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Polynomial Algebras and Smooth Functions in Banach Spaces

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Introduction

According to the fundamental Stone-Weierstrass theorem, if X is a finite dimensional real Banach space, then every continuous function on the unit ball B_X can be uniformly approximated by polynomials.
Before venturing out into infinite-dimensional Banach spaces, let us touch upon what polynomials on such spaces are like.

Definition. Let X, Y be Banach spaces,[1] $n \in \mathbb{N}$ and let $\mathcal{L}\left({}^{n}X; Y\right)$ be the space of n-linear Y-valued mappings on X.

- A mapping $P\colon X \longrightarrow Y$ is called an **n-homogeneous polynomial** if there exists $M \in \mathcal{L}\left({}^{n}X; Y\right)$ such that $P(x) = M(x, \dots, x)$ for all $x \in X$. For convenience, we also define 0-homogeneous polynomials as constant mappings from X to Y. We denote by $\mathcal{P}\left({}^{n}X; Y\right)$, $n \in \mathbb{N}_0$, the space of all n-homogeneous polynomials from X into Y. When the target space is the scalar field, we use a shortened notation $\mathcal{P}\left({}^{n}X\right) = \mathcal{P}\left({}^{n}X; Y\right)$.

- A mapping $P\colon X \longrightarrow Y$ is called a **polynomial of degree at most** n if there exist $P_k \in \mathcal{P}\left({}^{k}X; Y\right)$, $k = 0, \dots, n$, such that $P = \sum_{k=0}^{n} P_k$. If $P_n \neq 0$ we say that P has degree n. We denote by $\mathcal{P}^n(X; Y)$ the space of all polynomials of degree at most n.

- We denote by $\mathcal{P}(X; Y) = \bigcup_{n=0}^{\infty} \mathcal{P}^n(X; Y)$ the space of all polynomials.

- We say that $P \in \mathcal{P}\left({}^{n}X; Y\right)$ (resp. $\mathcal{P}^n(X; Y)$, $\mathcal{P}(X; Y)$) is **bounded** whenever $\|P\| = \sup_{x \in B_X} \|P(x)\| < +\infty$. We denote by $\left(\mathcal{P}\left({}^{n}X; Y\right), \|\cdot\|\right)$

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[1] Some among the following definitions hold for vector spaces. For the sake of conciseness, let us consider Banach spaces in this introduction, as the main results we hint at concern Banach spaces.

(resp. $(\mathcal{P}^n(X;Y), \|\cdot\|)$, $(\mathcal{P}(X;Y), \|\cdot\|)$) the normed linear space of all continuous[2] n-homogeneous polynomials (resp. all continuous polynomials of degree at most n, all continuous polynomials).

- We denote by $\mathcal{A}_n(X)$ the algebra generated by $\mathcal{P}^n(X)$.

For infinite dimensional Banach spaces the statement of the Stone-Weierstrass Theorem is false, even if we replace continuous functions by the uniformly continuous ones (which is a natural condition that coincides with continuity in the finite dimensional setting): in fact, on every infinite-dimensional Banach space X there exists a uniformly continuous real function not approximable by continuous polynomials (see [53]).
The natural problem of the proper generalization of the result for infinite dimensional spaces was posed by Shilov [59] (in the case of a Hilbert space). Aron [2] (see also Aron and Prolla [8]) observed that the uniform closure on B_X of the space of all polynomials of the finite type, denoted by $\mathcal{P}_f(X)$, which consists of all polynomials admitting a formula $P(x) = \sum_{j=1}^{n} \langle \phi_j, x\rangle^{n_j}, \phi_j \in X^*, n_j \in \mathbb{N}$, is precisely the space of all functions which are weakly uniformly continuous on B_X (Theorem 3.1.1):

Theorem ([2], [8])**.** *Let X, Y be Banach spaces. Then* $\overline{\mathcal{P}_f(X;Y)} = \mathcal{C}_{wu}(B_X;Y)$.

Since there exist infinite dimensional Banach spaces such that all bounded polynomials are weakly uniformly continuous on B_X (e.g. c_0 or more generally all Banach spaces not containing a copy of ℓ_1 and such that all bounded polynomials are weakly sequentially continuous on B_X), this result gives a very satisfactory solution to the problem.
Unfortunately, most Banach spaces, including $L_p, p \in [1, \infty)$, do not have the special property used in [8]. In this case, no characterization of the uniform limits of polynomials is known.
But the problem has a more subtle formulation as well. Let us consider the algebras $\mathcal{A}_n(X)$ consisting of all polynomials which can be generated by finitely many algebraic operations of addition and multiplication, starting from polynomials on X of degree not exceeding $n \in \mathbb{N}$. Of course, such polynomials can have arbitrarily high degree. The first mentioned result can

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[2] Boundedness and continuity are equivalent.

now be formulated as stating that $\overline{\mathcal{A}_1(X)}$ consists precisely of all functions which are weakly uniformly continuous on B_X. It is clear that, if n is the lowest degree such that there exists a polynomial P in $\mathcal{P}({}^nX)$ which is not weakly uniformly continuous, then

$$\overline{\mathcal{A}_1(X)} = \overline{\mathcal{A}_2(X)} = \cdots = \overline{\mathcal{A}_{n-1}(X)} \subsetneq \overline{\mathcal{A}_n(X)}.$$

The problem of what happens from n on has been studied in several papers, notably [53], [41] and [29]. The natural conjecture appears to be that once the chain of equalities has been broken, it is going to be broken at each subsequent step.

The proof of this latter statement given in [41], for all classical Banach spaces, based on the theory of algebraic bases, is unfortunately not entirely correct, as was pointed out by our colleague Michal Johanis. It is not clear to us if the theory of algebraic bases developed therein can be salvaged. Fortunately, the main statement of this theory, Lemma 1.5.4, can be proved using another approach. The complete proof, which will appear in [22], can be found in Chapter 2. Most of the results in this area which used [41] are therefore safe.

The aforementioned statement coincides with the following

Lemma. *For every $n \in \mathbb{N}$, there exists an $\varepsilon > 0$ such that, for every $m \geqslant M(n)$,*

$$\sup_{\sum\limits_{i=1}^{m} |x_i| \leqslant 1} |p(x_1, \dots, x_m) - s_{n+1}(x_1, \dots, x_m)| \geqslant \varepsilon,$$

for every p from the algebra $S_n(\mathbb{R}^m)$ generated by subsymmetric polynomials of degree at most n.

The above quantitative lemma implies the following

Theorem. *Let X be an infinite dimensional Banach space, and $P \in \mathcal{P}({}^nX)$ be a polynomial with the following property: for every $N \in \mathbb{N}$ and $\varepsilon > 0$, there exists a normalized finite basic sequence $\{e_j\}_{j=1}^N$ such that*

$$\sup_{\sum_{j=1}^N |a_j| \leqslant 1} \left| P\left(\sum_{j=1}^N a_j e_j\right) - \sum_{j=1}^N a_j^n \right| \leqslant \varepsilon.$$

Then $P \notin \overline{\mathcal{A}_{n-1}(X)}$.

This fundamental criterion, in combination with some new results on the asymptotic behaviour of polynomials on infinite dimensional spaces, is one of the keys to prove our main result (see [21]), dealt with in Chapter 3 (Theorem 3.2.1).

Theorem. *Let* X *be a Banach space and* m *be the minimal integer such that there is a non-compact* $P \in \mathcal{P}({}^m X; \ell_1)$*. Then* $n \geqslant m$ *implies* $\mathcal{P}({}^n X) \not\subset \overline{\mathcal{A}_{n-1}(X)}$*.*

Theorem 3.2.1 implies, together with the positive results of [2] and [8] (see the first theorem mentioned above), plus the corollary below (see [6]), all previously known results in this area (all confirming the above conjecture) as special cases.

Corollary ([6])**.** *Let* X*,* Y *be Banach spaces and suppose that* X *does not contain a subspace isomorphic to* ℓ_1*. Then* $\mathcal{P}_{wu}({}^n X; Y) = \mathcal{P}_{wsc}({}^n X; Y)$*.*

For example, in the following cases it can be easily inferred that the algebra chain is broken, from a certain point (which we can determine) onward, at each subsequent step: if X is a Banach space admitting a non-compact linear operator $T \in \mathcal{L}(X; \ell_p)$, $p \in [1, \infty)$ (see [41]); if $X = L_p([0,1])$, $1 \leqslant p \leqslant \infty$, or $X = \ell_\infty$ or $X = C(K)$, where K is a non-scattered compact; if $\ell_1 \hookrightarrow X$ ([41]); if $X = \ell_p$, $1 \leqslant p < \infty$; if X^* has type q; if X has an unconditional FDD, $\ell_1 \not\hookrightarrow X$ and there exists a $P \in \mathcal{P}({}^n X)$ which is not weakly sequentially continuous...

In Chapter 4 we also give solutions to three other problems posed in the literature, which are concerning smooth functions rather than polynomials, but which belong to the same field of study of smooth mappings on a Banach space.

The first result is a construction of a non-equivalent C^k-smooth norm on every Banach space admitting a C^k-smooth norm, answering a problem posed in several places in the literature, e.g. in [12].

Theorem. *Let* X *be an infinite dimensional Banach space admitting a* C^k*-smooth norm,* $k \geqslant 2$*. Then* X *admits a decomposition* $X = Y \oplus Z$*, where*

Y is infinite dimensional and separable. In particular, X admits a non-complete C^k-smooth renorming.

We solve a question in [11] by proving that a real Banach space admitting a separating real analytic function whose holomorphic extension is Lipschitz in some strip around X admits a separating polynomial.

Theorem. *Let X be a real Banach space which admits a real analytic separating function whose complex extension exists and is Lipschitz on some strip around X, i.e. on $X + 2rB_{X^{\mathbb{C}}} \subset X^{\mathbb{C}}$, for some $r > 0$. Then X is superreflexive and admits a separating polynomial.*

Eventually, we solve a problem posed by Benyamini and Lindenstrauss in [14], concerning the extensions of uniformly differentiable functions from the unit ball into a larger set, preserving the values in some neighbourhood of the origin. More precisely, we construct an example of a uniformly differentiable real-valued function f on the unit ball of a certain Banach space X, such that there exists no uniformly differentiable function g on λB_X, for any $\lambda > 1$, which coincides with f in some neighbourhood of the origin. To do so, we construct suitable renormings of c_0, based on the theory of $\mathcal{W}$-spaces.

Example. There exist countably many norms $\{\|\cdot\|_m\}_{m=2}^{\infty}$ on c_0 such that, if we set $X = \oplus_{\ell_2} \sum_{m=2}^{\infty}(c_0, \|\cdot\|_m)$, then there exists a uniformly differentiable function $f : B_X \to \mathbb{R}$ which cannot be extended to a uniformly differentiable function on any λB_X, $\lambda > 1$, preserving its original values in some neighbourhood of 0.

Chapter 1

Background and preliminary results

In this chapter we collect some background results concerning polynomials in Banach spaces, which will be used in the sequel. We refer to [36] for the standard notation concerning Banach spaces and to [30] for the standard notation concerning polynomials. For the sake of conciseness we will omit the proofs that are not directly involved in the papers [21] and [22], which the reader can find (along with further references) in the comprehensive monograph on smoothness in Banach spaces [44].

We also employ numerous classic tools and results in Banach Space Theory (such as Gâteaux and Fréchet differentiability, Tsirelson's space, James-Gurarii theorem, Bessaga-Pełczyński theorem, Dunford-Pettis property, Rosenthal's ℓ_1 theorem, biorthogonal systems, lifting properties, Asplund and weak-Asplund spaces, projectional resolutions of the identity, Weakly Lindelöf Determined (WLD) spaces, finite-dimensional decompositions (FDD), type and cotype, superreflexivity, Lipschitz mappings, holomorphy...): should the reader need to delve into these topics, we suggest consulting [35], [46], [36] and [44], where they are widely treated.

1.1 Polynomials

By $\mathbb{N}_0$ we denote the set $\mathbb{N} \cup \{0\}$, i.e. the non-negative integers. The canonical basis of $\mathbb{R}^N$ will be denoted by $\{e_j\}_{j=1}^N$.

Definition 1.1.1. Let $X_1, X_2, \ldots, X_n, Y$ be vector spaces.

- We say that a mapping $M\colon X_1 \times \cdots X_n \longrightarrow Y$ is **n-linear** if it is linear in each coordinate, that is $x \mapsto M(x_1, \ldots, x_{k-1}, x, x_{k+1}, \ldots, x_n)$ is a linear mapping from X_k into Y for each $x_1 \in X_1, \ldots x_n \in X_n$ and each $k \in \{1, \ldots, n\}$.
- By $L\left(X_1, \ldots X_n; Y\right)$ we denote the vector space of all n-linear mappings from $X_1 \times \cdots \times X_n$ to Y. Whenever $X_k = X$, $1 \leqslant k \leqslant n$, we use the short notation $L\left({}^nX; Y\right)$.
- A map is called **multilinear** if it is n-linear for some $n \in \mathbb{N}$. A 2-linear mapping will also be called **bilinear**.
- We say that $M \in L\left({}^nX; Y\right)$ is **symmetric** if $M(x_1, \ldots, x_n) = M\left(x_{\pi(1)}, \ldots, x_{\pi(n)}\right)$ for every permutation π of $\{1, \ldots, n\}$ and every $x_1, \ldots, x_n \in X$.
- By $L^s\left({}^nX; Y\right)$ we denote the vector space of all n-linear symmetric mappings from X^n to Y.

Definition 1.1.2. Let $X_1, X_2, \ldots, X_n, Y$ be normed linear spaces.

- We say that $M \in L\left(X_1, \ldots, X_n; Y\right)$ is a **bounded n-linear mapping** if
$$\|M\| := \sup_{x_1 \in B_{X_1}, \ldots, x_n \in B_{X_n}} \|M(x_1, \ldots, x_n)\| < +\infty.^{1}$$
- By $\left(\mathcal{L}\left(X_1, \ldots, X_n; Y\right); \|\cdot\|\right)$, resp. $\left(\mathcal{L}\left({}^nX; Y\right); \|\cdot\|\right)$, resp. $\left(\mathcal{L}^s\left({}^nX; Y\right); \|\cdot\|\right)$, we denote the normed linear space of all respective n-linear bounded mappings. For bounded n-linear forms, we use the shortened notation $\mathcal{L}\left({}^nX\right) = \mathcal{L}\left({}^nX; \mathbb{K}\right)$.

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[1] It is straightforward that $\|\cdot\|$ defines a norm on the subspace of $L\left(X_1, \ldots, X_n; Y\right)$ consisting of bounded multilinear mappings.

Remark 1.1.3. Let $M \in \mathcal{L}(X_1, \dots, X_n; Y)$. Then by homogeneity we have

$$\|M(x_1, \dots, x_n)\| \leqslant \|M\| \, \|x_1\| \cdots \|x_n\| \quad \text{for } x_j \in X, j = 1, \dots, n.$$

It turns out that for multilinear mappings an analogous result to that of continuity of linear functionals holds, i.e. polynomials are continuous mappings whenever they have at least one point of continuity.

Proposition 1.1.4. *Let $X_1, \dots, X_n, Y$ be normed linear spaces and $M \in \mathcal{L}(X_1, \dots, X_n; Y)$. The following are equivalent:*

- *(i) M is bounded;*
- *(ii) M is Lipschitz on bounded sets;*
- *(iii) M is continuous;*
- *(iv) M is bounded on a neighbourhood of some point.*

A particular property comes in handy: homogeneous polynomials are in a canonical one-to-one correspondence with the symmetric multilinear forms via the Polarization formula, as the following proposition states.

Proposition 1.1.5 (Polarization formula)**.** *Let X, Y be vector spaces and $M \in L(^nX; Y)$. Then*

$$M^s(x_1, \dots, x_n) = \frac{1}{2^n n!} \sum_{\varepsilon_j = \pm 1} \varepsilon_1 \cdots \varepsilon_n M\left(a + \sum_{j=1}^{n} \varepsilon_j x_j, \dots, a + \sum_{j=1}^{n} \varepsilon_j x_j\right)$$

for every $a, x_1, \dots, x_n \in X$. In particular, if M is symmetric, then it is uniquely determined by its values $M(x, \dots, x)$, $x \in X$, along the diagonal.

Definition 1.1.6. Let X, Y be vector spaces and $n \in \mathbb{N}$.

- A mapping $P \colon X \longrightarrow Y$ is said to be an n-**homogeneous polynomial** if there exists an n-linear mapping $M \in L(^nX, Y)$ such that $P(x) = M(x, \dots, x)$. We use the notation $P = \widehat{M}$. For the sake of convenience, we also define 0-homogeneous polynomials as constant mappings from X to Y.
- We denote by $P(^nX; Y)$, $n \in \mathbb{N}_0$, the vector space of all n-homogeneous polynomials from X into Y.

Suppose X, Y are normed linear spaces, $n \in \mathbb{N}_0$.

- We say that $P \in P\left({}^n X; Y\right)$ is a **bounded polynomial** if

$$\|P\| = \sup_{x \in B_X} \|P(x)\| < +\infty.$$

- We denote by $\left(\mathcal{P}\left({}^n X; Y\right), \|\cdot\|\right)$ the normed linear space of all n-homogeneous bounded polynomials from X into Y. When the target space is the scalar field, we use the shortened notation $\mathcal{P}\left({}^n X\right) = \mathcal{P}\left({}^n X; \mathbb{K}\right)$.

For a given n-homogeneous polynomial P, the n-linear mapping M that gives rise to P is not uniquely determined. In particular, the symmetrized n-linear mapping leads to the same polynomial: for every $M \in L\left({}^n X; Y\right)$, we have $\widehat{M} = \widehat{M^s}$. However, the following fundamental result holds.

Proposition 1.1.7 (Polarization formula, [17], [51])**.** *Let X, Y be vector spaces and $n \in \mathbb{N}$. For every $P \in P\left({}^n X; Y\right)$, there exists a unique symmetric n-linear mapping $\check{P} \in L^s\left({}^n X; Y\right)$ such that $P(x) = \check{P}(x, \ldots, x)$. It satisfies the formula*

$$\check{P}(x_1, \ldots, x_n) = \frac{1}{2^n n!} \sum_{\varepsilon_j = \pm 1} \varepsilon_1 \cdots \varepsilon_n P\left(a + \sum_{j=1}^n \varepsilon_j x_j\right),$$

where $a \in X$ can be chosen arbitrarily. Moreover, if X, Y are normed linear spaces and P is bounded, then $\check{P}$ is also bounded and we have

$$\|P\| \leqslant \left\|\check{P}\right\| \leqslant \frac{n^n}{n!} \|P\|.$$

On the other hand, for every $m > n$ and $a, x_1, \ldots, x_m \in X$, the following holds:

$$\sum_{\varepsilon_j = \pm 1} \varepsilon_1 \cdots \varepsilon_m P\left(a + \sum_{j=1}^m \varepsilon_j x_j\right) = 0.$$

Most of the time we will be concerned with the restrictions of the polynomials whose domain is a Banach space X to a suitable subspace $Y \hookrightarrow X$ with a Schauder basis $\{e_j\}_{j=1}^\infty$ (resp. a finite-dimensional Banach space). In this case it is possible, and very useful, to rely on the concrete representation of (the restriction of) the polynomial using the monomial expansion in terms of the vector coordinates.

