one oscillation in a number of ways:

- they can plot (or analyze the corresponding data row) the graphs of position, velocity and acceleration as a function of time, to determine whether these are sinusoidal-like functions. In fact position and velocity as functions of time have the same sinus-like shape, but they are shifted of a quarter of a period. Furthermore, the acceleration vs time graph is still a sinus-like function and

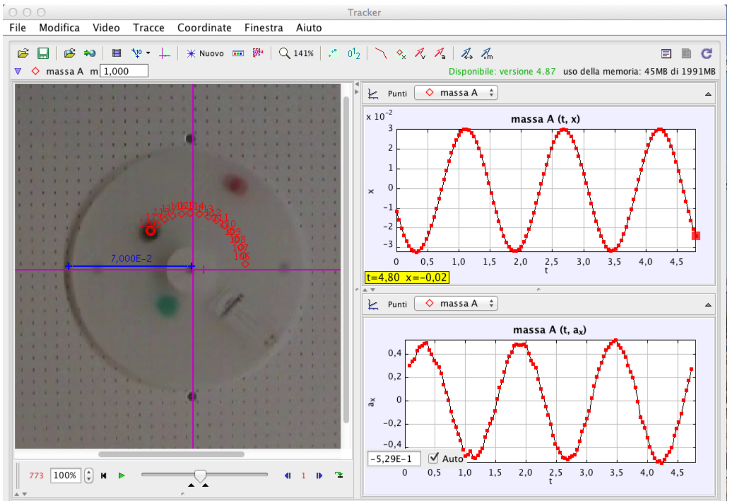


Figure 5.31.: Screenshot of the tracking of a uniform circular motion. Here we tracked a point marked on a disk. The disk was rotating around a pivot, on an air table.

results, at each time, opposite to the displacement one according to F = - k x law stating harmonic motion;

- they can directly measure the period (and thus the frequency) on different sections of the position vs time waveform or on the corresponding data row to verify whether the period remains constant regardless of the amplitude. So they can verify the important property of harmonic motion that the frequency of the oscillation is amplitude independent, that is it is fixed by the parameters of the system. For instance, if we track the motion of a pendulum or a vertical mass-spring, the amplitude of oscillation registered by the sonar decreases with time due to the air friction. In this case the restoring force no longer depends only on position, but also on velocity. This situation has not yet been faced by students. Nonetheless, from an experimental point of view, for small amplitudes and for not too large time intervals, the damping is very small so that we can neglect the dissipative contribution and consider the force as being dependent only on position thus giving a precise sense to measurements of the period. Obviously, waiting a long enough time, the amplitude of oscillations decreases and the damping becomes evident. One can thus perform a new measurement of the period of our motion in a new situation when the amplitude has diminished, but always remaining in the approximation of friction-less motion. The students could measure the period (and consequently the frequency) of the oscillation directly in different sections of the diagram with different amplitudes and verify it is constant;

- they can plot the graph of the acceleration as a function of the position to see if it is a straight line passing through the origin and laying in the second and

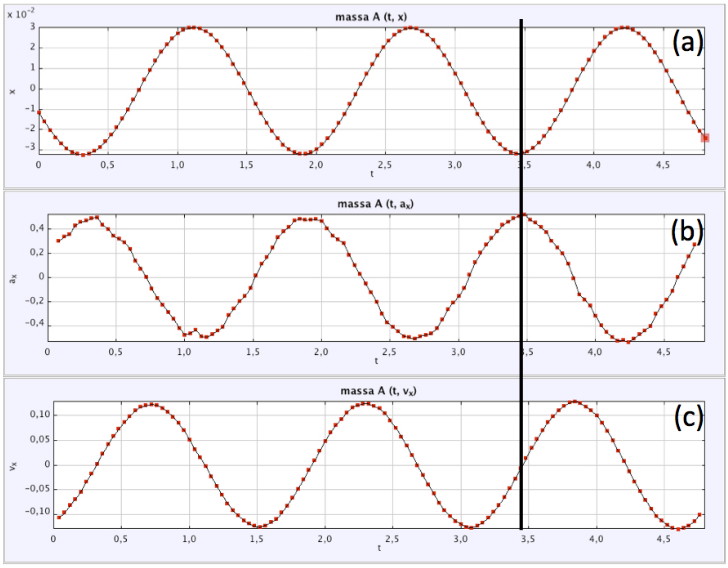


Figure 5.32.: Projection of a circular motion along the x axis: (a) the projection of the position along the x axis as a function of time; (b) the acceleration a_x as a function of time and (c) the velocity v_x as a function of time. The vertical bar is to facilitate the comparison between the graphs.

fourth quadrant, thus representing $a = -\omega^2 x$;

- where the experiment allows it, they can listen to the sound during the oscillation in order to determine "by ear" if the frequency remains constant or changes over time. For instance, this is very simple in the seesaw on a flat pivot and in the vertical bouncing ball experiments, where the amplitude of oscillation decreases over time while the frequency increases producing a sound that becomes higher over time.[5]
- They can analyze the motion waveform and see if it is generated by a single sharp frequency (the frequency in a harmonic oscillation is unique and constant and it is determined only by the design of the experiment). Both the data loggers, namely Loggerpro and Tracker can automatically provide another powerful tool to confirm that the frequency of the harmonic oscillation is fixed: the FFT (Fast Fourier Transform) tool. The FFT of the motion waveform results a sharp line at the same frequency the students found directly by measuring the period on the diagram, in the case of harmonic oscillation. In Fig. 5.33 are reported the FFT for a seesaw on a round pivot (harmonic motion) and a seesaw on a flat pivot (anharmonic motion). It is evident that in

[5] this technique is very useful to exclude the Harmonicity of the oscillation but it is not conclusive to confirm it. In fact, if we consider the bouncing disk experiment, in which the amplitude of the oscillation is fixed by the setup, the time between two consecutive bounces remains constant over time, so does the frequency and thus the produced sound.

the case (a) of the round pivot the motion is characterized by a sharp frequency indicating a sinus-like motion. On the other hand, in the case of flat pivot the frequency is not neat: it is spread over an interval ov values indicating a frequency that varies over time. Moreover, it is also evident the presence of higher bands of frequencies typical of non-linear phenomena. Of course 11th and 12th grade students cannot posses the mathematical background for understanding how FFT works in detail. They just know it is a tool, a kind of button to push, that is able to find all the frequencies present in a waveform. To make this clear to students we can show them, with a simulation, the complicated waveform resulting from the sum of two (and three) sinusoidal function with different frequency. Then applying the FFT to the waveform we obtained the frequencies we mixed. The Loggerpro program has the tools to produce such simulations.

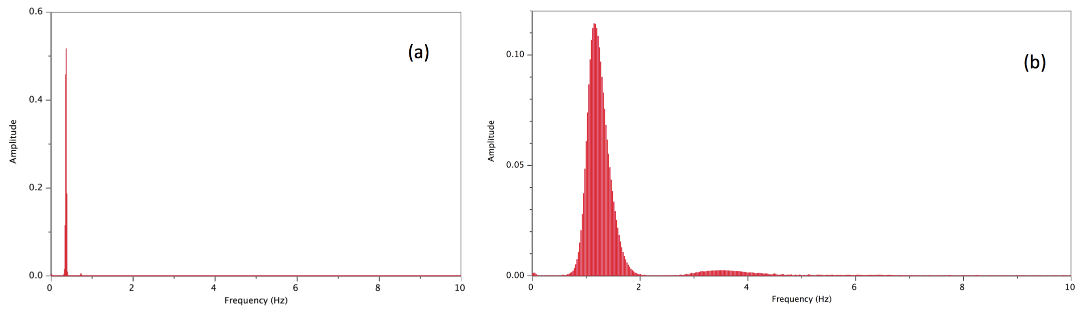


Figure 5.33.: Comparison between the FFT of: (a) the seesaw on a round pivot and (b) the seesaw on a flat pivot. The graphs have been obtain by Loggerpro.

5.3.2.2. A couple of examples

We report here, for the seek of brevity, just a couple of examples of the graphs obtained and analyzed by students: the vertical mass-spring and the seesaw on a flat pivot. In Fig. 5.34 it is reported the comparison between the graphs of position, velocity and acceleration as a function of time. The vertical black line helps to compare the values of these quantities at a common time. It is evident that the graph of the velocity is shifted of a quarter of a period with respect to the graph of the position and that position and acceleration are opposite at each time. This is a confirmation that the graphs represent sinusoidal functions as expected in the harmonic motion. In the graphs only the first five seconds of the experiment are shown to have waveforms expanded enough for the analysis. By the way, the experiment lasted sixty seconds. During this time, friction caused the reduction of the amplitude of oscillation of about one third. Nonetheless, the students could measure that the period of the oscillation remained constant. A further evidence that this motion is harmonic comes from the graphs of Fig. 5.35. In fact the graph of the acceleration as a function of the position is the one required by our definition of harmonic motion and the FFT shows a single sharp frequency indicating that the oscillation is sinusoidal.

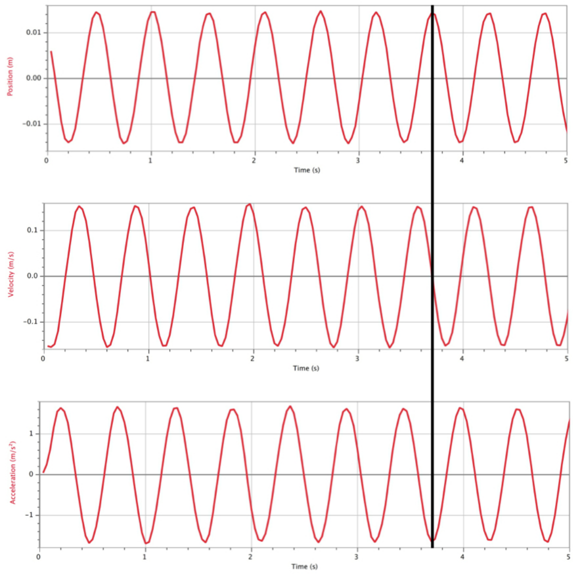


Figure 5.34.: The mass-spring system: comparison between the graphs of position, velocity and acceleration as a function of time. The data were obtained by a sonar detector and processed by Loggerpro software.

The analysis of the same graphs for the seesaw on a flat pivot shows that in this case the motion is not harmonic. In Fig. 5.36 are reported the graphs for the vertical position of one point at the edge of the seesaw, the velocity and the acceleration as a function of time. At a first sight students can think that the graph of the vertical position vs time is a sinusoidal function even if affected by damping. Some measurements, directly on the graph or on tha data raw show clearly that the period decreases rapidly with decreasing amplitude. It is evident that this is not a harmonic oscillation if one looks at the graphs of the velocity and the acceleration vs time. The straight segments in the graph of the velocity show that the motion has a uniform change of speed thus the graph of vertical position vs time is not sinusoidal. According to the laws of motion with uniform change of speed such graph is the result of many arcs of parabola linked together. The graph of the acceleration is piecewise constant. The acceleration simply change sign on each of the tracts. The few dots vertically distributed among two adjacent tracts are not real experimental points, they are the consequence of the calculation algorithm used by Tracker: in fact just positions are measured while velocity and acceleration are calculated. A further evidence that this motion is not harmonic comes from the graphs of Fig. 5.37. In fact the plot of the acceleration as a function of the position is given by two regions in which the acceleration is almost constant and thus not satisfying the definition of harmonic motion. Moreover, the FFT graph shows the presence of many broadened

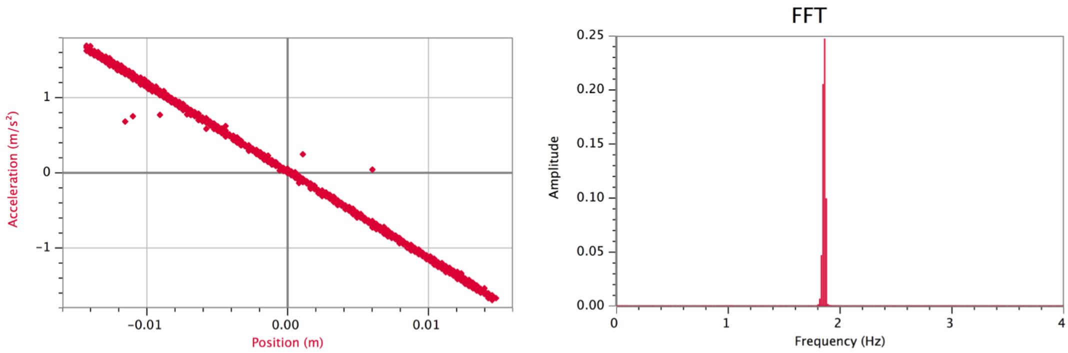


Figure 5.35.: The mass-spring system: the graph of the acceleration as a function of position (left) and the FFT graph (right). The data were obtained by a sonar detector and processed by Loggerpro software. The few points out of the line are an artifact due to a disturbance at the switching on and switching off the sonar detector. For this experiment a new version of the sonar detector was used.

frequencies multiple of the first dominant frequency (the higher harmonics).

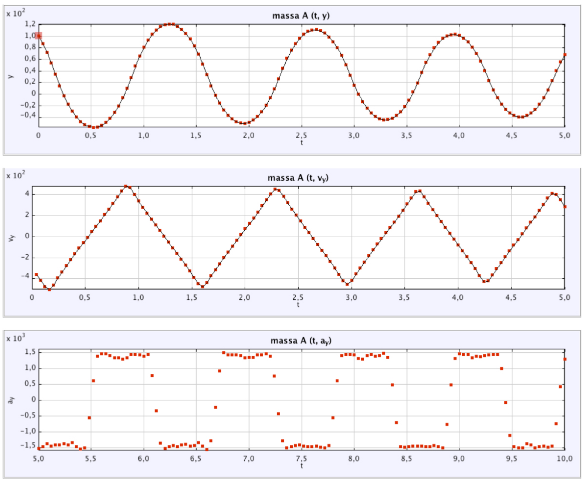


Figure 5.36.: The seesaw on a flat pivot: comparison between the graphs of position, velocity and acceleration as a function of time. The data were obtained by tracking the video of the experiment with Tracker video-analysis software.

5.3.3. Coupled oscillators and Normal Modes

Until this moment students have been dealing only with single oscillators, studied their characteristics and determined the cases in which the oscillators perform a Harmonic motion. At this point the teacher can introduce a new problem by the question: “what if we consider a system made of two coupled oscillators?”. Of course

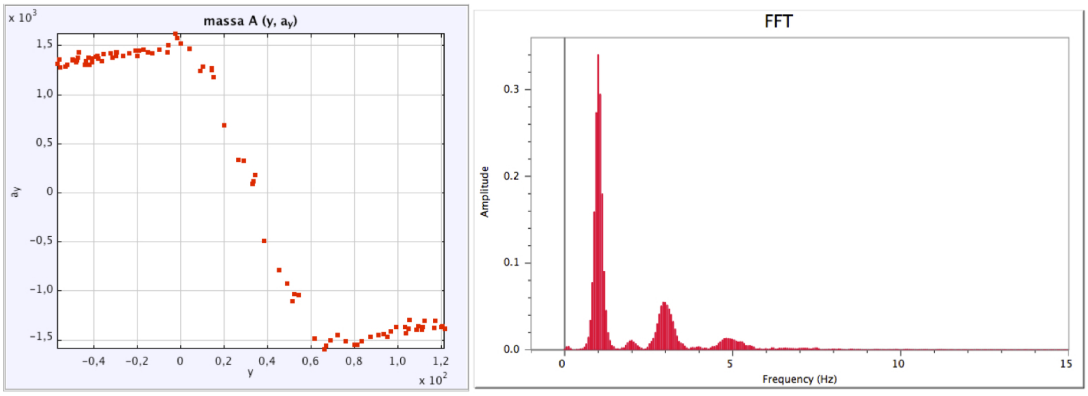


Figure 5.37.: The seesaw on a flat pivot: the graph of the acceleration as a function of position (left) and the FFT graph (right). The data were obtained by tracking the video of the experiment with Tracker video-analysis software. The points distributed vertically are due to the calculation algorithm.

a brief discussion on what we mean with the word *coupled* is needed. It is also necessary to point out that we are to consider just couplings that are not too much strong. After a few example of what *coupling* means the teacher asks the students to observe (and then try and repeat) the first experiment on coupled oscillators as to let the students discuss on the behaviour of such a system and introduce the concept of *normal modes of oscillation*. The general idea is to introduce the normal modes of two coupled oscillators, and extend the concept to three, five and many oscillator up to the case of a continuous system of oscillators. This is done through the performing of real experiments and simulations as well.

5.3.3.1. The two coupled pendulums and the two coupled mass-spring system

Two coupled pendulums. The system consists of two physical pendulums coupled by a soft spring as to have the pendulums vertical and the spring unstretched at rest position. In Fig. 5.38 are reported a picture and the scheme for the system of two coupled pendulums. In a first moment the teacher moves one of the pendulums from the equilibrium position and let the system evolve asking the students to describe what they see. What happens in time is that the pendulum that initially was moving slowly decreases the amplitude of its oscillations coming to a stop for a moment. Simultaneously, the other pendulum, that was motionless at the beginning, starts oscillating and increases its oscillation reaching the maximum amplitude exactly at the moment the first pendulum comes to a stop. The cycle repeats indefinetly[6]. In our experience the typical comment students do is: "*it is strange, it is as like the two pendulums exchange their motion the one with each other. It is as if one pushed the other, and this restrained the first and then they reversed roles*". If it

[6]actually the friction with air slowly reduces the amplitude of the oscillations until stop them.

does not emerge fron the discussion, the teacher has to guide the students toward the consideration that the exchange of motion between the pendulums is due to continuous to and fro exchange of mechanical energy between them. The teacher can cover one of the two pendulums with a panel to remark this exchange of energy[7]: in fact if we can see just one pendulum we see a continuous gain and loss in velocity and thus in kinetic energy. Therefore even if one can see just one pendulum he can say that it is coupled with at least one other pendulum because it is exchanging energy with it(them). Now the teacher can ask the students if they can imagine some peculiar motion configuration for the system. In our experience, surprisingly, many students easily come to the conclusion that there are two possible special movements: the one with the two pendulums that have the same displacement, same direction and same velocity at each time[8]. The other one, with the pendulums that have same displacement and velocity but opposite motion direction at each time[9]. The students can easily realize that if one starts with a normal mode, the system continues moving that way, never switching to the other mode. In this case, if the teacher covers one of the pendulums with a panel, the students can't tell whether it is coupled to other pendulums or not. In fact, in this particular motion configurations the two pendulums do not exchange energy and perform harmonic oscillations both at the same frequency. Such a statement can be easily verified via the data-logging techniques: both Tracker and LoggerPro are good for the purpose. Students can see that no matter the initial amplitude in thes particular cases, the frequency are fixed (as expected for a harmonic motion).

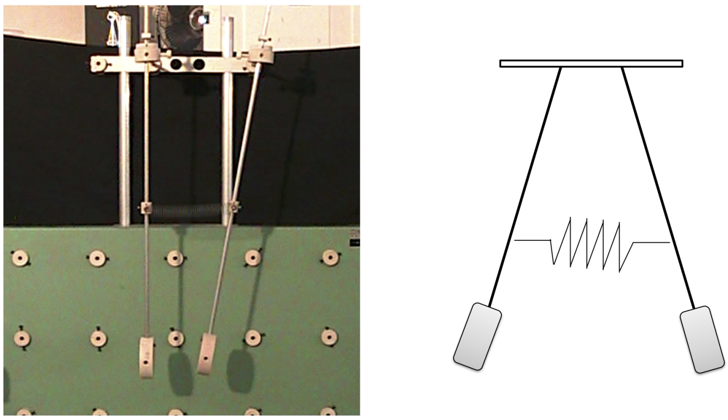


Figure 5.38.: Two coupled pendulums: on the left the picture of the actual system and on the right the scheme. Both the picture and the scheme represent the system in a generic motion configuration at a generic time.

[7]if we can see just one pendulum we can only say that it is coupled with other pendulums if there is an oscillating variation in its kinetic energy. We cannot guess if it is coupled with a single other pendulum or many others.

