WEYL COMPATIBLE TENSORS

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Abstract. We introduce the new algebraic property of Weyl compatibility for symmetric tensors and vectors. It is strictly related to Riemann compatibility, which generalizes the Codazzi condition while preserving much of its geometric implications. In particular it is shown that the existence of a Weyl compatible vector implies that the Weyl tensor is algebraically special, and it is a necessary and sufficient condition for the magnetic part to vanish. Some theorems (Derdziński and Shen, Hall) are extended to the broader hypothesis of Weyl or Riemann compatibility. Weyl compatibility includes conditions that were investigated in the literature of general relativity (as in McIntosh et al.) A simple example of Weyl compatible tensor is the Ricci tensor of an hypersurface in a manifold with constant curvature.

1. Introduction

The geometry of Riemannian or pseudo-Riemannian manifolds of dimension $n \geq 3$ is intrinsically described by $\mathscr{N} = \frac{1}{12}n(n-1)(n-2)(n+3)$ algebraically independent scalar fields, constructed from the Riemann and the metric tensors. The same counting is provided by the Weyl, Ricci and metric tensors. The Weyl tensor bears the symmetries of the Riemann tensor, with the extra property of being traceless[1]:

$$C_{jkl}{}^m = R_{jkl}{}^m + \frac{1}{n-2}(\delta_{[j}{}^m R_{k]l} + R_{[j}{}^m g_{k]l}) - \frac{1}{(n-1)(n-2)} R\delta_{[j}{}^m g_{k]l}. \tag{1}$$

The trace condition $C_{jab}{}^j = 0$ reduces the parameters of the Riemann tensor by the number $\frac{1}{2}n(n+1)$ that is accounted for by considering the Ricci tensor as algebraically independent. The two tensors are linked by functional relations as the following one [1, 2]:

$$-\nabla_m C_{abc}{}^m = \frac{n-3}{n-2}\left[\nabla_{[a}R_{b]c} - \frac{1}{2(n-1)}\nabla_{[a}g_{b]c}R\right]. \tag{2}$$

In the coordinate frame that locally diagonalizes the Ricci and the metric tensors (the latter with diagonal elements ± 1), the parameters that survive are the components of the Weyl tensor and the n eigenvalues of the Ricci tensor, whose number is precisely $\mathscr{N}$ [3]. This choice of fundamental tensors offers advantages, as in the classification of manifolds and in general relativity.

Debever and Penrose [4] proved that in four-dimensional space-time manifolds the equation

$$k_{[b}C_{a]rs[q}k_{n]}k^r k^s = 0 \tag{3}$$

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[1]Conventions: $X_{[ab]} =: X_{ab} - X_{ba}$, $R_{ab} = R_{amb}{}^m$, $u^2 = u^a u_a$.

always admits four null solutions (principal null directions). When two or more coincide, the Weyl tensor is named *algebraically special*, and the condition for degeneracy is

$$k_{[b}C_{a]rsq}k^r k^s = 0 \tag{4}$$

The degeneracies classify space-time manifolds in classes that coincide with the Petrov types [5]. For $n > 4$, Milson et al. showed that eq. (3) may not have solution at all [6]. They introduced the notion of *Weyl aligned null directions* (WAND): a null vector k is a WAND if there is a null frame including it, such that the Weyl scalars of maximal boost weight vanish. The order of alignment provides the backbone of a classification of Lorentzian manifolds ([6] Table III, and [7]).

The Einstein equations of general relativity link the energy momentum tensor T_{ij} to the Ricci tensor and the curvature scalar, but not to the Weyl tensor:

$$R_{ij} - \frac{R}{2}\, g_{ij} = 8\pi\, T_{ij}. \tag{5}$$

In $n = 4$ the Weyl tensor may be replaced by two symmetric tensors, the electric and magnetic components, and the identity (2) for the Weyl tensor translates into Maxwell-like equations for the components [8, 9]. The construction was extended to $n > 4$ [10].

In the study of Derdziński and Shen's theorem [11, 12] on the restrictions imposed by a Codazzi tensor on the structure of the Riemann tensor, we introduced the new algebraic notion of *Riemann compatible tensors* [13]. Most of the statements valid for (pseudo) Riemannian manifolds equipped with a nontrivial Codazzi tensor, such as the vanishing of Pontryagin forms, were shown to persist [14].

This paper is about Weyl compatibility of symmetric tensors, a property which is broader than Riemann compatibility. The existence of a Weyl compatible tensor has consequences on the Petrov type of the manifold and the electric and magnetic components of the Weyl tensor. Since the classifications of Lorentzian manifolds are based on vectors, our presentation of Weyl compatibile vectors crosses in several points definitions and properties discussed by other authors, but having a different origin.

The definitions and main properties of Riemann and Weyl compatibile symmetric tensors are reviewed in Sect. 2. Tensors $u_i u_j$ naturally define Riemann and Weyl compatibility for vectors, which is discussed in Sect. 3 with various new results, such as the extension of Derdziński and Shen's theorem [11] and of Hall's theorem [15], and a sufficient condition for the vanishing of Pontryagin forms. Next, Riemann (Weyl) *permutable* vectors are considered, and results of McIntosh and others [16, 17, 18] for the special case $R_{ijkl}u^l = 0$ are reobtained and extended.

