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Maybe I would not have written this thesis if my uncle Paride in the faraway 1994 had not enthusiastically tried to explain to an eight-year-old girl some special relativity and its paradoxes.

Abstract

Superconformal field theories have been widely explored over the last years, especially because of their large amount of symmetry which allows to derive exact results. The most prominent example within this class is the four-dimensional maximally supersymmetric $\mathcal{N} = 4$ Super Yang-Mills theory, which played a pivotal role in the proposal of the AdS/CFT correspondence, a conjectured weak/strong duality relating superconformal gauge theories to string theories on curved backgrounds. Soon after the formulation of the correspondence the presence of integrable structures was discovered both in planar $\mathcal{N} = 4$ SYM and in its string counterpart, and since then integrability in superconformal theories became one of the main research topics in theoretical physics. Interestingly, integrability in $\mathcal{N} = 4$ was later shown to be intimately related to the presence of the so called dual conformal symmetry, a hidden symmetry of the planar amplitudes that puts even stronger constraints on their structure. This suggests that some crucial aspects of a theory can be investigated through the computation of its scattering amplitudes, which can thereby provide a powerful tool also for the study of less supersymmetric superconformal field theories. In particular it is essential to understand whether some of the beautiful properties of $\mathcal{N} = 4$ survives when supersymmetry is not maximal.

In this thesis we present computations of massless scattering amplitudes in two different not maximally supersymmetric conformal theories: $\mathcal{N} = 2$ superconformal QCD in four dimensions and $\mathcal{N} = 6$ Chern-Simons matter theory (ABJM) in three dimensions. In $\mathcal{N} = 2$ SCQCD we compute all possible four-point amplitudes at one loop and the two-loop amplitude with fundamental fields as external legs. In ABJM we extend the two-loop computation of four-point scattering amplitudes and of the Sudakov form factor beyond the planar limit. We also discuss our results in relation to the corresponding ones in $\mathcal{N} = 4$ SYM, paying particular attention to the possible presence of dual conformal invariance and to maximum transcendentality.

Contents

Introduction

The aesthetic appeal of symmetry has been a precious guide for the development of modern theoretical physics. In particular twentieth-century physics has witnessed the triumph of symmetry and its precise formulation in theoretical language. The principles of special and general relativity were indeed motivated by the appeal of symmetry and the modern theory of elementary particles, the so-called standard model, is based on the concept of local gauge symmetry. The understanding of phase transitions and critical phenomena draws a great deal on the concept of broken symmetry. In particular broken gauge symmetries are central for our understanding of weak interactions, superconductivity and cosmology.

Conformal symmetry and supersymmetry played an important role in theoretical physics since the early 1970s. Both of these symmetries emerged for the first time in the context of string theory. Since its first appearance string theory turned out to be a useful tool in the analysis of strongly coupled systems: in those years physicist were trying to understand the strong force and string theory was formulated as a theory of hadrons. Later QCD was discovered and string theory was abandoned as a theory of strong interactions, becoming, on the other hand, the most promising candidate for a quantum theory of the gravitational interaction. Unlike other proposed theories of quantum gravity, it also offered a unified description of all fundamental interactions. To reach the goal of quantizing gravity, string theory gives up the concept of point-like particles as the elementary constituents of matter and it introduces extended objects, such as strings and branes. String theory requires the existence of extra spatial dimensions to the four known, but their number cannot be arbitrary. In fact, in order to have a quantum model without conformal anomalies, the theory on the worldsheet must be a conformal field theory living in ten dimensions. Moreover string theory needs supersymmetry in order to avoid inconsistencies such as the presence of tachyons and ghosts in the spectrum.

Conformal invariance is an immediate extension of scale invariance, which is the symmetry under dilatations of space. Conformal transformations are nothing but dilations by a scaling factor that is a function of position, namely a local dilatation. Though conformal symmetry cannot be an exact symmetry of gauge theories which describe interactions of elementary particles, it plays a central role in string theory. Conformal invariance is relevant also in other areas of research, as in statistical mechanics or in cosmology, in particular for the description of phase transitions and critical points.

Supersymmetry first appeared in the context of string theory, and at that time was considered mainly as a purely theoretical tool. Shortly after it was realized that supersymmetry can be a symmetry of quantum field theory, thus relevant to elementary particle physics. There are several reasons to consider supersymmetric theories suitable for describing elementary particle physics. The main one is that radiative corrections tend to be less important in supersymmetric theories, due to cancellations between fermionic and bosonic corrections. As a result it would be possible to solve theoretical puzzles, as the hierarchy problem, namely the existence of a big gap between the Planck scale and the scale of electroweak symmetry breaking, or the issue of renormalization of quantum gravity. Although supersymmetry could solve these problems, it cannot be the full answer since it cannot be exactly realized in nature: it must be broken at experimentally accessible energies since otherwise one certainly would have detected many of the additional particles that it predicts. On the other hand supersymmetric models are often easier to solve than non-supersymmetric ones since they are more constrained by the higher degree of symmetry. Thus they can be used as toy models where certain analytic results can be obtained and as a qualitative guide to the behaviour of more realistic theories. For example the study of supersymmetric versions of QCD has given some insights in the strong coupling dynamics, responsible for phenomena like quark confinement. Supersymmetry has been considered also outside the realm of elementary particle physics and it has found applications in condensed matter systems, in particular in the study of disordered systems.

Superconformal field theories are field theories which manifest conformal symmetry and supersymmetry as well. The most prominent example of such theories is the four-dimensional maximally supersymmetric $\mathcal{N} = 4$ Super Yang-Mills (SYM) theory, which has been widely studied in the last years. One of the main reasons is its role in the proposal of the AdS/CFT correspondence, which claims the equivalence between type IIB superstrings on the $AdS_5 \times S^5$ background and $\mathcal{N} = 4$ Super Yang-Mills theory. This duality is very interesting because it relates opposite regimes of the two theories, so that the strong coupling behaviour of one of the two can be studied by performing a simpler perturbative analysis on the opposite side of the correspondence. In particular, the strong coupling regime of $\mathcal{N} = 4$ SYM corresponds to the supergravity limit of the string theory living in a AdS space. The drawback of this strong/weak nature of the duality is

that it makes the conjecture very difficult to prove, because in general only perturbative computations are simple enough to deal with. Up to now, several strong tests have been performed, all confirming the validity of the correspondence, but a rigorous proof is still missing. The original formulation of the AdS/CFT correspondence has been extended to a more general duality relating any conformal field theory in d dimensions to a gravity theory on an Anti de Sitter space in $d+1$ dimensions.
Another relevant reason for being interested in superconformal field theories is that, soon after the formulation of the AdS/CFT correspondence, it was discovered the presence of integrable structures both in the planar $\mathcal{N}=4$ SYM theory and in its string counterpart. Later the search for integrable models was extended to other superconformal theories.

The scattering amplitudes subfield of particle physics occupies the space between collider phenomenology and string theory, thus making some mathematical insights from purely theoretical studies useful for the physics accessible by experiments. From the theoretical point of view scattering amplitudes are excellent objects to investigate since in many cases they can be computed analytically and manifest remarkable properties. In particular the study of scattering amplitudes in superconformal field theories has recently unveiled the existence of hidden symmetries and unexpected properties. Once again $\mathcal{N}=4$ Super Yang-Mills theory played a pivotal role and turned out to be the perfect playground to provide important insights into quantum field theory. One of the most surprising novelty compared to the non supersymmetric case is that planar scattering amplitudes of $\mathcal{N}=4$ SYM theory enjoy an additional dynamical symmetry, which is not present in the Lagrangian formulation and which constrains the form of the amplitudes to be much simpler than a naive analysis might suggest. This hidden symmetry, called dual conformal invariance, can be related to a duality between planar amplitudes and Wilson loops, which are important observables in a gauge theory, and was first suggested in the strong coupling string description. Dual conformal symmetry is also related to the integrability features of $\mathcal{N}=4$ SYM, thus being an important symmetry to be investigated in other models. The general structure of infrared divergences arising in the loop corrections to the massless amplitudes is well understood for a generic Yang-Mills theory, since it was discovered that soft and collinear divergences have a universal form, and the divergent part of higher loop amplitudes can be predicted from lower loop ones. Furthermore, as a consequence of dual conformal symmetry, the loop corrections to the four and five-point amplitudes were shown to be completely fixed in a form that can be recovered from the one-loop amplitude. This is the so-called exponential ABDK/BDS ansatz. Starting from six external particles, dual conformal invariance constrains the amplitudes only up to an undetermined function of the conformal cross ratios which violates the ABDK/BDS exponentiation. Nevertheless the duality with Wilson loops was shown to be preserved. One more remarkable property of $\mathcal{N} = 4$ SYM amplitudes, with an unclear origin, is that they exhibit uniform and maximal transcendentality weight. This means that the quantum corrections of the amplitudes, evaluated in the dimensional reduction scheme, have coefficients which are related in a precise way to the Riemann zeta function. This maximal transcendentality property was first observed for the anomalous dimension of twist-2 operators and then it was found that it is surprisingly enjoyed by all the known observables of the theory. While scattering amplitudes in $\mathcal{N} = 4$ SYM have been widely studied in past years, scattering amplitudes in other superconformal theories have received less attention.

The aim of this thesis is the investigation of scattering amplitudes in superconformal theories with less amount of supersymmetry compared to the maximal supersymmetric $\mathcal{N} = 4$ Super Yang-Mills theory. In particular we will present computations of scattering amplitudes in the four-dimensional $\mathcal{N} = 2$ superconformal QCD theory and in the three-dimensional $\mathcal{N} = 6$ Chern-Simons matter theory (ABJM).

The $\mathcal{N} = 2$ superconformal QCD theory is a $\mathcal{N} = 2$ supersymmetric Yang-Mills theory with gauge group $SU(N)$ coupled to $N_f = 2N$ hypermultiplets in the fundamental representation of the gauge group. Several aspects of this theory have been studied in the last years, in particular the ones concerning integrability. After some promising hints of integrability, it was eventually demonstrated that the theory is not an integrable model. However the possibility that the closed subsector $SU(2,1|2)$ built only with adjoint fields is integrable still remains open. An indirect way to test the appearance of integrable structures is to study the on-shell observables of the theory. In this context the expectation values of Wilson loops at weak coupling were studied, showing that any closed Wilson loop starts deviating from the corresponding $\mathcal{N} = 4$ SYM results at three-loop order. Scattering amplitudes have been less analyzed. The only known result concerns the one-loop scattering amplitude with adjoint fields as external particles. In this case it was found that the result matches that of $\mathcal{N} = 4$ SYM and thus consists of a dual conformal invariant and maximal transcendental expression. Nothing was known about amplitudes in more general sectors of the theory and at higher-loop order.

For this reason in this thesis we analyze scattering amplitudes in $\mathcal{N} = 2$ SCQCD with also fundamental particles as external fields and up to two loops. More precisely we compute one-loop and two-loop four-point scattering amplitudes in the planar limit, as presented in the published paper [1]. We work in $\mathcal{N} = 1$ superspace formalism and perform direct super Feynman diagram computations within dimensional reduction scheme. At one-loop order we provide a complete classification of the amplitudes, which can be divided in three independent sectors according to the color representation of the external particles. The pure adjoint sector consists of amplitudes with external fields belonging to the $\mathcal{N} = 2$ vector multiplet. In this sector we confirm the results of the previous work, since we obtain exactly the same expressions of the corresponding

$\mathcal{N} = 4$ SYM amplitudes, demonstrating the presence of dual conformal symmetry and maximal transcendentality. This agrees with the conjectured integrability of the closed subsector $SU(2,1|2)$. Outside the adjoint sector there is no reason to expect amplitudes to be dual conformal invariant. We show that in the mixed and fundamental sectors, with external fields in the fundamental representation, even if dual conformal invariance is broken, the results still exhibit maximal transcendentality weight. In order to check these properties beyond the one-loop perturbative order, we compute the simplest two-loop amplitude in the fundamental sector. We end up with a result that does not exhibit maximal transcendentality and is not dual conformal invariant. A very non trivial check of our two-loop result is the fact that it reproduces the expected factorized structure of the infrared divergences predicted for the general scattering of massless particles.

We carry on with the investigation of scattering amplitudes in superconformal theories studying the properties of scattering amplitudes in another superconformal model, which is formulated in three spacetime dimensions. This is the three-dimensional $\mathcal{N} = 6$ ABJM theory, which is a $\mathcal{N} = 6$ superconformal Chern-Simons theory with gauge group $U(M)_K \times U(N)_{-K}$ coupled to matter. In the past few years much progress has been achieved in the perturbative analysis of the ABJM model. A relevant result concerns the emergence of an integrable structure, which looks surprisingly similar to the $\mathcal{N} = 4$ SYM one. Again, an indirect way of testing integrability comes from the study of on-shell scatterng amplitudes and Wilson loops. In this context a lot of results have been recently carried out. First, the tree-level four- and six-point amplitudes have been found explicitly and Yangian invariance has been established for all point amplitudes. At loop level explicit computations are available for the four- and six-point case up to two loops. The one-loop four-point amplitude vanishes in the dimensional regularization scheme, while at two loops the planar amplitude can be written as a sum of dual conformal invariant integrals and it has been found to coincide with the expectation value of a four polygon Wilson loop. This points towards the fact that a Wilson loop/scattering amplitude duality might exist, even if the interpretation of the duality is less straightforward with respect to the four dimensional case. Since in the ABJM model all the odd legs amplitudes are forced to vanish by gauge invariance, the next relevant n-point amplitude is the six-point one, which has been shown not to vanish at one loop, contrary to the hexagon Wilson loop. This suggests that if a Wilson loop/scattering amplitude duality exists it must be implemented with a proper definition of Wilson loop. At two loops the six-point amplitude has been computed analytically, showing some similarity with the one-loop amplitude in $\mathcal{N} = 4$ SYM, even if the identification experienced for the four-point amplitude gets spoiled. There is another class of interesting objects - the form factors - which offer a bridge between correlation functions and scattering amplitudes. They have been widely studied in $\mathcal{N} = 4$ SYM, but recently an analysis of the form factors has been initiated also for the ABJM model, where computations have been performed through unitarity cuts and component Feynman diagrams formalism.

All remarkable properties detailed above have been found to hold in the large N limit of ABJM, which seems to be the regime where interaction simplifies in such a way that dualities and integrability can occur. Nevertheless, it is interesting to look at what happens to the subleading corrections of Wilson loops, amplitudes and form factors. For scattering amplitudes in four dimensions their complete evaluation including subleading partial amplitudes is constrained by underlying relations. These relations, called BCJ, hinge on the color-kinematic duality, which states that the scattering amplitude of a Yang-Mills theory can be given in a representation where the purely kinematic part has the same algebraic properties of the corresponding color factors; the color-kinematic duality is particularly interesting because it also relates gauge theory amplitudes with gravity amplitudes through the BCJ double-copy procedure. In three dimensions a proposal for BCJ like relations governed by a three algebra structure has been suggested. Despite checks at tree level, it would be interesting to understand how it applies to loop amplitudes. This, in fact, requires their knowledge at finite N. Furthermore, the computation of ABJM scattering amplitudes at finite N would allow to get informations about scattering amplitudes in a another theory closely-related to ABJM. This theory is the three dimensional superconformal BLG theory which is a maximally supersymmetric theory with $\mathcal{N} = 8$; it can be realized as the ABJM theory with gauge group $SU(2) \times SU(2)$. In BLG the planar limit cannot be taken, therefore the inspection of BLG amplitudes at loop level inevitably requires working at finite N.

For these reasons in this thesis we perform computations of four-point scattering amplitude and of the Sudakov form factor up to two loops at finite N, as presented in the published paper [2]. We perform computations using the $\mathcal{N} = 2$ superspace approach and the super Feynman diagrams within dimensional reduction scheme. In order to get the full amplitude we have to adjust the color factors of the two-loop amplitude computed in the planar limit and to calculate a new non-planar diagram. As a by-product, we also provide the full four-point scattering amplitude of BLG theory. Infrared divergences appear in the subleading contributions as poles in the dimensional regularization parameter. However, in contrast with the $\mathcal{N} = 4$ SYM amplitude where subleading terms have milder divergences, the three dimensional amplitude exhibits the same degree of divergence in both the leading and subleading parts. We use the information collected in the computation of the amplitude to evaluate the Sudakov form factor at any value of N. Taking the planar limit our result matches the form factor given in a prevous work and extends it to finite N. All results exhibit uniform transcendentality weight, suggesting that the maximal transcendentality principle likely applies to ABJM at finite N.

The structure of this thesis is the following. In the first two chapters we lay the fundations for understanding the results accomplished in Chapters 3 and 4. Chapter 1 is dedicated to superconformal gauge theories. In particular we discuss conformal symmetry and supersymmetry and we present the most representative superconformal theories, which are the well known supersymmetric $\mathcal{N} = 4$ SYM, the $\mathcal{N} = 2$ SCQCD and $\mathcal{N} = 6$ ABJM theories. It is worth presenting $\mathcal{N} = 4$ SYM because we will frequently refer to it since it is the most studied superconfomal theory, while the other two theories are the ones in which we perform computations. Chapter 2 is dedicated to the main known results concerning scattering amplitudes. In the first part of the chapter we present the main properties at tree level and at loop level of scattering amplitudes in non-supersymmetric Yang-Mills theories with gauge group $SU(N)$, which include QCD specifying $N = 3$. The second part of the chapter is devoted to showing how these properties are improved when supersymmetry is present and in particular we focus on the maximally supersymmetric $\mathcal{N} = 4$ Yang-Mills theory in four spacetime dimensons. In Chapter 3 we present one-loop and two-loop computations of four-point scattering amplitudes in planar $\mathcal{N} = 2$ SCQCD, while in Chapter 4 we extend a previous computation of planar two-loop four-point scattering amplitudes and of planar Sudakov form factor outside the planar limit in ABJM theory. Conclusions are drawn in Chapter 5, where we suggest further developments and outlooks of our research. Technical aspects such as superspace conventions, the explicit form of the superconformal algebra and the computation of Feynman loop integrals are summarized in Appendices A and B.

CHAPTER 1

Superconformal gauge theories

The recent developments in high energy physics have put great emphasis on gauge theories, indeed the standard theory of fundamental interactions is completely formulated in this framework. Gauge theories are the natural quantum theories of vector fields, which mediate the dynamics of the known subatomic particles. The unified way in which the mathematical formalism of gauge theory describes the electromagnetic, the weak and the strong interactions, and the consequent discovery of the corresponding bosons has put gauge theories in their actual preeminent position in high energy physics.
A gauge theory is defined as a theory where the action is invariant under a continuous group symmetry that depends on spacetime, i.e under a continuous local symmetry. Symmetry principles, both global and local, are a fundamental feature of modern particle physics. At the classical level, global symmetries account for many of the conserved quantities we observe in nature, while gauge symmetries decribe in a compact way the interactions of the basic constituents of matter. At the quantum level symmetries facilitate the study of the ultraviolet behaviour of field theory models and their renormalization.

Conformal symmetry and supersymmetry have been instrumental in the developments of modern theoretical physics. Conformal symmetry can be thought of as a local scale invariance. While at classical level a lot of theories enjoy conformal invariance, at quantum level this symmetry is broken in many quantum gauge theories, e.g. those which describe fundamental interactions among elementary particles. Supersymmetry under quite general assumptions is the largest possible extension of the Poincarè group. It is the supreme symmetry: it unifies spacetime symmetries with internal symmetries, fermions with bosons. At the quantum level, renormalizable supersymmetric models exhibit improved ultraviolet behaviour because of cancellations between fermionic and bosonic contributions. At the present time there is no direct experimental evidence that supersymmetry is a fundamental symmetry of nature. Nevertheless supersymmetric models have received a lot of attention in the last years. From the theoretical point of view one of the main reasons of interest is the improved ultraviolet behaviour. In particular supersymmetry allows to build models which keep the conformal symmetry even at quantum level, which implies that the theory is completely free of ultraviolet divergences. For example maximally extended $\mathcal{N} = 4$ Super Yang-Mills theory is a very well known four-dimensional theory that is finite to all orders of perturbation theory. The interest in superconformal field theories has widely increased in the late 1990s, when the AdS/CFT correspondence between superconformal field theory and superstring theory on AdS space has been conjectured.

In the first part of this chapter we will introduce the conformal symmetry and the supersymmetry, at classical level and at quantum level. Then we will present the most studied superconformal theory, the $\mathcal{N} = 4$ Super Yang-Mills theory. This theory, formulated in the late 1970's, has been first analyzed as a relative of QCD with an improved high energy behaviour due to its symmetries. It is worth discussing this theory because we will often use it as a benchmark in the analysis of our results. Then we will present two more superconformal theories: the four dimensional $\mathcal{N} = 2$ Superconformal QCD and the three dimensional ABJM. In this chapter we present their actions, while in Chapter 3 and 4 we will compute scattering amplitudes in these theories. Finally we will briefly introduce one of the main reasons for studying superconformal field theories, which is the AdS/CFT correspondence.

1.1 Superconformal theories

Theories without scales or dimensionful parameters are classically scale invariant. It means that the action of the theory is invariant under the simultaneous dilatation of spacetime coordinates:

$$x_\mu \to x'_\mu \qquad (\mathrm{d}x)^2 \to (\mathrm{d}x')^2 = \lambda^2 (\mathrm{d}x)^2 \tag{1.1}$$

and an appropriate rescalation of the fields involved:

$$\phi(x) \to \lambda^\Delta \phi(\lambda x) \tag{1.2}$$

where Δ is the so-called scaling dimension of the field and its value depends on the theory under consideration. A lot of physically relevant theories are classically scale invariant theory, e.g. the Yang-Mills theory coupled to massless fermions and scalars; as we will see usually scale invariance is broken by quantum corrections.
Invariance under scale transformations, under mild conditions, implies invariance under the bigger group of conformal transformations. A conformal transformation in d-dimensional spacetime is a change of coordinates that rescales the line element:

$$x_\mu \to x'_\mu \qquad (\mathrm{d}x)^2 \to (\mathrm{d}x')^2 = \Omega^2(x)(\mathrm{d}x)^2 \tag{1.3}$$

where $\Omega(x)$ is an arbitrary function of the coordinates. Scale transformations are a particular case of conformal transformations with Ω constant. The set of conformal transformations manifestly forms a group, identified with the noncompact group $SO(d, 2)$,

which has the Poincarè group as a subgroup, since the latter corresponds to the special case $\Omega = 1$. The epithet conformal derives from the property that the transformations rescale lengths but preserve the angles between vectors.
An important hint for the presence of conformal invariance [1] comes from the stress energy tensor of the theory. In fact, the vanishing of the trace of the stress energy tensor is a sufficient condition for conformal invariance: it is possible to show that in a classically conformal theory the tracelessness of the energy-momentum tensor implies the invariance of the action under conformal transformation, but the converse is not true. Under certain conditions the energy-momentum tensor of a theory with scale invariance can be made traceless: only in these special cases the full conformal invariance is a consequence of scale invariance and Poincarè invariance.

The conformal group is not the only possible extension of the Poincarè group. In fact another extension of spacetime symmetry exists and it relates two basic classes of elementary particles: bosons, which have an integer-valued spin, and fermions, which have a half-integer spin. This kind of spacetime symmetry was not considered until 1970s, because of the Coleman-Mandula theorem [4], which is a no-go theorem that states that spacetime and internal symmetries cannot be combined in any but a trivial way. In 1974 Haag, Lopuszanski and Sohnius found a way out of the Coleman-Mandula theorem and they demonstrated [5] that spacetime and internal symmetries can be consistently combined in one only way: supersymmetry. Supersymmetry algebra is the most general super-Poincaré algebra, namely a graded Lie algebra, which contains, in addition to the Poincarè generators, $\mathcal{N}$ fermionic spinorial generators $Q_{a\alpha}$, where $a = 1, ..., \mathcal{N}$ is an extra index which label the different generators. There can be also at most $\frac{1}{2}\mathcal{N}(\mathcal{N}-1)$ complex central charges Z_{ab}, which commute with all generators. The $\mathcal{N} = 1$ case is called simple supersymmetry, whereas the $\mathcal{N} > 1$ case is called extended supersymmetry. The supersymmetry algebra is presented in A.2.
In the presence of supersymmetry, the conformal group is enhanced to a supergroup obtained from $SO(d,2)$ by adding the supercharges $Q_{a\alpha}$ and the internal R-symmetry that rotates them. We also need to add the so-called conformal supercharges $S_{a\alpha}$, which are required to close the superconformal algebra. The supersymmetry algebra and the superconformal algebra are presented in Appendix A.2. As we will see below supersymmetry improves the ultraviolet behaviour of quantum field theory, and thus is an helpful symmmetry which allows to construct conformal quantum field theories.

It is important to notice that conformal invariance at quantum level generally does not follow from conformal invariance at the classical level. In a quantum theory, computing quantum correction to the classical quantities, ultraviolet divergences may arise. Of course physical quantities are finite and therefore divergences must appear only at

[1] For an exhaustive dissertation on classical and quantum conformal theories see [3].

intermediate stages of calculations and get cancelled. We need first to regularize the divergences, namely to make them finite so that we can manipulate them. We perform this by introducing an ultraviolet cut-off M. Since physical quantities must not be affected by divergences, physical theories must be renormalizable, which means that divergences can be removed by suitable redefinition of the coupling constants and the fields. The operation of renormalization introduces a scale in the theory, which breaks the conformal invariance. We will see that there can be exception to the break of the conformal symmetry at particular valued of the parameters, which constitute a renormalization-group fixed points.
Renormalization refers to a mathematical apparatus that allows systematic investigation of the changes of a quantum field theory in relation to the variation of the energy scale. In particular we are interested here in renormalizable theories. In the simplest case of a theory characterized by a single coupling constant, renormalizability can be stated in the following way: a physical quantity G will be given in such a theory as a power expansion in the coupling g, which we will assume to be dimensionless, with possibly ultraviolet divergent coefficients. We can write:

$$G = G(g, M, s_1, ..., s_n) \tag{1.4}$$

where G depends upon the coupling g, the ultraviolet cut-off M , and some invariants $s_1, ..., s_n$ constructed out of the momenta and masses involved in the process in question. Renormalizability means that it is possible to define a renormalized coupling $g_{ren} = g + c_1 g^2 + ...$, with $c_i = c_i(M/\mu)$ in such a way that:

$$G(g, M, s_1, ..., s_n) = \tilde{G}(g_{ren}, \mu, s_1, ..., s_n) \tag{1.5}$$

This equation means that the physical quantity can be expressed in term of the renormalized coupling g_{ren}, the finite scale μ and the invariants, in terms of a finite function. The finite scale μ has to be introduced in order for the dimensionless coefficients c_i to depend upon the dimensional quantity M. In other words, in eq. (1.5) all the divergences have been reabsorbed in the renormalized coupling. We can write:

$$g = g(g_{ren}, M/\mu) \qquad g_{ren} = g_{ren}(g, M/\mu) \tag{1.6}$$

and:

$$G(g(g_{ren}, M/\mu), M, s_1, ..., s_n) = \tilde{G}(g_{ren}, \mu, s_1, ..., s_n) \tag{1.7}$$

Therefore, renormalizability means that the redefinition of the coupling of the form (1.7) makes all physical quantities independent of the cut-off. Renormalization states that up to any order in perturbation theory we can remove all ultraviolet divergences from a physical quantities just by a redefinition of the coupling constant.

The behaviour of the coupling constant as a function of the energy scale μ is of particular interest, since it determines the strenght of the interaction and the conditions under which perturbation theory is valid. This dependence on the energy scale is known as the running of the coupling parameter, a fundamental feature of scale-dependence in quantum field theory. This scale dependence is encoded in the renormalization group equation:

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} g = \beta(g) \tag{1.8}$$

This differential equation describes the flow of the coupling constant g_{ren} as a function of the renormalized scale μ; the beta function is the rate of the renormalization group flow. So, eq. (1.8) allows to investigate perturbatively the running of the coupling constant at short distances (or in momentum space at high energy scales, which we refer to as the ultraviolet regime), or at large distances (low energy scales, which we refer as the infrared regime).

Beta functions are usually computed in some kind of approximation scheme. An example is perturbation theory, where one assumes that the coupling parameters are small. One can then make an expansion in powers of the coupling parameters and truncate the higher-order terms. It is possible to show that for a pertubative expansion of the beta function, the coefficient of the first term is universal, i.e. scheme-independent [6]. The one-loop beta function can be derived for a general theory with fermions and bosons in arbitrary representations of the gauge group [7]. The result for a general gauge theory with two-component spinors and real scalars transforming in a representation of a gauge group $SU(N)$ reads:

$$\begin{aligned} \beta(g) &= -\frac{g^3}{16\pi^2}\beta_0 + \mathcal{O}(g^4) \\ \beta_0 &= \left(\frac{11}{3}C(gauge) - \frac{2}{3}\sum T(fermions) - \frac{1}{6}\sum T(scalars)\right) \end{aligned} \tag{1.9}$$

where the sum runs over the number of fermions or scalars; C and T are invariants of the gauge group for the representation of the gauge fields, fermions and scalars. For fields transforming in the adjoint representation of the gauge group $C = N$ and $T = N$, while for fields in the fundamental (or antifundamental) representation $T = 1/2$.
Considering eq. (1.8), it is clear that, as a matter of principle, three behaviours are possible in the region of small g:

$$\begin{aligned} &(1)\ \ \beta(g) > 0 \\ &(2)\ \ \beta(g) < 0 \\ &(3)\ \ \beta(g) = 0 \end{aligned}$$

Example of quantum field theories that exhibit each of these behaviours are known, as we now show.

1. A positive sign for the beta function indicates a running coupling that increase at large momenta and decrease at small momenta. Thus the short-distance behaviour

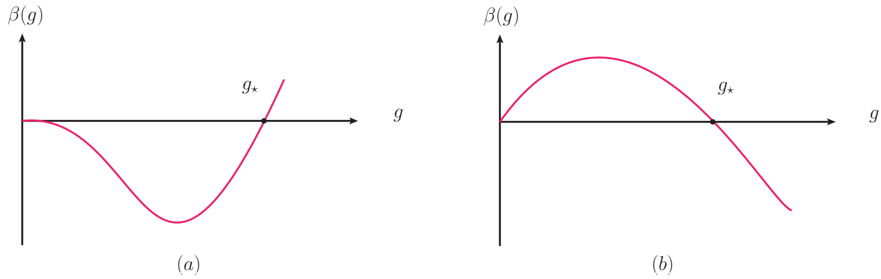


Figure 1.1:

of the theory can not be computed using perturbation theory. Examples of theories with such behaviour are QED, which describes quantum electrodynamcs, and the theory which involves scalar particles with a potential proportional to ϕ^4. The lack of a consistent high energy description for these theories makes them ill-defined as quantum field theories and it is possible to consider them only as low-energy effective theories.

2. A negative sign for the beta function indicates that the running coupling constant becomes large in the large-distance regime and it becomes small at large momenta or short distances. Theories where the coupling constant tends to zero at logarithmic rate as the momentum scale increase are called asymptotically free theories. In theories of this class the short-distance behaviour is completely solvable by perturbation theory and, though ultraviolet divergences appear in every order of perturbation theory, the renormalization group states that the sum of these divergences is completely harmless. The most important physical theory which exhibits a beta function with this behaviour is QCD, a quantum field theory that describes the strong interactions of elementary particle physics. Also the pure Yang-Mills theory with a general gauge group $SU(N)$, which is classically scale invariant, exhibits this behaviour at quantum level.

3. When the beta function vanishes, it means that the coupling constant does not flow. In these theories the running coupling constant is independent of the renormalization scale and it is free of ultraviolet divergences. Such theories are called conformal quantum field theories, since in these case the classical conformal invariance is not broken at quantum level by the introduction of a renormalization scale. These theories play a central role string theory, but are relevant also in other areas of research, as in statistical mechanics or in cosmology, in particular for the description of phase transition and critical points. We will discuss the importance of conformal quantum theories in the next paragraph and in the next sections.

We are interested in the last class of theories, the conformal quantum field theories, because in such theories the conformal invariance at quantum level gives many constraints.

In fact, in addition to the lack of ultraviolet divergences there are constraints on the correlation functions: in particular the one, two and three point functions are completely fixed by conformal invariance. Furthermore, if supersymmetry is present, the superconformal theory is completely solvable since conformal invariance constrains the form of one, two and three-point functions and supersymmetry relates them to n-point functions.
Conformally invariant interacting quantum field theories can be obtained in two ways.

- A possibility is that the theory is a finite theory, without divergences at all. In this case $\beta(g) = 0$ for all values of g and there is no renormalization group flow. Since g can have an arbitrary value we have a line (or manifold if there is more than one coupling constant) of fixed points. The standard example in this class of theories is the maximally supersymmetric $\mathcal{N} = 4$ Super Yang-Mills theory, which we will present in the next section. Another example is $\mathcal{N} = 2$ superconformal QCD, which will be discussed in section 1.3. We already mentioned that supersymmetry improves the ultraviolet behaviour of the theory and thus it is helpful to construct conformal quantum field theory.

- Another possibility is that an asymptotically free theory develops an interacting infrared stable fixed point $g_\star$ of the renormalization group equation, as shown in Fig. 1.1(a). This fixed point is such that in eq. (1.8) the beta function vanishes: $\beta(g_\star) = 0$. Examples of such theories are the Yang-Mills theory with gauge group $SU(N)$ coupled to $N_f = 16N$ fermions [8]. The three-dimensional ABJM theory, which will be discussed in section 1.4, can be also considered as a fixed point of a more general theory, the $\mathcal{N} = 3$ supersymmetric Yang-Mills-Chern-Simons theory. The possibility that a theory with a positive beta function flows to an ultraviolet fixed point, as shown in Fig. 1.1(b), also exists.

1.2 $\mathcal{N} = 4$ Super Yang-Mills

The four dimensional $\mathcal{N} = 4$ Super Yang-Mills (SYM) is a maximally supersymmetric gauge theory with gauge group $SU(N)$ [9]; it is the first and best studied superconformal theory. The pivotal role of this model in the context of modern theoretical physics can be summarize saying that "$\mathcal{N} = 4$ SYM is the harmonic oscillator of the 21st century". We will explore its role in the context of scattering amplitudes in the next chapter.

The matter content of $\mathcal{N} = 4$ SYM is completely fixed by the high number of supersymmetries. The theory includes, as physical particles, one spin-1 Yang-Mills vector, four spin-$\frac{1}{2}$ two-component spinors organized into the **4** representation of the R-symmetry group $SU(4)$ and six spin-0 particles, organized into the **6** representation of the R-symmetry group. All the particles are massless and transform under the adjoint representation of the gauge group $SU(N)$.

A compact alternative to the component field approach is given by the $\mathcal{N} = 1$ superspace approach (see [10] for an exhaustive dissertation and Appendix A.1 for conventions). Superspace is an extension of ordinary spacetime: in addition to the usual Minkowki coordinates $x^{\alpha\dot{\alpha}}$ (the cooordinates written in the spinorial language are related to x^{μ} through the Pauli matrices, as explained in Appendix A.1), superspace includes extra anticommuting coordinates in the form of two-component spinors $\theta^{\alpha}, \bar{\theta}^{\dot{\alpha}}$. Points in superspace are thus labelled by $z^A = (x^{\alpha\dot{\alpha}}, \theta^{\alpha}, \bar{\theta}^{\dot{\alpha}})$. To get (off-shell) representations of supersymmetry on physical fields we consider superfields, which are functions of superspace coordinates: $F(x, \theta, \bar{\theta})$, and transform in a representation of the Lorenz group. General superfields are not irreducible representations of supersymmetry. In order to construct irreducible representation constraints are needed. Now we are going to present two type of superfields which are irreducible representations of $\mathcal{N} = 1$ supersymmetry: the chiral and the vector superfileds, which are the basic tools to construct supersymmetric models.
The chiral superfield Φ is a scalar superfield defined by: $\bar{D}_{\dot{\alpha}}\Phi = 0$, where $\bar{D}_{\dot{\alpha}}$ is a covariant derivative (defined in (A.12)), i.e. a fermionic derivative which is invariant under supersymmetry transformation and covariant under Lorenz rotations. Superfields can be expanded in a Taylor series with respect to the anticommuting coordinates; because the square of an anticommuting quantity vanishes, this series has only a finite number of terms. The coefficients of the expansion are ordinary component fields. The chiral superfield, introducing the new variable $y^{\alpha\dot{\alpha}} = x^{\alpha\dot{\alpha}} + \frac{i}{2}\theta^{\alpha}\bar{\theta}^{\dot{\alpha}}$, has the following expansion:

$$\Phi(y) = \phi(y) + \theta^{\alpha}\psi_{\alpha}(y) - \theta^2 F(y) \tag{1.10}$$

where the complex components of the chiral superfields are defined by: $\phi = \Phi|_{\theta=0}$, $\psi_{\alpha} = D_{\alpha}\Phi|_{\theta=0}$ and $F = D^2\Phi|_{\theta=0}$. So a chiral superfield describes the $\mathcal{N} = 1$ chiral multiplet, which contains a scalar ϕ and one fermion ψ_{α}. These are the only two physical fields; the field F turns out to be an auxiliary field, namely a non-dynamical field which can be removed from the action using its equation of motion.
The vector superfield V is a real scalar superfield: $V = V^{\dagger}$. The vector superfield in the Wess-Zumino gauge describes in the correct way the $\mathcal{N} = 1$ vector multiplet; it has the following expansion:

$$V = \theta^{\alpha}\bar{\theta}^{\dot{\alpha}}A_{\alpha\dot{\alpha}} - \bar{\theta}^2\theta^{\alpha}\lambda_{\alpha} - \theta^2\bar{\theta}^{\dot{\alpha}}\lambda_{\dot{\alpha}} + \theta^2\bar{\theta}^2 D' \tag{1.11}$$

where the physical fields $A_{\alpha\dot{\alpha}}$ and λ_{α} are respectively the vector field and a fermion, called gaugino; the field D' is an auxiliary field.

In superspace supersymmetry is manifest: the supersymmetry algebra is represented by translations and rotations involving both the spacetime and the anticommuting coordinates. A further advantage is that superfields automatically include, in addition to the dynamical degrees of freedom, certain unphysical fields, like auxiliary fields, needed classically for the off-shell closure of the supersymmetry algebra, and compensating fields, i.e. fields that can be algebraically gauged away but are important for quantization.

It is convenient to describe the $\mathcal{N}=4$ SYM using the $\mathcal{N}=1$ superspace formalism, where the matter content is organized in terms of one real vector superfield V and three chiral superfields Φ_i, organized into the **3** representation of $SU(3)\subset SU(4)$. With this formalism the classical action (using the conventions of [10]) is:

$$S=\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\mathrm{tr}\left(e^{-gV}\bar{\Phi}_i e^{gV}\Phi^i\right)+\frac{1}{g^2}\int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\mathrm{tr}\left(W^\alpha W_\alpha\right) + \frac{ig}{3!}\int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\epsilon^{ijk}\,\mathrm{tr}\left(\Phi_i\left[\Phi_j,\Phi_k\right]\right)+\frac{ig}{3!}\int \mathrm{d}^4x\,\mathrm{d}^2\bar{\theta}\,\epsilon^{ijk}\,\mathrm{tr}\left(\bar{\Phi}_i\left[\bar{\Phi}_j,\bar{\Phi}_k\right]\right) \tag{1.12}$$

where $W_\alpha=i\bar{D}^2\left(e^{-gV}D_\alpha e^{gV}\right)$ is the superfield strength of the vector superfield V and g is the coupling constant which governs the interactions. All the superfields transform in the adjoint representation of the gauge group: $V=V^aT_a$, $\Phi^i=\Phi_i^aT_a$, where $i=1,2,3$ and T_a are $SU(N)$ matrices in the fundamental representation. The theory exhibits an $\mathcal{N}=4$ supersymmetry and a global R-symmetry $SU(4)$. Furthermore the classical theory is scale invariant. Supersymmetry enhances the conformal symmetry group to the superconformal symmetry group $SU(2,2|4)$.

