# Explicit investment rules with time-to-build and uncertainty*

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**Abstract**

We establish explicit socially optimal rules for an irreversible investment decision with time-to-build and uncertainty. Assuming a price sensitive demand function with a random intercept, we provide comparative statics and economic interpretations for three models of demand (arithmetic Brownian, geometric Brownian, and the Cox-Ingersoll-Ross). Committed capacity, that is, the installed capacity plus the investment in the pipeline, must never drop below the best predictor of future demand, minus two biases. The discounting bias takes into account the fact that investment is paid upfront for future use; the precautionary bias multiplies a type of risk aversion index by the local volatility. Relying on the analytical forms, we discuss in detail the economic effects. For example, the impact of volatility on the optimal investment is negligible in some cases. It vanishes in the CIR model for long delays, and in the GBM model for high discount rates.

**Keywords:** optimal capacity; irreversible investments; singular stochastic control; time-to-build; delay equations.

**AMS Classification**: 93E20, 49J40, 91B38.

**JEL Classification**: C61; D92; E22.

## 1 Introduction

How to track demand when the time-to-build retards capacity expansion? When to invest and how much? We answer these questions with a model of irreversible investment. The objective of the decision-maker is to maximize the expected discounted micro-economic social surplus, i.e., the sum of the consumers' net surplus and of the firms' profit. We are able to show in particular that the solution is implementable as a competitive equilibrium. We are able to calculate explicit, compact, decision rules.

In many capitalistic industries, construction delays are essential. In this paper, we focus on electricity generation. In this sector, construction delays can be considerable: they could be only one year for a small wind-farm but could be three years for a gas turbine and eight to ten years for a nuclear plant. The scenarios of the evolution of demand with their trends, their drag force, and their stochastic parts require particular attention. To this purpose, we develop the comparative statics and economic interpretations for three demand models applied to electricity generation. The intercept of the price sensitive demand function follows either an arithmetic Brownian motion as in Bar-Ilan et al. [2002],

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or a geometric Brownian motion as in Bar-Ilan and Strange [1996] and Aguerrevere [2003], or the Cox-Ingersoll-Ross (CIR) model. The latter is a mean-reverting process, and, to our knowledge, no real options investment model exists in the literature with time-to-build and a process of this type. The basic existence and regularity results are provided in a companion paper (Federico and Pham [2014]), but we simplify the specification for the sake of calculability.

An exact decision rule facilitates the clear understanding of the effects at play. The decision rule stipulates what the committed capacity should be, that is, the installed capacity plus capacity under construction. The action rule, given the current conditions, is that the committed capacity must not fall below the best predictor of demand after the delay, minus two biases. The first bias is a pure discounting bias unrelated to uncertainty: because the investment is paid for upfront but only produces after the delay, the required committed capacity is reduced. The second one is a precautionary bias where a risk aversion index is multiplied by local volatility.

We also illustrate the practical importance of a possible saturation of the demand with the CIR model. Indeed, one can observe on Figure 1 that the electricity consumption in several developed countries slows down and seems to reach some ceiling. The saturation is clearer for per capita electric consumption. We show that the investors' behavior is very different depending on whether demand is above or below the long-run average, or target. When demand is above the target, the investor is almost insensitive to the current demand, except if the return speed is very slow. Below the target, the comparison between the time-to-build and the expected time-to-target is critical: if the time-to-build is longer, then the optimal committed capacity is practically the target itself minus the biases; if the time-to-build is shorter, then the investors observe the process and invest progressively.

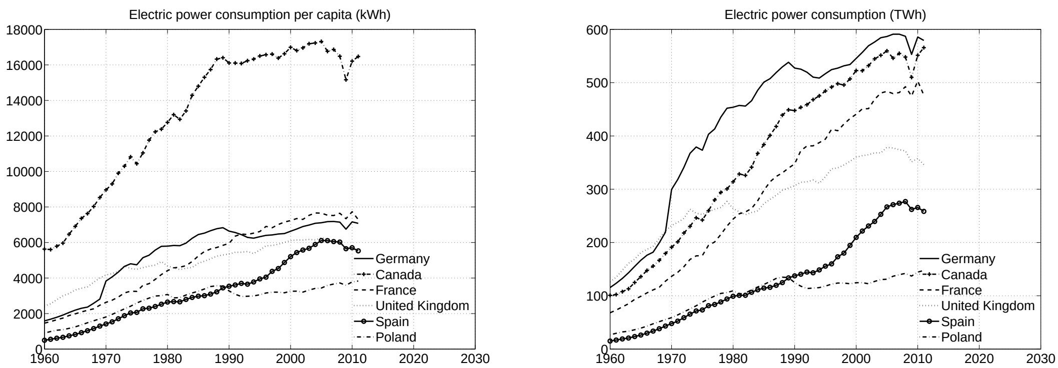


Figure 1: (Left) Electric power consumption per capita. (Right) Electric power consumption. Source: World Bank.

The literature on the topic provides a number of insights. Table 1 provides a tentative classification. The competitive pressure matters: competition kills the value of waiting and thus accelerates investment. Grenadier [2000, 2002] and Pacheco de Almeida and Zemsky [2003] follow this line of thought. We exclusively use a competitive market and show that this effect is completely internalized. The seminal work McDonald and Siegel [1986] on the option to wait in the case of irreversible decisions shows that uncertainty has a negative effect on investment. Strong support for this result is that with greater volatility, investment is triggered by a higher current product price, i.e. a smaller probability of a market downturn. Several extensions provide conditions under which this result does not hold or might be mitigated. Construction delays, that is, the time between the decision and the availability of the new capacity, have attracted the attention of economists. In particular,

| Paper | Objective | Competition | Investment |
|---|---|---|---|
| Majd and Pindyck 1987 | firm | no | irreversible |
| Bar-Ilan and Strange 1996 | firm | no | reversible |
| Grenadier 2000 | firm | perfect | irreversible |
| Bar-Ilan et al. 2002 | planner | no | irreversible |
| Grenadier 2002 | firm | imperfect | irreversible |
| Aguerrevere 2003 | planner/firm | perfect/imperfect | irreversible with flexible production |

Table 1: Papers on investment with uncertainty and time-to-build.

the models in Bar-Ilan and Strange [1996], Bar-Ilan et al. [2002], and Aguerrevere [2003] exhibit situations where an increase in uncertainty leads to an increase in investment.

The models that exhibit a positive effect on investment from an increase in uncertainty, do so only for a specific range of parameters. Besides, the quantitative effects are very small. Bar-Ilan et al. [2002] show in their simulations that when the uncertainty on demand is multiplied by five, then the investment threshold moves only by 1%. And as the authors themselves point out, the investment thresholds are nearly independent of the level of uncertainty. The large effects found in Majd and Pindyck [1987] are reconsidered in Milne and Whalley [2000].

In Aguerrevere [2003], a paper with which we share most of the modeling choices, the production is flexible, although the capacity accumulation is not. Investors keep the choice to produce only when it is profitable, and thus the rigidity of investment is attenuated by the option to produce or not. The capacity reserves are all the more profitable the longer the time-to-build. In consequence, uncertainty tends to increase the investment rate. This paper is significant because of the way it integrates meaningful economic questions, and the numerical simulations are instructive.

As far as electricity production is concerned, the flexibility of the base production is limited either for technological reasons (nuclear plants) or because the fixed cost per idle period are important (coal- or gas-fired power plants). In which case, the cost difference between producing or not is narrow. Our approach fills a gap in the literature.

This paper is organized as follows. Section 2 describes and justifies our modeling approach. Solutions and general properties are provided in Section 3. We give the expression of the decision rule and we show that the solution to the optimization program can be decentralized as a competitive equilibrium. The economic analysis of the joint effect of time-to-build and uncertainty is given in Section 4 for the a geometric Brownian motion, and in Section 5 for the CIR model. Section 6 concludes.

For information on the popular arithmetic Brownian motion application, please see Appendix A.

## 2 The model

We set up a model of an irreversible investment decision in which the objective is to maximize the expected discounted social surplus, i.e., the sum of the consumers' net surplus and of the firms' profit. This economic objective has a simple mathematical expression: it amounts to tracking the current demand of electricity using a quadratic penalty.

**1.** The inverse demand function at date $t$ is

$$p_t(Q) = \eta + \theta(D_t - Q), \tag{1}$$

with $\eta \geq 0$ and $\theta > 0$, where $p$ is the price and $Q$ is the output.[1] The (quasi) intercept $(D_t)_{t\geq 0}$ is a diffusion that satisfies the SDE

$$\begin{cases} \mathrm{d}D_t = \mu(D_t)\mathrm{d}t + \sigma(D_t)\mathrm{d}W_t, \\ D_0 = d, \end{cases} \tag{2}$$

where $(W_t)_{t\geq 0}$ is a Brownian motion on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. Without loss of generality, we suppose that the filtration $(\mathcal{F}_t)_{t\geq 0}$ is the one generated by the Brownian motion $W$ and enlarged by the $\mathbb{P}$-null sets.

