ON A NONLINEAR ELLIPTIC SYSTEM WITH SYMMETRIC COUPLING

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Abstract

Multiplicity results are proved for the nonlinear elliptic system

$$\begin{cases} -\Delta u + g(v) &= 0 \\ -\Delta v + g(u) &= 0 \qquad \text{in } \Omega, \\ \quad u = v &= 0 \qquad \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and $g : \mathbb{R} \longrightarrow \mathbb{R}$ is a nonlinear C^1-function which satisfies addtional conditions. No assumption of symmetry on g is imposed.
Extensive use is made of a global version of the Lyapunov-Schmidt reduction method due to Castro and Lazer (see [C] and [CL]), and of symmetric versions of the Mountain Pass Theorem (see [AR] and [R]).

Key Words and phrases: Elliptic system, Lyapunov-Schmidt reduction method, Mountain Pass Theorem.

1

1 Introduction

It is well-known that a symmetry in a differential equation often generates the existence of multiple solutions. Consider e.g. the superlinear and subcritical equation

$$-\Delta u = f(u) \ , \quad \text{in} \ \ \Omega \ , \ u|_{\partial\Omega} = 0 \ , \tag{2}$$

where $f \in C(\mathbb{R})$ is a superlinear and subcritical nonlinearity. If $f(u)$ is an odd function, then the equation has the symmetry $u \mapsto -u$. Using the concept of index theories (e.g. the Krasnoselskii genus), one shows that this symmetry implies that the equation has infinitely many solutions.

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In this article we consider a semilinear elliptic system in which the symmetry is not given by an odd nonlinearity, but by a *symmetric coupling*. We consider systems of the following form

$$\left\{ \begin{array}{rll} -\Delta u + g(v) & = 0 & \\ -\Delta v + g(u) & = 0 & \text{in } \Omega, \\ u = v & = 0 & \text{on } \partial\Omega, \end{array} \right. \tag{3}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with smooth boundary and $g : \mathbb{R} \longrightarrow \mathbb{R}$ is a C^1-function satisfying some assumptions to be specified later, but is not required to be odd. Note that this system allows the following symmetry:

$$T_1 : (u, v) \mapsto (v, u).$$

Indeed, looking at the associated functional (supposing it is well-defined)

$$J(u, v) = \int_\Omega \nabla u \cdot \nabla v + \int_\Omega G(u) + \int_\Omega G(v) \ , \tag{4}$$

where $G(s) = \int_0^s g(t)dt$ is the primitive of g, we see that this functional is invariant under the group action $T = \{id, T_1\}$.
Thus, one may try to proceed similarly as for equation (2) by defining a suitable index. However, one encounters two major problems. First, the functional is strongly indefinite due to the first term in the functional. Second, the group T has an infinite-dimensional fixed point space, given by the pairs of functions of the form $\{(u, u)\}$. We overcome these difficulties by performing an infinite dimensional Lyapunov-Schmidt reduction (following Castro-Lazer [CL]). Surprisingly, the resulting reduced functional $\widetilde{J}$ has the classical $\mathbb{Z}_2$-symmetry $\{id, -id\}$ (although, as we emphasize, no oddness assumption is taken for the nonlinearity), and so classical variational methods for the existence of multiple solutions can be employed.

We will denote by $0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$ the sequence of eigenvalues of $-\Delta$ with zero Dirichlet boundary condition in Ω. Also, $\{\varphi_j\}_j$ will denote an orthonormal basis, in $H_0^1(\Omega)$, of eigenfunctions of $-\Delta$ in Ω with Dirichlet boundary condition. We will study the existence of multiple solutions for problem (3) under three different sets of conditions. For the first two sets, we assume g satisfies

- (g_0) $g(0) = 0$ and
- (g_1) $\inf_{t\in\mathbb{R}} g'(t) > -\lambda_1$.

First, we consider the *superlinear setting*, in which we assume

- (g_2) There exists a positive constant C such that
  $|g(t)| \leq C(1 + |t|^p)$, where $p \in (1, \frac{N+2}{N-2})$ for all $t \in \mathbb{R}$, and
- (g_3) There exists $R > 0$ such that $0 < \mu G(t) \leq tg(t)$, for $|t| > R$, where $\mu > 2$.

Secondly, we also consider the *asymptotically linear setting*, in which g is assumed to satisfy

- (g_4) $g'(\infty) := \lim_{|t|\to\infty} \frac{g(t)}{t} \in (\lambda_k, \lambda_{k+1})$ for some $k \geq 1$.

Our main results read as follows.

Theorem A. *(superlinear case) If g satisfies $(g_0)-(g_3)$, problem* (3) *has infinitely many solutions.*

We observe that conditions (g_2) and (g_3) include the "classical" nonlinearity $g(t) = t|t|^{p-1}$. But we emphasize that Theorem A holds true for a more general kind of nonlinearities, e.g. $g(t) = (t^+)^p - (t^-)^q$, for $t \in \mathbb{R}$ and $1 < p, q < (N+2)/(N-2)$, without any further restriction on p and q.

In the asymptotically linear framework we have the following analogue of Theorem A.

Theorem B. *(asymptotically linear case) Assume g satisfies $(g_0)-(g_1)$ and (g_4). If, in addition, $g'(0) < \lambda_j$ for $j \le k$, then problem* (3) *has (at least)* $2(k-j+1)$ *nontrivial solutions.*

On the other hand, we consider a third setting, in which we only assume

(g_5) $\sup_{t\in\mathbb{R}} g'(t) < \lambda_1$.