The quantization procedure (described in details in [10]) of the classical action (1.12) requires the introduction of a gauge fixing and corresponding ghost terms:

$$S_{gf}=\int d^4xd^4\theta\,\mathrm{Tr}\left(-\frac{1}{\alpha}(D^2V)(\bar{D}^2V)+(c'+\bar{c}')\mathrm{L}_{\frac{gV}{2}}\left[c+\bar{c}+\coth\mathrm{L}_{\frac{gV}{2}}(c-\bar{c})\right]\right) \tag{1.13}$$

where $\mathrm{L}_{\frac{gV}{2}}X=\frac{g}{2}[V,X]$; it is convenient to use the supersymmetric Fermi-Feynman gauge $\alpha=1$. To low orders in V the action is:

$$S=\mathrm{tr}\int \mathrm{d}^4x\,\mathrm{d}^4\theta\left(\bar{\Phi}_i\Phi^i-\frac{1}{2}V\,\Box\,V+\bar{c}'c-c'\bar{c}\right) + \mathrm{tr}\int \mathrm{d}^4x\,\mathrm{d}^4\theta\left(g[\bar{\Phi}_i,V]\Phi^i+\frac{1}{2}gV\{D^\alpha V,\bar{D}^2D_\alpha V\}+\frac{1}{2}g(c'+\bar{c}')[V,c+\bar{c}]\right)+\ldots + \frac{ig}{3!}\int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\epsilon^{ijk}\,\mathrm{tr}\left(\Phi_i\left[\Phi_j,\Phi_k\right]\right)+\frac{ig}{3!}\int \mathrm{d}^4x\,\mathrm{d}^2\bar{\theta}\,\epsilon^{ijk}\,\mathrm{tr}\left(\bar{\Phi}_i\left[\bar{\Phi}_j,\bar{\Phi}_k\right]\right) \tag{1.14}$$

After gauge fixing we can identify the Feynman rules for the propagators, which are:

$$\langle V^aV^b\rangle=-\frac{\delta(\theta_1-\theta_2)}{p^2}\,\delta^{ab} \tag{1.15}$$

$$\langle\Phi_i^a\bar{\Phi}_j^b\rangle=\frac{\delta(\theta_1-\theta_2)}{p^2}\,\delta^{ab} \tag{1.16}$$

$$\langle\bar{c}'^ac^b\rangle=-\langle c'^a\bar{c}^b\rangle=\frac{\delta(\theta_1-\theta_2)}{p^2}\,\delta^{ab} \tag{1.17}$$

The Feynman rules for vertices can be immediately read from the expanded action (1.14).

We mentioned that the classical theory is scale invariant, but it turns out that the theory is conformal invariant also at the quantum level. We can easily compute the one-loop beta function from eq. (1.9), where the universal coefficient were presented. Since in $\mathcal{N} = 4$ SYM all the fields transform in the adjoint representation, all the Casimir are equal: $C = N$. Thus it emerges that fermions and scalars balance the negative contribution of the gauge fields:

$$\frac{11}{3} - \frac{2}{3} \times 4 - \frac{1}{6} \times 6 = 0 \tag{1.18}$$

and the one-loop beta function is zero. In order to know the other coefficients of the beta function perturbative calculations are needed. Both component and superfield calculations performed in the 1980s have estabilished that the beta function vanishes up to three loops [11, 12, 13, 14]. This astonishing result suggested that the theory might be actually conformally invariant. Few years later proofs that extend the conclusion to all order of perturbation theory were formulated [15, 16, 17] using superspace in the light-cone frame. In particular it was shown that $\mathcal{N} = 4$ SYM, in a certain form of the light-cone gauge, is completely free of ultraviolet divergences at any order of perturbation theory. It follows that the beta function vanishes in any gauge, to all orders of perturbation theory and there is no running of the coupling.

1.3 $\mathcal{N} = 2$ Superconformal QCD

Now we present the four dimensional theory $\mathcal{N} = 2$ superconformal QCD (SCQCD) theory; we will present in Chapter 3 computations of scattering amplitudes in this model.

The $\mathcal{N} = 2$ SCQCD belongs to a two-parameter family of $\mathcal{N} = 2$ superconformal theories [18] with gauge group $G = SU(N) \times SU(N)$. These theories involve gauge fields coupled to matter fields in the bifundamental representation of the gauge group. This two-parameter family is governed by the two coupling constants g ang $\hat{g}$, each of which is associated with a factor in the gauge group G. This is the classical action of the two-parameter family of superconformal theories written in terms of $\mathcal{N} = 1$ superfields:

$$\begin{aligned} S = &\int d^4x\, d^2\theta \left[\frac{1}{g^2}\mathrm{tr}(W^\alpha W_\alpha) + \frac{1}{\hat{g}^2}\mathrm{tr}(\hat{W}^\alpha \hat{W}_\alpha)\right] + \\ &+ \int d^4x\, d^4\theta\, \mathrm{tr}\left[e^{-gV}\,\bar{\phi}\, e^{gV}\,\phi + e^{-\hat{g}\hat{V}}\,\bar{\hat{\phi}}\, e^{\hat{g}\hat{V}}\,\hat{\phi} + \bar{Q}^I e^{gV} Q_I\, e^{-\hat{g}\hat{V}} + \bar{\tilde{Q}}_I\, e^{\hat{g}\hat{V}}\,\tilde{Q}^I e^{-gV}\right] + \\ &+ i\int d^4x\, d^2\theta \left[g\, \mathrm{tr}(\tilde{Q}^I \phi Q_I) - \hat{g}\, \mathrm{tr}(Q_I \hat{\phi} \tilde{Q}^I)\right] - i\int d^4x\, d^2\bar{\theta}\left[g\, \mathrm{tr}(\bar{Q}^I \bar{\phi} \bar{\tilde{Q}}_I) - \hat{g}\, \mathrm{tr}(\bar{\tilde{Q}}_I \bar{\hat{\phi}} \bar{Q}^I)\right] \end{aligned}$$

where $W_\alpha = i\bar{D}^2\left(e^{-gV} D_\alpha e^{gV}\right)$ and $\hat{W}_\alpha = i\bar{D}^2\left(e^{-\hat{g}\hat{V}} D_\alpha e^{\hat{g}\hat{V}}\right)$ are the chiral superfield strenghts of the vector superfields V and $\hat{V}$, that contain the gauge fields and transform in the adjoint representation of respectively the first and second factor of the gauge

group. The matter content of these theories can be summarize into the quiver diagram, depicted in Fig. 1.2. There are two chiral superfields ϕ and $\hat{\phi}$, which transform in the adjoint representation of the first term and of the second term of the gauge group G. It is customary to draw a node for each factor of the total gauge group, and a circular arrow for each chiral superfield in the adjoint representation of the gauge group. There are also N_f chiral superfields Q in the bifundamental representation of the gauge group and N_f chiral superfields $\tilde{Q}$ in the anti-bifundamental representation; these fields are indicated in the quiver diagram with an arrow between the nodes.
This two-parameter family of theories has two special limits: when the coupling constants are uqual to each other the theory is the $\mathbb{Z}_2$ orbifold of $\mathcal{N} = 4$ SYM; when one coupling constant is zero the theory is $\mathcal{N} = 2$ superconformal QCD.

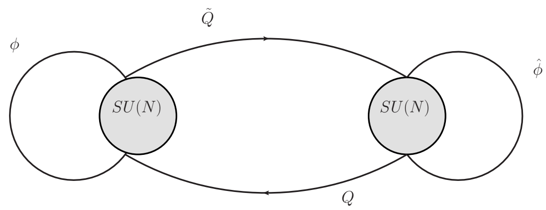


Figure 1.2: The quiver diagram for the two-parameter family of $\mathcal{N} = 2$ superconformal theories with gauge group $G = SU(N) \times SU(N)$

The field content of $\mathcal{N} = 2$ superconformal QCD can be conveniently expressed in terms of $\mathcal{N} = 1$ superfields (conventions are summarized in Appendix A.1). The classical Euclidean action reads:

$$S = \int \mathrm{d}^4x\mathrm{d}^4\theta \Big[\mathrm{tr}\big(e^{-gV}\bar{\Phi}e^{gV}\Phi\big) + \bar{Q}^I e^{gV} Q_I + \tilde{Q}^I e^{-gV} \bar{\tilde{Q}}_I \Big] + \frac{1}{g^2}\int \mathrm{d}^4x\mathrm{d}^2\theta\, \mathrm{tr}\big(W^\alpha W_\alpha\big) + \\ + ig\int \mathrm{d}^4x\mathrm{d}^2\theta\, \tilde{Q}^I\Phi Q_I - ig\int \mathrm{d}^4x\mathrm{d}^2\bar{\theta}\, \bar{Q}^I\bar{\Phi}\bar{\tilde{Q}}_I \tag{1.19}$$

where $W_\alpha = i\bar{D}^2(e^{-gV}D_\alpha e^{gV})$ is the superfield strength of the vector superfield V. The gauge group is $SU(N)$; there is a global symmetry group $U(N_f) \times SU(2)_R \times U(1)_R$, where $U(N_f)$ is the flavour symmetry and $SU(2)_R \times U(1)_R$ the R-symmetry group. If the number of flavours is tuned to be $N_f = 2N$ the theory becomes exactly superconformal. The superfield V contains the component gauge field and transforms in the adjoint representation of the gauge group $SU(N)$. The $\mathcal{N} = 1$ chiral superfield Φ also transforms in the adjoint representation of $SU(N)$ and combines with the superfield V into an $\mathcal{N} = 2$ vector multiplet. When needed adjoint indices will be denoted by $a, b, c, \ldots$. The rest of the matter is described in terms of the quark chiral scalar superfields Q_{iI} and $\tilde{Q}^{jJ}$, which transform respectively in the fundamental and antifundamental representation of $SU(N)$ and together form an $\mathcal{N} = 2$ hypermultiplet. We use $I, J = 1, \ldots, N_f$ as flavor indices and $i, j = 1, \ldots, N$ as (anti)fundamental color indices. A summary of the

field	$SU(N)$	$U(N_f)$	$U(1)_R$
V	Adj	1	0
Φ	Adj	1	1
$\bar{\Phi}$	Adj	1	-1
Q	$\square$	$\square$	0
$\bar{\tilde{Q}}$	$\square$	$\square$	0
$\bar{Q}$	$\bar{\square}$	$\bar{\square}$	0
$\tilde{Q}$	$\bar{\square}$	$\bar{\square}$	0

Table 1.1: Field content of $\mathcal{N} = 2$ SCQCD in terms of $\mathcal{N} = 1$ superfields. The global symmetry $SU(2)_R$ is not manifest in the $\mathcal{N} = 1$ superspace formulation.

field content of the theory is given in Table 1.1.

The $\mathcal{N} = 2$ SCQCD theory can be quantized in Euclidean space by path integration $\int \mathcal{D}\psi\, e^{S[\psi]}$ over all the fields ψ after performing gauge fixing in $\mathcal{N} = 1$ superspace. The standard procedure is described in details in [10] and results in adding to the action (1.19) the same gauge fixing term (1.13) that we have already introduced in the quantization of $\mathcal{N} = 4$ SYM action. After the gauge fixing procedure we expand the action, in order to extract the interaction terms up to the order needed in the computation carried out in Chapter 3:

$$\begin{aligned} S = \int \mathrm{d}^4x\mathrm{d}^4\theta \Bigg[&\mathrm{tr}\left(\bar{\Phi}\Phi + g\,\bar{\Phi}V\Phi - g\,\bar{\Phi}\Phi V + \frac{g^2}{2}\bar{\Phi}\Phi VV + \frac{g^2}{2}\bar{\Phi}VV\Phi - g^2\bar{\Phi}V\Phi V \right) + \\ &+ \bar{Q}^I Q_I + \tilde{Q}^I \bar{\tilde{Q}}_I + g\,\bar{Q}^I V Q_I - g\,\tilde{Q}^I V \bar{\tilde{Q}}_I + \frac{g^2}{2}\,\bar{Q}^I VVQ_I + \frac{g^2}{2}\,\tilde{Q}^I VV\bar{\tilde{Q}}_I + \\ &+ \mathrm{tr}\Bigg(-\frac{1}{2}V\,\square\,V + \frac{g}{2}\,V\{D^\alpha V, \bar{D}^2 D_\alpha V\} + \frac{g^2}{8}\,[V, D^\alpha V]\bar{D}^2[V, D_\alpha V] + \\ &+ \bar{c}'c - c'\bar{c} + \frac{g}{2}\,(c' + \bar{c}')[V, c + \bar{c}] + \frac{g^2}{12}\,(c' + \bar{c}')[V,[V, c - \bar{c}]] \Bigg) \Bigg] + \\ &+ ig \int \mathrm{d}^4x\mathrm{d}^2\theta\, \tilde{Q}^I \Phi Q_I - ig \int \mathrm{d}^4x\mathrm{d}^2\bar{\theta}\, \bar{Q}^I \bar{\Phi}\bar{\tilde{Q}}_I \;+\; \dots \end{aligned} \tag{1.20}$$

We can extract the Feynman rules for the propagators, which are:

$$\langle V^a V^b \rangle = \text{(wavy line)} = -\frac{\delta(\theta_1 - \theta_2)}{p^2}\,\delta^{ab} \tag{1.21}$$

$$\langle \Phi^a \bar{\Phi}^b \rangle = \text{(solid line)} = \frac{\delta(\theta_1 - \theta_2)}{p^2}\,\delta^{ab} \tag{1.22}$$

$$\langle Q_{iI} \bar{Q}^{jJ} \rangle = \langle \bar{\tilde{Q}}_{iI} \tilde{Q}^{jJ} \rangle = \text{(dashed line)} = \frac{\delta(\theta_1 - \theta_2)}{p^2}\,\delta_i^j \delta_I^J \tag{1.23}$$

$$\langle \bar{c}'^a c^b \rangle = -\langle c'^a \bar{c}^b \rangle = \text{(dotted line)} = \frac{\delta(\theta_1 - \theta_2)}{p^2}\,\delta^{ab} \tag{1.24}$$

where we introduced the graphic conventions we will extensively use in the computations presented in Chapter 3. Vertices can be immediately read from the expanded action (1.20).

Supersymmetric generalization of QCD with many favour were studied since the 1990s. We already mentioned that Yang-Mills theories coupled to a particular number of flavors can enjoy conformal invariance even at quantum level. Also $\mathcal{N} = 2$ SCQCD is a superconformal theory at quantum level. We can easily compute the one-loop beta function from eq. (1.9), where the universal coefficient was presented. In $\mathcal{N} = 2$ SCQCD the superfields V and Φ transform in the adjoint representation and their Casimir are equal: $C = N$, while Q and $\tilde{Q}$ transform in the (anti)fundamental representation with $T = 1/2$. Thus the first coefficient of the beta function is:

$$\left(\frac{11}{3} - \frac{2}{3} \times 2 - \frac{1}{6} \times 2\right) N + \left(-\frac{2}{6} \times 2 - \frac{1}{12} \times 4\right) N_f = 2N - N_f \tag{1.25}$$

Thus we find that the one-loop beta function is zero when $N_f = N$. In order to know the other coefficients of the beta function perturbative calculations are needed. In section 3.1 we will discuss the perturbative corrections to the propagators up to two-loops and we will show how the condition $N_f = 2N$ assures that the beta function identically vanishes.

1.4 $\mathcal{N} = 6$ Chern-Simons matter theory (ABJM)

ABJ(M) theory is a three-dimensional $\mathcal{N} = 6$ superconformal Chern-Simons theory with gauge group $U(M)_K \times U(N)_{-K}$ coupled to matter, formulated by Aharony, Bergman; Jafferis and Maldacena in 2008 [19]. When $M = N$, the theory is called ABJM, otherwise it is called ABJ. It describes two gauge vector multiplets, each of them in the adjoint representation of one of the two gauge groups, four complex scalars and their fermionic partners in the bifundamental representation, and their conjugates in the antibifundamental representation. The gauge sector is described by a Chern–Simons action with Chern-Simons level K. This is a topological field theory, thus the gauge fields are not propagating and they cannot enter in scattering processes as external particles but they can be coupled to matter fields which carry physical degrees of freedom.

It is particularly suitable for perturbative computations in ABJ(M) to use the $\mathcal{N} = 2$ superspace formalism, which has similarities with the four-dimensional $\mathcal{N} = 1$ superspace formalism. We summarize the three-dimensional superspace conventions in Appendix B.1, and we refer to [20] for details. One of the main difference with the four-dimensional one is that the Lorentz group in three-dimensional Minkowski space is $SL(2,\mathbb{R})$, rather than $SL(2,\mathbb{C})$. Because of this, as opposed to four-dimensional theories, there will be only the fundamental representation of the group and not also the antifundamental one. For this reason in three dimensions there is no difference between

dotted and undotted indices and new types of contractions are allowed. In spinorial language, a three-dimensional vector such as the spacetime coordinate is described by the symmetric second-rank spinor $x^{\alpha\beta}$, which is related to the usual x^μ by the gamma matrices given in Appendix B.1. The $\mathcal{N} = 2$ superspace is an extension of ordinary spacetime obtained adding anticommunting coordinates θ^α, to the ordinary spacetime coordinates; points in superspace are labelled by $(x^{\alpha\beta}, \theta^\alpha, \bar{\theta}^\beta)$. The basic objects are superfields, which are functions of the bosonic as well as of the fermionic coordinates and may carry external indices, according to representations of the symmetry algebra. As in the four-dimensional case, we recover the component fields, functions only of the spacetime coordinates, throught the Taylor expansion in θ, which is finite, due to the anticommuting nature of the fermionic coordinates.
The class of supersymmetric gauge theories in which we are interested in is constructed starting from two basic types of superfields: complex chiral superfields $\Phi(x, \theta, \bar{\theta})$ and real scalar superfields $V(x, \theta, \bar{\theta})$. Chiral superfields are defined such that they form the scalar multiplet, which is the scalar irreducible representation of supersymmetry. For this purpose they have to be constrained by the condition: $\bar{D}_\alpha \Phi = 0$. If we perform a Taylor expansion of the superfield introducing a new coordinate $y^{\alpha\beta} = x^{\alpha\beta} + \frac{i}{4}(\theta^\alpha\bar{\theta}^\beta + \theta^\beta\bar{\theta}^\alpha)$ we obtain:

$$\Phi(y, \theta) = C(y) + \theta^\alpha \varphi_\alpha(y) - \theta^2 f(y) \tag{1.26}$$

Thus the chiral superfield contains a complex scalar boson C, a two-component complex fermion φ_a, and an auxiliary field f.
Real unconstrained superfields can be used to describe gauge fields and their superpartners. The vector field in the Wess-Zumino gauge describes the $\mathcal{N} = 2$ vector multiplet:

$$V = \theta^\alpha\bar{\theta}^\alpha\sigma + \theta_\alpha(\gamma^\mu)^\alpha_\beta\bar{\theta}^\beta A_{\alpha\beta} + \theta^2\bar{\theta}^\alpha\bar{\chi}_\alpha + \bar{\theta}^2\theta^\alpha\chi_\alpha + \theta^2\bar{\theta}^2 d' \tag{1.27}$$

which contains a real scalar σ, a complex fermion χ_α and the gauge field $A_{\alpha\beta}$. The field d' turns out to be an auxiliary field.

In three dimensional $\mathcal{N} = 2$ formalism, the field content of the $\mathcal{N} = 6$ ABJ(M) theories with gauge group $U(M)_k \times U(N)_{-k}$ is given in terms of two vector superfileds $(V, \hat{V})$ in the adjoint representation of the first and the second group respectively. The vector superfields are coupled to four chiral superfields $(A^i)^a_{\bar{a}}$ and $(B_i)^{\bar{a}}_a$ carrying a fundamental index $i = 1, 2$ of a global $SU(2)_A \times SU(2)_B$ symmetry; they transform respectively in the bifundamental $(\mathbf{M}, \bar{\mathbf{N}})$ and in the antibifundamental $(\bar{\mathbf{M}}, \mathbf{N})$ representations of the gauge group $U(M)_K \times U(N)_{-K}$ (a and $\bar{a}$ are indices of the fundamental representation of the first and the second gauge groups respectively). When $M = N$ the theory is called ABJM, otherwise it is called ABJ.
The classical action of ABJ(M) theory written in $\mathcal{N} = 2$ formalism is:

$$\mathcal{S} = \mathcal{S}_{\text{CS}} + \mathcal{S}_{\text{mat}} \tag{1.28}$$

with

$$
\begin{aligned}
\mathcal{S}_{\rm CS} &= \frac{K}{4\pi} \int d^3x\, d^4\theta \int_0^1 dt \left\{ {\rm Tr}\Big[V\, \bar{D}^\alpha \left(e^{-tV} D_\alpha\, e^{tV} \right) \Big] - {\rm Tr}\Big[\hat{V}\, \bar{D}^\alpha \left(e^{-t\hat{V}} D_\alpha e^{t\hat{V}} \right) \Big] \right\} \\
\mathcal{S}_{\rm mat} &= \int d^3x\, d^4\theta\, {\rm Tr} \left(\bar{A}_i\, e^V A^i\, e^{-\hat{V}} + \bar{B}^i\, e^{\hat{V}} B_i\, e^{-V} \right) \\
&+ \frac{2\pi i}{K} \int d^3x\, d^2\theta\, \epsilon_{ik}\, \epsilon^{jl}\, {\rm Tr} \left(A^i B_j A^k B_l \right) + \frac{2\pi i}{K} \int d^3x\, d^2\bar{\theta}\, \epsilon^{ik}\, \epsilon_{jl}\, {\rm Tr} \left(\bar{A}_i \bar{B}^j \bar{A}_k \bar{B}^l \right) \qquad (1.29)
\end{aligned}
$$

The Chern-Simons level K is fixed to be an integer by gauge invariance of the action. Its inverse, $1/K$, plays the role of the gauge coupling constant and perturbation theory is valid for large values of K. Moreover, ABJ(M) theory admits a large M, N expansion, defining the 't Hooft coupling constants as $\lambda = M/K$ and $\hat{\lambda} = N/K$, which can be kept fixed while taking M, N and K large.

The quantization of the theory can be easily carried out in $\mathcal{N} = 2$ superspace. After performing gauge fixing (for details, see for instance [21]), the vector propagators are:

$$
\begin{aligned}
\langle V_b^a(1)\, V_d^c(2) \rangle &= \sim\!\sim\!\sim\!\sim\!\sim\!\sim = \frac{4\pi}{K} \frac{1}{p^2}\, \delta_d^a\, {\rm d}_b^c \times \bar{D}^\alpha D_\alpha\, \delta^4(\theta_1 - \theta_2) \\
\langle \hat{V}_{\bar{b}}^{\bar{a}}(1)\, \hat{V}_{\bar{d}}^{\bar{c}}(2) \rangle &= \sim\!\sim\!\sim\!\sim\!\sim\!\sim = -\frac{4\pi}{K} \frac{1}{p^2}\, \delta_{\bar{d}}^{\bar{a}}\, {\rm d}_{\bar{b}}^{\bar{c}} \times \bar{D}^\alpha D_\alpha\, \delta^4(\theta_1 - \theta_2) \qquad (1.30)
\end{aligned}
$$

whereas the matter propagators read:

$$
\begin{aligned}
\langle \bar{A}^{\bar{a}}{}_a(1)\, A^b{}_{\bar{b}}(2) \rangle &= \text{———————} = \frac{1}{p^2}\, \delta^{\bar{a}}_{\bar{b}}\, \delta_a{}^b \times D^2 \bar{D}^2\, \delta^4(\theta_1 - \theta_2) \\
\langle \bar{B}^a{}_{\bar{a}}(1)\, B^{\bar{b}}{}_b(2) \rangle &= \text{———————} = \frac{1}{p^2}\, \delta^a_b\, \delta_{\bar{a}}{}^{\bar{b}} \times D^2 \bar{D}^2\, \delta^4(\theta_1 - \theta_2) \qquad (1.31)
\end{aligned}
$$

where we have introduced the graphical convention, which will be used in computations presented in Chapter 4. The vertices employed in our two-loop calculations can be easily obtained expanding the action (1.29); they are given by:

$$
\begin{aligned}
&\int d^3x\, d^4\theta \Big[{\rm Tr}(\bar{A}_i V A^i) - {\rm Tr}(B_i V \bar{B}^i) + {\rm Tr}(\bar{B}^i \hat{V} B_i) - {\rm Tr}(A^i \hat{V} \bar{A}_i) + \\
&+ \frac{1}{2} {\rm Tr}(\bar{A}_i \{V, V\} A^i) + \frac{1}{2} {\rm Tr}(B_i \{V, V\} \bar{B}^i) + \frac{1}{2} {\rm Tr}(A_i \{\hat{V}, \hat{V}\} \bar{A}^i) + \\
&\quad + \frac{1}{2} {\rm Tr}(\bar{B}_i \{\hat{V}, \hat{V}\} B^i) - {\rm Tr}(\bar{B}^i \hat{V} B_i V) - {\rm Tr}(A^i \hat{V} \bar{A}_i V) \Big] + \\
&\quad + \frac{4\pi i}{K} \int d^3x\, d^2\theta \Big[{\rm Tr}(A^1 B_1 A^2 B_2) - {\rm Tr}(A^1 B_2 A^2 B_1) \Big] \; + \; {\rm h.c.} \qquad (1.32)
\end{aligned}
$$

Three-dimensional Chern-Simons theories coupled to matter are superconformal invariant theories at the classical level. At quantum level, since for $K >> N$ the theory is weakly coupled, a perturbative approach is available. The investigation of the conformal invariance at quantum level was performed up to two loops [22], showing that the theory is conformal invariant event at the quantum level. Furthermore it was shown

[19] that ABJM theory can be embedded in a $\mathcal{N} = 3$ supersymmetric Yang-Mills-Chern-Simons theory by adding a Yang-Mills term in the action (1.29) for the gauge fields. In the infrared region the Yang-Mills gauge coupling flows to infinity and the resulting theory is precisely the ABJM model.

1.5 The AdS/CFT correspondence

One of the most fascinating discoveries in the modern theoretical physics is the AdS/CFT correspondence [23], which relates gauge theories to gravity theories. According to this correspondence some conformal gauge theories are related to a string theory on a curved background. The first and well understood example of such a correspondence related the maximally supersymmetric $\mathcal{N} = 4$ SYM theory, with gauge group $SU(N)$, to type IIB superstring theory on a $AdS_5 \times S^5$ background; the four-dimensional gauge theory lives on the boundary of AdS_5 space (see [24] for a review).
This correspondence is a so-called strong/weak duality, since in the parameter range where one of the two theories is weakly coupled, and can be studied perturbatively, the other one is strongly coupled, and viceversa. On one hand, this fact makes the correspondence very interesting, because it would allow to investigate the non-perturbative regime of a theory by means of perturbative computations performed on the opposite side of the duality. On the other hand, however, it also makes the correspondence very difficult to prove. In fact, no rigorous proof of the conjecture exists at the moment, even if it has passed several non trivial checks.
In its strongest form, the correspondence claims the exact equivalence of the two theories for any values of the parameters. Weaker formulations exist in addition, that are more tractable as they concern particular simplified limits. The main example of such weaker versions is represented by the 't Hooft limit, in which $N \to \infty$ while the 't Hooft coupling $\lambda = \frac{g^2 N}{(4\pi)^2}$ is kept fixed. On the field theory side, non-planar contributions are suppressed in this regime, and perturbative computations can thus be performed in the planar limit, where all the non-planar contributions are neglected. When the field theory is strongly coupled, the string side can be approximated by classical type IIB supergravity on the $AdS_5 \times S_5$ background.

The correspondence can be illustrated in the following way. Let's consider a set of coincident D branes in type IIB superstring theory, as shown in Fig. 1.3. The theory contains an open string ending on the branes which interacts with closed strings. If we take the low energy limit, where the string lenght goes to zero, the open string does not interact anymore with the closed string and the system is decoupled: we find the four-dimensional $\mathcal{N} = 4$ Super Yang-Mills theory living on the brane and a free gravity theory outside. It is possible to consider the same system from a different point of view: D branes are massive charged objects, so a set of these massive objects can be thought of as

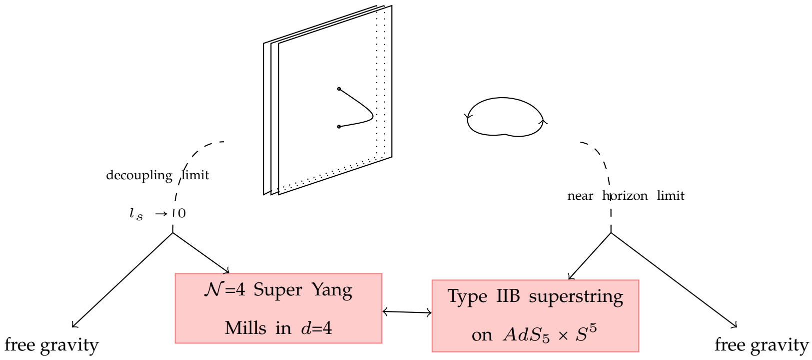


Figure 1.3: Schematic illustration of the AdS/CFT correspondence.

a generalization of a black hole. So D branes deform the space around them. In the low energy limit we find again two decoupled pieces: free gravity on one side and type IIB supergravity on $Ads_5 \times S^5$, which is the low energy limit of type IIB superstring theory, on the other side. We have found from both point of view two decoupled theories, and in both case one of the decoupled systems is free gravity. So, it is natural to identify the second system which appears in both description. We are thus led to the conjecture that at all energies $\mathcal{N} = 4$ SYM is dual to type IIB superstring on a $AdS_5 \times S^5$ background.

A first obvious check concern symmetries on the two sides. These are reported in Table 1.2. The symmetries on both sides are represented by the superconformal group $SU(2,2|4)$, while the isometry of the five-shere is the same of the R-symmetry group of $\mathcal{N} = 4$ SYM theory. In order to understand the correspondence it is important to notice that the field theory lives in a four-dimensional Minkowski space, which can be seen as the boundary of the Anti de Sitter space. The matching of the symmetries is an hint of duality. It is possible also to go further and find a matching of the parameters of the theories. The string theory parameters are the radius R of both AdS_5 and S^5 and the string theory coupling g_s, while the CFT ones are the super Yang-Mills coupling g_{ym} and N. These parameters can be matched into each other:

$$\frac{R^2}{l_s} = g_{ym}\sqrt{N} \sim \sqrt{\lambda} \qquad 4\pi g_s = \frac{\lambda}{N} \tag{1.33}$$

where l_s is the string lenght. Note that the perturbation analysis of the field theory can be trusted when the t'Hooft coupling $\lambda \sim g_{ym}^2 N$ is small. On the other hand the classical gravity description becomes reliable when g_s goes to zero and the radius of curvature R of AdS_5 and S^5 becomes large compared to the string lenght, so when R^2/l_s is large. It's clear from equation (1.33) that these two regime are incompatible. In fact the AdS/CFT is a weak/strong duality: the two theories are conjectured to be exactly the same, but when one side is weakly coupled the other is strongly coupled and viceversa.

$\mathcal{N}=4$ SYM	Type IIB on $Ads_5 \times S^5$
Conformal group $SO(4,2)$	Isometry group of AdS_5: $SO(4,2)$
32 supersymmetries	32 supersymmetries
R-Symmetry $SU(4)$	Isometry of S^5: $SO(6)=SU(4)$

Table 1.2: Summary of the symmetries present in $\mathcal{N}=4$ SYM an in type IIB supergravity on $Ads_5 \times S^5$.

To complete the picture of the correspondence we need a map between the observables in the two theories and a prescription for comparing physical quantities and amplitudes. A basic point is the following statement: a field in AdS space is associated with an operator in the CFT with the same quantum numbers and they know about each other via boundary couplings [25].

The original formulation of the AdS/CFT correspondence was later extended to other theories with less symmetries and to theories living in a different number of spacetime dimensions. The more general AdS/CFT correspondence is a conjectured duality relating any conformal field theory in d dimensions to a gravity theory on an Anti de Siter space in $d+1$ dimensions.

In the investigation of other concrete examples of duality ABJM played a crucial role. In fact, in the large N limit, the ABJM theory has been conjectured to be the AdS/CFT dual description of M-theory on an $AdS_4 \times S_7/Z_k$ background and for $k << N$ of a type IIA string theory on $AdS_4 \times CP_3$ [19]. For this reason, soon after its discovery the ABJM model has quickly become the ideal three-dimensional playground to study AdS/CFT as much as $\mathcal{N}=4$ SYM has been in the four-dimensional case.

Another well known correspondence is between the $\mathbb{Z}_2$ orbifold of $\mathcal{N}=4$ Super Yang-Mills theory, which corresponds to take $g=\hat{g}$ in the two-parameter family of superconformal theories presented in section 1.3, and type IIB superstring theory on $AdS_5 \times S^5/\mathbb{Z}_2$ background. The basic strategy to find this correspondence was to study orbifolds of Type IIB superstring on $AdS_5 \times S^5$ which preserve the AdS structure but break some of the supersymmetries [26]. At strong coupling the dual string description of $\mathcal{N}=2$ SCQCD seems much more problematic than those presented before. There are some proposal for the dual string/supergravity background which turn out to be either singular [27, 28, 29] or related to non critical models [30]. Any advancement on the field theory side might help claryfing the correct properties of the gravitational description.

1.6 Conclusions

In this chapter we have presented superconformal field theories and in particular $\mathcal{N} = 2$ SCQCD and $\mathcal{N} = 6$ ABJM. These are the models in which we are interested in since in Chapter 3 and 4 we will present computations of scattering amplitudes in these theories.

In order to study superconformal theories we have first introduced the basic concepts of supersymmetry and conformal symmetry, which form together the maximal extension of the Poincarè symmetry. Then we have shown that superconformal theories are particularly constrained by these symmetries, at classical and at quantum level. One of the most studied example of superconformal theory is $\mathcal{N} = 4$ SYM, which plays a primary role in the AdS/CFT correspondence, a conjectured duality between superconformal field theories and superstring theories. This correspondence raised the superconformal theories as a powerful tool to investigate gravity.

In the next chapter we will approach the main subject of this thesis, which are scattering amplitudes in superconformal gauge theories.

CHAPTER 2

General properties of scattering amplitudes

The scattering amplitudes of on-shell particles are perhaps the most basic quantities in any quantum field theory. They provide a link between models of nature and experimental data, being thus an indispensable tool for testing theoretical ideas about high energy physics. They also contain a wealth of off-shell information, making their evaluation an important alternative approach to direct off-shell calculations.

Tree-level scattering amplitudes in gauge theories are particularly simple, since they can be completely determined from lower point tree-level amplitudes by certain on-shell recursion relations. In the supersymmetric extensions of Yang-Mills theories these recursion relations can even be implemented at loop level.

On-shell loop amplitudes in massless theories always contain infrared divergences, due to exchange of soft gluons or virtual collinear splittings. The general structure of infrared divergences is well understood, since it was discovered that soft and collinear divergences have a universal form, and the divergent part of higher loop amplitudes can be predicted from lower loop ones. In supersymmetric theories the picture is simpler: the entire loop amplitude can be recovered from the one-loop one. Furthermore scattering amplitudes may exhibit larger symmetries than the Lagrangian, related to hidden symmetries of the theory, like Yangian symmetry and integrability.

Classical approaches to scattering amplitude calculations make use of Feynman diagrams. Symmetries, however, even those of the Lagrangian, are obscured in this approach, re-emerging only after all Feynman diagrams are assembled. For this reason, even at tree-level, the evaluation of multi-leg amplitudes can be quite involved. Multi-loop amplitudes have similar features. Nevertheless Feynman diagrams are a precious tool since scattering amplitudes at any loop order are in principle computable in terms of them; they can also be a guide for identifying new techniques.

The first part of this chapter is dedicated to the introduction of some preliminary concepts emerging in scattering amplitudes of Yang-Mills theory. In particular we will discuss the color structure of amplitudes, the tree-level recursion relations between amplitudes with a different number of external legs and the structure of infrared divergences emerging at loop order. The second part is instead dedicated to the properties of scattering amplitudes in supersymmetric gauge theories, in particular in $\mathcal{N} = 4$ Super Yang Mills theory. For this particular model we will present the extension of the tree-level recursion relations we presented for non supersymmetric theories. We will also introduce the conjectured structure of any loop amplitudes. After that we will present the dual conformal symmetry, which is a new conformal symmetry enjoyed by amplitudes, and the duality with Wilson loops. At the end we will present the main computation techniques available, and in particular the one we used to carry out the resuts presented in the next Chapters 3 and 4.

2.1 Amplitudes in Yang-Mills theory

2.1.1 Color decomposition

In this section we describe some common conventions for organizing the color structure of gauge theory amplitudes. In particular we will introduce the notion of color ordered partial amplitude, which is particularly convenient in the large N limit, in which planar diagrams dominate. Color ordered amplitudes emerge from a trace-based color decomposition.

Let's consider a Yang-Mills theory with gauge group $SU(N)$. In general, we consider two different $SU(N)$ representations for the external states: the adjoint representation, denoted with the adjoint index $a \in \{1, 2, \ldots, N^2-1\}$ for gluons, and the fundamental representation N, together with its conjugate representation $\bar{N}$, for quarks and anti-quarks respectively. Fundamental color indices are denoted by a lower $i_1, i_2 \cdots \in \{1, 2, \ldots, N\}$ and anti-fundamental indices are denoted by an upper $j_1, j_2 \cdots \in \{1, 2, \ldots, N\}$. The generators of $SU(N)$ in the fundamental representation are traceless hermitian $N \times N$ matrices, denoted by $(T^a)_i^{\,j}$. The color factor for a generic Feynman diagram in a $SU(N)$ Yang-Mills theory contains a factor of $(T^a)_i^{\,j}$ for each gluon-quark-antiquark vertex, and the $SU(N)$ structure constants f^{abc}, defined by $[T^a, T^b] = i f^{abc} T^c$, for each pure three-gluon vertex. Each four-gluon vertex has a contracted pair of structure constants $f^{abc} f^{cde}$; these are the only vertices we will use in our computations. The gluon and quark propagators contract many of the indices together with δ_{ab}, $\delta_i^{\,j}$ factors.

In order to identify all different types of color structure that can appear in a given amplitude we have to rewrite the structure constant f^{abc} in terms of T^a generators, using the relation:

$$f^{abc} = -i\,\mathrm{tr}([T^a, T^b] T^c) \qquad (2.1)$$

which follows from the definition of the structure constants. Conventions and normalizations concerning the generators of $SU(N)$ are summarized in A.1. After inserting repeatedly into the color factor for a typical Feynman diagram, one obtains a large number of traces of the generic form $\mathrm{tr}(\ldots T^a \ldots)\ \mathrm{tr}(\ldots T^a \ldots)\ldots \mathrm{tr}(\ldots)$. If the amplitude has external quark legs, then there will be also strings of T^a's terminated by fundamental indices, of the form $(T^{a_1}\ldots T^{a_m})_i^{\ j}$, one for each external quark-antiquark pair. The number of traces can be reduced considerably by repeated use of the $SU(N)$ Fierz identity:

$$(T^a)_{i_1}^{j_1}(T^a)_{i_2}^{j_2} = \delta_{i_1}^{j_2}\delta_{i_2}^{j_1} - \frac{1}{N}\delta_{i_1}^{j_1}\delta_{i_2}^{j_2} \tag{2.2}$$

The equation (2.2) is just the statement that the $SU(N)$ generators T^a form the complete set of traceless hermitian $N \times N$ matrices.

In the case of tree-level amplitudes with external states in the adjoint representation, such as n-gluon amplitudes, the color factors may be reduced to a single trace $\mathrm{tr}(T^{a_{\sigma(1)}}\ldots T^{a_{\sigma(n)}})$ for some permutation $\sigma \in S_n$ of the n-gluons. This reduction leads to the trace-based color decomposition for n-gluon tree amplitudes:

$$\mathcal{A}_n^{tree}(\{p_i, a_i\}) = \sum_{\sigma \in S_n/Z_n} \mathrm{tr}(T^{a_{\sigma_1}}\ldots T^{a_{\sigma_n}})\ A_n^{tree}(\sigma_1\ldots\sigma_n) \tag{2.3}$$

Here $\mathcal{A}_n^{tree}$ is the full amplitude, with dependence on the external gluon momenta p_i, $i = 1,\ldots,n$ and adjoint indices a_i. The color ordered *partial amplitudes* $A_n^{tree}(\sigma_1\ldots\sigma_n)$ have all the color factors removed, but contain the kinematic information. Cyclic permutations of the arguments of a partial amplitude, denoted by Z_n leave it invariant, because the associated trace is invariant under these operations. However, all $(n-1)!$ non-cyclic permutations, or orderings, of the partial amplitude appear in eq. (2.3). These permutations are denoted by $\sigma \in S_n/Z_n \equiv S_{n-1}$. Partial amplitudes $A_n^{tree}(1\ldots n)$ are simpler than the full amplitude because they only receives contributions from tree-level Feynman diagrams that can be drawn on a plane, in which the cyclic ordering of the external legs $1,2,\ldots,n$ matches the ordering of the arguments in A_n^{tree}. Therefore each partial amplitude can only have singularities in momentum invariants formed by squaring color-adjacent sums of momenta, such as $s_{i,i+1} \equiv (p_i + p_{i+1})^2$. In this way, the color decomposition disentangles the kinematic complexity of the full amplitude.
Similarly, tree amplitude $q\bar{q}gg\ldots g$ with two external quarks can be reduced to single strings of T^a matrices:

$$\mathcal{A}_n^{tree}(\{p_i, a_i\}) = \sum_{\sigma \in S_{n-2}} (T^{a_{\sigma_3}}\ldots T^{a_{\sigma_n}})^j{}_i\ A_n^{tree}(1_q 2_{\bar{q}}\sigma_3\ldots\sigma_n) \tag{2.4}$$

One might be worry that the color decomposition will lead to a huge proliferation in the number of partial amplitudes that have to be computed. Actually the decomposition of n-gluon amplitude, written in eq. (2.3) in terms of $(n-1)!$ single trace color structure, is overcomlete. In fact it is possible to find the following properties for the color ordered amplitudes: 1) cyclicity : $A_n(12\ldots n) = A_n(2\ldots n1)$; 2) reflection:

$\mathcal{A}_n(12\ldots n) = (-1)^n \mathcal{A}_n(n\ldots 21)$; 3) $U(1)$ decoupling identity:

$$\mathcal{A}_n^{tree}(123\ldots n) + \mathcal{A}_n^{tree}(132\ldots n) + \cdots + \mathcal{A}_n^{tree}(13\ldots n2) = 0 \tag{2.5}$$

Furthermore Kleiss and Kuijf found a linear relation between partial tree-level amplitudes [31, 32]. These relations combine with the other identities to reduce the number of indipendent n-gluon amplitudes to $(n-2)!$. However there are further linear relationships, called BCJ relations [33] that reduce the number of independent n-gluon color ordered tree amplitude to $(n-3)!$. The BCJ relations hinge on the color-kinematic duality, which states that the scattering amplitude of Yang-Mills theory can be given in a representation where the purely kinematic part has the same algebraic properties of the corresponding color factors. Bern, Carrasco, and Johansson showed that the assumption of color-kinematic duality implies a previously unknown set of relations among tree-level color ordered n-gluon amplitudes. The discovery of color-kinematic duality in gauge-theory amplitudes is particularly interesting because it also relates gauge theory amplitudes with gravity amplitudes through the BCJ double-copy procedure.

