EXISTENCE AND CONCENTRATION OF SEMI-CLASSICAL SOLUTIONS FOR DIRAC EQUATIONS WITH CRITICAL NONLINEARITIES

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Abstract. We study the semi-classical ground states of the Dirac equation with critical nonlinearity:

$$-i\hbar\alpha \cdot \nabla w + a\beta w + V(x)w = W(x)\big(g(|w|) + |w|\big)w$$

for $x \in \mathbb{R}^3$. The Dirac operator is unbounded from below and above so the associate energy functional is strongly indefinite. We develop an argument to establish the existence of least energy solutions for $\hbar$ small. We also describe the concentration phenomena of the solutions as $\hbar \to 0$.

Key words. nonlinear Dirac equation, semiclassical states, concentration

AMS subject classifications. 35Q40, 49J35

1. Introduction. This paper aims to study the existence and concentration phenomena of semiclassical ground states for the stationary Dirac equation with critical nonlinearities:

$$-i\hbar\sum_{k=1}^{3}\alpha_k\partial_k w + a\beta w + V(x)w = W(x)\big(g(|w|) + |w|\big)w \tag{1.1}$$

with $w : \mathbb{R}^3 \to \mathbb{C}^4$. Equation (1.1) is a first order partial differential equation on $\mathbb{R}^3$; the relevant Sobolev embedding is $H^{1/2}(\mathbb{R}^3, \mathbb{C}^4) \subset L^3(\mathbb{R}^3, \mathbb{C}^4)$, i.e. the Sobolev critical growth is 3. Thus the term $|w|w$ has critical growth, while $g(|w|)w$ is assumed to be superlinear and subcritical as $|w| \to \infty$. In (1.1), $\hbar$ denotes Plank's constant, $\partial_k = \frac{\partial}{\partial x_k}$, $a > 0$ is a constant, $\alpha_1, \alpha_2, \alpha_3$ and β are 4×4 complex matrices:

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad k = 1, 2, 3$$

with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and $V, W : \mathbb{R}^3 \to \mathbb{R}$ are continuous functions. In the sequel, for notational convenience, we will write $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\alpha \cdot \nabla = \sum_{k=1}^{3} \alpha_k \partial_k$, as well as

$$\tau := \min V, \quad \mathscr{V} := \{x \in \mathbb{R}^3 : V(x) = \tau\},$$
$$\tau_\infty := \liminf_{|x|\to\infty} V(x),$$
$$\pi := \max W, \quad \mathscr{W} := \{x \in \mathbb{R}^3 : W(x) = \pi\},$$
$$\pi_\infty := \limsup_{|x|\to\infty} W(x).$$

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Our arguments depend crucially on these numbers and sets.

Equation (1.1) or the more general one

$$-i\hbar\alpha \cdot \nabla w + a\beta w + M(x)w = F_w(x, w),\qquad(1.2)$$

arises when one seeks standing wave solutions of the nonlinear Dirac equation

$$-i\hbar\partial_t\psi = ic\hbar\alpha \cdot \nabla\psi - mc^2\beta\psi - V(x)\psi + G_\psi(x, \psi).\qquad(1.3)$$

Such equations have been widely used to build relativistic models of extended particles by means of nonlinear Dirac fields. Different functions G model various types of self-couplings [25]. Assuming that $G(x, e^{i\theta}\psi) = G(x, \psi)$ for all $\theta \in [0, 2\pi]$, a standing wave solution of (1.3) is a solution of the form $\psi(t, x) = e^{\frac{i\mu t}{\hbar}}w(x)$. It is clear that $\psi(t, x)$ solves (1.3) if and only if $w(x)$ solves (1.2) with $a = mc$, $M(x) = V(x)/c + \mu I_4$ and $F(x, w) = G(x, w)/c$.

Several papers have been devoted to the study of the existence of solutions of (1.2) under various hypotheses on the potential functions and the nonlinearity (see [18] for a review). In [5] the authors studied the problem with $M(x) \equiv \omega \in (-a, a)$ and the nonlinearity (the so-called Soler model)

$$F(w) = \tfrac{1}{2}H(\tilde{w}w) , \ H \in C^2(\mathbb{R}, \mathbb{R}) , \ H(0) = 0, \ \tilde{w}w := (\beta w, w)_{\mathbb{C}^4} ,$$

by using shooting methods. Such kind of nonlinearities were later studied in [17], where for the first time variational methods were applied to such problems (in fact, [17] also considered certain more general super-linear subcritical $F(w)$ independent of x). If the equation is periodic, that is, $M(x)$ and $F(x, w)$ depend periodically on x, by using a critical point theory the paper [7] established also the existence and multiplicity of solutions of (1.2) with scalar potentials of the type $M(x) = V(x)\beta$. Concerning non-periodic potentials (typically, Coulomb-type potentials), [15] considered some asymptotically linear nonlinearities, and [16] treated superlinear and subcritical nonlinearities with mainly the limits of $M(x)$ and $F(x, w)$ existing as $|x| \to \infty$.

For small $\hbar$, the standing waves are referred to as semi-classical states. To describe the transition from quantum to classical mechanics, the existence of solutions $w_\hbar, \hbar$ small, possesses an important physical interest. Recently, the paper [14] studied the existence of a family of ground states of the problem

$$-i\hbar\alpha \cdot \nabla w + a\beta w = W(x)|w|^{q-2}w, \quad (q \in (2, 3))$$

for all $\hbar$ small, and showed that the family concentrates around the maxima of $W(x)$ as $\hbar \to 0$.

To our knowledge, there are no results studying the existence and concentration phenomenon of semiclassical solutions for nonlinear Dirac equations involving the critical exponent of the relevant Sobolev embedding. Since the pioneering paper by Brezis-Nirenberg [8] on elliptic boundary value problems involving the critical Sobolev exponent, there have been a large number of works in this direction. In particular, many papers have been devoted to studying critical Schrödinger equations (not semiclassical case), see, e.g. [30] and the references therein.

The main objective of this paper is to study such situations for critical Dirac equations. For describing our study we present first some special consequences of the more general results of the paper. First, consider the problem

$$-i\hbar\alpha \cdot \nabla u + a\beta u = W(x)\big(|u|^{q-2} + |u|\big)u.\qquad(1.4)$$

THEOREM 1.1. *Let $q \in (2,3)$ and assume that W satisfies*
  (W_0) $W \in C^1(\mathbb{R}^3, \mathbb{R})$, $\inf W > 0$, *and* $\pi > \pi_\infty$.
Then there exists $\pi_0 > 0$ such that, if $\pi_\infty > \pi_0$, for sufficiently small $\hbar > 0$, (1.4) possesses a least energy solutions $w_\hbar \in \bigcap_{s \geq 2} W^{1,s}$. If additionally ∇W is bounded, then $w_\hbar$ satisfies:
  (a_1) *There exists a maximum point $x_\hbar$ of $|w_\hbar|$ with $\lim_{\hbar \to 0} dist(x_\hbar, \mathscr{W}) = 0$, such that, for some $c, C > 0$*

$$|w_\hbar(x)| \leq C \exp\Big(-\frac{c}{\hbar}|x - x_\hbar| \Big). \tag{1.5}$$

  (a_2) *Setting $v_\hbar(x) := w_\hbar(\hbar x + x_\hbar)$, for any sequence $\hbar \to 0$, $v_\hbar$ converges in H^1 to a least energy solution of*

$$-i\alpha \cdot \nabla v + a\beta v = \pi\big(|v|^{q-2} + |v|\big)v.$$

  In fact, one can choose

$$\pi_0 = a\Big(\frac{6\gamma_q}{S^{3/2}} \Big)^{(q-2)/2(3-q)},$$

where here (and in the sequel) S is the best Sobolev embedding constant:

$$S|u|_{L^6}^2 \leq |\nabla u|_{L^2}^2 \quad \text{for } u \in H^1(\mathbb{R}^3),$$

and γ_q denotes the least energy of the ground state for the superlinear subcritical equation (which exists, see [17, 16])

$$-i\alpha \cdot \nabla u + \beta u = |u|^{q-2}u. \tag{1.6}$$

  We also consider the equation with linear potential

$$-i\hbar\alpha \cdot \nabla u + a\beta u + V(x)u = |u|^{q-2}u + |u|u. \tag{1.7}$$

THEOREM 1.2. *Let $q \in (2,3)$ and assume that V satisfies $\tau > -a$ and*
  (V_0) $V \in C^1(\mathbb{R}^3, \mathbb{R})$, $V(x) \leq 0$ *and* $\tau < \tau_\infty$.
Then there is $\tau_0 > 0$ such that, if $(a + \tau_\infty)^{12-5q} < \tau_0$, for sufficiently small $\hbar > 0$, (1.7) possesses a least energy solutions $w_\hbar \in \bigcap_{s \geq 2} W^{1,s}$. If additionally ∇V is bounded, and either $12 \geq 5q$ or $(a + \tau)^{12-5q} < \tau_0$, then $w_\hbar$ satisfies:
  (a_1) *There exists a maximum point $x_\hbar$ of $|w_\hbar|$ with $\lim_{\hbar \to 0} dist(x_\hbar, \mathscr{V}) = 0$, such that, for some $c, C > 0$, (1.5) holds.*
  (a_2) *Setting $v_\hbar(x) := w_\hbar(\hbar x + x_\hbar)$, for any sequence $\hbar \to 0$, $v_\hbar$ converges in H^1 to a least energy solution of*

$$-i\alpha \cdot \nabla v + a\beta v + \tau v = \big(|v|^{q-2} + |v|\big)v.$$

  In fact, one may take $\tau_0 = a^{12-5q}\pi_0^{-2(3-q)}$ where π_0 is the number defined above.

  What will happen if $\tau = -a$?
  THEOREM 1.3. *Let (V_0) be satisfied, $\tau = -a$ and $q \in (2,3)$. Then for sufficiently small $\hbar > 0$, (1.7) possesses a least energy solutions $w_\hbar \in \bigcap_{s \geq 2} W^{1,s}$. In addition, $w_\hbar \to 0$ in H^1 as $\hbar \to 0$.*

REMARK 1.4. *The concentration phenomena showed in Theorems* 1.1 *and* 1.2 *can be re-described as follows: There is* $\sigma > 0$ *such that, for any given* $\delta, r > 0$, *there exists* $\hbar_0 > 0$ *satisfying*

$$\sup_{x \in N_r(\mathscr{A})} |w_\hbar(x)| \geq \sigma$$

$$\sup_{x \in N_r^c(\mathscr{A})} |w_\hbar(x)| \leq \delta$$

provided $0 < \hbar \leq \hbar_0$, *where* $\mathscr{A} = \mathscr{V}$ *in case* (V_0) *and* $\mathscr{A} = \mathscr{W}$ *in case* (W_0), $N_r(\mathscr{A})$ *denotes a neighborhood of* $\mathscr{A}$ *with radius* r, *and* $N_r^c(\mathscr{A}) = \mathbb{R}^3 \setminus N_r(\mathscr{A})$.

Our argument is variational: the semiclassical solutions are obtained as critical points of an energy functional $F_\hbar$ associated to an equivalent problem of (1.1). Different from the Laplacian in the Schrödinger equation, the Dirac operator is unbounded from above and below. As a result, the functional $F_\hbar$ is strongly indefinite and, hence, possesses an infinite-dimensional linking structure instead of a Mountain-Pass. Our arguments will be based on a suitable functional analytic framework. The linking structure yields a minimax value $c_\hbar$ for $F_\hbar$. Since the problem is posed on the whole space $\mathbb{R}^3$, $F_\hbar$ does not satisfy the general Palais-Smale condition, and so it cannot be directly concluded that $c_\hbar$ is a critical value. This will be checked via a reduced functional $I_\hbar$ with Nehari manifold $\mathscr{N}_\hbar$, which is such that $c_\hbar$ is nothing but the minimum of $I_\hbar$ on $\mathscr{N}_\hbar$. Comparing with [14], since the solutions depend not only on the linear potential but also on the nonlinear one, the present arguments are more delicate. One new ingredient is a comparison of $c_\hbar$ with the least energy of a class of limit problems. Another is an estimate for $c_\hbar$ through a discussion about some auxiliary functionals. And the third is a boundedness estimate for the maximum of the semiclassical ground states. Since either the linear part or the nonlinear one is not invariant under the $\mathbb{R}^3$-group action, such an estimate enables us to establish the concentration phenomena.

2. The main results. Now we describe more precisely the main results. Writing $\varepsilon = \hbar$, we are concerned with the equation

$$-i\varepsilon\alpha \cdot \nabla w + a\beta w + V(x)w = W(x)\big(g(|w|) + |w|\big)w. \tag{2.1}$$

On the nonlinear field, writing $G(|w|) := \int_0^{|w|} g(s)s\,ds$, we consider the following hypotheses:

(g_1) $g(0) = 0$, $g \in C^1(0, \infty)$, $g'(s) > 0$ *for* $s > 0$, *and there exist* $p \in (2, 3)$, $c_1 > 0$ *such that* $g(s) \leq c_1(1 + s^{p-2})$ *for* $s \geq 0$;
(g_2) *there exist* $q > 2$, $\theta > 2$ *and* $c_0 > 0$ *such that* $c_0 s^q \leq G(s) \leq \frac{1}{\theta} g(s)s^2$ *for all* $s > 0$.

Clearly, the power function $g(s) = s^{q-2}$ for $s \geq 0$ satisfies these assumptions. Set

$$\mathcal{R}_q := \Big(\frac{S^{3/2} c_0^{2/(q-2)}}{6\gamma_q}\Big)^{q-2},$$

where q and c_0 are the constants from (g_2), and γ_q is the least energy of (1.6). On the linear fields we will use the following hypotheses:

(P_0) $V, W \in C^1(\mathbb{R}^3, \mathbb{R})$, $V(x) \leq 0$, $\inf W > 0$;
($\hat{P}_0$) $\tau > -a$ *and* $\big(\frac{a+\tau_\infty}{a}\big)^{12-5q}\big(\frac{a}{\pi_\infty}\big)^{2(3-q)} < \mathcal{R}_q$;
(P_1) $\pi > \pi_\infty$ *and* $\min_{x \in \mathscr{W}} V(x) \leq \tau_\infty$; *or*

(P_2) $\tau < \tau_\infty$ and $\max\limits_{x \in \mathscr{V}} W(x) \geq \pi_\infty$.

These conditions suffice for the existence of semiclassical solutions. In order to show the concentration phenomenon we require a further technical assumption:

(P_3) ∇V and ∇W are bounded, and

$$either \ \ 5q \leq 12 \ , \ \ or \ \ \Big(\frac{a+\tau}{a}\Big)^{12-5q}\Big(\frac{a}{\pi}\Big)^{2(3-q)} < \mathcal{R}_q \ .$$

THEOREM 2.1. Let (g_1)-(g_2), (P_0)-$(\hat{P}_0)$, and either (P_1) or (P_2) be satisfied. Then, for sufficiently small $\varepsilon > 0$, (2.1) possesses a least energy solution $w_\varepsilon \in \bigcap_{s \geq 2} W^{1,s}$.

If additionally (P_3) also holds and $\mathscr{V} \cap \mathscr{W} \neq \emptyset$, then w_ε satisfies:

(a_1) There exists a maximum point x_ε of $|w_\varepsilon|$ with $\lim\limits_{\varepsilon \to 0} dist(x_\varepsilon, \mathscr{V} \cap \mathscr{W}) = 0$, such that, for some $c, C > 0$

$$|w_\varepsilon(x)| \leq C \exp\Big(-\frac{c}{\varepsilon}|x - x_\varepsilon|\Big). \tag{2.2}$$

(a_2) Setting $v_\varepsilon(x) := w_\varepsilon(\varepsilon x + x_\varepsilon)$, v_ε converges in H^1 (up to subsequences) to a least energy solution of

$$-i\alpha \cdot \nabla v + a\beta v + \tau v = \pi\big(g(|v|) + |v|\big)v. \tag{2.3}$$

More generally, also if $\mathscr{V} \cap \mathscr{W} \neq \emptyset$ is not necessarily satisfied, we can describe certain concentration phenomena. To this end we introduce the following notations: in case (P_1) set $\tau_w = \min_{x \in \mathscr{W}} V(x)$ and

$$\mathscr{A}_w := \{x \in \mathscr{W} : V(x) = \tau_w\} \cup \{x \notin \mathscr{W} : V(x) < \tau_w\};$$

in case (P_2) set $\pi_v = \max_{x \in \mathscr{V}} W(x)$ and

$$\mathscr{A}_v := \{x \in \mathscr{V} : W(x) = \pi_v\} \cup \{x \notin \mathscr{V} : W(x) > \pi_v\}.$$

Obviously, $\mathscr{A}_v$ and $\mathscr{A}_w$ are bounded. Moreover, if $\mathscr{V} \cap \mathscr{W} \neq \emptyset$ then $\mathscr{A}_v = \mathscr{A}_w = \mathscr{V} \cap \mathscr{W}$.