[8]This is the first normal mode of this system and it is usually called *pendular mode*

[9]This is the second normal mode of our system and it is usually called *breathing mode*

This experiment (and the ones to follow), together with the data logging techniques, turn out to be particularly useful because it allow: i) to easily introduce some particular (a student said “spectacular”) motion configurations of the entire system: the normal modes; ii) to recognize that when such a complex system oscillates in one of its normal modes, there is no energy exchange between the single parts (oscillators) of the system; iii) to see that every casual motion configuration of the system is simply a superposition of its normal modes. Now students are ready for the definition of normal modes. As a result of observation of the experiment and the measurements performed by students we can state the following definition for the normal modes: *the normal modes of oscillation for a system of coupled oscillators, are special motion configurations in which every part of the system (each oscillator) performs harmonic oscillations with the same frequency and maintains a fixed phase relation with the other parts.* So in the case of two coupled pendulums, in the first normal modes the pendulums have same frequency and phase while in the second normal mode they have same frequency and opposite phase. The data-logging analysis can easily show that the higher the mode, the higher the frequency so each mode is characterized by its own frequency. An important point that the teacher has to clarify is that a system made of coupled oscillators has as many normal modes as the number of its degrees of freedom. In our experiments, that are monodimensional chains of oscillators, the number of the degrees of freedom always corresponds to the number of the oscillators that are coupled. All these properties of normal modes can be easily verified by the data logging techniques. In Fig. 5.39 are reported the graphs of position vs time, acceleration vs position and the FFT graph, obtained by sonar detection of the motion of one of the pendulums as the whole system moves in the first of its normal modes (the so called *pendular mode*). In Fig. 5.40 are depicted the same graphs for the second normal mode (the *breathing mode*). At last, in Fig. 5.41 the same graphs but with the system in a casual motion configuration, different from each of its normal modes. The Fig. 5.39 and Fig. 5.40 show clearly that the motion of the normal mode is harmonic. In fact the waveform for the position vs time is sinusoidal. This is confirmed by the graph of the acceleration vs position that is the one expected by the definition of harmonic motion. Moreover, the FFT graphs present a single sharp line corresponding to a well defined frequency. The graphs of Fig. 5.41 show the typical beats phenomenon due to the superposition of the two modes. In fact the waveform for position vs time is the same students have seen before, collecting sound from two tuning forks with a few Hz different frequencies. The FFT graph clearly shows that such a complex waveform is generated by just two well defined frequencies: the frequencies corresponding to the normal modes of the system. No matter how many times with different initial conditions we repeat the experiment: the result is always the same, what may change is only the contribution of each frequency. The graph of acceleration vs position tents to differ from the one of Fig. 5.39 and Fig. 5.40as the motion in a random configuration is no more harmonic.

The students can afford the study of the two coupled pendulums experiment also

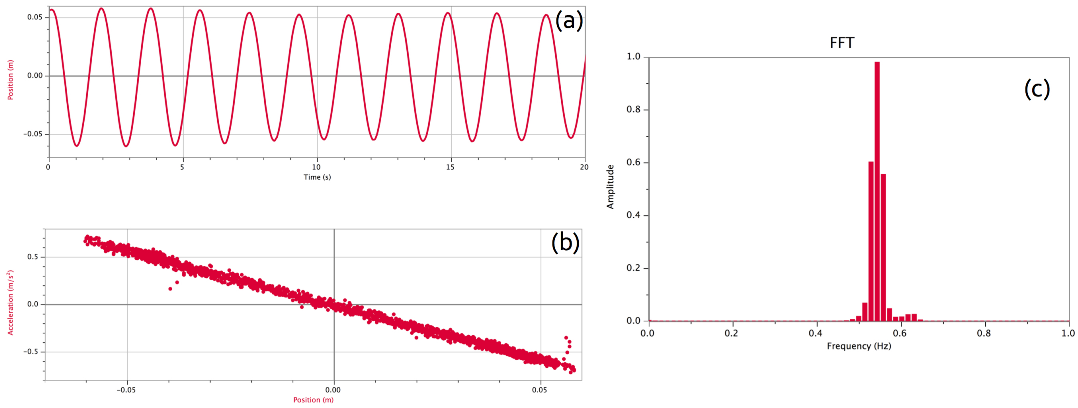


Figure 5.39.: Two coupled pendulums: the first normal mode. The graphs for a single pendulum: (a) position vs time; (b) acceleration vs position and (c) FFT

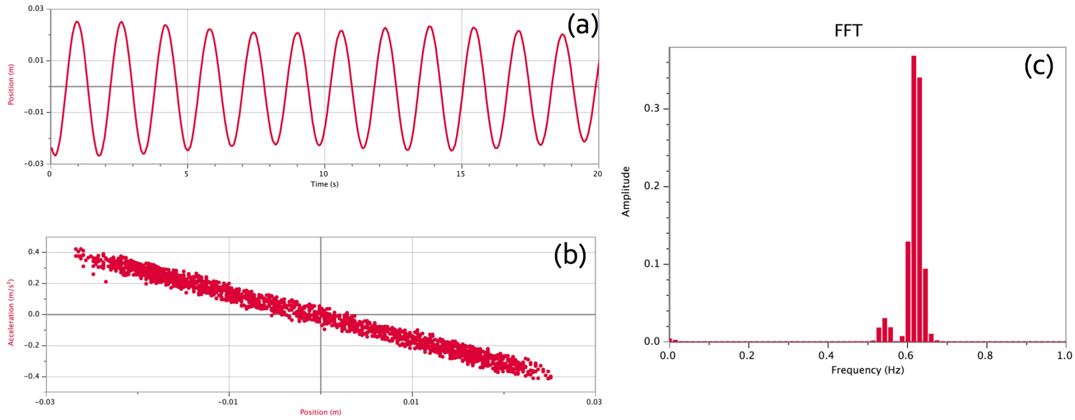


Figure 5.40.: Two coupled pendulums: the second normal mode. The graphs for a single pendulum: (a) position vs time; (b) acceleration vs position and (c) FFT

via the Tracker video analysis. In this case it is possible to track simultaneously either pendulums and compare the waveforms due to the beats as shown in Fig. 5.42. They can realize that when at the beginning only one pendulum is displaced, there is a complete energy transfer between the two pendulums during the motion. In fact, zero amplitude of oscillation for the first pendulum corresponds to maximum amplitude of oscillation for the second pendulum and vice versa, as looked like at eye observation. This video analysis software allows also to track the center of mass motion. Here students can see that when the system is excited randomly each part of the system (each pendulum) describes a motion that is the linear combination of the two normal modes. This motion is not harmonic and in general neither periodic, while the motion of the center of mass results always harmonic (see Fig. 5.43) and its frequency corresponds to the frequency of the first normal mode. This is a

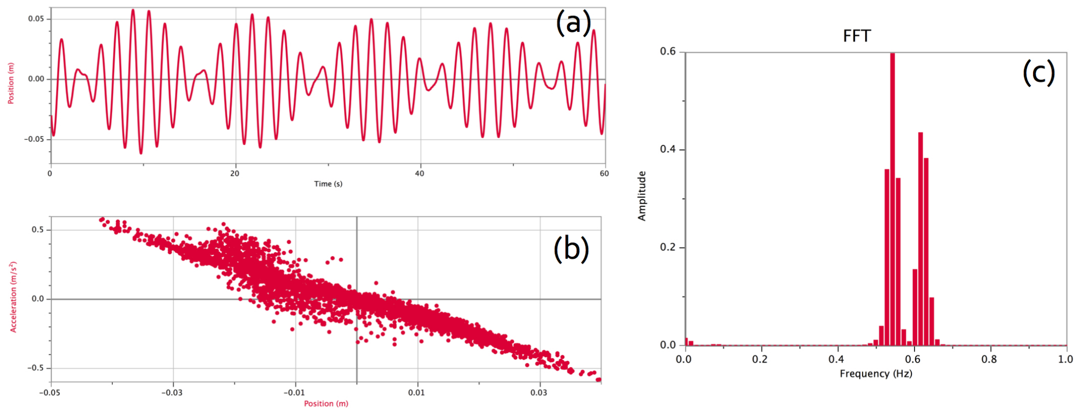


Figure 5.41.: Two coupled pendulums: a random motion configuration. The graphs for a single pendulum: (a) position vs time; (b) acceleration vs position and (c) FFT

consequence (see next paragraph) of the fact that the first normal coordinate is the almost the sum of the coordinates of each pendulum.

Two coupled mass-spring systems. The system consists of two up to four masses coupled by identical springs. The students can start the experiment with two masses and than implement the measure up to four masses. As in the case of the coupled pendulums, at first the system can be put into motion simply displacing the masses but it is also possible to drive it by a mechanical vibrator. In our experimental setup the chain is disposed vertically for convenience. The upper end is bound to a T-rod while the lower end is bound to the pivot of an electromechanical vibrator (Pasco SF-9324 model). The vibrator is coupled with a sine wave generator (Pasco WA-9867 model) to be frequency tunable with 0.1 Hz resolution in the range from 0.0 to 800.0 Hz. The scheme of the setup is reported in Fig. 5.44.

Also in this case students are asked to try and guess in how many ways the system can oscillate when excited by the vibrator and to find out some "special ways of movement". In our experience most students were able to identify the two normal modes regarding the two masses and three springs system. On the contrary, most students found difficult to predict normal modes when the system was more complex (three or four masses). To overcome this difficulty we can try to make the students to analyse an analogue and simpler system: a vibrating string with fixed ends. The first four normal modes (here also called *stationary waves*) are shown in Fig. 5.45.

This results a very useful analogy to predict the normal modes of one-dimension system as the coupled pendulums and mass-springs:

n coupled oscillators are represented by n equally spaced points on a string

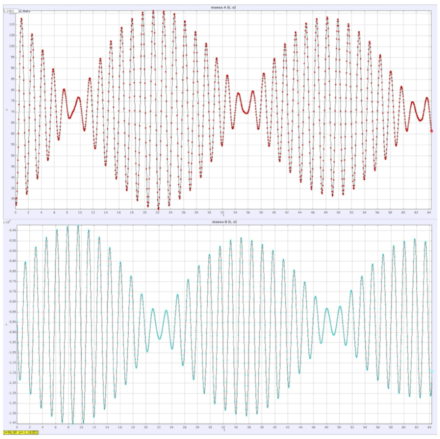


Figure 5.42.: two coupled pendulums: graphs of the position vs time, beats. The graphs have been obtained by tracking the video of the experiment. The exchange of energy between the two pendulums is almost total, in fact, to the zero amplitude points for the first pendulum, correspond the maximum amplitude points for the second pendulum and vice versa.

the n^{th} normal mode configuration of the oscillators is recognizable by n^{th} stationary waves on the string, as Fig. 5.46clearly shows.

In the case of mass-spring oscillators, this graphic analogy allows students not only to predict the motion configuration of each normal mode but also to have a hint of the relative amplitude of oscillators in that mode. In addition, slow motion video analysis and simulations of the system can help students to recognize unexpected motion details. For instance, in the case of tree masses, the displacement amplitude and velocity of the central mass is different from the one of the other masses even in the first normal mode. If not, this mass would not be subject to an elastic force an couldn't perform harmonic motion.

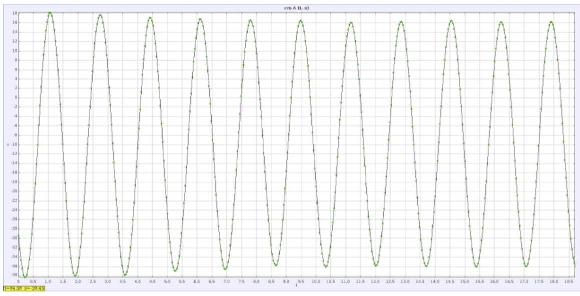


Figure 5.43.: two coupled pendulums: graphs of the position vs time for the center of mass. The motion of the centre of mass is harmonic and has the frequency of the first normal mode. The graph has been obtained by tracking the video of the experiment.

5.3.3.2. Calculation of the frequencies of the normal modes in the simple case of two coupled oscillators (two masses and three springs)

The mathematical treatment of normal modes for 11th and 12th grade students must be as simple as possible. So we will not use calculus but only ordinary algebra combined with the results of section 8.3.2.1 and limit ourselves to the case of two simple coupled oscillators. Namely two mass-spring systems coupled by a soft spring. Let us consider two equal masses m and three springs arranged as in Fig. 5.47. For simplicity of calculation the elastic constant k is the same for all the springs. Basically the system is formed by two identical mass-spring oscillators which masses are coupled by the central spring as in Fig. 5.47.

Let x_1 and x_2 measure the displacement of the left and right masses from their respective equilibrium positions. With such positions, the $F = m\,a$for the two masses are, for the left mass:

$$F_1 = -kx_1 + k\left(x_2 - x_1\right) \tag{5.13}$$

for the right mass:

$$F_2 = -k\left(x_2 - x_1\right) - kx_2 \tag{5.14}$$

If a_1 and a_2 are the acceleration of the left and right masses respectively, we can wright:

$$\begin{cases} ma_1 = -kx_1 + k\left(x_2 - x_1\right) \\ ma_2 = -k\left(x_2 - x_1\right) - kx_2 \end{cases} \tag{5.15}$$

It is clear that the two masses cannot move with harmonic motion because in these

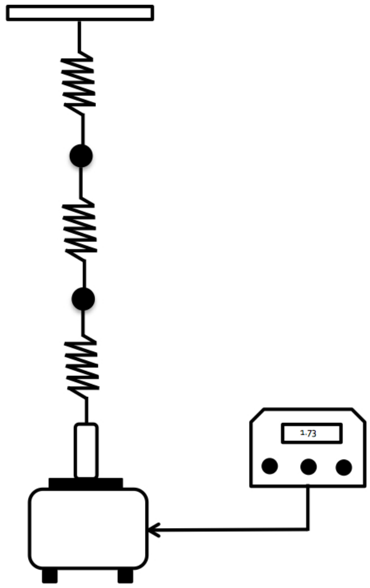


Figure 5.44.: Two coupled mass-spring systems: scheme of the experimental setup

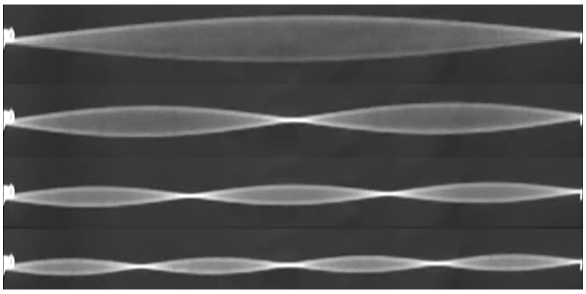


Figure 5.45.: The oscillating string with fixed ends. In figure are photraps of the first normal modes.

equations there isn't proportionality between the acceleration and the position. In fact the two equations are coupled: in the equation for the first mass appears also the position coordinate of the second mass and vice versa. We can try to uncouple the two equation by

$$\begin{cases} m(a_1 + a_2) = -k(x_1 + x_2) \\ m(a_1 - a_2) = -3k\,(x_1 - x_2) \end{cases} \tag{5.16}$$

We now introduce the new coordinates:

$$\begin{cases} X_A = x_1 + x_2 \\ X_B = x_1 - x_2 \end{cases} \tag{5.17}$$

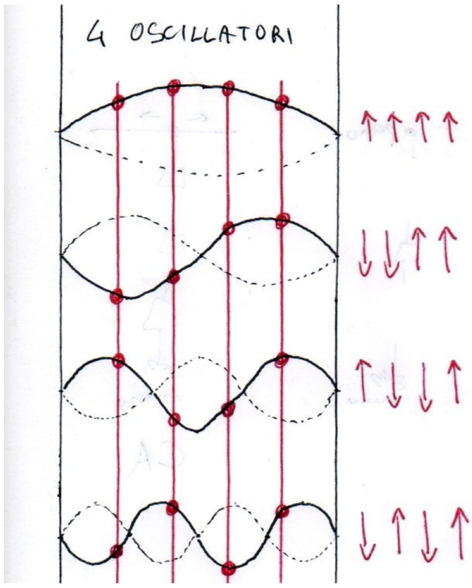


Figure 5.46.: The sketch a student made to use the analogy with stationary waves to predict the shape of the four normal modes of a system made of four coupled oscillators.

as $v = \frac{\Delta x}{\Delta t}$ when Δt tends to zero,

$$V_A = \frac{\Delta X_A}{\Delta t} = \frac{\Delta\left(x_1 + x_2\right)}{\Delta t} = \frac{\Delta x_1}{\Delta t} + \frac{\Delta x_2}{\Delta t} = v_1 + v_2 \tag{5.18}$$

and

$$a_A = \frac{\Delta V_A}{\Delta t} = \frac{\Delta\left(v_1 + v_2\right)}{\Delta t} = \frac{\Delta v_1}{\Delta t} + \frac{\Delta v_2}{\Delta t} = a_1 + a_2 \tag{5.19}$$

and in the same way:

$$V_B = v_1 - v_2 \tag{5.20}$$

and

$$a_B = a_1 - a_2 \tag{5.21}$$

The system 5.16 becomes:

$$\begin{cases} a_A = -\frac{k}{m} & X_A \\ a_B = -3\frac{k}{m} & X_B \end{cases} \tag{5.22}$$

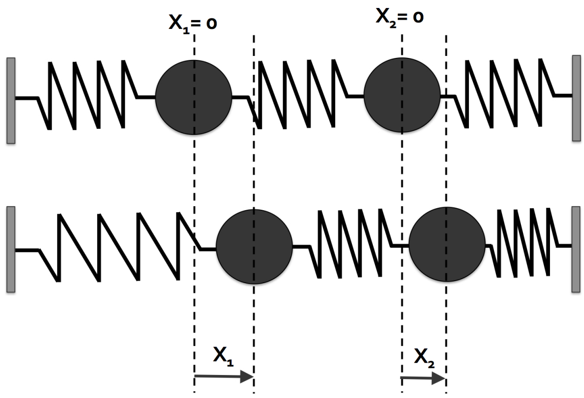


Figure 5.47.: Two masses coupled by springs.

The equations 5.22 are no more coupled and are therefore independent the one from each other and can be solved separately. For this reason, the new coordinates X_A and X_B are called *normal coordinates [Fis, , Fitzpatrick, 2013].* Each of equations 5.22 represents the well known equation for the harmonic motion. The first with $\omega_A^2 = k/m$ and the second with $\omega_B^2 = 3k/m$. The solutions of the equations 5.22 are therefore:

$$X_A(t) = A\sin(\omega_A t + \phi_A) \tag{5.23}$$

and

$$X_B(t) = B\sin(\omega_B t + \phi_B) \tag{5.24}$$

If we now want to go back to the old coordinates $x_1(t)$ and $x_2(t)$, from the system 5.17 we have:

$$\begin{cases} x_1 = \frac{1}{2}(X_A + X_B) \\ x_2 = \frac{1}{2}(X_A - X_B) \end{cases} \tag{5.25}$$

and substituting the equations 5.23 and 5.24 we have:

$$x_1(t) = \frac{1}{2}A\sin(\omega_A t + \phi_A) + \frac{1}{2}B\sin(\omega_B t + \phi_B) \tag{5.26}$$

and

$$x_2(t) = \frac{1}{2}A\sin(\omega_A t + \phi_A) - \frac{1}{2}B\sin(\omega_B t + \phi_B) \tag{5.27}$$

The equations 5.26 and 5.27 represent all the infinite possible laws of motion for the two masses. Anyway each possible motion configuration is given by the superposition of the normal modes of the system. The pendulums will choose only one law of motion as to satisfy the initial conditions (corrsponding to $t = 0$). The initial conditions are given by the values of the initial positions $x_1(0)$ and $x_2(0)$ and the initial velocity $v_1(0)$ and $v_2(0)$. For example, if we choose as initial conditions:

$$x_1(0) = x_2(0) = 0$$

$$v_1(0) = v_2(0) = v_0$$

we obtain the first normal mode. The situation corresponds to pushing with the same velocity the two masses simultaneously, starting from the rest position. On the other hand, if we choose as initial conditions:

$$x_1(0) = x_2(0) = 0$$

$$v_1(0) = -v_2(0)$$

we obtain the second normal mode. The situation corresponds to pushing with opposite (but equal in value) velocities the two masses simultaneously, starting from the rest position. Any other choice of initial conditions gives rise to a motion that is a "mix"[10] of this two normal modes. In this case we can observe the beats phenomenon. It is interesting to stress with students that in this case, even though the motion of each mass is given by the sum "someway" of the two normal modes[11], the resulting motion not only is not harmonic but neither it is periodic. This amazing property comes from the fact that the frequency of the two normal modes of the system are incommensurable. In fact $\frac{\omega_B}{\omega_A} = \sqrt{3}$.