We observe that under condition (g_5), system (3) is equivalent to the system

$$\begin{cases} -\Delta u &= h(v) \\ -\Delta v &= h(u) \qquad \text{in} \quad \Omega, \\ \quad u &= v = 0 \qquad \text{on} \quad \partial\Omega, \end{cases} \tag{5}$$

where $h = -g$ satisfies $\inf h' > -\lambda_1$. We point out that (5) is the very analogue in systems of the single-equation problem (2). In this direction we prove the following result which shows that system (3) (or, equivalently, system (5)) has a *strong hidden symmetry.*

Theorem C. *Assume g satisfies (g_5). Then (u, v) is a solution of* (3) *if and only if $u \equiv v$ and*

$$-\Delta u + g(u) = 0 \ , \quad in \quad \Omega \ , \ u|_{\partial\Omega} = 0 \ . \tag{6}$$

In other words, under condition (g_5), solving system (3) *is equivalent to solving the single-equation problem* (6).

System (3) is Hamiltonian and our approach to it is variational, i.e. we define an energy functional $J : H_0^1(\Omega) \times H_0^1(\Omega) \longrightarrow \mathbb{R}$ by

$$J(u, v) = \int_\Omega (\nabla u \cdot \nabla v + G(u) + G(v))\, d\zeta,$$

where $G(t) := \int_0^t g(s)ds$. Assuming either (g_2) or (g_4), this functional is of class C^1 (see [R]) and

$$\partial_u J(u, v)\varphi = \int_\Omega (\nabla\varphi \cdot \nabla v + g(u)\varphi)\, d\zeta, \qquad \forall u, v, \varphi \in H_0^1(\Omega), \tag{7}$$

and

$$\partial_v J(u, v)\psi = \int_\Omega (\nabla u \cdot \nabla\psi + g(v)\psi)\, d\zeta, \qquad \forall u, v, \psi \in H_0^1(\Omega). \tag{8}$$

Thus, because of classical regularity theory (see [GT]), critical points of J agree with classical solutions of problem (3). We then prove Theorem A and B showing the existence of critical points of J. Because of the form of the system

$$(u,v) \text{ is a solution of (3) if and only if } (v,u) \text{ is a solution of (3)}, \tag{9}$$

as can be easily verified. This fact provides some symmetry on the functional J when it is written in appropriate coordinates.

The paper is organized as follows: in Section 2 we recall the Castro-Lazer version of the Lyapunov-Schmidt reduction method in an abstract setting. We then show that our functional J satisfies the conditions of such setting. In Section 3 we prove Theorem A and in Section 4 we prove Theorem B. In proving them, we recall and use appropriate symmetric versions of the Mountain Pass Theorem of Ambrosetti and Rabinowitz. Finally, in Section 5 we prove Theorem C.

2 Preliminaries

We begin by stating a global version of the Lyapunov-Schmidt method (see [C] and [CL]).

Lemma 2.1. *Let H be a real separable Hilbert space. Let Z and W be closed subspaces of H such that $H = Z \oplus W$. Let $J : H \longrightarrow \mathbb{R}$ a function of class C^1. If there exist $m > 0$ and $\sigma > 1$ such that*

$$\langle \nabla J(\boldsymbol{z}+\boldsymbol{w}) - \nabla J(\boldsymbol{z}+\boldsymbol{w}_1), \boldsymbol{w}-\boldsymbol{w}_1\rangle \geq m\|\boldsymbol{w}-\boldsymbol{w}_1\|_H^\sigma \quad \forall \boldsymbol{z}\in Z \quad \forall \boldsymbol{w},\boldsymbol{w}_1 \in W \tag{10}$$

then:

(i) There exists a continuous function $\phi : Z \to W$ such that

$$J(\boldsymbol{z}+\phi(\boldsymbol{z})) = \min_{\boldsymbol{w}\in W} J(\boldsymbol{z}+\boldsymbol{w}).$$

Moreover, given $\boldsymbol{z}\in Z$, $\phi(\boldsymbol{z})$ is the unique element of W such that

$$\langle \nabla J(\boldsymbol{z}+\phi(\boldsymbol{z})), \boldsymbol{w}\rangle = 0 \qquad \forall \boldsymbol{w}\in W. \tag{11}$$

(ii) The functional $\widetilde{J} : Z \to \mathbb{R}$, defined by $\widetilde{J}(\boldsymbol{z}) := J(\boldsymbol{z}+\phi(\boldsymbol{z}))$ for $\boldsymbol{z}\in Z$, is of class C^1. Moreover,

$$D\widetilde{J}(\boldsymbol{z})\boldsymbol{h} = \langle \nabla \widetilde{J}(\boldsymbol{z}), \boldsymbol{h}\rangle = \langle \nabla J(\boldsymbol{z}+\phi(\boldsymbol{z})), \boldsymbol{h}\rangle \qquad \forall \boldsymbol{z},\boldsymbol{h}\in Z. \tag{12}$$

(iii) Given $\boldsymbol{z}\in Z$, $\boldsymbol{z}$ is a critical point of $\widetilde{J}$ if and only if $\boldsymbol{z}+\phi(\boldsymbol{z})$ is a critical point of J.