**2.** There is a time lag $h > 0$ between the date of the investment decision and the date when the investment is completed and becomes productive. Thus, the investment decision at time $t$ brings additional capacity at time $t + h$.

**3.** At time $t = 0$, there is an initial stream of pending investments initiated in the interval $[-h, 0)$ that are going to be completed in the interval $[0, h)$. The function that represents the cumulative investment planned in the interval $[-h, s]$, $s \in (-h, 0)$, is a nonnegative non-decreasing càdlàg function. Therefore, the set where this function lives is

$$\mathcal{I}^0 = \{I^0 : [-h, 0) \to \mathbb{R}^+,\ s \mapsto I^0_s \text{ càdlàg, non-decreasing}\}. \tag{3}$$

We set

$$I^0_{0^-} = \lim_{s\uparrow 0} I^0_s, \qquad I^0 \in \mathcal{I}^0. \tag{4}$$

**4.** The decision variable is represented by a càdlàg nondecreasing and nonnegative $(\mathcal{F}_t)_{t\geq 0}$-adapted process $(I_t)_{t\geq 0}$, where $I_t$ represents the cumulative investment in the interval $[0, t]$. Hence, the set of admissible strategies, which we denote by $\mathcal{I}$, is the set

$$\mathcal{I} = \{I : \mathbb{R}^+ \times \Omega \to \mathbb{R}^+,\ I \text{ càdlàg, } (\mathcal{F}_t)_{t\geq 0}\text{-adapted, nondecreasing}\}. \tag{5}$$

By setting $I_{0^-} := 0$, formally $\mathrm{d}I_t$ is the investment at time $t \geq 0$.

**5.** Given $I^0 \in \mathcal{I}^0$, $I \in \mathcal{I}$, we set

$$\bar{I}_t = \begin{cases} I^0_t, & t \in [-h, 0), \\ I^0_{0^-} + I_t, & t \geq 0. \end{cases} \tag{6}$$

Then we assume that the production capacity $(K_t)_{t\geq 0}$ is the the càdlàg process following the controlled dynamics driven by the state equation

$$K_t = k + \bar{I}_{t-h}, \quad \forall t \geq 0. \tag{7}$$

The equality above can be seen as a very special controlled locally deterministic equation with delay in the control variable.

Significantly, the randomness in (7) enters only through $I$, and there are no stochastic integrals.

**6.** The objective is to minimize over $I \in \mathcal{I}$ the functional

$$F(k, d, I^0; I) = \mathbb{E}\left[\int_0^{+\infty} e^{-\rho t}\left(\frac{1}{2}(K_t - D_t)^2\mathrm{d}t + q_0\mathrm{d}I_t\right)\right], \tag{8}$$

where $q_0 > 0$ is the unit investment cost.

[1] Aguerrevere [2003] takes a similar form and discusses its flexibility.

**Economic interpretation of the objective.** Given (1), the standard micro-economic connection between the demand function and the instantaneous net consumers' surplus $S_t$ at date $t$ is:

$$S_t = \int_0^{K_t} (\eta + \theta(D_t - q))\ dq - p_t K_t. \tag{9}$$

This is the sum of the values given to each unit consumed at date $t$ minus the price paid for them. Remark that if we interpret $\eta$ as the unit production cost, and if there is some fixed cost $f$ per year, the instantaneous producer's profit $\pi_t$ is $(p_t - \eta)K_t - f$. The social (or total) instantaneous surplus $\mathrm{TS}_t = S_t + \pi_t$ is:

$$\mathrm{TS}_t = \theta \int_0^{K_t} (D_t - q)\, dq - f \tag{10}$$

$$= \underbrace{-\frac{\theta}{2}(K_t - D_t)^2}_{\text{Depends on control}} \ + \underbrace{\frac{\theta}{2}D_t^2 - f}_{\text{Doesn't}}\,. \tag{11}$$

Finally, the program is the maximization of the expected discounted sum of such instantaneous social surpluses minus the investment costs:

$$\max_{I \in \mathcal{I}} \mathbb{E}\left[\int_0^{+\infty} e^{-\rho t}(\mathrm{TS}_t\, \mathrm{d}t - q_0 \mathrm{d}I_t).\right] \tag{12}$$

Indeed, if we normalize $\theta$ to 1 and get rid of the part not depending on the control in $\mathrm{TS}_t$, we retrieve the optimization problem we have set at point 6 above.

In the following, we will exploit the fact that the investment process and the instantaneous demand function (1) generate a spot price process:

$$p_t := \eta + D_t - K_t. \tag{13}$$

It reflects the marginal cost plus a term, which can be negative, that measures tension in the market.

**Diffusion process.** The process $D$ satisfies the following conditions:[2] we assume that the coefficients $\mu, \sigma : \mathbb{R} \to \mathbb{R}$ in (2) are continuous with sublinear growth and regular enough to ensure the existence of a unique strong solution to (2). Further, we assume that this solution takes values in an open set $\mathcal{O}$ of $\mathbb{R}$ and that it is non-degenerate over this set, that is, $\sigma^2 > 0$ on $\mathcal{O}$. In the example we shall discuss in the next section, the set $\mathcal{O}$ will be $\mathbb{R}$ or $(0, +\infty)$. We observe that, due to the assumption of sublinear growth of $\mu, \sigma$, standard estimates in SDEs (see, e.g., Krylov [1980, Ch. II]) show that there exist $\kappa_0, \kappa_1$ depending on $\mu, \sigma$ such that

$$\mathbb{E}\left[|D_t|^2\right] \le \kappa_0(1 + |d|^2)e^{\kappa_1 t}, \qquad t \ge 0. \tag{14}$$

# 3 Solution

The problems with delay are by nature of infinite dimension. Referring to our case, the functional $F$ defined in (8) depends not only on the initial $k$ but also on the past of the control $I^0$, which is a function. Nevertheless, the problem can be reformulated in terms of another one-dimensional state variable not affected by the delay. We rewrite the objective functional to introduce a new state variable, the so-called *committed capacity*.

[2] A reference for the theory of one-dimensional diffusions is Karatzas and Shreve [1991].

The idea of the reformulation in control problems with delay is contained in Bar-Ilan et al. [2002] (cf. also Bruder and Pham [2009]) in the context of optimal stochastic impulse problems. Here, we develop this idea for singular stochastic control. It is worth stressing that, unlike Bar-Ilan et al. [2002], we simplify the approach by working not on the value function of the optimization problem but directly on the basic functional.

### 3.1 Reduction to a problem without delay

For the case of the domain for the couple of variables $(k, d)$ of our problem, the set is:[3]

$$\mathcal{S} = \mathbb{R} \times \mathcal{O}. \tag{15}$$

Define the committed capacity as the capacity $h$ units of time later, i.e.,

$$C_t := K_{t+h} = c + I_t, \tag{16}$$

where $c := k + I^0_{0^-}$. Notice that, unlike (7), (16) represents a controlled dynamics not containing the delay in the control variable.

From now on, the dependence of $K$ on $k, I^0, I$; the dependence of $C$ on $c, I$; and the dependence of $D$ on $d$ is denoted respectively as $K^{k,I^0,I}$, $C^{c,I}$, and $D^d$.

The crucial facts that allow the removal of the delay are the following.

1. The committed capacity is $(\mathcal{F}_t)_{t\geq 0}$-adapted. This is due to the special structure of the controlled dynamics of $K$ that makes $K^{k,I^0,I}_{t+h}$ known given the information $\mathcal{F}_t$.
2. Within the interval $[0, h)$, the control $I$ does not affect the dynamics of $K^{k,I^0,I}$, which is (deterministic and) fully determined by $I^0$. In other words, $K^{k,I^0,I^{(1)}}_t = K^{k,I^0,I^{(2)}}_t$ for every $t \in [0, h)$ and every $I^{(1)}, I^{(2)} \in \mathcal{I}$. Therefore, we can write without ambiguity $K^{k,I^0}_t$ for $t \in [0, h)$ to refer to the "controlled" process $K$ within the interval $[0, h)$.