5.3.3.3. From two coupled oscillators to the continuous of infinite oscillators

The experiments and the mathematical treating, together with the simulations and the analysis of the slow-motion videos, that are reported above, are sufficient for the students to understand what a normal mode is and which are the properties of normal modes. In particular, the analogy with standing waves of Fig. 5.46, in our experience, has proved to be powerful for the prediction of the normal modes also for very complicated systems (many coupled oscillators). Now we have to implement the experiments and the data-logging analysis to systems of increasing complexity. The general idea is to increase the number of coupled oscillators up to the continuous chain of oscillators: the elastic string that is virtually the result of coupling an infinite number of oscillators (each point of the string). In this perspective, the experiments that we propose to students (to perform and analyse) are: *three coupled pendulums; five coupled pendulums; twenty coupled torsional pendulums* (the shive

[10] technically it is said a superposition of normal modes

[11] and thus of two motions that are periodic and more than simply periodic: they are harmonic!

wave machine); *three and four coupled mass-spring systems* and at last the *elastic string*. We propose also some qualitative experiments for the normal modes in two dimensions: the *Cladni plates* and for the normal modes in a non-mechanical case: two coupled *oscillating circuits*. For the sake of brevity we describe here just the systems with coupled pendulums as those with the mass-springs are equivalent. We just point out that in the case of mass-spring systems, many applet simulations are available on the internet [Falstad, 2014, Fis,] that are extremely helpful for students to verify their predictions.

So the further step is to perform a quantitative analysis via the data logging. Students have to try to put into motion the three coupled pendulums (Fig. 5.48) in the first, the second and the third normal mode and obtained the respective frequencies via the FFT (Fig. 5.49). Then they have to put into motion the system in many randomly chosen different ways. From the analysis of the waveform, in both cases of normal-mode configuration and random motion configuration, the students can see that when the system oscillates in one of its normal mode, each of its parts (pendulums in this case) oscillates with harmonic motion at the same frequency and with a fixed phase relation with the others. The amplitude of oscillation of each pendulums doesn't change, except for friction with the air, to indicate that there is no energy exchange between parts of the system. In addition, the higher the mode, the higher the frequency. In other words, they find the same results seen in the case of just two pendulums.

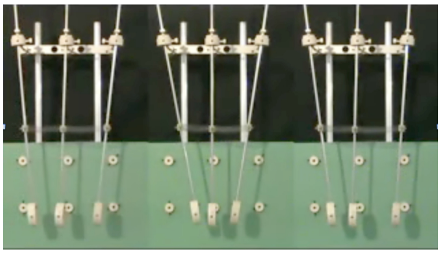


Figure 5.48.: A screenshot of the video of three coupled pendulums. From left to right: the first normal mode; the second normal mode; the third normal mode.

On the other hand, if the system is put into motion randomly, students can see that there is energy exchange between the pendulums. In fact the motion waveform of each pendulum clearly presents the beat phenomenon and the amplitude of oscillation varies in time. The more relevant didactic issue here is that, if we perform the FFT of each pendulum waveform, we obtain exactly the same frequencies of the normal modes previously measured (see Fig. 5.50). Each frequency peak, given by the FFT, has, in general, a different amplitude according to the way the normal modes superimpose, depending on the initial conditions. This allows to show to students that all the oscillations of the system are a linear combination of its normal modes.

In Fig. 5.51 it is shown a system of five coupled pendulums while in Fig. 5.52 are reported the waveforms of each pendulum. It is also shown in Fig. 5.53 the FFT

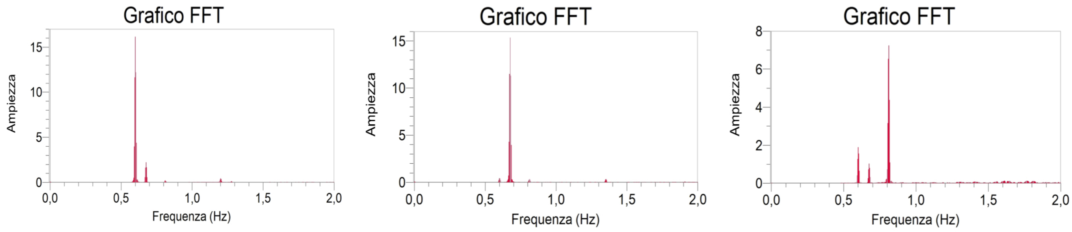


Figure 5.49.: The three coupled pendulums system: the FFT graph for the three normal modes. From left to the right: the frequency of thi first, second and third normal modee. In the graphs appear a little contribution also ot he other modes because it is very difficult for students to put the system into motion exactly in the coosen mode.

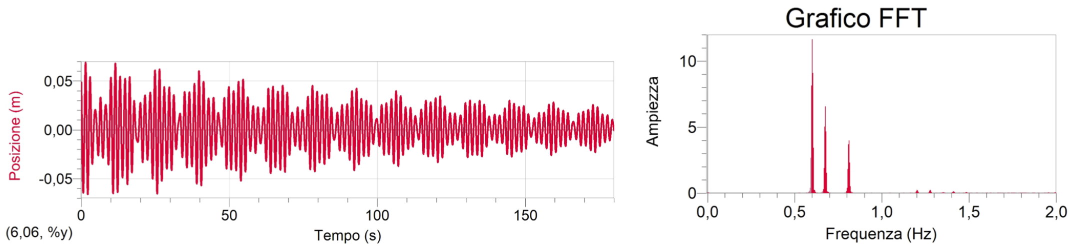


Figure 5.50.: The three coupled pendulums. On the left: the motion waveform corresponding to a random excitation of the system, with the typical lobes of the beats. On the right: the FFT with the frequencies of the normal modes superimposed.

that some students performed for one of these waveforms with the frequencies of the five normal modes mixing.

All these modes superimpose to give the motion of each pendulum. Moreover, when the system is excited randomly, the motion of each pendulum, being a linear combination of harmonic motions (the normal modes) is no more harmonic and generally neither periodic. In this case Tracker allows to plot the motion waveform for the centre of mass (Fig. 5.54) of the system which appear to be harmonic as expected from the fact that the first normal coordinate is given by the sum of the coordinates of each pendulum.

Now students can implement the number of coupled oscillator tracking the motion of a system of 50 torsional pendulums: the *Shive wave machine*. The *Shive machine* is a system of many torsional pendulums, as in Fig. 5.55. In our case we reduced the system to 18 pendulums to have them spaced enough. This was required for better data logging. In fact the sonar detector can't distinguish between two objects if they are too close. This experiments turns out to be of didactic interest because it can facilitate the conceptual transition from the discrete to the continuous case

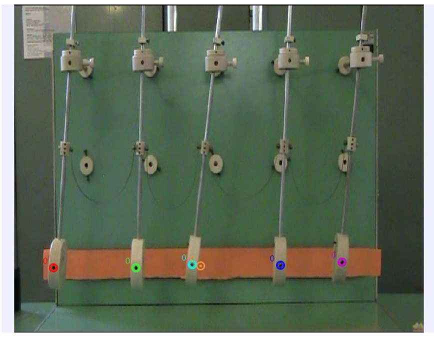


Figure 5.51.: A sceenshot of the video of five coupled pendulums

(for instance the vibrating string). In Fig. 5.56 it is reported the motion waveform and the corresponding FFT graph obtained by a group of students who tracked the complex motion of one pendulum. This data collection has been performed with the sonar and the Logger Pro software, but comparable results have been obtained with Tracker as well. The FFT, as depicted in Fig. 5.56, shows the frequencies of all the eighteen normal modes of the system. In this case it results evident that the first four normal modes are those that mostly contribute to the motion of the tracked pendulum. Furthermore, the more the number of pendulums the more the normal modes tend to be equally spaced in frequency. In fact, in the limit case of a continuous system, as the vibrating string, the frequency of each mode is an integer multiple of the frequency of the first normal mode. The teacher should point out to students that even with a limited number of oscillator, as eighteen in our case, the shape of the normal modes of the system (see Fig. 5.55) tend to be very similar to those of the elastic strings depicted in Fig. 5.45.

At the end of the path the teacher can perform, together with students the measure of the oscillating voltage of two coupled oscillating circuits. Namely two circuits with a capacitance in series with an inductance. Of course electric circuits are to early a topic to be full treated. Nonetheless this is an important experiment and easy to perform that can convince students that normal modes are not confined to the mechanics but are a conceptual organizer referring to many areas. An example of the exchange of energy between the two circuits with the usual beats waveform is reported in Fig. 5.57.

The last experiment to propose is the visualization of the normal modes of a bi-dimensional continuos system: the *Cladni plates*. They consits of plated with different shapes that are vibrated by a mechanical vibrator driven by a variable frequency generator. Students can observe the many configurations of normal modes forming as exciting frequencies increase. Neither quantitative nor qualitative analysis can be done, but a simple observation of characteristic standing forms appearing, as can be seen in Fig. 5.58.

5.4. Summary of the path

- Brainstorming on oscillations and periodic phenomena
- Examples of oscillating objects: a vertically bouncing ball, a simple pendulum, a disk bouncing between two elastic edges on an air table, a vertical mass on a spring, a seesaw on flat pivot, a seesaw on a round pivot, the semi-circular track, the Galileo oscillator.
- Grouping of the oscillations following common characteristics.
- Research of a stable equilibrium point and analysis of the total force acting along the trajectory; grouping of oscillations in dependence of the presence of a stable equilibrium point and a restoring force along the trajectory.
- Training to the use of the data-logging systems.
- Definition of harmonic motion.
- Integration of the differential equation of the harmonic motion via the projection of a uniform circular motion technique. Peculiar features of the harmonic motion (sinusoidal motion law, independence of the frequency from the amplitude etc...).
- Revisitation of the previous experiments and identification of the harmonic oscillation via different techniques (analysis of the graphs etc.).
- Introduction of the concept of natural frequency and resonance via experiments and videos.
- *Two coupled oscillators and normal modes*: two coupled pendulums and two coupled mass-spring systems: real experiments with data-logging, applet simulations and videos in slow-motion.
- Formal definition of *normal modes and calculation of the frequencies and of normal modes in the simple case of two coupled mass-spring systems.*
- Many coupled oscillatore to the continuous case: three coupled oscillators (pendulums and mass-springs): real experiments with data-logging, applet and video in slow-motion; four coupled mass-springs: real experiments and applet simulation and video in slow-motion; five coupled pendulum: real experiment with data logging; eighteen coupled torsional pendulums (Shive machine): real experiment with data-logging; the string and the slinky spring (transverse and longitudinal normal modes in continuous systems): real qualitative experiments and applet simulations.
- Normal modes in the non mechanical system of two coupled LC circuits: qualitative experiment.
- Normal modes in two dimensions (membranes): the *Cladni Plates:* qualitative experiments.

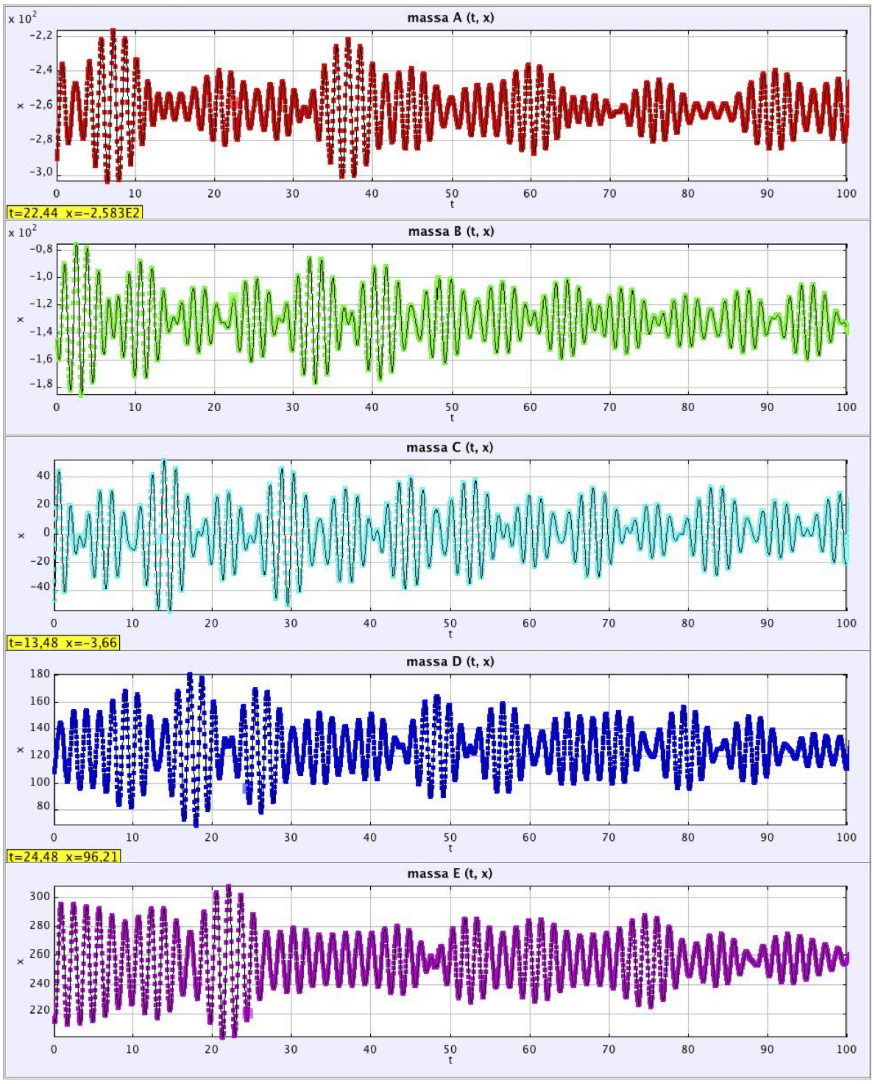


Figure 5.52.: The waveform of motion for each pendulum of a system of five in the case of random excitation. The colors are in agreement of the colors labeling the single pendulums in Fig. 5.51.

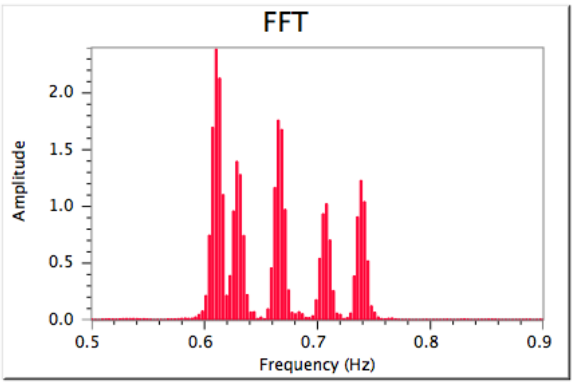


Figure 5.53.: The FFT graph corresponding to the first motion waveform of Fig. 5.52, namely to the red labeled pendulum of Fig. 5.51.

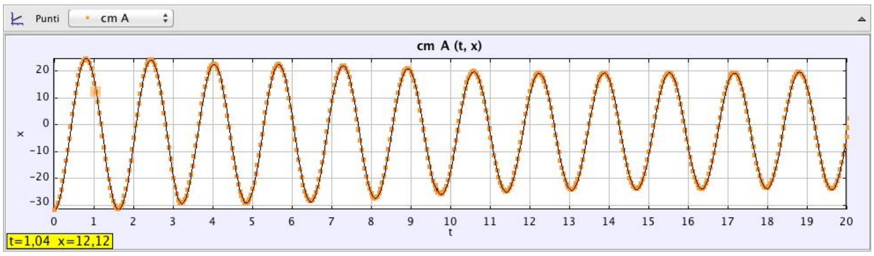


Figure 5.54.: The motion waveform for the center of mass of the system of five coupled pendulums, in a random motion configuration.

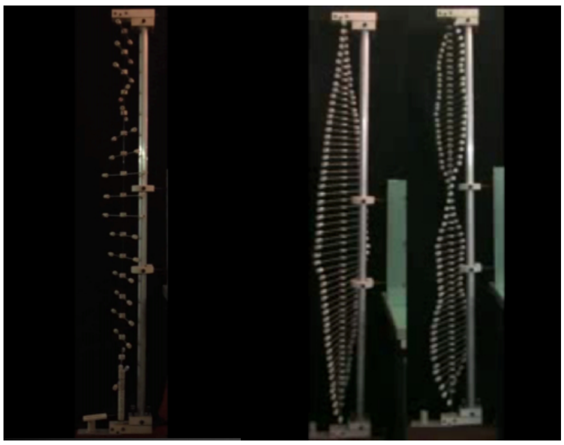


Figure 5.55.: A screenshot of the video for the Shive wave machine. From left to right: random motion configuration, first and second normal modes.

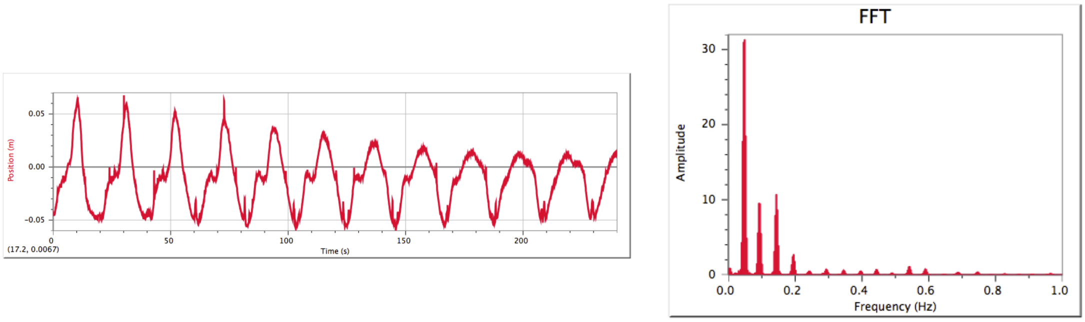


Figure 5.56.: The motion waveform (left) and the corresponding FFT analysis (right) for a 18-pendulums *Shive machine*.

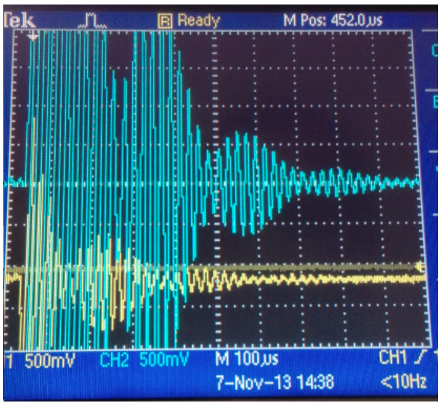


Figure 5.57.: The beats between two oscillating circuits: sceenshot from the oscilloscope display.

Figure 5.58.: Two dimension systems: the normal modes in Cladni plates.

6. The vertical mass-spring pendulum: an example of parametric oscillator. A path for undergraduate students

6.1. Introduction

As we have seen in previous chapter, when two or more oscillators are linearly-coupled the modes of oscillations are someway *"normal"*, that is: if we switch on just one of the modes the system will continue oscillating in that precise mode, the other modes will remain switched off.[1]From a mathematical point of view we can say that the normal modes form a basis for the oscillation of the system. In other words, any possible configuration of oscillation isn't but a linear combination of just the normal modes of the system itself. This is not true when the oscillators are not linearly-coupled. In such cases the system still have its peculiar modes of oscillation but these are no more "normal" in the sense that they are not independent the ones from each other. This is evident from the fact that if we switch on one mode of oscillation, in a matter of time it will transfer energy to the other modes. A continue to and fro exchange of energy between modes will happen as shown by the beats phenomenon. A well known example of oscillator in which the modes of oscillation are not-linearly coupled is the parametric oscillator[2]. The vertical spring-mass pendulum is an example of parametric oscillator[3]. We present here a teaching sequence on parametric oscillator for undergraduate physics students. The parametric instability is a condition that can eventually be reached while studing the vertical mass-spring oscillator if certain conditions are met. The vertically oscillating spring-mass system used to study harmonic oscillator physics in a first-year laboratory course has to be pre-designed in order to reliably reproduce the motion described in textbooks. In a laboratory organized so that students can assemble their own apparatus as they see

[1]In this sense the term *Normal* means that the modes are independent the one from each other and form a basis for the oscillation. Switching on a single pure mode will cause the system to oscillate in that precise mode indefinitely. Of course after a certain time the oscillation will come to a end because of ineliminable friction.

[2]the parametric oscillator is an oscillator in which at least one parameter oscillates in time

[3]more properly the spring-mass pendulum is an auto-parametric oscillator. In fact we refer to a parametric oscillator when....

fit (picking components from a pile), the spring-mass system is not likely to turn out so that it reproduces the idealized harmonic motion. There is a good chance they will come up with a system that shows complex dynamics. Furthermore, to test the spring-mass physics law $\omega^2 = k/m$, students have to apply different masses, thus often running into parametric instability. Hence, studying the spring-mass system in a laboratory where students pick the parts addresses both harmonic and parametric behavior. We believe the spring-mass experiment can be treated in class as a harmonic oscillator and then developed towards the parametric oscillator. This teaching sequence allows students to approach the challenging, complex physics content of parametric behavior.