Color decomposition at loop level are equally straightforward. The general color decomposition is similar to the previous one, except that multiple color traces may be now generated. The general l-loop color decomposition simplifies a lot in the large N limit, in fact the leading terms in the planar limit can be written compactly in the form:

$$\mathcal{A}_n^{(l)}(\{p_i, a_i\})|_{N\to\infty} = \sum_{\sigma\in S_n/Z_n} \mathrm{tr}(T^{a_{\sigma_1}}\ldots T^{a_{\sigma_n}})\mathcal{A}_n^{(l)}(\sigma_1\ldots\sigma_n) \tag{2.6}$$

The general color composition for the l-loop amplitude outside the planar limit can be found in [34].

2.1.2 Parke-Taylor formula and MHV amplitudes

Here we introduce a different classification of gluons amplitudes, based on the helicity of gluons. In a work done in 1980s, Parke and Taylor [35] found that, when considering the scattering of many gluons, certain classes of amplitudes vanish at tree-level; in particular when fewer than two gluons have negative helicity and all the rest have positive helicity they found:

$$A_n(1^+2^+\ldots n^+) = 0 \qquad \text{and} \qquad A_n(1^-2^+\ldots n^+) = 0 \tag{2.7}$$

So $A_n(1^-2^-3^+\ldots n^+)$ is the "first" non-vanishing gluon amplitude, in the sense that having fewer negative helicity gluons gives a vanishing amplitude. More negative helicity states are also allowed, but one needs at least two positive helicity states to get a non-vanishing result, except for $n = 3$. The amplitudes with two negative helicity gluons $A_n(1^-2^-3^+\ldots n^+)$ are called maximally helicity violating (MHV) amplitudes. These amplitudes are called MHV because at tree-level they violate helicity conservation to the maximum extent possible. The amplitudes with $K+2$ negative helicity gluons are called

N^KMHV amplitudes.

MHV gluon amplitudes may be calculated very efficiently by means of the Parke-Taylor formula, since they have an extremely simple form in terms of momentum bilinears:

$$A_n(1^+ \dots i^- \dots j^- \dots n^+) = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \dots \langle n1 \rangle} \tag{2.8}$$

where $\langle ij \rangle = \epsilon^{\alpha\beta} \lambda_{i\alpha} \lambda_{j\beta}$ and $\lambda_{i\alpha}$ are spinors [1] which satisfy the massless Weyl equation and are associated to outgoing gluons with positive helicity. A rigorous derivation of the Parke-Taylor result was given by Berends and Giele [37]. The compactness of eq. (2.8) makes MHV amplitudes extremely attractive. In fact computing n-point amplitude using Feynman diagrams can be really demanding when n is large, since the number of diagrams grows fast as the number of external particles increases. In case of MHV amplitudes it is possible to avoid such complicate calculation and get the result from (2.8) in a simple way.

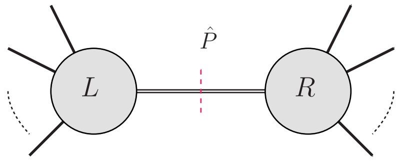
2.1.3 BCFW recursion relations

Let's now focus on the properties of tree-level scattering amplitudes in a Yang-Mills theory. A remarkable discovery consist of the presence of recursion relations among tree-level amplitudes with a different number of external legs. In particular recursion relations provide a method for building higher-point amplitudes from lower-point amplitudes. In 1988, Berends-Giele developed off-shell recursion relations [37] to construct n-point parton amplitudes from building blocks with one leg off-shell. This off-shell method remains useful as an algorithm for efficient numerical evaluation of scattering amplitudes. Here we focus on the newer recursive methods, the so-called BCFW recursion relations fomulated in 2005 by Britto, Cachazo, Feng and Witten, whose building blocks are themselves on-shell amplitudes. These on-shell recursion relations are elegant as they use input only from gauge invariant objects and they are very powerful for elucidating the mathematical structure of on-shell scattering amplitudes. The key idea is to use the power of complex analysis to exploit the analytic properties of on-shell scattering amplitudes.

Let's consider a n-point tree-level amplitude: $\mathcal{A}_n(p_1, \dots, p_n)$, with $p_i^2 = 0$. The momentum conservation is imposed: $\sum_{i=1}^n p_i^{\alpha\dot{\alpha}} = 0$. The amplitude is a rational function, with product of propagators at the denominator. The idea is to introduce a linear shift in the complex variable z for the external momenta and to study, with the tools of complex analysis, the new amplitude $\hat{\mathcal{A}}_n(z)$. The BCFW shift is the particular shift on the momenta p_i and p_j:

$$p_i \to \hat{p}_i = p_i + zq \qquad p_j \to \hat{p}_j = p_j - zq \tag{2.9}$$

[1]For insights into spinor helicity formalism, see [36].



with q such that: $q^2 = 0$ and $q \cdot p_i = q \cdot p_j = 0$. Note that the momentum conservation and the on-shell condition hold also for shifted momenta. So it is possible to study the tree-level amplitude as function of shifted momenta:

$$\mathcal{A}_n(p_1, \ldots, p_n) \to \hat{\mathcal{A}}_n(p_1, \ldots, \hat{p}_i, \ldots, \hat{p}_j, \ldots, p_n) \tag{2.10}$$

The shifted amplitude $\hat{\mathcal{A}}_n$ is a function with simple poles in $z_\star$, which satisfy: $\hat{P}^2 = (P + z_\star q)^2 = 0$, where P is the unshifted propagator. The poles are located at: $z_\star = -P^2/(2P \cdot q)$ and the shifted propagators can be written as $\hat{P}^2 = -P^2(z - z_\star)/z_\star$. So $\hat{\mathcal{A}}_n(z)$ has only simple poles, located away from the origin. Let's look at $\frac{\hat{\mathcal{A}}_n(z)}{z}$ in the complex plane. Pick a countour that surrounds all the simple poles. Using the residue's theorem we find:

$$\oint \mathrm{d}z \frac{\hat{\mathcal{A}}_n(z)}{z} = \mathrm{Res}\left[\frac{\hat{\mathcal{A}}_n(z)}{z}, 0\right] + \sum_{z_\star \neq 0} \mathrm{Res}\left[\frac{\hat{\mathcal{A}}_n(z)}{z}, z_\star\right] \tag{2.11}$$

If we assume that $\hat{\mathcal{A}}_n(z) \to 0$ for $z \to \infty$, the left hand side of eq. (2.11) vanish. In the right hand side, at a $z_\star$ pole, the shifted propagator $\hat{P}^2$ goes on-shell. In this case the shifted amplitude factorizes into two on-shell part: $\hat{\mathcal{A}}_L$ and $\hat{\mathcal{A}}_R$, as shown in the figure 2.1.3. The value of the residues is:

$$\mathrm{Res}\left[\frac{\hat{\mathcal{A}}_n(z)}{z},\ 0\right] = \mathcal{A}_n \tag{2.12}$$

$$\begin{aligned}\mathrm{Res}\left[\frac{\hat{\mathcal{A}}_n(z)}{z}, z_\star\right] &= \lim_{z \to z_\star}\left[\frac{z - z_\star}{z}\hat{\mathcal{A}}_L \frac{1}{\hat{P}^2}\hat{\mathcal{A}}_R\right] = \lim_{z \to z_\star}\left[\frac{z - z_\star}{z}\hat{\mathcal{A}}_L \frac{-z_\star}{P^2(z - z_\star)}\hat{\mathcal{A}}_R\right] \\ &= -\hat{\mathcal{A}}_L(z_\star)\frac{1}{P^2}\hat{\mathcal{A}}_R(z_\star)\end{aligned} \tag{2.13}$$

So, inserting the residues in eq. (2.11), we find:

$$\mathcal{A}_n = \sum \hat{\mathcal{A}}_L(z_\star)\frac{1}{P^2}\hat{\mathcal{A}}_L(z_\star) \tag{2.14}$$

This is the general form of on-shell recurrence relations: an n-point amplitude is expressed as a sum of products of on-shell amplitudes $\hat{\mathcal{A}}_L$ and $\hat{\mathcal{A}}_R$, which have fewer external legs. The sum is over all possible factorization channels. There is also an implicit sum over all possible particle states that can be exchanged on the internal line. The recursive formula (2.14) gives a manifest gauge invariant construction of scattering amplitudes.

In carrying out the result (2.14) we made the assumption that $\hat{\mathcal{A}}_n(z)$ vanishes when z is large. It is possible to justify this assumption in a pure Yang-Mills theory using the background field method [38]. Since the kinematical structure of all three-point amplitudes for on-shell massless particles is uniquely fixed by Poincare invariance and locality, we can assert that it is possible to construct recursively every n-point gluon tree amplitude from the input of just the three-point gluon amplitude.

The BCFW shift is not the only shift that can be constructed. For example it is possible to introduce a linear shift in the complex variable z for all momenta, and not just for two of them, as in the BCFW case. This particular all-line shift produces recursion relations that allow to construct all non-MHV amplitudes from the MHV ones. In particluar the N^KMHV tree amplitude is written as a sum over all tree-level diagrams with $K+1$ MHV vertices. This construction of the amplitude is called the MHV vertex expansion, or CSW expansion, and it can be viewed as the closed form solution to the all-line shift recursion relations. However this construction was discovered by Cachazo, Svreck and Witten [39] in 2004, before the introduction of recursion relations from complex shifts. The authors used the geometric interpretation of MHV amplitudes in twistor space to develop a tool for "sewing" MHV amplitudes together (with some off-shell continuation) to build diagrams out of MHV vertices.

2.1.4 Dealing with IR divergences

Up to now we have focused exclusively on tree-level amplitudes. The loop corrections to the tree-level results are highly relevant for our understanding of the mathematical structure of the scattering processes.

An l-loop amplitude can be written schematically as:

$$\mathcal{A}_n^{(l)} = \sum_i \int \left(\prod_{j=1}^{l} \frac{\mathrm{d}^d k_j}{(2\pi)^d} \right) \frac{1}{S_i} \frac{n_i c_i}{\prod_{\alpha_i} p_{\alpha_i}^2} \tag{2.15}$$

where the sum runs over all possible l-loop Feynman diagrams i. For each diagram k_j label the l-loop momenta, α_i label the propagators and n_i the kinematic numerators, which are contractions of external and loop momenta. The symmetry factors are represented by S_i, while the constants c_i capture the couplings and the color factors.

The analytic structure of loop amplitude is more complicated than the tree-level one. In fact the loop amplitude may have ultraviolet and infrared divergences. The origin and the structure of ultraviolet divergences in quantum field theories is well understood. They can be absorbed into a renormalization of the parameters of the theory. The fact that physical results must be independent of the UV regulator introduced in intermediate steps of a calculation gives rise to powerful constraints, which are summarized by

the renormalization group equations of the theory.
Perturbative results for on-shell scattering amplitudes in theories with massless fields also contain infrared singularities, which originate from loop-momentum configurations where particle momenta become soft or collinear. These singularities cancel in physical observables, which are insensitive to soft and collinear emissions. Infrared divergences are usually regulated using the dimensional regularization method, where the integrations over the loop momenta are performed in $d - 2\epsilon$ dimensions, where ϵ is called the dimensional regularization parameter and it is assumed to be small and negative. Feynman integrals can be analytically computed using this method; the infrared divergences manifest themselves as poles in ϵ.

The general structure of infrared divergences of QCD is well understood from the 1990s in the context of dimensional regularization. Magnea and Sterman in 1990 and later Sterman and Tejeda-Yeomans in 2003 found that soft and collinear divergences have a universal form and they can be factorized from each other and from a hard, short distance part of the amplitude [40, 41]. So the amplitude can be written as a product of functions that organize the contributions of momentum regions relevant to infrared poles in the scattering amplitude in the following way:

$$\mathcal{A}_n^{(l)} = h\left(p_i, \frac{Q^2}{\mu^2}, \alpha_s(\mu), \right) \times J\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu), \epsilon\right) \times S\left(p_i, \frac{Q^2}{\mu^2}, \alpha_s(\mu), \epsilon\right) \tag{2.16}$$

where μ is the renormalization scale, Q is the physical scale associated with the scattering process for external momenta p_i, and α_s is the running coupling constant. The functions in eq. (2.16) are: i) a process-dependent function h, that describe the short distance dynamics of the hard scattering; this function has no infrared singularities, but generically depends on the particle types, colors and kinematics; ii) the jet function J, which captures the collinear divergences, depends only on the type of external particles, but not on the full amplitude kinematics, and describes the perturbative evolution of the incoming and outgoing particles; iii) soft divergences are described by S; they come from exchange of long-wavelenght gluons. The soft function S does not depend on the particle types, but only on their momenta and color quantum numbers.

An important step toward an universal all-order result for the infrared singularity in QCD was made in 1998 by Catani [42], who correctly predicted the singularities of two-loop amplitudes apart from the $1/\epsilon$ pole term. Catani carried out his results working in color space. In this formalism an amplitude can be written as an internal product between $\mathcal{C} \equiv |c_1, \dots, c_n\rangle$, a unit vector in the n-parton color space, and $A_n \equiv |A_n(p_1, \dots, p_n)\rangle$, an abstract vector in colour space which is a color singlet:

$$A_n^{c_1 \dots c_n} \equiv (\mathcal{C}, A_n) = \langle c_1, \dots, c_n | A_n(p_1, \dots, p_n)\rangle \tag{2.17}$$

where c_i are the color indices of the n-parton color amplitude $A_n^{c_1 \cdots c_n}$. For gluons we have: $c_i = 1, \ldots, N^2 - 1$, while for quarks and antiquarks: $c_i = 1, \ldots, N$.

The one-loop amplitude has poles in $1/\epsilon$ and $1/\epsilon^2$. The coefficients of these poles are universal and are given by the following formula:

$$A_n^{(1)}(p_i) = I^{(1)}(\epsilon, p_i) A_n^{tree}(p_i) + A_n^{(1\,fin)}(p_i) \tag{2.18}$$

The contribution $A_n^{(1\,fin)}$ on the right-hand side is finite for $\epsilon \to 0$ and, hence, in (2.18) all one-loop singularities are factorized in colour space with respect to the tree-level amplitude A_n^{tree}. The singular dependence is embodied in the factor $I^{(1)}$ that acts as colour-charge operator onto the colour vector A_n^{tree}. Its explicit expression in terms of the colour charges of the n-partons can be find in [42]. Here we present the leading term in ϵ:

$$I^{(1)}(\epsilon, p_i) = -\frac{1}{2\epsilon^2} \sum_i \eta_i + \mathcal{O}(1/\epsilon) \tag{2.19}$$

where η_i is a color coefficient equal to $(N^2-1)/2N$ if the particle i is a quark or antiquark, or to N if the particle i is a gluon.

At two loops the amplitude has $1/\epsilon^4$, $1/\epsilon^3$, $1/\epsilon^2$ and $1/\epsilon$ poles. Because of the increasing degree of singularities it is not a priori guaranteed that all of them can be controlled by a universal factorization formula as in the one-loop case. Nevertheless Catani showed that a factorization formula exists and can be written in the following form:

$$\begin{aligned} A_n^{(2)}(p_i) = {} & I^{(1)}(\epsilon, p_i) A_n^{(1)}(p_i) + I^{(2)}(\epsilon, p_i) A_n^{tree}(p_i) + \\ & + A_n^{(2\,fin)}(p_i) \end{aligned} \tag{2.20}$$

The main features of the right hand side of eq. (2.20) are the following: the first term contains poles of the type $1/\epsilon^n$, with $n = 1, \ldots, 4$, coming from the single and double poles of $I^{(1)}$ and $A_n^{(1)}$. The second term contains a new operator $I^{(2)}$, given as follows:

$$\begin{aligned} I^{(2)}(p_i) = {} & -\frac{1}{2} I^{(1)}(\epsilon, p_i) \left(I^{(1)}(\epsilon, p_i) + \frac{4\pi\beta_0}{\epsilon} \right) \\ & + \frac{e^{\gamma_E \epsilon} \Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)} \left(\frac{2\pi\beta_0}{\epsilon} + K \right) I^{(1)}(2\epsilon, p_i) + H^{(2)}(\epsilon, p_i) \end{aligned} \tag{2.21}$$

where $\beta_0 = 11/3N - 2/3N_f$ is the first coefficient of the QCD beta function with gauge group $SU(N)$ coupled to N_f quarks, while $K = (67/18 - \pi^2/6)N - 5/9N_f$ and $H^{(2)}$ is an unknown function which contains only single poles. The last term $A_n^{(2\,fin)}$ is a non singular reminder analogous of $A_n^{(1\,fin)}$. Using the factorization formula (2.20) all the coefficients of the poles $1/\epsilon^4$, $1/\epsilon^3$, $1/\epsilon^2$ can be explicitly evaluated in terms of the one-loop operator $I^{(1)}$, the first coefficient of the beta function and the constant K.

In 2009 Becher and Neubert [43, 44] proposed a generalization of Catani's result, valid for an arbitrary on-shell n-point scattering amplitude. Using their formula they derived the three-loop coefficients of the $1/\epsilon^n$ pole terms (with $n = 1, \ldots, 6$) for an arbitrary n-point scattering amplitude in massless QCD, generalizing the Catani's two-loop result (2.20). They found the all-order structure of infrared singularities observing that the IR singularities of on-shell amplitudes in massless QCD are in a one-to-one correspondence to the UV poles of a certain operator matrix elements in a soft-collinear effective theory.

2.2 Amplitudes in $\mathcal{N}$ = 4 Super Yang-Mills

2.2.1 Superamplitudes

When supersymmetry is present there is a general enhancement of the tree-level and loop properties of scattering amplitudes we found in the previous section. The exceptional simplicity and numerous hidden symmetries of $\mathcal{N} = 4$ Super Yang-Mills theory have made it a playgroung for theorists interested in scattering amplitudes. Since the tree-level amplitudes of QCD coincide with tree-level amplitude of $\mathcal{N} = 4$ Super Yang-Mills theory, the early discoveries about QCD were also discoveries of $\mathcal{N} = 4$ SYM.

We want to study how the presence of supersymmetry affects the amplitudes in $\mathcal{N} = 4$ Super Yang-Mills theory, presented in 1.2. We can think of n-point amplitude with all outgoing particles as the S-matrix element $\langle 0|\mathcal{O}_1(p_1)\ldots\mathcal{O}_n(p_n)|0\rangle$ in which the n annihilation operators $\mathcal{O}_i(p_i)$, with $i = 1, \ldots, n$ act to the left on the out-vacuum. If the vacuum is supersymmetric: $Q|0\rangle = Q^\dagger|0\rangle = 0$, then for any set of n annihilation (or creation) operators we have:

$$\begin{aligned} 0 &= \langle 0|[Q^\dagger, \mathcal{O}_1(p_1)\ldots\mathcal{O}_n(p_n)]|0\rangle \\ &= \sum_{i=1}^{n}(-1)^{\sum_{j<i}|\mathcal{O}_j|}\langle 0|\mathcal{O}_1(p_1)\ldots[Q^\dagger, \mathcal{O}_i(p_i)]\ldots\mathcal{O}_n(p_n)|0\rangle \end{aligned} \tag{2.22}$$

and similarly for Q. In eq. (2.22) we had to take into account that a minus sign is picked up from every time $Q^\dagger$ passes by a fermionic operator: so $|\mathcal{O}|$ is 0 when the operator is bosonic and 1 if fermionic. Using the action of supersymmetry generators on free asymptotic states, the equation (2.22) will describe a linear relation among scattering amplitudes whose external states are related by supersymmetry. Such relations are called *supersymmetry Ward identities*. They were first studied in 1977 by Grisaru and Pendleton [45] and have since then had multiple applications. So, we found that amplitudes with external states related by supersymmetry are related to each other through the supersymmetric Ward identities.

We already introduced N^KMHV amplitudes in the previous section, as the gluon amplitudes with $K + 2$ negative helicity gluons. We can define the N^KMHV sector of $\mathcal{N} = 4$ SYM to be the one with all amplitudes connected to the N^KMHV gluon amplitude via supersymmetry. So, in $\mathcal{N} = 4$ SYM all MHV amplitudes are proportional to

$A_n(g^- g^- g^+ \dots g^+)$: since the supermultiplet in $\mathcal{N} = 4$ SYM is CPT self-conjugate, supersymmetry generators connect all the states, both positive and negative helicity states. Furthermore MHV gluon amplitudes can be calculated by means of the Parke-Taylor formula (2.8), so in $\mathcal{N} = 4$ SYM one single amplitude determines the entire MHV class.

Since $\mathcal{N} = 4$ SYM is a superconformal theory, we should clarify what we mean by scattering matrix. Since loop scattering amplitudes are IR divergent, one way to deal with this is to regulate the divergences using dimensional regularization method, namely considering the theory in $d = 4 - \epsilon$ dimension. The dimensional regularization breaks the conformal and the dual conformal symmetry slightly, because the integration measure is no longer four-dimensional (see [46] for details). In this way the scattering matrix is well defined, but the on-shell symmetries are not manifest. An alternative consists of considering the theory on the Coulomb branch of the moduli space, where the scalars acquire vevs in such a way that full supersymmetry is preserved, and define the $\mathcal{N} = 4$ SYM scattering matrix as the zero-vev limit of the Coulmb branch scattering matrix.

2.2.2 Super-BCFW recursion relations

Superamplitudes can be constructed with a supersymmetric version of the BCFW shift. The BCFW shift introduced in 2.1.3 preserves the on-shell conditions $p_i^2 = 0$ and momentum conservation $\sum_{i=1}^{n} p_i^{\alpha\dot{\alpha}} = 0$, but it is clear that it does not preserve supermomentum conservation. This can be remedied by a small modification of the BCFW shift, which include a shift in the Grassmann variables [47, 48]. In order to keep manifest the on-shellness condition, it is common to introduce the super BCFW recursion relation using the on-shell superspace formalism [2] of $\mathcal{N} = 4$ SYM. Let's introduce four Grassmann variables η_A, labeled by the $SU(4)$ index $A = 1, 2, 3, 4$. The new variables allow to collect the 16 physical states into an $\mathcal{N} = 4$ on-shell chiral superfield $\mathcal{Z}$:

$$\mathcal{Z} = g^+ + \eta_A \lambda^A - \frac{1}{2}\eta_A \eta_B S^{AB} - \frac{1}{3!}\eta_A \eta_B \eta_C \lambda^{ABC} + \eta_1 \eta_2 \eta_3 \eta_4 g^- \tag{2.23}$$

Thus it is possible to express the on-shell superamplitude in terms of the on-shell superfields: $\mathcal{A}_n(Z_1 \dots Z_n)$ and to extract any amplitude from the on-shell one using the Grassmann differential operators $\frac{\partial}{\partial \eta_{iA}}$, which act on the superfileds Z_i of the on-shell superamplitude. For example:

$$\mathcal{A}_n(g^- g^- g^+ \dots g^+) = \left(\prod_{A=1}^{4} \frac{\partial}{\partial \eta_{1A}} \right) \left(\prod_{B=1}^{4} \frac{\partial}{\partial \eta_{2B}} \right) \mathcal{A}_n(Z_1 Z_2 \dots Z_n)\,|_{\eta_{iC=0}, i>2} \tag{2.24}$$

With this notation the BCFW supershift is defined as follows:

$$p_i \to \hat{p}_i = p_i + zq \qquad p_j \to \hat{p}_j = p_j - zq \qquad \hat{\eta}_{iA} = \eta_{iA} + z\eta_{jA} \tag{2.25}$$

[2]There is a connection between the $\mathcal{N} = 4$ on-shell superspace formalism and the $\mathcal{N} = 1$ off-shell one, used in all other parts of this thesis. For an explicit derivation see [46].

where η_{iA} are the on-shell Grassmann variables associated to the on-shell superfield Z_i. In this way the supermomentum is invariant under the supershift. With this choice the recursion relations that result from the BCFW supershift are recovered as in 2.1.3: they involve diagrams with two vertices connected by an internal line with on-shell momentum $\hat{P}$. As in the non-supersymmetric case we must sum over all possible states that can be exchanged in the internal line: in this case this include all 16 states of $\mathcal{N} = 4$ SYM. The superamplitude version of the implicit helicity sum presented in eq. (2.14) is realized in the following way:

$$\mathcal{A}_n = \left(\prod_{A=1}^{4} \frac{\partial}{\partial \eta_{\hat{P}_A}}\right) \left[\hat{\mathcal{A}}_L \frac{1}{P^2} \hat{\mathcal{A}}_R\right]_{\eta_{\hat{P}_A}=0} = \int \mathrm{d}^4 p\, \hat{\mathcal{A}}_L \frac{1}{P^2} \hat{\mathcal{A}}_R \tag{2.26}$$

where $\eta_{\hat{P}_A}$ is the Grassmann variable associated with the internal line. The super-BCFW recursion relations can be solved to give closed-form expressions for all tree-level superamplitude, both MHV and non MHV [49]. For MHV amplitudes the super-BCFW method is a modern tool to derive the supersymmetric generalization of Parke-Taylor formula.

2.2.3 BDS ansatz

When we introduced $\mathcal{N} = 4$ SYM theory we mentioned that it is a conformal theory, since there is no running of the coupling. This means that all ultraviolet divergences cancel in the on-shell scattering amplitude, order by order in perturbation theory. The loop amplitudes still have infrared divergences, as in a typical theory with massless particles as explained in section 2.1.4. The fact that the IR divergences factorize is a general statement about massless gauge theories and so it is valid also in presence of supersymmetry.

For QCD we presented the Catani's result (2.20), in which the singularities of a two-loop four-point amplitude up to $1/\epsilon$ were factorized in a universal way. It is possible to extend to $\mathcal{N} = 4$ SYM the results presented in section 2.1.4 about the general structure of infrared divergences. We will focus on MHV loop amplitudes in planar $\mathcal{N} = 4$ SYM. Since a special property of MHV loop amplitudes is that all their leading singularities are proportional to the MHV tree-level amplitude, the interesting quantity is the reduced amplitude $\mathcal{M}_n^{(l)} = \mathcal{A}_n^{(l)}/\mathcal{A}_n^{tree}$, namely the l-loop amplitude divided by the tree-level one. The general structure of the one-loop n-point reduced amplitude in planar $\mathcal{N} = 4$ SYM is:

$$\mathcal{M}_n^{(1)} = \lambda\, I_n^{(1)}(\epsilon) + \mathcal{M}_n^{(1\,fin)} \tag{2.27}$$

where $\lambda = g^2 N/(4\pi)^2$ contains the information about the coupling and the color factors, while $\mathcal{M}_n^{(1\,fin)} = \lambda\, C^{(1)}$ is the finite part, where $C^{(1)}$ is a constant. We captured the divergences of the planar one-loop n-point amplitude in the function $I_n^{(1)}(\epsilon)$, which has the following expression[3]:

$$I_n^{(1)}(\epsilon) = -\frac{1}{\epsilon^2}\sum_{i=1}^{n}\left(\frac{\mu^2}{s_{i,i+1}}\right)^{\epsilon} \tag{2.28}$$

where μ is the IR scale of dimensional regularization and the invariants $s_{i,i+1} = (p_i + p_{i+1})^2$ are related to the Mandelstam variables.
It is possible to write the general all-loop structure of infrared divergences of the n-point reduced amplitude of $\mathcal{N} = 4$ SYM in a very compact way. In fact, thanks to the simpler structure of $\mathcal{N} = 4$ SYM amplitudes compared to the QCD ones, it is found that the infrared divergences exponentiates, which means that the loop corrections exhibit an iterative structure, which can be summarized in the following expression:

$$\mathcal{M}_n\Big|_{IR} = \exp\left[\sum_{l=1}^{\infty}\lambda^l f^{(l)}(\epsilon) I_n^{(1)}(l\epsilon)\right] \tag{2.29}$$

The function $I_n^{(1)}(l\epsilon)$ is defined in (2.28) and it contains the divergences of the one-loop amplitude with the substitution $\epsilon \to l\epsilon$. We introduced the function $f^{(l)}(\epsilon) = \Gamma_{cusp}^{(l)} + l\epsilon\Gamma_{col}^{(l)}$, thus the leading infrared divergence is governed by $\Gamma_{cusp}(\lambda) = \sum_l \lambda^l \Gamma_{cusp}^{(l)}$, the cusp anomalous dimension, a quantity which is so-called because it arises as the leading ultraviolet divergence of Wilson loops with light-like cusps, as we will see in the next section. The function $\Gamma_{col}(\lambda) = \sum_l \lambda^l \Gamma_{col}^{(l)}$ governs the subleading infrared divergence and it is sometimes called the "collinear" anomalous dimension. For the one-loop n-point amplitude it is found that: $\Gamma_{cusp}^{(1)} = 1$ and $\Gamma_{col}^{(1)} = 0$.

The general structure of the finite part is not known for QCD. In $\mathcal{N} = 4$ SYM instead it was found a strong enhancement of the QCD results. In fact the analytical expression for the two-loop four-point reduced amplitude $\mathcal{M}_4^{(2)}$ in planar $\mathcal{N} = 4$ SYM was shown by Anastasiou, Bern, Dixon and Kosower (ABDK) [50] to be expressible in terms of the one-loop reduced amplitude as:

$$\mathcal{M}_4^{(2)} = \frac{1}{2}\left(f^{(1)}(\epsilon)\mathcal{M}_4^{(1)}(\epsilon)\right)^2 + \lambda f^{(2)}(\epsilon)\mathcal{M}_4^{(1)}(2\epsilon) + \lambda^2 C^{(2)} + \mathcal{O}(\epsilon) \tag{2.30}$$

where $f^{(1)}(\epsilon) = 1$, $f^{(2)}(\epsilon) = -\zeta(2) - \zeta(3)\epsilon - \zeta(4)\epsilon^2$, while $C^{(2)} = -\zeta(2)^2/2$ is a constant, and $\zeta(s)$ is the Riemann zeta function evaluated at positive integer s, with $\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$.
By explicit calculation of the three-loop four-point amplitude Bern, Dixon, and Smirnov (BDS) [51] found that the iterative structure continues:

$$\mathcal{M}_4^{(3)} = -\frac{1}{3}\left(f^{(1)}(\epsilon)\mathcal{M}_4^{(1)}(\epsilon)\right)^3 + f^{(1)}(\epsilon)\mathcal{M}_4^{(1)}(\epsilon)\mathcal{M}_4^{(2)}(\epsilon) + \lambda^2 f^{(3)}(\epsilon)\mathcal{M}_4^{(1)}(3\epsilon) + \lambda^3 C^{(3)} + \mathcal{O}(\epsilon) \tag{2.31}$$

[3]Note that we obtain the function $I_n^{(1)}(\epsilon)$, given in (2.28), with a slightly redefinition of the Catani's operator (2.19), namely including the kinematic factor and not the color factor.

where $f^{(3)}(\epsilon)=\frac{11}{2}\zeta(4)+\epsilon(6\zeta(5)+5\zeta(2)\zeta(3))+\epsilon^2(c_1\zeta(6)+c_2\zeta(3)^2)$, with c_1 and c_2 rational numbers, and $C^{(3)}$ is a constant, which can be found explicitly in [51].
The two- and three-loop results indicate an exponentiation structure for the full amplitude, and not only for the infrared divergent part. This motivates the *ABDK/BDS ansatz* for the full MHV amplitude in planar $\mathcal{N}=4$ SYM:

$$\mathcal{M}^{(BDS)}(\epsilon)=\exp\left[\sum_{l=1}^{\infty}\left(\lambda^{l-1}f^{(l)}(\epsilon)\mathcal{M}^{(1)}(l\epsilon)+\lambda^l C^{(l)}+\mathcal{O}(\epsilon)\right)\right] \tag{2.32}$$

where $C^{(l)}$ is the finite part and $f^{(l)}(\epsilon)$ is the so-called scaling function, which is a three-term series in ϵ:

$$f^{(l)}(\epsilon)=f_0^{(l)}+\epsilon f_1^{(l)}+\epsilon^2 f_2^{(l)} \tag{2.33}$$

We already encountered $f_0^{(l)}=\Gamma_{cusp}^{(l)}$ and $f_1^{(l)}=l\,\Gamma_{col}^{(l)}$. The coefficients of $f^{(l)}(\epsilon)$ and $C^{(l)}$ are independent of the number of legs and are to be determined by matching to explicit computations. In the known cases they are polynomial in the Riemann zeta function evaluated at positive integers.
The eq. (2.32) is an all order ansatz for the form of the finite part of the n-point MHV scattering amplitude in the planar limit. Expanding the exponential, this relation implies that one can obtain every loop correction recursively by the one-loop one. The key ingredient of ABDK/BDS ansatz is the dual conformal symmetry, that we will discuss in the next section, which constrains the structure of the amplitude. The way one would go about testing the ABDK/BDS exponentiation Ansatz is by direct calculation of the n-point l-loop amplitudes. In addition to the matching of the ansatz with the four-point amplitude up to three loops, it has been shown numerically that it correctly reproduces the five-point two-loop amplitude [52, 53]. It is very interesting that something new happens at six- and higher-point: while the ABDK/BDS ansatz matches the IR divergent structure, it does not fully produce the correct finite part. In those cases the ABDK/BDS ansatz determines the finite part of the amplitude only up to a function of dual conformal cross-ratios of the external momenta. This function is called the remainder function and it is defined as:

$$r_n^{(l)}=\mathcal{M}_n^{(l)}-\mathcal{M}_n^{(BDS)} \tag{2.34}$$

where $\mathcal{M}_n^{(l)}$ is the actual MHV l-loop amplitude and $\mathcal{M}_n^{(BDS)}$ is the $\mathcal{O}(\lambda^l)$ terms in the expansion of the exponential Ansatz. The remainder function does not show up for $n=4,5$ because in those cases there are no available conformal cross-ratios. The first indication of the remainder function came from a strong coupling calculation by Alday and Maldacena [54, 55] who proposed to use the AdS/CFT correspondence to calculate the reduced amplitude. Subsequently, it was verified numerically that a remainder function is needed for the six-point two-loop MHV amplitude, and its analytic form was calculated [56, 57, 58].

Another interesting property of loop scattering amplitudes in $\mathcal{N}=4$ SYM is that they respect the *maximum transcendentality principle*, formulated for the first time by Kotikov and Lipatov in 2001 [59, 60]. This principle states that the quantum corrections of the amplitudes, evaluated in the dimensional reduction scheme, have coefficients which are related in a precise way to the Riemann zeta function. In particular it means that, assigning degree of transcendentality (DoT) -1 to the dimensional regularization parameter ϵ, the l-loop reduced amplitude has uniform degree of transcendentality $2\,l$:

$$\mathcal{M}_n^{(l)} \sim \sum_{k=0}^{\infty} \frac{\epsilon^k \alpha}{\epsilon^{2l}} \qquad\qquad DoT(\alpha) = k$$

where the coefficients α are proportional to a transcendental number with degree of transcendentality k, such that every term of the series has a global degree of transcendentality $2\,l$. The coefficients α are related to the coefficients which appear in the ABDK/BDS ansatz (2.32), which are polynomial in the Riemann zeta function evaluated at positive integers. In particular the coefficients $f_i^{(l)}$ (with $i = 0, 1, 2$) of the scaling function have degree of transcendentality equal to $2l - 2 + i$ and the constants $C^{(l)}$ equal to $2l$. In fact $\zeta(2), \zeta(3), \zeta(4)$ are recurring coefficients with degree degree of transcendentality respectively two, three and four [4].
The maximum transcendentality property was first observed for the anomalous dimension of twist-2 operators [59], and then it was found that it is enjoyed by all known observables of the theory. For scattering amplitudes, whenever analytic results are available, the uniform transcendentality property holds in the planar case as well as outside the planar limit. It is still unclear the origin of this property and if it is related to supersymmetry or to special features of the $\mathcal{N}=4$ SYM model.

2.3 Duality with Wilson loops and dual conformal invariance

We discussed in the previous section that loop scattering amplitudes in planar $\mathcal{N}=4$ SYM theory are constrainted to be (almost) comletely fixed by the one-loop amplitude. In order to understand in a deeper way the ABDK/BDS ansatz, it is useful to study the symmetries of scattering amplitudes. In fact we will discuss in this section that planar $\mathcal{N}=4$ SYM scattering amplitudes reveal a remarkable symmetry structure. In addition to the superconformal symmetry, manifest in the Lagrangian formulation of the theory, the planar amplitudes exhibit an additional hidden symmetry, which motivates the ABDK/BDS ansatz.

Since $\mathcal{N}=4$ SYM is a superconformal field theory we should expect that this is reflected in the structure of scattering amplitudes. It is possible to show [46, 61] that the action of the Poincarè generators on the scattering amplitude annihilate the amplitude,

[4] More precisely we note that $\zeta(n)$ with odd n has not been proven yet to be transcendental. Here, following what has been extensively done in literature, we assume a degree of transcendentality n for the $\zeta(n)$ numbers.

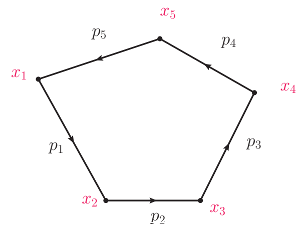


Figure 2.1: Geometrical interpretation of the momentum conservation for a five-point amplitude in the dual space.

once the momentum conservation is imposed. Furthermore we already discussed that the annihilation of the superamplitude by the supersymmetry generators encodes the supersymmetry Ward identity. In order to take into account also the conformal symmetry of the theory we can use the operation of inversion $\mathcal{I}$, which acts on the spacetime coordinates $y^{\alpha\dot{\alpha}}$ in the following way:

$$\mathcal{I}[y^{\alpha\dot{\alpha}}] = -\frac{y^{\alpha\dot{\alpha}}}{y^2} \tag{2.35}$$

and generates the (super)conformal symmetry group from the (super)Poincarè group. It is possible to show that the additional generators of the (super)conformal group vanish when they act on the superamplitude. So the superamplitude is invariant under the full superconformal group. At at loop level, when dimensional regularization is used to treat infrared divergences, there is a breakdown of the conformal symmetry.