Given these observations, we have the following:

**Proposition 1.**

$$F(k, d, I^0; I) = \mathbb{E}\left[\int_0^{+\infty} e^{-\rho t}(g(C^{c,I}_t, D^d_t)\mathrm{d}t + q_0 \mathrm{d}I_t)\right] + J(k, d, I^0), \tag{17}$$

*where*

$$J(k, d, I^0) = \frac{1}{2}\mathbb{E}\left[\int_0^h e^{-\rho t}\left(K^{k,I^0}_t - D^d_t\right)^2 \mathrm{d}t\right], \tag{18}$$

*and* $g : \mathcal{S} \to \mathbb{R}^+$ *is defined by*

$$\begin{aligned} g(c, d) :&= \frac{1}{2}e^{-\rho h}\mathbb{E}\left[(c - D^d_h)^2\right] \\ &= \frac{1}{2}e^{-\rho h}(c^2 - 2\beta_0(d)c + \alpha_0(d)), \end{aligned} \tag{19}$$

*where*

$$\alpha_0(d) := \mathbb{E}\big[|D^d_h|^2\big], \qquad \beta_0(d) := \mathbb{E}[D^d_h]. \tag{20}$$

[3]The real problem is meaningful for $k \geq 0$; nevertheless, it is convenient from the mathematical point of view to allow the case of $k < 0$. Because the problem is irreversible and starts from $k \geq 0$, the capital remains nonnegative.

*Proof.* Using the definition of $g$, the time-homogenous property of $D$, we have:

$$\begin{aligned}
\mathbb{E}\left[g(C_t^{c,I}, D_t^d)\right] &= \frac{1}{2}e^{-\rho h}\mathbb{E}\left[\mathbb{E}\left[(c' - D_h^{d'})^2\right]\Big|_{c'=C_t^{c,I},\ d'=D_t^d}\right] \\
&= \frac{1}{2}e^{-\rho h}\mathbb{E}\left[\mathbb{E}\left[(C_t^{c,I} - D_{t+h}^d)^2 \,|\, \mathcal{F}_t\right]\right] \\
&= \frac{1}{2}e^{-\rho h}\mathbb{E}\left[(C_t^{c,I} - D_{t+h}^d)^2\right] \\
&= \frac{1}{2}e^{-\rho h}\mathbb{E}\left[(K_{t+h}^{k,I^0,I} - D_{t+h}^d)^2\right]. \qquad (21)
\end{aligned}$$

Therefore, (8) can be rewritten as

$$\begin{aligned}
F(k,d,I^0;I) &= \mathbb{E}\left[\int_{[0,h)} e^{-\rho t}\left(\frac{1}{2}\left(K_t^{k,I^0,I} - D_t^d\right)^2 \mathrm{d}t + q_0 \mathrm{d}I_t\right)\right] \\
&\quad + \mathbb{E}\left[\int_{[h,+\infty)} e^{-\rho t}\left(\frac{1}{2}\left(K_t^{k,I^0,I} - D_t^d\right)^2 \mathrm{d}t + q_0 \mathrm{d}I_t\right)\right] \\
&= \mathbb{E}\left[\int_{[0,h)} e^{-\rho t}\left(\frac{1}{2}\left(K_t^{k,I^0,I} - D_t^d\right)^2 \mathrm{d}t + q_0 \mathrm{d}I_t\right)\right] \\
&\quad + \mathbb{E}\left[\int_0^{+\infty} e^{-\rho(t+h)}\left(\frac{1}{2}\left(K_{t+h}^{k,I^0,I} - D_{t+h}^d\right)^2 \mathrm{d}t + q_0 \mathrm{d}I_{t+h}\right)\right] \\
&= \mathbb{E}\left[\int_0^{+\infty} e^{-\rho t}(g(C_t^{c,I}, D_t^d)\mathrm{d}t + q_0 \mathrm{d}I_t)\right] + J(k,d,I^0). \qquad (22)
\end{aligned}$$

□

Thus, the functional $J(k,d,I^0)$ defined in Proposition 1 does not depend on $I \in \mathcal{I}$. Therefore, by setting

$$G(c,d;I) := \mathbb{E}\left[\int_0^{+\infty} e^{-\rho t}(g(C_t^{c,I}, D_t^d)\mathrm{d}t + q_0 \mathrm{d}I_t)\right], \qquad (23)$$

the original optimization problem of minimizing $F(k,d,I^0;\cdot)$ over $\mathcal{I}$ is equivalent to the optimization problem *without delay*

$$v(c,d) := \inf_{I\in\mathcal{I}} G(c,d;I) \ \text{ subject to (16) and (2)}. \qquad (24)$$

**Remark.** We consider only the case of a fixed time-to-build. It would be more realistic to assume that the time-to-build is uncertain at the time the investment is launched. This can be modeled by a family of random variables indexed in time (e.g., the date when the investment is launched). Assuming that there is a maximum value $H > 0$ for the time-to-build, one can consider a family of random variables $(h_s)_{s\geq -H}$, each taking values in $[0,H]$, with $h_s$ representing the random time-to-build corresponding the investment launched at time $s$. The expression of the capacity $K$ corresponding to (7) would be

$$K_t \ = \ k_0 + \int_{-H}^t \mathbf{1}_{\{h_s \leq t-s\}} \mathrm{d}\bar{I}_s.$$

However, even assuming independence between $(h_s)_{s\geq -H}$ and $D$, it seems not possible – or, at least, not straightforward – to define a variable like the committed capacity, as the one we are able to define here in the case of a fixed time-to-build, to solve the problem.

### 3.2 Solution characterization

In the sequel, to give sense to the problem (i.e., to guarantee finiteness), we make the standing assumption that the discount factor $\rho$ satisfies

$$\rho \ > \ \max(\kappa_1, 0), \tag{25}$$

where $\kappa_1$ is the constant appearing in (14). This assumption guarantees that there is some $\kappa$ depending on $\mu, \sigma$ s.t.

$$0 \ \leq \ v(c,d) \ \leq \ \kappa \ (1 + |c|^2 + |d|^2), \qquad \forall (c,d) \in \mathcal{S}. \tag{26}$$

In particular, it implies that the value function $v$ is finite and locally bounded.

Federico and Pham [2014] prove the following facts.[4]

1. $v$ is convex with respect to the variable $c$
2. $v$ is differentiable with respect to $c$, and $v_c$ is continuous in $\mathcal{S}$
3. The function $d \mapsto v_c(c,d)$ does not increase for each $c \in \mathbb{R}$
4. $v_c \geq -q_0$

In view of these facts, there is now the *continuation region*

$$\mathcal{C} := \{(c,d) \in \mathcal{S} \mid v_c(c,d) > -q_0\}, \tag{27}$$

and the *action region*

$$\mathcal{A} := \{(c,d) \in \mathcal{S} \mid v_c(c,d) = -q_0\}. \tag{28}$$

Therefore, $\mathcal{C}$ and $\mathcal{A}$ are disjoint and $\mathcal{S} = \mathcal{C} \cup \mathcal{A}$. Due to the continuity of $v_c$, the continuation region is an open set of $\mathcal{S}$, while the action region is a closed set of $\mathcal{S}$. Moreover, due to the monotonicity of $v_c(c,\cdot)$ and to the convexity of $v(\cdot, d)$, $\mathcal{C}$ and $\mathcal{A}$ can be rewritten as

$$\mathcal{C} = \{(c,d) \in \mathcal{S} \mid c > \hat{c}(d)\}, \quad \mathcal{A} = \{(c,d) \in \mathcal{S} \mid c \leq \hat{c}(d)\}, \tag{29}$$

where $\hat{c} : \mathcal{O} \to \mathbb{R}$ is a non-decreasing function. See Figure 2.

The latter function is the optimal boundary for the problem, in the sense that it characterizes the optimal control as follows. The optimal control consists of keeping the state process $(C, D)$ within the closure of the continuation region $\bar{\mathcal{C}}$. By continuity of trajectories of $D$ and continuity of the optimal boundary $\hat{c}$, this is obtained as follows.

1. At time $t = 0$:
   (a) If $(C_0, D_0) = (c,d) \notin \bar{\mathcal{C}}$, i.e., $c < \hat{c}(d)$, then the optimal investment $\mathrm{d}I_0$ is finite and equal to $\hat{c}(d) - c > 0$
   (b) If $(C_0, D_0) = (c,d) \in \mathrm{Int}(\mathcal{C})$, i.e., $c > \hat{c}(d)$, then no investment is done
   (c) If $(C_0, D_0) = (c,d) \in \partial\mathcal{C}$, i.e., $c = \hat{c}(d)$, then an infinitesimal investment is done in order to reflect vertically and upwards the process $(C, D)$ at the boundary
2. At time $t > 0$ just the last two actions (b)-(c) described above for time $t = 0$ are possible.

[4] Federico and Pham [2014] deal with reversible problems. We can apply their results by taking an infinite cost of disinvestment. The irreversible case with a profit maximization criterion is studied with similar generality in Ferrari [forth.].

In the case when $(C_t, D_t) \in \partial\mathcal{C}$, the description of the optimal control above is informal. Theorem 1 below gives the rigorous and explicit solution.

The form of the solution is typical of singular stochastic control when there are no fixed costs of investment and the uncontrolled state process (here the demand $D$) has continuous trajectories: except at time $t = 0$, when a jump (a finite size investment) is possible, the remaining part of the optimal control does not contain finite size investments. From a practical point of view, this is a bit unsatisfactory, since actually one would like to know how much to invest. On the other hand, as we have said, this solution is typical of the model, whose value is that it allows analytical tractability.