First, consider the equation with the nonlinear potential W leading the behavior.

THEOREM 2.2. Let (g_1)-(g_2), (P_0)-$(\hat{P}_0)$ and (P_1) be satisfied. Then, for sufficiently small $\varepsilon > 0$, (2.1) possesses a least energy solution $w_\varepsilon \in \bigcap_{s \geq 2} W^{1,s}$. If additionally (P_3) also holds, then w_ε satisfies:

(a_1) There exists a maximum point x_ε of $|w_\varepsilon|$ with $\lim\limits_{\varepsilon \to 0} dist(x_\varepsilon, \mathscr{A}_w) = 0$, such that (2.2) holds. for some constants $c, C > 0$.

(a_2) Setting $v_\varepsilon(x) := w_\varepsilon(\varepsilon x + x_\varepsilon)$, for any sequence $x_\varepsilon \to x_0$ as $\varepsilon \to 0$, v_ε converges in H^1 to a least energy solution of

$$-i\alpha \cdot \nabla v + a\beta v + V(x_0)v = W(x_0)\big(g(|v|) + |v|\big)v. \tag{2.4}$$

Next, consider the equation with the linear potential V leading the behavior.

THEOREM 2.3. Let (g_1)-(g_2), (P_0)-$(\hat{P}_0)$ and (P_2) be satisfied. Then for sufficiently small $\varepsilon > 0$, (2.1) possesses a least energy solution $w_\varepsilon \in \bigcap_{s \geq 2} W^{1,s}$. If in addition also (P_3) holds, then w_ε satisfies

(a_1) There exists a maximum point x_ε of $|w_\varepsilon|$ with $\lim\limits_{\varepsilon \to 0} dist(x_\varepsilon, \mathscr{A}_v) = 0$, such that, for some $c, C > 0$, (2.2) holds.

(a_2) *Setting $v_\varepsilon(x) := w_\varepsilon(\varepsilon x + x_\varepsilon)$, for any sequence $x_\varepsilon \to x_0$ as $\varepsilon \to 0$, v_ε converges in H^1 to a least energy solution of* (2.4).

It is clear that Theorem 2.1 is a consequence of Theorems 2.2 and 2.3.

REMARK 2.4. *We point out that if both* (P_1) *and* (P_2) *are satisfied, then in general it happens that* (2.1) *possesses two families of semiclassical ground states, one concentrating on $\mathscr{A}_v$ and another on $\mathscr{A}_w$.*

Finally, we consider the case (P_2) with $\tau = -a$.

THEOREM 2.5. *Let* (g_1)−(g_2), (P_0) *and* (P_2) *be satisfied. Assume $\tau = -a$. Then, for sufficiently small $\varepsilon > 0$,* (2.1) *possesses a least energy solution $w_\varepsilon \in \bigcap_{s \geq 2} W^{1,s}$. Moreover, $w_\varepsilon \to 0$ in H^1 as $\varepsilon \to 0$.*

Theorem 2.2 applies to the equation (2.1) with (V_0) and $W(x)$ being constant, in particular, Theorem 1.1 is a corollary of Theorem 2.2. Likely, Theorem 2.3 applies to the equation (2.1) with (W_0) and $V(x)$ being constant, so Theorem 1.2 is a particular case of Theorems 2.3. It is clear that Theorem 1.3 is a special case of Theorem 2.5.

REMARK 2.6. *The formulas in* ($\hat{P}_0$) *and* (P_3) *are equivalent, respectively, to*

$$\pi_\infty > a\Big(\mathcal{R}_q^{-1}\Big(\frac{a + \tau_\infty}{a}\Big)^{12-5q}\Big)^{1/2(3-q)},$$

$$\pi > a\Big(\mathcal{R}_q^{-1}\Big(\frac{a + \tau}{a}\Big)^{12-5q}\Big)^{1/2(3-q)}.$$

(The latter follows from the former if $12 \geq 5q$; this is why we made the assumption only if $12 < 5q$.)

Observe that, setting $u(x) = w(\varepsilon x)$, $V_\varepsilon(x) = V(\varepsilon x)$ and $W_\varepsilon(x) = W(\varepsilon x)$, the equation (2.1) is equivalent to the following

$$-i\alpha \cdot \nabla u + a\beta u + V_\varepsilon(x)u = W_\varepsilon(x)\big(g(|u|) + |u|\big)u \qquad (2.5)$$

We will in the sequel focus on these equivalent problems.

3. Variational setting. In what follows by $|\cdot|_q$ we denote the usual L^q-norm, and $(\cdot, \cdot)_2$ the usual L^2-inner product. Let $A_a = -i\alpha \cdot \nabla + a\beta$ denote the selfadjoint operator on $L^2(\mathbb{R}^3, \mathbb{C}^4)$ with domain $\mathcal{D}(A_a) = H^1(\mathbb{R}^3, \mathbb{C}^4)$. A Fourier analysis shows that $\sigma(A_a) = \sigma_c(A_a) = \mathbb{R} \setminus (-a, a)$ where $\sigma(\cdot)$ and $\sigma_c(\cdot)$ denote the spectrum and continuous spectrum. Thus the space L^2 possesses the orthogonal decomposition:

$$L^2 = L^- \oplus L^+, \quad u = u^- + u^+$$

so that A_a is negative definite (resp. positive definite) in L^- (resp. L^+). Let $E := \mathcal{D}(|A_a|^{1/2}) = H^{1/2}$ be equipped with the inner product

$$(u, v) = \Re(|A_a|^{1/2}u, |A_a|^{1/2}v)_2$$

and the induced norm $\|u\| = (u, u)^{1/2}$, where $|A_a|$ and $|A_a|^{1/2}$ denote respectively the absolute value of A_a and the square root of $|A_a|$. Since $\sigma(A_a) \subset \mathbb{R} \setminus (-a, a)$, one has

$$a|u|_2^2 \leq \|u\|^2 \qquad \text{for all } u \in E. \qquad (3.1)$$

Note that this norm is equivalent to the usual $H^{1/2}$-norm, hence E embeds continuously into L^q for all $q \in [2, 3]$ and compactly into L_{loc}^q for all $q \in [1, 3)$. It is clear that E possesses the following decomposition

$$E = E^- \oplus E^+ \quad \text{with} \quad E^\pm = E \cap L^\pm,$$

orthogonal with respect to both $(\cdot,\cdot)_2$ and $(\cdot,\cdot)$ inner products. This decomposition induces also a natural decomposition of L^q, hence there is $b_q > 0$ such that

$$b_q|u^\pm|_q^q \le |u|_q^q \quad \text{for all } u \in E. \tag{3.2}$$

In the sequel, set $\mathbb{R}^+ = [0,\infty)$, and define, for any $u \in E^+ \setminus \{0\}$,

$$E_u := E^- \oplus \mathbb{R}u, \quad \hat{E}_u := E^- \oplus \mathbb{R}^+u\,.$$

Let S and S_p denote the folllowing Sobolev embedding constants:

$$\begin{aligned} S|u|_6^2 &\le |\nabla u|_2^2 \\ S_p|u|_p^2 &\le \int_{\mathbb{R}^3} |\nabla u|^2 + |u|^2 \end{aligned} \quad, \quad \text{for all } u \in H^1\,.$$

LEMMA 3.1. *Let $A_\delta := -i\alpha \cdot \nabla + \delta\beta$ for all $\delta \ne 0$. Denoting $p_t = 6/(3-2t)$ for $t \in [0,1]$, one has $|u|_{p_t} \le S^{-t/2}\||A_\delta|^t u|_2$.*

Proof. See [7]. We consider two interpolation couples $\{Y_0, Y_1\}$ and $\{X_0, X_1\}$ with $Y_0 = L^2$, $Y_1 = L^6$, and $X_0 = L^2 = \mathcal{D}(|A_\delta|^0)$, $X_1 = \mathcal{D}(|A_\delta|)$. Let $Y_+ = Y_0 + Y_1$ be equipped with the norm

$$\|u\|_+ := \inf\{|v|_2 + |w|_6 : \ u = v + w,\ v \in Y_0,\ w \in Y_1\},$$

and $X_+ = X_0 + X_1$ be equipped with the norm

$$\|u\|_+ := \inf\{|v|_2 + \|w\|_{X_1} : \ u = v + w,\ v \in X_0,\ w \in X_1\}.$$

Consider the (complex) interpolation spaces $Y_t = L^{p_t}$ and $X_t = \mathcal{D}(|A_\delta|^t)$. Let $\imath : X_+ \to Y_+$ be the embedding operator. Then $\imath(X_1) \subset Y_1$ and the Sobolev embedding theorem yields $\|\imath\|_{\mathcal{L}(X_1,Y_1)} \le S^{-1/2}$. The Calderón-Lions interpolation theorem ([26]) now implies $\imath(X_t) \subset Y_t$ and

$$\|\imath\|_{\mathcal{L}(X_t,Y_t)} \le \|\imath\|_{\mathcal{L}(X_0,Y_0)}^{1-t}\|\imath\|_{\mathcal{L}(X_1,Y_1)}^t = \|\imath\|_{\mathcal{L}(X_1,Y_1)}^t = S^{-t/2},$$

ending the proof.   □

REMARK 3.2. *1) We note in particular that $S^{1/2}|u|_3^2 \le \||(-i\alpha \cdot \nabla + \delta\beta)|^{1/2}u|_2^2$ for any δ and all $u \in E$.*
2) One verifies similarly that, for any $p \in [2,3)$,

$$\delta^{(6-2p)/p}\, S_{2p/(4-p)}^{1/2}\, |u|_p^2 \le \||A_\delta|^{1/2}u\|_2^2.$$

$$S_{2p/(4-p)}^{1/2}\, |u|_p^2 \le \||A_1|^{1/2}u\|_2^2.$$

In virtue of the assumptions $(g_1) - (g_2)$, for any $\delta > 0$ with $\delta < (a-\tau)/4$, there exist $r_\delta > 0, c_\delta > 0$ and $c'_\delta > 0$ such that

$$\begin{cases} g(s) < \delta & \text{for all } 0 \le s \le r_\delta; \\ G(|u|) \ge c_\delta|u|^\theta - \delta|u|^2 & \text{for all } u \in \mathbb{C}^4; \\ G(|u|) \le \delta|u|^2 + c'_\delta|u|^p & \text{for all } u \in \mathbb{C}^4. \end{cases} \tag{3.3}$$

Denote $f(s) := g(s) + s$ for $s \geq 0$, and

$$F(|u|) := \int_0^{|u|} f(s)s \, ds = G(|u|) + \frac{1}{3}|u|^3.$$

On E we set $a(u,v) := \left(|A_a|^{1/2}(u^+ - u^-), |A_a|^{1/2}v\right)_2 = (u^+ - u^-, v)$, and $a(u) = a(u,u) = \|u^+\|^2 - \|u^-\|^2$. In the sequel, for convenience, we will write also $\int_{\mathbb{R}^3} \langle A_a u, v \rangle$ for $a(u,v)$. Define the functional

$$\Phi_\varepsilon(u) = \frac{1}{2}a(u) + \frac{1}{2}\int_{\mathbb{R}^3} V_\varepsilon(x)|u|^2 - \int_{\mathbb{R}^3} W_\varepsilon(x)F(|u|)$$

$$= \frac{1}{2}\left(\|u^+\|^2 - \|u^-\|^2\right) + \frac{1}{2}\int_{\mathbb{R}^3} V_\varepsilon(x)|u|^2 - \int_{\mathbb{R}^3} W_\varepsilon(x)F(|u|).$$

Plainly, $\Phi_\varepsilon \in C^2(E, \mathbb{R})$.

LEMMA 3.3. *Critical points of Φ_ε are solutions of* (2.5).

Proof. Observe that, for any $u, v \in E$,

$$\frac{d}{ds}\Phi_\varepsilon(u + sv)\Big|_{s=0}$$

$$= \Re a(u,v) + \Re \int_{\mathbb{R}^3} V_\varepsilon(x)\langle u, v \rangle - \Re \int_{\mathbb{R}^3} W_\varepsilon(x)f(|u|)\langle u, v \rangle$$

$$= (u^+ - u^-, v) + \Re \int_{\mathbb{R}^3} \left(V_\varepsilon(x) - W_\varepsilon(x)f(|u|)\right)\langle u, v \rangle.$$

Let $u \in E$ be a critical point of Φ_ε. For any real vector $v \in C_0^\infty(\mathbb{R}^3, \mathbb{R}^4)$ one has formally

$$0 = (u^+ - u^-, v) + \Re \int_{\mathbb{R}^3} \left(V_\varepsilon(x) - W_\varepsilon(x)f(|u|)\right)\langle u, v \rangle$$

$$= \left(\Re(A_a u) + V_\varepsilon(\Re u) - W_\varepsilon f(|u|)(\Re u), \ v\right)_2$$

and

$$0 = (u^+ - u^-, iv) + \Re \int_{\mathbb{R}^3} \left(V_\varepsilon(x) - W_\varepsilon(x)f(|u|)\right)\langle u, iv \rangle$$

$$= \left(\Im(A_a u) + V_\varepsilon(\Im u) - W_\varepsilon f(|u|)(\Im u), \ v\right)_2.$$

Hence, for any general $v \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^4)$, there holds

$$0 = \left(A_a u + V_\varepsilon u - W_\varepsilon f(|u|)u, \ v\right)_2$$

which implies that u is a weak solution of (2.5). Now a standard regularity argument shows that u is in fact a solution of (2.5).   □

Note that the functional $\Psi_\varepsilon(u) := \int_{\mathbb{R}^3} W_\varepsilon(x)F(|u|)$ is weakly sequentially lower semi-continuous, and Φ_ε' is weakly sequentially continuous.

It is easy to check by using (3.1) – (3.3) that the functional Φ_ε possesses the linking structure:

LEMMA 3.4. *Φ_ε possesses the linking structure:*

1) *There exist $r > 0$ and $\rho > 0$ both independent of ε such that $\Phi_\varepsilon|_{B_r^+}(u) \geq 0$ and $\Phi_\varepsilon|_{S_r^+} \geq \rho$, where $B_r^+ = \{u \in E^+ : \|u\| \leq r\}$ and $S_r^+ = \{u \in E^+ : \|u\| = r\}$;*

2) *For any $e \in E^+ \setminus \{0\}$, there exist $R_e > 0$ and $C = C_e > 0$ both independent of ε such that $\Phi_\varepsilon(u) < 0$ for all $u \in \hat{E}_e \setminus B_R$ and $\max \Phi_\varepsilon(\hat{E}_e) \leq C$.*

Define the following minimax value (see [23, 28, 6])

$$c_\varepsilon := \inf_{e \in E^+ \setminus \{0\}} \max_{u \in E_e} \Phi_\varepsilon(u) = \inf_{e \in E^+ \setminus \{0\}} \max_{u \in \hat{E}_e} \Phi_\varepsilon(u).$$

As a consequence of Lemma 3.4 we have

LEMMA 3.5. *There is $C > 0$ independent of ε such that $\rho \leq c_\varepsilon < C$.*

Proof. By 1) of Lemma 3.4 and the definition of c_ε one has $c_\varepsilon \geq \rho$. Take $e \in E^+$ with $\|e\| = 1$. It follows from 2) of Lemma 3.4 the following

$$c_\varepsilon \leq C \equiv C_e,$$

ending the proof. □

Recall that a sequence $\{u_n\} \subset E$ is said to be a $(PS)_c$, $c \in \mathbb{R}$, sequence for Φ_ε if $\Phi_\varepsilon(u_n) \to c$ and $\Phi'_\varepsilon(u_n) \to 0$, and Φ_ε is said to satisfy the $(PS)_c$ condition if any $(PS)_c$ sequence for Φ_ε has a convergent subsequence. With Lemma 3.4 and by a linking argument it follows that Φ_ε has a $(PS)_{c_\varepsilon}$ sequence (see e.g. [13, 28]). Obviously, if Φ_ε satisfies the $(PS)_c$ condition then c_ε is a critical value. Unfortunately, since there is no compact embedding from $H^{1/2}(\mathbb{R}^3)$ into $L^p(\mathbb{R}^3)$, the (PS) condition does not in general hold, and we have to go through a more delicate analysis.