Assuming (g_1) and either (g_2) or (g_4), we intend to apply Lemma 2.1 to the functional $J : H_0^1(\Omega)\times H_0^1(\Omega) \longrightarrow \mathbb{R}$ defined as

$$J(u,v) = \int_\Omega (\nabla u\cdot\nabla v + G(u) + G(v))\, d\zeta,$$

where $G(t) := \int_0^t g(s)ds$. First, it is well-known that assuming either (g_2) or (g_4), this functional is of the class C^1 (see [R]) and

$$\partial_u J(u,v)\varphi = \int_\Omega (\nabla\varphi \cdot \nabla v + g(u)\varphi)\, d\zeta, \qquad \forall u, v, \varphi \in H_0^1(\Omega), \tag{13}$$

and

$$\partial_v J(u,v)\psi = \int_\Omega (\nabla u \cdot \nabla \psi + g(v)\psi)\, d\zeta, \qquad \forall u, v, \psi \in H_0^1(\Omega). \tag{14}$$

Let us take $H = H_0^1(\Omega)\times H_0^1(\Omega)$ equipped with the inner product $\langle (u_1, v_1), (u_2, v_2)\rangle = \langle u_1, u_2\rangle_{H_0^1} + \langle v_1, v_2\rangle_{H_0^1}$. Here, $\langle f_1, f_2\rangle_{H_0^1} = \int_\Omega \nabla f_1 \cdot \nabla f_2$. Let us define $W := \{\mathbf{w} = (w, w) : w \in H_0^1(\Omega)\}$ and $Z := \{\mathbf{z} = (z, -z) : z \in H_0^1(\Omega)\}$. Then $H_0^1(\Omega) \times H_0^1(\Omega) = Z \oplus W$. Let us verify (10). Let $\mathbf{z} \in Z$ and $\mathbf{w}, \mathbf{w}_1 \in W$. Then

$$\begin{aligned}
&\langle \nabla J(\mathbf{z}+\mathbf{w}) - \nabla J(\mathbf{z}+\mathbf{w}_1), \mathbf{w}-\mathbf{w}_1\rangle \\
&= \langle \nabla J(z+w, -z+w) - \nabla J(z+w_1, -z+w_1), (w-w_1, w-w_1)\rangle \\
&= [\partial_u J(z+w, -z+w) - \partial_u J(z+w_1, -z+w_1)](w-w_1) \\
&\quad + [\partial_v J(z+w, -z+w) - \partial_v J(z+w_1, -z+w_1)](w-w_1) \\
&= 2\int_\Omega |\nabla(w-w_1)|^2 + \int_\Omega [g(z+w) - g(z+w_1)](w-w_1) \\
&\quad + \int_\Omega [g(-z+w) - g(-z+w_1)](w-w_1).
\end{aligned}$$

Because of (g_1), there exists $\epsilon \in (0, \lambda_1)$ such that $g'(t) \geq -\lambda_1 + \epsilon$ for all $t \in \mathbb{R}$. Thus, the Mean Value Theorem, the previous identities, and Poincarï¿½'s Inequality give us

$$\begin{aligned}
&\langle \nabla J(\mathbf{z}+\mathbf{w}) - \nabla J(\mathbf{z}+\mathbf{w}_1), \mathbf{w}-\mathbf{w}_1\rangle \\
&\geq 2\int_\Omega |\nabla(w-w_1)|^2 + 2(-\lambda_1+\epsilon)\int_\Omega (w-w_1)^2 \\
&\geq 2\int_\Omega |\nabla(w-w_1)|^2 + 2\frac{(-\lambda_1+\epsilon)}{\lambda_1}\int_\Omega |\nabla(w-w_1)|^2 \\
&= 2\frac{\epsilon}{\lambda_1}\int_\Omega |\nabla(w-w_1)|^2 = \frac{\epsilon}{\lambda_1}\|\mathbf{w}-\mathbf{w}_1\|_H^2.
\end{aligned}$$

We have then verified the hypotheses of Lemma 2.1. Thus, there exist a continuous function $\mathbf{w} \equiv \phi : Z \longrightarrow W$ and a functional $\widetilde{J} : Z \longrightarrow \mathbb{R}$ which satisfy (i), (ii) and (iii). Because of (iii), our concern becomes the existence of critical points of the functional $\widetilde{J}$.
Observe that, given $\mathbf{z} = (z, -z) \in Z$, $\mathbf{w}(\mathbf{z}) = (w(z), w(z))$ and

$$\begin{aligned}
\widetilde{J}(\mathbf{z}) &= J(z+w(z), -z+w(z)) \\
&= \int_\Omega [|\nabla w(z)|^2 - |\nabla z|^2 + G(z+w(z)) + G(-z+w(z))]\, d\zeta.
\end{aligned} \tag{15}$$

The symmetry of problem (3) expressed by condition (9) is translated into the following lemma.

Lemma 2.2. *If g satisfies (g_1) and either (g_2) or (g_4), then the function $\boldsymbol{w} \equiv \phi$ and the functional $\widetilde{J}$ are even.*

Proof. Let $\mathbf{z} = (z, -z) \in Z$. First, let us verify that

$$\langle \nabla J(-z + w(z), z + w(z)), (\varphi, \varphi)\rangle = 0 \ , \qquad \forall\ \varphi \in H_0^1(\Omega)$$

which, by uniqueness in (i) of Lemma 2.1, implies that $\mathbf{w}(\mathbf{z}) = \mathbf{w}(-\mathbf{z})$. Indeed, observe that

$$\begin{aligned}
&\langle \nabla J(-z + w(z), z + w(z)), (\varphi, \varphi)\rangle \\
&= \partial_u J(-z + w(z), z + w(z))\varphi + \partial_v J(-z + w(z), z + w(z))\varphi \\
&= \int_\Omega \nabla\varphi \cdot \nabla(z + w(z)) + g(-z + w(z))\varphi\, d\zeta + \int_\Omega \nabla(-z + w(z)) \cdot \nabla\varphi + g(z + w(z))\varphi\, d\zeta \\
&= \int_\Omega \nabla\varphi \cdot \nabla(-z + w(z)) + g(z + w(z))\varphi\, d\zeta + \int_\Omega \nabla(z + w(z)) \cdot \nabla\varphi + g(-z + w(z))\varphi\, d\zeta \\
&= \partial_u J(z + w(z), -z + w(z))\varphi + \partial_v J(z + w(z), -z + w(z))\varphi \\
&= \langle \nabla J(z + w(z), -z + w(z)), (\varphi, \varphi)\rangle = 0 \ , \qquad \forall\ \varphi \in H_0^1(\Omega).
\end{aligned}$$