In practice, the solution should be read as follows: the rule is to keep the committed capacity $C_t$ always greater than (or equal to) the value $\hat{c}(D_t)$, and to do the minimal effort to obtain that. In this sense, clearly our mathematical model is only a theoretical approximation of the reality.

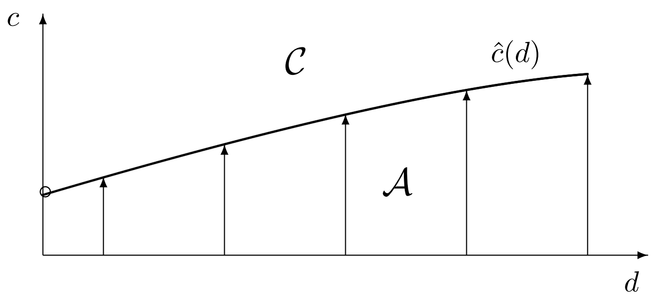


Figure 2: Continuation region ($\mathcal{C}$) and action region ($\mathcal{A}$) in the demand-committed capacity space.

We have an explicit characterization of $\hat{c}$, that is, of the optimal control that is provided by the following result.

**Theorem 1.** *The optimal boundary is explicitly written as*

$$\hat{c}(d) = \beta_0(d) - q_0\rho e^{\rho h} + \frac{1}{2}\sigma^2(d)\,\frac{\beta''(d)\psi'(d) - \beta'(d)\psi''(d)}{\psi'(d)}, \tag{30}$$

*where* $\beta_0(d)$ *is defined in* (20) *as* $\mathbb{E}[D_h^d]$,

$$\beta(d) := \int_0^{+\infty} e^{-\rho t}\mathbb{E}[\beta_0(D_t^d)]\mathrm{d}t, \tag{31}$$

*and* $\psi$ *is the strictly increasing fundamental solution to the linear ODE*

$$[\mathcal{L}\phi](d) := \rho\phi(d) - \mu(d)\phi'(d) - \frac{1}{2}\sigma^2(d)\phi''(d) = 0, \qquad d \in \mathcal{O}. \tag{32}$$

*The unique optimal control for the problem* (24) *is the process*

$$I_t^* = \left[\hat{c}\left(\sup_{0\le s\le t} D_s^d\right) - c\right]^+. \tag{33}$$

*Proof.* Theorem 4.2 and Corollary 5.2 of Federico and Pham [2014] state the above claim[5] with

$$\hat{c}(d) = \rho\left[\beta(d) - \frac{\psi(d)}{\psi'(d)}\beta'(d) - q_0 e^{\rho h}\right], \tag{34}$$

[5]Note however that here we have the term $e^{\rho h}$ multiplying $q_0$. This is due to the fact that our function $g$ is equal to the function $g$ in Section 5 of Federico and Pham [2014] up to the constant $e^{-\rho h}$.

Therefore, if (34) can be rewritten in the form (30), then it is more suitable for interpretation.

To this purpose, because $\psi$ solves the ODE (32), we have

$$\hat{c}(d) = \rho\beta(d) - \mu(d)\beta'(d) - \frac{1}{2}\sigma^2(d)\frac{\psi''(d)}{\psi'(d)}\beta'(d) - q_0\rho e^{\rho h}. \tag{35}$$

On the other hand, it is well-known from the connection between the linear ODE and the one-dimensional diffusions that the function $\beta$ solves the nonhomogeneous ODE (32) with the forcing term $\beta_0$:

$$\mathcal{L}\beta = \beta_0. \tag{36}$$

Hence, combining (35) and (36), the expression (30) follows. □

The social optimum we have characterized assumes that all decisions are controlled by one agent, the fictitious social planner. Proposition 2 states that, if investors with the same irreversible technology are many, and if they take the price process as given, then they collectively behave like the social planner would like to see them behaving. Consumers are not strategic, as we have shown in the economic interpretation of the objective (Section 2). In other terms, a competitive equilibrium implements the social optimum. This is the second fundamental theorem of welfare economics, adapted to our stochastic infinite time-horizon modeling.

**Proposition 2.** *Let $p_t^{k,I^0,d,*} := \eta + D_t^d - K_t^{k,I^0,*}$ be the spot price process (see equation 13) for initial conditions $k, I^0$ and $d$, where $K^{k,I^0,*}$ is the optimal capital process (i.e. corresponding to the optimal control $I^*$ provided by the solution of the optimization problem).*

*If competitive firms have linear cost, with $\eta$ the unit production cost and $q_0$ the unit investment cost, the expected net present value of a unit investment for a price-taking firm is*

$$\mathbb{E}\left[\int_h^{+\infty} e^{-\rho t}\left(p_t^{k,I^0,d,*} - \eta\right) \mathrm{d}t\right] - q_0 \leq 0, \tag{37}$$

*with the equality holding if and only if $(k + I^0_{0-}, d) \in \mathcal{A}$.*

The proof is in Appendix B. Competitive investment is null if the LHS is strictly negative, whereas any investment is optimal (in particular the socially optimal one) in case of equality.

### 3.3 Interpretation of the optimal boundary

The optimal boundary $\hat{c}(d)$ defined by (30) and the optimal control defined by (33) are easily amenable to interpretations. The optimal boundary is composed of three terms:

$$\hat{c}(d) = \beta_0(d) - b_\rho - b_\sigma(d). \tag{38}$$

1. $\beta_0(d)$ is what $d$ is expected to be $h$ years later: one commits to what demand is expected to be when the investment becomes operative.
2. The discounting bias $b_\rho = q_0\rho e^{\rho h}$ expresses the fact that the investment is paid right away, whereas the cost of the insufficient capacity is discounted.

   This effect can be retrieved with a heuristic non-stochastic ($\sigma = 0$) version of the model where irreversibility constraints are ignored. Suppose that instead of following the demand, the investor wants $K_t$ to follow $D_t - \Delta$. This is easy to implement in a deterministic world, and the investor suffers the quadratic loss

$$\int_h^{+\infty} e^{-\rho t} \left(\frac{1}{2}\Delta^2\right) \mathrm{d}t = \frac{1}{2}\Delta^2 \frac{e^{-\rho h}}{\rho}, \tag{39}$$

see "Depends on control" in equation (11). The advantage of this underinvestment is that, at date 0, the investor saves, once for all, $q_0\Delta$. The total investment cost is therefore

$$q_0 \int_0^{+\infty} e^{-\rho t} \mathrm{d}D_t - q_0\Delta. \tag{40}$$

If the investor wants to choose the optimal $\Delta$, he only minimizes

$$\frac{1}{2}\Delta^2 \frac{e^{-\rho h}}{\rho} - q_0\Delta. \tag{41}$$

The minimizing $\Delta$ is precisely the discounting bias $q_0 \rho e^{\rho h}$.

3. The precautionary bias

$$b_\sigma(d) := \frac{1}{2}\sigma^2(d) \left[\beta'(d)\frac{\psi''(d)}{\psi'(d)} - \beta''(d)\right] \tag{42}$$

gives the security margin due to the stochastic nature of the demand process. It is practically null if, for example, $\sigma(d)$ is close to 0.

The calculations go one step further if we assume the affine drift $\mu(d) = ad + b$. Then we have

$$\beta_0(d) = de^{ah} - bh\frac{1 - e^{ah}}{ah}. \tag{43}$$

The ratio must be taken as $-1$ when $a = 0$. Therefore, $\beta'' = 0$ in this case, and

$$b_\sigma(d) = \frac{1}{2}\sigma^2(d)\frac{e^{ah}}{\rho - a}\frac{\psi''(d)}{\psi'(d)}. \tag{44}$$

For the latter term $b_\sigma(d)$:

- The delay has an impact only if $a \neq 0$. The sign of $a$ determines the impact of the delay: the uncertainty about the future grows (diminishes) when $h$ increases if $a > 0$ ($a < 0$), which justifies a bigger (smaller) bias.
- The factor $\sigma^2(d)$ is local, it takes into account the local risk only.
- The factor $\frac{\psi''(d)}{\psi'(d)} > 0$ takes into account the global risk.[6] This term is a kind of absolute risk aversion related to the dynamics of $D$, not the delay.