Motivated by Ackermann [1] (see also [16, 23, 28]), we consider, for a fixed $u \in E^+$, the map $\phi_u : E^- \to \mathbb{R}$ defined by

$$\phi_u(v) = \Phi_\varepsilon(u + v).$$

Observe that, for any $v, w \in E^-$ (recalling that $-\tau = |V|_\infty$),

$$\phi''_u(v)[w, w] = -\|w\|^2 + \int_{\mathbb{R}^3} V_\varepsilon(x)|w|^2 - \Psi''_\varepsilon(u + v)[w, w]$$

$$\leq -\frac{a + \tau}{a}\|w\|^2 - \Psi''_\varepsilon(u + v)[w, w],$$

and in addition

$$\phi_u(v) \leq \frac{a - \tau}{2a}\|u\|^2 - \frac{a + \tau}{2a}\|v\|^2.$$

Therefore, there is a unique $h_\varepsilon(u) \in E^-$ such that

$$\phi_u(h_\varepsilon(u)) = \max_{v \in E^-} \phi_u(v).$$

It is clear that

$$0 = \phi'_u(h_\varepsilon(u))v$$

$$= -(h_\varepsilon(u), v) + \Re \int_{\mathbb{R}^3} V_\varepsilon(x)\langle u + h_\varepsilon(u), v\rangle - \Psi'_\varepsilon(u + h_\varepsilon(u))v$$

for all $v \in E^-$, and

$$v \neq h_\varepsilon(u) \quad \Leftrightarrow \quad \Phi_\varepsilon(u + v) < \Phi_\varepsilon(u + h_\varepsilon(u)).$$

For any $u \in E^+$ and $v \in E^-$, setting $z = v - h_\varepsilon(u)$ and $\ell(t) = \phi_u(h_\varepsilon(u) + tz)$, one has $\ell(1) = \phi_u(v), \ell(0) = \phi_u(h_\varepsilon(u)$ and $\ell'(0) = 0$. Thus $\ell(1) - \ell(0) = \int_0^1 (1-t)\ell''(t)dt$. This implies that

$$
\phi_u(v) - \phi_u(h_\varepsilon(u))
$$
$$
= \int_0^1 (1-t)\phi_u''\big(h_\varepsilon(u) + tz)\big)[z,\, z]dt
$$
$$
= -\int_0^1 (1-t)\bigg(\|z\|^2 - \int_{\mathbb{R}^3} V_\varepsilon(x)|z|^2 + \int_{\mathbb{R}^3} W_\varepsilon(x)f(|u + h_\varepsilon(u) + tz|)|z|^2
$$
$$
+ \int_{\mathbb{R}^3} W_\varepsilon(x)\frac{f'(|u + h_\varepsilon(u) + tz|)}{|u + h_\varepsilon(u) + tz|}\big(\Re\langle u + h_\varepsilon(u) + tz,\, z\rangle\big)^2 \bigg) dt
$$

hence,

$$
\Phi_\varepsilon(u + h_\varepsilon(u)) - \Phi_\varepsilon(u + v)
$$
$$
= \frac{1}{2}\Big(\|z\|^2 - \int_{\mathbb{R}^3} V_\varepsilon(x)|z|^2 \Big)
$$
$$
+ \int_0^1 \int_{\mathbb{R}^3} (1-t)W_\varepsilon(x)\bigg(f(|u + h_\varepsilon(u) + tz|)|z|^2 \tag{3.4}
$$
$$
+ \frac{f'(|u + h_\varepsilon(u) + tz|)}{|u + h_\varepsilon(u) + tz|}\big(\Re\langle u + h_\varepsilon(u) + tz, z\rangle\big)^2 \bigg).
$$

Define $I_\varepsilon : E^+ \to \mathbb{R}$ by

$$
I_\varepsilon(u) = \Phi_\varepsilon(u + h_\varepsilon(u))
$$
$$
= \frac{1}{2}\big(\|u\|^2 - \|h_\varepsilon(u)\|^2\big) + \frac{1}{2}\int_{\mathbb{R}^3} V_\varepsilon(x)|u + h_\varepsilon(u)|^2 - \Psi_\varepsilon(u + h_\varepsilon(u)).
$$

Set

$$
\mathcal{N}_\varepsilon := \{u \in E^+ \setminus \{0\} :\ I_\varepsilon'(u)u = 0\}
$$

LEMMA 3.6. *For any $u \in E^+ \setminus \{0\}$, there is a unique $t = t(u) > 0$ such that $t(u)u \in \mathcal{N}_\varepsilon$.*

Proof. See [1, 14, 16].   □

LEMMA 3.7. *We have*

$$
c_\varepsilon = \inf_{u \in \mathcal{N}_\varepsilon} I_\varepsilon(u).
$$

Proof. Indeed, given $e \in E^+$, if $u = v + se \in E_e$ with $\Phi_\varepsilon(u) = \max_{z \in E_e} \Phi_\varepsilon(z)$ then the restriction $\Phi_\varepsilon|_{E_e}$ of Φ_ε on E_e satisfies $(\Phi_\varepsilon|_{E_e})'(u) = 0$ which implies $v = h_\varepsilon(se)$ and $I_\varepsilon'(se)(se) = \Phi_\varepsilon'(u)(se) = 0$, i.e. $se \in \mathcal{N}_\varepsilon$. Thus $\inf I_\varepsilon(\mathcal{N}_\varepsilon) \le c_\varepsilon$. On the other hand, if $w \in \mathcal{N}_\varepsilon$ then $(\Phi_\varepsilon|_{E_w})'(w + h_\varepsilon(w)) = 0$ so $c_\varepsilon \le \max_{u \in E_w} \Phi_\varepsilon(u) = I_\varepsilon(w)$. Thus $\inf I_\varepsilon(\mathcal{N}_\varepsilon) \ge c_\varepsilon$. This proves the desired conclusion.   □

LEMMA 3.8. *For any $e \in E^+ \setminus \{0\}$, there is $T_e > 0$ independent of $\varepsilon > 0$ such that $t_\varepsilon \le T_e$ for $t_\varepsilon > 0$ satisfying $t_\varepsilon e \in \mathcal{N}_\varepsilon$.*

Proof. Since $I_\varepsilon'(t_\varepsilon e)(t_\varepsilon e) = 0$ one sees that the restriction of Φ_ε satisfies $(\Phi_\varepsilon|_{E_e})'(t_\varepsilon e + h_\varepsilon(t_\varepsilon e)) = 0$. Thus

$$
\Phi_\varepsilon(t_\varepsilon e + h_\varepsilon(t_\varepsilon e)) = \max_{w \in E_e} \Phi_\varepsilon(w).
$$

This, together with Lemma 3.7 and 2) of Lemma 3.4, implies the desired conclusion. □

Let $\mathscr{K}_\varepsilon := \{u \in E : \Phi'_\varepsilon(u) = 0\}$ be the critical set of Φ_ε. It is easy to see that if $K_\varepsilon \setminus \{0\} \neq \emptyset$ then

$$c_\varepsilon = \inf\{\Phi_\varepsilon(u) : u \in \mathscr{K}_\varepsilon \setminus \{0\}\}$$

(see an argument of [16]). Using the same iterative argument of [17, Proposition 3.2] one obtains easily the following

LEMMA 3.9. *If $u \in \mathscr{K}_\varepsilon$ with $|\Phi_\varepsilon(u)| \leq C_1$ and $|u|_2 \leq C_2$, then, for any $s \in [2, \infty)$, $u \in W^{1,s}(\mathbb{R}^3)$ with $\|u\|_{W^{1,s}} \leq \Lambda_s$ where Λ_s depends only on C_1, C_2 and s.*

Let $\mathscr{S}_\varepsilon$ be the set of all least energy solutions of Φ_ε. If $u \in \mathscr{S}_\varepsilon$ then $\Phi_\varepsilon(u) = c_\varepsilon$ and a standard argument shows that $\mathscr{S}_\varepsilon$ is bounded in E, hence, $|u|_2 \leq C_2$ for $u \in \mathscr{S}_\varepsilon$, some $C_2 > 0$ independent of ε. Therefore, as a consequence of Lemmas 3.7 and 3.9 we see that, for each $s \in [2, \infty)$, there is $C_s > 0$ independent of ε such that

$$\|u\|_{W^{1,s}} \leq C_s \quad \text{for all } u \in \mathscr{S}_\varepsilon. \tag{3.5}$$

This, together with the Sobolev embedding theorem, implies that there is $C_\infty > 0$ independent of ε with

$$|u|_\infty \leq C_\infty \quad \text{for all } u \in \mathscr{S}_\varepsilon. \tag{3.6}$$

4. Preliminary results. For proving our main theorems, we need some results on related autonomous equations. Recall that $G(|u|) \geq c_0|u|^q, q \in (2,3)$. For any $\mu \in (-a, a)$ and $\nu > 0$, consider the equation

$$-i\alpha \cdot \nabla u + a\beta u + \mu u = \nu\big(g(|u|) + |u|\big)u, \quad u \in H^1(\mathbb{R}^3, \mathbb{C}^4). \tag{4.1}$$

Its solutions are critical points of the functional

$$\Gamma^*_{\mu\nu}(u) := \frac{1}{2}\int_{\mathbb{R}^3}\langle(-i\alpha\cdot\nabla + a\beta + \mu)u,\, u\rangle - \nu\int_{\mathbb{R}^3}\left(G(|u|) + \frac{1}{3}|u|^3\right)$$

defined for $u \in E$. Let $\gamma^*_{\mu\nu}$ denote the linking level of $\Gamma^*_{\mu\nu}$.

PROPOSITION 4.1. *$\gamma^*_{\mu\nu}$ is attained provided $\mu \leq 0$ and*

$$\left(\frac{a+\mu}{a}\right)^{12-5q}\left(\frac{a}{\nu}\right)^{2(3-q)} < \mathcal{R}_q. \tag{4.2}$$

In order to prove this proposition we require a series of discussions.

4.1. The subcritical equation. Consider

$$-i\alpha \cdot \nabla u + a\beta u + \mu u = \nu g(|u|)u, \quad u \in H^1(\mathbb{R}^3, \mathbb{C}^4). \tag{4.3}$$

The solutions are critical points of the functional

$$\Gamma_{\mu\nu}(u) := \frac{1}{2}\left(\|u^+\|^2 - \|u^-\|^2\right) + \frac{\mu}{2}\int_{\mathbb{R}^3}|u|^2 - \nu\int_{\mathbb{R}^3}G(|u|)$$

defined for $u = u^- + u^+ \in E = E^- \oplus E^+$. Denote the critical set, the least energy, and the set of least energy solutions of $\Gamma_{\mu\nu}$ as follows

$$\begin{aligned}
\mathscr{L}_{\mu\nu} &:= \{u \in E : \Gamma'_{\mu\nu}(u) = 0\}, \\
\gamma_{\mu\nu} &:= \inf\{\Gamma_{\mu\nu}(u) : u \in \mathscr{L}_{\mu\nu} \setminus \{0\}\}, \\
\mathscr{R}_{\mu\nu} &:= \{u \in \mathscr{L}_{\mu\nu} : \Gamma_{\mu\nu}(u) = \gamma_{\mu\nu},\, |u(0)| = |u|_\infty\}.
\end{aligned}$$

The following lemma is from [16].

LEMMA 4.2. *There hold the following*

i) $\mathcal{L}_{\mu\nu} \neq \emptyset$, $\gamma_{\mu\nu} > 0$, *and* $\mathcal{L}_{\mu\nu} \subset \bigcap_{s \geq 2} W^{1,s}$;

ii) $\gamma_{\mu\nu}$ *is attained, and* $\mathcal{R}_{\mu\nu}$ *is compact in* $H^1(\mathbb{R}^3, \mathbb{C}^4)$;

iii) *there exist* $C, c > 0$ *such that*

$$|u(x)| \leq C \exp(-c|x|) \quad \text{for all } x \in \mathbb{R}^3, \ u \in \mathcal{R}_{\mu\nu}.$$

As before we introduce the following notations:

$$\mathcal{J}_{\mu\nu} : E^+ \to E^-, \quad \Gamma_{\mu\nu}(u + \mathcal{J}_{\mu\nu}(u)) = \max_{v \in E^-} \Gamma_{\mu\nu}(u + v);$$

$$J_{\mu\nu} : E^+ \to \mathbb{R}, \quad J_{\mu\nu}(u) = \Gamma_{\mu\nu}(u + \mathcal{J}_{\mu\nu}(u));$$

$$\mathcal{M}_{\mu\nu} := \{u \in E^+ \setminus \{0\} : J'_{\mu\nu}(u)u = 0\}.$$

Plainly, critical points of $J_{\mu\nu}$ and $\Gamma_{\mu\nu}$ are in one to one correspondence via the injective map $u \to u + \mathcal{J}_{\mu\nu}(u)$ from E^+ into E. Clearly, $J_{\mu\nu}$ has the Mountain-pass structure.

Notice that, similar to (3.4), for $u \in E^+$, $v \in E^-$ and $z = v - \mathcal{J}_{\mu\nu}(u)$, there holds

$$\Gamma_{\mu\nu}(u + \mathcal{J}_{\mu\nu}(u)) - \Gamma_{\mu\nu}(u + v)$$
$$= \frac{1}{2}\left(\|z\|^2 - \mu|z|_2^2\right) + \nu \int_0^1 \int_{\mathbb{R}^3} (1-t)\left(g(|u + \mathcal{J}_{\mu\nu}(u) + tz|)|z|^2 \right. \quad (4.4)$$
$$\left. + \frac{g'(|u + \mathcal{J}_{\mu\nu}(u) + tz|)}{|u + \mathcal{J}_{\mu\nu}(u) + tz|}\left(\Re\langle u + \mathcal{J}_{\mu\nu}(u) + tz, z\rangle\right)^2\right).$$

It is not difficult to check that, for each $u \in E^+ \setminus \{0\}$, there is a unique $t = t(u) > 0$ such that $tu \in \mathcal{M}_{\mu\nu}$ (see Lemma 3.6, or [1, 16]).

LEMMA 4.3. *Let* $u \in \mathcal{M}_{\mu\nu}$ *be such that* $J_{\mu\nu}(u) = \gamma_{\mu\nu}$. *Then*

$$\max_{w \in E_u} \Gamma_{\mu\nu}(w) = J_{\mu\nu}(u).$$

Proof. Clearly, since $u + \mathcal{J}_{\mu\nu}(u) \in E_u$,

$$J_{\mu\nu}(u) = \Gamma_{\mu\nu}(u + \mathcal{J}_{\mu\nu}(u)) \leq \max_{w \in E_u} \Gamma_{\mu\nu}(w).$$

On the other hand, for any $w = v + su \in E_u$,

$$\Gamma_{\mu\nu}(w) = \frac{1}{2}\|su\|^2 - \frac{1}{2}\|v\|^2 + \frac{\mu}{2}|v + su|_2^2 - \nu \int_{\mathbb{R}^3} G(|v + su|)$$
$$\leq \Gamma_{\mu\nu}(su + \mathcal{J}_{\mu\nu}(su)) = J_{\mu\nu}(su).$$

Thus, since $u \in \mathcal{M}_{\mu\nu}$,

$$\max_{w \in E_u} \Gamma_{\mu\nu}(w) \leq \max_{s \geq 0} J_{\mu\nu}(su) = J_{\mu\nu}(u),$$

giving the conclusion.   ☐

LEMMA 4.4. *Let* $\mu_j \in (-a, a)$ *and* $\nu_j > 0$, $j = 1, 2$, *with* $\min\{\mu_2 - \mu_1, \ \nu_1 - \nu_2\} \geq 0$. *Then* $\gamma_{\mu_1\nu_1} \leq \gamma_{\mu_2\nu_2}$. *If additionally* $\max\{\mu_2 - \mu_1, \ \nu_1 - \nu_2\} > 0$ *then* $\gamma_{\mu_1\nu_1} < \gamma_{\mu_2\nu_2}$. *In particular,* $\gamma_{\mu_1\nu_j} < \gamma_{\mu_2\nu_j}$ *if* $\mu_1 < \mu_2$, *and* $\gamma_{\mu_j\nu_1} > \gamma_{\mu_j\nu_2}$ *if* $\nu_1 < \nu_2$.