Hence, given $z \in H_0^1(\Omega)$,

$$\begin{aligned}
\widetilde{J}(-\mathbf{z}) &= J(-z + w(-z), z + w(-z)) \\
&= J(-z + w(z), z + w(z)) \\
&= \textstyle\int_\Omega (|\nabla w(z)|^2 - |\nabla(-z)|^2 + G(-z + w(z)) + G(z + w(z)))d\zeta \\
&= J(z + w(z), -z + w(z)) \\
&= \widetilde{J}(\mathbf{z}).
\end{aligned}$$

□

Remark 1: Observe that from condition (g_1) and Lemma 2.1, we conclude that the set of candidates to be solutions of (3) is contained in the graph $\{\mathbf{z} + \mathbf{w}(\mathbf{z}) : \mathbf{z} \in Z\}$. From condition (g_0) we have $\mathbf{w}(\mathbf{0}) = \mathbf{0}$. Hence, combining these two facts, we observe that under $(g_0) - (g_1)$ the unique solution (u, v) of (3) with $u \equiv v$, i.e living in the set of fixed points of the action group, is the trivial one. Compare this with Theorem C.

3 Proof of Theorem A

Throughout this section we assume g satisfies (g_0), (g_1), (g_2) and (g_3). To prove Theorem A we make use of the following version of the Symmetric Mountain Pass Theorem (see e. g. [R]). We recall that if E is a Banach space and $I \in C^1(E, \mathbb{R})$, a sequence $\{e_n\}$ in E is a (PS)-sequence for the functional I, provided that

$$\forall n \in \mathbb{N},\ \ |I(e_n)| \le C \quad \text{and} \quad DI(e_n) \longrightarrow 0,\ n \to \infty. \tag{16}$$

The functional I is said to satisfy the (PS)-condition on E if every (PS)-sequence in E has a convergent subsequence.

Theorem 3.1. *Let $E = E_1 \oplus E_2$ be an infinite dimensional Banach space, where E_1 is a finite dimensional subspace. Let us assume $I \in C^1(E, \mathbb{R})$ is even, satisfies the Palais-Smale condition and $I(0) = 0$. Assume, in addition, I satisfies:*

(I_1) *There exist positive constants* α *and* ρ *such that* $I|_{\partial B_\rho \cap E_2} \geq \alpha$.

(I_2) *For each finite dimensional subspace* $X \subset E$ *there exists an* $R = R(X) > 0$ *such that* $I|_{X \setminus B_R(0)} \leq 0$.

Then I possesses an unbounded sequence of critical values.

We apply Theorem 3.1 to the functional $-\widetilde{J}$. To this end, let $j \in \mathbb{N}$ such that $g'(0) < \lambda_j$. We take $E_1 := \langle (\varphi_1, -\varphi_1) \dots, (\varphi_{j-1}, -\varphi_{j-1}) \rangle \subset Z$ and $E_2 = E_1^\perp \subset Z$.

Claim 1: Under assumptions (g_0)-(g_3) functional $-\widetilde{J}$ satisfies (I_1).
Proof. Let us consider the functional $F : H_0^1(\Omega) \longrightarrow \mathbb{R}$ defined as

$$
\begin{aligned}
F(z) &= -J(z,-z) = \int_\Omega (|\nabla z|^2 - G(z) - G(-z))\, d\zeta \\
&= \int_\Omega (\frac{1}{2}|\nabla z|^2 - G(z))\, d\zeta + \int_\Omega (\frac{1}{2}|\nabla(-z)|^2 - G(-z))\, d\zeta.
\end{aligned}
$$

Because of hypothesis (g_0) and the variational characterization of λ_j (see [R] or [CV]), $F|_{\langle \varphi_1 \dots, \varphi_{j-1} \rangle^\perp}$ has a strict local minimum at zero and there exist positive constants α and ρ such that

$$
F(z) \geq \alpha \qquad \forall z \in \partial B_\rho \cap \langle \varphi_1 \dots, \varphi_{j-1} \rangle^\perp \subset H_0^1(\Omega).
$$

Hence, for each $\mathbf{z} = (z, -z) \in \partial B_{\sqrt{2}\rho} \cap E_2 \subset Z$,

$$
-\widetilde{J}(\mathbf{z}) = - \min_{w \in H_0^1(\Omega)} J(z+w, -z+w) \geq -J(z,-z) = F(z) \geq \alpha. \qquad \square
$$