Definition 1.1.8. Let $n \in \mathbb{N}$. For a multi-index $\alpha \in \mathbb{N}_0^n$ we denote its **order** by $|\alpha| = \sum_{j=1}^n \alpha_j$. We denote the set of multi-indices of order $d \in \mathbb{N}_0$ by

$$\mathcal{I}(n,d) = \big\{\alpha \in \{0,\dots,d\}^n : \ |\alpha| = d\big\}.$$

In order to treat infinite dimensional Banach spaces, we extend the definition also to the case when $n = \infty$, setting

$$\mathcal{I}(\infty,d) = \left\{\alpha \in \{0,\dots,d\}^{\mathbb{N}} : \ |\alpha| = \sum_{j=1}^{\infty} \alpha_j = d\right\}.$$

For $n, d \in \mathbb{N}$ we denote $\mathcal{I}^+(n,d) = \{\alpha \in \mathcal{I}(n,d) : \alpha_j > 0, j = 1,\dots,n\}$ and $\mathcal{I}^+(d) = \bigcup_{n=1}^d \mathcal{I}^+(n,d)$.

For $n \in \mathbb{N}$ we denote

$$\mathcal{N}(n) = \{\rho \in \mathbb{N}^n : \rho_1 < \rho_2 < \cdots < \rho_n\}.$$

For $n \in \mathbb{N}$ we have $|\mathcal{I}(n,d)| = \binom{n+d-1}{n-1}$.[2]

A given $(k_j)_{j=1}^d \in \{1,\dots,n\}^d$ determines a unique $\alpha \in \mathcal{I}(n,d)$ by the relation

$$\alpha = (|\{j : k_j = 1\}|, |\{j : k_j = 2\}|, \dots, |\{j : k_j = n\}|)\,. \tag{1.1}$$

Conversely, a given $\alpha \in \mathcal{I}(n,d)$ determines a unique $k(\alpha) = (k_1(\alpha),\dots,k_d(\alpha))$, $k_1(\alpha) \leqslant \cdots \leqslant k_d(\alpha)$, such that (1.1) holds.

Given $x = (x_1,\dots,x_n) \in \mathbb{K}^n$ and $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathcal{I}(n,d)$ we use the standard multi-index notation

$$x^\alpha = \prod_{l=1}^n x_l^{\alpha_l} = \prod_{j=1}^d x_{k_j(\alpha)}.$$

The case $n = \infty$ is similar and corresponds to multi-indices whose domain is $\mathbb{N}$. More precisely, for a fixed Schauder basis $\{e_j\}_{j=1}^\infty$ of X, with a dual basis $\{x_j^*\}_{j=1}^\infty \subset X^*$, $\alpha \in \mathcal{I}(\infty,d)$ and $x = \sum_{j=1}^\infty x_j e_j$,

$$x^\alpha = \prod_{\alpha_l \neq 0} x_l^{\alpha_l} = \prod_{\alpha_l \neq 0} \langle x_l^*, x\rangle^{\alpha_l}.$$

Note that $x \mapsto x^\alpha \in \mathcal{P}(^d\mathbb{K}^n)$ for any $\alpha \in \mathcal{I}(n,d)$.

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[2] It represents the number of distributions of d identical balls into n distinct boxes.

Given $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathcal{I}(n, d)$, we denote $\alpha! = \alpha_1! \times \cdots \times \alpha_n!$. We also use the corresponding multinomial coefficient by

$$\binom{d}{\alpha} = \binom{d}{\alpha_1, \dots, \alpha_n} = \frac{d!}{\alpha_1! \cdots \alpha_n!} = \frac{d!}{\alpha!}.$$

We also put a partial ordering on multiindices defined as follows. If $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathcal{I}(n, d)$, $\beta = (\beta_1, \dots, \beta_n) \in \mathcal{I}(n, p)$, $p \leqslant d$, and $\alpha_j \geqslant \beta_j$ holds for all $j \in \{1, \dots, n\}$ then we say that $\alpha \geqslant \beta$, and we also denote $\alpha - \beta = (\alpha_1 - \beta_1, \dots, \alpha_n - \beta_n) \in \mathcal{I}(n, d - p)$.

Proposition 1.1.9 (Multinomial formula)**.** *Let X, Y be vector spaces, $d \in \mathbb{N}, P \in P\left({}^d X; Y\right)$ and $x_1, \dots, x_n \in X$. Then*

$$P(x_1 + \cdots + x_n) = \sum_{\alpha \in \mathcal{I}(n,d)} \binom{d}{\alpha} \check{P}\left({}^{\alpha_1} x_1, \dots, {}^{\alpha_n} x_n\right).$$

The next proposition asserts that the abstract definition of homogeneous polynomials coincides on $\mathbb{K}^n$ with the classical definition that uses coordinates. Note that in this case all homogeneous polynomials are automatically bounded.

Proposition 1.1.10. *Let $n, d \in \mathbb{N}$ and Y be a vector space over $\mathbb{K}$. A mapping $P \colon \mathbb{K}^n \longrightarrow Y$ is a d-homogeneous polynomial if and only if there exist $\{y_\alpha\}_{\alpha \in \mathcal{I}(n,d)} \subset Y$ such that $P(x) = \sum_{\alpha \in \mathcal{I}(n,d)} x^\alpha y_\alpha$. Moreover, each y_α is uniquely determined by*

$$y_\alpha = \binom{d}{\alpha} \check{P}\left({}^{\alpha_1} e_1, \dots, {}^{\alpha_n} e_n\right),$$

where $\{e_j\}_{j=1}^n$ is the canonical basis of $\mathbb{K}^n$.

In the special case $Y = \mathbb{K}$, this reduces to the familiar formula

$$P(x) = \sum_{\alpha \in \mathcal{I}(n,d)} a_\alpha x^\alpha,$$

where the coefficients $a_\alpha \in \mathbb{K}$.

Proposition 1.1.11. *Let X be a normed linear space with a Schauder basis $\{e_j\}_{j=1}^\infty$, Y a vector space, $d \in \mathbb{N}$ and $P \in P\left({}^d X; Y\right)$. Denote $X_0 =$*

span$\{e_j\}_{j=1}^{\infty}$. *Then there is a unique collection of vectors* $\{y_\alpha\}_{\alpha\in\mathcal{J}(n,d)} \subset Y$ *such that the formula*

$$P(x) = \sum_{\alpha\in\mathcal{J}(n,d)} x^\alpha y_\alpha \tag{1.2}$$

holds for every $x \in X_0$. *The coefficients* y_α *are given by*

$$y_\alpha = \binom{n}{\alpha}\check{P}\left({}^{\alpha_1}e_1, {}^{\alpha_2}e_2, \dots\right).$$

Conversely, any $\{y_\alpha\}_{\alpha\in\mathcal{J}(n,d)} \subset Y$ *uniquely determines a polynomial* $P \in P\left({}^{d}X_0; Y\right)$ *by formula* (1.2).

Definition 1.1.12. Let X, Y be vector spaces and $n \in \mathbb{N}_0$.

- A mapping $P\colon X \longrightarrow Y$ is called a **polynomial of degree at most** n if there are $P_k \in P\left({}^{k}X; Y\right)$, $k = 0, \dots, n$, such that $P = \sum_{k=0}^{n} P_k$. If $P_n \neq 0$, we say that P has degree n and we use the notation $\deg P = n$.[3]
- We denote by $P^n\left(X; Y\right)$ the space of all polynomials of degree at most n. We denote by $P(X;Y) = \bigcup_{n=0}^{\infty} P^n(X;Y)$ the space of all polynomials.

Suppose X, Y are normed linear spaces, $n \in \mathbb{N}_0$.

- A mapping $P\colon X \longrightarrow Y$ is called a **bounded polynomial of degree at most** n if there are $P_k \in \mathcal{P}\left({}^{k}X; Y\right)$, $k = 0, \dots, n$, such that $P = \sum_{k=0}^{n} P_k$.
- We denote by $\mathcal{P}^n\left(X; Y\right)$ the space of all bounded polynomials of degree at most n. We denote by $\mathcal{P}(X;Y) = \bigcup_{n=0}^{\infty} \mathcal{P}^n(X;Y)$ the space of all bounded polynomials.

1.2 Differentiability

In this section we briefly list some facts concerning the derivative of polynomials.

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[3] Note that $\deg P$ is well-defined, as the homogeneous summands of a polynomial are uniquely determined.

Fact 1.2.1. Let $P \in \mathcal{P}({}^dX;Y)$, $v \in X$. The directional derivative

$$\frac{\partial P}{\partial v}(x) = \lim_{\lambda \to 0} \frac{P(x+\lambda v) - P(x)}{\lambda}$$

is easily shown to be a polynomial in x which satisfies the formula

$$\frac{\partial P}{\partial v}(x) = d \cdot \check{P}(v, {}^{d-1}x) \in \mathcal{P}({}^{d-1}X;Y).$$

By induction, for a fixed $\alpha \in \mathcal{I}(\infty, p)$, $p \leqslant d$, where $\alpha_i = 0$ $(i > k)$ and $y_1, \dots, y_k \in X$, we get

$$\frac{\partial^p P}{\partial^{\alpha_1} y_1 \dots \partial^{\alpha_k} y_k}(x) = \frac{d!}{(d-p)!} \check{P}({}^{\alpha_1}y_1, \dots, {}^{\alpha_k}y_k, {}^{d-p}x) \in \mathcal{P}({}^{d-p}X;Y). \quad (1.3)$$

Fact 1.2.2. Let X be a Banach space with a Schauder basis $\{e_j\}_{j=1}^{\infty}$, Y be a Banach space, $P \in \mathcal{P}({}^dX;Y)$. There is a unique set of vectors $y_\alpha^\rho \in Y, \alpha = (\alpha_1, \dots, \alpha_k) \in \mathcal{I}(\infty, d)$, $\rho_j \in \mathbb{N}$, $1 \leqslant \rho_1 < \rho_2 < \dots < \rho_k$,

$$y_\alpha^\rho = \frac{1}{\alpha_1! \cdots \alpha_k!} \frac{\partial^d P}{\partial^{\alpha_1} e_{\rho_1} \dots \partial^{\alpha_k} e_{\rho_k}}(0), \quad (1.4)$$

such that the formula

$$P\left(\sum_{j=1}^{\infty} x_j e_j\right) = \sum_{\alpha \in \mathcal{I}(\infty,d)} \sum_{1 \leqslant \rho_1 < \dots < \rho_k} x_{\rho_1}^{\alpha_1} \cdots x_{\rho_k}^{\alpha_k} y_\alpha^\rho \quad (1.5)$$

holds for every finitely supported vector $x \in X$. In the special case $Y = \mathbb{R}$ the coefficients are just real numbers a_α^ρ.

1.3 Symmetric and sub-symmetric polynomials

This section is motivated by the following fact: the concept of sub-symmetric polynomials on $\mathbb{R}^N$ can be used to capture the essential information on the behaviour of a given general polynomial.

Definition 1.3.1. A Schauder basis $\{e_j\}_{j=1}^{\infty}$ of a Banach space X is called **symmetric** if there exists $K > 0$ such that for any bijection $\sigma\colon \mathbb{N} \to \mathbb{N}$, the formal linear operator $I_\sigma(\sum_{j=1}^{\infty} a_j e_j) = \sum_{j=1}^{\infty} a_{\sigma(j)} e_j$ is an isomorphism of X such that $\|I_\sigma\| \left\|I_\sigma^{-1}\right\| < K$.

A Schauder basis $\{e_j\}_{j=1}^{\infty}$ of a Banach space X is called **spreading invariant** if there exists $K > 0$ such that for any increasing mapping $\sigma\colon \mathbb{N} \to \mathbb{N}$, the formal linear operator $I_\sigma(\sum_{j=1}^{\infty} a_j e_j) = \sum_{j=1}^{\infty} a_j e_{\sigma(j)}$ is an isomorphism into a subspace of X such that $\|I_\sigma\| \left\|I_\sigma^{-1}\right\| < K$.
A spreading invariant and unconditional basis is called **sub-symmetric**.

We remark that a symmetric basis is automatically unconditional.
A subset U of a Banach space X with a Schauder basis $\{e_j\}_{j=1}^{\infty}$ is called symmetric (resp. spreading invariant) if for any bijection $\sigma\colon \mathbb{N} \to \mathbb{N}$ (resp. for any increasing mapping $\sigma\colon \mathbb{N} \to \mathbb{N}$), $I_\sigma(U) \subset U$.

Definition 1.3.2. Let $\{e_j\}_{j=1}^{\infty}$ be a Schauder basis of a Banach space X, $U \subset X$ be symmetric (resp. spreading invariant) and $f : U \to Y$ be a function. If

$$f\left(\sum_{j=1}^{\infty} a_j e_j\right) = f\left(\sum_{j=1}^{\infty} a_j e_{\sigma(j)}\right), \quad \sum_{j=1}^{\infty} a_j e_j \in U,$$

for any bijection $\sigma\colon \mathbb{N} \to \mathbb{N}$ (resp. for any increasing mapping $\sigma\colon \mathbb{N} \to \mathbb{N}$), then we say that f is **symmetric** (resp. **sub-symmetric**) on U.

These notions will typically be applied to functions whose domain is a Banach space with a symmetric (resp. spreading invariant) basis or a subspace of a space with a Schauder basis consisting of finitely supported vectors.
We use the same terminology also for functions acting on $X = \mathbb{R}^n$, with the fixed and linearly ordered linear basis $\{e_j\}_{j=1}^{n}$. In this case the notion of subsymmetric is reduced to the identity $f(x) = f(y)$ being valid for every pair $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ of elements of $\mathbb{R}^n$ such that the sequences formed by all non-zero coordinates of x and y coincide (e.g. $x = (2, 0, 0, 1.5, \pi, 0)$ $y = (0, 2, 1.5, 0, 0, \pi)$).

Definition 1.3.3.

- For a given $d \in \mathbb{N}$ denote

$$\mathfrak{J}(d) = \left\{ \alpha = (\alpha_1, \dots, \alpha_k) : \ k \in \mathbb{N}, \alpha_j \in \{1, \dots, d\}, \sum_{j=1}^{k} \alpha_j = d \right\}.[4]$$

---

[4] For the sake of completeness, we also set $\mathfrak{J}(0) = \{\varnothing\}$.

- Given $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathcal{J}(d)$ we let
$$P_\alpha\left(\sum_{j=1}^{\infty} x_j e_j\right) = \sum_{1 \leqslant \rho_1 < \cdots < \rho_k} x_{\rho_1}^{\alpha_1} \cdots x_{\rho_k}^{\alpha_k}, \tag{1.6}$$
for all finitely supported $\sum_{j=1}^{\infty} x_j e_j \in c_{00}$, and set $P_\varnothing = 1$. Clearly, P_α is a subsymmetric polynomial. Polynomials which satisfy (1.6) are called **standard** or **elementary**.[5]

- Further, we denote $s_d = P_{(d)}$, i.e.
$$s_d(x) = \sum_{j=1}^{\infty} x_j^d$$
for all finitely-supported vectors $x = \sum x_j e_j$. Each s_d is a symmetric polynomial and it is called a **power sum symmetric** polynomial.

Remark 1.3.4. The standard polynomials form a linear basis of the finite dimensional linear space of all d-homogeneous subsymmetric (and not necessarily bounded) polynomials on span$\{e_j\}$.

More precisely, we have the following well-known fact.

Fact 1.3.5. Let X be the linear span of a Schauder basis $\{e_j\}_{j=1}^{\infty}$ (resp. $X = \mathbb{R}^n$) and Y a vector space. If a polynomial $P \in \mathcal{P}^d(X;Y)$ is subsymmetric, then, for fixed $\alpha = (\alpha_1, \dots, \alpha_k)$, the constants y_α^ρ do not depend on the choice of $\rho = \rho_1 < \cdots < \rho_k$. In particular, the following equality holds
$$P\left(\sum_{j=1}^{\infty} x_j e_j\right) = \sum_{k=0}^{d} \sum_{\alpha \in \mathcal{J}(k)} P_\alpha\left(\sum_{j=1}^{\infty} x_j e_j\right) y_\alpha \tag{1.7}$$
for all finitely supported $\sum_{j=1}^{\infty} x_j e_j \in X$ (resp. for all $x \in \mathbb{R}^n$).[6]

We will also rely on a finite dimensional version of the above result.

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[5] This terminology applies also to the case when $X = \mathbb{R}^n$.

[6] The coefficients y_α are given by $y_\alpha = \binom{d}{\alpha}\check{P}\left({}^{\alpha_1}e_1, \dots, {}^{\alpha_n}e_n\right)$, where $\alpha \in \mathcal{I}(d)$.

1.4 Spreading models

Definition 1.4.1. Given a set X, we let $X^{(n)}$ be the set of all subsets of X of cardinality n. We say that a system of k disjoint sets $\{S_i\}_{i=1}^k$ forms a **partitioning** of $X^{(n)}$ whenever $X^{(n)} = \bigcup_{i=1}^k S_i$.

Proposition 1.4.2 (Ramsey)**.** *Let $k, n \in \mathbb{N}$. Then for every partitioning $\{S_i\}_{i=1}^k$ of $\mathbb{N}^{(n)}$ there exists $i \in \{1, \ldots, k\}$ and an infinite set $M \subset \mathbb{N}$ such that $M^{(n)} \subset S_i$.*

This result can be reformulated in the following ways:

- Let n be a natural number. Let ψ be a mapping from $\mathbb{N}^{(n)}$ to some finite set C. Then there is an infinite subset M of $\mathbb{N}$ such that ψ is constant on $M^{(n)}$.
- If a coloring (with a finite number of colors) of sets of natural numbers of a given length n is defined, then there is an infinite subset M of $\mathbb{N}$ such that all subsets of M of length n have the same color.

Proposition 1.4.3 (Ramsey)**.** *Let $k, n, m \in \mathbb{N}$. Then there exists $M = M(k, n, m)$ such that, for every partitioning $\{S_i\}_{i=1}^k$ of $\{1, \ldots, M\}^{(n)}$, there exists $i \in \{1, \ldots, k\}$ and a subset $A \subset \{1, \ldots, m\}$, $|A| = m$, such that $A^{(n)} \subset S_i$.*

We will now list some basic facts concerning the spreading model construction for a Banach space X, which leads to a Banach space with a subsymmetric basis which captures the asymptotic behaviour of infinite sequences in X.

Definition 1.4.4. Let $K \geqslant 1$. We say that a sequence $\{x_n\}_{n=1}^{\infty}$ in a normed linear space is **K-spreading** if

$$\left\| \sum_{j=1}^{k} a_j x_{m_j} \right\| \leqslant K \left\| \sum_{j=1}^{k} a_j x_{n_j} \right\|$$

whenever $k \in \mathbb{N}$, $a_1, \ldots, a_k$ are any scalars and $m_j, n_j \in \mathbb{N}$ are such that $m_1 < m_2 < \cdots < m_k$, $n_1 < n_2 < \cdots < n_k$.