The superconformal symmetry is not the only symmetry of $\mathcal{N} = 4$ SYM planar scattering amplitudes. In fact there is an unexpected additional symmetry called *dual conformal symmetry*. Let's see how it emerges. There is an important part of the Poincarè symmetry, the translations, which is not manifest. In momentum space translation invariance corresponds to momentum conservation, which is always ensured through the additional requirement (a delta function):

$$\delta^4\left(\sum_{i=1}^{n} p_i^{\alpha\dot{\alpha}}\right) \qquad \delta^4\left(\sum_{i=1}^{n} \theta_i\right) \tag{2.36}$$

Geometrically the momentum conservation implies that the vectors $p_i^{\alpha\dot{\alpha}}$ close into a closed contour. We can think of momentum conservation as points $x_i^{\alpha\alpha}$ in the dual space [62] satisfying:

$$p_i^{\alpha\dot{\alpha}} = x_i^{\alpha\dot{\alpha}} - x_{i+1}^{\alpha\dot{\alpha}} \equiv x_{i,i+1}^{\alpha\dot{\alpha}} \tag{2.37}$$

The dual coordinates are not spacetime coordinates: they are dual momentum variables, with the dimension of a mass. In dual space momentum conservation simply corresponds to the periodicity condition:

$$x_{n+1} = x_1 \tag{2.38}$$

as shown in Fig. 2.1, where for massless particles the edges of the n-edge polygon are lightlike. The ordering of the external line is the color ordering of the theory. Note that the operation of drawing the dual graph is only possible for planar diagrams.
Let's consider the n-point tree-level amplitude written in dual coordinates. Since the defining relation of dual coordinates (2.37) is invariant under translations, the amplitude is guaranteed to be translational invariant in the dual space. The dual superconformal property of the amplitude can be extracted by studying how the amplitude transforms under the dual inversion acting on the dual coordinates:

$$\mathcal{I}[x_i^{\alpha\dot{\alpha}}] = -\frac{x_i^{\alpha\dot{\alpha}}}{x_i^2} \tag{2.39}$$

which generates the dual conformal symmetry group from the Poincarè group of the dual space. Of course to complete the description of dual space in $\mathcal{N} = 4$ SYM we would need to introduce also dual fermionic coordinates and to study the action of the inversion operator on them. Anyway, it was conjecture in [63], and then demonstrated using super-BCFW recursion relations in [47], that the full tree-level superamplitude is dual superconformal invariant.
If one combines the set of generators of the superconformal group and those of the dual superconformal group, an infinite dimensional algebra is obtained, called Yangian. So, planar superamplitudes of $\mathcal{N} = 4$ SYM are Yangian invariant [64]. This symmetry is an indicator of the integrability of the model in the planar limit.

This unexpected additional symmetry is present also beyond the tree-level. In fact it was found [62, 65, 66] that the loop integrals contributing to the loop reduced amplitude $\mathcal{M}_n$ are dual conformal covariant at all loops orders so far explored. Furthermore the whole reduced amplitude is dual conformal invariant.
Let's illustrate this statement with an example. We consider the four-point one-loop reduced amplitude of $\mathcal{N} = 4$ SYM. By a diagrammatic computation it is found that the analytical expression for the reduced amplitude is:

$$\mathcal{M}_4^{(1)} = \lambda\, st\, I_{\text{box}} \tag{2.40}$$

where s and t are the Mandeltam variables and I_{box} is the one-loop scalar box integral presented in Fig. 2.2(a):

$$I_{\text{box}} = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{k^2(k-p_1)^2(k-p_1-p_2)^2(k+p_4)^2} \tag{2.41}$$

The evaluation of this integral using dimensional regularization is presented in (A.20) (note that inserting eq. (A.20) in $\mathcal{M}_4^{(1)}$ we found the infrared structure presented in eq.(2.27)). The dual coordinates are obtained using eq. (2.37):

$$p_1 = x_{12} \qquad p_2 = x_{23} \qquad p_3 = x_{34} \qquad p_4 = x_{41} \tag{2.42}$$

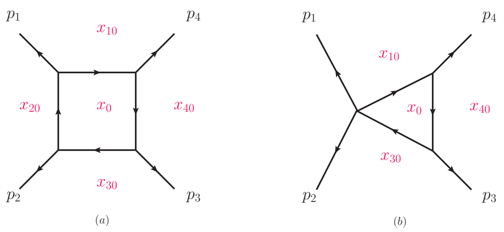


Figure 2.2: Scalar box and triangle integral, emerging in the computation of the four-point amplitude at one-loop.

We have to introduce a further dual coordinate x_0 associated to the loop variable k, which describe the "internal" zone and is defined as $k = x_1 - x_0 \equiv x_{10}$. The integral written in the dual coordinates is:

$$I_{\text{box}} = \int \frac{d^4 x_0}{(2\pi)^4} \frac{1}{x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2} \tag{2.43}$$

as shown in Fig. 2.2(a). If we consider conformal inversion of the dual coordinates:

$$\mathcal{I}[x_i] = -\frac{x_i}{x_i^2} \tag{2.44}$$

then the integral transforms covariantly:

$$I_{\text{box}} \to (x_1^2 x_2^2 x_3^2 x_4^2) \int \frac{d^4 x_0}{(2\pi)^4} \frac{1}{x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2} \tag{2.45}$$

The transformation of the Mandelstam variables under the dual inversion is:

$$s\,t \to \frac{s\,t}{x_1^2 x_2^2 x_3^2 x_4^2} \tag{2.46}$$

So we conclude that the reduced amplitude $\mathcal{M}_4^{(1)}$ is dual conformal invariant. As we already mentioned, the (dual) conformal invariance of the integral is slightly broken when the infrared divergences of the integral are treated using dimensional regularization.
In four-dimensional theories, at one-loop order, all tensor and scalar integrals which emerge from the computation of the massless scattering amplitude can be reduced by imposing the on-shell condition and integration by parts to a combination of the two following integrals: the scalar triangle and the box integrals. Although the triangle integral is not present in one-loop contributions to scattering amplitudes in $\mathcal{N} = 4$ SYM, in Chapter 3, when we will present the computations of four-point scattering amplitudes in $\mathcal{N} = 2$ SCQCD, we will encounter one-loop reduced amplitudes which receive contributions from these two types of integrals. Thus it is interesting to know the behaviour under dual inversion of the scalar triangle integral, depicted in Fig. 2.2(b):

$$I_{\text{triangle}} = \int \frac{d^4 k}{(2\pi)^4} \frac{1}{(k-p_1)^2 (k-p_1-p_2)^2 (k+p_4)^2} \tag{2.47}$$

The analytical evaluation of this integral using dimensional regularization is presented in eq. (A.19). The integral written in the dual coordinates, as shown in Fig. 2.2(b), becomes:

$$I_{\text{triangle}} = \int \frac{\mathrm{d}^4 x_0}{(2\pi)^4} \frac{x_{20}^2}{x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2} \tag{2.48}$$

If we consider conformal inversion of the dual coordinates:

$$I_{\text{triangle}} \to (x_1^2 x_3^2 x_4^2) \int \frac{\mathrm{d}^4 x_0}{(2\pi)^4} \frac{1}{x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2} \frac{x_{20}^2}{x_0^2} \tag{2.49}$$

we conclude that the integral does not transform covariantly. When the triangle integral contributes to the one-loop amplitude it breaks the dual conformal invariance.

To discuss the consequences of dual conformal symmetry further it is convenient to introduce a dual description for the scattering amplitudes. We have shown in eq. (2.37) that it is possible to associate a collection of dual coordinate x_i with a massless MHV amplitude, and each coordinate is light-like separated from its neighbours. The collection of points in the dual space therefore naturally defines a piecewise light-like polygonal contour C_n, as shown in Fig. 2.1. A natural object that can be associate with such a contour in a gauge theory is the *Wilson loop* (WL), introduced in 1974 by Wilson [67] in an attempt to find a nonperturbative formulation of quantum chromodynamics (QCD). The Wilson loop is a gauge invariant object defined as the trace of a path-ordered exponential of a gauge field A_μ transported along a closed line C_n:

$$\mathcal{W}_n = \mathrm{tr}\Big(\mathcal{P} \exp ig \oint_{C_n} \mathrm{d}x^\mu A_\mu \Big) \tag{2.50}$$

Here, in contrast to the situation for the scattering amplitude, the dual space is being treated as the actual configuration space of the gauge theory, i.e. the theory in which we compute the Wilson loop is local in this space.

A lot is known about the structure of a Wilson loop in planar $\mathcal{N} = 4$ super Yang-Mills theory. In particular the evaluation of the expectation value of a light-like Wilson loop, performed in dimensional regularization, has received a lot of attention. It was found that the integration in the vicinity of the cusps present at the points x_i produces ultraviolet divergences, whose origin is related to the small distance behaviour in the dual space [68, 69]. The ultraviolet divergences of the expectation value of the Wilson loop have the following form:

$$\langle W_n \rangle\big|_{UV} = \exp\left[-\sum_{l=1}^{\infty} \lambda^l \left(\frac{\Gamma_{cusp}^{(l)}}{(l\epsilon)^2} + \frac{\Gamma^{(l)}}{l\epsilon} \right) \sum_{i=1}^{n} (-\mu_{UV}^2 x_{i,i+2}^2)^{l\epsilon} \right] \tag{2.51}$$

where $\Gamma_{cusp}(\lambda) = \sum_l \lambda^l \Gamma_{cusp}^{(l)}$ is the cusp anomalous dimension [70, 71], and it governs the leading ultraviolet divergences of WL with light-like cusps; μ_{UV} is the UV scale of dimensional regularization. Such divergences are intimatly related to the infrared divergences of scattering amplitudes. The ultraviolet divergent structure of (2.51) matches precisely the infrared divergent structure of MHV scattering amplitude in planar $\mathcal{N}=4$ SYM, presented in (2.29), upon changing the spacetime coordinates to dual coordinates. This is the first connection between scattering amplitudes and Wilson loops. It was found that this connection runs deeper than just the leading infrared divergence. In fact there is a lot of evidence that in the planar theory the finite part of scattering amplitudes and the finite part of the expectation value of the Wilson loop are identical up to an additive constant. The identification of the two finite parts was first made at strong coupling [55] where the AdS/CFT correspondence can be used to study the theory. In this regime the identification is a consequence of a particular T-duality transformation of the string sigma model which maps the AdS background into a dual AdS space. Shortly afterwards the identification was made in perturbation theory, suggesting that such a phenomenon is actually a non-perturbative feature. The matching was first observed at four points and one-loop [68] and generalised to n-points in [72]. Two-loop calculations then followed [69]. In each case the duality relation was indeed verified.

An important point is that dual conformal symmetry finds a natural home within the duality between amplitudes and Wilson loops. It is simply the ordinary conformal symmetry of the Wilson loop defined in the dual space. Moreover, since this symmetry is a Lagrangian symmetry from the point of view of the Wilson loop, its consequences can be derived in the form of Ward identities [69]. Importantly, conformal transformations preserve the form of the contour, i.e. light-like polygons map to light-like polygons. Thus the conformal transformations effectively act only on a finite number of points (the cusp points x_i) defining the contour. A very important consequence of the conformal Ward identity is that the finite part of the Wilson loop is fixed up to a function of conformally invariant cross-ratios. In the cases of four and five edges, there are no such cross-ratios available due to the light-like separations of the cusp points. This means that the conformal Ward identity has a unique solution up to an additive constant. Remarkably, the solution coincides with the ABDK/BDS all-order ansatz for the corresponding scattering [73].

It seems very likely that the agreement between MHV amplitudes and light-like polygonal Wilson loops will continue to an arbitrary number of points, to all orders in the coupling. While the agreement between Wilson loops and scattering amplitudes is fascinating it is clearly not the end of the story. Firstly the duality we have described applies only to the MHV amplitudes, so it would be interesting to understand what happens for the N^KMHV ones, which reveal a much richer structure than their MHV counterparts. Even without regard to a dual Wilson loop, a greater investigation of N^KMHV amplitudes would be useful, in order to understand if the dual conformal symmetry plays a role in the N^KMHV case.

2.4 Computation techniques

This section is dedicated to the main available techniques used to compute scattering amplitudes. In principle it is straightforward to compute tree and loop amplitudes by drawing the standard Feynman diagrams and evaluating them, using standard reduction tecniques for loop integrals. Symmetries, however, even those of the Lagrangian, are obscured in this approach, re-emerging only after all Feynman diagrams are assembled.

When supersymmetry is present we have seen that the superfield description is convenient. In particular we will use the $\mathcal{N} = 1$ superfield description in four spacetime dimensions to write the classical action of $\mathcal{N} = 4$ SYM and $\mathcal{N} = 2$ SCQCD and the analogous $\mathcal{N} = 2$ superfield description in three dimensions to write the action of $\mathcal{N} = 6$ ABJM. However, the greatest advantages of superfields appear at the quantum level, with the use of supersymmetric Feynman diagrams, which are Feynman diagrams with ordinary fields substituted by superfields. In fact there are algebraic simplifications in super Feynman diagrams calculations due to the compactness of notation, the decrease in the number of indices (e.g., the vector field is hidden inside the scalar superfield V), and the automatic cancellation of component graphs related by supersymmetry. The latter in particular would require separate calculations in components. The investigation of the divergence structure of scattering amplitudes is also very much facilitated by the use of superfields.

In these thesis we carried out the computation of four-point scattering amplitudes using super Feynman diagrams, with the $\mathcal{N} = 1$ or the $\mathcal{N} = 2$ superfield description depending on the theory in consideration. The general method we use to obtain results can be summarized in the following steps:

1. We read the contributions to the partial amplitude by considering the supersymmetric effective action, which can be evaluated through super Feynman diagrams. Once we have extracted the Feynman rules from the supersymmetric action written in terms of superfields, we draw all the super Feynman diagrams selecting the ones which contribute to the four-point scalar supervertex associated to the chosen external configuration. The diagrams have to be suitably chosen to respect the color ordering of the partial amplitude we want to compute.

2. The first step in the computation of a super Feynman diagram is the so-called D-algebra procedure (see [10] for details). This is a graphical method in which the covariant derivatives D, $\bar{D}$ acting on propagators are rearranged by means of integrations by parts at the vertices. For each integration by parts, in general, several

terms will be produced. These integrations by parts end when every term has been reduced to an expression which is local in the superspace.

3. We want to present the explicit results for the scattering of component fields, so we have to extract components out of the superfields. We can extract the four–point component amplitude with scalar fields as external particles performing the projection $\int \mathrm{d}^4x\, \mathrm{d}^4\theta \cdots = \int \mathrm{d}^4x\, \bar{D}^2 D^2\, \ldots|_{\theta=0}$ on the superspace results. The other component amplitudes can be easily obtained by choosing different projections of the superspace results.

4. For each diagram, we are then left with a linear combination of standard momentum space integrals, possibly with numerators, which can be simplified by completion of squares and using on-shell symmetries.

5. The contributions of the different diagrams is then summed up. In some case the final result is expressed as a linear combination of master integrals, using the integration by part (IBP) reduction technique (see [74] for details).

6. Finally, each (master) integral is expanded in terms of the dimensional regularization parameter $\epsilon = 2 - d/2$ and the total result is presented as a series in the infrared divergent poles.

Super Feynman diagrams are not the only tool to compute scattering amplitudes. An historically well estabilished alternative tool for computing scattering amplitudes is the *generalized unitarity method* [75]. It is based on the unitarity of the scattering matrix $S^\dagger S = 1$: writing the S-matrix as $S = 1 + iT$, where T represents the interactive part, unitarity requires $-i(T - T^\dagger) = T^\dagger T$. Examining this constraint order by order in perturbation theory, it states that the imaginary part of the T matrix at a given order is related to the product of lower order results. This is equivalent to take loop propagators on-shell, operation called unitarity cut. One can reconstruct the integrand by analyzing different sets of unitarity cuts, exploiting analyticity. Reconstructing the full loop amplitude from systematic application of unitarity cuts is called the generalized unitarity method.

In general at one loop unitarity cuts are matched with a decomposition of the amplitude in terms of a set of scalar integrals - boxes, triangles, bubbles and sometimes tadpoles - in order to determine the coefficients of the integrals. In $\mathcal{N} = 4$ SYM, the high degree of supersymmetry implies that only box integrals have non-vanishing coefficients. Using unitarity, an infinite sequence of one-loop amplitudes could be determined in $\mathcal{N} = 4$ SYM from just the product of two tree-level MHV amplitudes.

Generalized unitarity can also be applied at the multi-loop level. In principle it can be used for any gauge (or gravitational) theory. As an example, the two- loop four-gluon scattering amplitude in QCD have been computed in this way. In practice the method has been pushed the furthest in $\mathcal{N} = 4$ SYM. The basic techniques of multi-loop generalized unitarity are reviewed in [76, 77, 78].

In section 2.1.3 we discussed how recursion relations can be used to compute any tree-level n-point amplitude in $\mathcal{N} = 4$ SYM. This was the background that led to the developements of new tools for computing amplitudes, using on-shell building blocks. In fact recently there were extensions of the BCFW method to planar loop amplitudes. Since loop amplitudes have a complicated analytic structure, the focus was originally on the loop integrand, [79, 80] which is a rational function with poles at the location of shifted propagators. The identification of loop momentum is natural in the dual space. For a better formulation of the recursion relations twistor and momentum twistor are needed. Once planar loop integrand is defined in the dual space it is natural to use the momentum supertwistor and set up the BCFW shift. The recursion relations for loop integrand can be solved in supersymmetric gauge theories in the planar limit.

The fundamental physical idea behind the BCFW description of an amplitude at all loop orders is that any amplitude can be fully reconstructed from the knowledge of its singularities; and the singularities of an amplitude are determined entirely by on-shell data.

Recently it was found that the BCFW recursion relations of loop integrand are closely tied to remarkable mathematical structure known as the *positive Grassmannian* [81], in a way that makes the conformal and dual conformal invariance of the theory completely manifest. The Grassmannian representation of on-shell processes might be in the future an alternative tool to standard Feynman diagrams technique.

2.5 Conclusions

In this chapter we have laid the foundations for understanding the computations and the results presented in the next two Chapters 3 and 4.

We have presented the main known results for scattering amplitudes in a Yang-Mills theory with gauge group $SU(N)$ and in its maximally supersymmetric extension, which is $\mathcal{N} = 4$ SYM. At the tree-level we discussed the color decomposition and the MHV classification. Furthermore we introduced the BCFW recursion relations, which allow to construct any tree-level n-point amplitude from lower point ones. Then we presented the main results for loop amplitudes: the factorization of IR divergences and the Catani's formula for Yang-Mills theory and the ABDK/BDS ansatz for $\mathcal{N} = 4$ SYM. The latter is motivated by a new symmetry of superamplitudes, the dual conformal symmetry, which find a natural home with the scattering amplitudes/Wilson loop duality.

Even though (super)Feynman diagrams remain the standard tecnique for computing scattering amplitudes, the discovery of these properties have led to the developments of alternative efficient tecniques to calculate scattering amplitudes.

In the next chapters we will deal with scattering amplitudes in other superconformal theories than $\mathcal{N} = 4$ SYM, and we will present new results which help to shed light on the properties of the scattering amplitude in those theories.

CHAPTER 3

Scattering amplitudes in $\mathcal{N} = 2$ superconformal QCD

The study of scattering amplitudes in $\mathcal{N} = 4$ Super Yang-Mills theory has unveiled the existence of hidden symmetries and unexpected properties, as we have seen in the previous chapter. Once again $\mathcal{N} = 4$ SYM theory played a pivotal role and turned out to be the perfect playground to provide important insights into quantum field theory. The investigation on the origin of such properties has led to study theories with less amount of supersymmetry. The aim of this chapter is the investigation of the properties of scattering amplitudes in $\mathcal{N} = 2$ superconformal QCD theory, presented in 1.3. We recall that this model is an $\mathcal{N} = 2$ supersymmetric Yang-Mills theory with gauge group $SU(N)$ coupled to $N_f = 2N$ fundamental hypermultiplets. The condition on the number of flavour of the fundamental fields is necessary to ensure exact conformal invariance.

Several aspects of $\mathcal{N} = 2$ SCQCD have been analyzed in the past few years. In the context of integrability, the dilatation operator at one loop was constructed first in the sector of operators made of elementary scalar fields [82] and then for the full theory [83]. Later on, through a diagrammatic analysis, the dilatation operator of the scalar sector was shown to deviate from the one of $\mathcal{N} = 4$ SYM at three loops [84]. After some first promising clues, it was definitely demonstrated that the Hamiltonian for the full theory is not integrable [85]. However the possibility that the closed $SU(2,1|2)$ subsector built only with adjoint fields is exactly integrable still remains open. In [86] it was claimed that in this subsector, present in all $\mathcal{N} = 2$ superconformal models, the integrable structure becomes exactly the one of $\mathcal{N} = 4$ SYM by substituting the $\mathcal{N} = 4$ coupling with an effective coupling. A weak coupling expansion of the $\mathcal{N} = 2$ SCQCD effective coupling was presented in [87]. Integrability from the perspective of the scattering amplitudes/Wilson loop duality has been far less analyzed. Expectation values of Wilson loops have been studied at weak coupling by taking the diagrammatic difference with $\mathcal{N} = 4$ SYM [88]. It was shown that light-like polygonal Wilson loops (actually any closed WL) start deviating from the corresponding $\mathcal{N} = 4$ SYM results at three-loop order, confirming the prediction coming from the localization matrix model construction of [89]. A more general analysis including the strong coupling behaviour of the matrix model was performed in [90]. Scattering amplitudes in $\mathcal{N} = 2$ SCQCD have been computed at one-loop order only in the adjoint sector using unitarity [91]. It was shown that in this sector the result matches that of $\mathcal{N} = 4$ SYM and thus consists of a dual conformal invariant and maximal transcendental expression. Nothing is known so far about amplitudes in more general sectors of the theory and at higher-loop order.

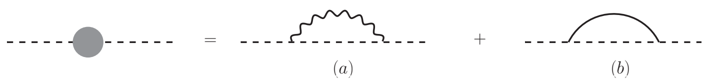
In this chapter we compute one-loop and two-loop four-point scattering amplitudes in planar $\mathcal{N} = 2$ SCQCD. The results were presented in the published paper [1]. We work in $\mathcal{N} = 1$ superspace formalism and perform direct super Feynman diagram computations within dimensional reduction scheme. At one-loop order we provide a complete classification of the amplitudes, which can be divided in three independent sectors according to the color representation of the external particles. The pure adjoint sector consists of amplitudes with external fields belonging to the $\mathcal{N} = 2$ vector multiplet. In this sector we confirm the results of the previous work [91], since we obtain exactly the same expressions of the corresponding $\mathcal{N} = 4$ SYM amplitudes, demonstrating the presence of dual conformal symmetry and maximal transcendentality. This agrees with the conjectured integrability of the closed subsector $SU(2,1|2)$. As a byproduct, we also provide a direct Feynman diagram derivation of the $\mathcal{N} = 4$ SYM result first derived long ago by stringy arguments [92]. Outside the adjoint sector there is no reason to expect amplitudes to be dual conformal invariant. We show that in the mixed and fundamental sectors, with external fields in the fundamental representation, even if dual conformal invariance is broken, the results still exhibit maximal transcendentality weight. We thus show that at one-loop order the maximal transcendentality property of the amplitudes is not a consequence of dual conformal invariance. In order to check these properties beyond the one-loop perturbative order, we computed the simplest two-loop amplitude in the fundamental sector. We end up with a result that does not exhibit maximal transcendentality and is not dual conformal invariant. A very non trivial check of our two-loop result is the fact that it reproduces the expected factorized structure of the infrared divergences predicted for general scattering of massless particles [42, 41].

The chapter is organized as follows. In the first section we compute one-loop and two-loop corrections to propagators and vertices, which will be used in the amplitudes computations. Then we discuss the general features of four-point scattering amplitudes and we present the one-loop amplitudes in the three independent sectors. After that we perform the computation of the two-loop amplitude in the pure fundamental sector. Several technical aspects such as superspace conventions and computations of the integrals are collected in Appendix A.

3.1 Propagator and vertex corrections

In this section we present the one- and two-loop planar corrections to propagators and vertices which are relevant for our computation. In particular, we will discuss how the finite corrections to propagators guarantee that the theory is conformal, at least up to two loops.

At one loop the corrections to the fundamental chiral superfield propagator $\langle Q_{iI}\bar{Q}^{jJ}\rangle$ are represented by the following diagrams:



(3.1)

The evaluation of these diagrams gives the results:

$$(a) = -g^2 N\,\delta_i^j\,\delta_I^J\,\mathcal{D} \qquad\qquad (b) = g^2 N\,\delta_i^j\,\delta_I^J\,\mathcal{D} \tag{3.2}$$

where δ_i^j and δ_I^J take into account the conditions on the color and flavor indices coming from the vertices and propagators in (1.20). We labeled with $\mathcal{D}$ the contribution to the effective action coming from the evaluation of the super Feynman diagrams in superspace:

$$\mathcal{D} = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \int \mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}\,A(p)\,Q_{iI}(p,\theta)\bar{Q}^{jI}(-p,\bar{\theta}) \tag{3.3}$$

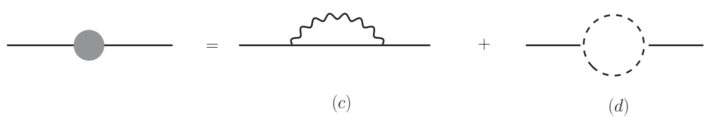
where $A(p)$ take into account the integration in the loop variable:

$$A(p) = \int \frac{\mathrm{d}^d k}{(2\pi)^d}\,\frac{1}{k^2(k+p)^2} = \frac{\Gamma(\epsilon)\Gamma^2(1-2\epsilon)}{(4\pi)^{2-\epsilon}\Gamma(2-2\epsilon)}\,\frac{1}{(p^2)^{-\epsilon}} \tag{3.4}$$

In the left side of eq. (3.4) we have inserted the analytic evaluation of the loop integral in dimensional regularization and p is the incoming momentum. So the one-loop fundamental chiral propagator vanishes:

$$(a) + (b) = 0$$

for any value of N_f. An identical result is obtained for the one-loop correction to the $\langle \bar{\tilde{Q}}_{iI}\tilde{Q}^{jJ}\rangle$ propagator. The one-loop correction to the adjoint chiral superfield $\langle\Phi^a\bar{\Phi}^b\rangle$ receives contributions from the following diagrams:



(3.5)

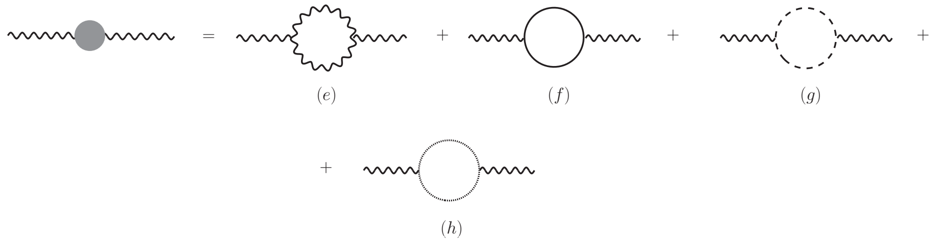
where:

$$(c) = -2g^2 N\,\delta^{ab}\,\mathcal{D} \qquad\qquad (d) = g^2 N_f\,\delta^{ab}\,\mathcal{D} \tag{3.6}$$

where $\mathcal{D}$ has the same exppression as in (3.3), but with the superfileds $Q_{iI}(p,\theta)\bar{Q}^{jJ}(-p,\bar{\theta})$ substituted by $\Phi^a(p,\theta)\bar{\Phi}^b(-p,\bar{\theta})$. So the one-loop correction to the adjoint propagator is:

$$(c) + (d) = g^2(N_f - 2N)\delta^{ab}\,\mathcal{D}$$

and vanishes if and only if the superconformal condition holds: $N_f = 2N$. The one-loop correction to $\langle V^a V^b\rangle$ propagator receives contributions from the following diagrams:



(3.7)

where the loop in diagram (g) can be constructed from two type of vertices, which involve Q or $\tilde{Q}$ superfields (remind the action (1.20)). The evaluation of the diagrams gives the following results:

$$(e) = g^2 N\left(-\frac{5}{2}\mathcal{P}_{1/2} + \frac{1}{2}\mathcal{P}_0\right) \qquad (f) = g^2\, N\mathcal{P}_{1/2} \tag{3.8}$$

$$(g) = g^2\, N_f\, \mathcal{P}_{1/2} \qquad (h) = g^2 N\left(-\frac{1}{2}\mathcal{P}_{1/2} - \frac{1}{2}\mathcal{P}_0\right) \tag{3.9}$$

where we defined:

$$P_{1/2} = \int \frac{d^4p}{(2\pi)^4}\, A(p) \int d^2\theta\, d^2\bar{\theta}\, V(p,\theta)\, \frac{D^\alpha \bar{D}^2 D_\alpha}{8}\, V(-p,\bar{\theta}) \tag{3.10}$$

$$P_0 = -\int \frac{d^4p}{(2\pi)^4}\, A(p) \int d^2\theta\, d^2\bar{\theta}\, V(p,\theta)\, \frac{\{D^2,\bar{D}^2\}}{16}\, V(-p,\bar{\theta}) \tag{3.11}$$

where $A(p)$ is defined in eq. (3.4). So the one-loop correction to the vector propagator is:

$$(e) + (f) + (g) + (h) = g^2\ (N_f - 2N)\ \mathcal{P}_{1/2}$$

and vanishes if and only if the superconformal condition holds: $N_f = 2N$.
To summarize we found that the diagrams which correct the Q propagator in the first line cancel each other for every value of N_f, while in the case of the Φ and V propagators the corrections exactly sum up to zero, when the superconformal condition is imposed. So all one-loop self energies are identically zero at the conformal point. The finiteness theorem for the superpotential assures that there are no infinte or finite corrections to the interaction terms, so to estabilish that the beta function $\beta(g)$ vanishes it sufficient to show that the corrections to the self-energies are finite. Thus the finite corrections to the propagators at one loop are enough to ensure conformal invariance at one loop order, where the condition $N_f = 2N$ has been used non-trivially as shown above.

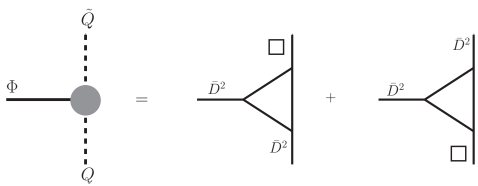
We recover the same one-loop result as in $\mathcal{N}$ = 4 SYM [9]. This is not surprising, since the propagators of $\mathcal{N}$ = 2 SCQCD are corrected by the same diagrams correcting the corresponding propagators of $\mathcal{N}$ = 4 SYM. The only difference is that in $\mathcal{N}$ = 2 SCQCD there are one adjoint chiral superfield and $2N_f$ fundamental superfields, while in $\mathcal{N}$ = 4 SYM there are just three adjoint chiral superfields. If we substitute in $\mathcal{N}$ = 2 SCQCD the fundamental matter loops depicted in diagram (g) with adjoint ones we find that $(f) + (g) = 3\, g^2\, N\mathcal{P}_{1/2}$. So we exactly recover the $\mathcal{N}$ = 4 SYM results when $N_f = 2N$.

At two loops the quantum corrections to the chiral fundamental and adjoint superfield propagators vanish [84]:

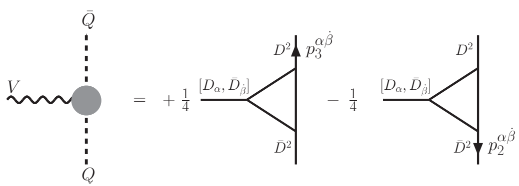
(3.12)

where the external lines can be a chiral adjoint or fundamental superfields. The finiteness of (3.12) is enough to ensure that the theory is conformal at two loops.

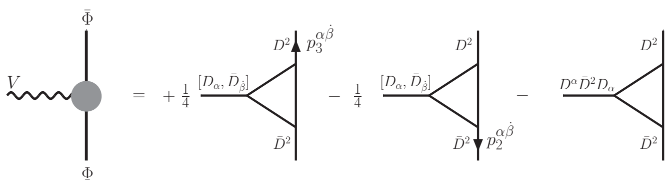
In our computations we also need the following one-loop vertex corrections (an overall factor $g^3 N$ is stripped out):



(3.13)

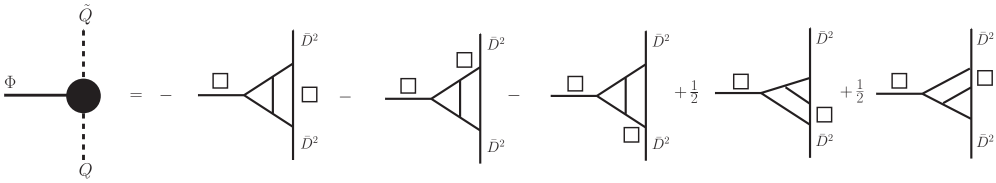


(3.14)



(3.15)

where we depicted only diagrams which contribute at leading color order. The symbol □ indicates the inverse of propagator with the opposite sign. In the first correction above we omitted also an overall factor i. We follow here the representation of [84], where the diagrams are evaluated off-shell and the expansions can be directly inserted in higher loop supergraph structures. At two loops we need the chiral vertex correction (we omit an overall ig^5N^2):



(3.16)

In this case the full off-shell expansion of the vertex gets lenghty [84]. We reported here only the terms giving a non-vanishing contribution to the amplitude in Fig.3.7, namely the ones which survive after taking on-shell momenta for the external fields Q and $\tilde{Q}$.

3.2 One-loop amplitudes

In this section we present the computations of scattering amplitudes in perturbation theory in the planar limit $N \to \infty$ with the 't Hooft coupling $\lambda = \frac{g^2 N}{(4\pi)^2}$ kept finite. More precisely, in order to preserve conformal invariance the number of flavours is also sent to infinity and thus the model is studied in the so called Veneziano limit with $N_f = 2N$.

The complete set of four-point amplitudes of the theory can be obtained by means of supersymmetry transformations from superamplitudes involving only the chiral scalar superfields Φ and Q as external particles. In fact, in the $\mathcal{N} = 1$ superfields language, supersymmetry rotates the Φ and V superfield components inside the $\mathcal{N} = 2$ vector multiplet and the Q and $\tilde{Q}$ ones in the $\mathcal{N} = 2$ hypermultiplet. We thus can classify the four-point superamplitudes into three independent sectors according to the color representation of the external superfields: four adjoint scalar superfields, two adjoint scalars and a quark/antiquark pair and finally two quark/antiquark pairs. Different amplitudes inside each sector are related by supersymmetry.

We perform standard perturbative computations directly with the $\mathcal{N} = 1$ off-shell Lagrangian (1.29), following the general method presented in 2.4. We recall here the main steps. We select a process in a particular sector and we discuss the partial amplitudes color decomposition. We work directly with traces using the trace based color decomposition of the amplitudes, selecting only the Feynman diagrams which contribute to the chosen color configuration, as mentioned in section 2.1.1 (see also [34] for a review of the method). We then present the loop results for the subamplitudes, obtained performing the following steps:

- At first we read the contributions to the partial amplitude by considering the effective action of the model. More precisely, we draw super Feynman diagrams contributing to the four-point scalar supervertex associated to the chosen external configuration, where the diagrams have to be suitably chosen to respect the color ordering.

- We then perform D-algebra on the selected superdiagrams. In order to extract the four-point component amplitude with scalar fields as external particles we perform the projection $\int d^4x\, d^4\theta \cdots = \int d^4x\, \bar{D}^2 D^2\ \ldots|_{\theta=0}$ on the superspace results.
- For each diagram, we are then left with a linear combination of standard bosonic integrals with numerators, which can be simplified by completion of squares and using on-shell symmetries.
- The contributions of the different diagrams is then summed up and the final result is expressed, using the integration by part reduction technique, as a linear combination of master integrals (see [74] for details).
- Finally, each master integral is expanded in terms of the dimensional regularization parameter $\epsilon = 2 - d/2$ and the total result is presented as a series in the infrared divergences poles.

At one-loop order we provide a complete classification of the four-point scattering amplitudes, computing one-loop amplitudes in each sector. In general we define with $(ABCD)$ a process where we treat all the particles as outgoing:

$$0 \to A(p_1) + B(p_2) + C(p_3) + D(p_4)$$

with light-like momentum assignments as in parentheses and momentum conservation given by $p_1 + p_2 + p_3 + p_4 = 0$. We define Euclidean Mandelstam variables as:

$$s = (p_1 + p_2)^2 \qquad t = (p_2 + p_3)^2 \qquad u = (p_1 + p_3)^2 = -t - s$$

Diagrammatically we start with the particle A in the upper left corner and proceed with the ordering counterclockwise.

3.2.1 Adjoint subsector

In the purely adjoint sector we first focus on the process $(\Phi\bar{\Phi}\Phi\bar{\Phi})$. Since we deal with adjoint external particles of a $SU(N)$ gauge theory, the color decomposition of the planar amplitude is the same as in the four-gluon scattering (see 2.1.1 and in particular eq. (2.6)):

$$A^{(l)}(\{p_i, a_i\}) = \sum_{\sigma \in S_4/Z_4} Tr(T^{a_{\sigma_1}} T^{a_{\sigma_2}} T^{a_{\sigma_3}} T^{a_{\sigma_4}}) \mathcal{A}^{(l)}(\sigma_1\sigma_2\sigma_3\sigma_4) \tag{3.17}$$

where the sum is performed over non-cyclic permutations inside the trace. This gives rise to six *a priori* independent color ordered subamplitudes which might be further reduced by exploiting the symmetries of the process. These subamplitudes only receives contributions from diagrams with the specified ordering of the external particles. In any case, all the different subamplitudes divided by the corresponding tree-level contributions are expected to yield the same result since they can be mapped by $\mathcal{N} = 2$ supersymmetry to proper gluon MHV amplitudes, which do not depend on the gluon

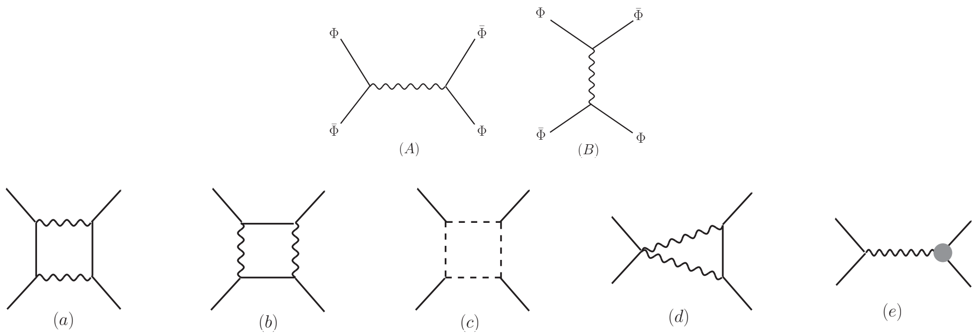


Figure 3.1: Tree level and one-loop non-vanishing planar diagrams contributing to the process $(\Phi\bar{\Phi}\Phi\bar{\Phi})$. The grey bullet in diagram (e) stands for the one-loop vertex insertion.

ordering inside the trace. We therefore expect that different orderings produce identical results.

As we will explain below from a diagrammatic point of view it is instructive to compute two non trivial orderings $\mathcal{A}(\Phi(1)\bar{\Phi}(2)\Phi(3)\bar{\Phi}(4))$ and $\mathcal{A}(\Phi(1)\bar{\Phi}(2)\bar{\Phi}(3)\Phi(4))$.