Claim 2: Under assumptions (g_0)-(g_3) the functional $-\widetilde{J}$ satisfies (I_2).
Proof. Let X be a finite dimensional subspace of Z. Then, there exists a constant $\gamma_X > 0$ such that $\|z\|^2 \leq \gamma_X \|z\|_{L^2}^2$ for all $\mathbf{z} = (z, -z) \in X$. Using hypothesis ($g_3$) and integrating,

$$
G(t) \geq a|t|^\mu - b
$$

where $a > 0$ and $b > 0$ are constants. Since $\mu > 2$, given any $\alpha > 0$, there exists a constant C_α such that

$$
a|t|^\mu - b \geq \frac{\alpha}{2} t^2 + C_\alpha
$$

(for this, simply consider $h(t) := a|t|^\mu - \frac{\alpha}{2}t^2 - b$, which is bounded below and continuous). Thus,

$$
G(t) \geq \frac{\alpha}{2} t^2 + C_\alpha \qquad \forall t \in \mathbb{R}.
$$

Therefore, given $\mathbf{z} = (z, -z) \in X$, $\mathbf{w}(\mathbf{z}) = (w(z), w(z))$,

$$
G(z + w(z)) + G(-z + w(z)) \geq \frac{\alpha}{2}(z + w(z))^2 + \frac{\alpha}{2}(-z + w(z))^2 + 2C_\alpha.
$$

We then have

$$
\begin{aligned}
-\widetilde{J}(\mathbf{z}) &= \int_\Omega [|\nabla z|^2 - |\nabla w(z)|^2 - G(z + w(z)) - G(-z + w(z))]\, d\zeta \\
&\leq \gamma_X \int_\Omega z^2 d\zeta - \alpha \int_\Omega z^2 d\zeta - \alpha \int_\Omega (w(z))^2 d\zeta - 2\widehat{C}_\alpha \\
&\leq (\gamma_X - \alpha) \int_\Omega z^2 - 2\widehat{C}_\alpha \ .
\end{aligned}
$$

Thus, taking $\alpha > \gamma_X$, we have that

$$-\widetilde{J}(\mathbf{z}) \longrightarrow -\infty\,, \quad \text{as } \|\mathbf{z}\| \to \infty,\, \mathbf{z} \in X.$$

Since, X is arbitrary we have verified (I_2). ■

It remains to show that $\widetilde{J}$ satisfies the Palais-Smale condition.

Lemma 3.1. *Under the assumptions* (g_0)*-*(g_3) *the functional* $\widetilde{J}$ *satisfies the (PS)-condition.*

Proof. Observe that from (11) and (12), it suffices to verify that J satisfies the Palais-Smale condition. Let $\{(u_n, v_n)\}_n \subset H_0^1(\Omega) \times H_0^1(\Omega)$ be a (PS)-sequence. We want to extract a strongly convergent subsequence. Due to the form of DJ, the compactness on the Sobolev Embeddings and Vainberg's Lemma (see e.g. [MZ]), we just have to prove that $\{u_n\}_n$ and $\{v_n\}_n$ are bounded sequences in $H_0^1(\Omega)$.

Condition (16) implies that there exists a sequence $\{\varepsilon_n\}_n$, $\varepsilon_n > 0$ and $\varepsilon_n \to 0^+$ so that

$$|DJ(u_n, v_n)[\phi, \psi]| \leq \varepsilon_n(\|\phi\| + \|\psi\|), \quad \forall \phi, \psi \in H_0^1(\Omega). \tag{17}$$

We take as test functions $\phi = \frac{1}{2}u_n$ and $\psi = \frac{1}{2}v_n$ to get

$$\begin{aligned} C \quad &+ \frac{\varepsilon_n}{2}(\|u_n\| + \|v_n\|) \\ &\geq \tfrac{1}{2}DJ(u_n, v_n)[u_n, v_n] - J(u_n, v_n) \\ &= \int_\Omega \{-G(v_n) - G(u_n)\} + \frac{1}{2}\int_\Omega \{g(u_n)u_n + g(v_n)v_n\} \\ &\geq \frac{1}{2}\int_\Omega \{g(v_n)v_n - \mu G(v_n)\} + \frac{1}{2}\int_\Omega \{g(u_n)(u_n) - \mu G(u_n)\} \\ &\qquad + \Big(\frac{\mu}{2} - 1\Big)\int_\Omega \{G(v_n) + G(u_n)\}. \end{aligned}$$

So, changing the constant C if necessary, we find by (g_3) that

$$\int_\Omega G(u_n) + G(v_n) \leq C\,[1 + \varepsilon_n(\|u_n\| + \|v_n\|)]. \tag{18}$$

Since $\{J(u_n, v_n)\}_n$ is bounded, we can choose a large positive constant C such that

$$\left|\int_\Omega \nabla u_n \cdot \nabla v_n + \int_\Omega G(u_n) + G(v_n)\right| \leq C. \tag{19}$$

Because of hypothesis (g_3), $|G(t)| - G(t) = 0$, for every $|t| \geq R$, so it is a bounded function. Thus, we get from (18) and (19) that

$$\begin{aligned} \left|\textstyle\int_\Omega \nabla u_n \cdot \nabla v_n\right| &\leq \textstyle\int_\Omega |G(u_n)| + |G(v_n)| + C \\ &\leq \textstyle\int_\Omega G(u_n) + G(v_n) + C \\ &\leq C[1 + \varepsilon_n(\|u_n\| + \|u_n\|)]. \end{aligned} \tag{20}$$

From (17), testing against $[\phi, \psi] = [u_n, v_n]$, we obtain

$$\left| 2\int_\Omega \nabla u_n \cdot \nabla v_n + \int_\Omega g(u_n)u_n + g(v_n)v_n \right| \le \varepsilon_n(\|u_n\| + \|u_n\|).$$

So, by (20) we obtain

$$\int_\Omega g(u_n)u_n + g(v_n)v_n \le C[1 + \varepsilon_n(\|u_n\| + \|u_n\|)]. \tag{21}$$