Remark 1.4.5. From Rosenthal's ℓ_1-theorem it follows that any K-spreading sequence in a Banach space X is either equivalent to the canonical basis of ℓ_1 or it is weakly Cauchy: indeed, the linear operator $T\colon \operatorname{span}\{x_{n_j}\} \longrightarrow \operatorname{span}\{x_j\}$ such that $T\left(x_{n_j}\right) = x_j$ is bounded and hence $w - w$ uniformly continuous.

Proposition 1.4.6 ([13])**.** *Let $\{e_n\}$ be a K-spreading sequence in a Banach space X. Then $\{e_n\}$ is a basic sequence if and only if it is not weakly convergent to a non-zero element of X. If moreover $\{e_n\}$ is weakly null, then $\{e_n\}$ is an unconditional basic sequence.*

Remark 1.4.7.

- A symmetric basis is automatically unconditional and in fact sub-symmetric (see [60]).
- If $\{e_n\} \subset X$ is a sub-symmetric basis that is K-spreading, then the sequence $\{f_n\} \subset X^*$, biorthogonal to $\{e_n\}$, is a sub-symmetric basic sequence that is $2CK$-spreading, where C is the unconditional basis constant of $\{e_n\}$.

Definition 1.4.8. Let $\{x_n\}$ be a sequence in a Banach space X. We say that a sequence $\{e_n\}$ in a Banach space Y is a **spreading model** of the sequence $\{x_n\}$ if for every $\varepsilon > 0$ and $k \in \mathbb{N}$ there is $N \in \mathbb{N}$ such that

$$(1-\varepsilon)\left\|\sum_{j=1}^{k} a_j e_j\right\| \leqslant \left\|\sum_{j=1}^{k} a_j x_{n_j}\right\| \leqslant (1+\varepsilon)\left\|\sum_{j=1}^{k} a_j e_j\right\|$$

for all $N \leqslant n_1 < n_2 < \cdots < n_k$ and all scalars $a_1, \ldots, a_k$.
If $\varepsilon_k = \frac{1}{2^k}$, $N_k = 2^k$ we call $\{x_{n_j}\}_{j=1}^{\infty}$ a **characteristic subsequence** of $\{x_n\}_{n=1}^{\infty}$.

Theorem 1.4.9 (Brunel, Sucheston, [19])**.** *Let X be a Banach space and suppose that $\{x_n\} \subset X$ is a bounded sequence such that $\{x_n\}_{n \in \mathbb{N}}$ is not relatively compact. Then $\{x_n\}$ has a subsequence with a spreading model.*

The proof is based on a repeated use of the finite Ramsey theorem, and can be found in e.g. in [36], p. 294.

Proposition 1.4.10. *Let X be a Banach space and $\{x_n\} \subset X$ a weakly null sequence with a spreading model $\{e_n\}$. Then $\{e_n\}$ is a sub-symmetric basic sequence with the unconditional basis constant at most* 2.

The relation (1.8) below is the fundamental result of the theory of spreading models. It can be obtained from the previous results by passing to subsequences and diagonalizing.

Proposition 1.4.11 (Brunel, Sucheston, see [13])**.** *Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers decreasing to zero, $\{N(k)\}_{k=1}^{\infty}$ be an increasing sequence of natural numbers and $\{x_n\}_{n=1}^{\infty}$ be a normalised basic sequence in a Banach space X. Then there exists a subsequence $\{y_n\}_{n=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ and a Banach space $(Y, |||\cdot|||)$ with a spreading invariant basis $\{e_n\}_{n=1}^{\infty}$, such that, for all $k \in \mathbb{N}$ and all scalars a_j, $j = 1, \dots, N(k)$,*

$$(1-\varepsilon_k)\left|\left|\left|\sum_{j=1}^{N(k)} a_j e_j\right|\right|\right| \leqslant \left\|\sum_{j=1}^{N(k)} a_j y_{n_j}\right\| \leqslant (1+\varepsilon_k)\left|\left|\left|\sum_{j=1}^{N(k)} a_j e_j\right|\right|\right|, \tag{1.8}$$

whenever $k \leqslant n_1 < \dots < n_{N(k)}$.

The following additional result will be made use of later on. We prefer to omit the standard proof of the estimate (1.9) concerning sub-symmetric polynomials, which can be obtained by modifying the proof of Theorem 1.4.9, by working simultaneously with the original norm $\|\cdot\|$ and P, and keeping in mind that d-homogeneous polynomials form a closed set in the topology of uniform convergence on the unit ball.

Theorem 1.4.12. *Let X be a Banach space, $P \in \mathcal{P}(^dX)$ and let Y be the Banach space whose existence is guaranteed by Proposition 1.4.11. Then there exists a sub-symmetric polynomial $R \in \mathcal{P}(^dY)$ such that, for all $k \in \mathbb{N}$, we have*

$$R\left(\sum_{j=1}^{N(k)} a_j e_j\right) - \varepsilon_k \leqslant P\left(\sum_{j=1}^{N(k)} a_j y_{n_j}\right) \leqslant R\left(\sum_{j=1}^{N(k)} a_j e_j\right) + \varepsilon_k, \tag{1.9}$$

whenever $k \leqslant n_1 < \dots < N(k), \sum_{j=1}^{N(k)} a_j y_{n_j} \in B_X$.

It is clear that in general a basic sequence may admit many non-isomorphic spreading models. We say that Y is a spreading model of X provided Y results as a spreading model built on some normalised basic sequence in X. The Ramsey theorems and the theory of spreading models allow us to infer the following useful result.

Theorem 1.4.13. *Let $d, n \in \mathbb{N}$ and $\varepsilon > 0$. There exists $N = N(d, n, \varepsilon)$ such that, for every $P \in \mathcal{P}({}^d\ell_1^N)$, $\|P\| \leqslant 1$, there exists $A \subset \{1, \dots, N\}$, $|A| = n$, and a subsymmetric polynomial $Q \in \mathcal{P}\left({}^dY\right)$ such that $\|P\restriction_Y - Q\| < \varepsilon$, where $Y = \operatorname{span}\{e_k\}_{k \in A}$ and $\{e_k\}_{k=1}^N$ is the canonical basis of ℓ_1^N.*

Proof. Given $P \in \mathcal{P}\left({}^d\ell_1^N\right)$ with $\|P\| \leqslant 1$, there are $a_{\alpha,\rho} \in \mathbb{R}$ such that $P = \sum_{\alpha \in \mathcal{I}^+(d)} R_\alpha$, where

$$R_\alpha(x) = \sum_{\rho \in \mathcal{N}(k,N)} a_{\alpha,\rho} x_{\rho_1}^{\alpha_1} \cdots x_{\rho_k}^{\alpha_k} \tag{1.10}$$

for $\alpha \in \mathcal{I}^+(k, d)$. By combining formulas (1.3), (1.4), (1.5) and the Polarization formula, we see that each $a_{\alpha,\rho}$ is such that $|a_{\alpha,\rho}| < \binom{d}{\alpha} \left\|\check{P}\right\| \leqslant d^d$.

We show that, for any $n \in \mathbb{N}$, $\varepsilon > 0$, $K > 0$ and $\alpha \in \mathcal{I}^+(d)$, there is $N = N_\alpha(n, \varepsilon, K)$ such that, for any polynomial $R \in \mathcal{P}\left({}^d\ell_1^N\right)$ of the form (1.10), with $|a_{\alpha,\rho}| \leqslant K$, and for all $\rho \in \mathcal{N}(k, N)$, there is $A \subset \{1, \dots, N\}$, $|A| = n$, and $c \in \mathbb{R}$ such that $\|R\restriction_Y - cP_\alpha^n\| < \varepsilon$, where $Y =\operatorname{span}\{e_k\}_{k \in A}$.

It is clear that we may take

$$N(n, d, \varepsilon) = N_{\alpha^v}\left(\dots N_{\alpha^2}\left(N_{\alpha^1}\left(n, \frac{\varepsilon}{v}, d^d\right), \frac{\varepsilon}{v}, d^d\right) \dots, \frac{\varepsilon}{v}, d^d\right),$$

where $\alpha^1, \dots, \alpha^v$ is an enumeration of $\mathcal{I}^+(d)$.

So fix $\alpha \in \mathcal{I}^+(k, d)$, $n \in \mathbb{N}$, $\varepsilon > 0$ and $K > 0$. Let $\delta = \frac{\varepsilon}{2n!}$ and $M = \left[\frac{K}{\delta}\right]$. By Ramsey's theorem there is $N \in \mathbb{N}$ such that, for every $2(M+1)$-colouring of k-subsets (i.e. subsets of cardinality k) of $\{1, \dots, N\}$, there is $A \subset \{1, \dots, N\}$, $|A| = n$, such that all k-subsets of A have the same colour.

Now, given $R \in \mathcal{P}\left({}^d\ell_1^N\right)$ of the form (1.10) with $|a_{\alpha,\rho}| \leqslant K$ for all $\rho \in \mathcal{N}(k, N)$, we put $m(\rho) = \left[\frac{a_{\alpha,\rho}}{\delta}\right] \in \{-M-1, -M, \dots, M\}$.

Note that $|a_{\alpha,\rho} - \delta m(\rho)| < \delta$. Each $\rho \in \mathcal{N}(k, N)$ uniquely determines a k-subset of $\{1, \dots, N\}$ and vice versa, therefore the function m induces a $2(M+1)$-colouring of the k-subsets of $\{1, \dots, N\}$. Let $A \subset \{1, \dots, N\}$,

$|A| = n$, be such that there is $m_0 \in \mathbb{N}$ satisfying $m(\rho) = m_0$ for all $\rho \subset A$. Then

$$\left| R\left(\sum_{j\in A} x_j e_j\right) - \delta m_0 P_\alpha^n\left(\sum_{j\in A} x_j e_j\right)\right| \leqslant \delta \sum_{\rho\subset A} \left|x_{\rho_1}^{\alpha_1}\cdots x_{\rho_k}^{\alpha_k}\right| \leqslant \delta\binom{n}{k} < \varepsilon$$

whenever $\left\|\sum_{j\in A} x_j e_j\right\| \leqslant 1$. □

1.5 Algebras

In this thesis we are going to work with algebras $\mathcal{A}$ of polynomials on a Banach space X, i.e. subsets of $\mathcal{P}(X)$ that are closed with respect to addition, pointwise multiplication, and scalar multiplication.

Definition 1.5.1. Given an algebra $\mathcal{A} \subset \mathcal{P}(X)$, we say that the set $B \subset \mathcal{A}$ **generates** the algebra $\mathcal{A}$ if $\mathcal{A}$ is the smallest algebra containing B, i.e. it is the intersection of all algebras containing B.
It is easy to see that B generates $\mathcal{A}$ if and only if for every $p \in \mathcal{A}$ there is a finite set $\{b_1, \ldots, b_l\} \subset B$ and a polynomial $P \in \mathcal{P}\left(\mathbb{R}^l\right)$ such that $p = P(b_1, \ldots, b_l)$.

Definition 1.5.2. Let X be a Banach space.

- We denote by $\mathcal{A}_n(X)$ the algebra generated by polynomials from $\bigcup_{i=0}^{n} \mathcal{P}_n(X)$.
- The space of subsymmetric d-homogeneous polynomials on $\mathbb{R}^N$ will be denoted by $H_d\left(\mathbb{R}^N\right)$.
- We denote by $S_k\left(\mathbb{R}^N\right)$ the algebra of subsymmetric polynomials generated by the set of polynomials $\bigcup_{l=0}^{k} H_l\left(\mathbb{R}^N\right)$.

Remark 1.5.3.

- Given $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathcal{J}(d)$ and $N \geqslant k$, we let

  $$P_\alpha^N \left(\sum_{j=1}^{N} x_j e_j \right) = \sum_{1 < \rho_1 < \cdots < \rho_k} x_{\rho_1}^{\alpha_1} \cdots x_{\rho_k}^{\alpha_k} \tag{1.11}$$

  and set $P_\varnothing^N = 1$. For $N \geqslant d$, the polynomials P_α^N, for $\alpha \in \mathcal{J}(d)$, form a linear basis of $H_d\left(\mathbb{R}^N\right)$.

- As we pointed out, if $k \leqslant N$, then $H_k(\mathbb{R}^N)$ has a linear basis consisting of P_α^N, $\alpha \in \mathscr{J}(k)$ (see (1.11)). In other words, $P \in H_k(\mathbb{R}^n)$ has the unique standard form

  $$P(x_1, \dots, x_N) = \sum_{\alpha \in \mathscr{J}(k)} a_\alpha P_\alpha^N(x_1, \dots, x_N), \quad a_\alpha \in \mathbb{R}. \tag{1.12}$$

- The spaces of subsymmetric polynomials $H_k(\mathbb{R}^k)$ and $H_k(\mathbb{R}^N)$, $N > k$, are canonically isomorphic, as their linear bases can be indexed with the same set $\mathscr{J}(k)$.

The following result is the key lemma for proving plenty of results in [41]. Unfortunately, as we mentioned in the introduction, the theory of algebraic bases developed there is not entirely correct. Fortunately, the core of this theory, Lemma 1.5.4, can be proved otherwise. Its proof is treated in Chapter 2.

Lemma 1.5.4. *For every $n \in \mathbb{N}$, there exists an $\varepsilon > 0$ such that, for every $m \geqslant M(n)$,*

$$\sup_{\sum_{i=1}^{m} |x_i| \leqslant 1} |p(x_1, \dots, x_m) - s_{n+1}(x_1, \dots, x_m)| \geqslant \varepsilon,$$

for every p in the algebra $S_n(\mathbb{R}^m)$, generated by subsymmetric polynomials of degree at most n.

The above quantitative lemma implies the following fundamental criterion.

Theorem 1.5.5. *Let X be an infinite dimensional Banach space, $n \in \mathbb{N}$ and $P \in \mathcal{P}(^nX)$ be a polynomial with the following property: for every $N \in \mathbb{N}$ and $\varepsilon > 0$ there exists a normalized finite basic sequence $\{e_j\}_{j=1}^N$ such that*

$$\sup_{\sum_{j=1}^N |a_j| \leqslant 1} \left| P\left(\sum_{j=1}^N a_j e_j\right) - \sum_{j=1}^N a_j^n \right| \leqslant \varepsilon.$$

Then $P \notin \overline{\mathcal{A}_{n-1}(X)}$.

Proof. Denote by $S^n(\ell_1^m)$ the algebra generated by all sub-symmetric polynomials on ℓ_1^m of degree at most n. By Lemma 1.5.4, there are $m \in \mathbb{N}$ and $\varepsilon > 0$ such that $\|Q - s_n\| \geqslant 3\varepsilon$ for all $Q \in S^{n-1}(\ell_1^m)$. Let $P \in \mathcal{P}(^nX)$ be the polynomial whose existence is guaranteed by the assumptions of the theorem. We claim that $P \notin \overline{\mathcal{A}_{n-1}(X)}$.
By contradiction, suppose that there exist $P_1, \ldots, P_k \in \mathcal{P}^{n-1}(X)$ and $r \in \mathcal{P}(\mathbb{R}^k)$ such that, for $R = r \circ (P_1, \ldots, P_k)$, we have $\|P - R\| < \varepsilon$. Put $K = 1 + \max_j \|P_j\|$ and let $0 < \eta \leqslant 1$ be such that $|r(u) - r(v)| < \varepsilon$, whenever $u, v \in KB_{\ell_\infty^k}$, $\|u - v\|_{\ell_\infty^k} < \eta$. Using Theorem 1.4.13 recursively kn times, we find $N \in \mathbb{N}$ such that, for any linearly independent $\{e_j\}_{j=1}^N \subset S_X$, there exist $A \subset \{1, \ldots, N\}$, $|A| = m$, and sub-symmetric polynomials $Q_1, \ldots, Q_k \in \mathcal{P}^{n-1}(Y)$ such that $\|P_j\restriction_Y - Q_j\| < \eta$, $j = 1, \ldots, k$, where $Y = \operatorname{span}\{e_j\}_{j\in A}$ with ℓ_1-norm.
Let $\{e_j\}_{j=1}^N$ be the linearly independent set from the assumptions of the theorem and $A \subset \{1, \ldots, N\}$, $Q_1, \ldots, Q_k \in \mathcal{P}^{n-1}(Y)$ as above. Note that since $\{e_j\}$ is normalized, $\|R\restriction_Y - P\restriction_Y\|_Y \leqslant \|R\restriction_Y - P\restriction_Y\|_X < \varepsilon$. Put $Q = r \circ (Q_1, \ldots, Q_k)$. Then $Q \in S^{n-1}(\ell_1^m)$ and $\|Q - s_n\| \leqslant \|Q - R\restriction_Y\| + \|R\restriction_Y - P\restriction_Y\| + \|P\restriction_Y - s_n\| < 3\varepsilon$, which is a contradiction.

□

1.6 Tensor products

This section is aimed at collecting basic definitions and elementary facts concerning tensor products. Tensor products offer an important point of view on polynomials and multilinear mappings.

Definition 1.6.1. Let $X_1, \ldots, X_n$ be vector spaces over $\mathbb{K}$.

- By Λ we denote the vector space of all *formal* linear combinations $\sum_{k=1}^{N} a_k \left(x_1^k \otimes \cdots \otimes x_n^k\right), a_k \in \mathbb{K}, x_j^k \in X_j$.
- By Λ_0 we denote the linear subspace of Λ spanned by the vectors

  $$a(x_1 \otimes \cdots \otimes x_n) - (x_1 \otimes \cdots \otimes ax_k \otimes \cdots \otimes x_n)$$

  and

  $$(x_1 \otimes \cdots \otimes (x_k + y_k) \otimes \cdots \otimes x_n) - (x_1 \otimes \cdots \otimes x_k \otimes \cdots \otimes x_n) - (x_1 \otimes \cdots \otimes y_k \otimes \cdots \otimes x_n),$$

  where $k \in \{1, \dots, n\}, x_j, y_j \in X_j, a \in \mathbb{K}$.
- The quotient space $\frac{\Lambda}{\Lambda_0}$ is called **tensor product** of $X_1, \dots, X_n$ and will be denoted by

  $$X_1 \otimes \cdots \otimes X_n = \bigotimes_{j=1}^{n} X_j.^{7}$$

Remark 1.6.2.

- By the definition of Λ_0, each $z \in X_1 \otimes \cdots \otimes X_n$ has a representation

  $$z = \sum_{j=1}^{k} x_1^j \otimes \cdots \otimes x_n^j.$$

  An element of $X_1 \otimes \cdots \otimes X_n$ that admits a representation $x_1 \otimes \cdots \otimes x_n$ is called **elementary tensor**.
- Given $\phi_j \in X_j'$,[8] the function

  $$\sum_{j=1}^{k} a_j \left(x_1^j \otimes \cdots \otimes x_n^j\right) \mapsto \sum_{j=1}^{k} a_j \phi_1(x_1^j) \cdots \phi_n(x_n^j) \tag{1.13}$$

  is a linear form on the vector space Λ.