# 4 Geometric Brownian Motion

## 4.1 The optimal boundary

In the case where the demand follows a geometric Brownian motion (GBM):

$$\mathrm{d}D_t = \mu D_t \mathrm{d}t + \sigma D_t \mathrm{d}W_t, \quad \mu \in \mathbb{R}, \; \sigma > 0, \tag{45}$$

[6] Rogers and Williams [2000, Prop. (50.3), Ch. V (p.292)] show that $\psi$ strictly increases and is convex.

with initial datum $d > 0$, the minimal constant $\kappa_1$ for which (14) is verified is $2\mu + \sigma^2$. Therefore, according to (25), we assume that

$$\rho > 2\mu + \sigma^2. \tag{46}$$

In this case $\mathcal{O} = (0, +\infty)$ and

$$\beta_0(d) = e^{\mu h} d \quad \text{and} \quad \beta(d) = \frac{e^{\mu h}}{\rho - \mu} d. \tag{47}$$

Moreover,

$$[\mathcal{L}\phi](d) = \rho\phi(d) - \mu d\phi'(d) - \frac{1}{2}\sigma^2 d^2 \phi''(d), \quad \phi \in C^2(\mathcal{O}; \mathbb{R}), \tag{48}$$

and the fundamental increasing solution to $\mathcal{L}\phi = 0$ is

$$\psi(d) = d^m, \tag{49}$$

where $m$ is the positive root of the equation

$$\rho - \mu m - \frac{1}{2}\sigma^2 m(m-1) = 0. \tag{50}$$

Due to Theorem 1, we have

$$\hat{c}(d) = de^{\mu h} - q_0\rho e^{\rho h} - \frac{1}{2}\sigma^2 \frac{e^{\mu h}}{\rho - \mu}(m-1)d, \tag{51}$$

with

$$m = \frac{1}{\sigma^2}\left(\sqrt{\left(\mu - \tfrac{1}{2}\sigma^2\right)^2 + 2\rho\sigma^2} - \left(\mu - \tfrac{1}{2}\sigma^2\right)\right). \tag{52}$$

Further, (46) implies $m > 2$.

### 4.2 Comparative statics

Note that

$$\hat{c}(d) = Ad - q_0\rho e^{\rho h}, \text{ with } A = \frac{1}{2}\frac{e^{\mu h}}{\rho - \mu}\left(2\rho - \mu + \frac{1}{2}\sigma^2 - \sqrt{\left(\mu - \frac{1}{2}\sigma^2\right)^2 + 2\rho\sigma^2}\right). \tag{53}$$

The next result analyzes the sensitivity of the optimal boundary, and thus of the action region with respect to the parameters of the model.

**Proposition 3.** *The optimal boundary expressed by (53) has the following properties:*

1. $\frac{\partial \hat{c}(d)}{\partial q_0} < 0$
2. $A > 0$
3. $\frac{h}{A}\frac{\partial A}{\partial h} = \mu h$*, and it has the sign of* $\mu$
4. $\frac{\sigma}{A}\frac{\partial A}{\partial \sigma} = -\frac{\sigma^2}{\sqrt{\left(\mu - \frac{1}{2}\sigma^2\right)^2 + 2\rho\sigma^2}} < 0$
5. $\frac{\mu}{A}\frac{\partial A}{\partial \mu} = \mu h + \frac{1}{2}\frac{\mu}{\rho - \mu}\left(1 - \frac{\mu + \frac{1}{2}\sigma^2}{\sqrt{\left(\mu - \frac{1}{2}\sigma^2\right)^2 + 2\rho\sigma^2}}\right)$*, and it has the sign of* $\mu$

6. $\frac{\rho}{A}\frac{\partial A}{\partial \rho} = \frac{1}{2}\frac{\rho\sigma^2+\mu^2-\frac{1}{2}\mu\sigma^2-\mu\sqrt{\left(\mu-\frac{1}{2}\sigma^2\right)^2+2\rho\sigma^2}}{(\rho-\mu)\sqrt{\left(\mu-\frac{1}{2}\sigma^2\right)^2+2\rho\sigma^2}} > 0$

*Proof.* Properties 1, 3, and 4 are immediate.

The other properties involve the same square root for the denominator. The signs are determined in all of the cases by showing that the numerators can be rearranged and simplified to show that their signs depend only on the sign of $\rho(\rho-\mu)$, which is positive given (46). The terms have the same sign for all of the relevant parameters. □

Property 1 says that the investment decreases with respect to the investment cost. Property 2 says that the investment is responsive to the current demand. Property 3 shows the importance of $\mu$: when, e.g., $\mu > 0$, a longer delay means above all a higher future demand, hence a higher investment. Property 4 confirms that more uncertainty makes the investor more cautious.

Property 5 has a similar logic as property 3: the impact on future demand dominates. To refine the analysis, a focus on the precautionary bias only is useful. Note that

$$b_\sigma(d) = \frac{1}{2}\frac{e^{\mu h}}{\rho-\mu}\left(\sqrt{\left(\mu-\tfrac{1}{2}\sigma^2\right)^2+2\rho\sigma^2}-\left(\mu+\tfrac{1}{2}\sigma^2\right)\right) d > 0. \tag{54}$$

But,

$$\frac{\mu}{b_\sigma(d)}\frac{\partial b_\sigma(d)}{\partial \mu} = \mu\left(h-\frac{1}{2}\frac{1}{\rho-\mu}\frac{2\rho-\mu+\frac{1}{2}\sigma^2-\sqrt{\left(\mu-\frac{1}{2}\sigma^2\right)^2+2\rho\sigma^2}}{\sqrt{\left(\mu-\frac{1}{2}\sigma^2\right)^2+2\rho\sigma^2}}\right). \tag{55}$$

In the second factor, the first term is positive and the second one is negative. We take $\mu > 0$ for the discussion. The overall sign of the elasticity depends, for example, on $h$: if $h$ is small, then the elasticity is negative (the precautionary bias decreases as $\mu$ increases); if $h$ is big, then the elasticity is positive (the precautionary bias increases).

The discount rate has two antagonistic effects on the optimal boundary: the discounting bias increases with respect to $\rho$ because the benefits of investment are discounted, and the precautionary bias decreases because the future costs are discounted. Indeed,

$$\frac{\rho}{b_\sigma(d)}\frac{\partial b_\sigma(d)}{\partial \rho} = \frac{1}{2}\frac{\rho}{\rho-\mu}\frac{\mu+\frac{1}{2}\sigma^2-\sqrt{\left(\mu-\frac{1}{2}\sigma^2\right)^2+2\rho\sigma^2}}{\sqrt{\left(\mu-\frac{1}{2}\sigma^2\right)^2+2\rho\sigma^2}} < 0. \tag{56}$$

Property 6 reflects only the latter effect.

### 4.3 Simulations

In the reference scenario is $\rho = 0.08$ year$^{-1}$, $\mu = 0.03$ year$^{-1}$, $\sigma = 0.06$ year$^{-1/2}$, and $q_0 = 1000$ MEuro·GW$^{-1}$. These values are grossly consistent with the behavior of demand in the countries pictured in Figure 1. We assume that there is no committed capacity at date 0, and that the demand starts at 10 GW.

Figure 3 shows the sensitivity of $\hat{c}(10\text{ GW})$ for a range of volatilities $\sigma$ and a range a drifts $\mu$. Other parameters are those of the reference scenario.

Figure 4 shows the optimal boundaries (Left) and the committed capacities for $h = 8$ and $h = 1$ for the same trajectory of demand, with $\sigma = 0.06$ and a starting point of $d = 10$ GW (Right). The committed capacities stop growing during the episode where demand is (fortuitously) stabilized. The committed capacity increases with the delay. For the long delay, the committed capacity is always ahead of the demand.

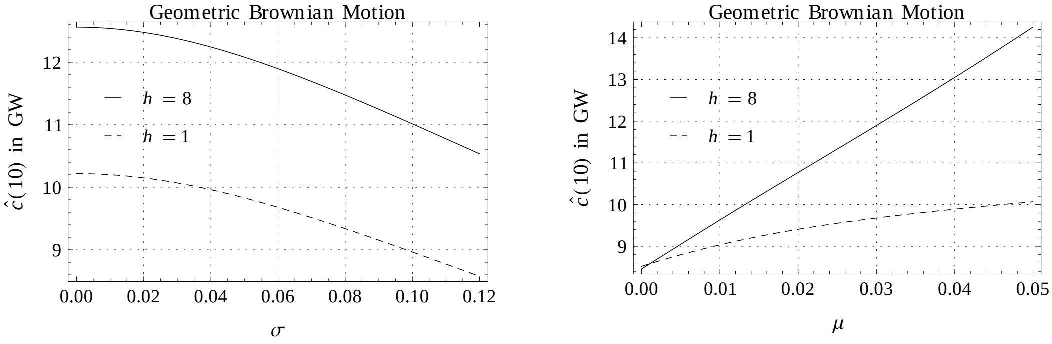


Figure 3: (Left) Investment threshold as a function of $\sigma$. (Right) investment threshold as a function of $\mu$.