Proof. Let $u \in \mathscr{L}_{\mu_2 \nu_2}$ with $\Gamma_{\mu_2 \nu_2}(u) = \gamma_{\mu_2 \nu_2}$ and set $e = u^+$. Then

$$\gamma_{\mu_2 \nu_2} = \Gamma_{\mu_2 \nu_2}(u) = \max_{w \in E_e} \Gamma_{\mu_2 \nu_2}(w).$$

Let $u_0 \in E_e$ be such that $\Gamma_{\mu_1 \nu_1}(u_0) = \max_{w \in E_e} \Gamma_{\mu_1 \nu_1}(w)$. One has

$$
\begin{aligned}
\gamma_{\mu_2 \nu_2} =& \Gamma_{\mu_2 \nu_2}(u) \geq \Gamma_{\mu_2 \nu_2}(u_0) \\
=& \Gamma_{\mu_1 \nu_1}(u_0) + \frac{1}{2}(\mu_2 - \mu_1)|u_0|_2^2 + (\nu_1 - \nu_2)\int_{\mathbb{R}^3} G(|u_0|) \\
\geq& \gamma_{\mu_1 \nu_1} + \frac{1}{2}(\mu_2 - \mu_1)|u_0|_2^2 + (\nu_1 - \nu_2)\int_{\mathbb{R}^3} G(|u_0|)
\end{aligned}
$$

as claimed. □

For later use, define, for $q \in (2, 3)$,

$$T_q := \inf_{u \in E^+ \setminus \{0\}} \max_{v \in E^-} \frac{a_1(u + v)}{|u + v|_q^2}$$

where $a_1(z) = \int_{\mathbb{R}^3} \langle A_1 z, z \rangle$, and consider the equation

$$-i\alpha \cdot \nabla u + \beta u = |u|^{q-2}u \tag{4.5}$$

with the least energy functional defined by

$$\Gamma_q(u) := \frac{1}{2}a_1(u) - \frac{1}{q}|u|_q^q$$

and the least energy denoted by γ_q. Set as before the induced map $\mathscr{J}_q : E^+ \to E^-$, the functional $J_q \in C^2(E^+, \mathbb{R}) : J_q(u) = \Gamma_q(u + \mathscr{J}_q(u))$, and the manifold $\mathscr{M}_q$.

LEMMA 4.5. *For any $q \in (2, 3)$, T_q is achieved at some u which is a least energy solution of the equation (4.5). Moreover,*

$$T_q = \left(\frac{2q\gamma_q}{q-2}\right)^{(q-2)/q}.$$

Proof. Set, for any $u \in E^+$,

$$\psi_u(v) = \frac{a_1(u + v)}{|u + v|_q^2} \quad \text{and} \quad T_q(u) = \max_{v \in E^-} \psi_u(v).$$

If $w \in E^-$ with $\psi_u(w) = T_q(u)$, then, for $v \in E^-$,

$$
\begin{aligned}
0 =& \psi_u'(w)v \\
=& \frac{2}{|u+w|_q^4}\Re\Big[a_1(u+w, v)|u+w|_q^2 \\
& - a_1(u+w)|u+w|_q^{2-q}\int_{\mathbb{R}^3}|u+w|^{q-2}(u+w)\overline{v}\Big]
\end{aligned}
$$

and

$$\psi_u''(w)[v,v] = \frac{2}{|u+w|_q^4} \Re \Big[a_1(v)|u+w|_q^2$$
$$- (q-2)a_1(u+w)|u+w|_q^{2-q} \int_{\mathbb{R}^3} |u+w|^{q-4} \big((u+w)\overline{v} \big)^2$$
$$+ (q-2)a_1(u+w)|u+w|^{2-2q} \Big(\int_{\mathbb{R}^3} |u+w|^{q-2}(u+w)\overline{v} \Big)^2$$
$$- a_1(u+w)|u+w|^{2-q} \int_{\mathbb{R}^3} |u+w|^{q-2}|v|^2 \Big]$$
$$\leq \frac{2}{|u+w|_q^4} \Re \Big[a_1(v)|u+w|_q^2$$
$$- a_1(u+w)|u+w|^{2-q} \int_{\mathbb{R}^3} |u+w|^{q-2}|v|^2 \Big].$$

Here we have used the estimate (by Hölder inequality)

$$\Big(\int_{\mathbb{R}^3} |u+w|^{q-2}(u+w)\overline{v} \Big)^2 \leq |u+w|_q^q \int_{\mathbb{R}^3} |u+w|^{q-4} \big((u+w)\overline{v} \big)^2.$$

and $a_1(u+w) > 0$ (since $T_q(u) > 0$). Therefore, ψ_u attains its maximum at a unique point.

Observe that

$$T_q = \inf_{u \in E^+ \setminus \{0\}} \max_{w \in E_u} \frac{a_1(w)}{|w|_q^2} = \inf_{u \in E^+ \setminus \{0\}} \max_{w \in \hat{E}_u} \frac{a_1(w)}{|w|_q^2}.$$

If the function

$$m_u(w) = \frac{a_1(w)}{|w|_q^2}$$

attains its maximum on E_u at $w \in \hat{E}_u$, setting $\hat{w} = m_u(w)^{1/(q-2)} w/|w|_q$, then $m_u(\hat{w}) = m_u(w)$ and, for any $v \in E_u$,

$$0 = m_u'(w)v = \frac{2m_u(w)^{1/(q-2)}}{|w|_q} \Re \Big[a_1(\hat{w}, v) - \int_{\mathbb{R}^3} |\hat{w}|^{q-2}\hat{w}\overline{v} \Big].$$

This implies that $\hat{w} \in \mathcal{M}_q$, consequently,

$$\gamma_q \leq \Gamma_q(\hat{w}) = \frac{q-2}{2q} \big(m_u(\hat{w}) \big)^{q/(q-2)} = \frac{q-2}{2q} \big(m_u(w) \big)^{q/(q-2)},$$

hence, $T_q \geq \Big(\frac{2q\gamma_q}{q-2} \Big)^{(q-2)/q}$.

On the other hand, let z be a least energy solution of (4.5), $z = z^+ + z^-$, $z^- = \mathscr{J}_q(z^+)$. Take $u = z^+$. One has $m_u(z) = |z|_q^{q-2}$. Plainly, one checks as above that, for $v \in E^-$,

$$\psi_u'(z^-)v = \frac{2}{|z|_q^2} \Re \Big[a_1(z^-, v) - \int_{\mathbb{R}^3} |z|^{q-2}z\overline{v} \Big] = 0$$

and

$$\psi_u''(z^-)[v,v] \le \frac{2}{|z|_q^4} \Re \left[a_1(v)|z|_q^2 - a_1(z)|z|^{2-q} \int_{\mathbb{R}^3} |z|^{q-2}|v|^2 \right].$$

Therefore,

$$T_q(z^+) = \frac{a_1(z)}{|z|_q^2} = |z|_q^{q-2} = \left(\frac{2q\gamma_q}{q-2} \right)^{(q-2)/q},$$

completing the proof. □

LEMMA 4.6. *If $g(s) = c_0 s^{q-2}$ and $\mu \le 0$, then the corresponding least energy of (4.3) denoted by $\gamma_{\mu\nu}(q)$ satisfies*

$$\gamma_{\mu\nu}(q) \le (a+\mu)^{\frac{2(3-q)}{q-2}} (c_0\nu)^{\frac{-2}{q-2}} \gamma_q. \tag{4.6}$$

Proof. Observe that, setting $z(x) = u(x/(a+\mu))$, (4.3) is equivalent to

$$-i\alpha \cdot \nabla z + \beta z + \frac{\mu}{a+\mu}(I-\beta)z = \frac{\nu}{a+\mu} g(|z|)z \tag{4.7}$$

with energy functional defined by

$$\Gamma_{\nu/\mu}(z) := \frac{1}{2} a_1(z) + \frac{\mu}{a+\mu}|z_2|_2^2 - \frac{\nu}{a+\mu} \int_{\mathbb{R}^3} G(|z|)$$

and the least energy denoted by $\gamma_{\nu/\mu}$, where $z = (z_1, z_2) \in \mathbb{C}^2 \times \mathbb{C}^2$. One has

$$\gamma_{\mu\nu} = (a+\mu)^{-2} \gamma_{\nu/\mu}. \tag{4.8}$$

Now assume $g(s) = c_0 s^{q-2}$ and $\mu \le 0$. We denote by $\gamma_{\nu/\mu}(q)$ the least energy corresponding to (4.7). Let z be a least energy solution of (4.5), $u = z^+$, and $e_q \in E_u$ with $\Gamma_{\nu/\mu}(e_q) = \max_{v \in E_u} \Gamma_{\nu/\mu}(v)$. Then by Lemma 4.5,

$$\begin{aligned}
\gamma_{\nu/\mu}(q) &\le \Gamma_{\nu/\mu}(e_q) \\
&= \frac{q-2}{2q} \left(\frac{a+\mu}{c_0\nu} \right)^{\frac{2}{q-2}} \left(\frac{a_1(e_q) + \frac{2\mu}{a+\mu}|e_{q2}|_2^2}{|e_q|_q^2} \right)^{\frac{q}{q-2}} \\
&\le \frac{q-2}{2q} \left(\frac{a+\mu}{c_0\nu} \right)^{\frac{2}{q-2}} T_q^{\frac{q}{q-2}} \\
&= \left(\frac{a+\mu}{c_0\nu} \right)^{\frac{2}{q-2}} \gamma_q.
\end{aligned}$$

This, jointly with (4.8), yields the desired conclusion (4.6). □

4.2. The critical equation (4.1). We now turn to the critical problem. Define $\mathscr{J}_{\mu\nu}^*$, $J_{\mu\nu}^*$ and $\mathscr{M}_{\mu\nu}^*$, $\gamma_{\mu\nu}^* := \inf_{\mathscr{M}_q} J_{\mu\nu}^*$ etc. as before.

Observe that, for $\mu \in \mathbb{R}$,

$$\int_{\mathbb{R}^3} \langle (A_a + \mu)u, u^+ - u^- \rangle \ge \begin{cases} \dfrac{a-\mu}{a} \||A_a|^{1/2}u|_2^2 & \text{if } \mu \ge 0 \\[2mm] \dfrac{a+\mu}{a} \||A_a|^{1/2}u|_2^2 & \text{if } \mu < 0 \end{cases} \tag{4.9}$$

LEMMA 4.7. $\gamma^*_{\mu\nu}$ *is attained if*

$$\gamma^*_{\mu\nu} < \ell^* := \frac{S^{3/2}}{6\nu^2}\Big(\frac{a+\mu}{a}\Big)^3.$$

Proof. Let $\{u_n\}$ be a $(PS)_c$ sequence with $c = \gamma^*_{\mu\nu}$: $\Gamma^*_{\mu\nu}(u_n) \to c$ and $\Gamma^{*\prime}_{\mu\nu}(u_n) \to 0$ as $n \to \infty$. It is not difficult to check that $\{u_n\}$ is bounded in E. By Lions' concentration principle [21], $\{u_n\}$ is either vanishing or non-vanishing.

Assume that $\{u_n\}$ is vanishing. Then $|u_n|_s \to 0$ for $s \in (2,3)$. By $(g_1), (g_2)$ one gets

$$\Gamma^*_{\mu\nu}(u_n) - \frac{1}{3}\Gamma^{*\prime}_{\mu\nu}(u_n)u_n \geq \frac{1}{6}\int_{\mathbb{R}^3} \langle(A_a + \mu)u_n, u_n\rangle + o(1)$$

or

$$\int_{\mathbb{R}^3} \langle(A_a + \mu)u_n, u_n\rangle \leq 6c + o(1).$$

Similarly,

$$\int_{\mathbb{R}^3} |u_n|^3 \leq \frac{6c}{\nu} + o(1).$$

Moreover,

$$\int_{\mathbb{R}^3} \langle(A_a + \mu)u_n, u_n^+ - u_n^-\rangle - \nu\int_{\mathbb{R}^3} |u_n|u_n(u_n^+ - u_n^-) \leq o(1).$$

Thus,

$$\big||A_a + \mu|^{1/2}u_n\big|_2^2 \leq \nu|u_n|_3^2|u_n^+ - u_n^-|_3 + o(1)$$

which, together with (4.9) and Lemma 3.1, implies

$$c \geq \ell^* = \frac{S^{3/2}}{6\nu^2}\Big(\frac{a+\mu}{a}\Big)^3,$$

a contradiction.

Therefore, $\{u_n\}$ is non-vanishing, that is, there exist $r, \delta > 0$ and $x_n \in \mathbb{R}^3$ such that, setting $v_n(x) = u_n(x + x_n)$, along a subsequence,

$$\int_{B_r(0)} |v_n|^2 \geq \delta.$$

Without loss of generality we assume $v_n \rightharpoonup v$. Then $v \neq 0$ and is a solution of (4.1). And so $\gamma^*_{\mu\nu}$ is attained. □

Now the combination of Lemma 4.6 and Lemma 4.7 gives the proof of Proposition 4.1.

Proof of Proposition 4.1. Observe that

$$\gamma^*_{\mu\nu} < \gamma_{\mu\nu} \leq \gamma_{\mu\nu}(q).$$

If

$$(a + \mu)^{\frac{2(3-q)}{q-2}} (c_0 \nu)^{\frac{-2}{q-2}} \gamma_q \; < \; \ell^*,$$

that is, (4.2) is satisfied, then $\gamma_{\mu\nu}^* < \ell^*$ so it is attained by Lemma 4.7.

As a consequence of Proposition 4.1 we see

LEMMA 4.8. *If μ and ν satisfy* (4.2) *then*

$$\mathscr{L}_{\mu\nu}^* := \{u \in E : \; \Gamma_{\mu\nu}^{*'}(u) = 0\} \neq \emptyset,$$
$$\gamma_{\mu\nu}^* := \inf\{\Gamma_{\mu\nu}^*(u) : \; u \in \mathscr{L}_{\mu\nu}^* \setminus \{0\}\},$$

and $\mathscr{R}_{\mu\nu}^* := \{u \in \mathscr{L}_{ab}^* : \; \Gamma_{\mu\nu}^*(u) = \gamma_{\mu\nu}^*, \; |u(0)| = |u|_\infty\}$ is compact in E.

Moreover, the following lemma is obvious.

LEMMA 4.9. *Let, for $j = 1, 2$, $\mu_j \in (-a, 0]$ and $\nu_j > 0$.*

i) *If $\min\{\mu_2 - \mu_1, \; \nu_1 - \nu_2\} \geq 0$ then $\gamma_{\mu_1\nu_1}^* \leq \gamma_{\mu_2\nu_2}^*$.*

2) *Assume $\min\{\mu_2 - \mu_1, \; \nu_1 - \nu_2\} > 0$, and either the pair (μ_1, ν_1) or the pair (μ_2, ν_2) satisfies* (4.2). *Then $\gamma_{\mu_1\nu_1}^* < \gamma_{\mu_2\nu_2}^*$.*

4.3. Auxiliary functionals. Assume that the sequence of functions $\hat{V}_\varepsilon$ and $\hat{W}_\varepsilon \in C \cap L^\infty(\mathbb{R}^3, \mathbb{R})$, $0 < \varepsilon \leq 1$, satisfy

($\star$) $\varpi := \sup_{\varepsilon, x} |\hat{V}_\varepsilon(x)| < a$, $\inf_{\varepsilon, x} \hat{W}_\varepsilon(x) > 0$; $\hat{V}_\varepsilon(x) \to \mu$ and $\hat{W}_\varepsilon(x) \to \nu$ uniformly on bounded sets of x as $\varepsilon \to 0$ with $\gamma_{\mu\nu}^*$ achieved (e.g. μ and ν satisfying (4.2)).