6.2. The mass-spring pendulum

The spring-mass system studied in undergraduate physics laboratories may show complex dynamics due to the simultaneous action of gravitational, elastic, and torsional forces, in addition to air friction. In this paper, we describe a laboratory exercise that caters to beginning students while giving those with more background an opportunity to explore more complex aspects of the motion. If students are not given predefined apparatus but are allowed to design the experiment setup, they may also learn something about physics thinking and experimental procedure. Using results thus produced, we describe a variety of spring-mass oscillation patterns, discussing the physics of the significant deviations from simple harmonic motion. The parametric oscillation behavior we have observed is reported and investigated. This study is based on analysis of motion waveforms.

A mass hanging from a spring is a common, easy-to-perform experiment often used to introduce first-year physics students to simply harmonic motion[Boscolo & Lowenstein, 2011]. Although the apparatus is simple and inexpensive, under certain conditions it turns out that the motion is not simple at all.[Olsson, 1976, Cayton, 1977, Christensen, 2004, Geballe, 1958, Galloni & Kohen, 1979, Armstrong, 1969, Cushing, 1984] Gravitational (pendulum), elastic (spring), and torsional forces [Olsson, 1976, Cayton, 1977, Geballe, 1958, Cushing, 1984] generate numerous, complex phenomena that result in surprising motions in the spring-mass system. These phenomena include multimode operation (many oscillating modes can be simultaneously active),[Cushing, 1984] parametric instability (oscillation instabilities when the system is driven at certain frequencies), and energy transfer between the spring-bouncing mode and the pendulum-swinging mode. A real spring-mass system behaves as a simple harmonic oscillator only under specific conditions: (i) the spring's mass must be negligible compared to the attached mass; (ii) the frequency of elastic oscillation must not resonate with the frequency of pendular swinging; and (iii) initial spring stretch must be strictly vertical.

A spring-mass system assembled with different values for mass and for spring constant can occasionally fall into a configuration where its elastic oscillation frequency

ω_k and its pendulum-oscillation frequency ω_p have a ratio of nearly two. In such cases, the motion is unstable and the mass passes from vertical to horizontal oscillation in apparently random fashion, showing parametric rather than harmonic oscillations. (A parametric oscillator is a harmonic oscillator that has a parameter oscillating in time.) In the spring-mass system, the gravitational force acting along the axis of the spring varies periodically with the pendular motion of the mass. This periodic action results in an exchange of energy and much more complex motion. The vertical oscillation amplitude decreases while the pendular amplitude increases, and vice-versa.

We noticed that an interaction between vertical and pendular oscillations sometimes occurred in our laboratory when students are free to choose the springs and masses themselves. The physical aspects of the configurations that are linked to parametric behavior required further research to understand. Thus, we performed a study of motion waveforms for a variety of frequency ratios that spanned the resonance value $\omega_k/\omega_p = 2$. The frequency ratio is selected at will according to the equation

$$\frac{\omega_k}{\omega_p} = \frac{\sqrt{k/m}}{\sqrt{g/\ell}}\,, \tag{6.1}$$

where k is the spring constant, m is the appended mass, g is the gravitational field strength, and $\ell = \ell_0 + mg/k$ is the equilibrium length of the vertical oscillating spring, ℓ_0 being the natural (unstretched) spring length.

In order to perform this experiment, students must deal with the gap between theory and practice, examine complex, multi-effect motion, and master experimental techniques. Moreover, they have to treat data statistically and purge the experiment of many "nuisances." In other words, the students are forced to develop the ability to manage unexpected experimental observations. The complexity of a phenomenon that is not fully understood requires a strategy for singling out the motion components, relaxing their interrelations so as to treat each component separately. Afterwards, these components can be combined to understand the complete motion. In short, this experiment turns out to be a useful, guided-research activity for students.

The theoretical work starts with the idealized equation of motion for the deviation $z(t)$ from the (vertical) equilibrium position for a mass on a spring

$$\frac{d^2z(t)}{dt^2} + \frac{C}{m}\frac{dz(t)}{dt} + \frac{k}{m}z(t) = 0, \tag{6.2}$$

where C is a damping coefficient. The experimental goal is to test the solution

$$z(t) \;=\; z_0\, e^{-\gamma\, t}\, \cos(\omega\, t), \tag{6.3}$$

where z_0 is the initial oscillation amplitude, $\gamma = C/2m$ is the damping constant, and

$$\omega^2 = \frac{k}{m} - \left(\frac{C}{2m}\right)^2 = \omega_k^2 - \gamma^2. \tag{6.4}$$

In these equations, the mass of the spring is assumed to be zero, which is not actually the case. The damping force is assumed to be proportional to velocity, although air friction against a moving object depends on its shape and is not (generally) linear with velocity. The investigation proceeds through the following steps: (a) test of Hooke's law $F = -k\,\Delta z$, (b) test of $\omega^2 = k/m$, (c) measurement of the damping time $\tau = 1/\gamma$, and (d) test of the sinusoidal solution Eq. (6.3).

In the following sections, we describe the procedure followed in laboratory work with students and point out its critical steps. In the final section, we present the experimental research work we carried out in preparation for teaching the course. This study is meant to provide a deeper grounding for those conducting a widely used experiment; it is also a possible topic for open-ended investigation or small, individual research projects. Complete investigation of the spring-mass system also requires studying forced oscillations,[Boscolo & Lowensteim, 2011] which is to be the subject of further research.

6.3. The setup

The apparatus was kept as simple as possible: just a spring and a weight oscillating vertically as shown in Fig. 6.1. A short piece of wire acts as a pivot. The z- and x-coordinates of the hanging mass, defined in relation to its equilibrium position, are tracked by the bottom (Sonar1) and lateral (Sonar2) ultrasonic motion detectors, respectively. The system is surrounded by plastic foam to absorb the direct sonar waves not reflected by the mass-bob. The latter is a pile of 20 g metal disks on a support, to which plasticine may be added to obtain the desired precise weight. The support is terminated by a diskette of 70-mm diameter to reflect the sound waves emitted by the bottom sonar as much as possible. This technique is necessary because the mass has wide lateral oscillations and wobbles around its pivot. Another spurious motion can be observed, namely: transverse spring vibration, possibly due to the mass wobbling. The value of the diskette diameter is a compromise between a larger size for ideal sonar detection and a smaller size in order to minimize both friction and spurious motion. These haphazard motions, superimposed on the bouncing and pendular oscillations, cause observed motion waveforms to be irregular. Distorted waveforms prevent us from attempting a simple analysis of the experiment and of its physics content. To minimize spurious motion, it is essential to start with smooth, precise initial mass displacement and to use small oscillation amplitudes.

The two sonars ought to measure vertical and transverse oscillation exclusively, but this is not always the case. Sonar1, which is used to detect vertical oscillation, partially detects transverse, pendular oscillation as well, as sketched in Fig. 6.2. Similarly, Sonar2, meant to detect the transverse motion, also detects vertical spring oscillation. This leads to waveforms in which an oscillation with smaller amplitude is superimposed on the main oscillation. The resulting distortion of the oscillation under study can be observed by restricting the oscillations to pure motions. For pendulum motion, the spring was replaced with a wire; for vertical motion, a thin metal stick was inserted along the spring axis. The observation of pure motions is useful as a basis to interpret the results of combined oscillations described in the following sections.

It is worth mentioning that vertical motion detection by Sonar1 is not affected by the rotation of the pendular oscillation plane, but this is not true with Sonar2. When the oscillation plane rotates by large angles, Sonar2 waveforms become almost useless. Good waveforms from Sonar1 are enough to study the motion, with the occasional help of some information from Sonar2.

A meterstick and a stopwatch are used to measure elongations and oscillation periods, respectively. Typical values of the spring parameters are $m_{\text{spring}} \simeq 4.8\,\text{g}$, $k \simeq 8\,\text{N/m}$, and rest length 18 cm. The effective mass values for the bob—equal to the hanging mass plus a third of the spring's mass[Boscolo & Lowenstein, 2011, Christensen, 2004, Cushing, 1984, Halliday et al., 1992] (discussed more fully below)—range from 40–80 g.

In class, students assemble the system using the bottom Sonar1 only. This paper consists mainly of research performed after the students had completed the lab, and was carried out to investigate their unexpected results.

6.4. Test of Hooke's law and $\omega^2 = k/m$

In our simple harmonic motion experiment, students determine the spring constant in the traditional way. A test of Hooke's law $F = -k\,\Delta z$ is performed by adding different masses to the spring and measuring the relative static elongations using a meterstick. The spring constant is obtained with a precision of a few percent. The test of $\omega^2 = k/m$ is performed using masses ranging from a maximum value, limited by spring damage, to a minimum value, determined by excessive vibrations. Some oscillations turn out to be very irregular, particularly when smaller masses are used. Hence, students are guided towards using a larger mass to obtain more regular oscillations. The oscillation period is measured using a stopwatch. The initial objective was to verify $\omega^2 = k/m$, rewritten in terms of the period T as

$$m = \frac{k}{4\pi^2} T^2. \tag{6.5}$$

A graph of the added mass versus T^2 shows the expected straight line with a small negative intercept on the y-axis, as shown in Fig. 6.3. This test shows the ef-

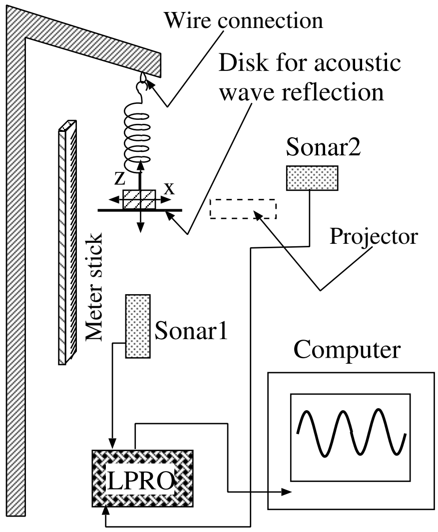


Figure 6.1.: Experiment layout. A meterstick and a light projector are used for measuring spring extension, while motion detectors (sonars) detect the motion. The digital data-acquisition tool LPRO (Vernier Logger Pro) acquires and sends the coordinates z(t) and x(t) to the computer. The top arm is bent to avoid reflecting the sonar signal.

fect of the non-negligible mass of the spring. A class discussion of the situation [Boscolo & Lowensteim, 2011] leads to the conclusion that the spring's mass cannot be neglected, and that the added mass should be replaced by an effective mass that can be written $m_e = m + m_s$, where m_s is the effective contribution of the spring's mass. Hence, Eq. (6.5) must be rewritten as

$$m = \frac{k}{4\pi^2}T^2 - m_s. \tag{6.6}$$

The y-intercept from the data then gives the expected value of one-third of the total spring mass ($m_s = m_{\text{spring}}/3$), as reported in the literature.[Christensen, 2004, Halliday et al., 1992] For the remainder of this paper, the pendulum mass will actually refer to the effective mass as defined here unless stated otherwise.

While students notice that lighter masses cause more complex oscillations, they also realize that the system does not show the expected simple harmonic oscillation and ask assistants for help. The new phenomenon of parametric oscillations comes to the fore for the first time in this context, and the complexity of the movement leads to further measurements with multiple motion detectors connected to a computer.

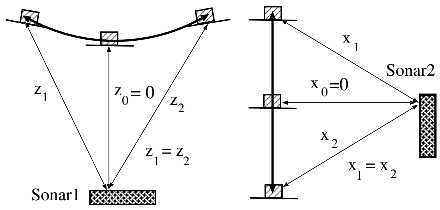


Figure 6.2.: Scheme of the sound-wave paths between sonars and appended masses going through transverse motions. Sonars measure the to-and-fro component of movement. Transverse motion is seen as a difference in length between the two triangle sides connected to a sonar.

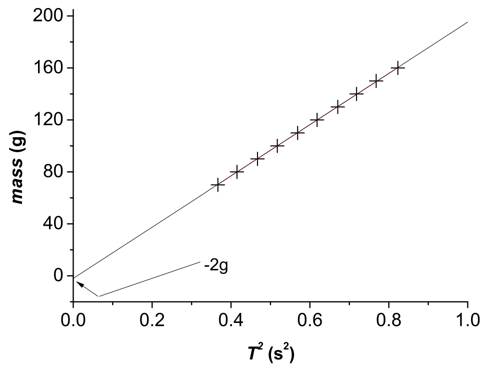


Figure 6.3.: Graph of 6.6, showing the negative intercept at the y axis.

6.5. Decay time and frequency measurements using motion waveforms

After completing the previous tests, students start taking measurements for the decay time $\tau = 1/\gamma$ in Eq. (6.3). The time interval between the beginning of the oscillation and the moment when the amplitude is reduced to $1/e$ of its initial value is measured using a stopwatch. Students use larger masses to get reasonably stable oscillations. They observe that τ depends on initial oscillation amplitude and on the value of the added mass (as expected from the definition $\tau = 2m/C$).

At this point, the study of the motion using waveforms is introduced, with the aim of measuring the two parameters τ and ω directly from the computer screen to check their previously obtained results. In theory, the value of decay time is constant for a given mass; nevertheless, we measure it for different sections of the same decay curve obtaining different values. Indeed, the decay does not appear to be exponential towards the end of the curve, instead showing decay times that increase with time. The very slow decay at the tail indicates an effective reduction of the damping coefficient; that is, a reduction of the effect of air resistance when the mass oscillates more slowly. As a result, the mathematical form of Eq. (6.3), based on a constant decay value, does not reproduce actual, observed motion. Initial fast decay indicates the possibility of energy transfer from the vertical motion to other motions. For the sake of thoroughness, we add that the assumed linear dependence of the damping time τ on the hanging mass ($\tau = 2m/C$) was tested, with better than 80 percent agreement. [Boscolo & Lowensteim, 2011]

After recording the variation in decay time along the waveform curves on the screen, students use the same graphs to measure oscillation frequency. They determine the time interval relative to a set of oscillations on the computer screen, then use a fast Fourier transform (FFT) tool, part of their data acquisition software, which provides the frequency spectrum. In this context, students are introduced to the concept and use of the FFT tool. After this measurement, students take a look at the waveforms obtained with different masses. The seemingly harmonic behavior with heavy masses, the very complex motion with elastic vertical and transverse pendular oscillation interchange, and the strong variation of waveform decay at the beginning of the motion make it clear that a new and more complete model for the system is needed to account for the many discrepancies between the simple motion predicted by the theory of harmonic oscillation and actual experimental observations at variance with expected results.

Separately, a few highly-motivated students then further studied the discrepancy between theory and experiment by fitting the experimental waveform to the function $f(t) = a(0)\exp(-t/\tau)\sin(\omega_0 t + \phi) + b$. The values of $a(0)$, τ, and ω_0 (initial amplitude, decay time, and oscillation frequency, respectively) are extracted from the waveform as explained above. Fitting the whole waveform turned out to be impossible; reasonably good fits were obtained only within sections of the decaying waveform. The farther the waveform section the longer the relative τ.

As an example, the fit of a mathematical model to a waveform using a 70 g mass gives the following results, which are in accord with the previous conclusion: (1) $\tau \sim 12\,\mathrm{s}$ for the first $10\,\mathrm{s}$ section, (2) $\tau \sim 17\,\mathrm{s}$ for the first $20\,\mathrm{s}$ (a longer section averages two different measured decay times), (3) $\tau \sim 20\,\mathrm{s}$ for the curve section $\Delta t = 10$–$30\,\mathrm{s}$, (4) $\tau \sim 30\,\mathrm{s}$ for the curve section $\Delta t = 20$–$60\,\mathrm{s}$, and (5) a much longer result at the curve tail. We checked the perfect correlation between the τ enhancement at a curve section and the relative frequency enhancement at that curve section, as expected theoretically from Eq. (6.4).

6.6. Measurements on harmonic and parametric motions

The remainder of this paper describes details of faculty members' research on motion coupling within the spring-mass system. This analysis is beyond the level of first-year undergraduate teaching. Typical waveforms obtained with a set of 70 g, 60 g, 50 g, and 40 g masses are shown in Fig. 6.4.

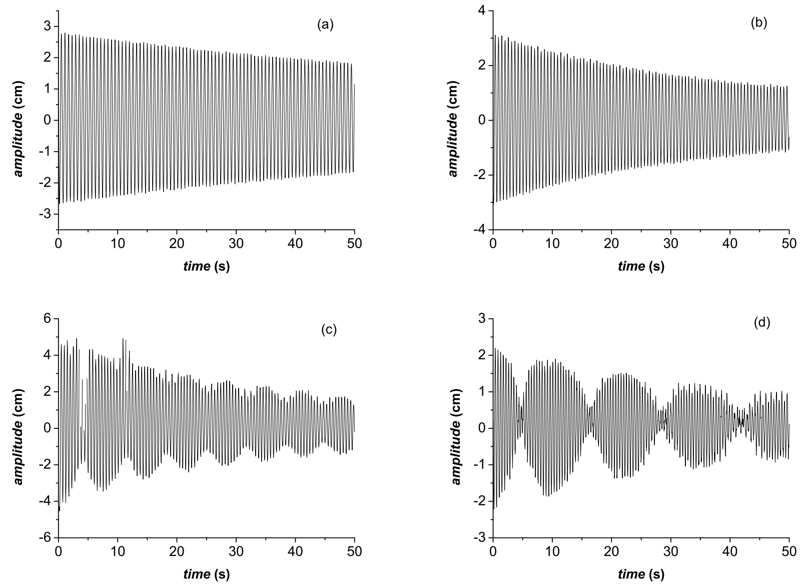


Figure 6.4.: Waveforms (a), (b), (c), and (d) corresponding to $70g$, $60g$, $50g$, and $40g$ masses, respectively. The system passes from out-of-resonance, with the $70g$ mass and $\omega_k/\omega_p \backsimeq 1.67$, to near-resonance, with the $40g$ mass and $\omega_k/\omega_p \backsimeq 2.04$. The waveforms evolve from a classical, damped sine wave towards complete modulation. The comb-like crests along the signal envelope, evident in frame (d), are generated by the particular way Sonar1 detects the transverse motion component.

Waveforms from heavier to lighter masses show a transition from an exponential decaying sinewave to partially modulated decaying sinewaves and, finally, to a multi-lobed, completely modulated sinewave. The same developments are observed when looking directly at the motion, with composite vertical-pendular motion for masses of intermediate value and the addition of frequent switches between vertical and pendular oscillations with the lightest mass. The intermediate case of a modulated-edge sinewave indicates that oscillation exchange between the two modes is partial. By counting the lobes, we can measure the frequency of oscillation-mode exchange. Another interesting observation is that the plane of pendular oscillation is not stable. It rotates by a few degrees in the case of low partial mode exchange. Conversely, when mode exchange is complete, the oscillation plane shifts by larger angles (up to 90 degrees) in an apparently random manner.

The study of waveforms leads us to conclude that there are two different classes of motion: harmonic and non-harmonic. The two types of motions refer to different physical phenomena.

Let us again look at the forces acting in the system, as depicted in Fig. 6.5. We note that the component along the spring axis of the force mg applied by the appended mass changes periodically during pendular oscillation and, in turn, alters spring extension. In particular, stretching-force oscillation causes the spring's equilibrium length to oscillate around its initial value ℓ:

$$\ell(t) = \ell + \delta\ell(t). \tag{6.7}$$

Therefore, pendular and spring-oscillation modes are coupled. Spring extension due to the weight makes a complete oscillation in half the time of one complete pendulum oscillation at resonance, that is when the ratio between the two oscillation periods T_k and T_p is 1/2.

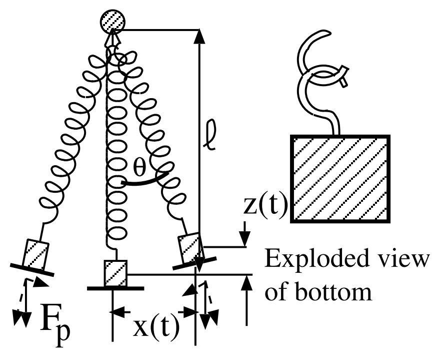


Figure 6.5.: Sketch of the applied force showing coupling between spring-bouncing and pendulum- swinging motions. The mass' center of gravity is not along the spring axis.