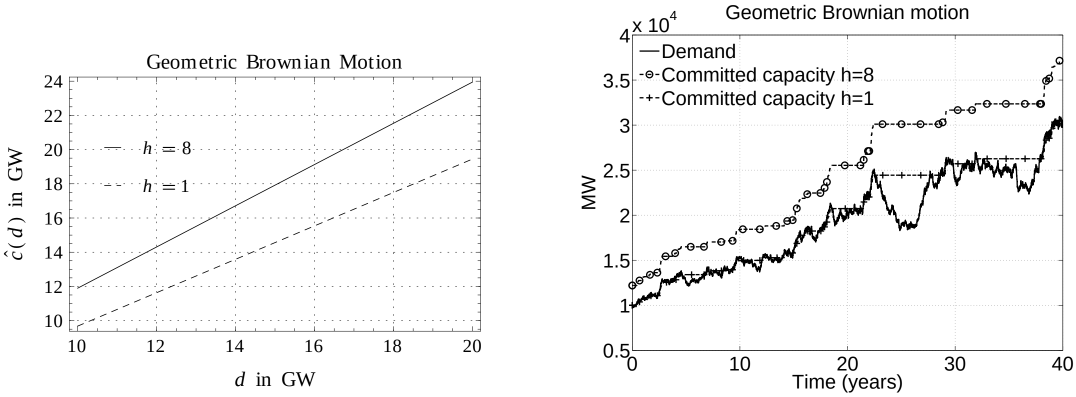


Figure 4: (Left) Investment boundaries. (Right) Demand, committed capacity behavior in the geometric case for $h = 8$ years and $h = 1$ year when $\sigma = 0.06$.

# 5 CIR model

## 5.1 The optimal boundary

For the case where the demand follows a Cox-Ingersoll-Ross model:

$$\mathrm{d}D_t = \gamma(\delta - D_t)\mathrm{d}t + \sigma\sqrt{D_t}\mathrm{d}W_t, \qquad \gamma, \delta, \sigma > 0, \tag{57}$$

then, under the assumption $2\gamma\delta \geq \sigma^2$, we have $\mathcal{O} = (0, +\infty)$. We suppose that this assumption is true. Also in this case (14) is verified with $\kappa_1 = \varepsilon$ for any $\varepsilon > 0$. Therefore, according to (25), we assume that $\rho > 0$.

This case has

$$\beta_0(d) = e^{-\gamma h}d + (1 - e^{-\gamma h})\delta \quad \text{and} \quad \beta(d) = e^{-\gamma h}\frac{d - \delta}{\rho + \gamma} + \frac{\delta}{\rho}. \tag{58}$$

Moreover,

$$[\mathcal{L}\phi](d) = \rho\phi(d) - \gamma(\delta - d)\phi'(d) - \frac{1}{2}\sigma^2 d\phi''(d), \quad \phi \in C^2(\mathcal{O};\mathbb{R}), \tag{59}$$

and the increasing fundamental solution to $\mathcal{L}\phi = 0$ is

$$\psi(d) = M(\rho/\gamma, 2\gamma\delta/\sigma^2, 2\gamma d/\sigma^2), \tag{60}$$

where $M$ is the confluent hypergeometric function of the first type.[7]

Hence,

$$\hat{c}(d) = e^{-\gamma h} d + (1 - e^{-\gamma h})\delta - q_0 \rho e^{\rho h} - \frac{1}{2}\sigma^2 \frac{e^{-\gamma h}}{\rho + \gamma} \frac{\psi''(d)}{\psi'(d)} \tag{61}$$

$$= \delta + e^{-\gamma h}(d - \delta) - q_0 \rho e^{\rho h} - e^{-\gamma h} \frac{\sigma^2}{2\gamma\delta + \sigma^2} \frac{M\left(2 + \frac{\rho}{\gamma}, 2 + \frac{2\gamma\delta}{\sigma^2}, \frac{2d\gamma}{\sigma^2}\right)}{M\left(1 + \frac{\rho}{\gamma}, 1 + \frac{2\gamma\delta}{\sigma^2}, \frac{2d\gamma}{\sigma^2}\right)} d. \tag{62}$$

## 5.2 Comparative statics

The analysis is done with a stylized version of the optimal boundary based on the following results.

**Proposition 4.** *The optimal boundary expressed by (62) verifies the following:*

1. *the tangent at $d = 0$ is the line*

$$\text{Tangent}(d) = \frac{\gamma\delta}{\gamma\delta + \frac{\sigma^2}{2}} e^{-\gamma h} d + \left(1 - e^{-h\gamma}\right)\delta - q_0 \rho e^{\rho h}; \tag{63}$$

2. *the asymptote when $d \to \infty$ is the line*

$$\text{Asymptote}(d) = \frac{\rho}{\rho + \gamma} e^{-\gamma h} d + \left(1 - \frac{\rho}{\rho + \gamma} e^{-\gamma h}\right)\delta - \frac{\sigma^2}{2\gamma}\frac{\rho}{\rho + \gamma} e^{-\gamma h} - q_0 \rho e^{\rho h}; \tag{64}$$

3. *the intersection between the two lines above is*

$$\left(\delta + \frac{\sigma^2}{2\gamma}, \delta - q_0 \rho e^{\rho h}\right). \tag{65}$$

*Proof.* The expression of the tangent line (63) at $d = 0$ immediately follows by the series expansion of $M$:

$$M(a, b, z) = \sum_{s=0}^{\infty} \frac{(a)_s}{(b)_s\, s!} z^s = 1 + \frac{a}{b} z + \frac{a(a+1)}{b(b+1)2!} z^2 + \cdots \tag{66}$$

To calculate the asymptote, we start from (34). Let $M(a, b; z)$ be the confluent hypergeometric function of the first type with parameters $a, b$. Then

$(i)$ $zM'(a, b; z) = a(M(a+1, b; z) - M(a, b; z))$ (here $M'$ is the derivative w.r.t. $z$)

$(ii)$ $M(a, b; z) \sim \frac{\Gamma(b)}{\Gamma(a)} e^z z^{a-b}$, when $z \to \infty$

Using $(i)$,

$$\frac{M(a, b; z)}{zM'(a, b; z)} = \frac{M(a, b; z)}{zM'(a, b; z)} = \frac{M(a, b; z)}{a(M(a+1, b; z) - M(a, b; z))} = \frac{1}{a\left(\frac{M(a+1, b; z)}{M(a, b; z)} - 1\right)}, \tag{67}$$

and using $(ii)$, we get

$$\lim_{z \to \infty} \frac{M(a, b, z)}{zM'(a, b; z)} = 0. \tag{68}$$

[7] See Abramowitz and Stegun [1965].

Thus, the slope of the asymptote of $\hat{c}$ is

$$\alpha := \lim_{d\to\infty} \frac{\hat{c}(d)}{d} = \lim_{d\to\infty} \frac{\rho\beta(d)}{d} = \frac{\rho}{\rho+\gamma} e^{-\gamma h} \tag{69}$$

and the its intersection with the $c$-axis is

$$\kappa := \lim_{d\to+\infty} (\hat{c}(d) - \alpha d). \tag{70}$$

Therefore,

$$\kappa = \delta \left(1 - \frac{\rho}{\rho+\gamma} e^{-\gamma h}\right) - \kappa_1 \frac{\rho}{\rho+\gamma} e^{-\gamma h} - q_0 \rho e^{\rho h}, \tag{71}$$

where

$$\kappa_1 := \lim_{d\to\infty} \frac{\psi(d)}{\psi'(d)}. \tag{72}$$

To compute the latter, $(i)$ is used to get

$$\frac{M(a,b;z)}{M'(a,b;z)} = \frac{z}{a\left(\frac{M(a+1,b;z)}{M(a,b;z)} - 1\right)}. \tag{73}$$

Then, the use of $(ii)$ and of the identity $a\Gamma(a) = \Gamma(a+1)$ yields

$$\lim_{z\to\infty} \frac{M(a,b;z)}{M'(a,b;z)} = \lim_{z\to\infty} \frac{z}{z-a} = 1. \tag{74}$$

Thus, given that the function of interest is $M(\rho/\gamma, 2\gamma\delta/\sigma^2, 2\gamma d/\sigma^2)$, we get $\kappa_1 = \frac{\sigma^2}{2\gamma}$ and the expression of the asymptote (64) follows.

Finally, the expression of the intersection (65) is a direct implication of points 1. and 2. of this Proposition. □

For the economic interpretations, $\hat{c}(d)$ has the stylized expression:

$$\min\left\{\text{Tangent}(d), \text{Asymptote}(d)\right\}. \tag{75}$$

The kink point $\left(\delta + \frac{\sigma^2}{2\gamma}, \delta - q_0\rho e^{\rho h}\right)$ is close to $(\delta, \delta)$ if the uncertainty is small compared to the convergence speed, and if $q_0$ is small.