Process $\Phi\bar{\Phi}\Phi\bar{\Phi}$

The color ordered subamplitude $\mathcal{A}(\Phi(1)\bar{\Phi}(2)\Phi(3)\bar{\Phi}(4))$ receives contributions at tree level from the processes depicted in Fig.3.1(A) and (B). The contribution of the diagrams to the four scalar superfield vertex of the effective action is:

$$\mathcal{S}^{(0)} = -g^2 \int d^4p_i\, d^4\theta \left(\frac{1}{s} + \frac{1}{t}\right) \mathrm{tr}(\Phi(p_1)\bar{\Phi}(p_2)\Phi(p_3)\bar{\Phi}(p_4))$$

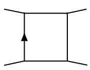
From these we extract the relevant color structure and, after projection to the purely scalar component of the superamplitude, we can read the tree level contribution:

$$\mathcal{A}^{(0)}(\phi(1)\bar{\phi}(2)\phi(3)\bar{\phi}(4)) = g^2 \left(\frac{u}{s} + \frac{u}{t}\right) \tag{3.18}$$

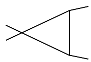
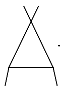
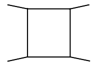
We now consider the planar one-loop corrections. The diagrams which contribute are listed in Fig.3.1(a)-(e). For each diagram we find first the contribution to the effective action by performing the D-algebra with on-shell conditions. For the diagram (a) we get:

$$\begin{aligned}\mathcal{S}^{(a)} = &- g^4 N\, \mathrm{tr}(T^aT^bT^cT^d) \int d^4p_i\, d^4\theta\, I_{\mathrm{triangle}}(s)\; \Phi_a(p_1)\bar{\Phi}_b(p_2)\Phi_c(p_3)\bar{\Phi}_d(p_4)\; + \\ &- g^4 N\, \mathrm{tr}(T^aT^bT^cT^d) \int d^4p_i\, d^4\theta\, I_{\mathrm{box}}^{\alpha\dot{\beta}}\; \Phi_a(p_1)\bar{\Phi}_b(p_2)D_\alpha\Phi_c(p_3)\bar{D}_{\dot{\beta}}\bar{\Phi}_d(p_4)\end{aligned}$$

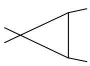
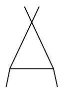
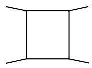
where $I_{\mathrm{triangle}}(s)$ and $I_{\mathrm{box}}^{\alpha\dot{\beta}}$ are defined in eq. (A.19) and (A.23) of Appendix A.3. At this point we can project down to the four-scalar component and directly read the contribution to the color ordered amplitude:

$$(a) = g^4 N$$

$$\Big(uk^2 - \mathrm{Tr}(k p_4 p_1 p_3)\Big) \qquad (3.19)$$

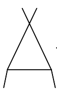
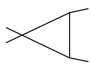
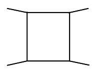
The numerator of the Feynman integral in (3.19) is spelled out explicitly whereas the denominator is represented pictorially together with an arrow indicating the integration variable k. Expanding the trace and completing the squares we can cast the final contribution in terms of a linear combination of scalar integrals:

$$(a) = g^4 N \Big[-(s+2t)$$

$$+\, t$$

$$+ \left(t^2 + \frac{st}{2}\right)$$

$$\Big] \qquad (3.20)$$

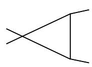
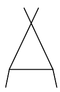
The contribution of diagram (b) can be immediately obtained from the one of diagram (a) by exchanging $s \leftrightarrow t$:

$$(b) = g^4 N \Big[s$$

$$-\,(t+2s)$$

$$+ \left(s^2 + \frac{st}{2}\right)$$

$$\Big] \qquad (3.21)$$

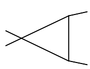
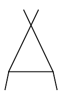
We proceed similarly for the remaining diagrams, performing D-algebra, component projection and reduction to scalar integrals. For the scalar box diagram (c) we obtain:

$$(c) = g^4 N_f \Big[-t$$

$$-\, s$$

$$+ \frac{st}{2}$$

$$\Big] \qquad (3.22)$$

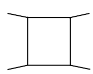
with $N_f = 2N$. For diagrams of type (d) we need to consider the four possible ways to draw the graph, which combine to:

$$(d) = g^4 N \Big[(s+t)$$

$$+ \ (s+t)$$

$$\Big] \qquad (3.23)$$

The diagram (e) represents the one-loop correction to the vertex. The vertex correction insertions were described in section 3.1. After taking into account the four possible insertions in the s- and t-channel diagrams we get an overall:

$$(e) = g^4 N \Big[(s+t)$$

$$+ \ (s+t)$$

$$\Big] \qquad (3.24)$$

Summing over all the contributions (3.20)-(3.24) it is easy to see that triangle integrals cancel out, leaving a final result which is proportional to the box integral:

$$\mathcal{A}^{(1)}(\phi(1)\bar\phi(2)\phi(3)\bar\phi(4)) \ = \ g^4 N\,(s+t)^2$$

$$=$$

$$= 2\,\frac{g^4 N}{(4\pi)^2}\left(\frac{u}{s}+\frac{u}{t}\right)\left\{-\frac{1}{\epsilon^2}\left(\frac{\mu}{s}\right)^{\epsilon} - \frac{1}{\epsilon^2}\left(\frac{\mu}{t}\right)^{\epsilon} + \frac{2}{3}\pi^2 + \frac{1}{2}\ln^2\frac{t}{s} + \mathcal{O}(\epsilon)\right\} \qquad (3.25)$$

where $\mu = 4\pi e^{-\gamma_\epsilon}\nu$, and ν is the IR scale of dimensional regularization. The reduced amplitude is then defined as the ratio between the one-loop (3.25) and the tree-level one (3.18):

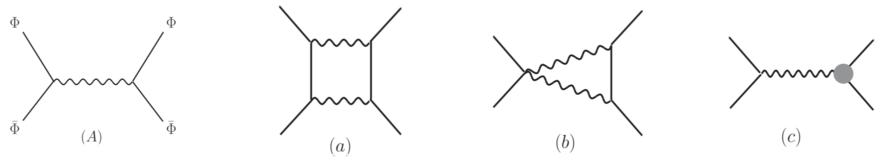


Figure 3.2: Tree level and one-loop diagrams contributing to $(\Phi\bar{\Phi}\bar{\Phi}\Phi)$ process.

$$\mathcal{M}^{(1)}(\phi(1)\bar{\phi}(2)\phi(3)\bar{\phi}(4)) = 2\lambda\left\{-\frac{1}{\epsilon^2}\left(\frac{\mu}{s}\right)^{\epsilon} - \frac{1}{\epsilon^2}\left(\frac{\mu}{t}\right)^{\epsilon} + \frac{2}{3}\pi^2 + \frac{1}{2}\ln^2\frac{t}{s}\right\} \tag{3.26}$$

where $\lambda = \frac{g^2 N}{(4\pi)^2}$. This confirms the result of [91] obtained via unitarity cuts method and it shows that in this sector the one-loop amplitudes are identical to the corresponding $\mathcal{N} = 4$ SYM ones. Therefore the amplitude in (3.26) is completely captured by a dual conformal invariant integral and respects the maximum transcendentality principle.

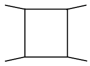
From a diagrammatic point of view the matching with $\mathcal{N} = 4$ SYM can be understood as follows. We could consider in $\mathcal{N} = 4$ SYM a four-point amplitude of adjoint scalar superfields with equal flavours $(\Phi_1\bar{\Phi}_1\Phi_1\bar{\Phi}_1)$. We note that diagrams (a), (b), (d) and (e) of Fig.3.1 can be drawn also for this process and are identical to the ones computed in $\mathcal{N} = 2$ SCQCD. In $\mathcal{N} = 4$ SYM diagram (c) is substituted with an analouge diagram with adjoint scalars circulating into the loop. This exactly reproduces the contribution of the fundamental loop of $\mathcal{N} = 2$ SCQCD when $N_f = 2N$. Therefore it would have been easy in this case to work taking the diagrammatic difference between the two models and to show that it is vanishing. It is a general feature of $\mathcal{N} = 2$ SCQCD diagrams that fundamental matter loops give the same results of $\mathcal{N} = 4$ SYM scalar adjoint loops.

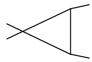
Process $\Phi\bar{\Phi}\bar{\Phi}\Phi$

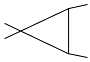
We now focus on the color ordered subamplitude $\mathcal{A}(\Phi(1)\bar{\Phi}(2)\bar{\Phi}(3)\Phi(4))$ for the process $(\Phi\bar{\Phi}\bar{\Phi}\Phi)$. From a diagrammatic point of view this is equivalent to consider the color ordered subamplitude $\mathcal{A}(\Phi(1)\bar{\Phi}(2)\bar{\Phi}(4)\Phi(3))$ for the process considered above $(\Phi\bar{\Phi}\Phi\bar{\Phi})$. At tree level only the diagram in Fig.3.2(A) contributes according to color ordered rules. After projection to the scalar component we obtain:

$$\mathcal{A}^{(0)}(\phi(1)\bar{\phi}(2)\bar{\phi}(3)\phi(4)) = -g^2\ \frac{t}{s} \tag{3.27}$$

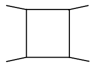
The non-vanishing planar one-loop diagrams are listed in Fig.3.2(a)-(c). For each diagram we perform D-algebra, component projections and master integrals expansion as detailed above and obtain:

$$(a) = g^4 N\, t^2$$  (3.28)

$$(b) = -g^4 N\, t$$  (3.29)

$$(c) = g^4 N\, t$$  (3.30)

Note that diagrams (b) and (c) now contribute in two ways, which can be obtained from the drawn diagrams by left/right reflection. The full amplitude then simply reads:

$$\mathcal{A}^{(1)}(\phi(1)\bar{\phi}(2)\bar{\phi}(3)\phi(4)) \;=\; g^4 N\, t^2$$  $=$

$$= 2\,\frac{g^4 N}{(4\pi)^2}\,\frac{t}{s}\left\{\frac{1}{\epsilon^2}\left(\frac{\mu}{s}\right)^{\epsilon} + \frac{1}{\epsilon^2}\left(\frac{\mu}{t}\right)^{\epsilon} - \frac{2}{3}\pi^2 - \frac{1}{2}\ln^2\frac{t}{s} + \mathcal{O}(\epsilon)\right\} \tag{3.31}$$

Taking the ratio with the tree-level amplitude (3.27) we immediately get:

$$\mathcal{M}^{(1)}(\phi(1)\bar{\phi}(2)\bar{\phi}(3)\phi(4)) = \mathcal{M}^{(1)}(\phi(1)\bar{\phi}(2)\phi(3)\bar{\phi}(4)) \tag{3.32}$$

as expected. We note that this ordering of the external fields gives rise to a smaller number of diagrams with respect to the ordering of section 3.2.1. Moreover, all the diagrams of Fig.3.2 display a corresponding diagram for the analogue process in $\mathcal{N} = 4$ SYM yielding the same result. It is then straightforward in this case to predict the final result. With this respect, since fundamental matter interaction does not play any role, our computation can be seen as a direct standard Feynman diagram confirmation of the $\mathcal{N} = 4$ SYM result, computed long ago by taking a low energy limit of a superstring [92] and then readily reproduced by unitarity methods.

3.2.2 Mixed adjoint/fundamental sector

We now consider amplitudes with two external fields in the fundamental/antifundamental representation of the gauge group $SU(N)$. Focusing on the process $(Q\bar{Q}\Phi\bar{\Phi})$, the color decomposition of planar amplitudes is given by:

$$A^{(l)}(Q\bar{Q}\Phi\bar{\Phi}) = \sum_{\sigma\in S_2} (T^{a_{\sigma_3}} T^{a_{\sigma_4}})^j{}_i \mathcal{A}^{(l)}(Q(1)\bar{Q}(2)\sigma_3\sigma_4) \tag{3.33}$$

There are two non trivial color structures given by strings of color indices starting with the antifundamental index of the $\bar{Q}$ field and ending with the fundamental index of the field Q. The two structures differs by a permutation of the color matrices of the adjoint fields. Once again we expect to obtain the same result for all the ordering of the reduced subamplitudes. Concerning the flavour structure of the amplitudes, it is easy to see that we only have non vanishing results for the quark Q_I and antiquark $\bar{Q}^J$ fields with equal flavours $I = J$. We therefore can omit the flavour indices in our expressions.

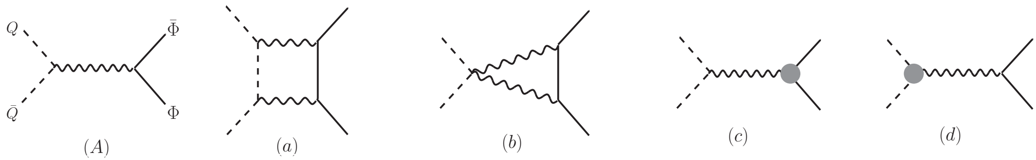


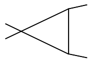
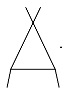
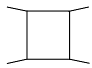
Figure 3.3: Tree level and one-loop diagrams for the process $(Q\bar{Q}\Phi\bar{\Phi})$.

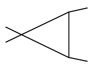
Process $Q\bar{Q}\Phi\bar{\Phi}$

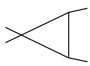
We consider first the subamplitude $\mathcal{A}(Q(1)\bar{Q}(2)\Phi(3)\bar{\Phi}(4))$. At tree level only the process in Fig.3.3(A) contributes and after projection we get:

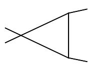
$$\mathcal{A}^{(0)}(q(1)\bar{q}(2)\phi(3)\bar{\phi}(4)) = g^2\,\frac{u}{s} \tag{3.34}$$

The diagrams giving non vanishing planar one-loop corrections are listed in Fig.3.3(a)-(d). These evaluate to:

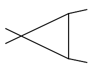
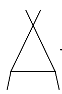
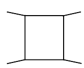
$(a) = g^4 N\Big[-(2t+s)$  $+\,t$  $+\left(t^2+\frac{st}{2}\right)$  $\Big]$ (3.35)

$(b) = g^4 N\ (s+t)$  (3.36)

$(c) = g^4 N\ \frac{(s+t)}{2}$  (3.37)

$(d) = -g^4 N\ \frac{(s+t)}{2}$  (3.38)

where we already combined in (3.36) the two possible permutations for diagrams of type (b). Summing over all the partial contributions we get:

$\mathcal{A}^{(1)}(q(1)\bar{q}(2)\phi(3)\bar{\phi}(4)) = g^4 N\Big[-t$  $+\,t$  $+\left(t^2+\frac{st}{2}\right)$  $\Big] =$

$$= \frac{g^4 N}{(4\pi)^2}\left\{\frac{u}{s}\left[-\frac{2}{\epsilon^2}\left(\frac{\mu}{t}\right)^{\epsilon} - \frac{1}{\epsilon^2}\left(\frac{\mu}{s}\right)^{\epsilon} + \frac{3}{4}\pi^2 + \frac{1}{2}\ln^2\frac{t}{s}\right] - \frac{t}{s}\left[\frac{\pi^2}{2} + \frac{1}{2}\ln^2\frac{t}{s}\right]\right\} \tag{3.39}$$

The reduced amplitude then reads:

$$\mathcal{M}^{(1)}(q(1)\bar{q}(2)\phi(3)\bar{\phi}(4)) = \lambda\left\{-\frac{2}{\epsilon^2}\left(\frac{\mu}{t}\right)^{\epsilon} - \frac{1}{\epsilon^2}\left(\frac{\mu}{s}\right)^{\epsilon} + \frac{3}{4}\pi^2 + \frac{1}{2}\ln^2\frac{t}{s} - \frac{t}{u}\left[\frac{\pi^2}{2} + \frac{1}{2}\ln^2\frac{t}{s}\right]\right\} \tag{3.40}$$

We first note that the dual conformal invariance which was present in the pure adjoint sector is lost. This is best seen by looking at the scalar integrals contributing to the amplitude in equation (3.39). Together with the dual conformal box, triangle integrals survive, inevitably breaking the dual conformal symmetry, as was shown in section 2.3.

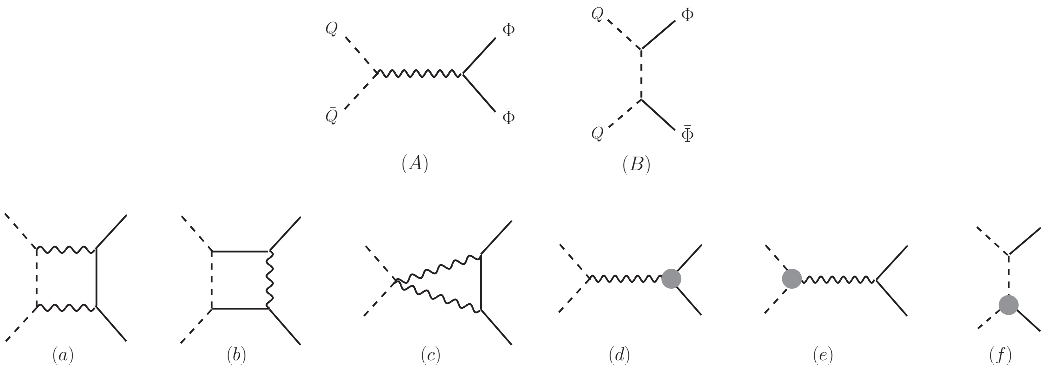


Figure 3.4: Tree level and one-loop diagrams for the process $(Q\bar{Q}\bar{\Phi}\Phi)$.

Nevertheless we notice that the result in (3.40) respects the maximal transcendentality principle. This explicitly shows that dual conformal invariance and maximal transcendentality are independent properties of the amplitudes at one-loop order.

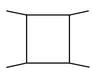
We further notice that for the chosen process different channels contribute asymmetrically. We might have considered the process with cyclically rotated fields $(\bar{q}\phi\bar{\phi}q)$. This would produce a result given by (3.40) with $s \leftrightarrow t$. It is amusing to note that if we had to sum over the two processes for the given subamplitude the result in (3.40) would be symmetrized in s and t giving an expression proportional to (3.26).

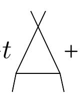
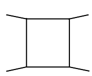
Process $Q\bar{Q}\bar{\Phi}\Phi$

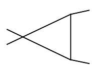
As a check of our computation we analyze the subamplitude $\mathcal{A}(Q(1)\bar{Q}(2)\bar{\Phi}(3)\Phi(4))$ for the process $(Q\bar{Q}\bar{\Phi}\Phi)$. This subamplitude is diagrammatically identical to the color ordered subamplitude $\mathcal{A}(Q(1)\bar{Q}(2)\bar{\Phi}(4)\Phi(3))$ for the process $(Q\bar{Q}\Phi\bar{\Phi})$. At tree level the diagrams in Fig.3.4(A) and (B) contributes to the scalar projection:

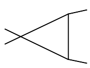
$$\mathcal{A}^{(0)}(q(1)\bar{q}(2)\bar{\phi}(3)\phi(4)) = g^2\,\frac{u}{s} \tag{3.41}$$

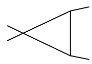
We now consider the planar one-loop corrections to the tree level amplitude. The diagrams which contribute are listed in Fig.3.4(a)-(f) and give:

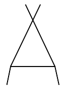
$$(a) = g^4 N\, t^2$$  (3.42)

$$(b) = g^4 N\Bigg[-t$$  $$+\ \frac{st}{2}$$  $$\Bigg]$$ (3.43)

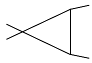
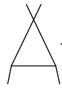
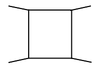
$$(c) = -g^4 N\, t$$  (3.44)

$$(d) = g^4 N\,\frac{t}{2}$$  (3.45)

$$(e) = -g^4 N\,\frac{t}{2}$$  (3.46)

$(f) = g^4 N \, 2t$  (3.47)

with diagrams of type (c) and (f) summed over the two possible choices. Summing over all the partial contributions we find:

$\mathcal{A}^{(1)}(q(1)\bar{q}(2)\bar{\phi}(3)\phi(4)) = g^4 N \Big[-t$  $+\, t$  $+ \left(t^2 + \frac{st}{2}\right)$  $\Big]$ (3.48)

This is exactly the result we found for in (3.39). By supersymmetry the same result holds for amplitudes involving $\tilde{Q}$ and $\bar{\tilde{Q}}$.

3.2.3 Fundamental sector

We now consider amplitudes with two pairs of quark/anti-quark superfields as external particles. We describe the color structure for the process $(Q\bar{Q}Q\bar{Q})$ and we remind that a similar description holds when substituting $\bar{Q}$ with $\tilde{Q}$ and/or Q with $\bar{\tilde{Q}}$. The planar amplitude can be decomposed as follows:

$$A^{(L)}(Q_I \bar{Q}^J Q_K \bar{Q}^M) = \delta_I^J \delta_K^M \, Q(1)_j \bar{Q}(4)^j \, Q(3)_i \bar{Q}(2)^i \, \mathcal{A}_1^{(L)}(q\bar{q}q\bar{q}) + \delta_I^M \delta_K^J \, Q(1)_i \bar{Q}(2)^i \, Q(3)_j \bar{Q}(4)^j \, \mathcal{A}_2^{(L)}(q\bar{q}q\bar{q}) \qquad (3.49)$$

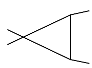
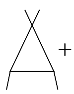
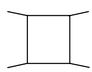
We thus have two independent color structures corresponding to the two possible ways of contracting the pairs of fundamental and antifundamental indices. For each color structure we only have a unique choice of flavour flow displayed in equation (3.49). We will omit the flavour indices in what follows.

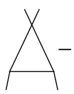
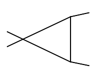
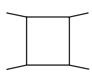
Process $Q\bar{Q}Q\bar{Q}$

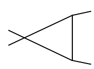
We compute the partial amplitude $\mathcal{A}_1(q\bar{q}q\bar{q})$. At tree level only the diagram depicted in Fig.3.5(A) gives a contribution:

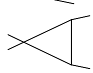
$$\mathcal{A}_1^{(0)}(q\bar{q}q\bar{q}) = g^2 \, \frac{u}{s} \qquad (3.50)$$

We now consider the planar one-loop corrections to the tree level amplitude. The diagrams which contribute are listed in Fig. 3.5(a)-(d) and give the following results:

$(a) = g^4 N \Big[-(s+2t)$  $+\, t$  $+ \left(t^2 + \frac{st}{2}\right)$  $\Big]$ (3.51)

$(b) = g^4 N \Big[-t$  $-\, s$  $+ \frac{st}{2}$  $\Big]$ (3.52)

$(c) = g^4 N \, (s+t)$  (3.53)

$(d) = -g^4 N \, (s+t)$  (3.54)

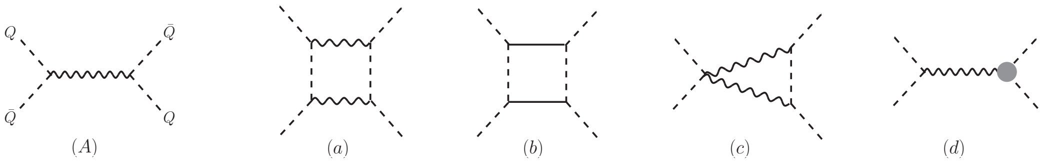
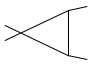
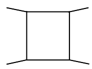


Figure 3.5: Tree level and one-loop diagrams for $(Q\bar{Q}Q\bar{Q})$.

where again we summed over the two left/right reflected diagrams of type (c) and (d). Summing over all partial contributions we find:

$\mathcal{A}_1^{(1)}(q\bar{q}q\bar{q}) = g^4 N \Big[-2(t+s)$  $+\left(t^2+st\right)$  $\Big] =$

$$= 2\,\frac{g^4 N}{(4\pi)^2}\,\frac{u}{s}\left\{-\frac{1}{\epsilon^2}\left(\frac{\mu}{t}\right)^{\epsilon}+\frac{7}{12}\pi^2+\frac{1}{2}\ln^2\frac{t}{s}+\mathcal{O}(\epsilon)\right\} \tag{3.55}$$

The ratio between the one-loop amplitude and the tree-level one is:

$$\mathcal{M}_1^{(1)}(q\bar{q}q\bar{q}) \;=\; 2\lambda\left\{-\frac{1}{\epsilon^2}\left(\frac{\mu}{t}\right)^{\epsilon}+\frac{7}{12}\pi^2+\frac{1}{2}\ln^2\frac{t}{s}+\mathcal{O}(\epsilon)\right\} \tag{3.56}$$

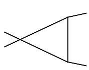
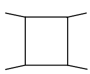
Once again we see that the result does not display dual conformal invariance whereas it respects maximal transcendentality. It is possible to show that the partial amplitude $\mathcal{A}_2^{(1)}$ is equal to $\mathcal{A}_1^{(1)}$ with the exchange $s \leftrightarrow t$.

Process $Q\tilde{Q}\bar{\tilde{Q}}\bar{Q}$

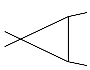
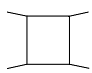
As a check of our result (3.56) we consider the process $(Q\tilde{Q}\bar{\tilde{Q}}\bar{Q})$, which is expected to provide an identical expression because of supersymmetry. We consider the color structure $Q(1)_j\bar{Q}(4)^j\,\bar{\tilde{Q}}(3)_i\tilde{Q}(2)^i$, which is the analogue of the one considered for the previous process. The amplitude corresponding to the tree level diagram depicted in Fig.3.6(A) is the following:

$$\mathcal{A}_1^{(0)}(q\tilde{q}\bar{\tilde{q}}\bar{q}) = -g^2 \tag{3.57}$$

We now consider the planar one-loop corrections to the tree level amplitude. The relevant diagrams are listed in Fig.3.6(a) and (b). The contributions of diagram (a) is:

$(a) = g^4 N \Big[-s$  $+\frac{st}{2}$  $\Big]$ (3.58)

The contribution of diagram (b) is equal to diagram (a). Summing the two diagrams above we find:

$\mathcal{A}_1^{(1)}(q\tilde{q}\bar{\tilde{q}}\bar{q}) = g^4 N \Big[-2s$  $+\,st$  $\Big] =$

$$= 2\,\frac{g^4 N}{(4\pi)^2}\left\{\frac{1}{\epsilon^2}\left(\frac{\mu}{t}\right)^{\epsilon}-\frac{7}{12}\pi^2-\frac{1}{2}\ln^2\frac{t}{s}+\mathcal{O}(\epsilon)\right\} \tag{3.59}$$

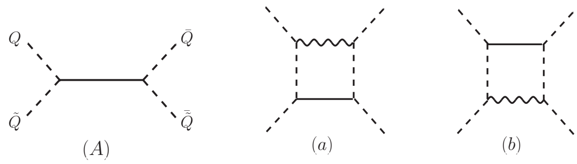


Figure 3.6: Tree level and one-loop diagrams for $(Q\tilde{Q}\bar{\tilde{Q}}\bar{Q})$ process.

Taking the ratio with the tree level result we obtain again the result in (3.56):

$$\mathcal{M}_1^{(1)}(q\tilde{q}\bar{\tilde{q}}\bar{q}) = \mathcal{M}_1^{(1)}(q\bar{q}q\bar{q}) \tag{3.60}$$

This process turns out to be the simplest from the computational point of view and thus it will be chosen for the two-loop analysis in the next section. Once again, one might want to consider the other color ordering or also reshuffled processes whose results can be obtained by suitable permutations of the Mandelstam variables.

3.3 Two-loop amplitudes

At two-loops the supergraph computation starts becoming cumbersome because of the increasing number of diagrams contributing to each process. There are some indications based on Feynman diagrammatics and integrability arguments that in the pure adjoint sector at two-loops the amplitude should be identical to that of $\mathcal{N}=4$ SYM. In fact, in [84] the dilatation operator of the theory has been found to coincide with that of $\mathcal{N}=4$ SYM up to two-loops in the purely scalar sector. Moreover, in [86] it has been argued that the sector built only with adjoint letters should be exactly integrable. If dual conformal invariance and the duality with light-like Wilson loops are a consequence of integrability, we then expect from the Wilson loop computation in [88] to obtain a result that deviates from the $\mathcal{N}=4$ SYM result only at three loop order. A diagrammatic check of this claim is in progress [93].

In the other two sectors nothing is known a priori and we expect a behaviour which is qualitative different from the $\mathcal{N}=4$ SYM case. From our one-loop detailed analysis it is easy to see that inside each sector the degree of complexity for different processes is very variable. It is therefore advisable to choose the special amplitude giving rise to less contributions. We present here the full result for the computationally easiest choice, the process of section 3.2.3 in the pure fundamental sector. We will see that the result is not dual conformal invariant and the maximal transcendentality principle is not respected at two–loop order.

3.3.1 Fundamental sector

We consider the process $(Q\tilde{Q}\bar{\tilde{Q}}\bar{Q})$ and compute the two-loop correction $\mathcal{A}_1^{(2)}(q\tilde{q}\bar{\tilde{q}}\bar{q})$ for the color structure $Q(1)_j\bar{Q}(4)^j\,\bar{\tilde{Q}}(3)_i\tilde{Q}(2)^i$. The diagrams which give a non-vanishing

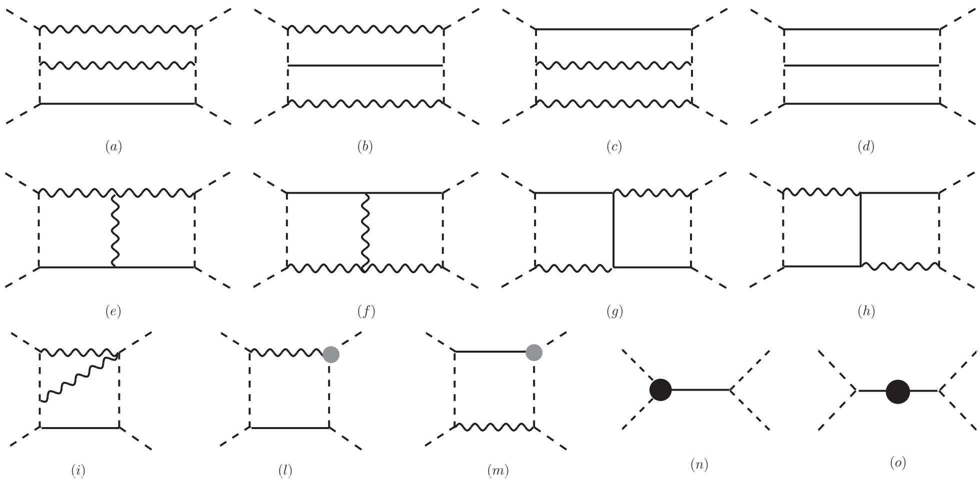
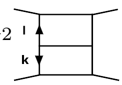
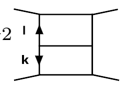
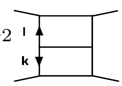


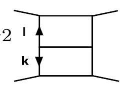
Figure 3.7: Non vanishing two-loop diagrams contributing to $Q\tilde{Q}\bar{\tilde{Q}}\bar{Q}$ amplitude. Gray and black bullets stand for one- and two-loop insertions respectively.

contribution are depicted in Fig.3.7. The diagrams (a)-(d) in the first line have the topology of vertical double boxes and we found useful to simplify their contributions by combining them properly. After performing the D-algebras and the projections to the purely scalar component amplitude we obtain:

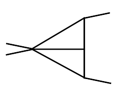
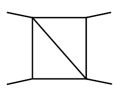
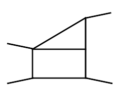
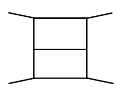
$(a) = g^6 N^2$  $\Big(- sk^2l^2 - l^2\mathrm{Tr}(p_2kp_4p_1) + k^2\mathrm{Tr}(lp_4p_1p_2) + \mathrm{Tr}(p_2k(p_3+p_3)lp_4p_1)\Big)$

$(b) = g^6 N^2$  $\Big(- sk^2l^2 + k^2\mathrm{Tr}(p_4p_1p_2l) + l^2\mathrm{Tr}(kp_3p_2p_1) + \mathrm{Tr}(p_4lkp_3p_2p_1)\Big)$

$(c) = g^6 N^2$  $\Big(- sk^2l^2 + l^2\mathrm{Tr}(p_1kp_3p_2) - k^2\mathrm{Tr}(lp_3p_2p_1) - \mathrm{Tr}(l(p_2+p_3)kp_3p_2p_1)\Big)$

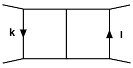
$(d) = g^6 N^2$  $\Big(- sk^2l^2 - k^2\mathrm{Tr}(p_1lp_3p_2) - l^2\mathrm{Tr}(kp_2p_1p_4) - \mathrm{Tr}(p_1lkp_2p_1p_4)\Big)$

where we again explicitly write the numerators and pictorially represent the denominators with loop variables k and l. The 6- and 4-gamma traces coming from the different contributions can be nicely combined, using A.9 and A.10, to produce a simple overall contribution:

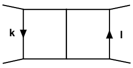
$(a) + (b) + (c) + (d) = -2s\Big($  $+$  $\Big) + 4st$  $- st^2$  (3.61)

where we omitted a g^6N^2 factor and completed the squares using the symmetries of the integrals to simplify the result. The diagram (e)-(h) in the second row of Fig.3.7

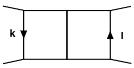
have the topology of horizontal double boxes and once again their contribution can be conveniently combined. After D-algebra and projection they give:

$(e) = \frac{1}{2}g^6N^2$  $\Big(-2k^2\mathrm{Tr}(lp_1p_2p_3) - 2\mathrm{Tr}(lp_1kp_2p_1p_3) + (k-p2)^2\mathrm{Tr}(lp_1p_2p_3) +$

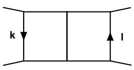
$- s(k-p_2)^2(l+p_3)^2 - s(k-p_2)^2l^2 + (l+p3)^2\mathrm{Tr}(kp_2p_1p_3)\Big)$

$(f) = \frac{1}{2}g^6N^2$  $\Big(2k^2\mathrm{Tr}(lp_2p_1p_4) - 2\mathrm{Tr}(lp_2kp_1p_2p_4) - (k+p1)^2\mathrm{Tr}(lp_2p_1p_4) +$

$- s(k+p_1)^2(l-p_4)^2 - s(k+p_1)^2l^2 - (l-p4)^2\mathrm{Tr}(kp_1p_2p_4)\Big)$

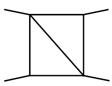
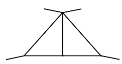
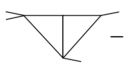
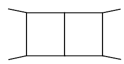
$(g) = g^6N^2$  $\Big(- s(k+p_1)^2(l+p_3)^2 + (k+p_1)^2\mathrm{Tr}(p_2lp_3p_1) - \mathrm{Tr}(p_2lp_3p_4kp_1) +$

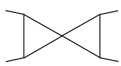
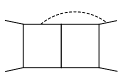
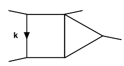
$- (l+p_3)^2\mathrm{Tr}(p_4p_2p_1k)\Big)$

$(h) = g^6N^2$  $\Big(- s(l-p_1)^2(k-p_2)^2 - (k-p_2)^2\mathrm{Tr}(p_1lp_4p_2) - \mathrm{Tr}(p_1lp_4p_3kp_2) +$

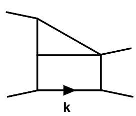
$+ (l-p_4)^2\mathrm{Tr}(p_1p_3kp_2)\Big)$

After expanding the traces and completing the squares the overall contribution massively simplifies to:

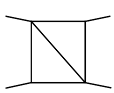
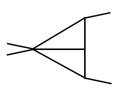
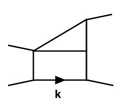
$(e) + (f) + (g) + (h) = \big(t - s\big)$  $-t$  $-6s$  $-s^2t$  $+$

$-s^2$  $+2s^2$  $+3s(k+p_3)^2$  (3.62)

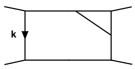
The diagram (i) drawn in Fig.3.7 contributes:

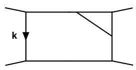
$(i) = \frac{1}{4}g^6N^2$  $\Big(sk^2 - \mathrm{Tr}(kp_3p_1p_2)\Big)$

We need to consider four diagrams of type (i) which, after expanding the traces and using symmetries of the integrals, can be combined to give:

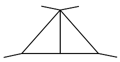
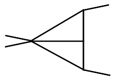
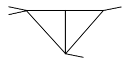
$(i) = \big(s + t\big)$  $+s$  $-t(k+p_4)^2$  (3.63)

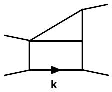
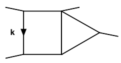
Now we compute one-loop vertex insertions of diagrams (l) and (m):

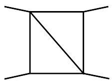
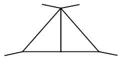
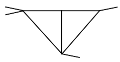
$$(l) = \frac{1}{4}g^6N^2$$  $$\Big(k^2\mathrm{Tr}(kp_1p_2p_4) + \mathrm{Tr}(kp_1kp_2p_1p_4) - (k+p_1+p_4)^2\mathrm{Tr}(p_3p_1p_2k) + 2s(k+p_1+p_4)^2k\cdot(k-p_2) + \mathrm{Tr}(p_4(k+p_1)p_3p_1p_2k) + s\mathrm{Tr}(p_4(p_1+k)(k-p_2)k)\Big)$$

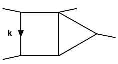
$$(m) = g^6N^2$$  $$\Big(s(k+p_1)^2(k+p_1+p_4)^2 + (k+p_1+p_4)^2\mathrm{Tr}(kp_1p_2p_4)\Big)$$

The total contribution coming from one-loop vertex insertions is given by four diagrams of type (l) and four diagrams of type (m). The overall results can be expressed as:

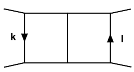
$$(l) = -t$$  $$-s$$  $$+4s$$  $$+$$

$$+t(k+p_4)^2$$  $$-s(k+p_3)^2$$  (3.64)

$$(m) = 2(s-t)$$  $$+2t$$  $$+4s$$  $$+$$

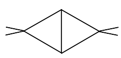
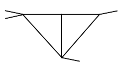
$$-2s(k+p_3)^2$$  (3.65)

The contributions (n) and (o) come from two-loop insertions of chiral vertex and propagator corrections. They give:

$$(n) = g^6N^2$$  $$\Big(sk^2l^2 + \frac{1}{2}s(k+p_1)^2l^2 + \frac{1}{2}s(k-p_2)^2l^2\Big)$$

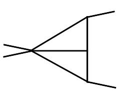
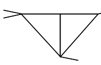
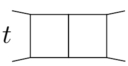
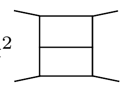
$$(o) = 0$$

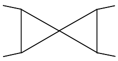
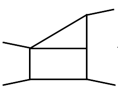
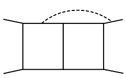
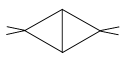
Combining the two vertex insertion, we then have an overall:

$$(n) = 2s$$  $$+2s$$  (3.66)

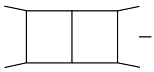
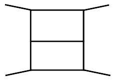
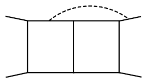
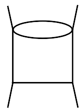
$$(o) = 0 \qquad (3.67)$$

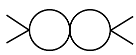
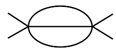
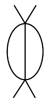
It is easy now to sum up pictorially the contributions (3.61)-(3.67) and get the final result (we omit the overall g^6N^2 factor):

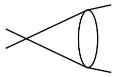
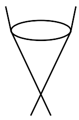
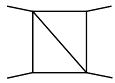
$$\mathcal{A}_1^{(2)}(q\tilde{q}\bar{\tilde{q}}\bar{q}) = -2s$$  $$+4s$$  $$-s^2t$$  $$-st^2$$  $$+$$

$$-s^2$$  $$+4st$$  $$+2s^2$$  $$+2s$$  (3.68)

We now have expressed the contributions coming from super Feynman diagrams in terms of scalar integrals and scalar integrals with irreducible numerators. Each of these integrals can now be expanded on the basis of two-loop master integrals using the formulas (A.38)-(A.44) in Appendix A.3. The full amplitude can then be written as the following linear combination on the master integral basis:

$\mathcal{A}_1^{(2)}(q\tilde{q}\bar{\tilde{q}}\bar{q}) = -\, s^2 t$  $-\, st^2$  $+\, 2s^2$  $-\, 24\, at$  $+$

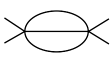
$-\, 4\,(a + a^2)$  $+\, \frac{4c - 18ac}{as}$  $-\, \frac{12c}{t}$  $+$

$+\, 4b$  $-\, 12b$  $+\, 12\,(s + t)$  (3.69)

where for convenience we defined the coefficients:

$$a = -\frac{1 - 2\epsilon}{2\epsilon} \qquad b = \frac{(1 - 2\epsilon)(1 - 3\epsilon)}{2\epsilon^2} \qquad c = -\frac{(1 - 2\epsilon)(1 - 3\epsilon)(2 - 3\epsilon)}{2\epsilon^3} \tag{3.70}$$

Looking carefully at the final result (3.69) and more generally at the expressions of the Feynman integrals contributing to each single diagram given in equations (A.38)-(A.44) of Appendix A.3, we notice the following remarkable property. With the exception of eq. (A.40), a given master integral in the linear combinations comes always multiplied by a fixed coefficient which is a function of the parameter ϵ. Expanding in ϵ the product between the coefficient and the corresponding master integral it is easy to verify that, even if the master integral itself contains terms of mixed transcendentality, the product always satisfy the maximal transcendentality property. Take for instance the sunset integral whose expansion is given in (A.25). It is clear that to orders which are relevant for the computation it does not preserve maximal transcendentality. Nevertheless in all the expansions (A.38)-(A.44), with the exception of eq. (A.40), it comes multiplied by the factor c defined in (3.70). Expanding the product we obtain:

$-\frac{(1 - 2\epsilon)(1 - 3\epsilon)(2 - 3\epsilon)}{2\epsilon^3}$  $= \frac{e^{-2\gamma_E \epsilon}}{(4\pi)^{4-2\epsilon}} \frac{1}{s^{-1+2\epsilon}} \left[\frac{1}{4\epsilon^4} - \frac{\pi^2}{24\epsilon^2} - \frac{8\zeta(3)}{3\epsilon} - \frac{19\pi^4}{480} \right]$

which respects the maximal transcendentality principle. It is clear from this analysis that the contribution to the final result (3.69) coming from the integral given in (A.40) is the only one that breaks the maximum transcendentality principle. The horizontal and vertical ladders in the first line of (3.69) are the only integrals respecting dual conformal symmetry, which is thus broken for the full amplitude as expected.