In Sect. 4 it is shown that the existence of a Weyl compatible vector is a sufficient condition for the Weyl tensor to be special, with results regarding the Penrose-Debever classification of space-times. The electric and magnetic components of the Weyl tensor are then considered, with the statements that existence of a Weyl compatible vector is necessary and sufficient for the Weyl tensor to be purely electric, and that Weyl permutability implies a conformally flat space-time.

In Sect. 5 it is shown the Ricci tensor of an hypersurface isometrically embedded in a pseudo-Riemannian space with constant curvature is Riemann and Weyl compatible.

In this paper we consider n-dimensional pseudo-Riemannian manifolds, with a Levi-Civita connection ($\nabla_i g_{jk} = 0$). Where necessary, we specialize to the metric signature $n-2$, i.e. to n-dimensional Lorentzian manifolds (space-times).

2. Riemann and Weyl compatible tensors

We briefly review the concept of compatibility for symmetric tensors, first introduced in [13], and investigated in [14]. Permutable tensors are then defined, as a special class.

Definition 2.1. A symmetric tensor b_{ij} is Riemann compatible if:

$$b_{am}R_{bcl}{}^m + b_{bm}R_{cal}{}^m + b_{cm}R_{abl}{}^m = 0. \tag{6}$$

The definition has a natural origin. Consider the vector-valued 1-form $B_l = b_{kl}dx^k$, where b is a symmetric tensor; its covariant exterior derivative is

$$DB_l = \frac{1}{2}\mathscr{C}_{jkl}\, dx^j \wedge dx^k,$$

where $\mathscr{C}_{ijk} =: \nabla_i b_{jk} - \nabla_j b_{ik}$ is the "Codazzi deviation tensor" defined in [14]. As it is well known, $DB_l = 0$ if and only if b_{kl} is a Codazzi tensor [19, 20]. In general, whether $DB_l \neq 0$ or not, another derivative gives

$$D^2 B_l = \frac{1}{3!}\left(\nabla_i \mathscr{C}_{jkl} + \nabla_j \mathscr{C}_{kil} + \nabla_k \mathscr{C}_{ijl}\right) dx^i \wedge dx^j \wedge dx^k.$$

The following identity links the Codazzi deviation to Riemann compatibility [14]:

$$\nabla_i \mathscr{C}_{jkl} + \nabla_j \mathscr{C}_{kil} + \nabla_k \mathscr{C}_{ijl} = b_{im}R_{jkl}{}^m + b_{jm}R_{kil}{}^m + b_{km}R_{ijl}{}^m \tag{7}$$

Therefore, $D^2B_l = 0$ (i.e. DB_l is closed) if and only if b is Riemann compatible. In particular, Codazzi tensors and Levi-Civita metrics, for which $\mathscr{C}_{jkl} = 0$, are Riemann compatible.

Example 2.2. *Consider the Ricci tensor. Its Codazzi deviation is* $\mathscr{C}_{abc} = \nabla_{[a}R_{b]c} = -\nabla_m R_{abc}{}^m$*, by the contracted Bianchi identity. The identity* (7) *turns out to be Lovelock's identity* [21]*:*

$$-(\nabla_a\nabla_m R_{bcd}{}^m + \nabla_b\nabla_m R_{cad}{}^m + \nabla_c\nabla_m R_{abd}{}^m) = R_{am}R_{bcd}{}^m + R_{bm}R_{cad}{}^m + R_{cm}R_{abd}{}^m. \tag{8}$$

Example 2.3. *Consider the change of metric from g to another Levi-Civita metric g', then it is $R_{abc}{}^m g'_{dm} + R_{abd}{}^m g'_{cm} = 0$. It follows that the metric g' is Riemann compatible. On $n = 4$ Lorentzian manifolds such metrics have been classified* [22] *(page 259).*

Example 2.4. *(see* [23]*, page 133) Consider a non degenerate Weyl connection i.e. $\tilde\nabla_a g_{bc} = \lambda_a g_{bc}$. The Codazzi deviation $\mathscr{C} = \lambda_a g_{bc} - \lambda_b g_{ac}$ has the form discussed in* [14]*, prop. 6.1. Then g is Riemann compatible if and only if λ is closed. If $\lambda_a = \tilde\nabla_a \sigma$, it is well known that there is a conformal map $g' = e^{2\sigma} g$ such that g' is Levi-Civita.*