Consider the equation

$$-i\alpha \cdot \nabla u + a\beta u + \hat{V}_\varepsilon(x)u = \hat{W}_\varepsilon(x)\big(g(|u|) + |u|\big)u \qquad (4.10)$$

for $u \in H^1(\mathbb{R}^3, \mathbb{C}^4)$. Denote

$$\hat{\Phi}_\varepsilon(u) = \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) + \frac{1}{2}\int_{\mathbb{R}^3} \hat{V}_\varepsilon(x)|u|^2 - \int_{\mathbb{R}^3} \hat{W}_\varepsilon(x)F(|u|),$$

As before, define the associated $\hat{c}_\varepsilon$, $\hat{\mathscr{N}}_\varepsilon$, etc.

Note that, setting $V_\varepsilon^0(x) = \hat{V}_\varepsilon(x) - \mu$ and $W_\varepsilon^0(x) = \nu - \hat{W}_\varepsilon(x)$, we have by definition

$$\hat{\Phi}_\varepsilon(u) = \Gamma_{\mu\nu}^*(u) + \frac{1}{2}\int_{\mathbb{R}^3} V_\varepsilon^0(x)|u|^2 + \int_{\mathbb{R}^3} W_\varepsilon^0(x)F(|u|). \qquad (4.11)$$

LEMMA 4.10. $\limsup_{\varepsilon \to 0} \hat{c}_\varepsilon \leq \gamma_{\mu\nu}^*$.

Proof. In virtue of Lemma 4.8, let $u = u^- + u^+ \in \mathscr{R}_{\mu\nu}^*$, a least energy solution of (4.1) and set $e = u^+$. It is clear that $e \in \mathscr{M}_{\mu\nu}^*$, $\mathscr{J}_{\mu\nu}^*(e) = u^-$ and $J_{\mu\nu}^*(e) = \gamma_{\mu\nu}^*$. There is a unique $t_\varepsilon > 0$ such that $t_\varepsilon e \in \hat{\mathscr{N}}_\varepsilon$. One has

$$\hat{c}_\varepsilon \leq \hat{I}_\varepsilon(t_\varepsilon e). \qquad (4.12)$$

It is clear that $\{t_\varepsilon\}$ is bounded, hence, without loss of generality we can assume $t_\varepsilon \to t_0$ as $\varepsilon \to 0$.

Observe that (4.11) induces that

$$
\begin{aligned}
&\Big(\hat{\Phi}_\varepsilon(t_\varepsilon e + \hat{h}_\varepsilon(t_\varepsilon e)) - \hat{\Phi}_\varepsilon(t_\varepsilon e + \mathscr{J}^*_{\mu\nu}(t_\varepsilon e))\Big) \\
&+ \Big(\Gamma^*_{\mu\nu}(t_\varepsilon e + \mathscr{J}^*_{\mu\nu}(t_\varepsilon e)) - \Gamma^*_{\mu\nu}(t_\varepsilon e + \hat{h}_\varepsilon(t_\varepsilon e))\Big) \\
&= \frac{1}{2}\int_{\mathbb{R}^3} V^0_\varepsilon(x)\big(|t_\varepsilon e + \hat{h}_\varepsilon(t_\varepsilon e)|^2 - |t_\varepsilon e + \mathscr{J}^*_{\mu\nu}(t_\varepsilon e)|^2\big) \\
&+ \int_{\mathbb{R}^3} W^0_\varepsilon(x)\big(F(|t_\varepsilon e + \hat{h}_\varepsilon(t_\varepsilon e)|) - F(|t_\varepsilon e + \mathscr{J}^*_{\mu\nu}(t_\varepsilon e)|)\big).
\end{aligned}
\tag{4.13}
$$

Since, denoting $z_\varepsilon = \mathscr{J}^*_{\mu\nu}(t_\varepsilon e) - \hat{h}_\varepsilon(t_\varepsilon e)$,

$$
|t_\varepsilon e + \hat{h}_\varepsilon(t_\varepsilon e)|^2 - |t_\varepsilon e + \mathscr{J}^*_{\mu\nu}(t_\varepsilon e)|^2 = |z_\varepsilon|^2 - 2\Re\langle t_\varepsilon e + \mathscr{J}^*_{\mu\nu}(t_\varepsilon e), z_\varepsilon\rangle
$$

and

$$
\begin{aligned}
&F(|t_\varepsilon e + \hat{h}_\varepsilon(t_\varepsilon e)|) - F(|t_\varepsilon e + \mathscr{J}^*_{\mu\nu}(t_\varepsilon e)|) \\
&= F(|t_\varepsilon e + \mathscr{J}^*_{\mu\nu}(t_\varepsilon e) - z_\varepsilon|) - F(|t_\varepsilon e + \mathscr{J}^*_{\mu\nu}(t_\varepsilon e)|) \\
&= - f(|t_\varepsilon e + \mathscr{J}^*_{\mu\nu}(t_\varepsilon e)|)\Re\langle t_\varepsilon e + \mathscr{J}^*_{\mu\nu}(t_\varepsilon e), z_\varepsilon\rangle + K_\varepsilon(x)
\end{aligned}
$$

with

$$
\begin{aligned}
K_\varepsilon(x) := \int_0^1 (1-s)\Big(&f(|t_\varepsilon e + \mathscr{J}^*_{\mu\nu}(t_\varepsilon e) - sz_\varepsilon|)|z_\varepsilon|^2 \\
&+ \frac{f'(|t_\varepsilon e + \mathscr{J}^*_{\mu\nu}(t_\varepsilon e) - sz_\varepsilon|)}{|t_\varepsilon e + \mathscr{J}^*_{\mu\nu}(t_\varepsilon e) - sz_\varepsilon|}\big(\Re\langle t_\varepsilon e + \mathscr{J}^*_{\mu\nu}(t_\varepsilon e) - sz_\varepsilon, -z_\varepsilon\rangle\big)^2\Big)ds
\end{aligned}
$$

we get from (4.13) (remark that $W^0_\varepsilon(x) \le \nu$)

$$
\begin{aligned}
&\Big(\hat{\Phi}_\varepsilon(t_\varepsilon e + \hat{h}_\varepsilon(t_\varepsilon e)) - \hat{\Phi}_\varepsilon(t_\varepsilon e + \mathscr{J}^*_{\mu\nu}(t_\varepsilon e))\Big) \\
&+ \Big(\Gamma^*_{\mu\nu}(t_\varepsilon e + \mathscr{J}^*_{\mu\nu}(t_\varepsilon e)) - \Gamma^*_{\mu\nu}(t_\varepsilon e + \hat{h}_\varepsilon(t_\varepsilon e))\Big) \\
&\le \frac{1}{2}\int_{\mathbb{R}^3} V^0_\varepsilon(x)|z_\varepsilon|^2 - \Re\int_{\mathbb{R}^3} V^0_\varepsilon(x)\langle t_\varepsilon e + \mathscr{J}^*_{\mu\nu}(t_\varepsilon e), z_\varepsilon\rangle \\
&\quad - \Re\int_{\mathbb{R}^3} W^0_\varepsilon(x)f(|t_\varepsilon e + \mathscr{J}^*_{\mu\nu}(t_\varepsilon e)|)\langle t_\varepsilon e + \mathscr{J}^*_{\mu\nu}(t_\varepsilon e), z_\varepsilon\rangle \\
&\quad + \nu\int_{\mathbb{R}^3} K_\varepsilon(x).
\end{aligned}
\tag{4.14}
$$

Remark that one has, similar to (4.4) (with z replaced by z_ε),

$$
\begin{aligned}
&\Gamma^*_{\mu\nu}(t_\varepsilon e + \mathscr{J}^*_{\mu\nu}(t_\varepsilon e)) - \Gamma^*_{\mu\nu}(t_\varepsilon e + \hat{h}_\varepsilon(t_\varepsilon e)) \\
&= \frac{1}{2}\big(\|z_\varepsilon\|^2 - \mu|z_\varepsilon|_2^2\big) + \nu\int_{\mathbb{R}^3} K_\varepsilon(x)
\end{aligned}
\tag{4.15}
$$

and, by the representation (3.4) with Φ_ε replaced by $\hat{\Phi}_\varepsilon$,

$$
\hat{\Phi}_\varepsilon(t_\varepsilon e + \hat{h}_\varepsilon(t_\varepsilon e)) - \hat{\Phi}_\varepsilon(t_\varepsilon e + \mathscr{J}^*_{\mu\nu}(t_\varepsilon e)) \ge \frac{1}{2}\Big(\|z_\varepsilon\|^2 - \int_{\mathbb{R}^3} \hat{V}_\varepsilon(x)|z_\varepsilon|^2\Big).
$$

Thus (4.14) (jointly with (g_1)) implies

$$\|z_\varepsilon\|^2 - \varpi|z_\varepsilon|_2^2$$
$$\leq -\Re \int_{\mathbb{R}^3} V_\varepsilon^0(x)\langle t_\varepsilon e + \mathscr{J}_{\mu\nu}^*(t_\varepsilon e), z_\varepsilon\rangle$$
$$\quad -\Re \int_{\mathbb{R}^3} W_\varepsilon^0(x)g(|t_\varepsilon e + \mathscr{J}_{\mu\nu}^*(t_\varepsilon e)|)\langle t_\varepsilon e + \mathscr{J}_{\mu\nu}^*(t_\varepsilon e), z_\varepsilon\rangle$$
$$\leq \int_{\mathbb{R}^3} |V_\varepsilon^0(x)||t_\varepsilon e + \mathscr{J}_{\mu\nu}^*(t_\varepsilon e)||z_\varepsilon|$$
$$\quad + c_1 \int_{\mathbb{R}^3} |W_\varepsilon^0(x)||t_\varepsilon e + \mathscr{J}_{\mu\nu}^*(t_\varepsilon e)||z_\varepsilon| \tag{4.16}$$
$$\quad + c_1 \int_{\mathbb{R}^3} |W_\varepsilon^0(x)||t_\varepsilon e + \mathscr{J}_{\mu\nu}^*(t_\varepsilon e)|^{p-1}|z_\varepsilon|$$
$$\leq c_2 \Big(\int_{\mathbb{R}^3} \big(|V_\varepsilon^0(x)| + |W_\varepsilon^0(x)|\big)^2 |t_\varepsilon e + \mathscr{J}_{\mu\nu}^*(t_\varepsilon e)|^2 \Big)^{1/2} |z_\varepsilon|_2$$
$$\quad + c_1 \Big(\int_{\mathbb{R}^3} |W_\varepsilon^0(x)|^{p/(p-1)}|t_\varepsilon e + \mathscr{J}_{\mu\nu}^*(t_\varepsilon e)|^p \Big)^{(p-1)/p} |z_\varepsilon|_p.$$

Since $t_\varepsilon \to t_0$ and e is exponentially decaying, we have for $q = 2, p$,

$$\limsup_{R\to\infty} \int_{|x|\geq R} |t_\varepsilon e + \mathscr{J}_{\mu\nu}^*(t_\varepsilon e)|^q = 0$$

which implies that

$$\int_{\mathbb{R}^3} \big(|V_\varepsilon^0(x)| + |W_\varepsilon^0(x)|\big)^2 |t_\varepsilon e + \mathscr{J}_{\mu\nu}^*(t_\varepsilon e)|^2$$
$$= \Big(\int_{|x|\leq R} + \int_{|x|>R} \Big) \big(|V_\varepsilon^0(x)| + |W_\varepsilon^0(x)|\big)^2 |t_\varepsilon e + \mathscr{J}_{\mu\nu}^*(t_\varepsilon e)|^2$$
$$\leq \int_{|x|\leq R} \big(|V_\varepsilon^0(x)| + |W_\varepsilon^0(x)|\big)^2 |t_\varepsilon e + \mathscr{J}_{\mu\nu}^*(t_\varepsilon e)|^2$$
$$\quad + c_3 \int_{|x|>R} |t_\varepsilon e + \mathscr{J}_{\mu\nu}^*(t_\varepsilon e)|^2$$
$$= o(1)$$

as $\varepsilon \to 0$, and similarly

$$\int_{\mathbb{R}^3} |W_\varepsilon^0(x)|^{p/(p-1)}|t_\varepsilon e + \mathscr{J}_{\mu\nu}^*(t_\varepsilon e)|^p = o(1)$$

as $\varepsilon \to 0$. Thus, since $\varpi < a$ by the assumption $(*)$, it follows from (4.16) that $\|z_\varepsilon\| = \|\hat{h}_\varepsilon(t_\varepsilon e) - \mathscr{J}_{\mu\nu}^*(t_\varepsilon e)\| \to 0$, that is, $\hat{h}_\varepsilon(t_\varepsilon e) \to \mathscr{J}_{\mu\nu}^*(t_0 e)$. Consequently,

$$\int_{\mathbb{R}^3} V_\varepsilon^0(x)|t_\varepsilon e + \hat{h}_\varepsilon(t_\varepsilon e)|^2 \to 0 \quad \text{and} \quad \int_{\mathbb{R}^3} W_\varepsilon^0(x)F(|t_\varepsilon e + \hat{h}_\varepsilon(t_\varepsilon e)|) \to 0$$

as $\varepsilon \to 0$. This, jointly with (4.11), implies

$$\hat{\Phi}_\varepsilon(t_\varepsilon e + \hat{h}_\varepsilon(t_\varepsilon e)) = \Gamma_{\mu\nu}^*(t_\varepsilon e + \hat{h}_\varepsilon(t_\varepsilon e)) + o(1)$$
$$= \Gamma_{\mu\nu}^*(t_0 e + \mathscr{J}_{\mu\nu}^*(t_0 e)) + o(1),$$

that is,

$$\hat{I}_\varepsilon(t_\varepsilon e) = J^*_{\mu\nu}(t_0 e) + o(1)$$

as $\varepsilon \to 0$. Recalling that by Lemma 4.3

$$J^*_{\mu\nu}(t_0 e) \le \max_{v \in E_e} \Gamma^*_{\mu\nu}(v) = J^*_{\mu\nu}(e) = \gamma^*_{\mu\nu},$$

we obtain, jointly with (4.12),

$$\lim_{\varepsilon \to 0} \hat{c}_\varepsilon \le \lim_{\varepsilon \to 0} \hat{I}_\varepsilon(t_\varepsilon e) = J^*_{\mu\nu}(t_0 e) \le \gamma^*_{\mu\nu}$$

as claimed.   □

Below, for $\mu \in [\tau, \tau_\infty]$ and $\nu \in [\pi_\infty, \pi]$, we set

$$V^\mu(x) := \max\{\mu, V(x)\},$$
$$W^\nu(x) := \min\{\nu, W(x)\},$$

and let $V^\mu_\varepsilon(x) = V^\mu(\varepsilon x)$, $W^\nu_\varepsilon(x) = W^\nu(\varepsilon x)$. Consider the functional

$$\Phi^{\mu\nu}_\varepsilon(u) = \frac{1}{2}\big(\|u^+\|^2 - \|u^-\|^2\big) + \frac{1}{2}\int_{\mathbb{R}^3} V^\mu_\varepsilon(x)|u|^2 - \int_{\mathbb{R}^3} W^\nu_\varepsilon(x)F(|u|)$$

with $\mathscr{N}^{\mu\nu}_\varepsilon$, $c^{\mu\nu}_\varepsilon$ and so on as before. By definition and Lemma 4.9,

$$\gamma^*_{\tau\pi} \le \gamma^*_{V(0)W(0)} \le \gamma^*_{V^\mu(0)W^\nu(0)}. \tag{4.17}$$

Moreover, observe that

$$\Phi^{\mu\nu}_\varepsilon(u) = \Gamma^*_{\mu\nu}(u) + \frac{1}{2}\int_{\mathbb{R}^3}(V^\mu_\varepsilon(x) - \mu)|u|^2 + \int_{\mathbb{R}^3}(\nu - W^\nu_\varepsilon(x))F(|u|).$$

This, together with Lemma 4.10, shows that if μ and ν satisfy (4.2) then

$$\gamma^*_{\mu\nu} \le c^{\mu\nu}_\varepsilon \quad \text{and} \quad \limsup_{\varepsilon \to 0} c^{\mu\nu}_\varepsilon \le \gamma^*_{V^\mu(0)W^\nu(0)}, \tag{4.18}$$

and particularly

$$\lim_{\varepsilon \to 0} c^{\mu\nu}_\varepsilon = \gamma^*_{\mu\nu} \quad \text{if } V(0) \le \mu \text{ and } W(0) \ge \nu. \tag{4.19}$$

5. Proofs of the main results. We are now giving the proofs of the main results on the critical equation:

$$-i\alpha \cdot \nabla u + a\beta u + V_\varepsilon(x)u = W_\varepsilon(x)f(|u|)u, \quad u \in H^1(\mathbb{R}^3, \mathbb{C}^4) \tag{5.1}$$

where $f(|u|) = g(|u|) + |u|$ with g satisfying (g_1) and (g_2). Recall that, by assumption, $V(x) \le 0$ hence $|V|_\infty = -\tau$. Observe that, by $(\hat{P}_0)$ and (4.2),

$$\gamma^*_{\mu\nu} < \ell^*, \quad \text{for} \quad \nu \in [\pi_\infty, \pi] \text{ and } \mu \le \tau_\infty \text{ sufficiently close to } \tau_\infty. \tag{5.2}$$

Note also that, for any $x_0 \in \mathbb{R}^3$, setting $\tilde{V}(x) = V(x + x_0)$ and $\tilde{W}(x) = W(x + x_0)$, if $\tilde{w}(x)$ is a solution of

$$-i\varepsilon\alpha \cdot \nabla\tilde{w} + a\beta\tilde{w} + \tilde{V}(x)\tilde{w} = \tilde{W}(x)f(|\tilde{w}|)\tilde{w},$$

then $w(x) := \tilde{w}(x - x_0)$ solves (5.1).