Inserting in (3.69) the expansions in ϵ of the master integrals of Appendix (A.3.2) and dividing by the tree level amplitude, the final result can be cast in the following form:

$$\mathcal{M}_1^{(2)}(q\tilde{q}\bar{\tilde{q}}\bar{q}) = \frac{e^{-2\gamma_E\epsilon}}{(4\pi)^{4-2\epsilon}t^{2\epsilon}}\Bigg[\frac{2}{\epsilon^4} - \frac{1}{\epsilon^2}\left(\frac{13\pi^2}{6} + 2\ln^2 x\right) - \frac{1}{\epsilon}\Bigg(2\pi^2\ln(1+x) + \frac{19}{3}\zeta(3) +$$
$$+\frac{2}{3}\ln^2 x(\ln x + 3\ln(1+x)) + 4\ln x\mathrm{Li}_2(-x) - 4\mathrm{Li}_3(-x)\Bigg) + 4(3\ln x - \ln(1+x))\zeta(3) +$$
$$+\frac{23}{60}\pi^4 + \frac{2}{3}\pi^2\ln x\ln(1+x) - (\pi^2 + \ln^2 x)\ln^2(1+x) + 4S_{2,2}(-x) - 4\ln x S_{1,2}(-x) +$$
$$+4\ln(1+x)\mathrm{Li}_3(-x) + \frac{2}{3}(\pi^2 - 6\ln x\ln(1+x))\mathrm{Li}_2(-x) + \frac{1}{6}(4\pi^2 + \ln^2 x)\ln^2 x\Bigg] \tag{3.71}$$

In order to get to the compact expression (3.71) we had to combine (generalized) polylogarithms with ones with inverse arguments using the identities listed in Appendix A.3.4. An important consistency check of our result is given by the fact that we exactly reproduce the exponential structure of the infrared poles which is expected for the scattering of massless particles in general gauge theories, as was explained in sections 2.1.4 and 2.2.3. In fact, if we extract the poles from the following general exponential expression for the two-loop amplitude:

$$\mathcal{M}^{(2)} = \frac{f_1(\epsilon)}{2}(\mathcal{M}^{(1)}(\epsilon))^2 + \lambda f_2(\epsilon)\mathcal{M}^{(1)}(2\epsilon)$$

we exactly reproduce our result with the choice:

$$f_1 = 1 \tag{3.72}$$
$$f_2 = -2\zeta(2) - \epsilon 14\zeta(3) + \mathcal{O}(\epsilon^2) \tag{3.73}$$

which is very reminiscent of the corresponding expansions for the scaling functions in $\mathcal{N}$ = 4 SYM. In particular in section 2.2.3 it was shown that the full two-loop amplitude exponentiates (see (2.30)) with scaling functions $f_1 = 1$ and $f_2 = -\zeta(2) - \epsilon\zeta(3) + \mathcal{O}(\epsilon^2)$. Nevertheless the the finite part of the two-loop amplitude (3.71) of $\mathcal{N}$ = 2 SCQCD does not exhibit exponential behaviour *à la* BDS as in the $\mathcal{N}$ = 4 SYM case.

3.4 Conclusions

In this chapter we have computed four-point scattering amplitudes in $\mathcal{N}$ = 2 SCQCD up to two loops in the Veneziano limit. At one loop we have considered all possible four-point scalar amplitudes, which can be classified into three independent sectors, according to the color representation of the external particles.

In the adjoint sector, namely when the external particles are four scalar fields in the adjoint representation of the gauge group, we found, in agreement with [91], that the one-loop result (3.26) coincides with the one for the planar $\mathcal{N} = 4$ SYM gluon scattering amplitude. So in this sector the one-loop result is dual conformal invariant and respects the maximum transcendentality principle. It would be important to go further and check if this connection with $\mathcal{N} = 4$ SYM survives at higher loops. In fact the difference between the expectation value of light-like Wilson loops evaluated in $\mathcal{N} = 4$ SYM and in $\mathcal{N} = 2$ SCQCD was computed and it was found a non vanishing term at three loops [88]. It would be interesting to check if this deviation is present also for scattering amplitudes, in order to understand if the Wilson loop/scattering amplitude duality is valid in this context. We left this computation for a future work [93].

We presented new results outside the adjoint sector. In the mixed sector, with two adjoint scalar fields and a quark/antiquark pair as external particles, we computed the one-loop scattering amplitude given in eq. (3.40). In the fundamental sector, with only fundamental fields as external particles, we presented results up to two loops, given in eq. (3.56) and eq. (3.71). In thise sectors we found that the loop results are not dual conformal invariant and do not respect the maximum transcendentality principle at two loops. It would be interesting to check the behaviour of higher loop corrections.

To check our two-loop result we analyzed its IR structure in the dimensional regularization scheme. We found that the IR structure is in agreement with the exponentiation of IR divergences which is predicted by the general analysis of [42, 41], with scaling functions (3.72) and (3.73) which are reminiscent of those of $\mathcal{N} = 4$ SYM. In contrast with planar scattering amplitudes in $\mathcal{N} = 4$ SYM, we found that the finite part of our two-loop result does not exponentiate, as suggested by the lack of dual conformal symmetry.

It would be interesting to extend our work to higher point scattering amplitudes. Finally, the generalization of our computations to the two parameter family of interpolating superconformal theories which connects $\mathcal{N} = 2$ SCQCD to the Z_2 orbifold of $\mathcal{N} = 4$ SYM through a parameter continuous deformation might lead to important insights into the connection with $\mathcal{N} = 4$ SYM.

CHAPTER 4

Scattering amplitudes in ABJM

In this chapter we carry on with the investigation of scattering amplitudes in superconformal theories. In particular the aim of this chapter is the investigation of the properties of scattering amplitudes in the three-dimensional $\mathcal{N} = 6$ ABJM theory, presented in 1.4. We recall that this model is a three-dimensional $\mathcal{N} = 6$ superconformal Chern-Simons theory with gauge group $U(M)_K \times U(N)_{-K}$ coupled to matter. A distinguished feature of this model compared to the well known $\mathcal{N} = 4$ SYM in four dimensions is that it is not maximally supersymmetric.

In the past few years much progress has been achieved in the perturbative analysis of three-dimensional Chern-Simons matter theories and especially of the ABJ(M) models. Beside its independent relevance in the context of AdS/CFT correspondence, the three-dimensional setup proves to be a good playground to check whether the mathematical structures exhibited by $\mathcal{N} = 4$ SYM have a counterpart in models which are *a priori* different in nature. A striking example is provided by the emergence of integrable structures in the spectral problem of the ABJM theory, which has been formulated along the lines of the $\mathcal{N} = 4$ SYM case and then extensively checked (see [94] for a review). With respect to integrability, the ABJM model looks surprisingly similar to $\mathcal{N} = 4$ SYM and independent features only become relevant at high perturbative orders [95, 96]. An indirect way of testing the appearance of integrable structures is to study the on-shell sector of the theory. In $\mathcal{N} = 4$ SYM a great effort has been devoted to the evaluation of scattering amplitudes, Wilson loops and form factors. These quantities have become important also in the context of AdS/CFT correspondence due to a number of remarkable stringy inspired properties they have been shown to possess. As we have discussed in Chapter 2 on-shell scattering amplitudes in $\mathcal{N} = 4$ SYM exhibit a duality with light-like Wilson loops, exponentiation, enhanced dynamical symmetries like dual conformal and Yangian invariance and color/kinematics duality. Following the seminal work of Van Neerven [97], the perturbative computation of form factors and their supersymmetric extensions have been performed in $\mathcal{N} = 4$ SYM up to three-loop order [98]-[99]. Form factors have also been studied at strong coupling [54, 100, 101] and conjectured to be dual to light-like periodic Wilson loops [54, 100, 102]. The existence of color/kinematics duality for form factors has also been proposed and verified in two and three-loop examples [103].

The three-dimensional picture seems to slightly depart from the four-dimensional case and, while a partial parallelism can still be traced, a precise definition of the above dualities requires more care. First, the tree-level four and six-point amplitudes have been found explicitly and Yangian invariance has been established [104, 105, 106] for all point amplitudes with the help of a three-dimensional version of BCFW recursion relations [107]. At loop level explicit computations are available for the four and six-point case up to two loops. The four-point one-loop complete superamplitude is of $\mathcal{O}(\epsilon)$ in dimensional regularization [108, 109, 110]. At two loops the planar amplitude can be written as a sum of dual conformal invariant integrals and has been found to coincide with the second order expansion of a light-like four-polygon Wilson loop [109, 110]. This points towards the fact that a Wilson loop/scattering amplitude duality might exist even if a strong coupling interpretation of the duality is less straightforward [111]-[112] with respect to the four-dimensional case [113, 114]. This result was generalized [20] to the less symmetric ABJ model [115] and evidence for an exponentiation *à la* BDS for the four-point ABJM amplitude was given at three loops [116, 117].
Beyond four points the connection with the four-dimensional case gets looser. In the ABJM model all the odd legs amplitudes are forced to vanish by gauge invariance. The six-point one-loop amplitude has been shown not to vanish [118, 119] contrary to the hexagon light-like Wilson loop [120, 121]. This suggests that if a scattering amplitude/Wilson loop duality exists it must be implemented with a proper definition of a (super)Wilson loop. At two loops the six-point amplitude has been computed analytically in [122] and shown to exhibit some similarity with the one-loop MHV amplitude in $\mathcal{N} = 4$ SYM, even if the identification experienced for the four-point amplitude gets spoiled.
Wilson loops in ABJM have been studied in the last years. The expectation value of light-like four-polygon Wilson loop has been shown to vanish at one loop for any number of cusps [120, 121] and calculated at two loops in the planar limit in [120, 123] for four cusps, and extended to n cusps in [124]. Interesting results concerning 1/2-BPS and 1/6-BPS Wilson loops were also carried out [125, 126, 127, 128, 129].
Very recently, an analysis of the form factors has been initiated also for the ABJM model, where computations for BPS operators have been performed through unitarity cuts [130] and component Feynman diagrams formalism [131]. All the remarkable properties detailed above have been found to hold in the large N limit of $\mathcal{N} = 4$ SYM and ABJM, which seems to be the regime where interaction simplifies in such a way that dualities and integrability can occur. Nevertheless, it is interesting to look at what happens to the subleading corrections of Wilson loops, amplitudes and form factors.

For scattering amplitudes in four dimensions their complete evaluation including subleading partial amplitudes is constrained by underlying BCJ relations [33]. These in turn are useful for determining gravity amplitudes as a double copy [132, 133, 134]. Moreover, interesting relations between the IR divergences of subleading $\mathcal{N} = 4$ SYM amplitudes and $\mathcal{N} = 8$ supergravity ones have been pointed out [135, 136, 137]. In three dimensions a proposal for BCJ like relations governed by a three algebra structure has been suggested [138]. Despite checks at tree level [139], it would be interesting to understand how it applies to loop amplitudes. This, in fact, requires their knowledge at finite N. Furthermore, the three-dimensional BLG theory [140]-[141], possessing $OSp(4|8)$ superconformal invariance is realized as a $SU(2) \times SU(2)$ theory [142], where the planar limit cannot be taken. Therefore inspection of BLG amplitudes at loop level inevitably requires working at finite N.

In this chapter we present the computation of four-point scattering amplitude and of the Sudakov form factor up to two loops at finite N in ABJ(M) theory. These results were presented in the published paper [2]. We perform the computations using the $\mathcal{N} = 2$ superspace approach, which makes possible to complete the computation in terms of a limited number of Feynman diagrams. To get the subleading contributions it is necessary to add a new non-planar diagram. As a by-product, we also provide the full four-point cattering amplitude ratio of BLG theory. Infrared divergences appear in the subleading contributions as poles in the dimensional regularization parameter. However, in contrast with the $\mathcal{N} = 4$ SYM amplitude where subleading terms have milder divergences, the three-dimensional amplitude exhibits a uniform leading ϵ^{-2} pole, both in the leading and subleading parts. As we will discuss, this can be understood as a consequence of the different color structures underlying amplitudes in the two cases.
We will use the information collected in the computation of the amplitude to evaluate the bilinear Sudakov form factor at any value of N. Indeed, it turns out that the superspace computation of this object can be reduced to a sum of s-channel contributions given by a subset of diagrams involved in the amplitude, albeit with different color factors. Taking the planar limit our result matches the form factor given in [130, 131] and extends it to finite N for both ABJM and ABJ theory. All the results we obtain exhibit uniform transcendentality two. This suggests that the maximal transcendentality principle [59, 143] likely applies to ABJ(M) at finite N.

The chapter is organized as follows. In the first section we summarize the results of the two-loop four-point scattering amplitude in planar limit, computed in [110]. Then, in the second section, we present the computation of the complete four-point scattering amplitude, obtained by adjusting the color factors and adding a non planar diagram, which produce a non-trivial Feynman integral solved explicitly in Appendix B.3. We also briefly discuss the structure of infrared divergences of the complete amplitudes and we recover the full four-point ampltude of BLG theory. After that we present the computation of the Sudakov form factors.

4.1 Four-point planar scattering amplitude

In this section we summarize the results presented in [110], which consist of the computation up to two loops of planar scattering amplitude of four chiral superfields, which in components give rise to the amplitude for two scalars and two chiral fermions. Amplitudes involving chiral matter external particles are the only non-trivial ones, since the vector fields are not propagating.
In particular we are interested in the four-point scattering process of the type $(A^i B_j A^k B_l)$, where the chiral superfields A transform in the bifundamental representation of the gauge group $U(N) \times U(N)$, while the chiral superfields B in the antibifundamental one. The indices i, j, k, l are fundamental indices of a global symmetry $SU(2)_A \times SU(2)_B$. It is important to notice that all ABJM amplitudes with four external particles are related by supersymmetric Ward identities [108]. As a consequence, the result for a particular component divided by its tree level counterpart is sufficient for reconstructing the whole superamplitude at that order.

The amplitudes presented in this chapter are computed following the general method described in 2.4 and using the $\mathcal{N} = 2$ superspace formalism. More precisely in this section the computations are carried out in the planar limit of ABJM theory, where N and K are large, but their ratio $\lambda = N/K << 1$ is kept fixed. In the next section we will present the computation outside the planar limit. In the planar limit it is possible to write the following color decomposition of the amplitude:

$$\mathcal{A}^{(l)}(A_{\bar{a}_1}^{a_1} B_{b_2}^{\bar{b}_2} A_{\bar{a}_3}^{a_3} B_{b_4}^{\bar{b}_4}) = \sum \mathcal{A}^{(l)}(\sigma_1\sigma_2\sigma_3\sigma_4)\delta_{b_{\sigma_2}}^{a_{\sigma_1}} \delta_{\bar{a}_{\sigma_3}}^{\bar{b}_{\sigma_2}} \delta_{b_{\sigma_4}}^{a_{\sigma_3}} \delta_{\bar{a}_{\sigma_1}}^{\bar{b}_{\sigma_4}} \tag{4.1}$$

where the sum is over exchange of even or odd sites between themselves.
In the planar sector, the loop contributions to the amplitude can be read from the single trace part of the effective superpotential:

$$\begin{aligned}\Gamma^{(l)}[A,B] = \int d^2\theta d^3p_1 \dots d^3p_4\,(2\pi)^3\,\delta^{(3)}(\textstyle\sum_i p_i) \times \\ \frac{2\pi i}{K}\epsilon_{ik}\epsilon^{jl}\,\mathrm{tr}\left(A^i(p_1)B_j(p_2)A^k(p_3)B_l(p_4)\right)\sum_i \mathcal{M}^{(l)}(p_1,\dots,p_4)\end{aligned} \tag{4.2}$$

where the sum runs over the diagrams i which contribute to the effective action. In (4.2) we have factorized the tree level expression, so that $\mathcal{M}^{(l)}(p_1,\dots,p_4)$ is the reduced amplitude, namely $\mathcal{A}_4^{(l)}/\mathcal{A}_4^{tree}$. In order to compute the complete amplitude the D-algebra is performed on the super Feynman diagrams to reduce them to local expressions in superspace, which contribute to (4.2). This operation is equivalent to the projection on

two scalar and two fermions. For each diagram we obtain contributions proportional to ordinary loop integrals, which are generally divergent, and we deal with them by dimensional regularization, $d = 3 - 2\epsilon$. Their explicit evaluation has been presented in [110, 20].
We take the external particles A, B massless, with outgoing momenta $p_1, \ldots, p_4$, with $p_i^2 = 0$. As usual, Mandelstam variables are defined by $s = (p_1 + p_2)^2, t = (p_1 + p_4)^2, u = (p_1 + p_3)^2$. In order to evaluate the diagrams we fix the convention for the upper-left leg to carry momentum p_1 and name the other legs counterclockwise. The total contribution from every single graph is given by summing over all possible permutations of the external legs accounting for the different scattering channels.

At tree level the amplitude is simply given by the diagram in Fig. 4.1 (a) associated to the classical superpotential in (1.32). Its explicit expression is:

$$\mathcal{A}_4^{tree}(A^i(p_1), B_j(p_2), A^k(p_3), B_l(p_4)) = \frac{2\pi i}{K}\epsilon^{ik}\epsilon_{jl} \tag{4.3}$$

At one loop it has been proved to vanish [108]. In $\mathcal{N} = 2$ superspace language a symmetry argument shows that the only diagram that can be constructed (Fig. 4.1b) leads to a vanishing contribution both off-shell and on-shell [144]. Wavy lines represent the gauge superfields of the two $U(N)$'s. The sum over all possible configurations of V and $\hat{V}$ has to be understood.

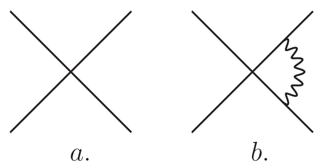


Figure 4.1: Diagrams contributing to the tree level and 1-loop four-point scattering amplitude.

At two loops, in the planar sector, the amplitude receives contribution from the diagrams depicted in Fig. 4.2, where the dark-gray blob represents the one-loop correction to the vector propagator and the light-gray blob the two-loop correction to the chiral propagator.
We begin by presenting the evaluation of diagram $2a$. After performing D-algebra, its s-channel contribution shown in Fig. 4.2 is given by a two-loop factorized Feynman integral:

$$\mathcal{D}_a^s = \mu^{4\epsilon}\int \frac{d^dk}{(2\pi)^d}\frac{d^dl}{(2\pi)^d}\frac{-(p_1+p_2)^2}{k^2\,(k+p_1+p_2)^2\,l^2\,(l-p_3-p_4)^2} = -G[1,1]^2\left(\frac{\mu^2}{s}\right)^{2\epsilon} \tag{4.4}$$

where μ is the mass scale of dimensional regularization and the G function is defined by:

$$G[a,b] = \frac{\Gamma(a+b-d/2)\Gamma(d/2-a)\Gamma(d/2-b)}{(4\pi)^{d/2}\Gamma(a)\Gamma(b)\Gamma(d-a-b)} \tag{4.5}$$

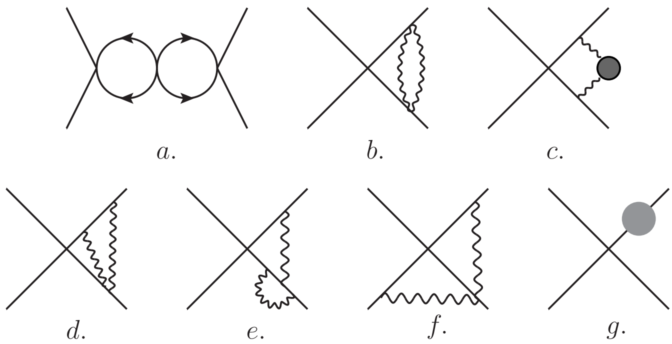


Figure 4.2: Diagrams contributing to the two-loop four-point scattering amplitude. The dark-gray blob represents one-loop corrections and the light-gray blob two-loop ones.

Taking into account all contributions of this type with color/flavor factors we obtain:

$$\mathcal{M}^{(a)} = -(4\pi\lambda)^2 G[1,1]^2 \left(\left(\frac{\mu^2}{s}\right)^{2\epsilon} + \left(\frac{\mu^2}{t}\right)^{2\epsilon} \right) = -3\zeta_2\lambda^2 + \mathcal{O}(\epsilon) \tag{4.6}$$

The contribution from diagram 2b, after D-algebra and with the particular assignment of momenta as in figure, is given by:

$$\mathcal{D}_b^{s_1} = \mu^{4\epsilon} \int \frac{d^d k}{(2\pi)^d} \frac{d^d l}{(2\pi)^d} \frac{2(p_3+p_4)^2}{l^2\,(l+k)^2\,(k-p_4)^2\,(k+p_3)^2} = \frac{2G[1,1]\Gamma(1+2\epsilon)\Gamma^2(-2\epsilon)}{(4\pi)^{d/2}\Gamma(1/2-3\epsilon)\,(s/\mu^2)^{2\epsilon}} \tag{4.7}$$

Therefore, summing over all four contributions we get:

$$\mathcal{M}^{(b)} = (4\pi\lambda)^2 \frac{G[1,1]\Gamma(1+2\epsilon)\Gamma^2(-2\epsilon)}{(4\pi)^{d/2}\Gamma(1/2-3\epsilon)} \left(\left(\frac{\mu^2}{s}\right)^{2\epsilon} + \left(\frac{\mu^2}{t}\right)^{2\epsilon} \right) \tag{4.8}$$

which is infrared divergent.

Diagram 2c, in contrast with the previous ones, is infrared divergent even when considered off-shell. This unphysical infrared divergence is cured by adding the 1PR diagram corresponding to two-loop self-energy corrections to the superpotential, depicted in Fig. 4.2g. In fact, the contribution from this diagram, when the correction is on the p_4 leg, yields:

$$\mathcal{D}_g^4 = -3\,G[1,1]\,G[1,3/2+\epsilon]\,(p_4^2)^{-2\epsilon} + 2G[1,1]^2\,(p_4^2)^{-2\epsilon} \tag{4.9}$$

The first term of this expression is infrared divergent even off-shell, but precisely cancels the infrared divergence of diagram 2c. The second term in (4.9) comes from a double factorized bubble and is finite when $d \to 3$, but since we take the momenta to be on-shell before expanding in ϵ, this piece vanishes on-shell. It turns out that after this cancelation between diagrams 2c and 2g the remainder is proportional to the integral corresponding to diagram 2b. Precisely, we have:

$$\mathcal{M}^{(c)} + \mathcal{M}^{(g)} = -3\mathcal{M}^{(b)} \tag{4.10}$$

Diagrams of type $2d$ can be evaluated by using Mellin-Barnes techniques. Specifically, with the momenta assignment as in figure, the D-algebra gives:

$$\mathcal{D}_d^{s1} = \mu^{4\epsilon} \int \frac{d^d k}{(2\pi)^d} \frac{d^d l}{(2\pi)^d} \frac{Tr(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma)\, p_4^\mu\, (p_3 + p_4)^\nu\, (k+p_4)^\rho\, (l-p_4)^\sigma}{(k+p_4)^2\, (k-p_3)^2\, (k+l)^2\, (l-p_4)^2\, l^2} \tag{4.11}$$

$$= -\frac{\Gamma^3(1/2-\epsilon)\Gamma(1+2\epsilon)\Gamma^2(-2\epsilon)}{(4\pi)^d \Gamma^2(1-2\epsilon)\Gamma(1/2-3\epsilon)\,(s/\mu^2)^{2\epsilon}} \tag{4.12}$$

and summing over the eight permutations multiplied by the corresponding flavor/color factors we obtain :

$$\mathcal{M}^{(d)} = -(4\pi\lambda)^2 \frac{2\Gamma^3(1/2-\epsilon)\Gamma(1+2\epsilon)\Gamma^2(-2\epsilon)}{(4\pi)^d \Gamma^2(1-2\epsilon)\Gamma(1/2-3\epsilon)} \left(\left(\frac{\mu^2}{s}\right)^{2\epsilon} + \left(\frac{\mu^2}{t}\right)^{2\epsilon} \right) \tag{4.13}$$

Using the identities derived in [144] it is possible to write diagram $2e$ as a combination of diagrams $2b$ and $2d$ plus a double factorized bubble which can be dropped when working on-shell. We find:

$$\mathcal{M}^{(e)} = 2\mathcal{M}^{(d)} + 4\mathcal{M}^{(b)} \tag{4.14}$$

The most complicated contribution comes from diagram $2f$, which involves a non-trivial function of the ratio s/t of kinematic invariants. Surprisingly, after some cancelations it turns out to be finite. The D-algebra for the specific choice of the external momenta as in figure results in the Feynman integral:

$$\mathcal{D}_f^{234} = \mu^{4\epsilon} \int \frac{d^d k}{(2\pi)^d} \frac{d^d l}{(2\pi)^d} \frac{-Tr(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma)\, p_4^\mu\, p_2^\nu\, k^\rho\, l^\sigma}{k^2\, (k-p_2)^2\, (k+l+p_3)^2\, (l-p_4)^2\, l^2} \tag{4.15}$$

which after taking the on-shell limit can be expressed exactly as a single one-fold Mellin-Barnes integral which is finite in the limit $\epsilon \to 0$:

$$\mathcal{D}_f^{234} = \frac{(1+s/t)\Gamma^3(1/2-\epsilon)}{(4\pi)^d \Gamma^2(1-2\epsilon)\Gamma(1/2-3\epsilon)(t/\mu^2)^{2\epsilon}} \times \tag{4.16}$$

$$\times \int_{-i\infty}^{+i\infty} \frac{d\mathbf{v}}{2\pi i} \Gamma(-\mathbf{v})\Gamma(-2\epsilon-\mathbf{v})\Gamma^*(-1-2\epsilon-\mathbf{v})\Gamma^2(1+\mathbf{v})\Gamma(2+2\epsilon+\mathbf{v}) \left(\frac{s}{t}\right)^{\mathbf{v}} \tag{4.17}$$

Taking into account the four permutations, flavor/color factors and expanding in ϵ we get:

$$\mathcal{M}^{(f)} = \lambda^2 \left(\tfrac{1}{2} \ln^2(s/t) + 3\zeta_2 \right) + \mathcal{O}(\epsilon) \tag{4.18}$$

Collecting all the partial results, the result for the planar four-point amplitude divided by its tree level counterpart up to $\mathcal{O}(\epsilon)$ terms reads

$$\mathcal{M}_4^{planar} \equiv \frac{\mathcal{A}_4^{(2)}|_{planar}}{\mathcal{A}_4^{(0)}} = \left(\frac{N}{K}\right)^2 \left(-\frac{(s/\mu'^2)^{-2\epsilon} + (t/\mu'^2)^{-2\epsilon}}{(2\epsilon)^2} + \frac{1}{2}\log^2\frac{s}{t} + 4\zeta_2 + 3\log^2 2 \right) \tag{4.19}$$

where s, t are the Mandelstam variables and $\mu'^2 = 8\pi e^{-\gamma_E}\mu^2$ with μ^2 the IR scale of dimensional regularization.

This result exhibits very interesting properties. First of all it matches the form of the two-loop correction to the four-cusped light-like Wilson loop [120], hinting at a possible Wilson loop/amplitude duality in ABJM. It has to be stressed that the Wilson loop/amplitude duality and dual conformal invariance are not supported at strong coupling by AdS/CFT arguments [55, 114], as it is not clear whether fermionic T-duality could be a symmetry of the dual string sigma model [111]-[112]. Moreover, the duality with the bosonic Wilson loop doesn't extend beyond four points since n-point amplitudes are no longer MHV for $n \geq 6$.
Another property of the two-loop result (4.19) is that it exhibits the same functional structure as the one-loop correction to the four-point amplitude in four-dimensional $\mathcal{N} = 4$ SYM theory [51, 68], provided the rescaling $\epsilon \to 2\epsilon$ there. This relation has been sharpened in [116], where an all-order in ϵ identity has been derived between the two objects. In the $\mathcal{N} = 4$ SYM case, the perturbative results for planar MHV scattering amplitudes can be expressed as linear combinations of scalar integrals that are off-shell finite in four dimensions and dual conformal invariant [68]. Precisely, once written in terms of dual variables, $p_i = x_{i+1} - x_i$, the integrands times the measure are invariant under translations, rotations, dilatations and special conformal transformations. In particular, invariance under inversion, $x^\mu \to x^\mu/x^2$, rules out bubbles and triangles and up to two loops, only square-type diagrams appear. Dual conformal invariance is broken on-shell by IR divergences that require introducing a mass regulator. Therefore, conformal Ward identities acquire an anomalous contribution [145]. A natural question which arises is whether the two-loop result (4.19) for three-dimensional ABJM models exhibits dual conformal invariance. In order to answer this question, we concentrate on the momentum integrals associated to the four diagrams in Fig. 4.2 which are the ones that eventually combine to lead to the final result (4.19). We study their behavior under dual conformal transformations when evaluated off-shell and in three dimensions. We first rewrite their expressions in terms of dual variables and then perform conformal transformations, the only non-trivial one being the inversion. Since under inversion $x_{ij}^2 \to \frac{x_{ij}^2}{x_i^2 x_j^2}$ and $d^d x_i \to \frac{d^d x_i}{(x_i^2)^d}$, it is easy to realize that, while in four dimensions the elementary invariant building block integrands are squares, in three dimensions they should be triangles. Therefore, it is immediate to conclude that the integrands associated to diagrams $4.2a - 4.2b$ cannot be invariant, since they contain bubbles. On the other hand, diagrams $4.2d$–$4.2f$ contain triangles but also non-trivial numerators which concur to make the integrand non-invariant under inversion. Despite dual conformal invariance seems not to be a symmetry of the integrals arising from our Feynman diagram approach, in the previous section we have showed that the on-shell amplitude, when written in dual space, has the same functional form of the light-like Wilson loop. As a consequence, on-shell the amplitude should possess dual conformal invariance, since Wilson loops inherit the ordinary conformal invariance of the ABJM theory, even though anomalously broken by UV divergences. As a consequence, expression (4.19) can be obtained as a combination

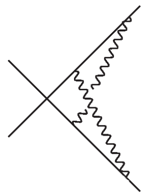


Figure 4.3: Nonplanar diagram contributing to the two-loop four-point scattering amplitude.

of manifestly dual conformally invariant integrals, as shown by a generalized unitarity computation [109].

4.2 Subleading contributions to the four-point amplitude

In this section we present our computations, which extend the previous result (4.19) to the case of finite N. This requires taking into account contributions from non-planar diagrams plus N-subleading terms in the color factor associated to each diagram in Fig. 4.1. Performing a preliminary color decomposition of the amplitude, we expect single trace contributions appearing with a leading N^2 and a subleading N^0 behaviour, and double trace contributions with a subleading N behaviour.

As in the planar case, at one loop it turns out that the momentum integrals, when evaluated in dimensional regularization are $\mathcal{O}(\epsilon)$ and therefore negligible.
At two loops, we have to consider the non-planar version of diagrams in Fig. 4.2 plus new genuinely non-planar graphs. In fact, it turns out that non-vanishing contributions only come from one new non-planar topology, depicted in Fig. 4.3.
Evaluating the complete color structure of each diagram in Fig. 4.2 it turns out that double trace terms always cancel. Therefore, taking into account all channels and exploiting the results for the loop integrals in [110, 20] we can list the contribution from each diagram to the single trace partial amplitude, divided by its tree level counterpart:

$$
\begin{aligned}
\mathcal{M}^{(a)} &= -G[1,1]^2\left(\frac{4\pi N}{K}\right)^2\left((s/\mu^2)^{-2\epsilon}+(t/\mu^2)^{-2\epsilon}\right)-4\,G[1,1]^2\left(\frac{4\pi}{K}\right)^2(u/\mu^2)^{-2\epsilon}\\
\mathcal{M}^{(b)} &= \frac{1}{2}\left(\frac{4\pi N}{K}\right)^2\left(\mathcal{D}_b(s)+\mathcal{D}_b(t)\right)+\left(\frac{4\pi}{K}\right)^2\left(\mathcal{D}_b(s)+\mathcal{D}_b(t)+3\,\mathcal{D}_b(u)\right)\\
\mathcal{M}^{(c)} &= -\frac{3}{2}\left(\frac{4\pi N}{K}\right)^2\left(\mathcal{D}_b(s)+\mathcal{D}_b(t)-2\mathcal{G}_d^p\right)+3\left(\frac{4\pi}{K}\right)^2\left(\mathcal{D}_b(s)+\mathcal{D}_b(t)-\mathcal{D}_b(u)-\mathcal{G}_d^p\right)\\
\mathcal{M}^{(d)} &= 2\left(\frac{4\pi N}{K}\right)^2\left(\mathcal{D}_d(s)+\mathcal{D}_d(t)\right)+4\left(\frac{4\pi}{K}\right)^2\left(\mathcal{D}_d(s)+\mathcal{D}_d(t)+3\,\mathcal{D}_d(u)\right)\\
\mathcal{M}^{(e)} &= 4\left(\frac{4\pi N}{K}\right)^2\left(\mathcal{D}_e(s)+\mathcal{D}_e(t)\right)-8\left(\frac{4\pi}{K}\right)^2\left(\mathcal{D}_e(s)+\mathcal{D}_e(t)-\mathcal{D}_e(u)\right)
\end{aligned}
$$

$$\begin{aligned}\mathcal{M}^{(f)} &= \left(\frac{N}{K}\right)^2 \left(\frac{1}{2}\log^2\frac{s}{t} + 3\,\zeta_2\right) + \frac{1}{K^2}\left(-\log^2\frac{s}{t} + \log^2\frac{u}{s} + \log^2\frac{t}{u} + 6\,\zeta_2\right) + \mathcal{O}(\epsilon)\\ \mathcal{M}^{(g)} &= -3\left(\frac{4\pi N}{K}\right)^2 \mathcal{G}_d^p + 3\left(\frac{4\pi}{K}\right)^2 \mathcal{G}_d^p \end{aligned} \tag{4.20}$$

The loop integrals $\mathcal{D}$ have been defined in [110, 20]. Upon evaluation, they are given by:

$$\begin{aligned}\mathcal{D}_b(s) &= \frac{2G[1,1]\Gamma(1+2\epsilon)\Gamma^2(-2\epsilon)}{(4\pi)^{d/2}\Gamma(1/2-3\epsilon)}(s/\mu^2)^{-2\epsilon}\\ \mathcal{D}_d(s) &= -\frac{\Gamma^3\left(\frac{1}{2}-\epsilon\right)\Gamma^2(-2\epsilon)\Gamma(2\epsilon+1)}{(4\pi)^d\Gamma\left(\frac{1}{2}-3\epsilon\right)\Gamma^2(1-2\epsilon)}(s/\mu^2)^{-2\epsilon}\\ \mathcal{D}_e(s) &= \mathcal{D}_d(s) + \frac{1}{2}\mathcal{D}_b(s)\end{aligned} \tag{4.21}$$

where $G[a,b]$ is defined in eq. (B.27). The integral $\mathcal{G}_d^p$ is the linear combination:

$$\mathcal{G}_d^p = \mathcal{G}_d(p_1) + \mathcal{G}_d(p_2) + \mathcal{G}_d(p_3) + \mathcal{G}_d(p_4) \tag{4.22}$$

where $\mathcal{G}_d$ is on-shell vanishing, but otherwise IR divergent:

$$\mathcal{G}_d(p_1) = \frac{G[1,1]G[1,3/2+\epsilon]}{(p_1^2)^{2\epsilon}} \tag{4.23}$$

It gets cancelled between diagrams (c) and (g).

For the non-planar contribution of Fig. 4.3 we still experience the cancellation of double trace contributions, once we take into account all possible configurations of gauge vector superfields. The contribution to the partial amplitude divided by the tree level counterpart reads:

$$\mathcal{M}^{(np)} = 4\left(\frac{4\pi}{K}\right)^2 \left(\mathcal{D}_{np}(s) + \mathcal{D}_{np}(t) + \mathcal{D}_{np}(u)\right) \tag{4.24}$$

where the non-planar Feynman integral:

$$\mathcal{D}_{np}(s) = -(\mu^2)^{2\epsilon}\int \frac{d^dk}{(2\pi)^d}\frac{d^dl}{(2\pi)^d}\frac{\mathrm{Tr}((k+l)\;k\;l\;(k+l)\;p_4\;p_3)}{k^2(k+l-p_3)^2(k+p_4)^2(l-p_3)^2\;(k+l+p_4)^2\;l^2} \tag{4.25}$$

is solved in Appendix B.3. Taking its leading contributions in the ϵ-expansion we obtain:

$$\mathcal{D}_{np}(s) = \frac{e^{-2\epsilon\gamma_E}(16\pi)^{2\epsilon}(s/\mu^2)^{-2\epsilon}}{64\pi^2}\left(\frac{1}{(2\epsilon)^2} - \frac{\pi^2}{24} - 4\log^2 2\right) \tag{4.26}$$

Now, summing these terms to the ones in eq. (4.20) suitably expanded in powers of ϵ, it is easy to see that simple poles can be reabsorbed into a redefinition of the regularization mass parameter $\mu'^2 = 8\pi e^{-\gamma_E}\mu^2$, which is the same for both planar and non-planar contributions. The complete four-point amplitude at two loops then reads up to $\mathcal{O}(\epsilon)$:

$$\mathcal{M}_4 = \left(\frac{N}{K}\right)^2 \left(-\frac{(s/\mu'^2)^{-2\epsilon} + (t/\mu'^2)^{-2\epsilon}}{(2\epsilon)^2} + \frac{1}{2}\log^2\frac{s}{t} + \frac{2\pi^2}{3} + 3\log^2 2\right) + \qquad (4.27)$$
$$+ \frac{1}{K^2}\left(\frac{2\,(s/\mu'^2)^{-2\epsilon} + 2\,(t/\mu'^2)^{-2\epsilon} - 2\,(u/\mu'^2)^{-2\epsilon}}{(2\epsilon)^2} + 2\,\log\frac{s}{u}\,\log\frac{t}{u} + \frac{\pi^2}{3} - 3\log^2 2\right)$$

We note that the result exhibits uniform transcendentality. This is a first indication that the maximal transcendentality principle [59, 143] could apply to ABJM amplitudes beyond the planar limit, as is the case in $\mathcal{N} = 4$ SYM in four dimensions [135].
We emphasize once again that along the calculation all double trace contributions have cancelled separately for each single topology, leading to a final result for the amplitude which contains only single trace terms. The technical reason for such a pattern can be traced back to the fact that, at least at this order, double trace contributions from diagrams with one gauge vector get compensated by analogous contributions where the other gauge vector runs inside the loops. Even if the disappearance of double trace structures from the final result was not *a priori* expected, it has a simple interpretation in terms of unitarity, as we are now going to discuss in detail. When constructing the whole two-loop four point amplitude from unitarity cuts, we have to take into account all two-particle cuts separating it into a one-loop and a tree level four-point amplitudes, as well as three-particle cuts dividing it into two five-point tree level amplitudes. Since the latter vanish, color structures do not emerge from three-particle cuts and we can focus on two-particle ones. We can concentrate for instance on the two-particle cut in the s-channel separating the amplitude into a one loop $\mathcal{A}_4^{(1)}(1,2,A,B)$ and a tree level $\mathcal{A}_4^{(0)}(B,A,3,4)$ four-points. In the color space and at any order in loops four-point amplitudes of the form $((A^i)^{i_1}_{\bar{i}_1}(B_j)^{\bar{i}_2}_{i_2}(A^k)^{i_3}_{\bar{i}_3}(B_l)^{\bar{i}_4}_{i_4})$ can be expanded on a basis of four independent structures, two single and two double traces (we remind that matter fields are in the bifundamental representation of the gauge group):

$$[1,2,3,4] = \mathrm{d}^{\bar{i}_2}_{\bar{i}_1}\,\mathrm{d}^{i_3}_{i_2}\,\mathrm{d}^{\bar{i}_4}_{\bar{i}_3}\,\mathrm{d}^{i_1}_{i_4} \quad , \quad [1,4,3,2] = \mathrm{d}^{\bar{i}_4}_{\bar{i}_1}\,\mathrm{d}^{i_3}_{i_4}\,\mathrm{d}^{\bar{i}_2}_{\bar{i}_3}\,\mathrm{d}^{i_1}_{i_2}$$
$$[1,2][3,4] = \mathrm{d}^{i_1}_{i_2}\,\mathrm{d}^{\bar{i}_2}_{\bar{i}_1}\,\mathrm{d}^{i_3}_{i_4}\,\mathrm{d}^{\bar{i}_4}_{\bar{i}_3} \quad , \quad [1,4][3,2] = \mathrm{d}^{i_1}_{i_4}\,\mathrm{d}^{\bar{i}_4}_{\bar{i}_1}\,\mathrm{d}^{i_3}_{i_2}\,\mathrm{d}^{\bar{i}_2}_{\bar{i}_3} \qquad (4.28)$$

Using the results of [130] we see that at one loop $\mathcal{A}_4^{(1)}(1,2,A,B)$ contains three possible structures:

$$[1,2,A,B] + [1,B,A,2] \quad , \quad [1,2][A,B] \quad \text{and} \quad [1,B][A,2] \qquad (4.29)$$

whereas the tree level amplitude $\mathcal{A}_4^{(0)}(B,A,3,4)$ enters with:

$$[3,4,B,A] - [3,A,B,4] \qquad (4.30)$$

Combining the traces from the two lower order amplitudes to obtain the two-loop structure, for the selected channel we find:

$$N\,([1,2,A,B]+[1,B,A,2])\times([3,4,B,A]-[3,A,B,4]) = \\ = N^2[1,2,3,4]-N[1,2][3,4]+N[1,2][3,4]-N^2[1,4,3,2]\,, \tag{4.31}$$

$$[1,2][A,B]\times([3,4,B,A]-[3,A,B,4]) = N\,[1,2][3,4]-N\,[1,2][3,4] \tag{4.32}$$

and:

$$[1,B][A,2]\times([3,4,B,A]-[3,A,B,4]) = [1,2,3,4]-[1,4,3,2] \tag{4.33}$$

From these relations it is easy to see that all double traces cancel. Repeating the same analysis for all channels one can prove the absence of double traces in the two-loop amplitude.