Compatibility was extended in [11] to generalized curvature tensors $K_{abc}{}^m$, i.e. tensors having the symmetries of the Riemann tensor with respect to permutations of indices, and the first Bianchi property. The Weyl tensor (1) is the most notable example. A symmetric tensor b_{ij} is *Weyl compatible* if:

$$b_{am}C_{bcl}{}^m + b_{bm}C_{cal}{}^m + b_{cm}C_{abl}{}^m = 0. \tag{9}$$

The following algebraic identity relates a symmetric tensor b_{ij} to the Weyl, Riemann and Ricci tensors [14]:

$$b_{im}C_{jkl}{}^m + b_{jm}C_{kil}{}^m + b_{km}C_{ijl}{}^m = b_{im}R_{jkl}{}^m + b_{jm}R_{kil}{}^m + b_{km}R_{ijl}{}^m \tag{10}$$
$$+\frac{1}{n-2}\left[g_{kl}(b_{im}R_j{}^m - b_{jm}R_i{}^m) + g_{il}(b_{jm}R_k{}^m - b_{km}R_j{}^m) + g_{jl}(b_{km}R_i{}^m - b_{im}R_k{}^m)\right].$$

Any contraction with the metric tensor gives zero; the identity is trivial if b is the metric tensor. An immediate consequence is:

Theorem 2.5. *An arbitrary symmetric tensor is Riemann compatible if and only if it is Weyl compatible and commutes with the Ricci tensor.*

Proof. If b is Riemann compatible, contraction of (6) with g^{cl} gives $b_{am}R_b{}^m - b_{bm}R_a{}^m = 0$, i.e. b commutes with the Ricci tensor. Then b is Weyl compatible by identity (10). The converse is obvious, by the same identity. □

In particular, Riemann and Weyl compatibility are equivalent for the Ricci tensor, or any symmetric tensor that commutes with it.

The existence of a nontrivial Weyl compatible tensor poses strong restrictions on the Weyl tensor. In [13] we proved a broad generalization of Derdzinski and Shen's theorem that holds both in Riemannian and pseudo-Riemannian manifolds. For the Weyl tensor it reads:

Proposition 2.6. *On a pseudo-Riemannian manifold with a Weyl compatible tensor b, if X, Y and Z are eigenvectors of b with eigenvalues λ, μ, ν ($b^i{}_jX^j = \lambda X^i$, etc.), and $\nu \neq \lambda, \mu$, then:*

$$C_{abcd}X^aY^bZ^c = 0. \tag{11}$$

By transvecting the relation with another vector one obtains a null Weyl scalar. Such scalars are relevant in the determination of Petrov types. The following example is an interesting illustration:

Example 2.7. *Consider the symmetric tensor $T_{ab} = \lambda(X_aY_b + X_bY_a) + \mu(Z_aW_b + Z_bW_a)$, with $\lambda + \mu \neq 0$, where all vectors are null and the only non-zero scalar products are $X_aY^a = -1$ and $Z_aW^a = 1$. X, Y are eigenvectors with eigenvalue $-\lambda$ while Z, W are eigenvectors with eigenvalue μ. If T is Weyl compatible, by Proposition 2.6 the four vector fields $X^aY^bZ^cC_{abcd}$, $X^aY^bW^cC_{abcd}$, $W^aZ^bX^cC_{abcd}$ and $W^aZ^bY^cC_{abcd}$ vanish. By transvecting with the same vectors, one obtains 9 nontrivial scalar fields.*
In a $n = 4$ space-time, the stress-energy tensor for the electromagnetic field written in the complex null-tetrad basis is [24] *(page 62)*

$$T_{ab} = 2\Phi_1\overline{\Phi}_1(k_al_b + k_bl_a + m_a\overline{m}_b + m_b\overline{m}_a)$$

It has the form above with $k = X$, $l = Y$, $m = Z$, $\overline{m} = Y$. Two Weyl scalars are easily seen to vanish: $\Psi_1 = k^al^bm^ck^dC_{abcd} = 0$, $\Psi_3 = k^al^b\overline{m}^cl^dC_{abcd} = 0$.

In [1] (Prop. 2.4) we showed that Lovelock's identity (8) remains valid if the Riemann tensor in the left hand side is replaced by certain K-tensors. If the Weyl tensor is considered together with Einstein's equations (5), one obtains a differential condition for the Weyl tensor that contains the energy-stress tensor T_{ij}:

$$\begin{aligned}&\nabla_i\nabla_m C_{jkl}{}^m + \nabla_j\nabla_m C_{kil}{}^m + \nabla_k\nabla_m C_{ijl}{}^m \\ &= -8\pi\frac{n-3}{n-2}\left(T_{im}C_{jkl}{}^m + T_{jm}C_{kil}{}^m + T_{km}C_{ijl}{}^m\right)\end{aligned}$$

The left hand side is the exterior covariant derivative $D\Pi_l$ of the 1-form $\Pi_l = \nabla_m C_{jkl}{}^m dx^j \wedge dx^k$. The following theorem follows:

Theorem 2.8. *On a n-dimensional space-time, the Ricci tensor and the energy-stress tensor are Weyl compatible if and only if $D\Pi_l = 0$.*

The condition $D\Pi_l = 0$ is satisfied in space-times that are conformally symmetric ($\nabla_i C_{jkl}{}^m = 0$), conformally recurrent ($\nabla_i C_{jkl}{}^m = \alpha_i C_{jkl}{}^m$) or conformally harmonic ($\nabla_m C_{jkl}{}^m = 0$). On these manifolds the Ricci and the energy-stress tensors are Weyl compatible [1, 20].