We can infer a useful criterion for distinguishing vectors in a tensor product.

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[7] This definition is motivated by the will to linearize multilinear mappings.

[8] This denotes the algebraic dual of X_j.

Proposition 1.6.3. *Let $X_1, \dots, X_n$ be vector spaces and $A_j \subset X_j'$ be subsets that separate the points of X_j, $j = 1, \dots, k$. Then $\sum_{j=1}^k a_j x_1^j \otimes \cdots \otimes x_n^j = 0$ in $X_1 \otimes \cdots \otimes X_n$ if and only if*

$$\sum_{j=1}^{k} a_j \phi_1(x_1^j) \cdots \phi_n(x_n^j) = 0$$

for every choice of $\phi_j \in A_j$.

Definition 1.6.4. By $\otimes$ we define the n-linear mapping $\otimes\colon X_1 \times \cdots \times X_n \longrightarrow \bigotimes_{j=1}^n X_j$ such that $\otimes(x_1, \dots, x_n) = x_1 \otimes \cdots \otimes x_n$.

Theorem 1.6.5 (Universality of tensor products – algebraic setting)**.** *Let $X_1, \dots, X_n, Y$ be vector spaces. For every n-linear mapping $M \in L(X_1, \dots, X_n; Y)$ there exists a unique linear operator $L_M \in L(X_1 \otimes \cdots \otimes X_n; Y)$ such that $M = L_M \circ \otimes$:*

$$\begin{tikzcd}
X_1 \times \cdots \times X_n \arrow[r, "M"] \arrow[d, "\otimes"'] & Y \\
X_1 \otimes \cdots \otimes X_n \arrow[ur, "L_M"'] &
\end{tikzcd}$$

The operator L_M satisfies

$$L_M(x_1 \otimes \cdots \otimes x_n) = M(x_1, \dots, x_n). \tag{1.14}$$

L_M is therefore called the ***linearization*** *of M.*

Theorem 1.6.6. *Let $X_1, \dots, X_n$ be vector spaces. For $M \in L(X_1, \dots, X_n; \mathbb{K})$ and $z = \sum_{j=1}^k x_1^j \otimes \cdots \otimes x_n^j \in X_1 \otimes \cdots \otimes X_n$ put*

$$\langle M, z \rangle = \sum_{j=1}^{k} M\left(x_1^j, \dots, x_n^j\right) = \sum_{j=1}^{k} L_M\left(x_1^j \otimes \cdots \otimes x_n^j\right) = L_M(z).$$

Then $\langle L(X_1, \dots, X_n; \mathbb{K}), X_1 \otimes \cdots \otimes X_n \rangle$ forms a dual pair.

We will now introduce an important example of natural norm on tensor products of Banach spaces (see [57]).

Definition 1.6.7. Let $X_1, \dots, X_n$ be normed linear spaces.

- The **projective tensor norm** π on $X_1 \otimes \cdots \otimes X_n$ is defined by the formula

  $$\pi(z) = \sup\left\{ |\langle M, z \rangle| : M \in \mathcal{L}(X_1, \dots, X_n; \mathbb{K}), \|M\| \leqslant 1 \right\}, \ z \in X_1 \otimes \cdots \otimes X_n.$$

- The **projective tensor product**, denoted by $X_1 \otimes_\pi \cdots \otimes_\pi X_n$, is the completion of the normed linear space $(X_1 \otimes \cdots \otimes X_n, \pi)$.

Proposition 1.6.8. *Let $X_1, \ldots, X_n$ be normed linear spaces. Then, for any $z \in X_1 \otimes_\pi \cdots \otimes_\pi X_n$ there exist bounded sequences $\left\{x_l^j\right\}_{j=1}^{\infty} \subset X_l, l = 1, \ldots, n$, such that $z = \sum_{j=1}^{\infty} x_1^j \otimes \cdots \otimes x_n^j$ is an absolute convergent series and*

$$\pi(z) = \inf \left\{ \sum_{j=1}^{\infty} \left\| x_1^j \right\| \cdots \left\| x_n^j \right\| : z = \sum_{j=1}^{\infty} x_1^j \otimes \cdots \otimes x_n^j \right\}.$$

Furthermore, $\pi(x_1 \otimes \cdots \otimes x_n) = \|x_1\| \cdots \|x_n\|$ for every $x_j \in X_j, j = 1, \ldots, n$.

This implies that $\otimes \colon X_1 \times \cdots \times X_n \longrightarrow X_1 \otimes_\pi \cdots \otimes_\pi X_n$ is a bounded n-linear mapping of norm 1. It follows that the projective norm is defined so that the universality property of the tensor product remains valid also in the topological sense:

Theorem 1.6.9 (Universality of the tensor product – topological setting)**.** *Let $X_1, \ldots, X_n, Y$ be normed linear spaces. For every $M \in \mathcal{L}\left(X_1, \ldots, X_n; Y\right)$ there exists a unique $L_M \in \mathcal{L}\left(X_1 \otimes_\pi \cdots \otimes_\pi X_n; Y\right)$ such that $M = L_M \circ \otimes$:*

$$\begin{tikzcd}
X_1 \times \cdots \times X_n \arrow[r, "M"] \arrow[d, "\otimes"'] & Y \\
X_1 \otimes_\pi \cdots \otimes_\pi X_n \arrow[ur, "L_M"'] &
\end{tikzcd}$$

The operator L_M satisfies (1.14) *and the mapping $M \mapsto L_M$ is an isometry of the spaces $\mathcal{L}\left(X_1, \ldots, X_n; Y\right)$ and $\mathcal{L}\left(X_1 \otimes_\pi \cdots \otimes_\pi X_n; Y\right)$.*

Note that, if $Y = \mathbb{K}$, we obtain the following (simple but important) duality relation.

Theorem 1.6.10. *Let $X_1, \ldots, X_n$ be normed linear spaces. Then*

$$\left(X_1 \otimes_\pi \cdots \otimes_\pi X_n; Y\right)^* = \mathcal{L}\left(X_1, \ldots, X_n; \mathbb{K}\right).$$

Whenever $n = 2$, observe that $\mathcal{L}\left(X_1, X_2; \mathbb{K}\right) = \mathcal{L}\left(X_1; X_2^*\right)$. This leads to an equivalent dual representation.

Fact 1.6.11. Let X, Y be normed linear spaces. Then

$$\left(X \otimes_\pi Y\right)^* = \mathcal{L}\left(X; Y^*\right),$$

where the evaluation is given by $\langle L, x \otimes y \rangle = L(x)(y)$.

We conclude this section by introducing symmetric tensor products, which turn out to have a close relationship with polynomials.

Definition 1.6.12. Let X be a normed linear space.

- The **symmetrization** $\otimes_s : X \times \cdots \times X \longrightarrow X \otimes \cdots \otimes X$ is a symmetric n-linear mapping given by

  $$\otimes_s(x_1, \ldots, x_n) = \frac{1}{n!} \sum_{\eta \in S_n} \otimes \left(x_{\eta(1)}, \ldots, x_{\eta(n)}\right) = \frac{1}{n!} \sum_{\eta \in S_n} x_{\eta(1)} \otimes \cdots \otimes x_{\eta(n)},$$

  where S_n is the set of all permutations of $\{1, \ldots, n\}$.

  We also use the notation $\otimes_s(x_1, \ldots, x_n) = x_1 \otimes_s \cdots \otimes_s x_n$ and $\otimes^n x = \otimes\left({}^n x\right) = x \otimes \cdots \otimes x$.

  The Polarization formula yields that $\otimes_s^n X$ =span$\{\otimes^n x : x \in X\}$.

- The space $\otimes_s^n X$ is called **symmetric tensor product** and the elements of $\otimes_s^n X$ are called **symmetric tensors**.

When $\otimes_s^n X$ is equipped with the projective norm inherited from its superspace $\otimes_\pi^n X$, its completion becomes a closed subspace $\otimes_{\pi,s}^n X$ of $\otimes_\pi^n X$. Then the linearization $\sigma_X^n : \otimes_\pi^n X \longrightarrow \otimes_{\pi,s}^n X$ of $\otimes_s$ is a projection of norm 1. Thus the following result holds.

Theorem 1.6.13 (Universality of the symmetric tensor product)**.** *Let X, Y be normed linear spaces. For every symmetric $M \in \mathcal{L}^s\left({}^n X; Y\right)$ there exists a unique $L_M \in \mathcal{L}\left(\otimes_{\pi,s}^n X; Y\right)$ such that $M = L_M \circ \otimes_s = L_M \circ \sigma_X^n \circ \otimes$.*

$$\begin{tikzcd}
 & X^n \arrow[dl, "\otimes"'] \arrow[d, "\otimes_s"'] \arrow[dr, "M"] & \\
\otimes^n X \arrow[r, "\sigma_X^n"'] & \otimes_s^n X \arrow[r, "L_M"] & Y \\
 & X \arrow[u, "\otimes^n"] \arrow[ur, "\widehat{M}"'] &
\end{tikzcd}$$

The mapping $M \mapsto L_M$ is an isometry of the spaces $\mathcal{L}^s\left({}^n X; Y\right)$ and $\mathcal{L}\left(\otimes_{\pi,s}^n X; Y\right)$.

Corollary 1.6.14. *Let X, Y be normed linear spaces. Then the spaces $\mathcal{P}\left({}^n X; Y\right)$ and $\mathcal{L}\left(\otimes_{\pi,s}^n X; Y\right)$ are canonically isomorphic.*

In particular,

$$\left(\otimes^n_{\pi,s} X\right)^* = \mathcal{P}\left({}^n X\right)$$

in the isomorphic sense, where the evaluation is given by $\langle P, \otimes^n x\rangle = P(x)$*. More generally,*

$$\left(\left(\otimes^n_{\pi,s} X\right) \otimes_\pi Y\right)^* = \mathcal{L}\left(\otimes^n_{\pi,s} X; Y^*\right) = \mathcal{P}\left({}^n X; Y^*\right)$$

in the isomorphic sense, where the evaluation is given by $\langle P, \otimes^n x \otimes y\rangle = P(x)(y)$*.*

1.7 Weak continuity and polynomials into ℓ_1

We will now provide the reader with a list of various notions of weak continuity, which play a key role in our investigations, some of which have been introduced and studied by R. M. Aron and his co-authors, e.g. in [3], [6] and [8]; see also [30].

Definition 1.7.1. Let X be a normed linear space, Y a Banach space and $U \subset X$ a convex set.

- By $\mathcal{C}(U;Y)$ we denote the space $C(U;Y)$ endowed with the locally convex topology τ_b of uniform convergence on CCB[9] subsets of U.[10]
- By $\mathcal{C}_w(U;Y)$ we denote the linear subspace of $\mathcal{C}(U;Y)$ consisting of all mappings that are $w - \|\cdot\|$ continuous on CCB subsets of U.
- By $\mathcal{C}_{wu}(U;Y)$ we denote the linear subspace of $\mathcal{C}(U;Y)$ consisting of all mappings that are $w - \|\cdot\|$ uniformly continuous on CCB subsets of U.[11]
- By $\mathcal{C}_{wsc}(U;Y)$ we denote the linear subspace of $\mathcal{C}(U;Y)$ consisting of all mappings that are $w - \|\cdot\|$ sequentially continuous on CCB subsets

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[9] A CCB set is a closed, convex and bounded subset of a normed linear space X.

[10] Note that if U is closed (for instance $U = X$), then the topology on $\mathcal{C}(U;Y)$ is the topology of uniform convergence on bounded subsets of U.

[11] $f \in \mathcal{C}_{wu}(U;Y)$ if and only if for any CCB set V and any $\varepsilon > 0$ there are $\delta > 0$ and $\phi_1, \ldots, \phi_k \in B_{X^*}$ such that $\|f(x) - f(y)\| < \varepsilon$ whenever $x, y \in V$ are such that $|\phi_j(x-y)| < \delta$ for $j = 1, \ldots, k$.

of U, i.e. that map weakly convergent sequences in CCB subsets of U to convergent sequences in Y.

- By $\mathcal{C}_{wsC}\left(U;Y\right)$ we denote the linear subspace of $\mathcal{C}\left(U;Y\right)$ consisting of all mappings that are $w-\left\|\cdot\right\|$ sequentially Cauchy-continuous on CCB subsets of U, i.e. that map weakly Cauchy sequences in CCB subsets of U to convergent sequences in Y.

- By $\mathcal{C}_{K}\left(U;Y\right)$ we denote the linear subspace of $\mathcal{C}\left(U;Y\right)$ consisting of all mappings that map CCB subsets of U to relatively compact sets in Y.

- By $\mathcal{C}_{wK}\left(U;Y\right)$ we denote the linear subspace of $\mathcal{C}\left(U;Y\right)$ consisting of all mappings that map CCB subsets of U to relatively weakly compact sets in Y.

Remark 1.7.2.

- When the range space is the scalar field, we simply omit it, e.g. $\mathcal{C}_{wsc}\left(U\right)=\mathcal{C}_{wsc}\left(U;\mathbb{K}\right)$.

- If we substitute CCB sets in the above definitions with bounded sets, $\mathcal{C}_{wsc}\left(U;Y\right)$ (resp. $\mathcal{C}_{wsC}\left(U;Y\right)$) are just $w-\left\|\cdot\right\|$ sequentially continuous (resp. $w-\left\|\cdot\right\|$ sequentially Cauchy-continuous) mappings on U.

- If X^* is separable, it is well-known that (B_X,w) is metrizable, thus $\mathcal{C}_{wsc}\left(U;Y\right)=\mathcal{C}_{w}\left(U;Y\right)$ and $\mathcal{C}_{wsC}\left(U;Y\right)=\mathcal{C}_{wu}\left(U;Y\right)$.

- $\mathcal{C}_{w}\left(U;Y\right)$, $\mathcal{C}_{wu}\left(U;Y\right)$, $\mathcal{C}_{wsc}\left(U;Y\right)$, $\mathcal{C}_{wsC}\left(U;Y\right)$ and $\mathcal{C}_{K}\left(U;Y\right)$ are closed subspaces of $\mathcal{C}\left(U;Y\right)$.

- If Y is any Banach space and U is any convex subset of a normed linear space X, the following inclusions hold true:

$$\begin{tikzcd}
 & \mathcal{C}_{K}\left(U;Y\right) \arrow[r, phantom, "\subset"] & \mathcal{C}_{wK}\left(U;Y\right) \\
\mathcal{C}_{wu}\left(U;Y\right) \arrow[ur, phantom, sloped, "\subset"] \arrow[r, phantom, "\subset"] \arrow[dr, phantom, sloped, "\subset"] & \mathcal{C}_{w}\left(U;Y\right) \arrow[r, phantom, "\subset"] & \mathcal{C}_{wsc}\left(U;Y\right) \\
 & \mathcal{C}_{wsC}\left(U;Y\right) \arrow[ur, phantom, sloped, "\subset"] &
\end{tikzcd}$$

Corollary 1.7.3 ([6])**.** *Let X be a normed linear space, Y a Banach space and $n \in \mathbb{N}$. Then*

$$\begin{aligned}\mathcal{L}_w\left({}^nX;Y\right) &= \mathcal{L}_{wu}\left({}^nX;Y\right),\\ \mathcal{L}_{wsc}\left({}^nX;Y\right) &= \mathcal{L}_{wsC}\left({}^nX;Y\right),\\ \mathcal{P}_w\left({}^nX;Y\right) &= \mathcal{P}_{wu}\left({}^nX;Y\right),\\ \mathcal{P}_{wsc}\left({}^nX;Y\right) &= \mathcal{P}_{wsC}\left({}^nX;Y\right).\end{aligned}$$

From this and the relations shown earlier we obtain the following inclusions:

$$\begin{tikzcd}
{} & \mathcal{P}_K\left(X;Y\right) \subset \mathcal{P}_{wK}\left(X;Y\right) \\
\mathcal{P}_{wu}\left(X;Y\right) = \mathcal{P}_w\left(X;Y\right) \arrow[ur, phantom, sloped, "\subset"] & \subset \mathcal{P}_{wsc}\left(X;Y\right) = \mathcal{P}_{wsC}\left(X;Y\right)
\end{tikzcd}$$

$$\begin{tikzcd}
{} & \mathcal{L}_{wK}\left(X;Y\right) \\
\mathcal{L}_{wu}\left(X;Y\right) = \mathcal{L}_K\left(X;Y\right) = \mathcal{L}_w \arrow[ur, phantom, sloped, "\subset"] & \subset \mathcal{L}_{wsc}\left(X;Y\right) = \mathcal{L}_{wsC}\left(X;Y\right)
\end{tikzcd}$$

Remark 1.7.4. It is not sufficient to check the $w - \|\cdot\|$ continuity of polynomials only at the origin,[12] as shown by the following example by Aron in [4].[13]

Let $P \in \mathcal{P}\left({}^3\ell_2\right)$ be defined as $P(x) = x_1 \sum_{n=2}^{\infty} x_n^2$. Then the restriction of P to any bounded set is weakly continuos at the origin, but P is not weakly sequentially continuous. Indeed, $e_1 + e_1 \xrightarrow{w} e_1$ but $P(e_1 + e_n) = 1$ and $P(e_1) = 0$.

Let X, Y be Banach spaces. Recall the duality relationship treated in the previous section:

$$((\otimes_{\pi,s}^n X) \otimes_\pi Y)^* = \mathcal{L}(\otimes_{\pi,s}^n X; Y^*) = \mathcal{P}({}^nX; Y^*). \tag{1.15}$$

As special cases, we of course have $(\otimes_{\pi,s}^n X)^* = \mathcal{P}({}^nX)$, $(X \otimes_\pi Y)^* = \mathcal{L}(X; Y^*)$. Recall a result by Bessaga and Pełczyński ([36] p. 206). Let X be a Banach space, $c_0 \hookrightarrow X^*$. Then X contains a complemented copy of ℓ_1 (and hence X^* actually contains a complemented copy of ℓ_∞). Applying this result to the duality relation (1.15) we get the next (probably known) result.

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[12] Unlike the $\|\cdot\| - \|\cdot\|$ continuity.

[13] See also [5].

Theorem 1.7.5. *Let X be a Banach space. The following are equivalent for $n \in \mathbb{N}$.*
1. $\mathcal{P}_K({}^nX;\ell_1) = \mathcal{P}({}^nX;\ell_1)$,
2. $c_0 \not\hookrightarrow \mathcal{P}({}^nX)$.