On the other hand, using again (17) and testing against $[\phi, \psi] = [0, u_n]$, we have

$$\left| \int_\Omega |\nabla u_n|^2 + g(v_n)u_n \right| \le \varepsilon_n \|u_n\|. \tag{22}$$

Now let us estimate the second term in left-hand side of inequality (22). Using Hölder inequality we have

$$\left| \int_\Omega g(v_n)u_n \right| \le \left(\int_\Omega |g(v_n)|^{1+\frac{1}{p}} \right)^{\frac{p}{1+p}} \left(\int_\Omega |u_n|^{1+p} \right)^{\frac{1}{1+p}} \tag{23}$$

Now note that for suitable positive constants c, d_1, d_2,

$$|g(t)|^{1+\frac{1}{p}} \le c\,|g(t)||t| + d_1 \le c\,g(t) + d_2. \tag{24}$$

Indeed, the first inequality in (24) follows from hypothesis (g_2), since

$$|g(t)|^{\frac{1}{p}} \le C\,|t| + d :$$

- for $|t| \ge 1$

$$\begin{aligned} |g(t)|^{1+\frac{1}{p}} &\le C\,|g(t)|\,|t| + d\,|g(t)| \\ &\le C\,|g(t)|\,|t| + d\,|g(t)|\,|t|. \end{aligned}$$

- for $|t| \le 1$ we see that $|g(t)|$ is simply bounded. So the first inequality in (24) holds. As for the second inequality in (24), we write

$$|g(t)|\,|t| = g(t) \cdot t + |g(t)|\,|t| - g(t) \cdot t,$$

and observe that, because of (g_3), $|g(t)|\,|t| - g(t)\cdot t = 0$, for $|t| \ge R$. So this difference remains bounded in $\mathbb{R}$ and the inequality holds.
From (21), (23) and (24), we get that

$$\begin{aligned} |\int_\Omega g(v_n)u_n| &\le \left(c\int_\Omega g(v_n)v_n + d_2\right)^{\frac{p}{1+p}} \|u_n\|_{L^{1+p}} \\ &\le \left(C[1 + \varepsilon_n(\|u_n\| + \|v_n\|)]\right)^{\frac{p}{1+p}} \|u_n\|. \end{aligned}$$

Then, by (22),

$$\int_\Omega |\nabla u_n|^2 \le \varepsilon_n \|u_n\| + (C[1 + \varepsilon_n(\|u_n\| + \|v_n\|)])^{\frac{p}{1+p}} \|u_n\|.$$

In a similar fashion, taking $[\phi, \psi] = [v_n, 0]$ in (17), we get the analogous estimate

$$\int_\Omega |\nabla v_n|^2 \leq \varepsilon_n \|v_n\| + (C[1 + \varepsilon_n(\|u_n\| + \|v_n\|)])^{\frac{p}{1+p}} \|v_n\|.$$

Joining these two estimates we obtain

$$\|u_n\|^2 + \|v_n\|^2 \leq \varepsilon_n(\|u_n\| + \|v_n\|) + C\ (\|u_n\| + \|v_n\|)^{\frac{2p+1}{1+p}} + K.$$

Since $\frac{2p+1}{1+p} < 2$, the sequence $\{(u_n, v_n)\}_n$ is bounded in H and the proof of the lemma is complete. □

4 Proof of Theorem B

Throughout this section we assume that g satisfies (g_0), (g_1) and (g_4). To prove Theorem B we make use of the following version of the Symmetric Mountain Pass Theorem (see e.g. [AR], [BBF], and [S]).

Theorem 4.1. *Let $E = E_1 \oplus E_2$ be a real Banach space, where E_1 is a finite dimensional subspace. Let $X \subset E$ be a finite dimensional subspace of E such that $\dim E_1 < \dim X$. Suppose that $I \in C^1(E, \mathbb{R})$ is an even functional, satisfying $I(\boldsymbol{0}) = 0$ and*

(I_1') *There exists a positive constant ρ such that $I|_{\partial B_\rho \cap E_2} \geq 0$.*

(I_2') *There exists $M > 0$ such that $\max_{\boldsymbol{z} \in X} I(\boldsymbol{z}) < M$.*

If I satisfies the Palais-Smale condition at level c, for every $c \in [0, M]$, then I possesses (at least) $\dim X - \dim E_1$ pairs of nontrivial critical points.

As in Section 3, we take $E_1 := \langle (\varphi_1, -\varphi_1) \dots, (\varphi_{j-1}, -\varphi_{j-1}) \rangle$ and $E_2 = E_1^\perp$. As we proved in the previous section, the fact that $-\widetilde{J}$ satisfies (I_1') comes from hypothesis (g_0) and the variational characterization of the eigenvalues, i.e. the local structure of the functional around zero in this case is similar to that of the superlinear setting.

Claim: Under hypotheses (g_0), (g_1) and (g_4), the functional $-\widetilde{J}$ satisfies (I_2').

Proof. Let us take $X = \langle (\varphi_1, -\varphi_1) \dots, (\varphi_k, -\varphi_k) \rangle$. Since $g'(\infty) > \lambda_k$, taking a number $\alpha \in (\lambda_k, g'(\infty))$ it follows that

$$G(t) > \frac{\alpha}{2} t^2 + C_\alpha \qquad \forall t \in \mathbb{R}.$$

The remaining of this proof is very similar to the proof of Claim 2 in Section 3 by simply using the inequality

$$\|x\|^2 \leq \lambda_k \int_\Omega x^2 \qquad \forall x \in \langle \varphi_1, ..., \varphi_k \rangle.$$

From this, given $\mathbf{z} = (z, -z) \in X$,

$$-\widetilde{J}(\mathbf{z}) \leq (\lambda_k - \alpha)\|z\|_{L^2}^2 + \widetilde{C}_\alpha \longrightarrow -\infty \text{ as } \|\mathbf{z}\| \to \infty, \mathbf{z} \in X\ .$$ □

It remains to show that $\widetilde{J}$ satisfies the Palais-Smale condition. In this case, we follow the ideas of the corresponding proof for the problem with one equation and asymptotic (nonresonant) nonlinearities, although our proof requires a bit more of technicalities.