McIntosh and others [18, 17] investigated symmetric tensors such that

$$R_{abc}{}^m b_{dm} + R_{abd}{}^m b_{cm} = 0 \tag{12}$$

which is equivalent to $[\nabla_a, \nabla_b]b_{cd} = 0$. The equation has a trivial solution $b_{ab} = \phi\, g_{ab}$, where ϕ is a scalar. Space-times that admit a nontrivial solution, such as Einstein static space-times, Gödel metric, Bertotti-Robinson metric, were investigated by McIntosh and Halford [18]. McIntosh and Hall proved [17] that in $n = 4$ vacuum space-times a nontrivial solution is $b_{ab} = \alpha g_{ab} + \beta u_a u_b$, where u has the property $R_{abc}{}^m u_m = 0$ and α is a scalar field.
Tensors b_{ij} with the property $b_{im}R_{jkl}{}^m = 0$ were studied in [16] in the case $b_{ij} = u_i u_j$.
The following definition includes the aforementioned cases:

Definition 2.9. A symmetric tensor is K-permutable, K-null or K-antipermutable if it has the property

$$b_{im}K_{jkl}{}^m = \omega\, b_{lm}K_{jki}{}^m, \qquad \omega = 1, 0, -1 \tag{13}$$

where K is the Riemann or the Weyl tensor. In terms of differential forms it is $b_{im}F_l{}^m = \omega\, b_{lm}F_i{}^m$, where $F_i{}^m = K_{abi}{}^m dx^a \wedge x^b$ is the 2-form associated to the tensor K.

Proposition 2.10. *If a symmetric tensor b has the property* (13) *(where K is the Riemann (Weyl) tensor) then b is Riemann (Weyl) compatible.*

Proof. In the definition of Riemann (Weyl) compatibility use (13) for each term: $b_{im}R_{jkl}{}^m + b_{jm}R_{kil}{}^m + b_{km}R_{ijl}{}^m = \omega\, b_{lm}(R_{jki}{}^m + R_{kij}{}^m + R_{ijk}{}^m) = 0$ by the first Bianchi identity. □

If (13) holds with the Riemann tensor, then b (being Riemann compatible) is Weyl compatible, but (13) may not hold with the Weyl tensor.
Derdziński-Shen's theorem for the Riemann tensor and Theorem 2.6 for the Weyl tensor become:

Proposition 2.11. *If a symmetric tensor b satisfies* (13)*, and X, Y are two eigenvectors, $b^i{}_j X^j = \lambda X^i$ and $b^i{}_j Y^j = \mu Y^i$ with $\lambda + \omega\mu \neq 0$, then: $K_{jklm}X^l Y^m = 0$.*

Proof. Contraction of (13) with $X^i Y^l$ gives $\lambda K_{jkl}{}^m X_m Y^l = \omega\mu K_{jki}{}^m X^i Y_m$, then $0 = (\lambda + \omega\mu)K_{jklm}X^l Y^m$. □

3. Riemann and Weyl compatible vectors

The notion of K-compatible tensor includes vectors u_i in a natural way, through the symmetric tensor u_iu_j:

Definition 3.1. A vector field u_i is K-compatible (where K is the Riemann, the Weyl or a generalized curvature tensor, [25], vol.1 page 198) if:

$$(u_iK_{jkl}{}^m + u_jK_{kil}{}^m + u_kK_{ijl}{}^m)u_m = 0 \tag{14}$$

Theorem 3.2. *A vector field u with $u^2 \neq 0$ is K-compatible if and only if there is a symmetric tensor D_{ij} such that:*

$$K_{abcm}u^m = D_{ac}u_b - D_{bc}u_a \tag{15}$$

Proof. Multiplication by u_d and cyclic summation over abd makes the right hand side vanish and K-compatibility is obtained. If u is K-compatible then multiplication of (14) by u^i gives $(u^2K_{jkl}{}^m - u_ju^iK_{ikl}{}^m + u_ku^iK_{ijl}{}^m)u_m = 0$ and we may set $D_{jl} = K_{ijlm}u^iu^m/u^2$. □

It is easily shown that u is an eigenvector of the symmetric tensor D. For the Weyl tensor, D will be identified with its electric component (see Sect. 4). On a Lorentzian manifold, the theorem with $K = C$ (Weyl tensor) follows from eq. 20 in [10].