5.1. Proof of Theorem 2.2. Assume $(P_0) - (\hat{P}_0)$ and (P_1) are satisfied. By virtue of the above observation, without loss of generality, we can assume that $0 \in \mathscr{W}$ such that $V(0) = \min_{x \in \mathscr{W}} V(x)$. Then $\pi = W(0)$ and $\kappa := V(0) \leq \tau_\infty$. Consider the functional Φ_ε.

LEMMA 5.1. *Assume (P_0)-$(\hat{P}_0)$ and (P_1) are satisfied. Then c_ε is attained for small ε.*

Proof. Given $\varepsilon > 0$, let $u_k \in \mathscr{N}_\varepsilon$ be a minimizing sequence: $I_\varepsilon(u_k) \to c_\varepsilon$. By the Ekeland variational principle we can assume that u_k is, in addition, a $(PS)_{c_\varepsilon}$ sequence for I_ε on $\mathscr{N}_\varepsilon$. A standard argument shows that u_k is in fact a $(PS)_{c_\varepsilon}$ sequence for I_ε on E^+ (see, e.g., [23, 30]). Then $w_k = u_k + h_\varepsilon(u_k)$ is a $(PS)_{c_\varepsilon}$ sequence for Φ_ε on E: $\Phi_\varepsilon(w_k) \to c_\varepsilon$ and $\Phi'_\varepsilon(w_k) \to 0$ as $k \to \infty$. It is easy to see that $\{w_k\}$ is bounded in E. We can assume without loss of generality that $w_k \rightharpoonup w_\varepsilon \in \mathscr{K}_\varepsilon$ in E. If $w_\varepsilon \neq 0$ then clearly $\Phi_\varepsilon(w_\varepsilon) = c_\varepsilon$. So we are going to check that $w_\varepsilon \neq 0$ for all $\varepsilon > 0$ small.

Assume by contradiction that there is a sequence $\varepsilon_j \to 0$ with $w_{\varepsilon_j} = 0$. Then $w_k \rightharpoonup 0$ in E and L^s for $s \in [2,3]$, $w_k \to 0$ in L^s_{loc} for $s \in (1,3)$, and $w_k(x) \to 0$ a.e. in $x \in \mathbb{R}^3$. Observe that $\tau_\infty \geq \kappa$ by (P_1). By $(\hat{P}_0)$, one can choose $\kappa \leq \mu \leq \tau_\infty$ satisfying

$$\left(\frac{a+\mu}{a}\right)^{12-5q}\left(\frac{a}{\nu}\right)^{2(3-q)} < \mathcal{R}_q \tag{5.3}$$

for any $\nu \in (\pi_\infty, \pi)$. Consider the functional $\Phi_{\varepsilon_j}^{\mu\nu}$. Remark that $V_\varepsilon^\mu(0) = \mu$ and $W_\varepsilon^\nu(0) = \nu$, thus

$$\gamma^*_{\mu\nu} = \lim_{\varepsilon_j \to 0} c_{\varepsilon_j}^{\mu\nu} \tag{5.4}$$

by (4.19). Let $t_k > 0$ be such that $t_k u_k \in \mathscr{N}_{\varepsilon_j}^{\mu\nu}$. Then $\{t_k\}$ is bounded and we may assume $t_k \to t_0$ as $k \to \infty$. By (P_1), the set $O_\varepsilon := \{x \in \mathbb{R}^3 : V_\varepsilon(x) < \mu \text{ or } W_\varepsilon(x) \geq \nu\}$ is bounded. Remark that $\Phi_{\varepsilon_j}(t_k u_k + h_{\varepsilon_j}^{\mu\nu}(t_k u_k)) \leq I_{\varepsilon_j}(u_k)$. We obtain

$$c_{\varepsilon_j}^{\mu\nu} \leq I_{\varepsilon_j}^{\mu\nu}(t_k u_k) = \Phi_{\varepsilon_j}^{\mu\nu}(t_k u_k + h_{\varepsilon_j}^{\mu\nu}(t_k u_k))$$

$$= \Phi_{\varepsilon_j}(t_k u_k + h_{\varepsilon_j}^{\mu\nu}(t_k u_k)) + \frac{1}{2}\int_{\mathbb{R}^3}\left(V_{\varepsilon_j}^\mu(x) - V_{\varepsilon_j}(x)\right)|t_k u_k + h_{\varepsilon_j}^{\mu\nu}(t_k u_k)|^2$$

$$+ \int_{\mathbb{R}^3}\left(W_{\varepsilon_j}(x) - W_{\varepsilon_j}^\nu(x)\right)F(|t_k u_k + h_{\varepsilon_j}^{\mu\nu}(t_k u_k)|)$$

$$\leq I_{\varepsilon_j}(u_k) + \frac{1}{2}\int_{O_{\varepsilon_j}}\left(\mu - V_{\varepsilon_j}(x)\right)|t_k u_k + h_{\varepsilon_j}^{\mu\nu}(t_k u_k)|^2$$

$$+ \int_{O_{\varepsilon_j}}\left(W_{\varepsilon_j}(x) - \nu\right)F(|t_k u_k + h_{\varepsilon_j}^{\mu\nu}(t_k u_k)|)$$

$$= c_{\varepsilon_j} + o(1)$$

as $k \to \infty$, hence,

$$c_{\varepsilon_j}^{\mu\nu} \leq c_{\varepsilon_j}. \tag{5.5}$$

On the other hand, let $v_k \in \mathscr{N}_{\varepsilon_j}^{\mu\nu}$ be a minimizing sequence for $I_{\varepsilon_j}^{\mu\nu}$ and set $z_k = v_k + h_{\varepsilon_j}^{\mu\nu}(v_k)$. Denoting $E_k = E^- \oplus \mathbb{R}^+ v_k$, since $V^\mu \geq V$ and $W^\nu \leq W$, one has

$$I_{\varepsilon_j}^{\mu\nu}(v_k) = \max_{z \in E_k} \Phi_{\varepsilon_j}^{\mu\nu}(z) \geq \max_{z \in E_k} \Phi_{\varepsilon_j}(z).$$

Consequently, $c_{\varepsilon_j} \leq c_{\varepsilon_j}^{\mu\nu}$. This, together with (5.4) and (5.5), yields

$$\gamma_{\mu\nu}^* = \lim_{\varepsilon_j \to 0} c_{\varepsilon_j}^{\mu\nu} = \lim_{\varepsilon_j \to 0} c_{\varepsilon_j}. \tag{5.6}$$

Now, since $\pi_\infty < \pi$, we can take $\pi_\infty < \nu_1 < \nu_2 < \pi$. We see by Lemma 4.9 and (5.6) the following

$$\gamma_{\mu\nu_1}^* < \gamma_{\mu\nu_2}^* = \lim_{\varepsilon_j \to 0} c_{\varepsilon_j} = \gamma_{\mu\nu_1}^*,$$

a contradiction.   □

Remark that, by Lemma 3.9, the critical point u_ε corresponding to c_ε satisfies $u_\varepsilon \in \cap_{s \geq 2} W^{1,s}$.

From the last argument of the proof of Lemma 5.1 (just below (5.5)), we see also the following

LEMMA 5.2. $\limsup_{\varepsilon \to 0} c_\varepsilon \leq \limsup_{\varepsilon \to 0} c_\varepsilon^{\mu\pi} = \gamma_{\mu\pi}^*$ *for any* $\mu \geq \kappa$ *satisfying* (5.3) *with* $\nu = \pi$.

REMARK 5.3. *It is not difficult to check that* $\mathscr{S}_\varepsilon$ *is compact for all small* $\varepsilon > 0$. *Indeed, assume by contraction that, for some* $\varepsilon_j \to 0$, $\mathscr{S}_{\varepsilon_j}$ *is not compact in* E. *Let* $u_n^j \in \mathscr{S}_{\varepsilon_j}$ *with* $u_n^j \rightharpoonup 0$ *as* $n \to \infty$. *As done in proving the above Lemma 5.1, one gets a contradiction.*

For the later use, letting $D = -i\alpha \cdot \nabla$, we write (5.1) as

$$Du = -a\beta u - V_\varepsilon(x)u + W_\varepsilon(x)f(|u|)u.$$

By Lemma 3.9, $u \in \cap_{s \geq 2} W^{1,s}$ for any $u \in \mathscr{K}_\varepsilon$. Acting the operator D on the two sides of the above representation and noting that $D^2 = -\Delta$ we get

$$\Delta u = (a^2 - V_\varepsilon^2(x))u + r_\varepsilon(x, |u|)u$$

where

$$r_\varepsilon(x, |u|) := DV_\varepsilon(x) + 2V_\varepsilon(x)W_\varepsilon(x)f(|u|) - W_\varepsilon(x)^2 f(|u|)^2$$
$$- \left(DW_\varepsilon(x)f(|u|) + \frac{f'(|u|)}{|u|}W_\varepsilon(x)\Re\langle u, Du\rangle \right)$$

Letting

$$\operatorname{sgn} u = \begin{cases} \dfrac{\overline{u}}{|u|} & \text{if } u \neq 0 \\[2mm] 0 & \text{if } u = 0, \end{cases}$$

by Kato's inequality ([11]), there holds

$$\Delta|u| \geq \Re\big[\Delta u(\operatorname{sgn} u)\big]$$

Observe that

$$\Re\left[DV_\varepsilon(x)u\frac{\overline{u}}{|u|} \right] = 0$$

and

$$\Re\left[\left(DW_\varepsilon(x)f(|u|) + \frac{f'(|u|)}{|u|}W_\varepsilon(x)\Re\langle u, Du\rangle \right)u\frac{\overline{u}}{|u|} \right] = 0.$$

Hence

$$\Re\left[r_\varepsilon(x,|u|)u\frac{\overline{u}}{|u|}\right] = \big(2V_\varepsilon(x) - W_\varepsilon(x)f(|u|)\big)W_\varepsilon(x)f(|u|)|u|.$$

We obtain

$$\Delta|u| \geq \big(a^2 - V_\varepsilon^2(x)\big)|u| + \big(2V_\varepsilon(x) - W_\varepsilon(x)f(|u|)\big)W_\varepsilon(x)f(|u|)|u|. \qquad (5.7)$$

This, together with (3.6), implies in particular that there is $\Lambda > 0$ satisfying

$$\Delta|u| \geq -\Lambda|u|.$$

It then follows from the sub-solution estimate [20, 27] that

$$|u(x)| \leq C_0 \int_{B_1(x)} |u(y)|\,dy \qquad (5.8)$$

with C_0 independent of x and $u \in \mathscr{K}_\varepsilon$, $\varepsilon > 0$, where $B_1(x) = \{y \in \mathbb{R}^3 : |y| \leq 1\}$.

Now we turn to the concentration. Thus assume that (P_3) also holds. Thus, by Proposition 4.1, $\gamma^*_{\tau\pi}$ is attained.

LEMMA 5.4. *Assume additionally that (P_3) also holds. Let $u_\varepsilon \in S_\varepsilon$. There is a maximum point y_ε of $|u_\varepsilon|$ such that $\lim_{\varepsilon\to0} dist(\varepsilon y_\varepsilon, \mathscr{A}_w) = 0$, and for any sequence $\varepsilon y_\varepsilon \to y_0$, $v_\varepsilon(x) := u_\varepsilon(x + y_\varepsilon)$ converges in H^1 to a least energy solution of*

$$-i\alpha \cdot \nabla v + a\beta v + V(y_0)v = W(y_0)f(|v|)v. \qquad (5.9)$$

Proof. The proof will be carried out in several steps.

Step 1) Given arbitrarily a sequence $\varepsilon_j \to 0$ as $j \to \infty$, let $u_j \in \mathscr{S}_j \equiv \mathscr{S}_{\varepsilon_j}$. Then $\{u_j\}$ is bounded. A concentration argument shows that $\{u_j\}$ is either varnishing or non-varnishing.

If $\{u_j\}$ is varnishing then $|u_j|_s \to 0$ for $s \in (2,3)$. Note that, by Lemma 5.2, $\limsup_{j\to\infty} c_{\varepsilon_j} \leq \gamma^*_{\mu\pi}$ for μ close sufficiently to τ_∞. Recall that (see Proposition 4.1 and Lemma 4.7)

$$\gamma^*_{\mu\pi} < \frac{S^{3/2}}{6\pi^2}\Big(\frac{a+\mu}{a}\Big)^3 < \frac{S^{3/2}}{6\pi^2_\infty}\Big(\frac{a+\tau_\infty}{a}\Big)^3. \qquad (5.10)$$

It is not difficult to check that As before (see Lemma 4.7) that $\Phi_{\varepsilon_j}^\infty(u_j) := \Phi_{\varepsilon_j}^{\tau_\infty\pi_\infty}(u_j) \to c$ (some $c \leq \gamma^*_{\mu\pi}$), and $\Phi_{\varepsilon_j}^{\infty\,'}(u_j) \to 0$. By (g_1) and (g_2) one gets

$$\int_{\mathbb{R}^3} \langle (A_a + V_{\varepsilon_j}^{\tau_\infty})u_j, u_j\rangle \leq 6c_{\varepsilon_j} + o(1)$$

and

$$\int_{\mathbb{R}^3} W_{\varepsilon_j}^{\pi_\infty}(x)|u_j|^3 \leq 6c_{\varepsilon_j} + o(1).$$

Moreover,

$$\int_{\mathbb{R}^3} \langle (A_a + V_{\varepsilon_j}^{\tau_\infty})u_j, u_j\rangle - \int_{\mathbb{R}^3} W_{\varepsilon_j}^{\pi_\infty}(x)|u_j|^3 \leq o(1).$$

Then (recalling that $V(x) \leq 0$ hence $|V_{\varepsilon_j}^{\tau_\infty}|_\infty = -\tau_\infty$)

$$\frac{a + \tau_\infty}{a} \|u_j\|^2 \leq \int_{\mathbb{R}^3} \langle (A_a + V_{\varepsilon_j}^{\tau_\infty}) u_j, u_j^+ - u_j^- \rangle$$
$$\leq (6 c_{\varepsilon_j})^{1/3} \pi_\infty^{2/3} S^{-1/2} \|u_j\|^2 + o(1),$$

that is,

$$c \geq \frac{S^{3/2}}{6 \pi_\infty^2} \Big(\frac{a + \tau_\infty}{a} \Big)^3,$$

contradicting to (5.10).