Proof. Suppose 2 fails. Since $(\otimes^n_{\pi,s}X)^* = \mathcal{P}({}^nX)$, ℓ_1 is complemented in $\otimes^n_{\pi,s}X$ by the Bessaga- Pełczyński theorem. Hence ℓ_1 is a range of a bounded linear operator from $\otimes^n_{\pi,s}X$ and 1 fails by the universality of the projective symmetric tensor product. On the other hand, if 1 fails, then there is a non-compact bounded linear operator $T : \otimes^n_{\pi,s}X \to \ell_1$.
Setting $B = T(B_{\otimes^n_{\pi,s}X})$, we claim that B contains B_{ℓ_1} (up to isomorphism). Indeed, since B is not relatively compact, $\overline{B}$ is not weakly compact ([36] p. 277). By the Eberlein-Šmulyan theorem, there exists a bounded sequence $\{x_n\} \subseteq B$ with no weakly convergent subsequences, which cannot be weakly Cauchy either (Schur). Thus, by Rosenthal's ℓ_1-theorem, $\{x_n\}$ admits a subsequence equivalent to the usual ℓ_1-basis, which proves the claim.
Finally, using the lifting property of ℓ_1 ([36] p. 238), ℓ_1 is a complemented subspace of $\otimes^n_{\pi,s}X$, whence 2 fails by duality. □

We will need two principles for passing to suitable sequences in the domain. The first one is based on an improvement of the classical result that ℓ_2 is a linear quotient of any Banach space containing a copy of ℓ_1.

Lemma 1.7.6. *Let X be a Banach space, $\ell_1 \hookrightarrow X$, $p \geqslant 2$. Then there exists $T \in \mathcal{L}(X;\ell_p)$ and a basic sequence $\{f_j\}$ in X equivalent to ℓ_1 basis such that $T(f_j) = e_j$ is the unit basis in ℓ_p.*

Proof. It suffices to prove the result for $p = 2$, since then we can compose T with the formal identity $Id : \ell_2 \to \ell_p$, which is a bounded linear operator. Let $L : \ell_2 \hookrightarrow L_1$ be an isomorphic embedding, $\{e_j\}$ be the basis of ℓ_2. By Pełczyński-Hagler, [46] p. 253, there is an isomorphic embedding $M : L_1 \hookrightarrow X^*$. So $\{y_j = M \circ L(e_j)\}$ is a weakly null sequence in X^*, which is equivalent to the ℓ_2 basis. There is a normalized sequence $\{\tilde{f}_j\} \in X^{**}$ biorthogonal to $M \circ L(e_j)$. By Goldstine's theorem we replace $\tilde{f}_j$ by $f_j \in B_X$ so that $\langle f_j, y_k\rangle = 0$, $k \leqslant j$, $\langle f_j, y_j\rangle = 1$. Since $\{y_j\}$ is weakly null, we can pass to subsequences so that $\{f_j, y_j\}$ is a biorthogonal system. Since $M^*(X) \subset$

L_∞ and $\{L(e_j), M^*(f_j)\}$ is a biorthogonal system in L_1, L_∞, by the DPP property of L_1, $\{M^*(f_j)\}$ does not contain a weakly Cauchy subsequence. By Rosenthal's ℓ_1-theorem, we may assume without loss of generality that it is an ℓ_1-basis. By the lifting property of ℓ_1, $\{f_j\}$ is an ℓ_1-basis. Finally, $R = L^* \circ M^* : X^{**} \to \ell_2$ is a quotient mapping such that $R(f_j) = e_j$. So $T = R\restriction_X\colon X \to \ell_2$ is the desired operator. □

In particular, let X be a Banach space, $\ell_1 \hookrightarrow X$. Then there is a $P \in \mathcal{P}(^2X; \ell_1)$ such that it takes a sequence $\{f_j\}$ in X, equivalent to an ℓ_1-basis, into $\{e_j\}$ a unit basis in the range ℓ_1.

Proposition 1.7.7. *Let X be a Banach space, $\ell_1 \hookrightarrow X$, $k \in \mathbb{N}, k \geqslant 2$. Then there exists a polynomial $P \in \mathcal{P}(^kX)$ and a basic sequence $\{f_j\}$ in X equivalent to an ℓ_1-basis such that*

$$P\left(\sum_{j=1}^{\infty} a_j f_j\right) = \sum_{j=1}^{\infty} a_j^k. \tag{1.16}$$

In particular, P is not weakly continuous at the origin and

$$\mathcal{P}_{wu}\left(^kX\right) \neq \mathcal{P}\left(^kX\right).$$

Proof. Let T and $\{f_j\}$ be as above and let $\{g_j\}$ be the sequence of the coordinate functionals on ℓ_k. Letting $P(x) = \sum_{j=1}^{\infty} \left(g_j\left(T(x)\right)\right)^k$ proves (1.16). Assume by contradiction that P is weakly continuous, so given $\varepsilon > 0$ there exist $\phi_1, \dots, \phi_n \in X^*$ and $\delta > 0$ such that $|\phi_j(x)| < \delta$, $j = 1, \dots, n$, implies $|P(x)| < \varepsilon$. We have that $\phi_j\restriction_{[f_j]} \in \ell_\infty$. By a simple argument there exist pairwise distinct indices m, l, r such that

$$|\phi_j(f_m) - \phi_j(f_l)|, |\phi_j(f_m) - \phi_j(f_r)| < \delta,\ j = 1, \dots, n.$$

So choosing $\varepsilon > 0$ small enough and letting $x = f_m - \frac{1}{2}f_l - \frac{1}{2}f_r$ clearly witnesses the contradiction. □

We will need a modification of a well-known principle for dealing with non-weakly sequentially continuous polynomials of minimal degree, which has been used many times in the literature (see e.g. [20] for its most general formulation). In our case, we replace the non-wsc property by the non-compactness and add the assumption $\ell_1 \not\hookrightarrow X$.

Lemma 1.7.8. *Let X, Y be Banach spaces, $\ell_1 \not\hookrightarrow X$, $\mathcal{P}({}^kX;Y) = \mathcal{P}_K({}^kX;Y)$ for all $k < n$ and $P \in \mathcal{P}({}^nX;Y) \backslash \mathcal{P}_K({}^nX;Y)$. Then there is a weakly null sequence $\{y_k\}_{k=1}^\infty$ such that $\{P(y_k)\}_{k=1}^\infty$ is not relatively compact.*

Proof. By Rosenthal's ℓ_1-theorem, there is a $\delta > 0$ and a weakly Cauchy sequence $\{x_k\}_{k=1}^\infty$ such that

$$\|P(x_k) - P(x_l)\| > \delta, \quad k \neq l \in \mathbb{N}. \tag{1.17}$$

By a simple application of the multilinearity of $\check{P}$,

$$P(x_k - x_l) = P(x_k) + \sum_{j=1}^{n-1} \binom{n}{j} (-1)^j \check{P}\left({}^jx_l, {}^{n-j}x_k\right) + (-1)^n P(x_l).$$

By assumption, all polynomials of degree less than n are compact, so for any fixed k, passing to a subset of indices $N_k \subset N_{k-1}, N_0 = \mathbb{N}$, there exist the limits

$$y_k^j = \lim_{l \in N_k} \check{P}\left({}^jx_l, {}^{n-j}x_k\right), \quad j = 1, \ldots, n-1.$$

Let M be the diagonal set of $N_k, k \in \mathbb{N}$. Next, fix for each $k \in M$, an m_k such that for all $j \in \{1, \ldots, n-1\}$

$$\left\| y_k^j - \check{P}\left({}^jx_l, {}^{n-j}x_k\right) \right\| < \frac{\delta}{20n^{n+1}}, \quad l \geqslant m_k, l \in M.$$

Then

$$\left\| P(x_k - x_l) - \left(P(x_k) + \sum_{j=1}^{n-1} \binom{n}{j} (-1)^j y_k^j + (-1)^n P(x_l) \right) \right\| < \frac{\delta}{20},$$

whenever $l \geqslant m_k, l \in M$.

Whence,

$$\left\| P(x_k - x_l) - P(x_k) - (-1)^n P(x_l) - \left(\sum_{j=1}^{n-1} \binom{n}{j} (-1)^j y_k^j \right) \right\| < \frac{\delta}{20},$$

for $l \geqslant m_k, l \in M$.

Thus

$$\|P(x_k - x_l) - P(x_p - x_r)\| \geqslant \|P(x_k) - (-1)^n P(x_l) - P(x_p) + (-1)^n P(x_r)\| - \frac{\delta}{10},$$

whenever $k, p \in \mathbb{N}$, $l, r \in M$, $l \geqslant m_k$ and $r \geqslant m_p$.

Suppose that $k, l, p \in \mathbb{N}$ are given and denote

$$z = (-1)^n P(x_k) + P(x_l) - (-1)^n P(x_p).$$

Using (1.17), there is an $r_{k,l,p} \in \mathbb{N}$ such that $\|P(x_r) - z\| \geqslant \frac{\delta}{2}$ for all $r \geqslant r_{k,l,p}$. Whence

$$\|P(x_k - x_l) - P(x_p - x_r)\| \geqslant \frac{\delta}{2} - \frac{\delta}{10} > \frac{\delta}{4},$$

whenever $k, p \in \mathbb{N}$, $l, r \in M$, $l \geqslant m_k$ and $r \geqslant \max\{m_p, r_{k,l,p}\}$.

Now it suffices to find $l_k \in M$ such that $l_k \geqslant \max\{m_k, r_{1,l_1,k}, \dots, r_{k-1,l_{k-1},k}\}$ and put $y_k = x_k - x_{l_k}$. Then $\{y_k\}$ is weakly null and $\{P(y_k)\}$ is a $\frac{\delta}{4}$-separated sequence. □

Proposition 1.7.9 ([44]). *Let X be a Banach space, $Y = \ell_p$, $1 \leqslant p < \infty$, or $Y = c_0$ and suppose there is a non-compact operator $T \in \mathcal{L}(X; Y)$. Then there are $S \in \mathcal{L}(X; Y)$ and a normalized basic sequence $\{x_n\} \subset X$ such that $S(x_n) = e_n$, $n \in \mathbb{N}$, where $\{e_n\}$ is the canonical basis of Y. If X does not contain ℓ_1, then $\{x_n\}$ may be chosen to be weakly null. If $X = \ell_1$, then S is in fact onto.*

Definition 1.7.10. Let $1 \leqslant p, q \leqslant \infty$. We say that a sequence $\{x_j\}_{j=1}^{\infty}$ in a Banach space over $\mathbb{K}$ has an **upper p-estimate** (resp. **lower q-estimate**) if there exists $C > 0$ such that for every $n \in \mathbb{N}$ and every $a_1, \dots, a_n \in \mathbb{K}$

$$\left\| \sum_{j=1}^{n} a_j x_j \right\| \leqslant C \left(\sum_{j=1}^{n} |a_j|^p \right)^{\frac{1}{p}}, \tag{1.18}$$

respectively

$$\left\| \sum_{j=1}^{n} a_j x_j \right\| \geqslant C \left(\sum_{j=1}^{n} |a_j|^q \right)^{\frac{1}{q}},$$

where the right-hand side is replaced by $\max_{j=1,\dots,n} |a_j|$ if $p = \infty$ or $q = \infty$.

Fact 1.7.11. Let X be a Banach space and $1 \leqslant p, q \leqslant \infty$. A sequence $\{x_j\}_{j=1}^{\infty} \subset X$ has an upper p-estimate if and only if the linear operator $T \colon \ell_p \longrightarrow X, T(e_j) = x_j$ is bounded. A sequence $\{x_j\}_{j=1}^{\infty} \subset X$ has a lower q-estimate if and only if the linear operator $T \colon \operatorname{span}\{x_j\} \longrightarrow \ell_q, T(x_j) = e_j$ is bounded. In case $p = \infty$ we replace ℓ_p by c_0 and analogously for $q = \infty$.

Corollary 1.7.12 ([38])**.** *Let* X *be a Banach space such that* X^* *is of type* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ *and let* $\{x_k\}_{k=1}^{\infty} \subset X$ *be a semi-normalized basic sequence. Then for each* $s > q$ *there is a subsequence* $\{x_{n_k}\}_{k=1}^{\infty}$ *such that there exists a bounded linear operator* $T\colon X \longrightarrow \ell_s$ *satisfying* $T(x_{n_k}) = e_k$, *where* $\{e_k\}$ *is the canonical basis of* ℓ_s. *Furthermore, there is a subsequence* $\{x_{n_k}\}_{k=1}^{\infty}$ *such that for each* $n \in \mathbb{N}$, $n > q$, *there is* $P \in \mathcal{P}(^nX)$ *such that* $P(x_{n_k}) = 1$ *for all* $k \in \mathbb{N}$.

Definition 1.7.13 ([47])**.** Let $1 \leqslant p \leqslant \infty$. We say that a Banach space X has the S_p**-property** (resp. the T_p**-property**) if every normalized weakly null sequence has a subsequence with an upper p-estimate (resp. lower q-estimate).

The S_∞ property is equivalent to saying that every normalized weakly null sequence contains a subsequence equivalent to the basis of c_0.

Theorem 1.7.14 ([54])**.** *Let* X, Y *be Banach spaces and* $P \in \mathcal{P}(^nX; Y)$. *If* $n < p < \infty$, *then* P *takes sequences with an upper* p*-estimate into sequences with an upper* $\frac{p}{n}$*-estimate.*

Corollary 1.7.15 ([40])**.** *Let* X *be a Banach space which enjoys the* S_p*-property,* $1 < p \leqslant \infty$. *If* $n < p$, *then*

$$\mathcal{P}^n(X) = \mathcal{P}^n_{wsC}(X).$$

The next result holds true, as ℓ_p and c_0 have properties S_p and S_∞ respectively.

Corollary 1.7.16 ([16], [54])**.** *Let* Γ *be any set,* $1 < p < \infty$ *and* $n \in \mathbb{N}$, $n < p$. *Then*

$$\mathcal{P}^n(\ell_p) = \mathcal{P}^n_{wu}(\ell_p),$$
$$\mathcal{P}(c_0) = \mathcal{P}_{wu}(c_0).$$

Conversely, if $n \geqslant p$, *then* $\sum_{j=1}^{\infty} x_j^n \in \mathcal{P}(^n\ell_p) \backslash \mathcal{P}_{wsc}(^n\ell_p)$.

Theorem 1.7.17 ([56])**.** *Let* X *be a normed linear space. The following are equivalent:*

(i) X *has the Dunford-Pettis property.*

(ii) $\mathcal{L}_{wK}(X;Y) \subset \mathcal{L}_{wsC}(X;Y)$ *for every Banach space* Y.

(iii) $\mathcal{L}_{wK}(^nX;Y) \subset \mathcal{L}_{wsC}(^nX;Y)$ *for every Banach space* Y *and every* $n \in \mathbb{N}$.

(iv) $\mathcal{P}_{wK}(X;Y) \subset \mathcal{P}_{wsC}(X;Y)$ *for every Banach space* Y.

The following result is a generalization of a well-known result (due to Aron and co-authors), which holds true for polynomials.

Theorem 1.7.18 ([20]). *Let* X, Y *be Banach spaces,* $\ell_1 \not\hookrightarrow X$ *and* $U \subset X$ *be a convex subset with non-empty interior. Then* $\mathcal{C}_w(U;Y) = \mathcal{C}_{wsc}(U;Y)$.

Theorem 1.7.19 ([20]). *Let* X, Y *be Banach spaces,* $\ell_1 \not\hookrightarrow X$ *and* $U \subset X$ *be a convex subset with non-empty interior. Then* $\mathcal{C}_{wu}(U;Y) = \mathcal{C}_{wsC}(U;Y)$.

Chapter 2

A corrigendum in the finite-dimensional setting

2.1 Contextualization

The main result of this thesis relies on [41], more precisely on the finite-dimensional quantitative Lemma 1.5.4 (Lemma 2 in the paper), which is also the principal tool for obtaining the results in [41] and which was obtained as a by-product of a new theory of algebraic bases for algebras of sub-symmetric polynomials on $\mathbb{R}^n$.

Unfortunately, the arguments in [41] contain a serious gap, which was recently spotted by our colleague Michal Johanis. More precisely, the power series on top of page 213 should have been correctly centered at the point $\left(x_1^0, \ldots, x_n^0\right)$, rather than at the origin. It is not clear to us at the present moment if this problem can be fixed, so the theory of algebraic bases developed in [41] remains to be only a conjecture.

In this chapter we give a different proof of the above-mentioned lemma. As a result, all the infinite dimensional applications stated in [41], as well as in several papers by various authors which have relied on our previous work (e.g. [28], [29]), remain valid. In fact, the strongest results concerning polynomial algebras are contained in the paper [21], which is also based on the lemma in question.

Let us now proceed with the corrected proof of Lemma 1.5.4.

2.2 Sub-symmetric polynomials on $\mathbb{R}^n$

Given $k, n \in \mathbb{N}$, $k \leqslant N$ and $\alpha \in \mathcal{I}^+(k,d)$, we define $P_\alpha^N \in \mathcal{P}\left({}^d\mathbb{R}^N\right)$ by

$$P_\alpha^N(x) = \sum_{1 \leqslant \rho_1 < \cdots < \rho_k \leqslant N} x_{\rho_1}^{\alpha_1} \cdots x_{\rho_k}^{\alpha_k}. \tag{2.1}$$

For $N \geqslant d$, the polynomials P_α^N, for $\alpha \in \mathscr{J}^+(d)$, form a linear basis of the space of subsymmetric d-homogeneous polynomials on $\mathbb{R}^N$. An important special case of these polynomials are the power sum symmetric polynomials $s_n^N(x) = P_{(n)}^N(x) = x_1^n + \cdots + x_N^n$.

Our main result concerns the properties of subsymmetric polynomials. However in its proof we need to work also with partial derivatives of the polynomials P_α^N and for this reason we consider also the polynomials P_α^N given by the formula (2.1), where $\alpha \in \mathcal{I}(k,d)$, $k \leqslant N$, using the convention that $x^0 = 1$ for every $x \in \mathbb{R}$.

We denote by $H^{n,K}\left(\mathbb{R}^N\right)$ the subspace of $\mathcal{P}^n\left(\mathbb{R}^N\right)$ generated by the polynomials P_α^N, $\alpha \in \bigcup_{d=0}^n \bigcup_{k=1}^K \mathcal{I}(k,d)$.

For formal reasons, we also put $P_\alpha^N = 0$ if $k > N$ and $P_\varnothing^N = 1$, both even for $N = 0$, further $\mathcal{I}(0,0) = \{\varnothing\}$ and $\mathbb{R}^0 = \{0\}$. Note that these definitions are consistent with (2.1), using the convention that a sum over an empty set is zero and a product over an empty set is equal to 1.

The following fact describes an important relation between the restriction of P_α^M to the first N coordinates and P_α^N. Note that for $M > N$ we consider canonically $\mathbb{R}^N$ as a subspace of $\mathbb{R}^M$.

Fact 2.2.1. Let $M, N, k, d \in \mathbb{N}_0$, $N < M$ and $\alpha \in \mathcal{I}(k,d)$ be such that $\alpha_m > 0$ and $\alpha_{m+1} = \cdots = \alpha_k = 0$ for some $0 \leqslant m \leqslant k$. Then

$$P_\alpha^M(x) = \sum_{j=m}^{k} \binom{M-N}{k-j} P_{(\alpha_1,\ldots,\alpha_j)}^N(x)$$

for every $x \in \mathbb{R}^N$. Conversely

$$P_\alpha^N(x) = \sum_{j=m}^{k} (-1)^{k-j} \binom{M-N+k-j-1}{k-j} P_{(\alpha_1,\ldots,\alpha_j)}^M(x)$$

for every $x \in \mathbb{R}^N$.