4.2.1 A comment on the IR divergences

The evaluation of the four-point amplitude at finite N reveals that IR divergences appear at two loops as double poles in ϵ, both in the leading and subleading terms. A comparison with the structure of IR divergences in $\mathcal{N}=4$ SYM amplitudes discloses a number of considerable differences.

First of all, while in $\mathcal{N}=4$ SYM amplitudes divergences already appear at one loop, in three dimensions the first singularity is delayed at second order. Based on this observation, in [109, 110, 20] a comparison between the planar four-point amplitude in $\mathcal{N}=4$ SYM at one loop and the same amplitude in ABJM at two loops has been discussed. A perfect identification between the two results, in particular for what concerns IR divergences, has been found upon rescaling $\epsilon\to 2\epsilon$ and formally identifying the mass scales. Instead, for finite N the subleading contributions spoil this identification.

To begin with, in $\mathcal{N}=4$ SYM double trace partial amplitudes appear already at one loop, while they are subleading in ϵ for the ABJM theory, at least up to two loops. Moreover, in the four-dimensional case subleading contributions to the amplitude have milder IR divergences compared to the leading ones [135]. In fact, the leading ϵ^{-2L} pole of a L-loop amplitude has been found to cancel in subleading contributions and the most subleading-in-color partial amplitude goes as ϵ^{-L}. More generally, it has been proved that N^k-subleading terms have at most ϵ^{-2L+k} poles. Instead, for ABJM theory cancellation of leading poles does not occur at two loops and the leading and subleading partial amplitudes have the same leading singularity $1/\epsilon^2$. This is basically due to the different color structures appearing in the two theories and can be better understood by constructing the two-loop operator which generates IR divergences in the ABJM theory, when applied to the tree level amplitude.

We can define an abstract color space spanned by the basis of four traces (4.28) onto which projecting chiral amplitudes of the form $((A^i)^{i_1}_{\bar{i}_1}(B_j)^{\bar{i}_2}_{i_2}(A^k)^{i_3}_{\bar{i}_3}(B_l)^{\bar{i}_4}_{i_4})$. In such a space the whole amplitude is thus represented as a four-vector. For instance, the tree level amplitude is proportional to $(1,-1,0,0)$. Following what has been done in four dimensions [42, 146], we define the operator $I^{(2)}(\epsilon)$ as a matrix acting on such a space and providing the IR divergences arising at second order coming from exchanges of two soft gluons between external legs

$$I^{(2)}(\epsilon) = -\frac{e^{-2\gamma_E\epsilon}(8\pi)^{2\epsilon}}{(2\epsilon)^2} \times \tag{4.34}$$
$$\times \begin{pmatrix} (N^2-2)(\mathtt{S}+\mathtt{T})+2\mathtt{U} & 0 & N(\mathtt{T}-\mathtt{U}) & N(\mathtt{S}-\mathtt{U}) \\ 0 & (N^2-2)(\mathtt{S}+\mathtt{T})+2\mathtt{U} & N(\mathtt{T}-\mathtt{U}) & N(\mathtt{S}-\mathtt{U}) \\ N(\mathtt{S}-\mathtt{U}) & N(\mathtt{S}-\mathtt{U}) & 2(N^2-1)\mathtt{S}+2(\mathtt{U}-\mathtt{T}) & 0 \\ N(\mathtt{T}-\mathtt{U}) & N(\mathtt{T}-\mathtt{U}) & 0 & 2(N^2-1)\mathtt{T}+2(\mathtt{U}-\mathtt{S}) \end{pmatrix}$$

where we have defined

$$\mathtt{S} = (s/\mu^2)^{-2\epsilon}\,, \qquad \mathtt{T} = (t/\mu^2)^{-2\epsilon}\,, \qquad \mathtt{U} = (u/\mu^2)^{-2\epsilon} \tag{4.35}$$

The action of such an operator on the tree level amplitude gives the structure of divergences for the complete four-point two-loop amplitude. In particular, when we apply it to the tree level vector $(1,-1,0,0)$, double trace contributions cancel. This stems for the absence of double trace contributions in the ABJM two-loop amplitude.

We note that the upper and lower 2×2 blocks on the right of matrix (4.34) are not required for our two-loop calculation. However, we have spelled them out for completeness: in principle, they might be required at higher orders if the IR divergences were to exponentiate in a similar manner to what happens in four dimensions.

It is interesting to compare this matrix with the analogous ones in QCD [146] and $\mathcal{N}=4$ SYM [135] at one loop. Apart from the different dimensions obviously due to the different dimensions of the corresponding color spaces, they share the same configuration of leading IR divergences: while the leading-in-N diagonal terms go like $1/\epsilon^2$, the subleading-in-N off-diagonal terms go like $1/\epsilon$. However, the different structure of the tree-level amplitudes allows for the appearance of $1/\epsilon$ divergent double trace contributions in four dimensions, which are not present in three dimensions.

4.2.2 BLG amplitude

BLG theory is the only model with $OSp(4|8)$ superconformal invariance in three dimensions. It can be realized as an ABJM theory with gauge group $SU(2)\times SU(2)$. Therefore, we can use the previous results to get the complete two-loop amplitude ratio.

Even though the gauge group is actually $SU(2)\times SU(2)$, rather than $U(2)\times U(2)$ as would be for the ABJM theory, it turns out that this does not affect the color structure of the amplitude. Indeed, although extra terms from the subleading part of the gluon contractions appear in individual diagrams (with color factor up to $\sim N^{-2}$), all such contributions drop out and the final result turns out to be the same as the one of the

ABJM case. Therefore, setting $N = 2$ in eq. (4.27) the result reads:

$$\mathcal{M}_4^{BLG} = \frac{1}{K^2}\left(\frac{-2\,(s/\mu'^2)^{-2\epsilon} - 2\,(t/\mu'^2)^{-2\epsilon} - 2\,(u/\mu'^2)^{-2\epsilon}}{(2\epsilon)^2} + \right.$$
$$\left. + \log^2\frac{s}{t} + \log^2\frac{s}{u} + \log^2\frac{t}{u} + 3\pi^2 + 9\log^2 2\right) + \mathcal{O}(\epsilon) \qquad (4.36)$$

It is very interesting to observe how leading and subleading contributions in (4.27) combine in order to give a result which is completely symmetric in any exchange of external labels. This is manifest in the IR divergent piece and in the finite term.

Multiplying this by the tree level four-point superamplitude [105], we obtain a two-loop superamplitude which is totally antisymmetric under any exchange of external labels. This is consistent with the fact that the theory possesses an underlying three algebra with a four-index structure constant f^{abcd} which is totally antisymmetric.

4.3 A superfield computation of the Sudakov form factor

The Sudakov form factor for the ABJ(M) theory in the planar limit has been evaluated up to two loops by Feynman diagrams [131] and by unitarity cuts [130]. In this section we exploit the previous results to provide an alternative evaluation of the Sudakov form factor based on a supergraph calculation and valid at any order in N.

In ordinary perturbation theory, the evaluation of form factors and scattering amplitudes are intimately connected whenever diagrams contributing to form factors can be obtained from diagrams contributing to amplitudes by simply collapsing free external matter legs into a bubble representing the operator insertion.

This operation is particularly effective in superspace, given the peculiar structure of diagrams contributing to the four-point chiral amplitudes. In fact, since loop contributions always arise from corrections to the quartic superpotential vertex, it turns out that collapsing two free external legs in the supergraphs of Figs. 4.2, 4.3 we generate all the two-loop corrections to the form factor of a quadratic matter operator. As a consequence, the loop integrals appearing in the two computations are exactly the same. Only combinatorics and color factors in front of them are different.

More precisely, for ABJ(M) theories we consider the following projection of the superfield form factor:

$$\Phi(s) = \langle\, A_1(p_1)\, B_1(p_2)\,|\,\mathrm{Tr}(A_1\, B_1)(p_1 + p_2)\,|\,0\,\rangle \qquad (4.37)$$

At one loop there is only one single diagram contributing, which comes from collapsing the one-loop diagram of the amplitude. As in the amplitude case [110, 20], the corresponding integral is $\mathcal{O}(\epsilon)$, therefore negligible in three dimensions.

At two loops, quantum corrections can be read from Figs. 4.2, 4.3 where we collapse two free external legs into the insertion of the operator $\mathrm{Tr}(A_1\, B_1)$. In this procedure we discard diagram $1(f)$ since it reduction simply does not exist, since it does not have two free external lines.

A simple evaluation of the relevant color factors emerging from each graph leads to the following results (we still indicate $(p_1+p_2)^2 \equiv s$):

$$\begin{aligned}
\Phi^{(a)} &= -\left(\frac{4\pi}{K}\right)^2 (M-N)^2 \frac{1}{64} \\
\Phi^{(b)} &= \frac{1}{4}\left(\frac{4\pi}{K}\right)^2 (M^2+N^2-4MN+2)\mathcal{D}_b(s) \\
\Phi^{(c)} &= \frac{1}{4}\left(\frac{4\pi}{K}\right)^2 \left(M^2+N^2-8MN+6\right)\left(\mathcal{D}_b(s)-2\mathcal{G}_d(p_1)-2\mathcal{G}_d(p_2)\right) \\
\Phi^{(d)} &= \left(\frac{4\pi}{K}\right)^2 (M^2+N^2-4MN+2)\mathcal{D}_d(s) \\
\Phi^{(e)} &= 2\left(\frac{4\pi}{K}\right)^2 (2MN-2)\mathcal{D}_e(s) \\
\Phi^{(g)} &= \frac{1}{2}\left(\frac{4\pi}{K}\right)^2 (M^2+N^2-8MN+6)\ \left(\mathcal{G}_d(p_1)+\mathcal{G}_d(p_2)\right) \\
\Phi^{(np)} &= \left(\frac{4\pi}{K}\right)^2 (-2MN+2)\mathcal{D}_{np}(s)
\end{aligned} \tag{4.38}$$

where $\mathcal{D}$ and $\mathcal{G}_d$ integrals are given in eqs. (4.21, 4.23, 4.26). We note that also in ordinary Feynman diagram approach, as it happens in unitarity based calculations, a non-planar diagram $\mathcal{D}_{np}$ contributes to determine the final result also in the planar limit.

Summing all the contributions we find:

$$\Phi_{ABJM}(s) = 2\left(\frac{4\pi}{K}\right)^2 (N^2-1)\ \left(\mathcal{D}_d(s)-\mathcal{D}_{np}(s)\right) \tag{4.39}$$

First, setting $M=N$ and inserting the results (4.21, 4.26) for the two integrals, we obtain the complete form factor at two loops for the ABJM theory

$$\boxed{\Phi_{ABJM}(s) = \frac{(N^2-1)}{4K^2}\left(-\frac{e^{-2\gamma_E\epsilon}(8\pi\mu^2)^{2\epsilon}s^{-2\epsilon}}{\epsilon^2}+\frac{2\pi^2}{3}+6\log^2 2\right)+\mathcal{O}(\epsilon)} \tag{4.40}$$

The leading contribution in N coincides with the result of [130] under the identification $K=4\pi k$ between the two Chern-Simons levels. For finite N, expression (4.40) represents the complete non-planar result. Curiously, the subleading part combines in such a way that it is proportional to the leading one.

In the generalized unitarity approach the planar two-loop contribution to the Sudakov form factor turns out to be given in terms of a single crossed triangle integral

$XT(s)$ (see eq. (4.14) in [130]). Comparing that result with the present one an interesting relation is obtained among the integrals:

$$XT(s) = 2\left(\mathcal{D}_d(s) - \mathcal{D}_{np}(s)\right) \tag{4.41}$$

More generally, for $M \neq N$, summing the previous contributions we obtain the complete form factor for the ABJ theory. In the planar limit, it reads:

$$\Phi_{ABJ}(s) = \frac{1}{2K^2}\left(\frac{e^{\gamma_E}\, s}{4\pi\mu^2}\right)^{-2\epsilon}\Bigg(-\frac{MN}{2\epsilon^2} - \log 2\,\frac{\left(M^2+N^2\right)}{2\epsilon} + \\ -\frac{1}{24}\pi^2\left(11M^2 - 30MN + 11N^2\right) + \log^2 2\left(M^2+N^2\right)\Bigg) + \mathcal{O}(\epsilon) \tag{4.42}$$

and agrees with the result of [131].

4.4 Conclusions

In this chapter we have presented the evaluation of physical observables in ABJ(M) theories for finite M, N. In particular, we have focused on the four-point scattering amplitude and on the Sudakov form factor up to two loops. Although the most interesting features like dualities and extra symmetries are expected to arise in the planar limit, the evaluation of quantities for finite ranks of the gauge groups gives useful information about the complete structure of IR (UV) divergences, also in connection with supergravity amplitudes. Moreover, from results at finite N we can read the quantum corrections to observables in the BLG model.

The complete two-loop four-point amplitude that we have obtained for ABJM possesses interesting properties. First of all, at least at two loops double trace partial amplitudes cancel completely in the final result. Moreover, subleading contributions share with the planar part the same degree of leading IR singularities. These are novelties if compared to the $\mathcal{N} = 4$ SYM case. In fact, given the different color structure of the theory, in four dimensions double trace divergent terms appear already at one loop. Furthermore, non-trivial cancellations of the leading poles in the non-planar part of the amplitudes occur, which do not seem to have an analogue in the three-dimensional ABJM model. In $\mathcal{N} = 4$ SYM, IR divergences associated to the most subleading-in-N terms have been conjectured to exponentiate and to give rise to the IR structure of the corresponding $\mathcal{N} = 8$ supergravity amplitudes obtained by the double-copy prescription [137]. It would be very interesting to investigate whether the IR divergent contributions exponentiate also for ABJM. This would necessarily require the evaluation of the IR divergent part of the amplitude, at least at the next non-trivial order, that is four loops.

The connection of BLG amplitudes with those of the corresponding $\mathcal{N} = 16$ supergravity via the color/kinematics duality of the gauge theory and the double-copy property of gravity is still to be widely investigated. As discussed in [138], BLG amplitudes can be written in such a way that BCJ-like relations [33] hold and in principle can be used to construct supergravity amplitudes as double copies of the gauge ones. In particular, this first requires expressing the whole amplitude in terms of a suitable basis of color factors related by Jacobi identities. At tree level the color/kinematics duality states that it is possible to rearrange the amplitude in such a way that the kinematic coefficients associated to those color structures obey a corresponding Jacobi identity.

We have used the information collected in the computation of the amplitude to evaluate the Sudakov form factor at any value of N. Indeed, it turned out that the superspace computation of this object can be reduced to a sum of contributions given by a subset of diagrams involved in the amplitude, with different color factors. Taking the planar limit our result matches the form factor given in [130, 131] and extends it to finite N for both ABJM and ABJ theory.

The non-planar contributions to ABJ(M) observables do not spoil the uniform transcendentality of the planar results. This is analogous to what has been observed for $\mathcal{N} = 4$ SYM amplitudes [135].

CHAPTER 5

Conclusions

Scattering amplitudes are fundamental observables in modern particle physics, since they connect theoretical physics with experiments. From the experimental point of view they represent the objects that have to be tested using colliders. From the theoretical point of view scattering amplitudes are interesting objects to investigate since in many cases they can be computed analytically and manifest remarkable properties. In this thesis we have focused on scattering amplitudes in superconformal gauge theories. Superconformal field theories have been widely explored over the last years. One of the main reasons is their special role in the formulation of AdS/CFT correspondence, which is a conjectured weak/strong duality first stated between the superconformal $\mathcal{N} = 4$ Super Yang-Mills theory and type IIB superstring on a $AdS_5 \times S^5$ background. After this first example other superconformal theories have been investigated and in particular the string dual for the superconformal $\mathcal{N} = 6$ ABJM was found. This correspondence is very interesting, because it would allow to study the non-perturbative regime of a theory by means of perturbative computations performed on the opposite side of the duality. Another relevant reason for being interested in superconformal field theories is that, soon after the formulation of the correspondence, some developments allowed to obtain a deep comprehension of the planar $\mathcal{N} = 4$ SYM theory and led to the discovery of integrable structures both in the gauge theory itself and in its string counterpart. Later on it was shown that also in planar ABJM theory integrable structures naturally show up; the search for integrable models was then extended to other superconformal theories.

Scattering ampliudes in superconformal $\mathcal{N} = 4$ SYM have remarkable properties: they can be computed analytically revealing a simple structure and their form is constrained by the symmetries of the model. They can be also related to other on-shell observables, the Wilson loops, through duality. It is important to learn more about scattering amplitudes to get insights into superconformal field theories. In particular the study of scattering amplitudes in theories different from $\mathcal{N} = 4$ SYM, such as theories which are not maximally supersymmetric, can be helpful in order to understand the origin of their remarkable properties.

In this thesis we have presented computations which represent a progress in the comprehension of scattering amplitudes in superconformal theories. Various aspects have been investigated, such as amplitudes with fundamental fields as external particles and color subleading contributions to the amplitudes. We have focused on two - not maximally - superconformal theories: $\mathcal{N} = 2$ SCQCD and $\mathcal{N} = 6$ ABJM, living respectively in four and three dimensions.

In Chapter 3 we have computed one-loop four-point massless scattering amplitudes in $\mathcal{N} = 2$ SCQCD for all the three indipendent processes, which correspond to the three indipendent color sectors of the theory. We have found that only in the adjoint sector the result coincides with those of $\mathcal{N} = 4$ SYM. In the other sectors, when fundamental particles are present, we found new results, which have no analogue in $\mathcal{N} = 4$ SYM. These results lack of dual conformal invariance, but they still respect the maximal transcendentality principle. We have computed the two-loop scattering amplitude for the process with fundamental external fields and we have found that the two-loop result is not dual conformal invariant and it does not respect the maximum transcendentality principle. As a check of our two-loop result we have analyzed its IR structure and we found that it is in agreement with the exponentiation of IR divergences with scaling function reminiscent of those of $\mathcal{N} = 4$ SYM.

In Chapter 4 we have extended the computation of four-point massless scattering amplitude and of the Sudakov form factor up to two loops outside the planar limit in ABJM theory. The complete two-loop four-point amplitude that we have obtained possesses interesting properties, which do not appear in $\mathcal{N} = 4$ SYM. First of all double trace partial amplitudes cancel completely in the final result; moreover subleading contributions share with the planar part the same degree of leading IR singularities. We have used the information collected in the computation of the amplitude to evaluate the Sudakov form factor at any value of N. All the ABJM results exhibit uniform transcendentality weight, suggesting that the maximal transcendentality principle applies to ABJM also at finite N.

The one-loop results of $\mathcal{N} = 2$ SCQCD and the two-loop results of ABJM, despite of being extracted from theories living in different spacetime dimensions, possess one common property: they all respect the maximum transcendentality principle. This property was first observed for the anomalous dimension of twist-2 operators of $\mathcal{N} = 4$ SYM and then it was found that it is enjoyed by all known observables of the theory. The maximum transcendentality principle is not respected instead in the two-loop amplitude of $\mathcal{N} = 2$ SCQCD with fundamental fields as external legs. The origin of the maximum transcendentality principle is still unclear and our results have shown that it is not directly related to the presence of maximal supersymmetry or to the planar limit. In particular our results suggest that the maximum transcendentality principle is a valid guideline only in $\mathcal{N} = 4$ SYM and in ABJM theories.

Another property investigated in our results is the dual conformal invariance of amplitudes, which is known to be indirectly related to the integrability of the theory in the planar limit. The planar two-loop four-point scattering amplitude in ABJM, which is an integrable model, manifests the dual conformal symmetry, but we found that outside the planar limit the complete amplitude loses this symmetry. It is known that $\mathcal{N} = 2$ SCQCD is not an integrable model and in fact we found that scattering amplitudes at one and two loops are not dual conformal invariant when external fundamental particles are present. The only dual conformal amplitude at one loop is the one with all external fields in the adjoint representation: in fact this amplitude is related to a subsector of the theory where the presence of integrability has been conjectured and has still to be checked.

Another important aspect of massless scattering amplitudes is the presence of infrared divergences. The well known structure of infrared divergences in planar Yang-Mills theories with and without supersymmetry has been a guide for testing the infrared part of the two-loop scattering amplitudes of $\mathcal{N} = 2$ SCQCD. In fact we found that the infrared part of our result is in agreement with the exponentiation of IR divergences predicted for a general Yang-Mills theory. On the other hand the exploration of subleading contributions to the two-loop amplitude in ABJM had enlightened new aspects in the behaviour of infrared divergences outside the planar limit, since we found that subleading contributions share with the planar ones the same degree of divergences of the leading infrared singularities.

It would be important to go further. One particular prospect concerns higher-loop amplitudes with adjoint external fields in $\mathcal{N} = 2$ SCQCD. In fact, since the difference between the expectation value of light-like Wilson loops evaluated in $\mathcal{N} = 4$ SYM and in $\mathcal{N} = 2$ SCQCD was computed and it was found a non vanishing term at three loops, it would be interesting to check if the scattering amplitude with adjoint external fields remains connected to the $\mathcal{N} = 4$ SYM one at higher loops. It would be significant to check if this deviation is present also for scattering amplitudes in order to understand if the Wilson loop/scattering amplitude duality is valid in this model. We left this computation for a future work [93].

More ambitiously it would be interesting to generalize our computations to the two parameter family of interpolating superconformal theories which connects $\mathcal{N} = 2$ SCQCD to the Z_2 orbifold of $\mathcal{N} = 4$ SYM through a continuous parameter deformation. In fact checking the presence of Wilson loop/scattering amplitudes and the presence of integrability in this two-parameter family and in the more general case of $\mathcal{N} = 2$ superconformal theories, might lead to important insights into superconformal quantum field theories and into the origin of the beautiful properties found in $\mathcal{N} = 4$ SYM.

APPENDIX A

Appendix: four-dimensional conventions

A.1 $\mathcal{N} = 1$ superspace conventions

In this thesis, when we have dealt with supersymmetric four-dimensional theories, we have used the four-dimensional $\mathcal{N} = 1$ superspace, described by coordinates $(x^{\alpha\dot{\alpha}}, \theta^{\alpha}, \bar{\theta}^{\dot{\alpha}})$, where $\alpha, \dot{\alpha} = 1, 2$. Here we summarize our conventions, which can be found in [10].
The simplest nontrivial representation of the Lorenz group $SL(2,\mathbb{C})$ is the two-component complex spinor representation $(\frac{1}{2}, 0)$, labeled by a Greek index, e.g. ψ^{α}, while the complex conjugate representation $(0, \frac{1}{2})$ is labeled by a dotted Greek index, e.g. $\psi^{\dot{\alpha}}$. So the vector representation $(\frac{1}{2}, \frac{1}{2})$ can be represented by a second rank hermitian spinor, with one undetted and one dotted indices, e.g. $x^{\alpha\dot{\alpha}}$.
Spinor indices are raised and lowered following NW-SE conventions:

$$\psi^{\alpha} = C^{\alpha\beta}\psi_{\beta} \qquad \psi_{\alpha} = \psi^{\beta}C_{\beta\alpha} \qquad \bar{\psi}^{\dot{\alpha}} = C^{\dot{\alpha}\dot{\beta}}\bar{\psi}_{\dot{\beta}} \qquad \bar{\psi}_{\dot{\alpha}} = \bar{\psi}^{\dot{\beta}}C_{\dot{\beta}\dot{\alpha}} \tag{A.1}$$

where $C_{\alpha\beta}$ is the antisymmetric matrix of $SL(2,\mathbb{C})$:

$$C^{\alpha\beta} = C^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \qquad C_{\alpha\beta} = C_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{A.2}$$

which satisfies the folowing relations:

$$C^{\alpha\beta}\,C_{\gamma\delta} \quad = \delta^{\alpha}{}_{\gamma}\,\delta^{\beta}{}_{\delta} - \delta^{\alpha}{}_{\delta}\,\delta^{\beta}{}_{\gamma} \tag{A.3}$$

Spinors are contracted according to:

$$\psi\chi \quad = \psi^{\alpha}\,\chi_{\alpha} = \chi^{\alpha}\,\psi_{\alpha} = \chi\psi \qquad \psi^2 = \tfrac{1}{2}\,\psi^{\alpha}\,\psi_{\alpha} \tag{A.4}$$

$$\bar{\psi}\bar{\chi} \quad = \bar{\psi}^{\dot{\alpha}}\,\bar{\chi}_{\dot{\alpha}} = \bar{\chi}^{\dot{\alpha}}\,\bar{\psi}_{\dot{\alpha}} = \bar{\chi}\bar{\psi} \qquad \bar{\psi}^2 = \tfrac{1}{2}\,\bar{\psi}^{\dot{\alpha}}\,\bar{\psi}_{\dot{\alpha}} \tag{A.5}$$

Objects that transform as vectors under the Lorenz group can be rewritten using spinor indices and Pauli matrices $(\sigma_{\mu})^{\alpha\dot{\beta}}$ in the following way:

$$\begin{array}{lll} \text{coordinates}: & x^{\mu} = (\sigma^{\mu})_{\alpha\dot{\beta}}\,x^{\alpha\dot{\beta}} & x^{\alpha\dot{\beta}} = \frac{1}{2}\,(\sigma_{\mu})^{\alpha\dot{\beta}}\,x^{\mu} \\ \text{derivatives}: & \partial_{\mu} = \frac{1}{2}\,(\sigma_{\mu})^{\alpha\dot{\beta}}\,\partial_{\alpha\dot{\beta}} & \partial_{\alpha\dot{\beta}} = (\sigma^{\mu})_{\alpha\dot{\beta}}\,\partial_{\mu} \\ \text{fields}: & V^{\mu} = \frac{1}{\sqrt{2}}\,(\sigma^{\mu})_{\alpha\dot{\beta}}\,V^{\alpha\dot{\beta}} & V^{\alpha\dot{\beta}} = \frac{1}{\sqrt{2}}\,(\sigma_{\mu})^{\alpha\dot{\beta}}\,V^{\mu} \end{array} \tag{A.6}$$

The Pauli matrices satisfy:

$$\sigma_\mu{}^{\alpha\dot\beta}\,\sigma^\nu{}_{\alpha\dot\beta} = 2\,\delta_\mu{}^\nu \qquad\qquad \sigma_\mu{}^{\alpha\dot\beta}\,\sigma^\mu{}_{\gamma\dot\eta} = 2\,\delta^\alpha{}_\gamma\delta^{\dot\beta}{}_{\dot\eta} \tag{A.7}$$

which imply the following trace identities:

$$\mathrm{tr}(\sigma^\mu\,\sigma^\nu) \equiv -(\sigma^\mu)^{\alpha\dot\beta}\,(\sigma^\nu)_{\alpha\dot\beta} = -2\,g^{\mu\nu} \tag{A.8}$$

$$\begin{aligned}\mathrm{tr}(\sigma^\mu\,\sigma^\nu\,\sigma^\rho\,\sigma^\eta) &\equiv (\sigma^\mu)^{\alpha\dot\beta}\,(\sigma^\nu)_{\gamma\dot\beta}\,(\sigma^\rho)^{\gamma\dot\delta}\,(\sigma^\eta)_{\alpha\dot\delta} = \\ &= 2\,(g^{\mu\nu}\,g^{\rho\eta} - g^{\mu\rho}\,g^{\nu\eta} + g^{\mu\eta}\,g^{\nu\rho})\end{aligned} \tag{A.9}$$

$$\begin{aligned}\mathrm{tr}(\sigma^\mu\,\sigma^\nu\,\sigma^\rho\,\sigma^\sigma\,\sigma^\tau\,\sigma^\eta) &\equiv (\sigma^\mu)^{\alpha\dot\beta}\,(\sigma^\nu)_{\gamma\dot\beta}\,(\sigma^\rho)^{\gamma\dot\delta}\,(\sigma^\sigma)_{\xi\dot\delta}\,(\sigma^\tau)^{\xi\dot\kappa}(\sigma^\eta)_{\alpha\dot\kappa} = \\ &= -g^{\mu\nu}\,\mathrm{tr}(\sigma^\rho\,\sigma^\sigma\,\sigma^\tau\,\sigma^\eta) + g^{\mu\rho}\,\mathrm{tr}(\sigma^\nu\,\sigma^\sigma\,\sigma^\tau\,\sigma^\eta) \\ &\quad -g^{\mu\sigma}\,\mathrm{tr}(\sigma^\nu\,\sigma^\rho\,\sigma^\tau\,\sigma^\eta) + g^{\mu\tau}\,\mathrm{tr}(\sigma^\nu\,\sigma^\rho\,\sigma^\sigma\,\sigma^\eta) \\ &\quad -g^{\mu\eta}\,\mathrm{tr}(\sigma^\nu\,\sigma^\rho\,\sigma^\sigma\,\sigma^\tau)\end{aligned} \tag{A.10}$$

where the trace over an odd number of Pauli matrices vanishes.
It follows that the scalar product of two vectors can be rewritten as

$$p\cdot k = \frac{1}{2}\,p^{\alpha\dot\beta}\,k_{\alpha\dot\beta} \tag{A.11}$$

Superspace covariant derivatives are defined as

$$D_\alpha = \partial_\alpha + \frac{i}{2}\,\bar\theta^{\dot\beta}\,\partial_{\alpha\dot\beta} \qquad , \qquad \bar D_{\dot\alpha} = \bar\partial_{\dot\alpha} + \frac{i}{2}\,\theta^\beta\,\partial_{\beta\dot\alpha} \tag{A.12}$$

and they satisfy the anticommutation relation

$$\{D_\alpha, \bar D_{\dot\beta}\} = i\partial_{\alpha\dot\beta}, \qquad \{D_\alpha, D_\beta\} = 0, \qquad \{\bar D_{\dot\alpha}, \bar D_{\dot\beta}\} = 0 \tag{A.13}$$

Integration in superspace is defined as: $\int d^2\theta = \frac{1}{2}\partial^\alpha\partial_\alpha$, $\int d^2\bar\theta = \frac{1}{2}\bar\partial^{\dot\alpha}\bar\partial_{\dot\alpha}$ and $\int d^4\theta = d^2\theta d^2\bar\theta$, such that we can project superfields to components using:

$$\begin{aligned}\int d^4x\,d^2\theta... &= \int d^4x\,D^2...|_{\theta=0} \qquad \int d^4x\,d^2\bar\theta... = \int d^4x\bar D^2...|_{\bar\theta=0} \\ \int d^4x\,d^4\theta... &= \int d^4x\,\bar D^2 D^2...|_{\theta=\bar\theta=0}\end{aligned} \tag{A.14}$$

In Chapter 3, when we perform computations in $\mathcal{N} = 2$ SCQCD, we defined the components of the chiral superfields of the theory as follows:

$$\Phi(x,\theta) = \phi(x) + \theta^\alpha\psi_\alpha(x) - \theta^2 F(x) \qquad Q(x,\theta) = q(x) + \theta^\alpha\lambda_\alpha(x) - \theta^2 G(x) \tag{A.15}$$

with a similar expansion for $\tilde Q$ and corresponding expressions for the conjugated superfields. We will need for our purpose only the lowest components of the scalar multiplets, which can be readily obtained by projections using (A.14) The vector superfield V in the Wess-Zumino gauge has the following expansion:

$$V = \theta^\alpha\bar\theta^{\dot\alpha}A_{\alpha\dot\alpha} - \bar\theta^2\theta^\alpha\lambda_\alpha - \theta^2\bar\theta^{\dot\alpha}\lambda_{\dot\alpha} + \theta^2\bar\theta^2 D' \tag{A.16}$$

The superfields V and Φ are in the adjoint representation of the gauge group, that is $V = V_a T^a$ and $\Phi = \Phi_a T^a$, where T^a are $SU(N)$ generators in the fundamental representation, which are traceless hermitian $N \times N$ matrices. The superfields Q and $\tilde{Q}$ are respectively in the fundamental and antifundamental representation of $SU(N)$. When needed, adjoint indices are denoted by $a, b, c, \ldots$, while fundamental indices are denoted by a lower $i, j, k, \ldots$ and antifundamental indices by a upper one. The generators of $SU(N)$ are normalized as $\mathrm{Tr}(T^a T^b) = \delta^{ab}$ and obey the following Fierz identity:

$$(T^a)_i^{\ j}(T^a)_k^{\ l} = \delta_i^l \delta_k^j - \frac{1}{N}\delta_i^j \delta_k^l$$

which can be used to reduce the number of traces.

A.2 Superconformal algebra

We report here the commutation and the anticommutation relations satified by the generators of the supersymmetry algebra and of the superconformal algebra in four spacetime dimensions with a generic $\mathcal{N}$. We follow the notations of [10].
The supersymmetry algebra contains, in addition to the generators of the Poincarè group $\{P_{\alpha\dot{\beta}}, J_{\alpha\beta}, \bar{J}_{\dot{\alpha}\dot{\beta}}\}$, $\mathcal{N}$ fermionic spinorial generators $Q_{a\alpha}$ and their hermitian conjugates, where $a = 1, ..., \mathcal{N}$ is an isospin index and at most $\frac{1}{2}\mathcal{N}(\mathcal{N}-1)$ complex central charges $Z_{ab} = -Z_{ba}$, which commute with all other generators. The supersymmetry algebra is:

$$\{Q_{a\alpha}, \bar{Q}^b{}_{\dot{\beta}}\} = \delta_a{}^b P_{\alpha\dot{\beta}} \tag{A.17a}$$

$$\{Q_{a\alpha}, Q_{b\beta}\} = C_{\alpha\beta} Z_{ab} \tag{A.17b}$$

$$[Q_{a\alpha}, P_{\beta\dot{\beta}}] = [P_{\alpha\dot{\alpha}}, P_{\beta\dot{\beta}}] = [\bar{J}_{\dot{\alpha}\dot{\beta}}, Q_{c\gamma}] = 0 \tag{A.17c}$$

$$[J_{\alpha\beta}, Q_{c\gamma}] = \frac{1}{2} i C_{\gamma(\alpha} Q_{c\beta)} \tag{A.17d}$$

$$[J_{\alpha\beta}, P_{\gamma\dot{\gamma}}] = \frac{1}{2} i C_{\gamma(\alpha} P_{\beta)\dot{\gamma}} \tag{A.17e}$$

$$[J_{\alpha\beta}, J^{\gamma\delta}] = -\frac{1}{2} i \delta_{(\alpha}{}^{(\gamma} J_{\beta)}{}^{\delta)} \tag{A.17f}$$

$$[J_{\alpha\beta}, \bar{J}_{\dot{\alpha}\dot{\beta}}] = [Z_{ab}, Z_{cd}] = [Z_{ab}, \bar{Z}^{cd}] = 0 \tag{A.17g}$$

The $\mathcal{N} = 1$ case is called simple supersymmetry, whereas the $\mathcal{N} > 1$ case is called extended supersymmetry. The set of transformations which mix the supercharges leaving the supersymmetry albegra invariant forms a group, called R-symmetry, group which in four dimensions is $SU(N)$.

When also conformal symmetry is present, the supersymmetry algebra becomes the superconformal algebra. The generators consist of generators of the Poincaré algebra, the special conformal boost generators $K_{\alpha\dot{\beta}}$ and the dilation generators Δ, which togheter form the the conformal group, plus $2\mathcal{N}$ spinor generators $\{Q_{a\alpha}, S^{a\alpha}\}$ (and their hermitian conjugates) and $\mathcal{N}^2$ further bosonic charges $\{A, T_a^b\}$ where $T_a{}^a = 0$. The superconformal algebra is:

$$\{Q_{a\alpha}, \bar{Q}^b{}_{\dot\beta}\} = \delta_a{}^b P_{\alpha\dot\beta} \qquad \{S^{a\alpha}, \bar{S}_b{}^{\dot\beta}\} = \delta_b{}^a K^{\alpha\dot\beta} \tag{A.18a}$$

$$\{Q_{a\alpha}, S^{b\beta}\} = -i\delta_a{}^b(J_\alpha{}^\beta + \frac{1}{2}\delta_\alpha{}^\beta\Delta) - \frac{1}{2}\delta_\alpha{}^\beta\delta_a{}^b(1-\frac{4}{N})A + 2\delta_\alpha{}^\beta T_a{}^b \tag{A.18b}$$

$$[T_a{}^b, S^{c\gamma}] = \frac{1}{2}(\delta_a{}^c S^{b\gamma} - \frac{1}{N}\delta_a{}^b S^{c\gamma}) \tag{A.18c}$$

$$[T_a{}^b, Q_{c\gamma}] = -\frac{1}{2}(\delta_c{}^b Q_{a\gamma} - \frac{1}{N}\delta_a{}^b Q_{c\gamma}) \tag{A.18d}$$

$$[A, S^{c\gamma}] = \frac{1}{2}S^{c\gamma} \qquad [\Delta, S^{c\gamma}] = -i\frac{1}{2}S^{c\gamma} \tag{A.18e}$$

$$[J_\alpha{}^\beta, S^{c\gamma}] = -\frac{1}{2}i\delta_{(a}{}^{|\gamma|}S^{c\beta)} \qquad [P_{\alpha\dot\alpha}, S^{c\gamma}] = -\delta_\alpha{}^\gamma\bar{Q}^c{}_{\dot\alpha} \tag{A.18f}$$

$$[A, Q_{c\gamma}] = -\frac{1}{2}Q_{c\gamma} \qquad [\Delta, Q_{c\gamma}] = i\frac{1}{2}Q_{c\gamma} \tag{A.18g}$$

$$[J_\alpha{}^\beta, Q_{c\gamma}] = \frac{1}{2}i\delta_\gamma{}^{(\beta}Q_{c\alpha)} \qquad [K^{\alpha\dot\alpha}, Q_{c\gamma}] = \delta_\gamma{}^\alpha\bar{S}_c{}^{\dot\alpha} \tag{A.18h}$$

$$[T_a{}^b, T_c{}^d] = \frac{1}{2}(\delta_a{}^d T_c{}^b - \delta_c{}^b T_a{}^d) \tag{A.18i}$$

$$[\Delta, K^{\alpha\dot\alpha}] = -iK^{\alpha\dot\alpha} \qquad [\Delta, P_{\alpha\dot\alpha}] = iP_{\alpha\dot\alpha} \tag{A.18j}$$

$$[J_\alpha{}^\beta, K^{\gamma\dot\gamma}] = -\frac{1}{2}i\delta_{(a}{}^{|\gamma|}K^{\beta)\dot\gamma} \qquad [J_\alpha{}^\beta, P_{\gamma\dot\gamma}] = \frac{1}{2}i\delta_\gamma{}^{(\beta}P_{\alpha)\dot\gamma} \tag{A.18k}$$

$$[J_{\alpha\beta}, J^{\gamma\delta}] = -\frac{1}{2}i\delta_{(a}{}^{(\gamma}J_{\beta)}{}^{\delta)} \tag{A.18l}$$

$$[P_{\alpha\dot\alpha}, K^{\beta\dot\beta}] = i(\delta_{\dot\alpha}{}^{\dot\beta}J_\alpha{}^\beta + \delta_\alpha{}^\beta\bar{J}_{\dot\alpha}{}^{\dot\beta} + \delta_\alpha{}^\beta\delta_{\dot\alpha}{}^{\dot\beta}\Delta) \tag{A.18m}$$

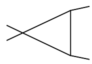
where all indices between parenthesis ( ) are to be symmetrized except those between vertical lines | |. The superconformal algebra contains the supersymmetry algebra as a subalgebra; however in the superconformal case there are no central charges.

A.3 Integrals

In Chapter 3, when computing one-loop and two-loop scattering amplitudes in $\mathcal{N} = 2$ SCQCD, after performing D-algebra and projection, we encountered several Feynman integrals. In this section we discuss how to deal with them. One-loop computations are easy enough to directly reduce by hand each integral into a sum of box and triangle scalar integrals, presented in A.3.1. At two loops, we find convenient to express the integrals in terms of a set of known master integrals by using the *Mathematica* package **FIRE** [147]. In (A.3.2) we introduce the two-loop master integral basis and the explicit expressions in dimensional regularization. In (A.3.3) we list the expansions of the amplitude integrals on the master basis. External momenta in the pictures are always labeled counterclockwise starting from the upper left corner and are always put on the mass shell $p_1^2 = p_2^2 = p_3^2 = p_4^2 = 0$.