Therefore $\{u_j\}$ is non-varnishing, that is, there exist a sequence $\{y_j'\} \subset \mathbb{R}^3$ and constants $r > 0, \delta > 0$ such that

$$\liminf_{j \to \infty} \int_{B_r(y_j')} |u_j|^2 \geq \delta.$$

Set

$$v_j(x) = u_j(x + y_j').$$

Then v_j solves, denoting $\hat{V}_{\varepsilon_j}(x) = V(\varepsilon_j(x + y_j'))$ and $\hat{W}_{\varepsilon_j}(x) = W(\varepsilon_j(x + y_j'))$,

$$-i\alpha \cdot \nabla v_j + a\beta v_j + \hat{V}_{\varepsilon_j}(x) v_j = \hat{W}_{\varepsilon_j}(x) f(|v_j|) v_j \tag{5.11}$$

with least energy (using the notations of the previous section)

$$\hat{c}_{\varepsilon_j} = \hat{\Phi}_{\varepsilon_j}(v_j)$$
$$:= \frac{1}{2} \int_{\mathbb{R}^3} \langle A_a v_j, v_j \rangle + \hat{V}_{\varepsilon_j}(x) |v_j|^2 - \int_{\mathbb{R}^3} \hat{W}_{\varepsilon_j}(x) F(|v_j|)$$
$$= \int_{\mathbb{R}^3} \hat{W}_{\varepsilon_j}(x) \bar{F}(|v_j|)$$

where $\bar{F}(|u|) = \frac{1}{2} f(|u|) |u|^2 - F(|u|)$. Plainly,

$$\hat{c}_{\varepsilon_j} = \hat{\Phi}_{\varepsilon_j}(v_j) = \Phi_{\varepsilon_j}(u_j) = c_{\varepsilon_j}.$$

Additionally, $v_j \rightharpoonup u \neq 0$ in E and $v_j \to u$ in L_{loc}^s for $s \in [1, 3)$.

Since V and W are bounded, we can assume without loss of generality that $V(\varepsilon_j y_j') \to V_0$ and $W(\varepsilon_j y_j') \to W_0$ as $j \to \infty$. Since ∇V is bounded, one sees that, given arbitrarily $r > 0$, for any $x \in B_r(0)$,

$$|V(\varepsilon_j x + \varepsilon_j y_j') - V(\varepsilon_j y_j')| = \Big| \int_0^1 \nabla V(\varepsilon_j y_j' + s\varepsilon_j x) \varepsilon_j x \, ds \Big| \leq |\nabla V|_\infty r \varepsilon_j.$$

This implies that $\hat{V}_{\varepsilon_j}(x) \to V_0$ as $j \to \infty$ uniformly on bounded sets of x. Similarly, $\hat{W}_{\varepsilon_j}(x) \to W_0$ as $j \to \infty$ uniformly on bounded sets of x. It then follows from (5.11) that, for any $\varphi \in C_0^\infty$,

$$0 = \lim_{j \to \infty} \int_{\mathbb{R}^3} \langle (A_a + \hat{V}_{\varepsilon_j}(x)) v_j - \hat{W}_{\varepsilon_j}(x) f(|v_j|) v_j, \varphi \rangle$$
$$= \int_{\mathbb{R}^3} \langle (A_a + V_0) u - W_0 f(|u|) u, \varphi \rangle,$$

Thus u solves

$$-i\alpha \cdot \nabla u + a\beta u + V_0 u = W_0 f(|u|)u \tag{5.12}$$

with the energy

$$\Gamma_{V_0 W_0}^*(u) := \frac{1}{2} \int_{\mathbb{R}^3} \langle A_a u, u \rangle + V_0 |u|^2 - \int_{\mathbb{R}^3} W_0 F(|u|)$$
$$= \int_{\mathbb{R}^3} W_0 \bar{F}(|u|) \geq \gamma_{V_0 W_0}^*.$$

By a Fatou's lemma,

$$\int_{\mathbb{R}^3} W_0 \bar{F}(|u|) \leq \lim_{j \to \infty} \int_{\mathbb{R}^3} \hat{W}_{\varepsilon_j}(x) \bar{F}(|v_j|)$$

which implies

$$\Gamma_{V_0 W_0}^*(u) \leq \lim_{j \to \infty} c_{\varepsilon_j} \leq \gamma_{V_0 W_0}^*.$$

Therefore,

$$\lim_{j \to \infty} c_{\varepsilon_j} = \Gamma_{V_0 W_0}^*(u) = \gamma_{V_0 W_0}^* \tag{5.13}$$

and

$$\lim_{j \to \infty} \int_{\mathbb{R}^3} \hat{W}_{\varepsilon_j}(x) \bar{F}(|v_j|) = \int_{\mathbb{R}^3} W_0 \bar{F}(|u|) = \gamma_{V_0 W_0}^*.$$

Let $\eta : [0, \infty) \to [0, 1]$ be a smooth function satisfying $\eta(s) = 1$ if $s \leq 1$, $\eta(s) = 0$ if $s \geq 2$. Define $\tilde{v}_j(x) = \eta(2|x|/j)u(x)$. One has

$$\|u - \tilde{v}_j\| \to 0 \quad \text{and} \quad |u - \tilde{v}_j|_s \to 0 \quad \text{as } j \to \infty \tag{5.14}$$

for $s \in [2, 3]$. Setting $z_j = v_j - \tilde{v}_j$, one checks easily that, along a subsequence,

$$\lim_{j \to \infty} \left| \int_{\mathbb{R}^3} \hat{W}_{\varepsilon_j}(x) \big(F(|v_j|) - F(|z_j|) - F(|\tilde{v}_j|) \big) \right| = 0 \tag{5.15}$$

and

$$\lim_{j \to \infty} \left| \int_{\mathbb{R}^3} \hat{W}_{\varepsilon_j}(x) \big(f(|v_j|)v_j - f(|z_j|)z_j - f(|\tilde{v}_j|)\tilde{v}_j \big)\overline{\varphi} \right| = 0 \tag{5.16}$$

uniformly in $\varphi \in E$ with $\|\varphi\| \leq 1$ (see [2, 13, 30]). Using the exponentially decay of u, (5.14), and the facts that $\hat{V}_{\varepsilon_j}(x) \to V_0$, $\hat{W}_{\varepsilon_j}(x) \to W_0$ as $j \to \infty$ uniformly on any bounded set of x, one checks easily the following

$$\int_{\mathbb{R}^3} \hat{V}_{\varepsilon_j}(x)\langle v_j, \tilde{v}_j \rangle \to \int_{\mathbb{R}^3} V_0 |u|^2;$$
$$\int_{\mathbb{R}^3} \hat{W}_{\varepsilon_j}(x) F(|\tilde{v}_j|) \to \int_{\mathbb{R}^3} W_0 F(|u|),$$

Consequently, by (5.15),

$$\hat{\Phi}_{\varepsilon_j}(z_j) = \hat{\Phi}_{\varepsilon_j}(v_j) - \Gamma_{V_0 W_0}^*(u)$$
$$+ \int_{\mathbb{R}^3} \hat{W}_{\varepsilon_j}(x)\big(F(|v_j|) - F(|z_j|) - F(|\tilde{v}_j|)\big) + o(1)$$
$$= o(1)$$

as $j \to \infty$, which implies that $\hat{\Phi}_{\varepsilon_j}(z_j) \to 0$. Similarly, by (5.16),

$$\hat{\Phi}_{\varepsilon}'(z_j)\varphi = \int_{\mathbb{R}^3} \hat{W}_{\varepsilon_j}(x)\big(f(|v_j|)v_j - f(|z_j|)z_j - f(|\tilde{v}_j|)\tilde{v}_j\big)\overline{\varphi} + o(1)$$
$$= o(1)$$

as $j \to \infty$ uniformly in $\|\varphi\| \leq 1$, which implies that $\hat{\Phi}_{\varepsilon_j}'(z_j) \to 0$. Therefore,

$$o(1) = \int_{\mathbb{R}^3} \hat{W}_{\varepsilon_j}(x)\bar{F}(|z_j|).$$

This, together with (g_2), shows

$$\int_{\mathbb{R}^3} \hat{W}_{\varepsilon_j}(x)f(|z_j|)|z_j|^2 \to 0.$$

Notice that $\{|z_j|_\infty\}$ is bounded so $\int_{\mathbb{R}^3} \hat{W}_{\varepsilon_j}(x)f(|z_j|)|z_j^+ - z_j^-|^2 \leq C^2$. As a consequence, we get

$$\left(1 + \frac{\tau}{a}\right)\|z_j\|^2 \leq \|z_j\|^2 + \Re \int_{\mathbb{R}^3} \hat{V}_{\varepsilon_j}(x)\langle z_j, z_j^+ - z_j^-\rangle$$
$$= \Phi_{\varepsilon_j}'(z_j)(z_j^+ - z_j^-) + \Re \int_{\mathbb{R}^3} \hat{W}_{\varepsilon_j}(x)f(|z_j|)\langle z_j, z_j^+ - z_j^-\rangle$$
$$\leq o(1) + C\left(\int_{\mathbb{R}^3} \hat{W}_{\varepsilon_j}(x)f(|z_j|)|z_j|^2\right)^{1/2}$$
$$= o(1),$$

that is, $\|z_j\| \to 0$ which, together with (5.14), yields $v_j \to u$ in E as $j \to \infty$.

In order to verify that $v_j \to u$ in H^1, observe that by (5.11) and (5.12)

$$A_a z_j = \hat{W}_{\varepsilon_j}(x)f(|v_j|)v_j - W_0 f(|u|)u - \big(\hat{V}_{\varepsilon_j}(x)v_j - V_0 u\big).$$

By the exponential decay of u and the uniform estimate (3.6), it is easy to show that $|A_a z_j|_2 \to 0$. Therefore $v_j \to u$ in H^1.

Step 2) $v_j(x) \to 0$ as $|x| \to \infty$ uniformly in $j \in \mathbb{N}$. Assume by contradiction that the conclusion does not hold. Then by the sub-solution estimate there exist $\sigma > 0$ and $x_j \in \mathbb{R}^3$ with $|x_j| \to \infty$ such that $\sigma \leq |v_j(x_j)| \leq C_0 \int_{B_1(x_j)} |v_j|$. Since $v_j \to u$ in H^1 one gets

$$\sigma \leq C_0 \int_{B_1(x_j)} |v_j| \leq C_0 \int_{B_1(x_j)} |v_j - u| + C_0 \int_{B_1(x_j)} |u|$$
$$\leq C'\left(\int_{\mathbb{R}^3} |v_j - u|^2\right)^{1/2} + C_0 \int_{B_1(x_j)} |u| \to 0,$$

a contradiction.

Step 3) $\{\varepsilon_j y'_j\}_j$ is bounded. Assume by contradiction that $\varepsilon_j |y'_j| \to \infty$ (along a subsequence). Then $V_0 \geq \tau_\infty$ and $W_0 < \pi$ (by (P_1)) which, together with Lemma 4.9 (noting that $\gamma^*_{V_0 W_0}$ attains), implies $\gamma^*_{V_0 W_0} > \gamma^*_{\mu\pi}$ for any $\mu \in [\tau, \tau_\infty]$. On the other hand, choosing μ close to τ_∞ satisfying (5.3), it follows from (5.13) and Lemma 5.2, $\lim_{\varepsilon_j \to 0} c_{\varepsilon_j} = \gamma^*_{V_0 W_0}$ and $\lim_{\varepsilon_j \to 0} c_{\varepsilon_j} \leq \gamma^*_{\mu\pi}$, hence $\gamma^*_{V_0 W_0} \leq \gamma_{\mu\pi}$, a contradiction. Therefore, we can assume $\varepsilon_j y'_j \to y_0$, $V_0 = V(y_0)$ and $W_0 = W(y_0)$. So, $u(x)$ is a least energy solution of (5.9). Now by *Step 2* it is easy to see that one may assume that $y_j = y'_j$ is a maximum point of $|u_j|$.

Step 4) $\{\varepsilon y_\varepsilon\}_\varepsilon$ is bounded. Assume by contradiction that there is $\varepsilon_j \to 0$ with $\varepsilon_j |y_j| \to \infty$ where y_j is a maximum point of $|u_j|$ ($y_j = y_{\varepsilon_j}$, $u_j = u_{\varepsilon_j}$). Repeating the above arguments one sees that any relative subsequence y'_j of y_j and $v_j(x) = u_j(x + y'_j)$ satisfies that $v_j(x) \to 0$ as $|x| \to \infty$ uniformly in $j \in \mathbb{N}$ and $\{\varepsilon_j y'_j\}$ is bounded (*Step 2), 3)*). Consequently, $\varepsilon_j |y_j - y'_j| \geq \varepsilon_j |y_j| - \varepsilon_j |y'_j| \to \infty$, particularly, $|y_j - y'_j| \to \infty$. Then, $\max |u_j| = |u_j(y_j)| = |v_j(y_j - y'_j)| \to 0$, a contradiction.

Step 5) $\lim_{\varepsilon \to 0} \mathrm{dist}(\varepsilon y_\varepsilon, A_w) = 0$. It is sufficient to check that $y_0 \in \mathscr{A}_w$. By virtue of Proposition 4.1 and (P_3), $\gamma^*_{\tau\pi}$ is archived, it hence follows from (4.18) that

$$\lim_{j \to \infty} c_{\varepsilon_j} = \lim_{j \to \infty} c_{\varepsilon_j}^{\tau\pi} \leq \gamma^*_{V^\tau(0) W^\pi(0)} = \gamma^*_{V(0) W(0)} = \gamma^*_{\kappa\pi},$$

which, together with (5.13), shows

$$\gamma^*_{V(y_0) W(y_0)} \leq \gamma^*_{\kappa\pi}.$$

Since $\pi \geq W(y_0)$ one has $V(y_0) \leq \kappa$. If $\pi = W(y_0)$, i.e., $y_0 \in \mathscr{W}$, there must be $V(y_0) = \kappa$ (because $\kappa = \min_{\mathscr{W}} V$). If $W(y_0) < \pi$ then we must have $V(y_0) < \kappa$. In conclusion, $y_0 \in \mathscr{A}_w$.

The proof is hereby complete. $\square$

LEMMA 5.5. *There exists $C > 0$ such that for all $j \in \mathbb{N}$*

$$|u_j(x)| \leq C e^{-\sqrt{\omega/2}|x - y_j|} \qquad \forall j \in \mathbb{N}$$

where $\omega = a^2 - \tau^2$

Proof. By the *Step 2* of the proof of Lemma 5.4 we may take $\delta > 0$ and $r > 0$ such that $|v_j(x)| \leq \delta$ and

$$\left| \Re \left[r_{\varepsilon_j}(x, |v_j|) v_j \frac{\overline{v_j}}{|v_j|} \right] \right| \leq \frac{\omega}{2} |v_j|$$

for all $|x| \geq r, j \in \mathbb{N}$. This, together with (5.7), implies

$$\Delta |v_j| \geq \frac{\omega}{2} |v_j| \quad \text{for all } |x| \geq r, j \in \mathbb{N}.$$

Let $\Gamma(y) = \Gamma(y, 0)$ be a fundamental solution to $-\Delta + \omega/2$ (see, e.g., [27]). Using the uniform boundedness, one may choose Γ so that $|v_j(y)| \leq \frac{\omega}{2} \Gamma(y)$ holds on $|y| = r$, all $j \in \mathbb{N}$. Let $z_j = |v_j| - \frac{\omega}{2} \Gamma$. Then

$$\Delta z_j = \Delta |v_j| - \frac{\omega}{2} \Delta \Gamma$$
$$\geq \frac{\omega}{2} \left(|v_j| - \frac{\omega}{2} \Gamma \right) = \frac{\omega}{2} z_j.$$

By the maximum principle we can conclude that $z_j(y) \leq 0$ on $|y| \geq r$. It is well known that there is $C' > 0$ such that $\Gamma(y) \leq C' \exp(-\sqrt{\omega/2}|y|)$ on $|y| \geq 1$. We see that

$$|v_j(y)| \leq C \exp(-\sqrt{\omega/2}|y|)$$

for all $y \in \mathbb{R}^3$ and all $j \in \mathbb{N}$, that is,

$$|u_j(x)| \leq C \exp(-\sqrt{\omega/2}|x - y_j|)$$

for all $x \in \mathbb{R}^3$ and all $j \in \mathbb{N}$

The proof is completed.   □

Proof. [Proof of Theorem 2.2]   Writing $\varepsilon = \varepsilon_j$ and going back to the equation (2.1) with the variable substitution $x \mapsto x/\varepsilon$, we set

$$w_\varepsilon(x) = u_\varepsilon(x/\varepsilon) \quad \text{and} \quad x_\varepsilon = \varepsilon y_\varepsilon.$$

Then w_ε is a least energy solution of (2.1) for all ε small, and $w_\varepsilon \in W^{1,s}$ for all $s \geq 2$ by Lemma 5.6. It is clear that x_ε is a maximum point of $|w_\varepsilon|$, and the conclusions (a_1) and (a_2) follow from Lemma 5.5 and Lemma 5.4, respectively.   □

5.2. Proof of Theorem 2.3. Assume (P_0)-$(\hat{P}_0)$ and (P_2) are satisfied. We can assume, without loss of generality, that $0 \in \mathscr{V}$ such that $W(0) = \max_{x \in \mathscr{V}} W(x)$. Then $V(0) = \tau$ and $\pi \geq \kappa := W(0) \geq \pi_\infty$. Consider the functional Φ_ε. We verify similarly to Lemma 5.1 the following existence.