Proof. The first relation follows from the following (recall that $x \in \mathbb{R}^N$, i.e. $x_{N+1} = \cdots = x_M = 0$ as per the aforementioned convention):

$$\begin{aligned}
P_\alpha^M(x) &= \sum_{1 \leqslant \rho_1 < \cdots < \rho_k \leqslant M} x_{\rho_1}^{\alpha_1} \cdots x_{\rho_m}^{\alpha_m} \\
&= \sum_{\substack{1 \leqslant \rho_1 < \cdots < \rho_k \leqslant M \\ \rho_m \leqslant N}} x_{\rho_1}^{\alpha_1} \cdots x_{\rho_m}^{\alpha_m} \\
&= \sum_{j=m}^{k} \sum_{\substack{1 \leqslant \rho_1 < \cdots < \rho_k \leqslant M \\ \rho_j \leqslant N \leqslant \rho j + 1}} x_{\rho_1}^{\alpha_1} \cdots x_{\rho_m}^{\alpha_m} \\
&= \sum_{j=m}^{k} \binom{M-N}{k-j} P_{(\alpha_1,\ldots,\alpha_j)}^N(x).
\end{aligned}$$

The second relation can be proved by induction on $k - m$. For $k - m = 0$ it follows immediately from the first one. For the induction step we use the first relation together with the inductive hypothesis to obtain

$$\begin{aligned}
P_\alpha^N(x) &= P_\alpha^M(x) - \sum_{j=m}^{k-1} \binom{M-N}{k-j} P_{(\alpha_1,\ldots,\alpha_j)}^N(x) \\
&= P_\alpha^M(x) - \sum_{j=m}^{k-1} \binom{M-N}{k-j} \sum_{l=m}^{j} (-1)^{j-l} \binom{M-N+j-l-1}{j-l} P_{(\alpha_1,\ldots,\alpha_l)}^M(x) \\
&= P_\alpha^M(x) - \sum_{l=m}^{k-1} \left(\sum_{j=l}^{k-1} (-1)^{j-l} \binom{M-N}{k-j} \binom{M-N+j-l-1}{j-l} \right) P_{(\alpha_1,\ldots,\alpha_l)}^M(x)
\end{aligned}$$

and the result now follows from the identity

$$\sum_{j=l}^{k} (-1)^{j-l} \binom{M-N}{k-j} \binom{M-N+j-l-1}{j-l} = 0.$$

Adding or removing a couple of zero summands, this is equivalent to

$$\sum_{p=0}^{M-N} (-1)^{k-l-p} \binom{M-N}{p} \binom{M-N+k-l-p-1}{M-N-1} = 0,$$

which is the Fréchet formula for the polynomial

$$t \mapsto \binom{M-N+k-l-t-1}{M-N-1}$$

of degree $M - N - 1$ (see [39] or [45] for a more recent proof). □

It is very important to notice that the previous fact covers all the special cases like $N < k \leqslant M$, $k > M$, $N = 0$, $m = 0$ or $k = 0$. Observe also that in particular in the subsymmetric case (i.e. $\alpha \in \mathcal{I}^{+}(d)$) we have $P_{\alpha}^{M}\restriction_{\mathbb{R}^N} = P_{\alpha}^{N}$. Hence for sub-symmetric polynomials the superscript N can be dropped. We will use this simplification for the polynomials $s_n^N = s_n$.
The next fact deals with the situation when we fix the first N coordinates of P_{α}^{M}.

Fact 2.2.2. Let $N, d \in \mathbb{N}_0$, $M, k \in \mathbb{N}$, $N < M$, $k \leqslant M$, $\alpha \in \mathcal{I}(k,d)$ and $y \in \mathbb{R}^N$. Then the polynomial $(x_1, \ldots, x_{M-N}) \mapsto P_{\alpha}^{M}(y_1, \ldots, y_N, x_1, \ldots, x_{M-N})$ belongs to $H^{d,\min\{k,M-N\}}\left(\mathbb{R}^{M-N}\right)$.

Proof.

$$\begin{aligned}
& P_{\alpha}^{M}(y_1, \ldots, y_N, x_1, \ldots, x_{M-N}) \\
&= \sum_{j=0}^{k} \sum_{\substack{1 \leqslant \rho_1 < \cdots < \rho_k \leqslant M \\ \rho_j \leqslant N \leqslant \rho_{j+1}}} y_{\rho_1}^{\alpha_1} \cdots y_{\rho_j}^{\alpha_j} x_{\rho_{j+1}-N}^{\alpha_{j+1}} \cdots x_{\rho_k - N}^{\alpha_k} \\
&= \sum_{\substack{0 \leqslant j \leqslant k \\ k-(M-N) \leqslant j \leqslant N}} P_{(\alpha_1,\ldots,\alpha_j)}^{N}(y) P_{(\alpha_{j+1},\ldots,\alpha_k)}^{M-N}(x_1, \ldots, x_{M-N}).
\end{aligned}$$

□

Let $k, d \in \mathbb{N}$, $\alpha \in \mathcal{I}(k,d)$, $k \leqslant N$, $x \in \mathbb{R}^N$ and $1 \leqslant l \leqslant N$. Then

$$\begin{aligned}
\frac{\partial P_{\alpha}^{N}}{\partial x_l}(x) &= \frac{\partial}{\partial x_l}\left(\sum_{j=1}^{k} \sum_{\substack{1 \leqslant \rho_1 < \cdots < \rho_k \leqslant N \\ \rho_j = l}} x_{\rho_1}^{\alpha_1} \cdots x_{\rho_k}^{\alpha_k} \right) \\
&= \sum_{\substack{j=1 \\ \alpha_j > 0}}^{k} \sum_{\substack{1 \leqslant \rho_1 < \cdots < \rho_{j-1} < l \\ l < \rho_{j+1} < \cdots < \rho_k \leqslant N}} x_{\rho_1}^{\alpha_1} \cdots x_{\rho_{j-1}}^{\alpha_{j-1}} x_l^{\alpha_j - 1} x_{\rho_{j+1}}^{\alpha_{j+1}} \cdots x_{\rho_k}^{\alpha_k} \\
&= \sum_{\substack{j=1 \\ \alpha_j > 0}}^{k} \alpha_j P_{(\alpha_1,\ldots,\alpha_{j-1})}^{l-1}(x_1, \ldots, x_{l-1}) x_l^{\alpha_j - 1} P_{(\alpha_{j+1},\ldots,\alpha_k)}^{N-l}(x_{l+1}, \ldots, x_N).
\end{aligned}$$

These partial derivatives have the following useful property:

Fact 2.2.3. Let $k, d, N \in \mathbb{N}$, $\alpha \in \mathcal{I}(k,d)$, $k \leqslant N$. Then $\sum_{l=1}^{N} \frac{\partial P_{\alpha}^{N}}{\partial x_l}$ belongs to $H^{d-1,k}\left(\mathbb{R}^N\right)$.

Proof.

$$\begin{aligned}\sum_{l=1}^{N}\frac{\partial P_\alpha^N}{\partial x_l} &= \sum_{l=1}^{N}\sum_{\substack{j=1\\ \alpha_j>0}}^{k}\sum_{\substack{1\leqslant\rho_1<\cdots<\rho_{j-1}<l\\ l<\rho_{j+1}<\cdots<\rho_k\leqslant N}} x_{\rho_1}^{\alpha_1}\cdots x_{\rho_{j-1}}^{\alpha_{j-1}} x_l^{\alpha_j-1} x_{\rho_{j+1}}^{\alpha_{j+1}}\cdots x_{\rho_k}^{\alpha_k}\\ &= \sum_{\substack{j=1\\ \alpha_j>0}}^{k}\alpha_j\sum_{l=1}^{N}\sum_{\substack{1\leqslant\rho_1<\cdots<\rho_k\leqslant N\\ \rho_j=l}} x_{\rho_1}^{\alpha_1}\cdots x_{\rho_{j-1}}^{\alpha_{j-1}} x_l^{\alpha_j-1} x_{\rho_{j+1}}^{\alpha_{j+1}}\cdots x_{\rho_k}^{\alpha_k}\\ &= \sum_{\substack{j=1\\ \alpha_j>0}}^{k}\alpha_j P^N_{(\alpha_1,\ldots,\alpha_{j-1},\alpha_j-1,\alpha_{j+1},\ldots,\alpha_k)}(x).\end{aligned}$$

□

We note that this fact does not hold with $\mathcal{J}^+(k,d)$ and the space of sub-symmetric polynomials in place of $\mathcal{J}(k,d)$ and $H^{d-1,k}\left(\mathbb{R}^N\right)$: this is the sole reason for considering the larger spaces $H^{n,K}\left(\mathbb{R}^N\right)$.

For each $x\in\mathbb{R}^N$ we naturally identify $DP_\alpha^N(x)$ with the vector

$$\left(\frac{\partial P_\alpha^N}{\partial x_1}(x),\ldots,\frac{\partial P_\alpha^N}{\partial x_N}(x)\right)\in\mathbb{R}^N.$$

Fact 2.2.4. Let $M,N,k,d\in\mathbb{N}$, $M>N$, $\alpha\in\mathcal{J}(k,d)$, $k\leqslant N$ and $x\in\mathbb{R}^N$. Then $DP_\alpha^N(x)$ is a linear combination of vectors

$$DP_\beta^M(x)\restriction_{\mathbb{R}^N}=\left(\frac{\partial P_\beta^M}{\partial x_1}(x),\ldots,\frac{\partial P_\beta^M}{\partial x_N}(x)\right)\in\mathbb{R}^N,$$

where $\beta\in\cup_{m=1}^{k}\mathcal{J}(m,d)$.

Proof. Let $1\leqslant m\leqslant k$ be such that $\alpha_m>0$ and $\alpha_{m+1}=\cdots=\alpha_k=0$. Fix $1\leqslant l\leqslant N$. If $\alpha_j>0$, then $m\geqslant j$ and hence, by Fact 2.2.1:

$$P^{N-l}_{(\alpha_{j+1},\ldots,\alpha_k)}(x_{l+1},\ldots,x_N)=\sum_{s=m}^{k}c_s P^{M-l}_{(\alpha_{j+1},\ldots,\alpha_s)}(x_{l+1},\ldots,x_N,0,\ldots,0),$$

where $c_s=(-1)^{k-s}\begin{pmatrix}M-N+k-s-1\\ k-s\end{pmatrix}$.

Therefore, using Fact 2.2.2 and the fact that $\alpha_{s+1} = \cdots = \alpha_k = 0$, if $m \leqslant s \leqslant k$ we obtain

$$\begin{aligned}\frac{\partial P^N_\alpha}{\partial x_l} &= \sum_{\substack{j=1\\ \alpha_j>0}}^{k} \alpha_j P^{l-1}_{(\alpha_1,\ldots,\alpha_{j-1})}(x_1,\ldots,x_{l-1}) x_l^{\alpha_j - 1} \cdot \\ &\quad \cdot \sum_{s=m}^{k} c_s P^{M-l}_{(\alpha_{j+1},\ldots,\alpha_s)}(x_{l+1},\ldots,x_N,0,\ldots,0) \\ &= \sum_{s=m}^{k} c_s \frac{\partial P^M_{(\alpha_1,\ldots,\alpha_s)}}{\partial x_l}(x),\end{aligned}$$

from which the statement follows. □

We will also make use of the following version of the Lagrange multipliers theorem.

Theorem 2.2.5. *Let $G \subseteq \mathbb{R}^n$ be an open set, $f \in C^1(G)$, $F \in C^1\left(G;\mathbb{R}^M\right)$ and assume that F has a constant rank. If the function f has a local extremum with respect to $M = \{x \in G : F(x) = 0\}$ at $a \in M$, then $Df(a)$ is a linear combination of $DF_1(a),\ldots,DF_m(a)$, where $F_1,\ldots,F_m$ are the components of the mapping F.*

Proof. Let $k = \operatorname{rank} F(x)$ for $x \in G$. Since DF is continuous, we may without loss of generality assume that $DF_1(x),\ldots,DF_k(x)$ are linearly independent for each $x \in G$. From the Rank theorem it follows that there are C^1-smooth functions g_j of k variables, $j = k+1,\ldots,m$ and a neighbourhood U of a such that $F_j(x) = g_j\left(F_1(x),\ldots,F_k(x)\right)$ for each $x \in U$, $j = k+1,\ldots,m$ (see e.g. [62], Proposition 8.6.3.1). Notice that $g_j(0,\ldots,0) = g_j\left(F_1(a),\ldots,F_k(a)\right) = F_j(a) = 0$, $j = k+1,\ldots,m$. Therefore $M \cap U = \{x \in U : F_1(x) = 0,\ldots,F_k(x) = 0\}$ and we may use the classical version of the Lagrange multipliers theorem. □

Now we are ready to prove the key lemma.

Lemma 2.2.6. *For every $n, K \in \mathbb{N}$ there are $N \in \mathbb{N}$ and $u, v \in \mathbb{R}^N$ such that $P(u) = P(v)$ for every $P \in H^{n,K}\left(\mathbb{R}^N\right)$ but $s_{n+1}(u) \neq s_{n+1}(v)$.*

Proof. The proof is based on the observation that

$$\sum_{l=1}^{N} \frac{\partial s_{n+1}}{\partial x_l}(x) = (n+1)s_n(x),$$

which, together with Fact 2.2.3, leads to an inductive proof.

For each fixed $k \in \mathbb{N}$ we prove the statement by induction on n.

So fix $K \in \mathbb{N}$ and denote

$$\mathcal{M}(n) = \bigcup_{1 \leqslant d \leqslant n} \bigcup_{1 \leqslant k \leqslant K} \mathcal{I}(k,d).$$

The space $H^{n,K}\left(\mathbb{R}^N\right)$ is generated by a constant function and polynomials P_α^N, $\alpha \in \mathcal{M}(n)$.

For $n = 1$ the functions P_α^N, $\alpha \in \mathcal{M}(n)$ are linear, so there is $N \in \mathbb{N}$ large enough such that $\bigcap_{\alpha \in \mathcal{M}(n)} \ker P_\alpha^N$ contains a non-zero element u. Then it suffices to take $v = 2u$.

The inductive step from $n-1$ to n will be proven by contradiction. So assume that for each $N \geqslant K$ and each $u, v \in \mathbb{R}^N$ satisfying $P_\alpha^N(u) = P_\alpha^N(v)$ for all $\alpha \in \mathcal{M}(n)$ we have $s_{n+1}(u) = s_{n+1}(v)$.

Now let

$$F^N \colon \mathbb{R}^N \longrightarrow \mathbb{R}^{|\mathcal{M}(n)|}$$

be the mapping whose components are the polynomials P_α^N, $\alpha \in \mathcal{M}(n)$, in some fixed order and let $\mathbb{A}_N(x)$ be its Jacobi matrix at $x \in \mathbb{R}^N$, i.e.

$$\mathbb{A}_N(x) = \left(\frac{\partial P_\alpha^N}{\partial x_l}(x)\right)_{\left(\substack{\alpha \in \mathcal{M}(n) \\ l = 1, \ldots, M}\right)}.$$

Note that the number of rows of the matrix of functions $\mathbb{A}_N$ does not depend on N. Thus there is $N \geqslant K$ and $y \in \mathbb{R}^N$ such that

$$\operatorname{rank} \mathbb{A}_N(y) = r = \max_{\substack{M \geqslant K \\ x \in \mathbb{R}^M}} \operatorname{rank} \mathbb{A}_M(x).$$

By the inductive hypothesis, there are $M > N$ and $g, h \in \mathbb{R}^{M-N}$ such that $P(g) = P(h)$ for all $P \in H^{h-1,K}\left(\mathbb{R}^{M-N}\right)$ but $s_n(g) \neq s_n(h)$. If we denote by $\mathbb{A}_M(x)\restriction_N$ the matrix consisting of the first N columns of the matrix $\mathbb{A}_M(x)$, then

$$r = \operatorname{rank} \mathbb{A}_N(y) \leqslant \operatorname{rank} \mathbb{A}_M(y)\restriction_N \leqslant \operatorname{rank} \mathbb{A}_M(y) \leqslant r,$$

where the first inequality follows from Fact 2.2.4.
Let $w_1^M, \dots w_r^M$ be the rows of $\mathbb{A}_M$ such that $w_1^M \restriction_N(y), \dots w_r^M \restriction_N(y)$ are linearly independent. Using the continuity of the entries of $\mathbb{A}_M$ it is easy to see that there is a neighbourhood $U \subseteq \mathbb{R}^M$ of y such that, for each $x \in U$, the vectors $w_1^M \restriction_N(x), \dots w_r^M \restriction_N(x)$ are linearly independent: therefore they form a basis of the space spanned by the rows of $\mathbb{A}_M \restriction_N$. Clearly the same holds for $w_1^M(x), \dots, w_r^M(x)$ and $\mathbb{A}_M(x)$.
Fix an arbitrary $z \in U$ and put

$$S = \left\{ x \in U : P_\alpha^M(x) = P_\alpha^M(z), \alpha \in \mathcal{M}(n) \right\}.$$

By our assumption, s_{n+1} is constant on S and so Theorem 2.2.5 implies that $Ds_{n+1}(z)$ is a linear combination of the rows of $\mathbb{A}_M(z)$. It follows that, for each $z \in U$, the vector $Ds_{n+1}(z)$ is a linear combination of $w_1^M(z), \dots, w_r^M(z)$.
Next, we put

$$u = y + c \sum_{j=1}^{M-N} g_j e_{N+j}, \; v = \sum_{j=1}^{M-N} h_j e_{N+j}$$

for some suitable $c \neq 0$ so that $u, v \in U$. Notice that, since $H^{n-1,K}\left(\mathbb{R}^{M-N}\right)$ is generated by homogeneous polynomials, we still have $P(cg) = P(ch)$ for all $P \in H^{n-1,K}\left(\mathbb{R}^{M-N}\right)$ but $s_n(cg) \neq s_n(ch)$. For a fixed $\alpha \in \mathcal{M}(n)$ and $1 \leqslant l \leqslant N$ consider the polynomial

$$P(x) = \frac{\partial P_\alpha^M}{\partial x_l}(y_1, \dots, y_N, x_1, \dots, x_{M-N}).$$

Then, by Fact 2.2.2, we have $P \in H^{n-1,K}\left(\mathbb{R}^{M-N}\right)$ and so $P(cg) = P(ch)$.
Therefore

$$w_j^M(u) \restriction_N = w_j^M(v) \restriction_N, \quad j = 1, \dots, r. \tag{2.2}$$

We have $Ds_{n+1}(u) = \sum_{j=1}^r \lambda_j w_j^M(u)$ and $Ds_{n+1}(v) = \sum_{j=1}^r \mu_j w_j^M(v)$ for some $\lambda_j, \mu_j \in \mathbb{R}$ and of course the same holds when we restrict to the first N coordinates of all of these vectors.
But since $Ds_{n+1}(u) \restriction_N = (n+1)(y_1^n, \dots, y_N^n) = Ds_{n+1}(v) \restriction_N$, combined with (2.2) and the fact that $w_1^M(u) \restriction_N, \dots, w_r^M(u) \restriction_N$ are linearly independent, we obtain $\mu_j = \lambda_j$, $j = 1, \dots, r$.