When $h$ and $\sigma$ are small, the tangent is the 45 degree line minus the discounting bias: committed capacity follows demand. The asymptote expresses a conservative behavior because the capacity increases by only $\frac{\rho}{\rho+\gamma}$ for each unit increase of demand.

With a large convergence speed compared to the volatility (a small $\sigma^2/\gamma$), the uncertainty has a negligible impact on the optimal boundary.

The tangent and the asymptote become flatter and flatter as $h$ increases: current conditions as measured by $d$ matter less when the delay is longer. The flattening effect is exponential. Reversion to the mean implies that as the delay increases, the current demand progressively loses relevance for the prediction of the future demand. No precautionary bias is needed at the limit for the large delays.

## 5.3 Simulations

The CIR model provides a rich setting to analyze the effects of the time-to-build, of the volatility, and of different convergence rates. The following reference parameters are: the discount rate is $\rho = 0.08$ year$^{-1}$, the long-term demand is $\delta = 20$ GW, the demand approaches this limit at a speed $\gamma = 0.8$ or $0.08$ year$^{-1}$, and $\sigma = 0.1$ or $0.05$ GW$^{1/2}$·year$^{-1/2}$. The investment cost is $q_0 = 1000$ MEuro·GW$^{-1}$. We assume that there is no committed capacity at date 0, and that the demand starts at 10 GW.

Figure 5 shows the sensitivity of $\hat{c}(10$ GW$)$. A range of volatilities $\sigma$ (with $\gamma = 0.08$) and a range of mean-reversion parameters $\gamma$ (with $\sigma = 0.1$) are explore. Other parameters are those of the reference scenario.

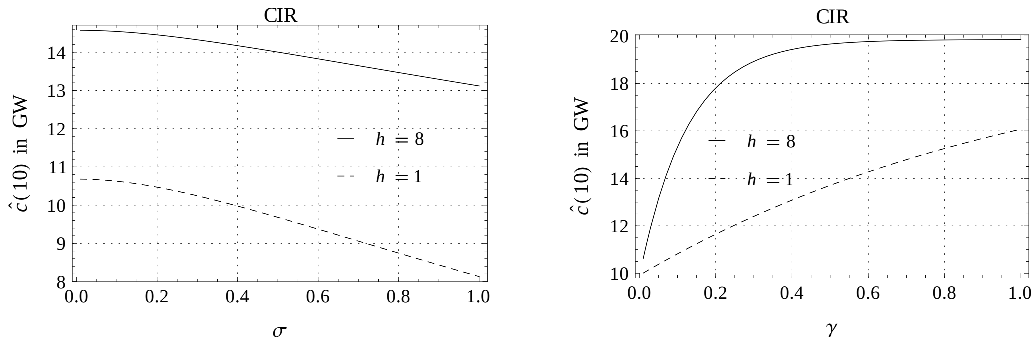


Figure 5: (Left) Investment threshold $\hat{c}(10)$ as a function of $\sigma$. (Right) investment threshold $\hat{c}(10)$ as a function of $\mu$.

We also consider four cases, where the delay $h = 1$ or 8, and the demand volatility $\sigma = 0.1$ or 0.05. Figure 6 (Left) gives the four boundaries. However, the two boundaries with $h = 8$ are almost completely flat and confounded. The other two have very close tangents and asymptotes and are hard to discern visually. Figure 6 (Right) shows the committed capacities for $h = 8$ and $h = 1$ for the same trajectory of demand with $\sigma = 0.1$. For a long delay, the committed capacity is immediately at the long-term value while for short delay, more time is taken. In both cases, once the long-term value has been reached, committed capacities barely increase.

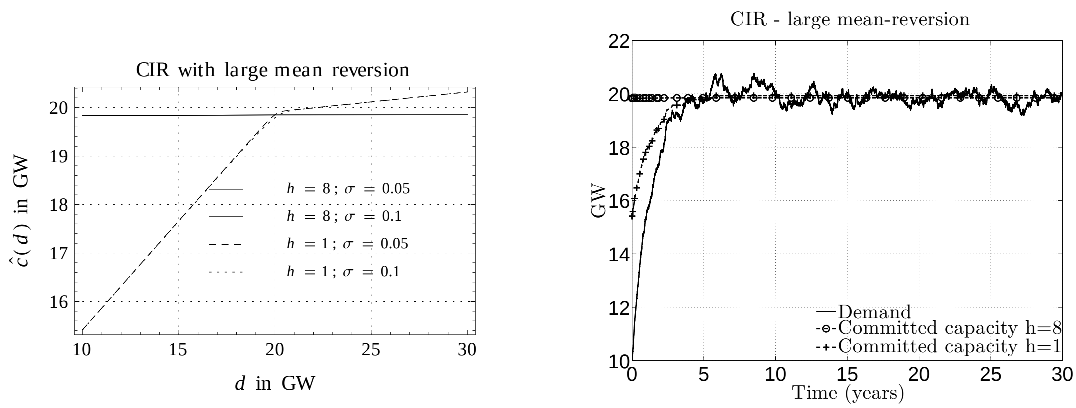


Figure 6: (Left) Investment boundaries. (Right) Demand, committed capacity behavior with the CIR model for an eight-year delay and a one-year delay in the case of a large mean-reversion ($\gamma = 0.8$).

Figure 7 (Left) shows four boundaries with the same parameters as in Figure 6 except that $\gamma = 0.08$. The boundaries have a less marked kink than with a faster convergence rate: boundaries are more like the 45 degree line because the demand evolves much more slowly, and they are much more alike in terms of positions and slopes.

Figure 7 (Right) shows the committed capacities for $h = 8$ and $h = 1$ for the same trajectory of demand with $\sigma = 0.1$. The committed capacities are more responsive to the current conditions because they are better predictors of the future demand than when $\gamma$ is large. This effect plays for demand levels below 20 or above. Longer delays go with greater committed capacities.

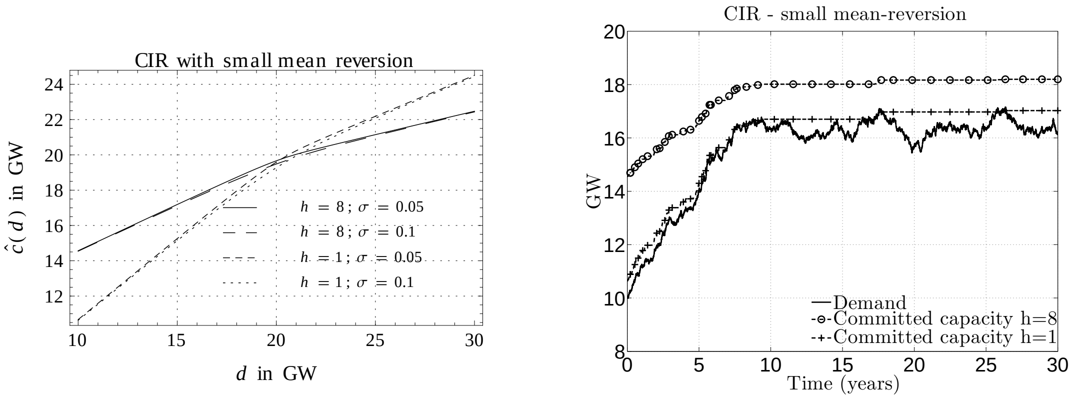


Figure 7: (Left) Investment boundaries. (Right) Demand, committed capacity behavior in the CIR case for an eight-year delay and a one-year delay in the case of a small mean-reversion ($\gamma = 0.08$).

## 6 Conclusion

Electricity demand has a random part and is price sensitive. Our minimization of an expected quadratic loss is founded on microeconomic theory, and our optimal solution can be implemented as a competitive equilibrium. In this paper where the delay between the investment decision and activation of the new capacity is accounted for, we have characterized the explicit decision rules for important classes of demand processes.

The benefits of closed-form solutions cannot be overstated, because we can show the interaction, in investors decisions, between the time-to-build and the uncertainty. In particular, we identify the base rule and the two corrective terms: the investor should invest if his or her committed capacity (i.e., the capacity in the pipeline) is below the best linear estimate of the future demand, the given demand today, and the delay minus a discounting bias and a precautionary bias determined by uncertainty and global risk aversion. The latter term varies substantially with the demand model.

In the arithmetic Brownian motion, the delay and the uncertainty have additive separate effects. In the geometric Brownian motion, the shocks are amplified exponentially so that with a longer delay, restricting the future capacity becomes more costly. On the other hand, the discounting bias is accentuated by the delay. The question of which of these opposite effects dominates the other as the delay increases can be addressed with our explicit expressions. In the CIR case, reversion to the mean implies that as the delay increases, the current demand progressively loses relevance for the prediction of the future demand. No precautionary bias is needed at the limit for the large delays.