The coupling between pendular oscillation and spring oscillation induces parametric instability between the two modes [Olsson, 1976, Cayton, 1977, Lai, 1984, Holzwarth & Malone, 20(when the resonance condition is met. The energy exchange between the modes, tested by the waveform envelope modulation, is observed up to a frequency ratio [Eq. (6.1)] of about 1.8. The farther from the resonance condition, the lower the energy exchange. The observed mode exchange within an interval of frequency ratio reproduces the common physical fact that the resonance curves have Gaussian-like form with a certain width. We can say that our resonance has a relatively large width. Moreover, out of resonance, our measurements show that the greater the amplitude of vertical oscillation, the more extensive mode exchange becomes, yielding a wider resonance curve. The resonance curve was also observed to enlarge as motion disorder increases, i.e. when spurious motion becomes significant compared with the two motions of spring oscillation and pendulum oscillation. Spurious motion is always present when oscillations are large. Both spurious motion and large oscillation lead to an increase of the mode coupling.

Motion due to oscillatory mode exchange (parametric motion) is initiated by either vertical or lateral shift of the appended mass, i.e. with either elastic or pendular oscillation excitation. Vertical motion represented by a waveform like that of Fig. 6.6 is generated by starting pendular oscillation with lateral displacement of the appended mass. This parametric motion suggests that the coupling between pendular and elastic oscillations is nonlinear. In the literature, equations describing nonlinear coupled pendulum and spring oscillations are given as[Olsson, 1976]

$$\frac{d^2x(t)}{dt^2} + \omega_p^2\, x(t) = c\, x(t)\, z(t), \tag{6.8}$$

$$\frac{d^2z(t)}{dt^2} + \omega_k^2\, z(t) = c\, \frac{x^2(t)}{2}, \tag{6.9}$$

where the coupling constant c is a function of the system parameters.[Olsson, 1976] Incidentally, damping is not considered in these equations. More refined motion equations that satisfactorily describe the results of our experiment are presented in Ref. 3. These equations can be obtained through Lagrangian formulation. This system can be solved analytically only in one particular case [Olsson, 1976] and numerically in all others.[Cayton, 1977].

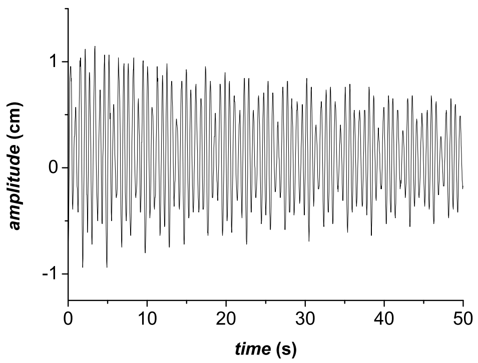


Figure 6.6.: Waveform of vertical oscillation obtained after a lateral initial displacement. The irregular shape of the waveform depends on the simultaneous presence of vertical and transverse oscillations, as seen by Sonar1. A $70g$ mass was used in this test.

The presence in the motion of different oscillation modes can be observed directly in the waveform spectrum lines obtained with the FFT tool. The spectra of the waveforms in Fig. 6.4, shown in Fig. 6.7, have the expected frequency lines for the oscillations those waveforms represent. The neater the waveform (i.e. the more ordered the motion), the neater the FFT. In each frame, the line corresponding to ω_k is largely dominant, and the line corresponding to ω_p is present, albeit with limited intensity. In disordered motion, the lines for spurious oscillation modes show up in the spectra. The two (c) and (d) spectra of Fig. 6.7 make the $2\omega_p$ harmonic stand out, because the pendular oscillation has large amplitude and Sonar1 doubles the pendular frequency, as explained in Fig. 6.2.

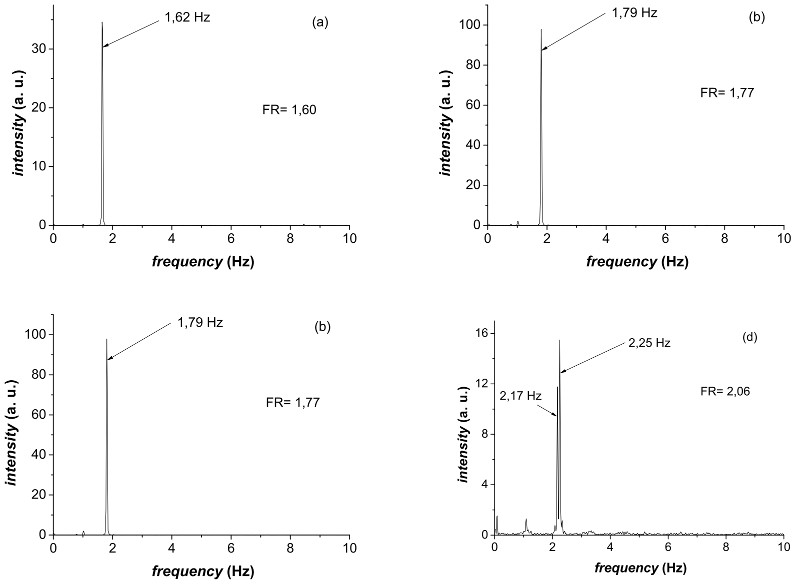


Figure 6.7.: Plots (a), (b), (c), and (d) show the FFT signals derived from $70g$, $60g$, $50g$, and $40g$ waveforms, respectively. FR is the actual value of the frequency ratio. The frequencies of some pronounced spectrum lines are labeled.

6.6.1. Behavior with mixed vertical and transverse excitation

Any excitation (initial mass shift with vertical and lateral components) that corresponds to a frequency ratio within the resonance width can start mixed vertical and pendular motion. Indeed, the coupling term on the right-hand side of motion equations (6.8) and (6.9) shows that an initial $x(0) \neq 0$ causes the onset of energy exchange between the two modes. It is almost impossible to apply a pure vertical or a pure lateral shift by hand.

The result obtained with mixed excitation is interesting: energy bouncing between the two modes occurs even with relatively heavy masses, i.e. with a frequency ratio that is out of resonance. The waveform obtained with 80 g mass (1.63 frequency ratio) proves to be modulated as shown in the first frame of Fig. 6.8. Its FFT spectrum has many sharp lines, as shown in the second frame of Fig. 6.8. A rather intriguing observation is that the system set itself in the peculiar stable motion shown in Fig. 6.9 after a few oscillation cycles—a behavior cited also in Ref 2. Results of mixed-excitation experiments enable us to state that nonlinear coupling between the two modes occurs even far from resonance.

Other observations worth reporting are:

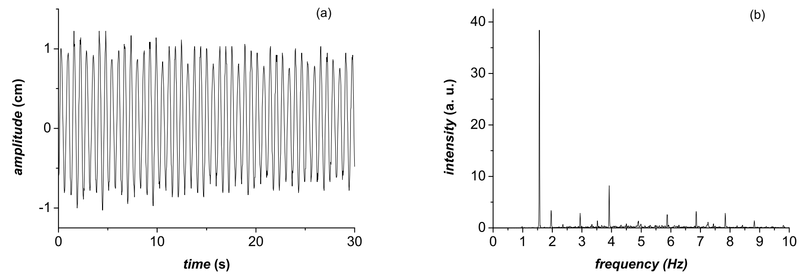


Figure 6.8.: Frame (a): Waveform obtained with an 80 mass and small diagonal displacement. The energy exchange between the two modes is clearly visible. Frame (b): Frequency spectrum. Many lines are present, besides the spring frequency at 1.55 Hz and the pendular frequency at 0.95 Hz (with very small amplitude). Second harmonic frequencies are visible. Other clear lines, at higher frequencies, correspond to wobble motions.

- (1) At resonance, both vertical and lateral mass displacements lead to similar system instability.
- (2) Near resonance conditions, both strong vertical and strong lateral mass displacements lead to substantially similar results in waveforms and spectra. This is not the case with small displacements: a small vertical displacement excites only vertical spring oscillation (no exchange with pendular oscillation), whereas a small lateral displacement excites both modes.
- (3) Mixed excitation produces complete energy transfer between the two modes even at the frequency ratio $\omega_k/\omega_p = 1.8$. The spectra are neater than in pure vertical excitation.

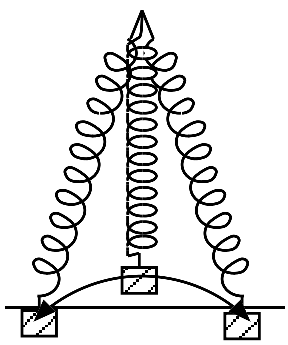


Figure 6.9.: Motion gure normally assumed by masses above $60g$.

6.6.2. Analogy to optics

Parametric instability is common in nonlinear optics. This phenomenon is employed extensively to produce and amplify laser waves at diverse desired frequencies.[Yariv, 1988, Cialdi et al., 2011] A pump wave ω_1 launched across a nonlinear crystal generates, from noise, a signal wave ω_2 along with a so-called idler wave ω_i. Among others, the main condition that must be satisfied is

$$\omega_1 = \omega_2 + \omega_i \,. \tag{6.10}$$

In our spring-mass system, the spectrum clearly shows the idler frequency,

$$\omega_i = \omega_k - \omega_p \,. \tag{6.11}$$

For example, the spectrum of the waveform yielded by vertically exciting a 70 g mass shows the idler line distinctly, as seen in Fig. 6.10. This figure reproduces a selected section of the spectrum recorded by Sonar2, which, for this type of excitation, detects low-amplitude components more cleanly.

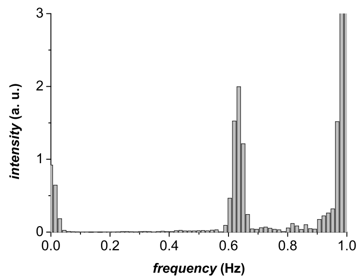


Figure 6.10.: The idler frequency line at $0.62 = (1.63 - 1.01)\,Hz$ from the FFT graph of the waveform generated by vertically exciting a $70g$ mass (see Fig. 6.8(a)), recorded by Sonar2.

7. Some Results of the Experimentations

7.1. The context

The path on oscillations and normal modes has been proposed to different contests of students. The final version has been tested on three classes of 11th grade students during curricular lessons, for a total of 86 students. All the three classes were composed of students of scientific orientation attending the third year (*liceo scientifico*). The entire experimentation had a duration of 21 hours spanned over 7 weeks.

7.2. The target and the steps of the experimetation

All the students were 11th grade students. They had a finite mathematical background (little trigonometry, second degree equations and no calculus) and they had never studied waves. The experimentation developed in three phases: i) the administration of an initial questionnaire (pre-test); ii) the implementation of the activities during curricular lessons; iii) the administration of a final questionnaire (post-test).

7.3. The questionnaire

A questionnaire has been administered to students before (pre-test) and after the experimentation (post-test). The post-test has been administered long after the conclusion of the experimentation: over one month after. The post-test and the pre-test are equal except for two additional questions in the final test related specifically to normal modes . So the initial test is composed of ten questions while the final one is composed of twelve questions. The students had one hour time for answering the pre-test and twenty additional minutes for answering the post-test. The first four questions had to deal with the relationship between forces and kinematic quantities with particular attention to the capability of students to represent on graphs these quantities in the experiments proposed. Questions five to seven refer to the properties of oscillating system, with a particular attention to the phenomenon of resonance. They are meant to see the pre-conception of students and their thinking on such topics. Questions eight and nine refer to periodic motions and are meant to see how students mean the concept of periodicity and how they can interpret periodicity through the graphs. As the final question for the pre-test, the tenth question is meant to see how students interpret different kind of motions and which are the properties they can identify as common to be able to group motions following common properties. Both the additional two questions of the post-test deal with normal modes. They are intended to evaluate the efficancy of the path and to see whether the experimental approach effective for student's comprehension of normal modes. The complete questionnaire is reported in appendix.

Question n° 1 This question has two main goals: the first is to investigate the ability of students to identify the force responsible for the oscillation of the pendulum and recognize it as a restoring force; the second, how they can manage graphs for F, v and a where the independent variable is a position variable instead of time. In the post-test 68% drew the force as restoring (both generic and elastic) while none did it in the pre-test. In fact in the pre-test drew the force as a sinusoid, same for the velocity and the acceleration. This happened (as emerged from the interview) because students were used to represent the kinematic quantities as a function of time. For the velocity, in the post-test, 66% represented the graph as an ellipse or a circle, with the correct point of inversion of the motion while none did in the pre-test. For the graph of the acceleration the 60% reported the same graph as the force. 3% did not answer the question.

Question n° 2 In this question the misconception of the force bound to the velocity and the same capability of managing graphs as in the previous question are investigated. In the post-test 50% of students drew an overall correct graph for the force and 42% a correspondent graph for the acceleration, while 33% continued drawing a variable force as the 80% of the pre-test. 17% did not answer in the post-test and 20% in the pre-test. For the velocity, the 40% of students could draw an overall correct graph in the post-test; 20% did not answer. In the pre-test only 5% provided a reasonable graph for the velocity and 30% did not answer.

Question n° 3 The questions investigate the role of the forces during the oscillation of a vertical mass-spring and their relationship with the variation of the velocity. In the post-test 73% of students drew correctly the forces and the resultant force with respect to the equilibrium point, 47% in the point of maximum extension of the spring and 44% in the point of maximum compression of the spring. The results for the pre-test were respectively: 60%, 40%, 37%. It is interesting to note that 46% of students represented an upward elastic force also in the case of maximum compression of the spring, as they were used to make exercises with elongated springs therefore pulling instead of pushing (this explanation emerged clearly from the interviews). In the pre-test, this misconception interested 57% of students.

Question n° 4 In this question we investigate the conception that students have about forces and motion, namely forces and velocity. The interesting result is that many students (43% in the post-test and 61% in the pre-test) believed that there must be an applied force if the body has a velocity. During the interviews, some of the students said that they misunderstood the question and that the force they drew was the weight of the ball. A couple of students instead were really convinced that if there is a velocity there must be a force in any case. One said: “how can a body have a velocity if there isn’t a force to push it?”

Question n° 5 In this question the characteristics of oscillators are investigated. In the pre-test we wanted to see how is the common thinking of students regarding the frequency of oscillation in relationship with the amplitude. In the post-test the question aims to see if the idea of frequency as an intrinsic characteristic of the oscillator has been learned. In the pre-test most students (about 80%) said that the frequency depends on the amplitude and about 10% answered that the frequency remains constant. In particular the more stretched the spring, the more high the frequency. Only two students related a bigger amplitude to a smaller frequency. 72% were convinced that also an initial velocity affects the frequency. In the post-test 89% of students affirm that the frequency does not depend on amplitude; 5% think that the more the amplitude the higher the frequency; 2% think that the more the amplitude, the smaller the frequency; 4% does not give an answer. In case of additional initial velocity the 13% of students think that the frequency gets higher but the motion is still harmonic.

Question n° 6 The question is similar to the prvious one but the example is from every day life. The percentages are very similar to those of the previous question. During the interviews it emerged that students tend to confuse the frequency of the sound with the intensity.

Question n° 7 This question aims to investigate the common thinking of students about a resonance phenomenon in the pre-test and to evaluate if students have learned the idea of resonance and normal modes of a single oscillator as the best way to exchange energy. In the pre-test just 8% of students could manage the question; 10% did not answer and the 82% answered that after a while both the mass-springs system oscillate at the same frequency of 1,4 Hz. In the post-test about 73% of students described well the resonance of the left oscillator and only 10% remained as in the pre-test.

Question n° 8 The question simply aims to see if students are able to extrapolate the concept of periodicity from a graph. In the pre-test 87% recognized the periodicity represented in the graph even if a few of these students confused the period with the semi-period while giving the value. 10% said that the graph does not represent a periodic motion because the plot is not sinusoidal; 3% did not answer. In the post-test all students affirmed the periodicity of the graph.

Question n° 9 The question is very similar to the previous one. The main difference is that here the graph is damped. In the pre-test 78% of students said it is not a periodic motion because different sections of the graph are not equal. About 16% said it is not periodic, without adding an explanation; about 6% did not answer. In the post test 73% answered that the graph represents a periodic motion damped by friction because just the amplitude is reduced, not the shape of motion. 8% said that the motion is periodic and also harmonic because the plot is a sinusoidal function. 15% answered that the motion is not periodic because different section of the graph are not superimposable. 4% did not answer.

Question n° 10 This question aims to see how students see different types of motion, namely which are the characteristics of the motions that they use to group into categories. It is interesting to see how, in the pre-test, about 80% of students grouped the motions taking into account the trajectories (for instance, the mass-spring with the bouncing ball because both describe a straight trajectory, and the pendulum with the seesaw on a flat pivot because both describe an arc); about 20% considered the duration of the motion, thus the damping (long-lasting such, for instance, the pendulum and the mass-spring and short-lasting such as the seesaw on a flat pivot and a Waltenhofen pendulum). None considered the acting force. In the post test 63% of students introduced the presence of a restoring force, thus the harmonic motion as a criterion to group the motions.

Questio n° 11 This question and the next one are present just in the post-test. Both are intended to evaluate the learning of the normal modes. The question is divided into five sub-questions. 98% of students were able to recognize that there are 4 normal modes because there are 4 degrees of freedom; 2% answered 5 normal modes because confused the number of masses with the number of springs (this emerged from the interviews). 100% were able to represent the first normal mode, 82% the second, 63% the third and 49% the forth. In many cases the errors were due to the low precision in plotting the scheme, especially for the higher modes. 76% of students answered that the masses perform harmonic motion when the system is in one of its normal modes. 91% answered that in a normal mode all the masses have the same frequency and that this is evident because the phase relationship between masses is constant over time and there are no beats. At last, 57% answered to the fifth sub-question that if the system is excited at a frequency different from its normal frequencies, they can see the beats phenomenon.

Question n° 12 This question is made to test the comprehension of the superposition of normal modes and the meaning of the FFT tool. Students had to pay attention to the values reported in the figures, in fact in the figure (d) we have represented the same system of the figures (a) e (c) but in a motion configuration in which only the first two normal modes are excited. 81% of students recognized that fig (a) and (c) represent a system of three pendulums that is in motion in a random way so the FFT graphs provides the frequency of the three normal modes mixing.

7.4. Some hints from brainstorming, discussions and interviews

In the initial brainstorming, students were asked to group the oscillating systems they observed by the naked eye. Most of them decided to put together oscillators with similar trajectories. For instance, the vertical mass-spring, the ball bouncing on the floor and the bouncing disk were grouped together “because all move along a straight line”; the simple pendulum, the rod tilting on a flat pivot and the ball running along a semi-circular rail, were grouped together “because they describe an arc”. In the final test, on the contrary, over 60% of the students grouped oscillators taking into account the forces acting on the system, being them restoring forces or not. Another interesting fact emerged from the initial brainstorming is that about 80% of the students thought that the oscillation frequency of a vertical mass-spring oscillator does depend on the initial displacement. In particular, they thought that the greater the initial amplitude, the greater the frequency. Some students said: “when the amplitude is bigger, the frequency is higher because the movement of the mass is faster”. A few students thought that the frequency of oscillation decreases with the initial displacement because: “the velocity is the same but the space is longer, so the oscillation takes more time”. Only less than 20% of the students decided that the frequency is constant, regardless the initial displacement. They stated this fact on the base of direct observation by naked eye: “looking at the oscillation I can’t see difference”. Analogous results and similar comments were obtained with the simple pendulum, despite the fact that almost all the students already knew the pendulum isochronism law. The situation greatly changed after the didactical intervention as at the end of the path nearly 90% of the students were able to recognize that the frequency of a harmonic oscillator is independent on the frequency. Regarding normal modes, while many students were able to imagine “some special motion configurations” of a system of two coupled pendulums before the topic was introduced and the experiments performed, only a couple of them were also able to predict the motion of the third normal mode of a system of three coupled pendulums. None could predict higher modes in more complex systems (five pendulums). Anyway, after introducing the graphic technique, the number of students able to predict the motion of all normal modes increased significantly. In the final test and in the interviews was proposed a question on a system of five coupled oscillators. All the students were able to describe the motion configuration of the first normal mode by words and/or by sketches. Over 80% described correctly the second normal mode, over 60% the third and the fourth and nearly 50% the fifth one. Most of the wrong answers on higher normal modes were due to inaccuracy in drawing the sketches.

7.5. Answers to the research questions

In the first research question we wonder if the dynamical choice of defining the harmonic motion through the linearization of the restoring force is more effective than the kinematical definition through the projection of a uniform circular motion on a diameter. We must say that the students of the three classes involved in the experimentation of the path had not received any kind of definition of harmonic motion before. So a comparison of effectiveness with other definition is not possible for these students. Anyway, in the final interviews almost all students were able to recognize the harmonicity/anharmonicity of motion they had already seen during the implementation of the path, on the basis of the analysis of the acting forces. About 80% could recognize the harmonicity/anharmonicity also in new cases, after analysing the forces. Some students referred: "I think it is not harmonic because I guess that the force is not a restoring one. I should perform a tracking to plot the graphs to be sure". This is somehow also a positive answer to the third research question. The same request was made to the 12th grade students of the extra-curricular lessons and of PLS activities. They had previously received the kinematic definition of harmonic motion. They were not able with such a definition, to state the harmonicity/anharmonicity of the experiments we proposed, exept in the case of the pendulum. Anyway one student said: "to tell the truth I said that the pendulum is harmonic because I was told before, but I don't know how to prove this by the definition". The situation improved definitely with the dynamic definition. We don't have the precise percentages because in extra-curricular lessons and in PLS activities there was not a systematic data collection. Also with the undergraduate students of PED course things did't change. In fact we proposed to prove the harmonicity/anharmonicty, in agreement with the kinematic definition, for the experiments: vertical mass-spring, seesaw on a flat and on a round pivot, bouncing disk, ball on a circular track and some other. None of the students was able to do that, while they could by the dynamic definition.