Eventually, from Fact 2.2.3 and Fact 2.2.2 it follows that

$$x \mapsto \sum_{l=1}^{M} w_j^M \left(y + \sum_{j=1}^{M-N} x_j e_{N+j} \right)_l \in H^{n-1,K}\left(\mathbb{R}^{M-N}\right), \quad j = 1, \ldots, r.$$

Therefore

$$\begin{aligned} (n+1)s_n(u) &= \sum_{l=1}^{M} \frac{\partial s_{n+1}}{\partial x_l}(u) = \sum_{j=1}^{r} \lambda_j \sum_{l=1}^{M} w_j^M(u)_l \\ &= \sum_{j=1}^{r} \lambda_j \sum_{l=1}^{M} w_j^M(v)_l = \sum_{l=1}^{M} \frac{\partial s_{n+1}}{\partial x_l}(v) \\ &= (n+1)s_n(v). \end{aligned}$$

Since $s_n(u) = s_n(y) + s_n(cg)$ and $s_n(v) = s_n(y) + s_n(ch)$, we get $s_n(cg) = s_n(ch)$, which is a contradiction. □

2.3 The corrected proof

Corollary 2.3.1. *For every $n \in \mathbb{N}$ there exist $N \in \mathbb{N}$ and $\varepsilon > 0$ such that for every $M \geqslant N$*

$$\sup_{x \in B_{\ell_1^M}} |p(x) - s_{n+1}(x)| \geqslant \varepsilon$$

for every p from the algebra generated by the sub-symmetric polynomials on $\mathbb{R}^M$ of degree at most N.

Proof. Applying Lemma 2.2.6 to $K = n$ we obtain $N \in \mathbb{N}$ and $u, v \in B_{\ell_1^N}$ such that $P(u) = P(v)$ for every $P \in H^{n,n}\left(\mathbb{R}^N\right)$ but $s_{n+1}(u) \neq s_{n+1}(v)$. We put $\varepsilon = \frac{1}{2}|s_{n+1}(u) - s_{n+1}(v)|$. Let $M \geqslant N$. Since all sub-symmetric polynomials from $\mathcal{P}^n\left(\mathbb{R}^N\right)$ are contained in $H^{n,n}\left(\mathbb{R}^N\right)$, from the remark after Fact 2.2.1 it follows that in particular $P(u) = P(v)$ for every sub-symmetric $P \in \mathcal{P}^n\left(\mathbb{R}^M\right)$. We conclude that $p(u) = p(v)$ for every p from the algebra generated by the sub-symmetric polynomials from $\mathcal{P}^n\left(\mathbb{R}^M\right)$. The statement now easily follows. □

Chapter 3

The main theorem

3.1 Introduction

Generally speaking, with a single exception when $\mathcal{P}(X) = \mathcal{P}_{wu}(X)$, there are no results giving a characterization of the uniform closure $\overline{\mathcal{P}(X)}^{\tau_b}$ in any infinite-dimensional space. The refinement of the problem is finding the characterization of $\overline{\mathcal{A}_n(X)}$ and this is wide open as well. The results in this section focus on the natural question when $\overline{\mathcal{A}_n(X)} = \overline{\mathcal{A}_{n+1}(X)}$ (the inclusion $\subset$ always holds). We are going to use the theory of sub-symmetric polynomials previously developed together with the asymptotic approach to polynomial behaviour to obtain rather general results showing that the inclusion $\supset$ is almost never satisfied.
We begin by formulating a positive result.

Theorem 3.1.1 ([2], [8])**.** *Let X, Y be Banach spaces, Then* $\overline{\mathcal{P}_f(X;Y)}^{\tau_b} = \mathcal{C}_{wu}(B_X;Y)$.

Proposition 3.1.2. *Let X be a Banach space such that it does not contain ℓ_1 and $\mathcal{P}\left({}^nX\right) = \mathcal{P}_{wsc}\left({}^nX\right)$. Then*

$$\mathcal{A}_1(X) = \mathcal{A}_2(X) = \cdots = \mathcal{A}_n(X).$$

Proof. The following chain of equalities holds true:

$$\mathcal{P}_w\left({}^nX\right) \underset{\text{Cor. 1.7.3}}{=} \mathcal{P}_{wu}\left({}^nX\right) \underset{\text{Teo. 1.7.18}}{=} \mathcal{P}_{wsc}\left({}^nX\right) \underset{\text{hyp.}}{=} \mathcal{P}\left({}^nX\right).$$

Thus

$$\mathcal{P}\left({}^{n}X\right) = \mathcal{P}_{wu}\left({}^{n}X\right) \subset \mathcal{C}_{wu}\left({}^{n}X\right) \underset{\text{Teo. 3.1.1}}{=} \overline{\mathcal{P}_{f}(X;Y)}^{\tau_b} \underset{\text{easy}}{\subset} \overline{\mathcal{A}_1(X)}.$$

Since $\overline{\mathcal{A}_1(X)}$ is an algebra, $\mathcal{A}_n(X) \subset \overline{\mathcal{A}_1(X)}$, from which the result follows.

□

All our results, which we are now going to present, go in the opposite direction and rely on the useful criterion investigated in Theorem 1.5.5.

3.2 The proof

Theorem 3.2.1. *Let X be a Banach space, and m be the minimal integer such that there is a non-compact $P \in \mathcal{P}({}^{m}X;\ell_1)$. Then $n \geqslant m$ implies $\mathcal{P}({}^{n}X) \not\subset \overline{\mathcal{A}_{n-1}(X)}$.*

Proof. If $\ell_1 \hookrightarrow X$ then it suffices to combine Theorem 1.5.5 and Proposition 1.7.7. For the rest of the proof we assume that $\ell_1 \not\hookrightarrow X$. Denote $\{f_j\}_{j=1}^{\infty}$ the canonical basis in ℓ_1, $P = (P_k)_{k=1}^{\infty} \in \mathcal{P}({}^{m}X;\ell_1)$, $P_k \in \mathcal{P}({}^{m}X)$.

We claim that by performing some adjustments to P, we may assume in addition that there exists a weakly null normalized basic sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ such that $P(x_j) = f_j$ for each j.

To this end, note that by Lemma 1.7.8 there exists a weakly null sequence $\{y_k\}_{k=1}^{\infty}$ in X such that $\{P(y_k)\}_{k=1}^{\infty}$ is not relatively compact, i.e. it contains a separated subsequence, which we call again $\{P(y_k)\}_{k=1}^{\infty}$. By [1] p. 22, by passing to a subsequence, we may assume that $\{y_k\}$ is a normalized basic sequence. As ℓ_1 is a Schur space, $\{P(y_k)\}_{k=1}^{\infty}$ contains no weakly null subsequences. By Rosenthal's ℓ_1 theorem, $\{P(y_k)\}_{k=1}^{\infty}$ contains a subsequence, again $\{P(y_k)\}$, equivalent to the ℓ_1-basis. By a well-known result (according to Bill Johnson, who has pointed out to us some very closely related other results), every sequence in ℓ_1, which is equivalent to the ℓ_1-basis, contains a further subsequence which spans a complemented subspace. Since we have been unable to find this result explicitly in the literature, let us indicate the idea of proof. Supposing that $\{z_k\}$ is the ℓ_1-basic sequence in ℓ_1, we may assume, by passing to a subsequence, that z_k is pointwise convergent to $u_0 \in \ell_1$ and that there exists a sequence of disjoint block vectors $\{u_k\}$

such that $\sum_k |z_k - u_0 - u_k| < \infty$. The case when $u_0 = 0$ is well-known [36] Prop. 4.45, so let us assume the contrary. Moreover, we may assume that the the norms of u_0 restricted to the supports of u_k form a fast decreasing sequence. Then, by the classical results [36] Thm. 4.23, Prop. 4.45, we have that the sequence $\{u_k\}_{k=0}^{\infty}$ is equivalent to the ℓ_1-basis, which is moreover complemented in ℓ_1. Hence, $u_0, z_1 - u_0, z_2 - u_0 \dots$ is also equivalent to a complemented ℓ_1 basis in ℓ_1. To finish, it suffices to find a suitable projection in this latter space, which takes $(a_0, a_1, \dots) \to (\sum_{k=1}^{\infty} a_k, a_1, a_2, \dots)$.

Hence there exists a weakly null and normalized basic sequence $\{x_k\} = \{y_{n_k}\} \subseteq \{y_n\}$ such that $P(x_k) = g_k$, where $\{g_k\}$ is a equivalent to an ℓ_1 basis which spans a complemented subspace in ℓ_1. Hence, composing P with the appropriate projection in ℓ_1, we substitute ℓ_1 with its complemented subspace $[g_k]_{k=1}^{\infty}$, and the claim follows.

Let $\{\phi_j\}_{j=1}^{\infty}$ be a bounded sequence in X^*, biorthogonal to $\{x_j\}_{j=1}^{\infty}$. We are mostly going to be interested in the behaviour of P restricted to $Y = \operatorname{span}\{x_j : j \in \mathbb{N}\} \hookrightarrow X$. For the sake of convenience, set $Y_{\{j: j \geqslant k+1\}} := \operatorname{span}\{x_j : j \geqslant k+1\}$. Note that we have $P(\lambda x_j) = \lambda^m f_j$. Formula (1.5) for the restriction of P to Y can be rewritten, by collecting the appropriate finitely many terms, into the following formula, which holds for all finitely supported vectors $x = \sum a_j x_j \in Y$:

$$P_k\left(\sum_{j=1}^{\infty} a_j x_j\right) = \sum_{\substack{p+q+r=m \\ \alpha \in \mathcal{I}(k-1,p)}} (a_1, \dots, a_{k-1})^{\alpha} a_k^q S_k^{\alpha,q,r}\left(\sum_{j=k+1}^{\infty} a_j x_j\right), \tag{3.1}$$

where $S_k^{\alpha,q,r} \in \mathcal{P}({}^r Y_{\{j: j \geqslant k+1\}})$. Note that, by the minimality assumption on m, for a fixed $0 \neq \beta = (\beta_1, \dots) \in \mathcal{I}(\infty, t)$, $t \leqslant p < m$ where $\beta_i = 0$, $i > k-1$, we have that

$$\frac{\partial^t}{\partial^{\beta_1} x_1 \dots \partial^{\beta_{k-1}} x_{k-1}} P = \left(\frac{\partial^t}{\partial^{\beta_1} x_1 \dots \partial^{\beta_{k-1}} x_{k-1}} P_j\right) : X \to \ell_1$$

is a compact $(m-t)$-homogeneous polynomial with range in ℓ_1. So, for a fixed β of the aforementioned type,

$$\lim_{j \to \infty} \left\| \frac{\partial^t}{\partial^{\beta_1} x_1 \dots \partial^{\beta_{k-1}} x_{k-1}} P_j \right\| = 0.$$

For $y = \sum_{i=1}^{k-1} a_i x_i \in [x_1, \dots, x_{k-1}]$ and $l > k-1$,

$$\frac{\partial^t}{\partial^{\beta_1} x_1 \dots \partial^{\beta_{k-1}} x_{k-1}} P_l(y + \sum_{j=l}^{\infty} a_j x_j)$$
$$= \sum_{\substack{p+q+r=m \\ \alpha \geqslant \beta \\ \alpha \in \mathcal{I}(k-1,p)}} \frac{\alpha!}{(\alpha-\beta)!} (a_i)^{\alpha-\beta} a_l^q S_l^{\alpha,q,r} \left(\sum_{j=l+1}^{\infty} a_j x_j \right).$$

We claim that for a fixed $0 \neq \beta = (\beta_1, \dots) \in \mathcal{I}(\infty, t)$, $t \leqslant m$ where $\beta_i = 0$, $i > k-1$, q, r such that $t + q + r = m$, we have that

$$\lim_{l \to \infty} \left\| S_l^{\beta,q,r} \right\|_{Y_{\{j : j \geqslant l+1\}}} = 0. \tag{3.2}$$

For the proof of the claim by contradiction, choose a maximal $\beta \in \mathcal{I}(k-1, t)$ which fails (3.2). Hence, for any (if it exists) $\alpha \in \mathcal{I}(k-1, p)$, $p > t$, q, r such that $p + q + r = m$,

$$\lim_{l \to \infty} \left\| S_l^{\alpha,q,r} \right\|_{Y_{\{j : j \geqslant l+1\}}} = 0. \tag{3.3}$$

Passing to a suitable subsequence of $l \to \infty$ (for simplicity assuming it is still indexed by $\mathbb{N}$) we conclude that there exists a normalized sequence of $v_l \in Y_{\{j : j \geqslant l+1\}}$ such that $b^{\alpha,q,r} = \lim_{l \to \infty} S_l^{\alpha,q,r}(v_l)$ exist for all α, q, r, and there is at least one non-zero term (with $\alpha = \beta$) among them. Moreover, if $\alpha \in \mathcal{I}(k-1, p)$, where $p > t$, then $b^{\alpha,q,r} = 0$. That means that, for a suitably chosen $y = \sum_{i=1}^{k-1} a_i x_i \in [x_1, \dots, x_{k-1}]$ and $a_l \in \mathbb{R}$, we have

$$\lim_{l \to \infty} \frac{\partial^t}{\partial^{\beta_1} x_1 \dots \partial^{\beta_{k-1}} x_{k-1}} P_l(y + a_l x_l + v_l) = \sum_{q+r=m-t} \beta! a_l^q b^{\beta,q,r} \neq 0,$$

which contradicts the minimality of m.

Fix an arbitrary sequence $\delta_k \searrow 0$. By passing to a fast enough growing subsequence of $\{x_j\}$ we can disregard in (3.1) all terms with $p \geqslant 1$, so that (using the short notation $S_k^{q,r} = S_k^{0,q,r}$)

$$\sup_{\left\| \sum_{j=1}^{\infty} a_j x_j \right\| \leqslant 1} \left\| P_k \left(\sum_{j=1}^{\infty} a_j x_j \right) - \sum_{q+r=m} a_k^q S_k^{q,r} \left(\sum_{j=k+1}^{\infty} a_j x_j \right) \right\| \leqslant \delta_k. \tag{3.4}$$

Let $\{\varepsilon_n^k\}_{n=1}^\infty, \varepsilon_n^k \searrow 0$, be decreasing sequences of real numbers and $\{N_k(j)\}_{j=1}^\infty$ be an increasing sequence of natural numbers. To start with, by applying Theorem 1.4.11, we may assume that $\{x_n\}_{n=1}^\infty$ is a characteristic sequence of its spreading model E with a sub-symmetric basis $\{e_n\}_{n=1}^\infty$.

By a repeated application of Theorem 1.4.11, there are nested subsequences $\mathbb{N} \supset M_1 \supset M_2 \supset \cdots$ of index sets so that the following holds: for a subsequence $\{x_n\}_{n\in M_k}$ of $\{x_n\}_{n=1}^\infty$, there is a sub-symmetric polynomial $R_k^{q,r} \in \mathcal{P}({}^rE)$, r and q such that, for all scalars a_j, $j = 1, \ldots, N_k(K)$,

$$R_k^{q,r}\left(\sum_{j=1}^{N_k(K)} a_je_j\right) - \varepsilon_K^k \leqslant S_k^{q,r}\left(\sum_{j=1}^{N_k(K)} a_jx_{n_j}\right) \leqslant R_k^{q,r}\left(\sum_{j=1}^{N_k(K)} a_je_j\right) + \varepsilon_K^k, \tag{3.5}$$

provided $K \leqslant n_1 < \cdots < n_{N_k(K)}$, $n_j \in M_k$ and $\left\|\sum_{j=1}^{N_k(K)} a_jx_{n_j}\right\| \leqslant 1$.

By (1.7),

$$R_k^{q,r}\left(\sum_{j=1}^{\infty} a_je_j\right) = \sum_{\alpha\in\mathcal{J}(r)} a_\alpha^{q,r,k}P_\alpha\left(\sum_{j=1}^{\infty} a_je_j\right) \tag{3.6}$$

for all finitely supported vectors.

By passing to a suitable diagonal sequence $M = \{m_i\}_{i=1}^\infty$ of the system $\{M_k\}_{k=1}^\infty$ and keeping in mind that the set $\{R_k^{q,r}\}_{k,q,r}$ in $\mathcal{P}(E)$ is uniformly bounded, we may also assume that there exist finite limits

$$b_\alpha^{q,r} = \lim_{k\to\infty} a_\alpha^{q,r,k}, \quad k \in M. \tag{3.7}$$

We consider the sub-symmetric polynomial $W^{q,r} \in \mathcal{P}({}^rE)$ defined by

$$W^{q,r}\left(\sum_{j=1}^{\infty} a_je_j\right) = \sum_{\alpha\in\mathcal{J}(r)} b_\alpha^{q,r}P_\alpha\left(\sum_{j=1}^{\infty} a_je_j\right). \tag{3.8}$$

We claim that $W^{q,r} = 0$, unless $r = 0$. Assuming the contrary, there is a finitely supported vector $v = \sum_{i=1}^T v_ie_i$ such that

$$\sum_{q+r=m, r\geqslant 1} W^{q,r}(v) = \delta \neq 0.$$

We may assume without loss of generality that $\delta > 0$. Hence, for a sliding finitely supported block vector $w_j = \sum_{i=1}^T v_{i+j}x_{m_{i+j}}$, by (3.5), (3.6), and

(3.7), we get that

$$\liminf_{j} \sum_{q+r=m, r\geqslant 1} S_l^{q,r}(w_j) > \frac{\delta}{2} \tag{3.9}$$

holds for all $l \in M$ large enough. But this contradicts again the minimality assumption on m. Indeed, we denote by U a w^*-cluster point of $\{P(x+w_k) : k \in \mathbb{N}\}$ in the dual Banach space $\mathcal{P}^m(X;\ell_1)$. In particular, for every x there is a subsequence $L \subset \mathbb{N}$ such that

$$U(x) = \lim_{j\to\infty, k\in L} P(x+w_j).$$

Let $U = U^0 + U^1 + \cdots + U^m = (U_k^0 + U_k^1 + \cdots + U_k^m)_{k=1}^\infty$ be the unique splitting of U into a sum of j-homogeneous summands U^j. Then by (3.9)

$$\sum_{i=0}^{m-1} U_k^i(a_k x_k) \geqslant \sum_{q+r=m, r\geqslant 1} a_k^q W^{q,r}(v) \geqslant \frac{\delta}{2},$$

a contradiction with the minimality of m.
This verifies the claim that $W^{q,r} = 0$, unless $r = 0$.
Combining all the previous results, we conclude that there is an infinite increasing sequence $M \subset \mathbb{N}$ and $c \neq 0$, such that the following holds: for any $\rho > 0$ and $N \in \mathbb{N}$, there is a finite set $\{t_1, \ldots, t_N\} \subset M$ such that

$$\left\| P_k\left(\sum_{i=1}^N a_j x_{t_i}\right) - c a_k^m \right\| < \rho, \quad k \in \{t_1, \ldots, t_N\}.$$

It is now clear that the polynomial $Q \in \mathcal{P}(^{m+l}X)$, $l \geqslant 0$, defined as

$$Q(x) = \sum_{j=1}^\infty \phi_j^l(x) P_j(x),$$

satisfies the condition laid out in Theorem 1.5.5, whence

$$Q \in \mathcal{P}(^{m+l}X) \backslash \overline{\mathcal{A}_{m+l-1}(X)}.$$

□

3.3 Corollaries

In this section we present several previously known results in this area, all implied by Theorem 3.2.1 together with the positive results of [2], [8] and [6] below.