# A Arithmetic Brownian Motion

## A.1 The Frontier

With an arithmetic Brownian model of demand, our model is a particular case of Bar-Ilan et al. [2002], where the fixed investment cost is null. The optimal strategy is simpler. The demand dynamics are:

$$\mathrm{d}D_t = \mu \mathrm{d}t + \sigma \mathrm{d}W_t, \qquad \mu \in \mathbb{R},\, \sigma > 0, \tag{76}$$

then $\mathcal{O} = \mathbb{R}$ and (14) is verified with $\kappa_1 = \varepsilon$ for each $\varepsilon > 0$. Therefore, according to (25), we assume that $\rho > 0$. Thus,

$$[\mathcal{L}\phi](d) = \rho\phi(d) - \mu\phi'(d) - \frac{1}{2}\sigma^2\phi''(d), \quad \phi \in C^2(\mathcal{O}). \tag{77}$$

The increasing fundamental solution to $\mathcal{L}\phi = 0$ is $\psi(d) = e^{\lambda d}$ where $\lambda$ is the positive solution to

$$\rho - \mu\lambda - \frac{1}{2}\sigma^2\lambda^2 = 0. \tag{78}$$

Because, in this case,

$$\beta_0(d) = d + \mu h \quad \text{and} \quad \beta(d) = \frac{\mu h}{\rho} + \frac{d}{\rho} + \frac{\mu}{\rho^2}. \tag{79}$$

Due to Theorem 1, $\hat{c}$ is affine:

$$\hat{c}(d) = d + \mu h - q_0\rho e^{\rho h} - \frac{\sqrt{\mu^2 + 2\rho\sigma^2} - \mu}{2\rho}. \tag{80}$$

## A.2 Comparative statics

Consider that

$$\frac{\partial^2 \hat{c}(d)}{\partial h \partial \sigma} = 0. \tag{81}$$

Whatever the time to build $h$, the investment is retarded in the same way by an increase in $\sigma$, and conversely. The additive separability eliminates the cross effects between the uncertainty and the delay with this model, contrary to Bar-Ilan et al. [2002].

An increase in uncertainty always retards investment:

$$\partial\hat{c}(d)/\partial\sigma = -\frac{\sigma}{\sqrt{\mu^2 + 2\rho\sigma^2}} < 0. \tag{82}$$

The variation of $\hat{c}(d)$ with respect to the time-to-build $h$ is

$$\partial\hat{c}(d)/\partial h = \mu - q_0\rho^2 e^{\rho h}. \tag{83}$$

The effect is to hasten investment if $\mu$ is relatively large. If $h$ is relatively large, then the cost of investment appears large compared to the future discounted damage, and investment is retarded. We retrieve the effects encountered in the case of the geometric Brownian motion.

Furthermore,

$$\partial\hat{c}(d)/\partial\mu = h + \frac{1}{2\rho}\left(1 - \frac{\mu}{\sqrt{\mu^2+2\rho\sigma^2}}\right) > 0, \tag{84}$$

and

$$\partial\hat{c}(d)/\partial\rho = -q_0(1+h\rho)e^{\rho h} + \frac{1}{2}\left(\frac{\mu^2+\rho\sigma^2-\mu\sqrt{\mu^2+2\rho\sigma^2}}{\rho^2\sqrt{\mu^2+2\rho\sigma^2}}\right). \tag{85}$$

In the latter expression, the first term is negative (the discounting bias is reinforced), whereas the second term is positive (the precautionary bias is attenuated). Thus, we get the same effects encountered in the case of the geometric Brownian motion.

### A.3 Simulations

On Figure 8 (Left), $b := \hat{c}(d) - d$ is given as a function of $\sigma$, for two contrasted values of $h$ (1 and 8 years). The other parameters are: $\rho = 0.08$ year$^{-1}$, $\mu = 0.3$ GW·year$^{-1}$, with an initial demand of 10 GW and demand, committed capacity, and installed capacity all equal at date 0 ($D_0 = C_0 = K_0$).

Figure 8 (Left) shows that the impact of the time-to-build with these values is much more important than the impact of uncertainty.

By and large, this result is in line with Bar-Ilan et al. [2002]. In their setting, increasing the time-to-build from one year to eight years reverses the relation between uncertainty and investment, which is possible only because they are not separable. Specifically, for a long delay, an increase in uncertainty hastens investments but decreases their level. But, these effects are very small (Bar-Ilan et al. [2002, pp. 85, Figure 2]).

The excess of committed capacity does not imply that the system will hold an excess of installed capacity. In fact, the reverse is observed in Figure 8 (Right). In the case of a delay of eight years, an excess of committed capacity as measured by the value of $b$ is 1.873 GW. But in eight years, the demand will grow on average 2.4 GW, which clearly indicates that the optimal strategy is to avoid excess installed capacity.

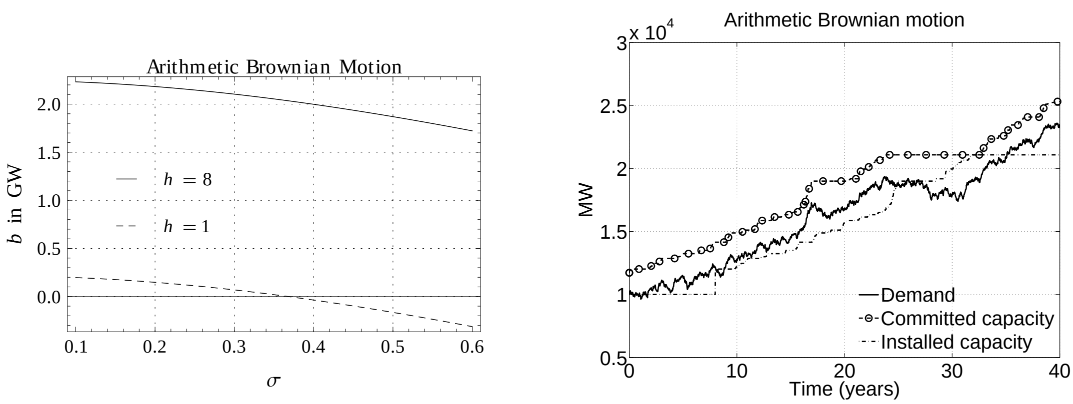


Figure 8: (Left) $b := \hat{c}(10) - 10$ as a function of $\sigma$. (Right) Demand, committed and installed capacity behavior for an eight-year delay, $\sigma = 0.6$ GW·year$^{-1/2}$.

# B Proof of Proposition 2

Let $c := k + I^0_{0-}$. We prove first that

$$v_c(c,d) = \mathbb{E}\left[\int_0^{+\infty} e^{-\rho t} g_c(C_t^{c,*}, D_t^d)\mathrm{d}t\right], \tag{86}$$

where $C^{c,*}$ is the optimal state process associated to the optimal control $I^*$ provided by Theorem 1. Let $I^* \in \mathcal{I}$ be optimal for $(c,d)$. Then

$$\frac{G(c+\varepsilon,d;I^*)-G(c,d;\ I^*)}{\varepsilon} \geq \frac{v(c+\varepsilon,d)-v(c,d)}{\varepsilon}. \tag{87}$$

On the other hand,

$$\frac{G(c+\varepsilon,d;I^*)-G(c,d:I^*)}{\varepsilon} = \mathbb{E}\left[\int_0^{+\infty} e^{-\rho t}\ \frac{g(C_t^{c,I^*}+\varepsilon,D_t^d)-g(C_t^{c,I^*},D_t^d)}{\varepsilon}\mathrm{d}t\right]. \tag{88}$$

Taking the limsup in (87) and taking into account (88), we get

$$\limsup_{\varepsilon\downarrow 0}\frac{v(c+\varepsilon,d)-v(c,d)}{\varepsilon} \leq \mathbb{E}\left[\int_0^{\infty} e^{-\rho t} g_c(C_t^{c,*},D_t^d)\mathrm{d}t\right]. \tag{89}$$

On the other hand, arguing symmetrically with $c-\varepsilon$, we get

$$\liminf_{\varepsilon\downarrow 0}\frac{v(c,d)-v(c-\varepsilon,d)}{\varepsilon} \geq \mathbb{E}\left[\int_0^{+\infty} e^{-\rho t} g_c(C_t^{c,*},D_t^d)\mathrm{d}t\right]. \tag{90}$$

Therefore, (89) and (90) assert (86).

Using the same argument as in (19) and taking into account (86) we get

$$v_c(c,d) = \mathbb{E}\left[\int_0^{+\infty} e^{-\rho(t+h)}(K_{t+h}^{k,I^0,*}-D_{t+h}^d)\mathrm{d}t\right] = \mathbb{E}\left[\int_h^{+\infty} e^{-\rho t}(K_t^{k,I^0,*}-D_t^d)\mathrm{d}t\right].$$

Now using the definitions of $\mathcal{A},\mathcal{C}$ and $p^{k,I^0,d,*}$, the claim follows.

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