The second research question was to assess the possibility of teaching normal modes to 11th grade students, that is to say without the basis of calculus. The results of questions eleven and twelve of the post-test show that the use of multiple representation, together with the data-logging techniques and the FFT can overcome the mathematical difficulties. These results have been confirmed in the final interviews, in which it also appeared clear that the use of the graphic analogy between normal modes and standing waves on a string is a very effective tool to predict the shape of normal modes even in many oscillators-systems. The decrease of correct prediction as a function of the increase of mode number was mainly due to the loss in accuracy when plotting the graph by hand. In fact the students missing the point in the post-test, were able to make the right prediction by plotting the "analogy" on graph paper sheets.

As regards the third and the forth research questions, at the beginning (during brainstorming and peer discussions) most students had great difficulty to draw the graphs of forces as a function of a positional coordinate. In the interviews many of them said that they had always represented kinematics quantities as a function of time and "found weird and difficult this new way of representation". As reported by students themselves, the tools of data-logging programs for the production of graphs "helped a lot". The comparison of the answer to question one and two of pre-test and post-test are enlightening. Surprisingly the introduction of the FFT as a tool encountered no particular problems. As emerged during the overall implementation of the path, in the post-test (question 12) and in the final interviews, students had no problem to understand what the FFT tool can provide and use it. One student said: "when we performed the simulation in which we summed two perfect sinuses with different frequencies and obtained that strange graph and then applied to that graph the FFT tool and we had back exactly the two frequencies of the two initial sinuses, I really understood what the FFT does".

Acknowledgments

So many people I have to thank that it would be impossible to do in a few lines. I would limit myself to thank here just the ones whom I have worked with during my PhD course, in the activities of the Physics Education Research Group, in the PLS activities and in all the other activities/works I have partecipated in. So, in a random order, thanks a lot to Enrico whose help and support have been of great importance to me. Thanks to Sara and the two Marina's for the support and collaboration. Thanks to all the teachers that let me enter their classes. Thanks to Paolo for the assistance in the laboratory. Thanks to Ilario, Stefano and Fabrizio for the really fruitful and pleasant cooperation and for teaching me so much physics. Thanks to Lino, Ivan and Federico for the support. And last but not least, thanks a lot to my tutor, Marco Giliberti and to Michela Cavinato, for teaching me so much, for the enlightening discussions and for their genius in addressing problems.

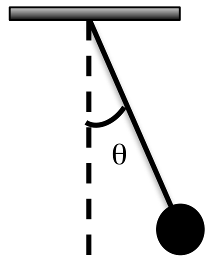
A. The questionnaire

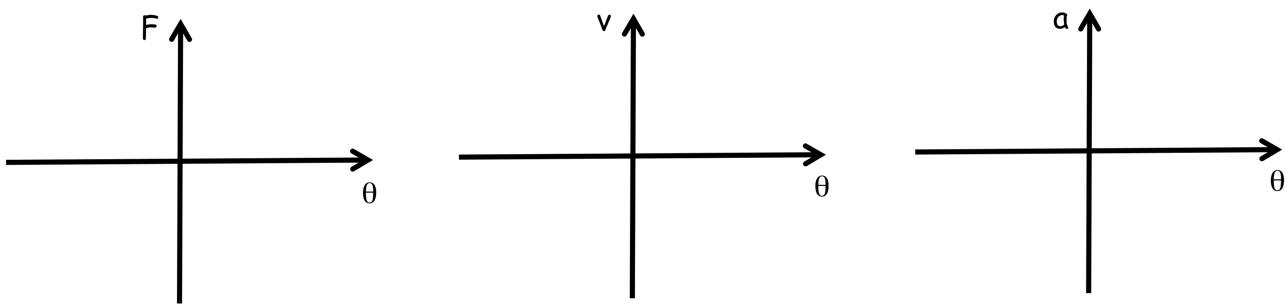
A pretest and a post test have been proposed to the students. The postest has been proposed long after (over one month) the conclusion of the activities as to evaluate the effectiveness of the path on long term.

Name: ______________________ School/Class: ___________________

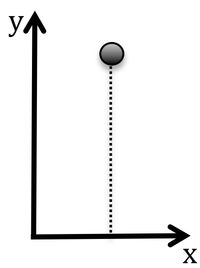
Test on Oscillations

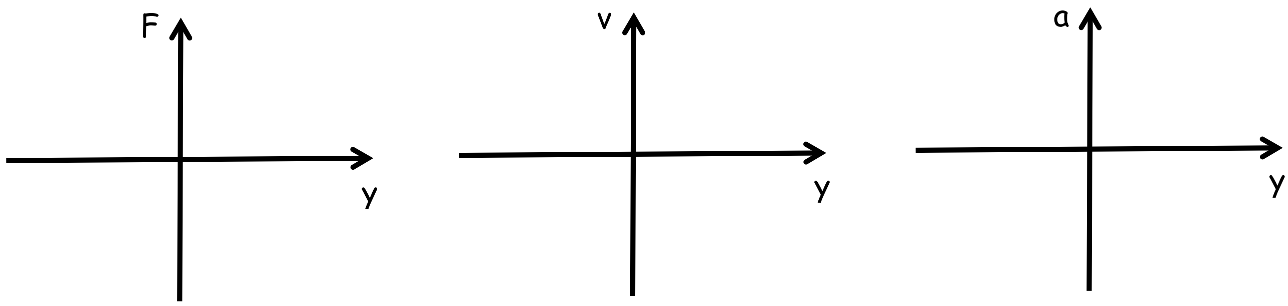
1. A pendulum is left free to oscillate in condition of small amplitude oscillations, without friction. If we state F the intensity of the resultant force acting on the bob of the pendulum, try to represent, over a period of oscillation, the graphs of force, velocity, and acceleration as a function of angular position θ.



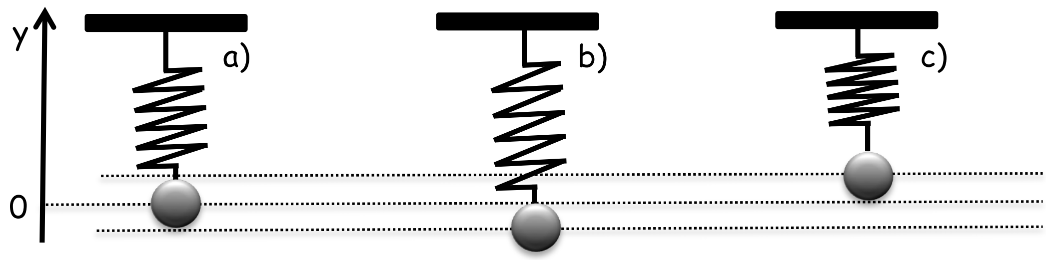


2. A ball is let free to fall from a certain height as shown in figure. Let us neglet the friction and assume that each impact with the ground is perfectly elastic so that the ball goes back to the starting altitude every time. Let us state F the total force acting on the ball. Try to represent the graphs of the force, the velocity and the acceleration as a function of the altitude y (considered a single complete bounce).



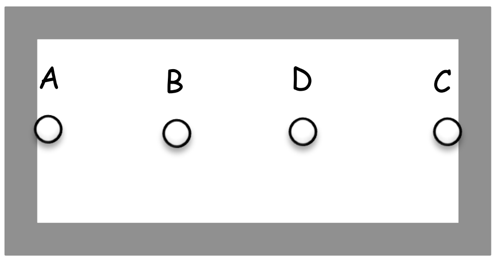


3. A mass appended to a spring, as shown in the figure, is let free to oscillalte. In the three figures are represented the following situations: a) the ball is in the equilibrium position; b) the ball is in the position of maximum extension of the spring; c) the ball is in the position of maximum compression of the spring.



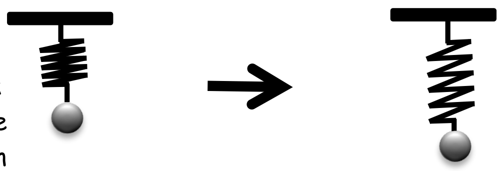
Draw, on the three figures, all the forces acting on the ball and their resultant.

4. A billiard ball performs elastic bounces between the edges of the table. The following figure shows the ball in four different moments: A and C during the impact with the edges; B at an intermediate point between the banks while moving towards the right and D while moving towards the left.



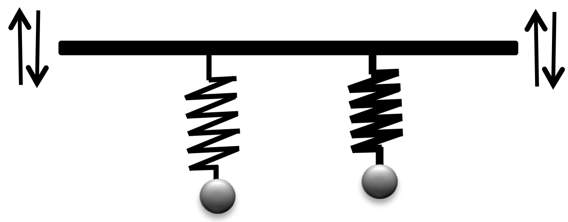
Draw on the figure the vector of the resultant force acting on the ball in the three situations (neglect any form of friction).

5. Consider a mass hanging at the free end of a spring as in the figure. The mass is initially in its equilibrium position. Imagine to displace the mass downwards, pulling it down. The system starts to oscillate with a certain frequency. Repeat the operation increasing each time the displacement. According to you, does the frequency of oscillation depend on how much has been pulled the spring? (if so, the frequency increases or decreases with increasing elongation of the spring?). What happens if an initial speed is impressed by hitting the mass from bottom to top?

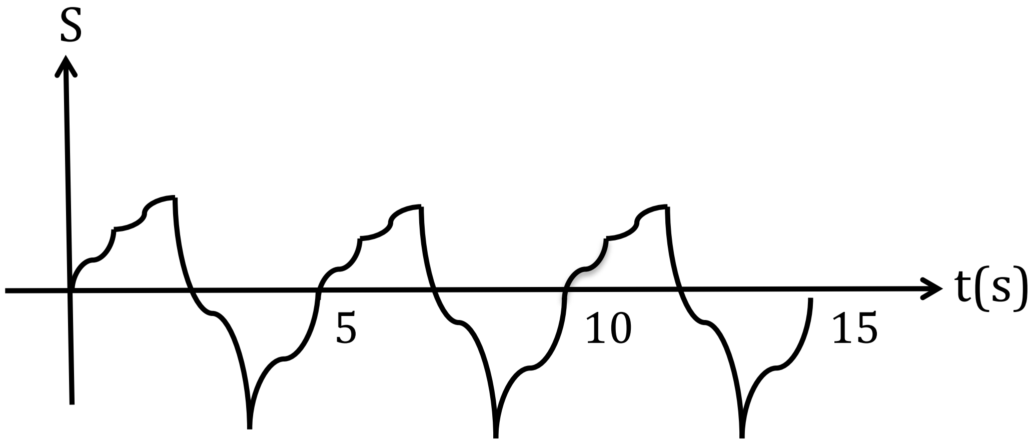


6. If you hit the edge of a crystal glass with a spoon, it makes a sound corresponding to the frequency of vibration of the glass. If you repeat the operation by hitting the glass in the same point but with greater force (be careful not to break it!) what do you expect to hear? A more pitched sound, a deepr sound (corresponding to a greater or lesser frequency of vibration) or the same sound as before?

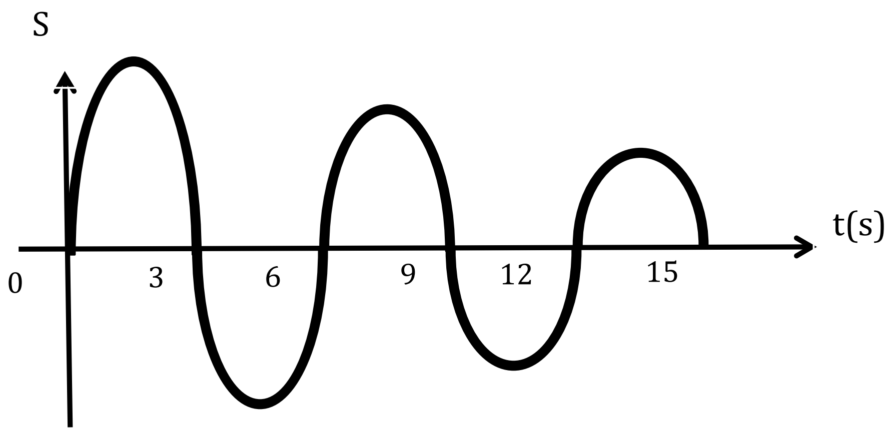
7. Two mass-spring systems are appended at the same rod as in the figure. The system on the left, if moved from the equilibrium position, oscillates with a frequency of 1.4 Hz. The system on the right oscillates with a frequency of 17 Hz. If the whole rod is swung up and down (see arrows in the figure) at a frequency of 1.4 Hz, do you expect to see in time?



8. Look at the graph in the figure. In your opinion, does it represent a periodic motion (please jusify the answer)? If your answer is yes, which is the period?



9. Look at the graph in the figure. In your opinion, does it represent a periodic motion (please jusify the answer)? If your answer is yes, which is the period?



10. Below are some examples of motion of bodies:

- Motion of a pendulum that swings around its equilibrium position
- Motion of a rubber ball bouncing vertically on the floor
- Moto described by the tip of the blade of a fan cooler
- Motion of a billiard ball bouncing between the edges of the table
- Motion of the Moon around the Earth
- Motion of a ball that rolls back and forth along a semi-circular track that is disposed vertically
- Motion of a mass appended at a vertical spring oscillating around its equilibrium position
- Motion described by the axis of a spinning top
- Motion traced by the end of the wiper of a car
- Motion described by a skewered chicken during cooking
- Motion of a mass attached to a spring arranged horizontally on a frictionless plane
- Motion described by the end of a seesaw, laying on a cube

Try to group these motions into three categories based on the features that you think they have in common. For each category indicate the criteria you have chosen.

MOTION TYPE 1	**MOTION TYPE 2**	**MOTION TYPE 3**

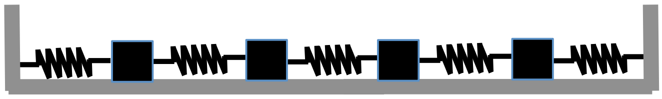
Common features:

MOTION TYPE 1:

MOTION TYPE 2:

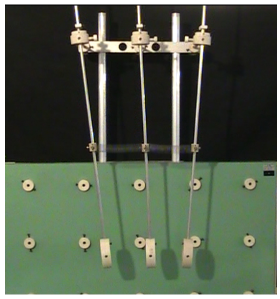
MOTION TYPE 3:

11. A system without friction is constituted by 4 masses connected with ideal springs. The masses are free to move horizontally, as shown in the figure:

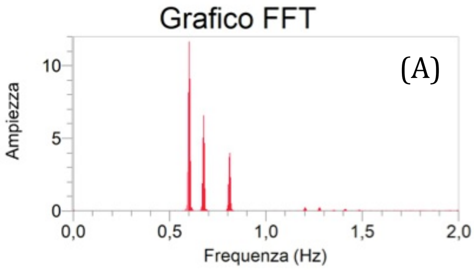


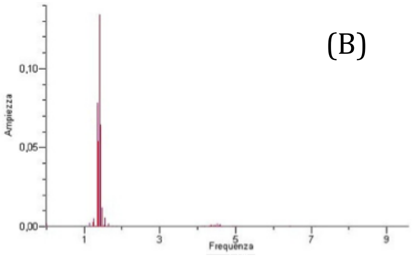
a. How many normal modes does the system have? (justify the answer)
b. Represent the normal modes by arrows which length be proportional to the oscillation amplitude.
c. When the system oscillates in one of its normal modes, what kind of motion does each mass perform?
d. Are the oscillation frequencies of each mass equal to each other when the system is in a normal mode? How can you determine?
e. What do you expect to see, in time, when the system is set in motion at a frequency different from that of one of its normal modes?

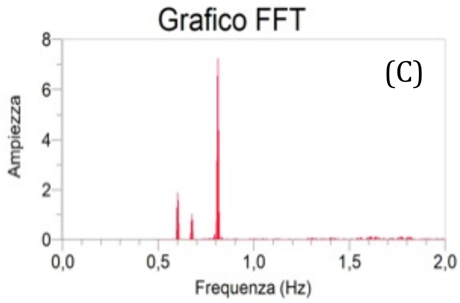
12. In the graphs of the figures below there are plotted the frequencies obtained by the analysis (Fourier transform) of the motion of a pendulum. Which, plot/s may refer to a pendulum belonging to the system of three pendulums coupled by two identical springs and put into motion in a completely random way? (justify the answer)

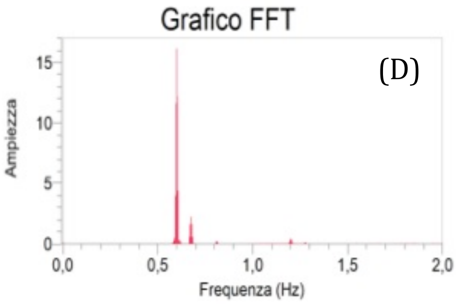


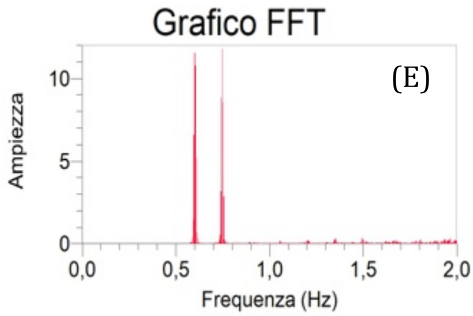
a. all the plots
b. none of the plots
c. the plots (A) e (C)
d. the plots (D) ed (E)
e. the plots (A), (C), (D) ed (E)
f. just the plot (C)
g. the plots (A), (C) e (D)











B. WORKSHOP COPYCAT

The copycat refers to a workshop based on the present path on oscillations that the Physics Education Group of the University of Milano performed at GIREP 2014 Conference, in Palermo on july 2014. I decided to attach the very last version of the copycat as it was at the moment of the performance. It is clearly a work in progress, in fact the text is not entirely translated into English.

The workshop is based on some theatrical strategies adopted in a two hour performance in order to show some meaningful experiments and the underlying useful ideas to describe a secondary school path on oscillations, that develops from harmonic motion to normal modes of oscillations and makes extensive use of video analysis, data logging, slow motions and applet simulations.

Theatre is an extremely useful tool to stimulate motivation starting from positive emotions. That is the reason why the theatrical approach to the presentation of physical themes has been explored by the group "Lo spettacolo della Fisica" [della fisica,] of the Physics Department of University of Milano for the last ten years [Carpineti et al., 2011] ; [Carpineti et al., 2006] and has been inserted also in the European FP7 Project TEMI (Teaching Enquiry with Mysteries Incorporated) [TEMI, 2014] which involves 13 different partners coming from 11 European countries, among which the Italian (Milan) group.

According to the TEMI guidelines, this workshop has a written script based on emotionally engaging activities of presenting mysteries to be solved while participants have been involved in nice experiments following the developed path.

WORKSHOP

INTRODUCTION

IN SCENA

Cubi neri 80x50x60; 70x40x50; 90x40x40 con sopra alcuni esperimenti. Altri sulla cattedra. Due alogene illuminano la scena. Sul palco 2/3 sgabelli. Cicloide, X^4, X^2, pendoli multipli *un po' a lato.*

Sulle sedie del pubblico sono presenti i kit TEMI, una sedia è senza kit con sopra una borsa

INIZIO

B, C, S sono impacchettati.

G non lo è e fa entrare il pubblico tutto assieme, poi esce.

- SLIDE: GOOD VIBRATIONS

MUSICA DEI BEACH BOYS

G quando tutto il pubblico è entrato, spegne la musica e entra.

G: Good afternoon to everybody, I'm Marco Giliberti from the university of Milan and I'm going to start this workshop on oscillations by giving a look on how harmonic motions is generally presented…

- SLIDE: Here you can see two typical definitions of harmonic motions

1) Disegno con proiezione sul diametro)

2) An object performs harmonic motions if it is acted upon by a force F = -kx

G: As you probably all already know, with definitions such as these, students do not go much further.