Corollary 3.3.1 ([6], see Theorem 3.1.1)**.** *Let X, Y be Banach spaces and suppose that X does not contain a subspace isomorphic to ℓ_1. Then* $\mathcal{P}_{wu}({}^nX;Y) = \mathcal{P}_{wsc}({}^nX;Y)$.

The next result was first formulated in [41].

Theorem 3.3.2. *Let X be a Banach space, $\ell_1 \hookrightarrow X$. Then*

$$\overline{\mathcal{A}_1(X)} \subsetneqq \overline{\mathcal{A}_2(X)} \subsetneqq \cdots$$

Proof. Combine Proposition 1.7.7 and Theorem 1.5.5. □

Corollary 3.3.3 ([41])**.** *Let X be a Banach space admitting a non-compact linear operator $T \in \mathcal{L}(X;\ell_p)$, $p \in [1,\infty)$. Then, letting $n = \lceil p \rceil$, we obtain*

$$\overline{\mathcal{A}_n(X)} \subsetneqq \overline{\mathcal{A}_{n+1}(X)} \subsetneqq \cdots . \tag{3.10}$$

Proof. By Proposition 1.7.9, we may assume that $T(B_X)$ contains the unit vectors in ℓ_p. It then suffices to compose T with the polynomial $P \in \mathcal{P}({}^n\ell_p;\ell_1)$, given by formula $(x_j) \to (x_j^n)$, to obtain a non-compact n-homogeneous polynomial from X into ℓ_1. It remains to apply Theorem 3.2.1. □

Corollary 3.3.4. *Let $X = L_p([0,1])$, $1 \leqslant p \leqslant \infty$, or $X = \ell_\infty$ or $X = C(K)$, where K is a non-scattered compact. Then*

$$\overline{\mathcal{A}_1(X)} \subsetneqq \overline{\mathcal{A}_2(X)} \subsetneqq \cdots$$

Proof. If $1 < p < \infty$, ℓ_2 is isomorphic to a complemented subspace of $L_p([0,1])$ ([36] p. 210), therefore we may use Corollary 3.3.3. The spaces $L_1([0,1])$, ℓ_∞, $L_\infty([0,1])$ and $\mathcal{C}(K)$, K non-scattered, contain ℓ_1 ([36]), therefore Theorem 3.3.2 applies. □

Corollary 3.3.5. *Given $1 \leqslant p < \infty$, we have the following:*

$$\overline{\mathcal{A}_1(\ell_p)} = \cdots = \overline{\mathcal{A}_{n-1}(\ell_p)} \subsetneqq \overline{\mathcal{A}_n(\ell_p)} \subsetneqq \overline{\mathcal{A}_{n+1}(\ell_p)} \subsetneqq \cdots$$

where $n - 1 < p \leqslant n$.

Proof. By [6] we know that $\mathcal{P}^n(\ell_p) = \mathcal{P}^n_{wu}(\ell_p)$ whenever $n < p$. So, using Theorem 3.1.1, we obtain that $\overline{\mathcal{A}_{n-1}(\ell_p)} = \overline{\mathcal{A}_1(\ell_p)}$. The rest follows readily from Corollary 3.3.3. □

Corollary 3.3.6. *Let X be a Banach space, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Assume that X^* has type q. Then for $n > p$ we have*

$$\overline{\mathcal{A}_1(X)} \subsetneqq \overline{\mathcal{A}_n(X)} \subsetneqq \overline{\mathcal{A}_{n+1}(X)} \subsetneqq \cdots .$$

Proof. By Corollary 1.7.12, there is a normalized basic sequence $\{y_k\}_{k=1}^\infty$ in X^* which has the upper q-estimate. Thus, $T\colon \ell_q \to X^*$, $T(e_k) = y_k$, is a non-compact bounded linear operator. Since T is weakly compact, $T^*\colon X \to \ell_p$ is a non-compact operator. An appeal to Lemma 3.3.3 finishes the argument. □

Corollary 3.3.7 ([29])**.** *Let X be a Banach space with an unconditional FDD, $\ell_1 \not\hookrightarrow X$, and suppose that n is the least integer such that there exists a $P \in \mathcal{P}(^nX)$ which is not weakly sequentially continuous. Then*

$$\overline{\mathcal{A}_1(X)} = \cdots = \overline{\mathcal{A}_{n-1}(X)} \subsetneqq \overline{\mathcal{A}_n(X)} \subsetneqq \overline{\mathcal{A}_{n+1}(X)} \subsetneqq \cdots .$$

Proof. It was shown in [29], by using the averaging technique from [9] as in [28], that under these assumptions $c_0 \hookrightarrow \mathcal{P}(^nX)$. □

3.4 Some open problems

In this section we list the main remaining open problems, which we have so far failed to solve, in spite of trying several approaches.

Problem 3.4.1. *Give a description of $\overline{\mathcal{P}(X)}$ for a general separable Banach space X.*

This problem is open even for $X = \ell_2$!

Problem 3.4.2. *Suppose that X does not contain ℓ_1 and $\mathcal{P}(X) \neq \mathcal{P}_{wsc}(X)$. Is there a non-compact bounded linear operator from X into ℓ_p for some $1 \leqslant p < \infty$?*

An important remaining problem is the following.

Problem 3.4.3. *Let X be a separable Banach space not containing ℓ_1 and let $n \in \mathbb{N}$ be the smallest integer such that $\mathcal{P}\left({}^nX\right) \neq \mathcal{P}_{wsc}\left({}^nX\right)$. Is then*

$$\overline{\mathcal{A}_1(X)} = \cdots = \overline{\mathcal{A}_{n-1}(X)} \subsetneq \overline{\mathcal{A}_n(X)} \subsetneq \overline{\mathcal{A}_{n+1}(X)} \subsetneq \cdots ?$$

The answer to this problem is positive provided the following problem has a positive answer (see [44]).

Problem 3.4.4. *Let X be a separable Banach space not containing ℓ_1 and let $n \geqslant 2$ be an integer such that $\mathcal{P}\left({}^nX\right) \neq \mathcal{P}_{wsc}\left({}^nX\right)$. Does then the space $\mathcal{P}\left({}^nX\right)$ contain c_0?*

Note that the opposite implication follows from Theorem 1.7.5.

Observe that if the dual X^* contains a subspace isomorphic to c_0 or a superreflexive space, then we can conclude that (3.10) holds for some n. Indeed, in this case either $\ell_1 \hookrightarrow X$ or, by using the James-Gurarii theorem ([36] p. 450), X admits a non-compact linear operator into some ℓ_p. This leaves us with two possibilities. If X fails (3.10) for every $n \in \mathbb{N}$, then either X^* is ℓ_1-saturated or it contains a Tsirelson-like subspace Y, in the sense that Y contains no copy of ℓ_1, c_0 or a superreflexive space.

Problem 3.4.5. *Let X be a Banach space such that X^* is ℓ_1-saturated. Is then $\mathcal{P}(X) = \mathcal{P}_{wsc}(X)$?*

Chapter 4

Some results on smooth functions

In this chapter we solve several open problems from the literature regarding the behavior of smooth functions on Banach spaces.

4.1 Non-complete C^k-smooth renormings

We begin with a problem posed in various papers, e.g. in [12] or [10], concerning the existence of a non-complete C^k-smooth renorming of a Banach space which admits a C^k-smooth equivalent norm, where $k \geqslant 2$. The non-complete C^k-smooth renorming plays an important role in some applications regarding the so-called smooth negligibility and the existence of C^k-smooth diffeomorphisms between certain subsets of the given Banach space X, see e.g. [31]. Our result can be used to simplify some parts of the theory of these mappings, in particular the techniques which bypass the use of the non-complete norm, used in [10], are no longer needed. We begin with an auxiliary result.

Theorem 4.1.1. *Let X be a Banach space with w^*-sequentially compact dual ball. If $c_0 \cong Y \hookrightarrow X$ then Y contains a further subspace $c_0 \cong Z \hookrightarrow Y$ such that Z is complemented in X.*

Proof. If $c_0 \hookrightarrow X$ then X^* has a quotient ℓ_1. By the lifting property, we also have $\ell_1 \hookrightarrow X^*$ is a complemented subspace, and moreover, the basis

$\{e_j\}$ of c_0 and $\{f_j\}$ of ℓ_1 in X^* form a biorthogonal system. Since B_{X^*} is w^*-sequentially compact, by passing to a subsequence we get that $f_j \to f$ in w^*-topology. So $\{g_{2j}\} = \{f_{2j} - f_{2j+1}\}$ is w^*-null, and also equivalent to an ℓ_1-basis, which is still biorthogonal to $\{e_{2j}\}$, again a c_0 basis sequence. Thus $T : X \to X$, $T(x) = \sum g_{2j}(x)e_{2j}$ is a projection, and c_0 is a complemented subspace of X. □

In fact, a more general version of the above result was shown by Schlumprecht in his PhD thesis [58]. The condition on X is quite common, e.g. all weak Asplund spaces or WLD spaces have it ([26], [35], [36]).

Theorem 4.1.2. *Let X be an infinite dimensional Banach space admitting a C^k-smooth norm, $k \geqslant 2$. Then X admits a decomposition $X = Y \oplus Z$ where Y is infinite dimensional and separable. In particular, X admits a non-complete C^k-smooth renorming.*

Proof. By Corollary 3.3 in [24] we have that either $c_0 \hookrightarrow X$ or X is superreflexive. Either way, using the previous Theorem 4.1.1 (or the existence of PRI on superreflexive spaces), $X = Y \oplus Z$ where Y is infinite dimensional and separable. But since every separable Banach space injects into c_0, it admits a non-complete C^∞-smooth norm. It follows that X admits a C^k-smooth noncomplete norm. □

We point out that for $k = 1$ the existence of a (nonequivalent) C^1-smooth norm on a given C^1-smooth Banach space (or even any Asplund space) X remains open.

4.2 Separating polynomials

In this section we present a theorem which solves a problem posed in [11], concerning an assumption used by these authors in the proof of their main result. Before we pass to the description of our result, let us recall that for every real Banach space X one may construct its complexified version $X^{\mathbb{C}}$, which is (as a real Banach space) isomorphic to $X \oplus X$. The complex norm on $X^{\mathbb{C}}$ is not uniquely determined, but this fact plays no role in our argument. We refer to the paper [52] for details.

Theorem 4.2.1. *Let X be a real Banach space which admits a real analytic separating function whose complex extension exists and is Lipschitz on some strip around X, i.e. on $X + 2rB_{X^{\mathbb{C}}} \subset X^{\mathbb{C}}$, for some $r > 0$. Then X is superreflexive and admits a separating polynomial.*

Proof. By contradiction. Let $f : B_X \to \mathbb{R}$ be a separating real analytic and Lipschitz function with $f(0) = 0, df(0) = 0$ and $\inf_{S_X} f > 1$, and such that the complex extension $\tilde{f} : B_X + rB_{X^{\mathbb{C}}} \to \mathbb{C}$ exists and is K-Lipschitz, $r > 0$. Denote $S = B_X + rB_{X^{\mathbb{C}}} \subset X^{\mathbb{C}}$. This implies that $\tilde{f}$ is bounded by $K + r$ on S.

By the Cauchy formula [55] Thm. 10.28 (for the second derivative of $\tilde{f}$)

$$d^2\tilde{f}(a)[h] = \frac{2}{2\pi i}\int_\gamma \frac{\tilde{f}(a+\zeta h)}{\zeta^3} d\zeta \tag{4.1}$$

holds for every $a, h \in B_X$, and the path $\gamma(t) = re^{it}$, $t \in [0, 2\pi]$. Noting that the denominator in the Cauchy formula is in absolute value r^3, we obtain that $d^2f(a)$ is uniformly bounded on B_X. Hence df is Lipschitz on B_X.

By a result of Fabian, Whitfield and Zizler in [37], Theorem 3.2 in [24] X is superreflexive. By a result of Deville, Thm 4.1 in [24] X has a separating polynomial. □

4.3 Extension of uniformly differentiable functions

In the last part of this section we give a solution to an extension problem, posed in the monograph of Benyamini and Lindenstrauss [14] p. 278, concerning uniformly differentiable functions on the unit ball of a Banach space X. Suppose that $f : B_X \to \mathbb{R}$ is a uniformly differentiable function in the interior of the unit ball B_X. Is there a uniformly smooth extension of f whose domain is the whole X, or at least some neighbourhood of B_X? A weaker version of this problem (if we expect a positive solution, i.e. the existence of some extension) would be to require that the extension coincides with f at least in some open neighbourhood of the origin. We will show that even the weaker version of the problem has a negative solution. Our solution is based on the application of the theory of $\mathcal{W}$-class of Banach spaces, which was developed in a series of papers [42], [43], [25] and [20] (this class was denoted by $\mathcal{C}$-class in the first three papers), which provides a link between uniform smoothness and weak continuity.

Definition 4.3.1. Let X, Y be normed linear spaces, $U \subset X$ open, $f \subset \mathcal{C}^k(U; Y)$ and $k \in \mathbb{N}$. We say that f is $\mathcal{C}^{k,+}$**-smooth** on U if $d^k f$ is uniformly continuous on U.

Definition 4.3.2. Let $\lambda \in (0, 1]$. We say that a Banach space X is a $\mathcal{W}_\lambda$**-space** (or that it belongs to the class $\mathcal{W}_\lambda$) if

$$\mathcal{C}^{1,+}(B_X) \subset \mathcal{C}_{wsC}\left(\lambda B_X\right)$$

(in the sense of restriction).[1] We say that a Banach space is a $\mathcal{W}$**-space** (or that it belongs to the class $\mathcal{W}$) if it is a $\mathcal{W}$-space for some $\lambda \in (0, 1]$.

Remark 4.3.3.

- Clearly, if X is a $\mathcal{W}_\lambda$-space, then it is a $\mathcal{W}_\xi$-space for every $0 < \xi < \lambda$. Conversely, if X is a $\mathcal{W}_\xi$-space for every $0 < \xi < \lambda$, then X is a $\mathcal{W}_\lambda$-space.
- Every Schur space is trivially a $\mathcal{W}_1$-space.
- If X is a $\mathcal{W}$-space, then $\mathcal{C}^{1,+}(X) \subset \mathcal{C}_{wsC}(X)$.
- It was shown in [20] that every $C(K)$ (K scattered) space is a $\mathcal{W}_1$-space. In particular, c_0 in the supremum norm is also a $\mathcal{W}_1$-space (this was shown in [42]).
- Being a $\mathcal{W}$-space is invariant under isomorphism, but the precise value of λ may change.

Proposition 4.3.4. *For every* $m \in \mathbb{N}, m \geqslant 2$, *there is an equivalent renorming of* c_0 *such that* $(c_0, \|\cdot\|_m)$ *belongs to* $\mathcal{W}_{\frac{1}{m}}$*-class, but it does not belong to* $\mathcal{W}_{\frac{1}{m-1}}$*-class.*

---

[1] X is a $\mathcal{W}_\lambda$-space if every uniformly differentiable function $f : B_X \to \mathbb{R}$ takes weakly Cauchy sequences in λB_X to convergent sequences.

Proof. The renorming $\|\cdot\|_m$ of c_0 is determined by its closed unit ball $B_m \subset c_0$,

$$B_m = \overline{\operatorname{conv}}\{\{\pm m e_j\}_{j=1}^{\infty} \cup B_{c_0}\}.$$

Clearly,

$$B_{c_0} \subset B_m \subset m B_{c_0}. \tag{4.2}$$

Note that if $x = (x_j) \in B_m$ then $\operatorname{card}\{j : |x_j| \geqslant 1 + \frac{1}{m}\} \leqslant m^2$. Indeed, suppose

$$x = \sum_{k=1}^{n} a_k m e_k + a_0 (\sum_{j=1}^{\infty} b_j e_j) = \sum_{j=1}^{\infty} x_j e_j, \text{ where } \sum_{k=0}^{n} a_k = 1 \text{ and } a_k \geqslant 0.$$

Letting $A = \{k : a_k \geqslant \frac{1}{m^2}\}$, clearly $\operatorname{card}(A) \leqslant m^2$. Now $|x_j| = |a_0 b_j + a_j m| \leqslant a_0 + m a_j < 1 + \frac{1}{m}$, unless $j \in A$. Choose $\phi_m : \mathbb{R} \to \mathbb{R}_0^+$ a C^∞-smooth even convex function, $\phi_m \left[-1 - \frac{1}{m}, 1 + \frac{1}{m}\right] = 0$, $\phi_m(t) > 0, t > 1 + \frac{1}{m}$, and such that both ϕ_m, ϕ_m' are $\frac{1}{m^2 2^{m+1}}$-Lipschitz. Let now $\Phi_m(x) = \sum_{j=1}^{\infty} \phi_m(x_j)$. It is clear from the previous discussion that Φ_m depends on at most m^2-coordinates in a neighbourhood of any interior point in B_m. Hence it is a uniformly differentiable symmetric function such that both $\Phi_m, d\Phi_m$ are $\frac{1}{2^{m+1}}$-Lipschitz. But $\Phi_m(t e_j) > 0$ for every $t > 1 + \frac{1}{m}, j \in \mathbb{N}$, hence Φ_m restricted to $\frac{1}{m-1} B_m$ does not take weakly null sequences into null sequences. □

Example 4.3.5. There is a Banach space X and a uniformly differentiable function $f : B_X \to \mathbb{R}$ which cannot be extended to a uniformly differentiable function on any λB_X, $\lambda > 1$, preserving its original values in some neighbourhood of 0.

Proof. Let $X = \oplus_{\ell_2} \sum_{m=2}^{\infty} (c_0, \|\cdot\|_m)$, $P_m : X \to (c_0, \|\cdot\|_m)$ be the canonical projections onto the direct summands. Let $f(x) = \sum_{m=2}^{\infty} \Phi_m \circ P_m(x)$. The functions f and *df* are 1-Lipschitz, so f is uniformly differentiable (even with a Lipschitz derivative). It is also clear that (4.2) implies that Φ_m cannot be extended to $(1 + \frac{1}{m-1}) B_m$, preserving its values on $\frac{1}{m-1} B_m$. Since m can be chosen arbitrary large, the result follows. □

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