Proposition 3.3. *A concircular vector field on pseudo-Riemannian manifold, $\nabla_k u_l = Ag_{kl} + Bu_ku_l$ with constants A and B, is Riemann (and therefore Weyl) compatible.*

Proof. The condition of concircularity implies that $R_{jkl}{}^m u_m = AB(u_jg_{kl} - u_kg_{jl})$, which has the form (15). □

Statements valid for compatible tensors [14] become stronger for compatible vectors, and new facts arise. For example, the generalized Derdziński and Shen's theorem has now a surprisingly simple proof, with no need of auxiliary $K-$tensors:

Theorem 3.4. *Let K be a generalized curvature tensor, and u be a K-compatible vector. 1) If $u^2 \neq 0$ and v, w are vectors orthogonal to u, $u_av^a = 0$, $u_aw^a = 0$, then:*

$$K_{abcd}w^av^bu^c = 0. \tag{16}$$

2) If $u^2 = 0$ and v is orthogonal to u, $u_av^a = 0$, then:

$$K_{abcd}u^av^bu^c = 0. \tag{17}$$

Proof. 1) The K-compatibility condition $(u_aK_{bcde} + u_bK_{cade} + u_cK_{abde})u^e = 0$ is contracted with $u^av^bw^c$:

$$(u^au_a)v^bw^cK_{bcde}u^e + u^a(u_bv^b)w^cK_{cade}u^e + u^av^b(w^cu_c)K_{abde}u^e = 0.$$

The last two terms cancel because of orthogonality. 2) The K-compatibility condition is contracted with u^av^b and equation $u_cK_{abde}u^av^bu^e = 0$ is obtained. The result follows if u is non zero. □

Remark 3.5. *For the Riemann tensor, $R_{abcd}v^aw^bu^c$ is the vector obtained through parallel transport of u along a parallelogram with infinitesimal vectors v and w. It is known that, if $R_{abcd}v^aw^bu^c = 0$ for any v and w, then it is $R_{abcd}u^c = 0$. If u is Riemann compatible, then it has zero variations along infinitesimal parallelograms with directions orthogonal to it.*

We here give a stronger version of Theorem 5.3 in [14] for the vanishing of the Pontryagin forms:

Theorem 3.6. *Let $X(1),\dots,X(n)$ be an orthonormal basis of a n-dimensional pseudo-Riemannian manifold, $X(a)_k X(b)^k = \pm\delta_{ab}$. If $X(3)\dots X(n)$ are Riemann compatible, then all Pontryagin forms vanish.*

Proof. Among three vectors, at least one is Riemann compatible. Then $R_{ij}{}^{kl}X(a)^i \wedge X(b)^j X(c)_k$ is always zero, by Theorem 3.4. This means that the column vectors of the matrix $R_{ij}{}^{kl}X(a)^i \wedge X(b)^j$ are orthogonal to all vectors $X(c)$ with $c \neq a, b$, i.e. they belong to the subspace spanned by $X(a)$ and $X(b)$. Because of the antisymmetry in k, l, it is necessarily $R_{ij}{}^{kl}X(a)^i \wedge X(b)^j = \lambda_{ab}X(a)^k \wedge X(b)^l$. This condition of pureness of the Riemann tensor implies the vanishing of all Pontryagin forms ([14] Theor. 5.2). □

The identity (10) relating Riemann and Weyl compatibility, is rewritten for vectors:

$$(u_a C_{bclm} + u_b C_{calm} + u_c C_{ablm})u^m = (u_a R_{bclm} + u_b R_{calm} + u_c R_{ablm})u^m + \frac{1}{n-2}\left[g_{cl}u_{[a}R_{b]m} + g_{al}u_{[b}R_{c]m} + g_{bl}u_{[c}R_{a]m}\right]u^m. \tag{18}$$

A first consequence is the restatement of Theorem 2.5 for vectors:

Proposition 3.7. *A vector field u is Riemann compatible if and only if it is Weyl compatible and $u_{[a}R_{b]}{}^m u_m = 0$, i.e. at every point u is either zero or an eigenvector of the Ricci tensor.*

A second consequence is the extension of a theorem by Hall, which he proved for null vectors in $n = 4$ space-times [15]. It is valid in any dimension and metric signature, and for vectors not necessarily null:

Theorem 3.8. *Consider the following conditions on a vector field u:*

$$A)\quad u_{[a}R_{b]clm}u^c u^m + u^2 R_{ablm}u^m = 0,$$
$$B)\quad u_{[a}C_{b]clm}u^c u^m + u^2 C_{ablm}u^m = 0,$$
$$C)\quad u_{[a}R_{b]m}u^m = 0.$$

Any two of these conditions imply the third one. In particular, if $u^2 \neq 0$ the stronger statement holds: A is true if and only if B and C are true.

Proof. Eq.(18) is contracted with u^c,

$$u_{[a}C_{b]clm}u^c u^m + u^2 C_{ablm}u^m = u_{[a}R_{b]clm}u^c u^m + u^2 R_{ablm}u^m + \frac{1}{n-2}\left[u_l u_{[a}R_{b]m}u^m + (g_{al}u_b - g_{bl}u_a)u^c u^m R_{cm} - u^2(g_{al}R_{bm} - g_{bl}R_{am})u^m\right]. \tag{19}$$

If condition C is true, its contraction with u^b gives $(u_a u^b R_{bm} - u^2 R_{am})u^m = 0$ and (19) becomes $u_{[a}C_{b]clm}u^c u^m + u^2 C_{ablm}k^m = u_{[a}R_{b]clm}u^c u^m + u^2 R_{ablm}u^m$. Therefore B and C imply A, or A and C imply B.
Suppose now that A is true; contraction of condition A by g^{al} gives $u^2 R_{bm}u^m - u_b(u^c u^m R_{cm}) = 0$, and (19) becomes:

$$u_{[a}C_{b]clm}u^c u^m + u^2 C_{ablm}u^m = \frac{1}{n-2}u_l u_{[a}R_{b]m}u^m \tag{20}$$