LEMMA 5.6. *Let (g_1)-(g_2), (P_0)-$(\hat{P}_0)$ and (P_2) be satisfied. Then c_ε is attained for all small $\varepsilon > 0$.*

Proof. Given $\varepsilon > 0$, let $u_k \in \mathscr{N}_\varepsilon$ be a minimizing sequence of I_ε and set $w_k = u_k + h_\varepsilon(u_k)$ which is a $(PS)_{c_\varepsilon}$ sequence for Φ_ε on E. We may assume $w_k \rightharpoonup w_\varepsilon \in \mathscr{K}_\varepsilon$ in E. If $w_\varepsilon \neq 0$ then clearly $\Phi_\varepsilon(w_\varepsilon) = c_\varepsilon$ and we are done.

Assume by contradiction that there is a sequence $\varepsilon_j \to 0$ with $w_{\varepsilon_j} = 0$. Then $w_k = u_k + h_{\varepsilon_j}(u_k) \rightharpoonup 0$ in E and L^s for $s \in [2,3]$, $w_k \to 0$ in L^s_{loc} for $s \in (1,3)$. and $w_k(x) \to 0$ a.e. for $x \in \mathbb{R}^3$. By $(\hat{P}_0)$, one can choose $\tau < \mu < \tau_\infty$ satisfying (5.3) for any $\nu \in [\pi_\infty, \pi]$. In particular, take $\nu = \kappa$ and consider the functional $\Phi^{\mu\kappa}_{\varepsilon_j}$. Since $V^\mu_{\varepsilon_j}(0) \leq \mu$ and $W^\kappa_{\varepsilon_j}(0) = \kappa$, and mu and κ satisfy (5.3), it follows from (4.19) that

$$\lim_{j \to \infty} c^{\mu\kappa}_{\varepsilon_j} = \gamma^*_{\mu\kappa}.$$

As before, letting $t_k > 0$ be such that $t_k u_k \in \mathscr{N}^{\mu\kappa}_{\varepsilon_j}$, and $O_\varepsilon := \{x \in \mathbb{R}^3 : V_\varepsilon(x) \leq \mu \text{ or } W_\varepsilon(x) > \kappa\}$, we obtain

$$c^{\mu\kappa}_{\varepsilon_j} \leq I^{\mu\kappa}_{\varepsilon_j}(t_k u_k) \leq I_{\varepsilon_j}(u_k) + \frac{1}{2}\int_{O_{\varepsilon_j}} \left(\mu - V_{\varepsilon_j}(x)\right)|t_k u_k + h^{\mu\kappa}_{\varepsilon_j}(t_k u_k)|^2$$

$$+ \int_{O_{\varepsilon_j}} \left(W_{\varepsilon_j}(x) - \kappa\right)F(|t_k u_k + h^{\mu\kappa}_{\varepsilon_j}(t_k u_k)|)$$

$$= c_{\varepsilon_j} + o(1)$$

as $k \to \infty$, hence, $c^{\mu\kappa}_{\varepsilon_j} \leq c_{\varepsilon_j}$, consequently,

$$\gamma^*_{\mu\kappa} \leq \lim_{j \to \infty} c_{\varepsilon_j}.$$

On the other hand, let $v_k \in \mathscr{N}_\varepsilon^{\mu\kappa}$ be a minimizing sequence for $I_\varepsilon^{\mu\kappa}$ and set $E_k = E^- \oplus \mathbb{R}^+ v_k$. Since $V^\mu \geq V$ and $W^\kappa \leq W$, one has

$$I_\varepsilon^{\mu\kappa}(v_k) = \max_{z \in E_k} \Phi_\varepsilon^{\mu\kappa}(z) \geq \max_{z \in E_k} \Phi_\varepsilon(z),$$

so $c_\varepsilon \leq c_\varepsilon^{\mu\kappa}$. Therefore, the above discussion yields

$$\gamma_{\mu\kappa}^* = \lim_{\varepsilon_j \to 0} c_{\varepsilon_j}^{\mu\kappa} = \lim_{\varepsilon_j \to 0} c_{\varepsilon_j}.$$

In particular, for any $\tau < \mu_1 < \mu_2 < \tau_\infty$ so that (5.3) holds for $\mu = \mu_1$ and $\mu = \mu_2$, one obtains the following

$$\gamma_{\mu_1\kappa}^* < \gamma_{\mu_2\kappa}^* = \lim_{\varepsilon_j \to 0} c_{\varepsilon_j} = \gamma_{\mu_1\kappa}^*,$$

a contradiction.   □

As a by-product of the above argument we have

LEMMA 5.7. $\limsup_{\varepsilon \to 0} c_\varepsilon \leq \limsup_{\varepsilon \to 0} c_\varepsilon^{\mu\kappa} = \gamma_{\mu\kappa}^*$ for all μ satisfying (5.3).
We also point out that the above argument applies directly to verify that $\mathscr{S}_\varepsilon$ is compact for all small $\varepsilon > 0$.

Now we study the concentration phenomenon under additionally (P_3). Such an assumption implies particularly that $\gamma_{\tau\kappa}^*$ is archived.

LEMMA 5.8. Assume additionally that (P_3) is satisfied. Let $u_\varepsilon \in \mathscr{S}_\varepsilon$. There is a maximum point y_ε of $|u_\varepsilon|$ such that $\lim\limits_{\varepsilon \to 0} \mathrm{dist}(\varepsilon y_\varepsilon, \mathscr{A}_v) = 0$, and for any sequence $\varepsilon y_\varepsilon \to y_0$, $v_\varepsilon(x) := u_\varepsilon(x + y_\varepsilon)$ converges in H^1 to a least energy solution of

$$-i\alpha \cdot \nabla v + a\beta v + V(y_0)v = W(y_0)h(|v|)v.$$

Proof. We are sketchy along the lines carried out previously for Lemma 5.4.

Step 1) Let $u_j \in \mathscr{S}_j \equiv \mathscr{S}_{\varepsilon_j}$ with $\varepsilon_j \to 0$. Then $\{u_j\}$ is bounded and non-vanishing. There exist a sequence $\{y_j'\} \subset \mathbb{R}^3$ and constants $r > 0, \delta > 0$ such that

$$\liminf_{j \to \infty} \int_{B_r(y_j')} |u_j|^2 \geq \delta.$$

Then $v_j(x) = u_j(x + y_j')$ is a least energy solution of

$$-i\alpha \cdot \nabla v_j + a\beta v_j + \hat{V}_{\varepsilon_j}(x)v_j = \hat{W}_{\varepsilon_j}(x)f(|v_j|)v_j$$

where $\hat{V}_{\varepsilon_j}(x) = V(\varepsilon_j(x + y_j'))$ and $\hat{W}_{\varepsilon_j}(x) = W(\varepsilon_j(x + y_j'))$. The least energy

$$\hat{c}_{\varepsilon_j} = \hat{\Phi}_{\varepsilon_j}(v_j) = \Phi_{\varepsilon_j}(u_j) = c_{\varepsilon_j}$$
$$= \int_{\mathbb{R}^3} \hat{W}_{\varepsilon_j}(x)\bar{F}(|v_j|).$$

Moreover, $v_j \rightharpoonup u \neq 0$ in E, $v_j \to u$ in L_{loc}^s for $s \in [1,3)$, and u solves

$$-i\alpha \cdot \nabla u + a\beta u + V_0 u = W_0 f(|u|)u$$

with the energy

$$\Gamma^*_{V_0 W_0}(u) = \int_{\mathbb{R}^3} W_0 \bar{F}(|u|) \geq \gamma^*_{V_0 W_0}.$$

By the Fatou's lemma one gets

$$\lim_{j \to \infty} c_{\varepsilon_j} = \Gamma^*_{V_0 W_0}(u) = \gamma^*_{V_0 W_0} \qquad (5.17)$$

and

$$\lim_{j \to \infty} \int_{\mathbb{R}^3} \hat{W}_{\varepsilon_j}(x) \bar{F}(|v_j|) = \int_{\mathbb{R}^3} W_0 \bar{F}(|u|) = \gamma^*_{V_0 W_0}.$$

Using these facts one sees as before that, in fact, $v_j \to u$ in H^1.

Step 2) $v_j(x) \to 0$ as $|x| \to \infty$ uniformly in $j \in \mathbb{N}$. This follows from (5.8) and the convergence of $v_j \to u$ in H^1.

Step 3) $\{\varepsilon_j y'_j\}_j$ is bounded. Assume by contradiction that $\varepsilon_j |y'_j| \to \infty$. Then $V_0 \geq \tau_\infty > \tau$ and $W_0 \leq \pi_\infty \leq \kappa$, which implies $\gamma^*_{V_0 W_0} \geq \gamma^*_{\tau_\infty \pi_\infty} > \gamma^*_{\mu \pi_\infty} \geq \gamma^*_{\mu \kappa}$ for any $\mu \in [\tau, \tau_\infty)$. On the other hand, choosing $\mu < \tau_\infty$ satisfying (5.3), it follows from (5.17) and Lemma 5.7, that $\lim_{\varepsilon_j \to 0} c_{\varepsilon_j} = \gamma^*_{V_0 W_0}$ and $\lim_{\varepsilon_j \to 0} c_{\varepsilon_j} \leq \gamma^*_{\mu \kappa}$, hence $\gamma^*_{V_0 W_0} \leq \gamma^*_{\mu \kappa}$, a contradiction. Therefore, we can assume $\varepsilon_j y'_j \to y_0$, $V_0 = V(y_0)$ and $W_0 = W(y_0)$. Now by *Step 2* it is easy to see that one may assume that $y_j = y'_j$ is a maximum point of $|u_j|$.

Step 4) $\{\varepsilon y_\varepsilon\}_\varepsilon$ is bounded.

Step 5) $\lim_{\varepsilon \to 0} \text{dist}(\varepsilon y_\varepsilon, \mathscr{A}_v) = 0$. It is sufficient to check that $y_0 \in \mathscr{A}_v$. By (P_3) and (4.18)

$$\lim_{j \to \infty} c_{\varepsilon_j} = \lim_{j \to \infty} c^{\tau\pi}_{\varepsilon_j} \leq \gamma^*_{V^\tau(0) W^\pi(0)} = \gamma^*_{V(0) W(0)} = \gamma^*_{\tau\kappa}$$

which, jointly with (5.17), implies

$$\gamma^*_{V(y_0) W(y_0)} \leq \gamma^*_{\tau\kappa}.$$

Since $V(y_0) \geq \tau$, one has $W(y_0) \geq \kappa$. If $V(y_0) = \kappa$, i.e. $y_0 \in \mathscr{V}$, then $W(y_0) = \kappa$ because $\kappa = \max_{\mathscr{V}} W$. If $y_0 \notin \mathscr{V}$ then there must be $W(y_0) > \kappa$. This proves that $y_0 \in \mathscr{A}_v$.

The proof is hereby complete.   □

Now repeating the arguments of Lemma 5.5 we obtain

LEMMA 5.9. *There exists $C > 0$ such that for all ε small*

$$|u_\varepsilon(x)| \leq C e^{-\sqrt{\omega/2}|x - y_\varepsilon|} \qquad \forall x \in \mathbb{R}^3.$$

Proof. [Proof of Theorem 2.3] Define

$$w_\varepsilon(x) = u_\varepsilon(x/\varepsilon) \quad \text{and} \quad x_\varepsilon = \varepsilon y_\varepsilon.$$

Then w_ε is a least energy solution of (2.1) for all ε small, x_ε is a maximum point of $|w_\varepsilon|$, and the conclusion (a_1) and (a_2) follow from Lemma 5.9 and Lemma 5.8.   □

5.3. Proof of Theorem 2.5. *Proof.* [Proof of Theorem 2.5] Consider the functional Φ_ε.

Step 1. For each $\varepsilon > 0$, there is $\delta_\varepsilon > 0$ such that $\|u^\pm\|^2 \pm \int_{\mathbb{R}^3} V_\varepsilon(x)|u^\pm|^2 \geq \delta_\varepsilon \|u^\pm\|^2$. This implies that Φ_ε possesses the linking structure. Let c_ε denote the linking level of Φ_ε.

Step 2. There is $\varsigma > 0$ independent of ε such that Φ_ε satisfies $(PS)_c$ condition for all $c < \varsigma$. The proof is somewhat usual, see [16, Lemma 3.6] or [13] for example.

Step 3. Setting $V^\mu(x) = \max\{\mu, V(x)\}$ for $\mu \in (0, \tau_\infty)$, $\nu = \inf W$, one has

$$\Phi_\varepsilon(u) \leq \Phi_\varepsilon^\mu(u) := \frac{1}{2} \int_{\mathbb{R}^3} \langle (-i\alpha \cdot \nabla + a\beta + V_\varepsilon^\mu(x))u,\, u \rangle - \int_{\mathbb{R}^3} W_\varepsilon(x)F(|u|)$$

$$\leq \tilde{\Phi}_\varepsilon^\mu(u) := \frac{1}{2} \int_{\mathbb{R}^3} \langle (-i\alpha \cdot \nabla + a\beta + V_\varepsilon^\mu(x))u,\, u \rangle - \nu c_0 \int_{\mathbb{R}^3} |u|^q$$

Let c_ε^μ and $\hat{c}_\varepsilon^\mu$ denote the linking levels respectively of Φ_ε^μ and $\hat{\Phi}_\varepsilon^\mu$. The arguments of Lemma 4.10 are applicable (using (4.4) instead of (4.15)) to $\hat{\Phi}_\varepsilon^\mu$, which yields

$$\limsup_{\varepsilon \to 0} \hat{c}_\varepsilon^\mu \leq \gamma_{\mu\nu}.$$

This, jointly with Lemma 4.6, implies that

$$\limsup_{\varepsilon \to 0} \hat{c}_\varepsilon^\mu \leq (a + \mu)^{\frac{2(3-q)}{q-2}} (c_0\nu)^{\frac{-2}{q-2}} \gamma_q.$$

Thus

$$\lim_{\varepsilon \to 0} \lim_{\mu \to -a} c_\varepsilon^\mu = 0$$

Take $\mu_0 > -a$ such that $c_\varepsilon^{\mu_0} < \varsigma$ for all ε small. Sice $c_\varepsilon \leq c_\varepsilon^\mu$, we see that, for all ε small, (2.1) has a ground state solution u_ε with $\Phi_\varepsilon(u_\varepsilon) = c_\varepsilon \to 0$ as $\varepsilon \to 0$.

Step 4. Finally, similarly to the last argument of the step 1 of Lemma 5.4, one checks that $u_\varepsilon \to 0$ in H^1. The proof is hereby complete. $\square$

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