Lemma 4.1. *Under assumptions (g_0), (g_1) and (g_4) the functional $\widetilde{J}$ satisfies the (PS)-condition.*

Proof. As before, from (11) and (12), it suffices to verify that J satisfies the Palais-Smale condition. We take a (PS)-sequence $\{(u_n, v_n)\}_n$ in $H_0^1(\Omega) \times H_0^1(\Omega)$ and again it is sufficient to prove that this sequence is bounded. In this case, we argue by contradiction. Let us assume that $\{\|(u_n, v_n)\|\}_n$ is not bounded. Passing to a subsequence, denoted the same for simplicity of notation, we can say that either $\|u_n\| \to \infty$ or $\|v_n\| \to \infty$. We claim that

(I) if $\|u_n\| \to \infty$, then there exists a subsequence $\|v_{n_k}\| \to \infty$, and

(II) if $\|v_n\| \to \infty$, then there exists a subsequence $\|u_{n_k}\| \to \infty$.

Indeed, let us prove (I) arguing by contradiction. If $\|u_n\| \to \infty$ and $\|v_n\| \leq C$, then, passing to a subsequence we have that

$$v_n \rightharpoonup v, \quad \text{in} \quad H_0^1(\Omega) \qquad \tfrac{u_n}{\|u_n\|} \rightharpoonup \bar{u}, \quad \text{in} \quad H_0^1(\Omega)$$

$$v_n \to v, \quad \text{in} \quad L^r(\Omega) \qquad \tfrac{u_n}{\|u_n\|} \to \bar{u}, \quad \text{in} \quad L^r(\Omega), \quad \text{for} \quad r \in [1, \tfrac{2N}{N-2}).$$

There exists a sequence $\{\varepsilon_n\}_n$, $\varepsilon_n > 0$ and $\varepsilon_n \to 0^+$ so that

$$|DJ(u_n, v_n)[\phi, \psi]| \leq \varepsilon_n(\|\phi\| + \|\psi\|), \quad \forall \phi, \psi \in H_0^1(\Omega). \tag{25}$$

Testing $\partial_v J(u_n, v_n)$ against $\frac{u_n}{\|u_n\|}$ and using (25) we get that

$$\left| \|u_n\| + \int_\Omega g(v_n) \frac{u_n}{\|u_n\|} \right| \leq \varepsilon_n.$$

From (g_4), $|g(t)| \leq C(1 + |t|)$ for all $t \in \mathbb{R}$. Using Vainberg's Lemma (see [MZ]) we have that

$$\int_\Omega g(v_n) \frac{u_n}{\|u_n\|} \longrightarrow \int_\Omega g(v)\, \bar{u}$$

and so we get

$$\|u_n\| \longrightarrow - \int_\Omega g(v)\, \bar{u}, \text{ as } n \to \infty.$$

This contradicts our initial assumption. We proceed in an analogue way to prove (II) and therefore the claim is proved.

Now, using the claim, and passing to a subsequence, we can assume without loss of generality that:

$$\|u_n\| \to \infty \quad \text{and} \quad \|v_n\| \to \infty.$$

Hence, there exist $u, v \in H_0^1(\Omega)$ such that

$$\tfrac{u_n}{\|u_n\|} \rightharpoonup \bar{u}, \quad \text{in} \quad H_0^1(\Omega) \qquad \tfrac{v_n}{\|v_n\|} \rightharpoonup \bar{v}, \quad \text{in} \quad H_0^1(\Omega)$$

$$\tfrac{u_n}{\|u_n\|} \to \bar{u}, \quad \text{in} \quad L^r(\Omega) \qquad \tfrac{v_n}{\|v_n\|} \to \bar{v}, \quad \text{in} \quad L^r(\Omega), \quad \text{for} \quad r \in [1, \tfrac{2N}{N-2}).$$

We claim that $\{\|u_n\|\}_n$ and $\{\|v_n\|\}_n$ go to infinity at the same rate. More precisely, we claim that

$$\lim_{n\to\infty} \frac{\|u_n\|}{\|v_n\|} = 1. \tag{26}$$

To prove this claim, we first test $\partial_u J(u_n, v_n)$ against $\frac{v_n}{\|v_n\|}$ and then divide by $\|u_n\|$ to get

$$\left| \frac{\|v_n\|}{\|u_n\|} + \int_\Omega \frac{g(u_n)}{\|u_n\|} \cdot \frac{v_n}{\|v_n\|} \right| \leq \frac{\varepsilon_n}{\|u_n\|}. \tag{27}$$

Assumption (g_4) implies that $g(t) = g'(\infty)t + \gamma(t)$, where $\gamma(t) = o(t)$, as $|t| \to \infty$. Then,

$$\int_\Omega \frac{g(u_n)}{\|u_n\|} \frac{v_n}{\|v_n\|} = g'(\infty) \int_\Omega \frac{v_n}{\|v_n\|} \frac{u_n}{\|u_n\|} + \int_\Omega \gamma(u_n) \frac{v_n}{\|v_n\| \|u_n\|}. \tag{28}$$