Validity of A and B imply that $u_l u_{[a} R_{b]m} u^m = 0$ i.e. C is true.
A stronger result holds if $u^2 \neq 0$. Contraction of (20) by u^l makes the left-hand-side vanish and condition C is true. Then, the same equation (20) states that also B is true, i.e. A implies B and C. □

Remark 3.9. *Condition B plays a special role in the classification of manifolds. Some cases where it holds are: 1)* $u^m R_{abcm} = 0$ ([16, 8])*; 2) k is a recurrent null vector,* $\nabla_a k_b = \lambda_a k_b$*, with* $\nabla_{[a}\lambda_{b]} = 0$ ([24] page 69)*; 3) Manifolds with constant curvature* ([24] page 101)*:*

$$R_{bclm} = \frac{R}{n(n-1)}(g_{bl}g_{cm} - g_{cl}g_{bm})$$

In cases 1,2 the vector is Riemann compatible.

Proof. 1) The relation implies $R_{am}u^m = 0$. Then the whole right hand side of (18) is zero and $(u_a C_{bclm} + u_b C_{calm} + u_c C_{ablm})u^m = 0$. Multiply by u^c and obtain A.
2) $[\nabla_a, \nabla_b]u_c = R_{abc}{}^m u_m$; because of recurrency and closedness, the left hand side is $\nabla_a(\lambda_b u_c) - \nabla_b(\lambda_a u_c) = (\lambda_b \nabla_a - \lambda_a \nabla_b)u_c = 0$. Then case 1) is obtained.
3) Contraction with g^{cm} shows that the manifolds are Einstein, then condition C holds. If u is a vector, obtain $u_a R_{bclm} u^c u^m = \frac{R}{n(n-1)} u_a (g_{bl}u^2 - u_b u_l)$; then $u_{[a,} R_{b]clm} u^m = u^2 \frac{R}{n(n-1)}(u_a g_{bl} - u_b g_{al}) = -u^2 R_{ablm} u^m$ i.e. condition B is true, and B and C imply A. □

In the same way that compatibility for tensors is translated to vectors, the definitions 13 become:

Definition 3.10. A vector is Riemann (Weyl) permutable if $R_{kl[i}{}^m u_{j]} u_m = 0$ ($C_{kl[i}{}^m u_{j]} u_m = 0$).

If $u^2 \neq 0$, this implies that u is an eigenvector of the curvature operator with an eigenvalue equal to zero: $R_{kljm}u^m = 0$ ($C_{kljm}u^m = 0$).
Next we define "Riemann-null" vectors:

$$R_{abc}{}^m u_m = 0. \tag{21}$$

Vectors of this sort with $u^2 = 0$ describe gravitational waves in Einstein's linearized theory (see [8] page 244). A complete classification of space-times that satisfy (21) is given in Theorem 1.1 of ref.[16].
Eq.(21) arises as the integrability condition for the equation $\nabla_a u_b + \nabla_b u_a = 2\lambda g_{ab}$ with constant λ and the constraint $\nabla_a u_b - \nabla_b u_a = 0$ (u is a *homothetic* vector, see [24] pp. 69, 564).

4. Null and time-like Weyl compatible vectors

$n = 4$ space-times were classified by Petrov according to the degeneracy of the eigenvalues of the self-dual part of the Weyl tensor, which solve an equation of degree four [5]. In type I spaces they are distinct, in type II spaces two coincide and two are distint, in type D spaces they coincide pairwise, in type III spaces three are equal and, finally, in type N spaces all eigenvalues coincide [8]. Type O spaces are conformally flat.
The same types arise in the classification by Bel and Debever [26, 27], which is

based on null vectors k (principal null directions) that solve increasingly restricted equations:

$$\text{type}\, I \qquad k_{[b}C_{a]rs[q}k_{n]}k^r k^s = 0 \tag{22}$$

$$\text{type}\, II, D \qquad k_{[b}C_{a]rsq}k^r k^s = 0 \tag{23}$$

$$\text{type}\, III \qquad k_{[b}C_{a]rsq}k^r = 0 \tag{24}$$

$$\text{type}\, N \qquad C_{arsq}k^r = 0 \tag{25}$$

$$\text{type}\, O \qquad C_{arsq} = 0 \tag{26}$$

When at least two linearly independent vectors k are degenerate, i.e. k meets condition (23), the Weyl tensor is termed *algebraically special* [28, 8]. The classification was generalized to $n > 4$ and includes the above relations [29, 30, 10, 7].