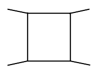
A.3.1 One-loop integrals

At one-loop order all tensor and scalar integrals can be reduced by completing the squares and integration by parts to a combination of the two following integrals: the scalar triangle and the box integrals. The analytic expression of the triangle integral and its expansion in the dimensional regularization parameter $\epsilon = 2 - d/2$ is:

$$I_{\text{triangle}}(s) =$$  $$= \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k^2 (k-p_3)^2 (k+p_4)^2} =$$

$$= \frac{\Gamma(3-d/2)\Gamma^2(d/2-2)}{s^{3-d/2}(4\pi)^{d/2}\Gamma(d-3)} = \frac{e^{-\gamma_E \epsilon}}{s^{1+\epsilon}\,(4\pi)^{2-\epsilon}} \left[\frac{1}{\epsilon^2} - \frac{\pi^2}{12} + \mathcal{O}(\epsilon)\right] \tag{A.19}$$

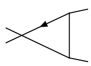
The scalar box integral is defined as:

$$I_{\text{box}} =$$  $$= \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k^2 (k-p_1)^2 (k-p_1-p_4)^2 (k+p_2)^2} \tag{A.20}$$

This integral can be easily evaluated with Mellin-Barnes representations and its ϵ expansion reads:

$$I_{\text{box}} = \frac{2\, e^{-\gamma_E \epsilon}}{st\,(4\pi)^{2-\epsilon}} \left[\left(\frac{1}{s^\epsilon} + \frac{1}{t^\epsilon}\right)\frac{1}{\epsilon^2} - \frac{2}{3}\pi^2 - \frac{1}{2}\ln^2\frac{t}{s} + \mathcal{O}(\epsilon)\right] \tag{A.21}$$

where the dependence on the dimensional regularization mass regulator μ is understood. At one-loop order, we also need the expressions for triangle and box integrals with numerators. One can directly evaluate the needed tensor integrals. The vector-triangle integral is:

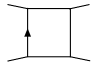
$$I^{\alpha\dot\beta}_{\text{triangle}} =$$  $$= \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k^{\alpha\dot\beta}}{k^2 (k+p_4)^2 (k-p_1-p_2)^2} =$$

$$= \frac{\Gamma(3-d/2)\Gamma(d/2-2)\Gamma(d/2-1)}{(4\pi)^{d/2} s^{3-d/2}\Gamma(d-2)} (p_1+p_2)^{\alpha\dot\beta} - \frac{\Gamma(3-d/2)\Gamma^2(d/2-2)}{(4\pi)^{d/2} s^{3-d/2}\Gamma(d-2)} p_4^{\alpha\dot\beta} \tag{A.22}$$

After the expansion in ϵ:

$$I^{\alpha\dot\beta}_{\text{triangle}} = \frac{e^{(2-\gamma_E)\epsilon}}{s^{1+\epsilon}\,(4\pi)^{2-\epsilon}} \left[-\frac{1}{\epsilon} + \mathcal{O}(\epsilon)\right](p_1+p_2)^{\alpha\dot\beta} + \frac{e^{(2-\gamma_E)\epsilon}}{s^{1+\epsilon}(4\pi)^{2-\epsilon}} \left[-\frac{1}{\epsilon^2} - 2 + \frac{\pi^2}{12} + \mathcal{O}(\epsilon)\right] p_4^{\alpha\dot\beta}$$

It is also useful to define the following vector-box integral:

$$I^{\alpha\dot\beta}_{\text{box}} =$$  $$= \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k^{\alpha\dot\beta}}{k^2 (k-p_1)^2 (k-p_1-p_4)^2 (k+p_2)^2} \tag{A.23}$$

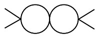
which can be evaluated as in the scalar case and expanded in ϵ:

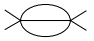
$$I^{\alpha\dot\beta}_{\text{box}} = \left[\frac{e^{-\gamma_E \epsilon}}{st\,(4\pi)^{2-\epsilon}} \left(\frac{1}{t^\epsilon}\frac{1}{\epsilon^2} - \frac{\pi^2}{12}\right) - \frac{\pi^2 + \ln^2\frac{t}{s}}{2(4\pi)^2 s(s+t)}\right](p_1-p_2)^{\alpha\dot\beta} +$$

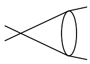
$$+ \left[\frac{e^{-\gamma_E \epsilon}}{st\,(4\pi)^{2-\epsilon}} \left(\frac{1}{s^\epsilon}\frac{1}{\epsilon^2} - \frac{7}{12}\pi^2 - \frac{1}{2}\ln^2\frac{t}{s}\right) + \frac{\pi^2 + \ln^2\frac{t}{s}}{2(4\pi)^2 s(s+t)}\right](p_1+p_4)^{\alpha\dot\beta} + \mathcal{O}(\epsilon)$$

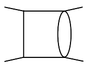
A.3.2 Two-loop master integrals

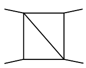
At two loops the integrals can be expressed as linear combinations on the following master integral basis:

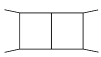
$$I_{spec}(s) \quad = \quad$$  $$= \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{1}{k^2(k+p_1+p_2)^2 l^2 (l-p_1-p_2)^2} \tag{A.24}$$

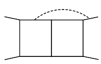
$$I_{sunset}(s) \quad = \quad$$  $$= \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{1}{k^2 l^2 (l-k-p_1-p_2)^2} \tag{A.25}$$

$$I_{tri}(s) \quad = \quad$$  $$= \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{1}{k^2(k+p_1+p_2)^2 l^2 (l-k+p_4)^2} \tag{A.26}$$

$$I_{mug}(s,t) \quad = \quad$$  $$= \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{1}{k^2(k+p_1)^2(k-p_2)^2 l^2 (l-k-p_1-p_4)^2} \tag{A.27}$$

$$I_{diag}(s,t) \quad = \quad$$  $$= \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{1}{k^2(k-p_2)^2 l^2 (l-p_4)^2 (l-k-p1-p4)^2} \tag{A.28}$$

$$I_{lad}(s,t) \quad = \quad$$  $$= \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{1}{k^2(k+p_1)^2(k-p_2)^2 l^2 (l+p_3)^2 (l-p_4)^2 (l-k-p1-p4)^2} \tag{A.29}$$

$$I_{vlad}(s,t) \quad = \quad$$  $$= \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{(k+p_1+p_4)^2}{k^2(k+p_1)^2(k-p_2)^2 l^2 (l+p_3)^2 (l-p_4)^2 (l-k-p_1-p_4)^2} \tag{A.30}$$

These can be expanded in dimensional regularization up to the needed order:

$$\begin{aligned} I_{spec}(s) = \frac{e^{-2\gamma_E \epsilon}}{(4\pi)^{4-2\epsilon} s^{2\epsilon}} \Bigg[& \frac{1}{\epsilon^2} + \frac{4}{\epsilon} + 12 - \frac{\pi^2}{6} + \epsilon \left(32 - \frac{2\pi^2}{3} - \frac{14}{3}\zeta(3) \right) + \\ & + \epsilon^2 \left(80 - 2\pi^2 - \frac{7\pi^4}{120} - \frac{56}{3}\zeta(3) \right) + \mathcal{O}(\epsilon^3) \Bigg] \end{aligned} \tag{A.31}$$

$$\begin{aligned} I_{sunset}(s) = \frac{e^{-2\gamma_E \epsilon}}{(4\pi)^{4-2\epsilon}} \frac{1}{s^{-1+2\epsilon}} \Bigg[& -\frac{1}{4\epsilon} - \frac{13}{8} + \epsilon \left(-\frac{115}{16} + \frac{\pi^2}{24} \right) + \epsilon^2 \left(-\frac{865}{32} + \frac{13\pi^2}{48} + \frac{8}{3}\zeta(3) \right) + \\ & + \epsilon^3 \left(-\frac{5971}{64} + \frac{115\pi^2}{96} + \frac{19\pi^4}{480} + \frac{52}{3}\zeta(3) \right) + \mathcal{O}(\epsilon^4) \Bigg] \end{aligned} \tag{A.32}$$

$$\begin{aligned} I_{tri}(s) = \frac{e^{-2\gamma_E \epsilon}}{(4\pi)^{4-2\epsilon}} \frac{1}{s^{2\epsilon}} \Bigg[& \frac{1}{2\epsilon^2} + \frac{5}{2\epsilon} + \frac{19}{2} + \frac{\pi^2}{12} + \epsilon \left(\frac{65}{2} + \frac{5\pi^2}{12} - \frac{13}{3}\zeta(3) \right) + \\ & + \epsilon^2 \left(\frac{211}{2} + \frac{19\pi^2}{12} - \frac{41\pi^4}{720} - \frac{65}{3}\zeta(3) \right) + \mathcal{O}(\epsilon^3) \Bigg] \end{aligned} \tag{A.33}$$

$$\begin{aligned} I_{mug}(s,t) = \frac{e^{-2\gamma_E \epsilon}}{(4\pi)^{4-2\epsilon}} \frac{1}{t^\epsilon s^{1+\epsilon}} \Bigg\{ & \frac{1}{\epsilon^3} + \frac{2}{\epsilon^2} + \frac{1}{\epsilon} \left(4 - \frac{\pi^2}{2} \right) + 8 - \pi^2 - \frac{32}{3}\zeta(3) + \\ & + \mathrm{Li}_3(-x) - \ln x \mathrm{Li}_2(-x) - \frac{1}{2}(\pi^2 + \ln^2 x)\ln(1+x) + \frac{1}{2}\pi^2 \ln x + \frac{1}{6}\ln^3 x + \end{aligned}$$

$$
\begin{aligned}
&+\epsilon\bigg[-2\mathrm{Li}_4(-x)+(2+\ln x+\ln(1+x))\mathrm{Li}_3(-x)+\\
&-\left(\frac{\pi^2}{6}+2\ln x+\ln x\ln(1+x)\right)\mathrm{Li}_2(-x)-S_{1,2}(-x)\ln x+S_{2,2}(-x)+\\
&+\left(\ln x-\ln(1+x)-\frac{64}{3}\right)\zeta(3)+16-2\pi^2-\frac{31}{180}\pi^4+\\
&-\frac{5}{24}\ln^4 x+\frac{\pi^2}{4}(-4-2\ln x-\ln(1+x))\ln(1+x)+\\
&+\frac{1}{6}(2+\ln x+4\ln(1+x))\ln^3 x+\frac{\pi^2}{6}\ln x(6+3\ln x+5\ln(1+x))+\\
&-\frac{1}{12}(8\pi^2+3\ln(1+x)(4+2\ln x+\ln(1+x))\ln^2 x\bigg]+\mathcal{O}(\epsilon^2)\bigg\}
\end{aligned}
\tag{A.34}
$$

$$
\begin{aligned}
I_{diag}(s,t)=&-\frac{e^{-2\gamma_E\epsilon}}{(4\pi)^{4-2\epsilon}}\frac{1}{s+t}\bigg[-\frac{1}{\epsilon^2}\left(\frac{\ln^2 x}{2}+\frac{\pi^2}{2}\right)+\frac{1}{\epsilon}\bigg(2\mathrm{Li}_3(-x)-2\ln x\mathrm{Li}_2(-x)+\\
&-(\ln^2 x+\pi^2)\ln(1+x)+\frac{2}{3}\ln^3 x+\ln s\ln^2 x+\pi^2\ln t-2\zeta(3)\bigg)-4\mathrm{Li}_4(-x)+\\
&+4\Big(\ln(1+x)-\ln s\Big)\mathrm{Li}_3(-x)+2\Big(\ln^2 x+2\ln s\ln x-2\ln x\ln(1+x)\Big)\mathrm{Li}_2(-x)+\\
&+2\left(\frac{2}{3}\ln^3 x+\ln s\ln^2 x+\pi^2\ln t-2\zeta(3)\right)\ln(1+x)+4(S_{2,2}(-x)-\ln xS_{1,2}(-x))+\\
&-(\ln^2 x+\pi^2)\ln^2(1+x)-\frac{1}{2}\ln^4 x-\frac{4}{3}\ln s\ln^3 x-\left(\ln^2 s+\frac{11}{12}\pi^2\right)\ln^2 x+\\
&-\pi^2\ln^2 s-2\pi^2\ln s\ln x+4\zeta(3)\ln t-\frac{\pi^4}{20}+\mathcal{O}(\epsilon)\bigg]
\end{aligned}
\tag{A.35}
$$

$$
\begin{aligned}
I_{lad}(s,t)=&-\frac{e^{-2\gamma_E\epsilon}}{(4\pi)^{4-2\epsilon}}\frac{1}{ts^{2+2\epsilon}}\bigg[-\frac{4}{\epsilon^4}+\frac{5\ln x}{\epsilon^3}-\frac{1}{\epsilon^2}\left(2\ln^2 x-\frac{5}{2}\pi^2\right)+\\
&-\frac{1}{\epsilon}\bigg(\frac{2}{3}\ln^3 x+\frac{11}{2}\pi^2\ln x-\frac{65}{3}\zeta(3)+4\mathrm{Li}_3(-x)-4\ln x\mathrm{Li}_2(-x)+\\
&-2\left(\ln^2 x+\pi^2\right)\ln(1+x)\bigg)+\frac{4}{3}\ln^4 x+6\pi^2\ln^2 x-\frac{88}{3}\zeta(3)\ln x+\frac{29}{30}\pi^4+\\
&-4\left(S_{2,2}(-x)-\ln xS_{1,2}(-x)\right)+44\mathrm{Li}_4(-x)-4\Big(\ln(1+x)+6\ln x\Big)\mathrm{Li}_3(-x)+\\
&+2\left(\ln^2 x+2\ln x\ln(1+x)+\frac{10}{3}\pi^2\right)\mathrm{Li}_2(-x)+\left(\ln^2 x+\pi^2\right)\ln^2(1+x)+\\
&-\frac{2}{3}\left(4\ln^3 x+5\pi^2\ln x-6\zeta(3)\right)\ln(1+x)+\mathcal{O}(\epsilon)\bigg]
\end{aligned}
\tag{A.36}
$$

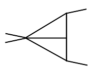
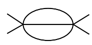
$$
\begin{aligned}
I_{vlad}(s,t)=&\frac{\Gamma[1+\epsilon]^2}{(4\pi)^{4-2\epsilon}}\frac{1}{s^{2+2\epsilon}}\bigg[\frac{9}{4\epsilon^4}-\frac{2\ln x}{\epsilon^3}-\frac{7\pi^2}{3\epsilon^2}+\frac{1}{\epsilon}\bigg(\frac{4}{3}\ln^3 x+\frac{14}{3}\pi^2\ln x+\\
&-4(\ln^2 x+\pi^2)\ln(1+x)+8\mathrm{Li}_3(-x)-8\ln x\mathrm{Li}_2(-x)-16\zeta(3)\bigg)+\\
&+20S_{2,2}(-x)-20\ln xS_{1,2}(-x)-28\mathrm{Li}_4(-x)+
\end{aligned}
$$

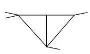
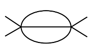
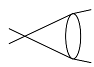
$$+\left(8\ln x+20\ln(1+x)\right)\mathrm{Li}_3\left(-x\right)+\left(6\ln^2 x-20\ln x\ln(1+x)-\frac{4\pi^2}{3}\right)\mathrm{Li}_2\left(-x\right)+$$
$$-\frac{4}{3}\ln^4 x-\frac{13}{3}\pi^2\ln^2 x+\left(\frac{16}{3}\ln^3 x+\frac{26}{3}\pi^2\ln x\right)\ln(1+x)+$$
$$-5(\ln^2 x+\pi^2)\ln^2(1+x)+\left(28\ln x-20\ln(1+x)\right)\zeta(3)-\frac{7\pi^4}{45}+\mathcal{O}(\epsilon)\Big] \qquad \text{(A.37)}$$

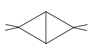
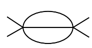
with $x = t/s$. Corresponding expressions can be written for t-channel integrals.

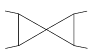
A.3.3 Two-loop expansions on master basis

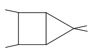
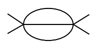
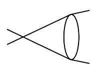
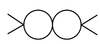
We list here the expansions on the master integral basis of the two-loop integrals encountered in computations carried out in Chapter 3. We present also the integrals which eventually get canceled in the sum but are still present at the level of single diagrams in order to make manifest the degree of transcendentality of each contribution.

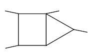
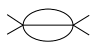
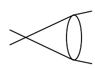
 $= \frac{c}{s^2}$  (A.38)

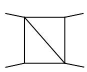
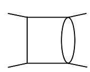
 $= -\frac{c}{s^2}$  $+\frac{b}{s}$  (A.39)

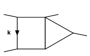
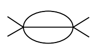
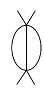
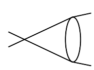
 $= \frac{2}{a}\Big[\frac{c}{s^2}$  $-\frac{a^2}{s}$  $\Big]$ (A.40)

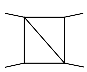
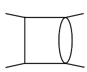
 $= \frac{4a^2}{s^2}$  (A.41)

 $= -\frac{6c}{s^3}$  $+\frac{3b}{s^2}$  $+\frac{4a^2}{s^2}$  (A.42)

 $= -3c\Big[\frac{1}{s^2 t}$  $+(s\leftrightarrow t)\Big] - 3\frac{b}{st}$  $+$

$+3\,\frac{s+t}{st}$  $-6\,\frac{a}{t}$  (A.43)

 $(k+p_3)^2 = -\frac{2c}{s^2}$  $-\frac{3c}{st}$  $-\frac{2b}{s}$  $+$

$+3\,\frac{s+t}{s}$  $-4a$  (A.44)

where the coefficients a, b, c are defined in (3.70). Corresponding integrals in the t-channel can be obtained by $s \leftrightarrow t$.

A.3.4 Polylogarithm identities

The two-loop amplitude is a function of standard polylogarithms and Nielsen generalized polylogarithms defined as:

$$S_{n,p}(z) = \frac{(-1)^{n+p-1}}{(n-1)!p!}\int_0^1 dt\ \frac{(\ln t)^{n-1}\ (\ln(1-zt))^p}{t} \qquad S_{n,1}(z) = \mathrm{Li}_{n+1}(z) \tag{A.45}$$

Following the literature (see e.g. appendix A of [51]), the final result can be simplified by using the following set of identities for the polylogarithms with inverse argument:

$$\begin{aligned}
\mathrm{Li}_2(-1/x) &= -\mathrm{Li}_2(-x) - \frac{\pi^2}{6} - \frac{1}{2}\ln^2 x \\
\mathrm{Li}_3(-1/x) &= \mathrm{Li}_3(-x) + \frac{\pi^2}{6}\ln x + \frac{1}{6}\ln^3 x \\
\mathrm{Li}_4(-1/x) &= -\mathrm{Li}_4(-x) - \frac{1}{24}\ln^4 x - \frac{\pi^2}{12}\ln^2 x - \frac{7\pi^4}{360} \\
S_{1,2}(-1/x) &= -S_{1,2}(-x) + \mathrm{Li}_3(-x) - \ln x\,\mathrm{Li}_2(-x) + \zeta(3) - \frac{1}{6}\ln^3 x \\
S_{2,2}(-1/x) &= S_{2,2}(-x) - 2\mathrm{Li}_4(-x) + \ln x\,\mathrm{Li}_3(-x) - \ln x\,\zeta(3) + \frac{1}{24}\ln^4 x - \frac{7\pi^4}{360}
\end{aligned} \tag{A.46}$$

In our case $x = t/s$ is a positive real number, thus the above identities hold for the whole domain of x.

APPENDIX B

Appendix: three dimensional conventions

B.1 $\mathcal{N} = 2$ superspace conventions

In three-dimensional spacetime the Lorentz group is $SL(2,\mathbb{R})$ and the corresponding fundamental representation acts on a real two component spinor, labeled by a Greek index, e.g. ψ^α. Thus a vector can be described by a symmetric second-rank spinor, e.g. $x^{\alpha\beta}$ or by a traceless second-rank spinor, e.g. $p_\alpha{}^\beta$. With this notation we can introduce the three-dimensional $\mathcal{N} = 1$ superspace, labeled by three spacetime coordinates $x^{\alpha\beta}$ and two anticommuting spinor coordinates θ^α, denoted collectively by $z^M = (x^{\alpha\beta}, \theta^\alpha)$. When an extended supersymmetry ($\mathcal{N} > 1$) is present there are $\mathcal{N}$ spinor coordinates: θ^α_I, where $I = 1, ..., \mathcal{N}$.

In this thesis, when we have dealt with three-dimensional theories with extended supersymmetry, we have used the three-dimensional $\mathcal{N} = 2$ superspace. In order to construct the $\mathcal{N} = 2$ formalism it is important to notice that the superspace has 2-component anticomuting coordinates θ^α_1 and θ^α_2. Then we can define new complex anticommuting coordinates:

$$\theta^\alpha = \theta^\alpha_1 - i\theta^\alpha_2, \qquad \bar{\theta}^\alpha = \theta^\alpha_1 + i\theta^\alpha_2 \tag{B.1}$$

With these new variables the $\mathcal{N} = 2$ superspace is described by $(x^{\alpha\beta}, \theta^\alpha, \bar{\theta}^\beta)$, where $\alpha, \beta = 1, 2$. We present now our conventions concerning the three-dimensional $\mathcal{N} = 2$ superspace, following the ones of [10] and [20]. Spinorial indices are raised and lowered as:

$$\psi^\alpha = C^{\alpha\beta}\psi_\beta \qquad \psi_\alpha = \psi^\beta C_{\beta\alpha} \tag{B.2}$$

where the C matrix:

$$C^{\alpha\beta} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \qquad C_{\alpha\beta} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{B.3}$$

obeys the relation:

$$C^{\alpha\beta} C_{\gamma\delta} \quad = \delta^\alpha{}_\gamma \, \delta^\beta{}_\delta - \delta^\alpha{}_\delta \, \delta^\beta{}_\gamma \tag{B.4}$$

Spinors are contracted according to:

$$\psi\chi = \psi^\alpha\,\chi_\alpha = \chi^\alpha\,\psi_\alpha = \chi\psi \qquad \psi^2 = \frac{1}{2}\,\psi^\alpha\,\psi_\alpha \tag{B.5}$$

Dirac matrices $(\gamma^\mu)^\alpha{}_\beta$ are defined to satisfy the algebra:

$$(\gamma^\mu)^\alpha{}_\gamma\,(\gamma^\nu)^\gamma{}_\beta = -g^{\mu\nu}\delta^\alpha{}_\beta + i\,\epsilon^{\mu\nu\rho}\,(\gamma_\rho)^\alpha{}_\beta \tag{B.6}$$

Trace identities needed for loop calculations can be easily obtained from the above algebra:

$$\mathrm{tr}(\gamma^\mu\,\gamma^\nu) = (\gamma^\mu)^\alpha{}_\beta\,(\gamma^\nu)^\beta{}_\alpha = -2\,g^{\mu\nu} \tag{B.7}$$

$$\mathrm{tr}(\gamma^\mu\,\gamma^\nu\,\gamma^\rho) = -(\gamma^\mu)^\alpha{}_\beta\,(\gamma^\nu)^\beta{}_\gamma\,(\gamma^\rho)^\gamma{}_\alpha = 2\,i\,\epsilon^{\mu\nu\rho} \tag{B.8}$$

$$\begin{aligned}\mathrm{tr}(\gamma^\mu\,\gamma^\nu\,\gamma^\rho\,\gamma^\sigma) &= (\gamma^\mu)^\alpha{}_\beta\,(\gamma^\nu)^\beta{}_\gamma\,(\gamma^\rho)^\gamma{}_\delta\,(\gamma^\sigma)^\delta{}_\alpha = \\ &= 2\,(g^{\mu\nu}\,g^{\rho\sigma} - g^{\mu\rho}\,g^{\nu\sigma} + g^{\mu\sigma}\,g^{\nu\rho})\end{aligned} \tag{B.9}$$

Using these matrices, vectors and second-rank spinors are exchanged according to:

$$\begin{array}{lll} \text{coordinates}: & x^\mu = (\gamma^\mu)_{\alpha\beta}\,x^{\alpha\beta} & x^{\alpha\beta} = \frac{1}{2}\,(\gamma_\mu)^{\alpha\beta}\,x^\mu \\ \text{derivatives}: & \partial_\mu = \frac{1}{2}\,(\gamma_\mu)^{\alpha\beta}\,\partial_{\alpha\beta} & \partial_{\alpha\beta} = (\gamma^\mu)_{\alpha\beta}\,\partial_\mu \\ \text{fields}: & A_\mu = \frac{1}{\sqrt{2}}\,(\gamma_\mu)^{\alpha\beta}\,A_{\alpha\beta} & A_{\alpha\beta} = \frac{1}{\sqrt{2}}\,(\gamma^\mu)_{\alpha\beta}\,A_\mu \end{array} \tag{B.10}$$

It follows that the scalar product of two vectors can be rewritten as:

$$p\cdot k \;=\; \frac{1}{2}\,p^{\alpha\beta}\,k_{\alpha\beta} \tag{B.11}$$

Superspace covariant derivatives are defined as:

$$D_\alpha = \partial_\alpha + \frac{i}{2}\,\bar\theta^\beta\,\partial_{\alpha\beta} \qquad , \qquad \bar D_\alpha = \bar\partial_\alpha + \frac{i}{2}\,\theta^\beta\,\partial_{\alpha\beta} \tag{B.12}$$

and they satisfy:

$$\{D_\alpha, \bar D_\beta\} = i\partial_{\alpha\beta} \qquad \{D_\alpha, D_\beta\} = 0 \qquad \{\bar D_\alpha, \bar D_\beta\} = 0 \tag{B.13}$$

Apart from the nature of the vector representation in (B.13), and that one does not make distinctions between dotted and undotted spinor indexes, (B.13) is the same algebra of covariant derivatives of $\mathcal{N}=1$ four-dimensional superspace (A.13) thus making Feynman supergraph rules very similar to the known rules, apart for new contractions which are now allowed, such as $\bar D^\alpha D_\alpha$ or $\bar\theta^\alpha\theta_\alpha$. So, the three-dimensional $\mathcal{N}=2$ formalism is the analogous of the four-dimensional $\mathcal{N}=1$ one, described in Appendix A.1. This is the main reason for using $\mathcal{N}=2$ formalism instead of the $\mathcal{N}=1$ one in three dimensions.

In Chapter 4, when we perform computations in ABJM theory, we defined the components of the chiral superfields of the theory as follows:

$$\begin{aligned} Z &= \phi(x_L) + \theta^\alpha\psi_\alpha(x_L) - \theta^2\,F(x_L) \\ \bar Z &= \bar\phi(x_R) + \bar\theta^\alpha\bar\psi_\alpha(x_R) - \bar\theta^2\,\bar F(x_R) \end{aligned} \tag{B.14}$$

The components of the real vector superfield $V(x,\theta,\bar{\theta})$ in the Wess-Zumino gauge are the gauge field $A_{\alpha\beta}$, a complex two-component fermion λ_α, a real scalar σ and an auxiliary scalar D, such that:

$$V = i\,\theta^\alpha\bar{\theta}_\alpha\,\sigma(x) + \theta^\alpha\bar{\theta}^\beta\,\sqrt{2}\,A_{\alpha\beta}(x) - \theta^2\,\bar{\theta}^\alpha\bar{\lambda}_\alpha(x) - \bar{\theta}^2\,\theta^\alpha\lambda_\alpha(x) + \theta^2\,\bar{\theta}^2\,D(x) \tag{B.15}$$

The vector superfields $(V,\hat{V})$ are in the adjoint representation of the two gauge groups $U(M)\times U(N)$, that is $V = V_A T^A$ and $\hat{V} = \hat{V}_A\hat{T}^A$, where T^A are the $U(M)$ generators and $\hat{T}^A$ are the $U(N)$ ones. The $U(M)$ generators are defined as $T^A = (T^0, T^a)$, where $T^0 = \frac{1}{\sqrt{N}}$ and T^a (with $a = 1,\ldots,M^2-1$) are a set of $M\times M$ hermitian matrices. The generators are normalized as $\mathrm{Tr}(T^AT^B) = \delta^{AB}$. The same conventions hold for the $U(N)$ generators.

For any value of the couplings, the action (1.28) is invariant under the following gauge transformations:

$$e^V \to e^{i\bar{\Lambda}_1}e^V e^{-i\Lambda_1} \qquad\qquad e^{\hat{V}} \to e^{i\bar{\Lambda}_2}e^{\hat{V}}e^{-i\Lambda_2} \tag{B.16}$$

$$A^i \to e^{i\Lambda_1}A^i e^{-i\Lambda_2} \qquad\qquad B_i \to e^{i\Lambda_2}B_i e^{-i\Lambda_1} \tag{B.17}$$

where Λ_1,Λ_2 are two chiral superfields parametrizing $U(M)$ and $U(N)$ gauge transformations, respectively. Antichiral superfields transform according to the conjugate of (B.17). The action is also invariant under the $U(1)_R$ R-symmetry group under which the A^i and B_i fields have $\frac{1}{2}$-charge.

B.2 Superconformal algebra

We report here the commutation and the anticommutation relations satified by the generators of the supersymmetry algebra and of the superconformal algebra in three spacetime dimensions with a generic $\mathcal{N}$. We follow the notations of [10].
Supersymmetry algebra is introduced by grading the Poincarè algebra in a non-trivial way by the use of anticommutators. This grading involves the enlargement of the group by the introduction of $I = 1,...,\mathcal{N}$ spinor supersymmetry generators Q^I_α in addition to the generators of the Poincarè group $\{P_{\alpha\beta}, M_{\gamma\delta}\}$. The supersymmetry algebra is:

$$[P_{\alpha\beta}, P_{\gamma\delta}] = 0 \tag{B.18a}$$
$$\{Q^I_\alpha, Q^J_\beta\} = 2\,P_{\alpha\beta}\,\delta^{IJ} \tag{B.18b}$$
$$[Q^I_\alpha, P_{\beta\gamma}] = 0 \tag{B.18c}$$

The set of transformations which mix the supercharges while leaving the (anti)commutation relations invariant forms a group, called R-symmetry group, which in three dimensions is $SO(N)$. The $\mathcal{N} = 1$ case is called simple supersymmetry, whereas the $\mathcal{N} > 1$

case is called extended supersymmetry.

For $\mathcal{N}=2$, the R symmetry group is $SO(2)$, which is isomorphic to $U(1)$. Notice that the latter is the R-symmetry group of the $\mathcal{N}=1$ four-dimensional algebra, described in Appendix A.2. To make the analogy between the $\mathcal{N}=2$ formalism in three dimensions and the $\mathcal{N}=1$ in four dimensions, we ca redifine supercharges in the following way:

$$Q_\alpha = \frac{1}{2}(Q_\alpha^{(1)} + iQ_\alpha^{(2)}), \qquad \bar{Q}_\alpha = \frac{1}{2}(Q_\alpha^{(1)} - iQ_\alpha^{(2)}) \tag{B.19}$$

In such way the new generators Q_α and $\bar{Q}_\alpha$ satisfy the analogue relations, presented in A.2, satisfied by the generators of $\mathcal{N}=1$ superspace in four dimensions:

$$\{Q_\alpha, \bar{Q}_\beta\} = P_{\alpha\beta} \qquad \{Q_\alpha, Q_\beta\} = 0 \tag{B.20}$$

apart from the nature of the vector representation, as was mentioned before.

When also the conformal symmetry is present, the supersymmetry algebra becomes the superconformal algebra, which can be obtained from the supersymmetry one adding the generators of the conformal group $SO(4,1)$: the dilation generator Δ and the special conformal boost transformations $K_{\alpha\beta}$.

B.3 Non-planar integral

We compute here the following non-planar integral:

$$\mathcal{D}_{np}(s) = -\int \frac{d^dk}{(2\pi)^d}\frac{d^dl}{(2\pi)^d}\frac{\mathrm{Tr}((k+l)\,k\,l\,(k+l)\,p_4\,p_3)}{k^2(k+l-p_3)^2(k+p_4)^2(l-p_3)^2\,(k+l+p_4)^2\,l^2} \tag{B.21}$$

which emerges in (4.24) as non-planar contribution to the four-point ABJM amplitude presented in section 4.2. We have dropped the $(\mu^2)^{2\epsilon}$ factor for convenience; in the calculation we will always make use of the on-shell conditions $p_i^2 = 0$.
We begin by making Feynman combining of $1/l^2$ and $1/(l-p_3)^2$ propagators:

$$-\int \frac{d^dk}{(2\pi)^d}\frac{d^dl}{(2\pi)^d}\frac{\mathrm{Tr}(p_4p_3(k+l)kl(k+l))}{k^2(k+p_4)^2(k+l+p_4)^2(k+l-p_3)^2}\int_0^1 d\alpha_2\frac{1}{[(l-\alpha_2p_3)^2]^2} \tag{B.22}$$

Performing the change of variables $l \to r-k$ and elaborating the numerator with simple algebra we can write the integrand as the sum of two terms:

$$\int \frac{d^dk}{(2\pi)^d}\frac{d^dr}{(2\pi)^d}\frac{r^2\left[\mathrm{Tr}(p_4p_3rk) - k^2s\right]}{k^2(k+p_4)^2(r+p_4)^2(r-p_3)^2}\int_0^1 d\alpha_2\frac{1}{[(r-k-\alpha_2p_3)^2]^2} \tag{B.23}$$

We are going to analyze the two pieces separately.

B.3.1 Integral 1)

In the first term we first concentrate on the k-integration and Feynman parametrize the $1/k^2$ and $1/(k+p_4)^2$ propagators. Performing a harmless shift $k \to k - \alpha_1 p_4$ we end up with:

$$\int \frac{d^d r}{(2\pi)^d} \frac{1}{(r+p_4)^2 (r-p_3)^2} \int_0^1 d\alpha_1 \int_0^1 d\alpha_2 \int \frac{d^d k}{(2\pi)^d} \frac{r^2 \,\mathrm{Tr}(p_4 p_3 r k)}{(k^2)^2 [(k - r - \alpha_1 p_4 + \alpha_2 p_3)^2]^2} \tag{B.24}$$

where the k-integration can be immediately performed, being a vector bubble integral, leading to:

$$\frac{1}{2} \int \frac{d^d r}{(2\pi)^d} \int_0^1 d\alpha_1 \int_0^1 d\alpha_2 \frac{(r^2)^2\, s - \alpha_2\, s\, r^2\, 2\, p_3 \cdot r}{(P^2)^{4-d/2} (r+p_4)^2 (r-p_3)^2} G[2,2] \tag{B.25}$$

Here we have defined:

$$P^2 = (\alpha_1 p_4 - \alpha_2 p_3 + r)^2 \tag{B.26}$$

and:

$$G[a,b] = \frac{\Gamma(a+b-d/2)\Gamma(d/2-a)\Gamma(d/2-b)}{(4\pi)^{d/2}\Gamma(a)\Gamma(b)\Gamma(d-a-b)} \tag{B.27}$$

Completing the squares in the numerator of (B.25) we obtain the sum of two scalar integrals:

$$\frac{1}{2} \int \frac{d^d r}{(2\pi)^d} \int_0^1 d\alpha_1 \int_0^1 d\alpha_2 \frac{\bar{\alpha}_2 (r^2)^2\, s + \alpha_2\, s\, r^2\, (r-p_3)^2}{(P^2)^{4-d/2} (r+p_4)^2 (r-p_3)^2} G[2,2] \tag{B.28}$$

where we have defined $\bar{\alpha} = 1 - \alpha$.

The second integral is very easy to compute. Setting $d = 3 - 2\epsilon$ and expanding the result in powers of the dimensional regulator, we obtain:

$$\frac{1}{2} G[2,2] \int \frac{d^d r}{(2\pi)^d} \int_0^1 d\alpha_1 \int_0^1 d\alpha_2 \frac{\alpha_2\, s\, r^2}{(P^2)^{4-d/2}(r+p_4)^2} = \frac{1}{64\pi^2} + \mathcal{O}(\epsilon) \tag{B.29}$$

The first integral requires a little bit more of effort. Using Mellin-Barnes representation allows to easily evaluate the α_i integrals. After a shift $(r^2)^2 \to (r^2)^{2-\delta}$ we obtain:

$$\begin{aligned}
&\frac{2^{4\epsilon}\pi^{2\epsilon}}{128\pi^3} \int \frac{du dv}{(2\pi i)^2} (-1)^v s^{-\delta-2\epsilon} \Gamma(-u)\Gamma(-v)\Gamma(-w)\Gamma(w+1)\Gamma(-u-w-\delta-2\epsilon) \\
&\Gamma\left(-\epsilon - \frac{1}{2}\right)^2 \Gamma(u+w+1)\Gamma(v+\delta-2)\Gamma(v+w+1)\Gamma\left(-u-\delta-\epsilon+\frac{3}{2}\right) \\
&\frac{\Gamma(-v-w-\delta-2\epsilon)\Gamma(-u-v-w-\delta-2\epsilon+1)\Gamma(u+v+w+\delta+2\epsilon+1)}{\Gamma(\delta-2)\Gamma(-2\epsilon-1)\Gamma\left(-\delta-3\epsilon+\frac{1}{2}\right)\Gamma(-u-\delta-2\epsilon+1)\Gamma(-u-\delta-2\epsilon+2)}
\end{aligned} \tag{B.30}$$

Expanding in δ and ϵ up to order zero terms one gets two remaining one-fold integrals:

$$\int \frac{du}{2\pi i} \Gamma(3/2-u)\Gamma(u) \left(\Gamma(-1+u)^{**}\Gamma(1-u) - 2\Gamma(u)^{*}\Gamma(-u)\right) \tag{B.31}$$

where asterisks denote how many of the first right (left) poles of the Γ functions have to be considered left (right), according to the notation of [74]. Such Barnes integrals can be solved by lemmas (D.12) and (D.37) of [74]. Summing the contributions gives:

$$\frac{1}{64\pi^2}\left[\frac{(16\pi)^{2\epsilon}s^{-2\epsilon}e^{2\epsilon(1-\gamma_E)}}{(2\epsilon)^2}-\frac{\pi^2}{24}-\frac{3}{2}-4\log 2\,(1+\log 2)\right] \tag{B.32}$$

B.3.2 Integral 2)

We now consider the second piece in eq. (B.23) with the shift $k \to k - p_4$:

$$\int \frac{d^d r}{(2\pi)^d}\,\frac{-r^2\,s}{(r+p_4)^2(r-p_3)^2}\int_0^1 d\alpha_2 \int \frac{d^d k}{(2\pi)^d}\,\frac{1}{k^2[(k-r-p_4+\alpha_2 p_3)^2]^2} \tag{B.33}$$

and perform the bubble k-integral:

$$\int_0^1 d\alpha_2\, G[1,2]\int \frac{d^d r}{(2\pi)^d}\,\frac{-r^2\,s}{(r+p_4)^2(r-p_3)^2[(r+p_4-\alpha_2 p_3)^2]^{3/2+\epsilon}} \tag{B.34}$$

Shifting $r^2 \to (r^2)^{1-\delta}$ and using Mellin-Barnes representation, for $d = 3-2\epsilon$ we have:

$$\begin{aligned}
&-\frac{s^{-\delta-2\epsilon}}{(4\pi)^{3-2\epsilon}}\,\frac{\Gamma\left(-\epsilon-\frac{1}{2}\right)\Gamma\left(\frac{1}{2}-\epsilon\right)}{\Gamma(\delta-1)\Gamma(-2\epsilon)\Gamma\left(-\delta-3\epsilon+\frac{1}{2}\right)}\int_{-i\infty}^{+i\infty}\frac{du\,dv}{(2\pi i)^2}\,(-1)^v\\
&\Gamma(-u)\Gamma(-v)\Gamma(u+1)\Gamma(v+1)\Gamma(v+\delta-1)\Gamma(-v-\delta-2\epsilon)\\
&\Gamma(-u-v-\delta-2\epsilon)\Gamma\left(-u-\delta-\epsilon+\frac{1}{2}\right)\frac{\Gamma(u+v+\delta+2\epsilon+1)}{\Gamma(-u-\delta-2\epsilon+1)}
\end{aligned} \tag{B.35}$$

Now, selecting poles that give an order δ^0 result leads to a one-fold Mellin-Barnes integral, which can be expanded in ϵ. The one-fold integral vanishes identically, leaving:

$$-\frac{1}{64\pi^2}\left(\frac{s^{-2\epsilon}e^{-2\epsilon\gamma_E}(4\pi)^{2\epsilon}}{2\epsilon}-1-2\log 2\right) \tag{B.36}$$

B.3.3 Sum

Summing the two contributions (B.32) and (B.36) it is interesting to observe that all terms of lower transcendentality cancel, leaving:

$$\mathcal{D}_{np}(s)=\frac{e^{-2\epsilon\gamma_E}(16\pi)^{2\epsilon}s^{-2\epsilon}}{64\pi^2}\left(\frac{1}{(2\epsilon)^2}-\frac{\pi^2}{24}-4\log^2 2\right) \tag{B.37}$$

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