Our experience with secondary students and also with graduate students in mathematics shows that the comprehension of the link between mathematics and physics in the study of oscillations is far from clear if we start that way.

The bottom line is that the kinematic definition of harmonic motion is not enough to understand the physics implied; and also at university, the dynamical definition, which stems from the analysis of the potential energy, is often un-effective.

SPACCHETTAMENTO E TEMI

- SLIDE: TEACHING ENQUIRY - PHYSICS CANNOT BE DELIVERED AS A PACKAGE

G: We cannot deliver students pre-packed definitions, because in this way we don't give them instruments and concepts to analyse and read the world around them.

Con fare eccessivo, ironico. **But now let me introduce our PER team of the university of Milano. In alphabetical order:**

B, C, S si spacchettano quando presentati:

the sweet and shy Sara Barbieri!

B non si spacchetta e fa cenno di no con la mano…

She never wanted to be here now, but, please, Sara, get out of that package! Let's make her an applause of encouragement!

And now the extraordinary, exuberant and smiling Marina Carpineti!

And last but not least our best man! The always working and joking Marco Stellato!

C: Thank-you Marco for your introduction, but now let's come back to our theme.

The definitions you still see on the slide prevents to grasp the importance of the harmonic motion as a conceptual organizer that should emerge from the choice/recognition of particular deep similarities/diversities among different types of periodic motions.

In the first part of this workshop we'll show you an approach to harmonic motion that allows to realize at a glance the anharmonicity/harmonicity of an oscillation and to understand the link with the mathematical aspects of the problem.

B intanto sgattaiola nel pubblico camminando come gatto Silvestro dove si siede sulla sedia con sopra la borsa.

S: This workshop will let you experience the path we prepared and experimented with 11^{th} and 12^{th} grade Italian students. That's why you have a kit: to be able to put herself in the shoes of our students. At the end you can take it home.

You'll have to change your character. Most of the time you'll be told a story in which we'll pretend you to be a secondary student. But sometimes you'll have to wear back your traditional habits of professors…

In fact, sometimes we'll take our time and make a Comment to discuss with you both the physics and the educational aspects.

- SLIDE: THE EUROPEAN PROJECT TEMI - TEACHING ENQUIRY WITH MYSTERIES INCORPORATED

G: Part of what we are going to show you is embedded in the work of the University of Milan Team working on the TEMI project.

TEMI is a 42-month science education project, funded by the European Commission under FP7 (Science in Society) which will help transform how sciences are taught in classrooms. It has 13 partners from all around Europe.

B fa partire le slide.

C: The work of this workshop is based on the 5E method of Bybee

- SLIDE: 5E

S: Oscillations are a great classic in secondary school. But, while harmonic motion is nearly universally presented, normal modes are not, even if they are at the heart of a lot of physics.

Here we present a sequence that starting from simple oscillations in a meaningful way, leads up to normal modes of oscillation.

PART I: HARMONIC MOTION

SINCRONIZZIAMO E PENDOLO MENTALE

ENGAGE

- SLIDE: ENGAGE

G: Let's start with the first E: Engage.

From now on please switch in the student mode…

S mette in azione il metronomo a 40, suono 2 e tiene vicino il microfono per far sentire il suono a tutti.

- SLIDE: SYNCHRONIZE THE OSCILLATIONS!

C: This is the starting point in our classroom path. So we now act like you were students… I partecipanti vengono invitati a sincronizzare con il metronomo il loro pendolo.

T=1.5s da cui la lunghezza giusta è l= 56 cm. I nostri pendoli hanno un nodino che non si vede, ma si sente al tatto, alla lunghezza giusta.

B, C, G e provano tutti con il loro pendolino uguale a quello che hanno i partecipanti.

C: Io l'ho trovato! E voi?

Andiamo dai workshopper… confrontiamo il nostro pendolo con quello loro… Improvvisazione

C: Ci riprovo!. L'ho trovato di nuovo… Sono una maga? No ho segnato sul mio pendolo la lunghezza giusta! Perché, infatti il periodo del pendolo dipende solo dalla sua lunghezza!

Utilizziamo la dipendenza della frequenza dalla lunghezza per farvi vedere quest'altro esperimento da mago…

S: Pendoli a forza del pensiero. *Improvvisazione*. Lei mi dica: quale pendolo vuole che faccia oscillare con la sola forza del pensiero?... E lei?

Raccomandiamo di effettuare l'esperimento con l'attrezzo alla debita distanza dal corpo perché la terza pallina potrebbe essere particolarmente pericolosa e dolorosa.

C: *Interrompendo S.* Very good Marco. Nice demonstration…

OSCLILLAZIONI VARIE

G: But let's go on, now. We'll show you some different periodic motions. We ask you to try to memorize them well, because we'll need them later.

Where is Sara?... Sara! Where are you. Please come here, you we need you now! *Va a prendere B. B riluttante si alza e va*

Please shed some light on each experiment, use the halogen lamp…

C spegne un'alogena l'alogena, S porge l'altra alogena a S che illumina ogni moto.

- SLIDE: *compaiono, illuminati uno dopo l'altro*
  ALTALENA – BOWL AND BALL – OSCILLATING TRACK – BOUNCING BALL – PHYSICAL PENDULUM – CICLOIDAL PENDULUM – WALTENHOFEN PENDULUM

C: *Con enfasi.* Let's start with the seesaw on a flat pivot. Please Marco…

S: *CUBO* See-saw on a flat large pivot. *… Blee Bla…*

See-saw on a flat small pivot.

C: An now the see-saw on a round pivot

S: *ALTRO CUBO*

C: And now another type of motion that will be run by our collegue specialised in…

B: *CATTEDR*A *lascia l'alogena a S dice pianissimo* the bowl *(sollevandole)* and the ball!

C: Puoi ripetere per favore?

B: *sempre pianissimo* the bowl and the ball.

C. Non ho sentito bene…

B: *Nell'orecchio di C* the bowl and the ball.

C. Ah! *Stentorea* the bowl and the ball!

C: An oscillating track

G: *CUBO oscillating track* Here is that track *mostra il track*

C: And now an experiment that students like very much: the bouncing ball in UV! con torcia UV

S: *Fa l'esperimento. Con torcia UV*.

C: The physical pendulum

G: *CATTEDRA physical pendulum*

C: Cicloidal pendulum

S: *CATTEDRA*

C: Waltenhofen pendulum

G: *CUBO Waltenhofen poco smorzato. Mostra com'è fatto.*

C: Mass-spring oscillator

C: *CATTEDRA*

C: And, finally, a bouncing disk on an air table. We have a video for that. *Fa partire il video*

VIDEO BOUNCING DISK.

And now: all together!

MUSICA GENTLE GIANT Facciamo partire tutti i moti insieme. B va allo sgabello più defilato sul palco.

S: Dear students, we are studying mechanics, and we are looking for a way of describing these oscillations. Obviously they are all different, each one is particular. Nonetheless they have some common points. Try to classify previous motions into 2 or 3 categories, putting together the ones that you believe have particular similarities from a mechanical point of view. And give a motivation of what you have done.

- SLIDE: «DIVIDE THE JUST SEEN OSCILLATIONS INTO 2 OR 3 CATEGORIES»

CATALOGAZIONE STUDENTI

Dopo pochi secondi: Done? *Con un sorriso. B si mette in silenzio e ferma in piedi poggiata al muro.*

Here we show you the answers you gave us…

- SLIDE: STUDENTS ANSWERS

DESCRIVIAMO LE FORZE

EXPLORE

- SLIDE: EXPLORE

C: And now comes another E: we are in the Explore phase

C: *Con gli occhiali, come una maestrina.* Every way of grouping can make sense. But look… you *(indicando un workshopper muto)* have written that a bouncing ball is similar to a bouncing disk because they are both bouncing!... And you *(indicando un altro workshopper muto)* are saying that you would put I the same group the bouncing ball and the see-saw on the large flat pivot because their motion is particularly damped. This shows how difficult it is to look at things from a physical perspective.

- SLIDE: GUIDANCE: «DESCRIBE THE FORCE ACTING IN EACH SYSTEM»

G: Now let's think all together. We are studying mechanics, right? What are the key concept of mechanics? I'd say: forces and motion, so it seems to me probably meaningful and also fruitful to look at the forces that generate previous motions.

Now we'll have a very qualitative approach, typical of an exploration phase.

Let's begin from a very simple motion.

Sara… You're the only one of us able to make a readable drawing, you now that… So, please, pick up the marker and go to the paper board.

Esperimento dal vero + disegno forza su lavagna a fogli. B è alla lavagna a fogli a disegnare le forze in funzione della posizione. Alla fine del disegno stacca il foglio e lo porge a chi le è vicino e non sta facendo l'esperimento che lo posiziona in maniera che sembri casuale, ma che poi serva alla catalogazione che porta al moto armonico.

G: *Performs mass-spring.* This is a mass-spring oscillator: there's an equilibrium position here… when I pull downward, the force acting on the mass pushes upwards. And if I push the mass upward the total force is acting downwards, towards the equilibrium position

Mentre G descrive l'esperimento B disegna in un grafico la stessa cosa, quando G si volta verso B …

We have also a slide.

- SLIDE LOGGER MASS-SPRING x vs t; a vs x.

It shows the position vs time and the acceleration vs position. The data have been taken by a Logger pro sonar system.

C: *Performs Pendolo*.

S: *Performs Moon ball che rimbalza*.

G: *Performs Oscillating circular track*. Consider a little piece of our circle put on the top… The details are very complicated. It's not easy to choose a coordinate, but look… let's choose a vertical axis. Again: there is an equilibrium position and if I push the circle upward the total force is acting downwards, towards the equilibrium position. On the contrary if a pull it downwards the force pushes the circle up. As before with the mass-spring.

- *SLIDE TRACKER GUIDA CIRCOLARE OSCILLANTE*

This is an artistic video made by a very peculiar students of ours…

S: What did students were able to do in our experimentations? Here are some typical plot they made

- SLIDE: DISEGNI TIPICI STUDENTI F(X) E DIFFICOLTA'

S: Commenta la slide con 3-4 disegni degli studenti.

G: *Fa il gesto di suonare il gong e anche il rumore.*

COMMENT PERCHE' COINVOLGERE GLI STUDENTI CON F(X)

- SLIDE: WHY INVOLVE STUDENTS WITH THE DIFFICULT TASK OF FINDING OUT F(X)?

C: Porre l'attenzione sulla forza che agisce in ogni oscillazione: se si pone l'attenzione sulla F(x), cioè sulla forza F in funzione della posizione x, si procede in un modo che è del tutto anomalo per gli studenti, che sanno a mala pena rappresentare le grandezze in funzione del tempo t.

G: It is indeed very difficult for students to draw graphs like the once seen before. But, in our experience, this operation increases students' ability of representing and also of reading graphs. In general, not only previous ones.

Moreover the path allows to recognise the anharmonicity or the harmonicity of a motion even without knowing its equation of motion or its solution. Even without knowing the details of the forces involved. As for the small oscillations on a cycloid or the seesaw on a round pivot.

Making students familiar with the concept of a force depending on position will also make it easier the introduction of the potential energy concept.

SPIEGAZIONE MOTO ARMONICO

EXPLAIN

- SLIDE: EXPLAIN

C: Let's pass to the third E: the Explain

- SLIDE: THE RESTORING FORCE: $F(x) = - kx$

S: *Guardando i grafici appesi al muro*

Now, as for a spell… you become high school students again.

This guided procedure we showed you, allows a new categorization based on the analysis of the forces acting on each oscillator, some oscillations are driven by a restoring force.

Let us suppose that the resulting force on our moving body is a restoring force, that is a force that gives rise to a motion with a stable equilibrium position. Let us also use the curvilinear coordinate *s* to make our description. Let the zero corresponding to the equilibrium position. In this case the graph of the component of the restoring force along the trajectory will lies in the second and in the fourth quadrant.

In this case the restoring force can be approximated by its tangent line in the origin, provided the amplitude of oscillation is small enough.

- SLIDE: $F_s = -ks$; $(k > 0)$

Therefore, every body subjected to a sufficiently regular restoring force, for small amplitude of oscillation, will obey the same equation of motion:

$F=-ks$

from which we immediately arrive to equation

$a=-k/m\ s$

that defines harmonic motion.

From which, with some efforts the isochronism follows.

QUALE E' ARMONICO?

G: Which of the following oscillations is harmonic?

- SLIDE: IS THAT OSCILLATION HARMONIC?

MUSICA DA TENSIONE

S: Domanda: E' armonico o no? Alzate la mano.

Improvvisazione. Moti armonici e non, facili da riconoscere tranne uno…

C: *Performs*. Pendolo cicloidale.

Sara please…

B: *Performs*. The bowl and the ball *poi torna al muro*

G: *Performs*. Slinky

S: *Performs*. Tubo a U con acqua

C: *Performs*. Cicloide e isocronismo

G: Pendolo di Galileo *(si spera che tutti sbaglino).*

G: *Fa il gesto di suonare il gong e anche il rumore.*

COMMENTO

EXTEND

- SLIDE: EXTEND

C: Now comes the moment for the fourth E: Extend.

COMMENT

C: The first goal is to make students able to recognize if a motion is harmonic or not even without knowing the exact expression of the acting forces, but simply by watching the oscillations and sometimes by listening to the sound generated by the oscillations themselves.

Later we will also discuss the role of the damping in relation to the concepts of anharmonicity/harmonicity.

S: Let us consider a one degree of freedom system subject to a restoring force. As already said the s-component of the restoring force, F_s, vs s lies in the second and in the fourth quadrant.

We would like to stress that the harmonic motion defined above is not necessarily rectilinear, as s is a curvilinear coordinate (for instance, the ends of a torsional pendulum describe an arc of circumference performing harmonic oscillations over a wide range of angles). It is also important to emphasize that F_s must not be confused with the intensity of the total acting force, but it is only its component along the direction of motion. This is a conceptual aspect for which particular care is needed in describing motions on curved trajectories. In fact, in these cases, the resultant force is different from zero even in the equilibrium position, because the contribution of the centripetal component has generally to be considered; while, on the contrary F_s is, indeed, null.

In conclusion, the path towards the previous definition leads us to say that the small oscillations of a one degree of freedom system are harmonic if in s=0

- SLIDE: 4-POINT CRITERION
  1. There is a stable equilibrium point
  2. The function F(x) is continuous
  3. The function F(x) is differentiable
  4. $F'(0) \neq 0$

(a) there is a stable equilibrium point;

(b) the function F_s is continuous;

(c) the function F_s is differentiable; (d) d F_s/ds different from 0.

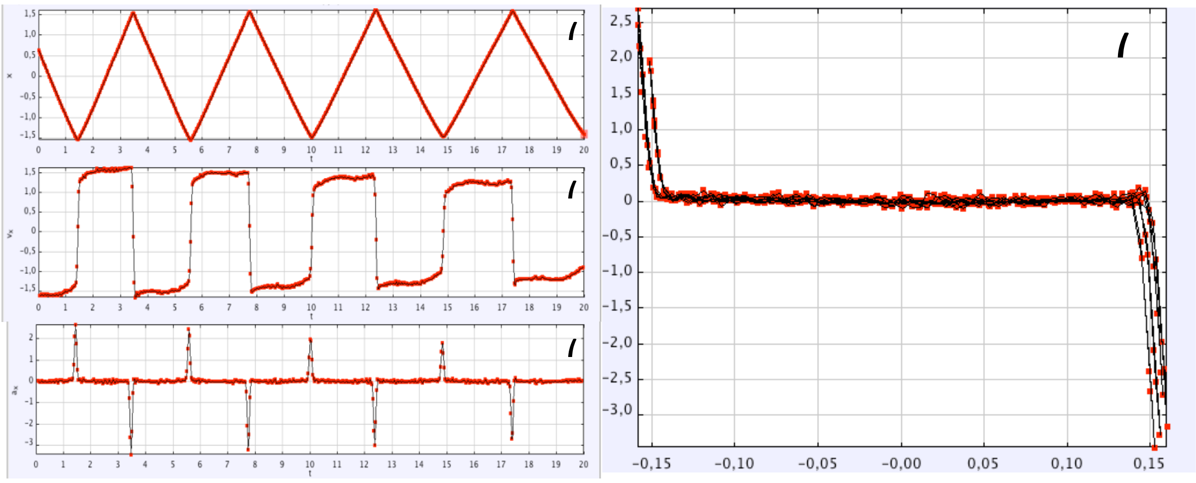
Obviously, condition (c) implies condition (b). Nevertheless we believe that, from a didactical point of view, keeping these conditions separate allows a clearer comprehension of the physics involved.

These point have been written for teachers. For students they must be simplified… That is we can draw one and only one tangent line in zero.

ARMONICO O NO?

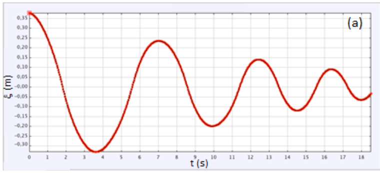
G: And now let’s use our four point criterion to detect harmonicity/an-harmonicity. Is the bouncing disk harmonic or not?

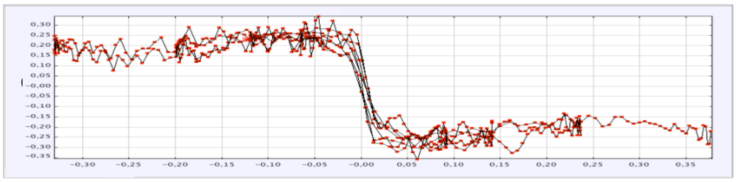
- SLIDE: BOUNCING DISK



G: No, because there is not a stable equilibrium point.

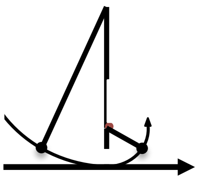
- SLIDE: GALILEO OSCILLATOR

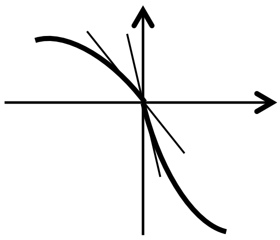




C: F(s) is not continuous

- SLIDE: THE INTERRUPTED PENDULUM





S: *Performs*. Interrupted pendulum.

F(x) is not differentiable.

For what concerns the period we have $T_{12} = T_1/2 + T_2/2$. Therefore the motion is isochronous, nonetheless it is not harmonic. The FFT shows it clearly.

- SLIDE FFT PENDOLO INTERROTTO

MUSICA TARTINI

G: And now a very subtle example: the case when only the condition (d) of our four-point criterion is not satisfied, that is when

F'(s=0)=0

In this situation the potential energy goes as s^4. The small oscillations are as those of a ball on this track with a X^4 profile.

You have your one in the kit back. You can take it out and play with us.

Let's see what happens.

Near the equilibrium point the track is nearly flat and the motion is nearly a uniform motion! Small oscillations are difficult to see.

Nonetheless when the amplitude is enough we can see that the motion is not isochronous.

In fact, as Sara will show you in a moment,

B va a nascondersi sotto la cattedra

if A is the amplitude of oscillation, it can be demonstrated that the period of oscillation T in proportional to 1/A!

Sara?... ok, I'll show you…

B va a nascondersi sotto la cattedra

- SLIDE PAPER X4 CON CALCOLO DEL PERIODO

G: What I've just said is true only if you can distinguish, if your detector can distinguish.

If you can detect the non differentiability of the force vs position then… ok! But if you are not able, as it might be in the case of an interrupted pendulum with very similar length… you wouldn't detect anarmonicity

The same is here: if your x^4 track is indistinguishable from an x^2 track, then the small oscillations will appear to you all harmonic.

All depends on what your eyes can see. And the eyes can see only what the mind is prepared to accept…

It has been probably a fortune that Galileo had not a precise chronometer otherwise probably he would never be able discover the isochronism of the pendulum…

Most of physics is a delicate balance of seeing in depth and of seeing from a large perspective…

DAMPING

G: *Fa il gesto di suonare il gong e anche il rumore.*

COMMENTO

COMMENT

C: And now something about damping. In fact maybe you remember that some students divided the motions in different categories depending and how evident the damping was…

- SLIDE: DAMPING

C: In additions to the conditions previously discussed, a useful way to recognize anharmonicity is to find an amplitude dependence of the period of oscillation. Since real motions are always damped, we have to exclude that this dependence comes from damping, instead of being due to an intrinsic anharmonicity.

C: Pendolo di Waltenhofen descrivere lo smorzamento

DISCO DI EULERO

G: And now the Euler disk…!

RISULTATI SPERIMENTAZIONE

EVALUATE

- SLIDE: EVALUATE

S: Finally the last E: the Evaluate phase

- SLIDE: RISULTATI SPERIMENTAZIONE TEST FINALE SULLA CLASSIFICAZIONE DEI MOTI.