Let's consider the above classification in the perspective of Weyl compatibility. According to the general definition (14), a vector u is Weyl compatible if

$$(u_a C_{bcdm} + u_b C_{cadm} + u_c C_{abdm})u^m = 0. \tag{27}$$

Theorem 4.1. *On a Lorentzian manifold, if a null vector k is Weyl compatible (or Riemann compatible), then the Weyl tensor is algebraically special.*

Proof. Multiply (27) by k^c and use the antisymmetry of Weyl's tensor:
$0 = (k_a C_{bcd}{}^m + k_b C_{cad}{}^m)k^c k_m = k_{[a}C_{b]cd}{}^m k^c k_m = -k_{[a}C_{b]cmd}k^c k^m.$ □

The theorem extends Theorem 1.1 in [16], for null vectors such that $R_{ijk}{}^m k_m = 0$.

Example 4.2. *If a space-time admits a null concircular vector, $\nabla_k u_l = Ag_{kl} + Bu_k u_l$, then the Weyl tensor is algebraically special (see Prop.3.3).*

Example 4.3. *In ref.*[7] *(Table 1), a n-dimensional space-time is type II(d) if the condition* (27) *holds. A null-dust n-dimensional space-time is characterized by the energy-momentum tensor $T_{ab} = \Phi^2 k_a k_b$ with null k (*[24] *eq.5.8). The condition $D\Pi_l = 0$ (see theorem 2.8) is verified if and only if the space-time is type II(d) (with respect to k).*

$n = 4$ space-times with a null Weyl-compatible vector are Petrov type II or D. In a type III space-time three principal directions coincide, i.e. there is a null vector such that $k_{[b}C_{a]rsq}k^r = 0$. This means that the null k is Weyl-permutable, a property that implies Weyl compatibility (see def. 3.10):

Proposition 4.4. *A null vector k satisfies* (24)*, which corresponds to $n = 4$ space-times of Petrov type III, if and only if it is Weyl-permutable.*

On a $n = 4$ space-time the 10 independent components of the Weyl tensor can be accounted for by two symmetric tensors. Given a vector u with $u^a u_a = -1$, the electric and magnetic components of the Weyl tensor are [9]:

$$E_{ab} = u^j u^m C_{jabm}\,, \qquad H_{ab} = u^j u^m \tilde{C}_{jabm} \tag{28}$$

where $\tilde{C}_{abcd} = \frac{1}{2}\epsilon_{abrs}C^{rs}{}_{cd}$ is the dual tensor. The two tensors are symmetric, traceless, and satisfy $E_{ab}u^b = 0$, $H_{ab}u^b = 0$. Then they each have 5 independent components, and completely describe the Weyl tensor.
If they are proportional, $\nu E = \mu H$ for some scalar fields μ and ν (including the case when one of them is zero), the space-time is type I, D or O [24] (page 73).
The following theorem was partly proven in [14] and is stated in [7] for any n:

Theorem 4.5. *On a Lorentzian manifold, a time-like vector u is Weyl-compatible if and only if $H = 0$.*

It follows that a $n = 4$ space-time with a Weyl compatible time-like vector is type I, D or O. This extends Theorem 1.1 in [16].

Example 4.6. *If a space-time admits a time-like concircular vector $\nabla_k u_l = Ag_{kl} + Bu_k u_l$, with constant A and B, then H vanishes.*

Theorem 4.7. *An $n = 4$ space-time with a time-like Weyl permutable vector is conformally flat, $C_{jkl}{}^m = 0$ (type O).*

Proof. Let E and H be the electric and magnetic components evaluated with u. If u is Weyl permutable, then it is Weyl compatible and $H = 0$. Let's show that also E is zero. Multiply the relation (27) for Weyl compatibility by u^j: $u^2 C_{kilm} u^m = u_k E_{il} - u_i E_{kl}$. Because u is Weyl permutable, it follows that $C_{kilm} u^m = 0$ (see after Def. 3.10); then $0 = u_k E_{il} - u_i E_{kl}$. Multiply by u^k and use $u^k E_{kl} = 0$ to obtain $E_{il} = 0$. □

In $n = 4$ the electric and magnetic components of the Weyl tensor can be generalized by replacing $u^i u^j$ by a symmetric tensor T^{ij}:

$$E_{ab} = T^{jm} C_{jabm}, \qquad H_{ab} = T^{jm} \tilde{C}_{jabm}. \tag{29}$$

It is easy to show that E and H are symmetric and traceless tensors.

Proposition 4.8.
1) If T is Weyl-compatible then E commutes with T (any n);
2) $H = 0$ if and only if T is Weyl compatible ($n = 4$).

The proof is based on the two identities:

$$E_{ab} T^b{}_c - T_a{}^b E_{bc} = -[T_{cb} C_{jam}{}^b + T_{ab} C_{cjm}{}^b + T_{jb} C_{acm}{}^b] T^{jm} \tag{30}$$

$$H^a{}_b = \frac{1}{6} [T_j{}^m C_{rsbm} + T_r{}^m C_{sjbm} + T_s{}^m C_{jrbm}] \epsilon^{jars} \tag{31}$$

Proof. $E_{ab} T^b{}_c - T_a{}^b E_{bc} = [C_{jabm} T^b{}_c - C_{jbcm} T_a{}^b] T^{jm}$
$= -[T^b{}_c C_{jamb} + T^b{}_a C_{cjmb} + T^b{}_j C_{acmb}] T^{jm} = -[T_{cb} C_{jam}{}^b + T_{ab} C_{cjm}{}^b + T_{jb} C_{acm}{}^b] T^{jm}$,
where the term added is identically zero. The other identity is:
$H^a{}_b = T_j{}^m \tilde{C}^{ja}{}_{bm} = \frac{1}{2} T_j{}^m C_{rsbm} \epsilon^{jars} = \frac{1}{6} [T_j{}^m C_{rsbm} \epsilon^{jars} + T_r{}^m C_{sjbm} \epsilon^{rasj}$
$+ T_s{}^m C_{jrbm} \epsilon^{sajr}] = \frac{1}{6} [T_j{}^m C_{rsbm} + T_r{}^m C_{sjbm} + T_s{}^m C_{jrbm}] \epsilon^{jars}$. □

5. Hypersurfaces

Let $\mathscr{M}_n$ be a hypersurface in a pseudo-Riemannian manifold $(\mathbb{V}_{n+1}, \tilde{g})$. The metric tensor (first fundamental form) is $g_{ij} = \tilde{g}(B_i, B_j)$, where $B_1 \dots B_n$ are the suitable tangent vectors. If N is the unit vector field normal to the hypersurface, it is $\tilde{g}(B_i, N) = 0$. The Riemann tensor is given by the Gauss equation [21]:

$$R_{jklm} = \tilde{R}_{\mu\nu\rho\sigma} B^\mu{}_j B^\nu{}_k B^\rho{}_l B^\sigma{}_m \pm (\Omega_{jl}\Omega_{km} - \Omega_{jm}\Omega_{kl})$$

with a symmetric tensor Ω_{ij} (second fundamental form) constrained by the Codazzi equation: $\nabla_k \Omega_{jl} - \nabla_j \Omega_{kl} = N^\mu \tilde{R}_{\nu\mu\rho\sigma} B^\nu{}_j B^\rho{}_l B^\sigma{}_k$.

If $\mathbb{V}_{n+1}$ is a constant curvature manifold, the equations simplify:

$$R_{jklm} = \frac{\tilde{R}}{n(n+1)}(g_{jl}g_{km} - g_{jm}g_{kl}) \pm (\Omega_{jl}\Omega_{km} - \Omega_{jm}\Omega_{kl}), \tag{32}$$

$$\nabla_k\Omega_{jl} - \nabla_j\Omega_{kl} = 0. \tag{33}$$

Theorem 5.1. *Let $\mathscr{M}_n$ be a hypersurface isometrically embedded in a pseudo-Riemannian space $\mathbb{V}_{n+1}$ with constant curvature. Then:*
1) Ω is Weyl compatible;
2) the eigenvectors of Ω are Weyl compatible;
3) the Ricci tensor is Weyl compatible.

Proof. 1) For a hypersurface that is isometrically embedded in a constant curvature space, Ω_{ij} is a Codazzi tensor, and then it is both Riemann and Weyl compatible.
2) Given the form (32) of the Riemann tensor, if $\Omega_{km}u^m = \lambda u_k$ then:

$$u_i u^m R_{jklm} = \mu u_i(u_k g_{jl} - u_j g_{kl}) \pm \lambda u_i(\Omega_{jl}u_k - u_j\Omega_{kl})$$

where, for shortness, $\mu = \tilde{R}/n(n+1)$. Summation over cyclic permutations of i, j, k cancels all terms in the right-hand-side, and one is left with Riemann compatibility: $u_i u^m R_{jklm} + u_j u^m R_{kilm} + u_k u^m R_{ijlm} = 0$.
3) The Ricci tensor of a hypersurface isometrically embedded in a constant curvature space is $R_{kl} = \pm(\Omega^2_{kl} - \Omega_p{}^p\Omega_{kl}) + k(n-1)g_{kl}$. Let us first show that Ω^2 is Riemann compatible. Evaluate the expression $(\Omega^2)_{im}R_{jkl}{}^m + (\Omega^2)_{jm}R_{kil}{}^m + (\Omega^2)_{km}R_{ijl}{}^m$ with the Riemann tensor (32). The first term is:

$$\Omega^2_{im}R_{jkl}{}^m = k\left(g_{jl}\Omega^2_{ik} - g_{kl}\Omega^2_{ij}\right) \pm \left(\Omega_{jl}\Omega^3_{ik} - \Omega_{kl}\Omega^3_{ij}\right)$$

After summing over cyclic permutations of ijk, all terms in the right hand side cancel. Therefore the tensor Ω^2 is Riemann compatible and thus Weyl compatible. Since the Ricci tensor is the sum of Riemann compatible terms, it is itself Riemann compatible, and thus Weyl compatible. □

If the Einstein's equations (5) are also considered, it follows that the energy-stress tensor T_{kl} is Riemann compatible (as it commutes with the Ricci tensor). In particular, if $T_{kl} = au_k u_l + bg_{kl}$ with $u^i u_i = -1$, then the Weyl tensor is purely electric; if $T_{ij} = \Phi^2 k_i k_j$ ($k^2 = 0$) then the space-time is type II_d.

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