Parte II: MODI NORMALI

PENDOLI MULTIPLI E PENDOLI ACCOPPIATI

ENGAGE

C: Now we can start with a synthesis of a new complete cycle of Es. Let’s begin with the Engage.

- SLIDE: ENGAGE

C: Let’s have a look at this different kind of structures.

S: Sara? Where are you? Sara, do you remember? We need you, please.

Come out of the table! What are you doing there?!

B esce da sotto la cattedra.

Vengono fatte oscillare due strutture di pendoli

MUSICA TARTINI

S: 3 pendoli accoppiati

B: Pendoli multipli

C: Con n oscillatori emergono comunque moti collettivi, ma c’è differenza?

ISOLIAMO DUE PENDOLI

EXPLORE

G: Let’s Explore in more details

- SLIDE: EXPLORE

In un caso c'è accoppiamento nell'altro no. Per far vedere questo un oscillatore deve essere evidenziato (con plastilina fluorescente in UV). Così si vede che sono diversi.

B performs prima pendoli multipli poi pendoli accoppiati

- SLIDE PRESA DATI E FFT.

G: let's concentrate on just one oscillator at a time. We'll do this by marking one oscillator with fluorescent plastilina.

You see? When the pendulums are not coupled the oscillation is harmonic. While in the other it is not.

Why? Because in the latter case there is not a fixed equilibrium point. In fact we have an equilibrium point when the spring is not stretched. And, during the motion, this happens in this position,… or this other position and so on…

2, 3 N MODI NORMALI SU PENDOLI E MASSE-MOLLE

S: E' possibile il moto armonico quando gli oscillatori sono accoppiati? Vediamo

Si mettono in oscillazione prima due pendoli poi tre… i tre pendoli nel primo e nel secondo modo normale. Così si muovono di moto armonico. Una presa dati mostra che questi moti particolari avvengono a frequenza fissata, indipendentemente dall'ampiezza dell'oscillazione.

Sono le uniche due possibilità? *Al pubblico* voi che ne dite?

G: Marco! Try to study this thing in a more systematic way!

B: 4 masse-molla *gradualmente aumenta la frequenza.*

S: Andiamo a guardare se la cosa vale anche per altri sistemi di oscillatori accoppiati: il sistema di 4 masse accoppiate con molle identiche. Se queste oscillazioni particolari avvengono a frequenze fissate ci chiediamo cosa avviene sollecitando dall'esterno il sistema con frequenza variabile. Prima di farlo proviamo a chiedere al pubblico se si aspetta di osservare configurazioni di moto particolari del tipo di quelle viste con i pendoli, quante possono essere e come sono fatte.

Masse molle. Qual è il terzo modo normale o il quarto ecc.

Primo modo è semplice e il secondo qual è? qual è il terzo? Difficile vero? Fra poco capiremo come fare

C: Gioco su come sono i modi più alti con il pubblico

Come sono? Lo vedremo dopo…

EXPLAIN DEI MODI NORMALI

EXPLAIN

S: Sistema per risolvere i modi normali 2X2.

Sistema ortonormale

Disegnino

C: VIDEO Enrico e trackeraggio

Come si muove il centro di massa? *Chiede al pubblico. Improvvisazione*

G: *Shive machine*

S: *Corda*

Spiegazione di come trovare l'n-esimo modo normale

MULTIRAPPRESENTAZIONI

COMMENT

S: Importanza delle multi-rappresentazioni: grafiche, algebriche, sperimentali, iconiche, ecc.

PIASTRE DI CHLADNI

S: Piastre di Chladni

MODI ACCOPPIATI

EXTEND

Modi accoppiati,

G: guida circolare con pallina

PERCHE' PARLARE DI MODI NORMALI NELLA SCUOLA

COMMENT

G: *comincia piano poi delira…*

I wander: can you really have a happy life without normal modes? I believe no, you cannot. Every time I look at things I cannot help but seeing a large amount of normal modes all in action.

They are in the small ripples of the water in the harbour.

They are in motion of the curtain cord.

They are in every sound we listen, in every music.

They are at the heart of quantum physics: what but normal modes of vector potential are the photons?!

And maybe that's why our universe has chosen the basis of normal modes in refracting prisms that produce the rainbow.

The Klein Gordon equation contains a harmonic term…

What, if not an expansion in a Fourier series is the Ptolemaic system

tutto è combinazione di (pochi) modi normali

Marco G: la natura sceglie la base dei modi normali (un prisma li sceglie). Il sistema Tolemaico è uno sviluppo in serie di Fourier. L'equazione di KG contiene un termine armonico. La fisica nasce con l'acustica, nutria dei modi normali. I modi normali sono moti armonici speciali nello spazio delle configurazioni.

Un possibile spunto teatrale: scena del pendolo di 1m che oscilla e il suo collegamento con l'accelerazione di gravità g (pigreco^2=g)

C: va a bloccare G…

Va bene è proprio ora, siamo giunti alla fine del nostro percorso

MUSICA BOOGIE

C, G, S cominciano a mettersi in posizione di saluto. B comincia a ballare…

Saluti e ringraziamenti

Bibliography

[tor,]

[PLS,] Piano lauree scientifiche at www.progettolaureescientifiche.eu.

[tra,] Tracker software and reference manual retrieved at https://www.cabrillo.edu/ dbrown/tracker/.

[Fis,] www.fisicaondemusica.unimore.it.

[Log,] www.vernier.com/products/software/lp, logger pro software and reference manual.

[Mos, 1996] (1996). *The Harmonic Oscillator in Moder Physics*. Netherlands: Harwood.

[Edr, 2003] (2003). *Educational Researcher*, 32(1).

[DBR, 2003] (2003). Dbr collective.

[Cab, 2014] (2014). Cabrillo university home page at https://www.cabrillo.edu.

[Dou, 2014] (2014). Douglas brown homepage at http://www.cabrillo.edu/ dbrown/.

[Ope, 2014] (2014). http://www.opensourcephysics.org/webdocs/tools.cfm?t=tracker.

[det, 2014] (2014). http://www.vernier.com/files/manuals/md-btd.pdf.

[PIS, 2014] (2014). Pisa tests at http://www.oecd.org/pisa/.

[TIM, 2014] (2014). Trends in international mathematics and science study at http://timss.bc.edu.

[YOU, 2014] (2014). youtube videos at http://www.youtube.com.

[Abd-El-Khalick & Lederman, 2000] Abd-El-Khalick, F. & Lederman, N., G. (2000). Improving science teachers' conceptions of nature of science: A criticla review of the literature. *International Journal of Science Education*, 22(7), 665–702.

[Amaldi, 2005] Amaldi, U. (2005). *fisica: idee ed esperimenti*, volume 1. Bologna: Zanichelli.

[Amaldi, 2011] Amaldi, U. (2011). *La fisica di Amaldi*, volume 1. Bologna: Zanichelli.

[Armstrong, 1969] Armstrong, H., L. (1969). The oscillating spring and weight - an experiment often misinterpreted. *Am. J. Phys.*, 37, 447–449.

[Arnold, 1979] Arnold, V., I. (1979). *Metodi Matematici della Meccanica Classica*. Roma: Editori Riuniti.

[Barbieri & Giliberti, 2012] Barbieri, S., R. & Giliberti, M. (2012). *Laboratorio aperto: Oscillazioni e Onde*. Milano: CUSL.

[Barrat & Strobel, 1981] Barrat, C. & Strobel, G., L. (1981). Sliding friction and the harmonic oscillator. *American Journal of Physics*, 49, 500–1.

[Battaglia, 2011] Battaglia, O., R. (2011). *Sperimentazione di una proposta didattica sulla distribuzione di Maxwell e prospettive di ricerca sul fattore di Boltzmann*. PhD thesis, Storia e Didattica delle Matematiche della Fisica e della Chimica, Palermo.

[Bergamaschini et al., 2007] Bergamaschini, M., E., Marazzini, P., & Mazzoni, L. (2007). *L'indagine del mondo fisico*, volume B. Milano: Carlo Signorelli editore.

[Bergomi & et al, 1997] Bergomi, N. & et al (1997). Teaching mechanical oscillations using an integrated curriculum. *International Journal of Science Education*, 19, 981–95.

[Boscolo & Lowensteim, 2011] Boscolo, I. & Lowensteim, R. (2011). The spring-mass experiment as a step from oscillations to waves: mass and friction issues and their approaches. *Lat. Am. J. Phys. Educ.*, 5, 409–417.

[Brown, 1992] Brown, A., L. (1992). Design experiments: Theoretical and methodological challanges in creating complex interventions in classroom settings. *The Journal of the Learning sciences*, 2(2), 141–178.

[Cacciamani, 2008] Cacciamani, S. (2008). *Imparare cooperando. Dal cooperative Learning alle comunità di ricerca*. Roma: Carocci.

[Caforio & Ferilli, 2006] Caforio, A. & Ferilli, A. (2006). *Nuova Physica 2000*, volume 1. Firenze: Le Monnier.

[Carpineti et al., 2006] Carpineti, M., Cavallini, G., Giliberti, M., Ludwig, N., Mazza, C., & Perini, L. (2006). Let's throw light on matter: a physics show for primary school. *Nuovo cimento della società italiana di fisica B*, 121(8), 901–911.

[Carpineti et al., 2011] Carpineti, M., Cavinato, M., Giliberti, M., Ludwig, N., & Perini, L. (2011). Theatre to motivatethe study of physics. *JCOM*, 10(1), 1–10.

[Cayton, 1977] Cayton, T., E. (1977). The laboratory spring-mass oscillator: an example of parametric instability. *Am. J. Phys.*, 45(8), 723–732.

[Christensen, 2004] Christensen, J. (2004). An improved calculation of the mass for the resonant spring pendulum. *Am. J. Phys.*, 72, 818–828.

[Cialdi et al., 2011] Cialdi, S., Boscolo, I., Castelli, F., Villa, F., Ferrari, G., & Giammarchi, M., G. (2011). Efficient two-step positronium laser excitation to rydberg levels. *Nucl. Instr. Meth. Phys. Res. B*, 269, 1527–1533.

[Clarke et al., 2006] Clarke, J., Dede, C., Ketelhut, D., J., & Nelson, B. (2006). A design-based research strategy to promote scalability for educational innovations. *Educational Technology*, 46(3), 27–36.

[Cobb et al., 2003] Cobb, P., Confrey, J., DiSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher*, 32(1), 9–13.

[Cohen et al., 2007] Cohen, L., Manion, L., & Morrison, K. (2007). *Research Methods in Education*. New York NY: Routledge, 6th edition.

[Collins, 1992] Collins, A. (1992). *Towards a Design Science of Education*, (pp. 15–22). New directions in Educational Technology. Springer: Berlin.

[Collins et al., 2004] Collins, A., Joseph, D., & Bielaczyc, K. (2004). Design research: Theoretical and methodological issues. *The Journal of the Learning Sciences*, 13(1), 15–42.

[Crowford, 1968] Crowford, F., S. J. (1968). *Waves*, volume 3 of *Berkeley Physics Course*, chapter 1-3. McGrow-Hill: New York NY.

[Cushing, 1984] Cushing, J., T. (1984). The spring-mass system revisited. *Am. J. Phys.*, 52, 925–932.

[della fisica,] della fisica, S. Available at http://spettacolo.fisica.unimi.it.

[Driver & Easly, 1978] Driver, R. & Easly, J., A. (1978). Pupils and paradigms: A review of literature related to concept development in adolescent science students. *Studies in Science Education*, (5), 61–84.

[Driver & Erickson, 1983] Driver, R. & Erickson, G., L. (1983). Theories-in-action: Some theoretical and empirical issues in the study of students' conceptual frameworks in science. *Studies in science education*, (10), 37–60.

[Duit, 2006] Duit, R. (2006). : Braga: ESERA summerschool.

[Duit, 2007] Duit, R. (2007). Science education research internationally: Conceptions, research methods, domaians of research. *Eurasia Journal of Mathematics, Science and Technology Education*, 3(1).

[Duit et al., 2005] Duit, R., Gropengieber, H., & Kattmann, U. (2005). *Towards science educationresearch that is relevant for improving prac tice: The Model of Educational Reconstruction*. Taylor and Francis: London, developing standards in research on science education edition edition.

[Duit et al., 2012] Duit, R., Gropengieber, H., Kattmann, U., Komorek, M., & Parchmann, I. (2012). *Science Education Research and Practice in Europe- Retrosspective and Prospective*, volume 5 of *Cultural perspectives in Science Education*, chapter 1, (pp. 13–37). SensePublisher.

[Duit & Treagust, 1998] Duit, R. & Treagust, D. (1998). *Learning in science: From behaviourisms towards social constructivism and beyond*, volume Inrenational Handbook of Science Education, (pp. 3–25). Kluwer Academic Publisher: Dordrecht, The Nederland.

[Duit & Treagust, 2003] Duit, R. & Treagust, D. (2003). Conceptual change: a powerful framework for improving science teaching and learning. *International Journal of Science Education*, 25(6), 671–688.

[Edwards & Hultsch, 1972] Edwards, T., W. & Hultsch, R., A. (1972). Mass distribution and frequencys of a vertical spring. *Am. J. Phys.*, 40, 445–449.

[Falstad, 2014] Falstad, P. (2014). www.falstad.com.

[Fazio & Montano, 2000] Fazio, M. & Montano, M., C. (2000). *Fisica per licei scientifici*, volume 1. Milano: Arnoldo Mondadori scuola.

[Fensham, 2001] Fensham, P. (2001). *Science content as problematic-Issues for research*, volume Research in science education: Past, Present and Future. Kluwer Academic Publisher: Dordrecht, The Nederland.

[Feynmann et al., 1964] Feynmann, R., P., Leighton, R., B., & Sands, M. (1964). *The Fynemann Lectures on Physics I*. New York, NY: Basic Books.

[Fitzpatrick, 2013] Fitzpatrick, R. (2013). *Oscillations and Waves: An introduction*. Boca Raton, FL: Taylor and Francis.

[Galloni & Kohen, 1979] Galloni, E., E. & Kohen, M. (1979). Influence of the mass of the spring on its static and dynamic effects. *Am. J. Phys.*, 47, 1076–1078.

[Geballe, 1958] Geballe, R. (1958). Statics and dynamics of a helical spring. *Am. J. Phys.*, 26, 287–290.

[Gil & Di Gregorio, 2003] Gil, S. & Di Gregorio, D. E. (2003). Nonisochronism in the interrupted pendulum. *American Journal of Physics*, 71, 1115–20 and references therein.

[Giliberti, 2007] Giliberti, M. (2007). *Elementi per una didattica della Fisica Quantistica*. Materiali didattici dell'indirizzo Fisico, Informatico e Matematico. Milano: CUSL.

[Giliberti et al., 2014] Giliberti, M., Stellato, M., Barbieri, S., Cavinato, M., Rigon, E., & Tamborini, M. (2014). Detecting anharmonicity at a glance. *European Journal of Physics*, 35(6).

[Gonzales-Buelga et al., 2007] Gonzales-Buelga, A., Wagg, D., J., & Neild, S., A. (2007). Parametric variation of a coupled pendulum-oscillator system using real time dynamic substructuring. *Struct. Control Health Monit.*, 14, 991–1012.

[Halliday et al., 1992] Halliday, D., Resnick, R., & Krane, K., S. (1992). *Physics*, volume 1, (pp. 336 problem n.25). John Wiley andSons: New York NY, 4th edition.

[Holzwarth & Malone, 2000] Holzwarth, D., E. & Malone, J. (2000). Pendulum period versus hanging-spring period. *Phys. Teach.*, 38, 47.

[Jenkins, 2001] Jenkins, E. (2001). *Research in science education in Europe: Retrospest and prospect*, volume Research in science education: Past, Present and Future. Kluwer Academic Publisher: Dordrecht, The Nederland.

[Kattmann et al., 1995] Kattmann, U., Duit, R., Gropengieber, H., & Komorek, M. (1995). : San Francisco Ca.: Narst annual meeting.

[Kvale, 1996a] Kvale, S. (1996a). *Interviews*, chapter 11. Sage.

[Kvale, 1996b] Kvale, S. (1996b). *Interviews*, chapter 14. Sage.

[Lai, 1984] Lai, H., M. (1984). On the recurrence of a resonant spring pendulum. *Am. J. Phys.*, 52, 219–223.

[Laing, 1967] Laing, R., D. (1967). *The Politics of Experience and the Birth of Paradise*. Penguin.

[Ligorio & Cacciamani, 2013] Ligorio, M., B. & Cacciamani, S. (2013). *Psicologia dell'educazione*. Roma: Carocci Editore, first edition.

[Martinez et al., 2010] Martinez, E., Carbonell, V., Flores, M., & Maya, J. (2010). Simulations as a new physics tool. *Computer Applications in Engineer Education*, 18, 757–761.

[McDermott et al., 1987] McDermott, L., C., Rosenquist, M., L., & VanZee, E., H. (1987). Students difficulties in connecting graphs and physics: Examples from kinematics. *Am. J. Phys.*, 55, 503–513.

[Mencuccini & Silvestrini, 2013] Mencuccini, C. & Silvestrini, V. (2013). *Fisica 1. Meccanica Termodinamica. Corso di fisica per le facoltà scientifiche. Con esempi ed esercizi*, volume 1 of *Fisica*. Napoli: Liguori Editore.

[Morin,] Morin, D. normal modes available at www.people.fas.harvard.edu/.../normalmodes.pdf.

[N., 1962] N., M. (1962). *Nonlinear Oscillations*. Princeton, New Jersey: D. Van Nostrand Company, Inc.

[Olsson, 1976] Olsson, M., G. (1976). Why does a mass on a spring sontimes misbehave? *Am. J. Phys.*, 44, 1211–1212.

[Onorato et al., 2013] Onorato, P., Mascheretti, P., & De Ambrosis, A. (2013). Studying the motion along cycloidal paths by means of digital video analysis. *European Journal of Physics*, 34, 921–30.

[Onorato et al.,] Onorato, P., Mascoli, D., & De Ambrosis, A. Damped oscillations and equilibrium in a mass-spring system subject to sliding friction forces: Integrating experimental and theoretical analyses. *American Journal of Physics*, 78, 1120–7.

[Papucci, 2008] Papucci, S. (2008). *Principi di Fisica*, volume 1. Milano: Hoepli.

[Pecori & Torzo, 2001] Pecori, B. & Torzo, G. (2001). Physics of the seesaw. *Phys. Teach.*, 39, 25–29.

[Phillips, 2000] Phillips, D., C. (2000). *Constructivism in education: Opinions and second opinions on controversial issues.* Chicago, IL: The University of Chicago Press.

[Simon, 1977] Simon, H., A. (1977). *Models of discovery and other topics in the method of science.* Dordrecht, The Nederlands: D. Reidel.

[Smith, 2010] Smith, W. (2010). *Waves and Oscillations-a prelude to quantum mechanics.* New York: Oxford University Press.

[Squire, 2005] Squire, K. (2005). What happens when video games enter the classroom? *Innovative journal of Online Education*, 1(6).

[Stellato et al., 2014a] Stellato, M., Giliberti, M., Rigon, E., & Tamborini, M. (2014a). A guided inquiry based teaching and learning sequence on oscillations based on experiments and data-logging techniques. In L. Dvorak & V. Koudelkova (Eds.), *ICPE-EPEC 2013 - The International Conference on Physics Education - Conference Proceedings* (pp. 405–416). Prague: ICPE-EPEC Matfiz Press.

[Stellato et al., 2014b] Stellato, M., Giliberti, M., Rigon, E., & Tamborini, M. (2014b). A high school teaching and learning sequence on normal modes. In M. F. Tasar (Ed.), *Proceedings of the World Conference on Physics Education WCPE2012* (pp. 695–704). Ankara, Turkey: Pegem Akademi.

[Svelto, 2010] Svelto, O. (2010). *Principles of Lasers.* Springer-Verlag, fifth edition.

[TEMI, 2014] TEMI (2014). Available at http://teachingmysteries.eu/en.

[Torzo, 2014] Torzo, G. (2014). http://www.labtrek.it/torzo/oliphys/8/galil8.htm.

[Wang & Hannafin, 2005] Wang, F. & Hannafin, M., J. (2005). Design-based research and technology-enhanced learning environments. *Educational Technology Research and Development*, 53(4), 5–23.

[Widodo, 2004] Widodo, A. (2004). *Constructivistic Oriented Lessons: The learning environments and the teaching sequences.* Peter Lang.

[Yariv, 1988] Yariv, A. (1988). *Quantum Electronics*, (pp. 411). John Wiley and Sons: New York NY, 3rd edition.