Now we show that

$$\int_\Omega \gamma(u_n) \frac{v_n}{\|v_n\| \|u_n\|} \longrightarrow 0.$$

Indeed, just observe that given $\varepsilon > 0$ arbitrary, there exists $T \geq 0$ such that

$$\left| \frac{\gamma(t)}{t} \right| < \varepsilon, \quad \text{for } |t| \geq T.$$

On the other hand, $\gamma(t) = g(t) - g'(\infty)t$ is continuous in $[-T, T]$ and so it is bounded in $[-T, T]$. Thus, it follows that

$$\begin{aligned} \int_\Omega \left| \gamma(u_n) \frac{v_n}{\|v_n\| \|u_n\|} \right| &\leq \int_{\{|u_n| > T\}} + \int_{\{|u_n| \leq T\}} \\ &\leq \varepsilon \int_\Omega \left| \frac{u_n}{\|u_n\|} \frac{v_n}{\|v_n\|} \right| + \frac{C_T}{\|u_n\|} \int_\Omega \left| \frac{v_n}{\|v_n\|} \right| \\ &\leq C\varepsilon + \frac{C_T}{\|u_n\|} C \\ &\leq 2C\varepsilon, \quad \text{for } n \text{ large enough.} \end{aligned}$$

Hence, we can take the limit in (28) to get

$$\int_\Omega \frac{g(u_n)}{\|u_n\|} \frac{v_n}{\|v_n\|} \longrightarrow \int_\Omega g'(\infty)\, \bar{u}\, \bar{v}.$$

This and (27) give

$$\frac{\|v_n\|}{\|u_n\|} \longrightarrow - \int_\Omega g'(\infty)\, \bar{u}\, \bar{v}. \tag{29}$$

Arguing in a similar fashion, but now testing $\partial_v J(u_n, v_n)$ against $\frac{u_n}{\|u_n\|}$, we also obtain

$$\frac{\|u_n\|}{\|v_n\|} \longrightarrow - \int_\Omega g'(\infty)\, \bar{u}\, \bar{v}, \tag{30}$$

which together with (29) implies that actually $\int_\Omega g'(\infty)\, \bar{u}\, \bar{v} = -1$ and therefore the claim is proved.

Let us now take $\phi \in H_0^1(\Omega)$. Using (25) we have that

$$\left| \int_\Omega \nabla \phi \cdot \nabla \left(\frac{v_n}{\|v_n\|} \right) + \frac{g(u_n)}{\|v_n\|} \phi \right| \longrightarrow 0. \tag{31}$$

Due to the weak convergence of $\frac{v_n}{\|v_n\|}$ to $\bar{v}$, we know that

$$\int_\Omega \nabla\phi \cdot \nabla\left(\frac{v_n}{\|v_n\|}\right) \longrightarrow \int_\Omega \nabla\phi \cdot \nabla\bar{v}. \tag{32}$$

On the other hand, (26) implies that

$$\int_\Omega \frac{g(u_n)}{\|v_n\|}\,\phi \longrightarrow \int_\Omega g'(\infty)\,\bar{u}\,\phi. \tag{33}$$

To see why this is true, it is enough to notice that

$$\int_\Omega \frac{g(u_n)}{\|v_n\|}\,\phi = \int_\Omega \frac{g(u_n)}{\|u_n\|} \cdot \frac{\|u_n\|}{\|v_n\|}\,\phi = \frac{\|u_n\|}{\|v_n\|}\int_\Omega \frac{g'(\infty)u_n + \gamma(u_n)}{\|u_n\|}\,\phi$$

and arguing as above, it can be proved that $\int_\Omega \frac{\gamma(u_n)}{\|u_n\|}\,\phi \longrightarrow 0$.

From (31), (32) and (33), we have proven that

$$\forall \phi \in H_0^1(\Omega) : \int_\Omega \nabla\bar{v} \cdot \nabla\phi + g'(\infty)\,\bar{u}\,\phi = 0. \tag{34}$$

Using (29) and reasoning analogously, we also get that

$$\forall \phi \in H_0^1(\Omega) : \int_\Omega \nabla\bar{u} \cdot \nabla\phi + g'(\infty)\,\bar{v}\,\phi = 0. \tag{35}$$

From relations (34) and (35), testing both integrals against $\phi = \bar{v} + \bar{u}$ we obtain

$$\int_\Omega |\nabla(\bar{u} + \bar{v})|^2 = -g'(\infty)\int_\Omega (\bar{v} + \bar{u})^2.$$

Since $g'(\infty) > 0$, $\bar{v} = -\bar{u}$. Replacing this in any of the relations (34) or (35) we get that $\bar{u} = -\bar{v} \in H_0^1(\Omega)$ is a weak solution, and actually a classical one, to the problem

$$\begin{cases} -\Delta u = g'(\infty)\,u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

This, as well as (29) and (30), imply that $g'(\infty) = \lambda_j$ for some $j \in \mathbb{N}$. This contradicts hypothesis (g_4). Hence, a contradiction is reached assuming that $\{\|(u_n, v_n)\|\}_n$ is unbounded, and the conclusion of the lemma follows. □

5 Proof of Theorem C

Assume condition (g_5). Let us assume (u, v) is a solution of (3). Multiply the first equation in (3) by $u - v$, and then multiply the second equation by $u - v$. Taking the difference of both results, we get

$$\int_\Omega |\nabla(u - v)|^2 + (g(v) - g(u))(u - v) = 0$$

or, equivalently,

$$\int_\Omega |\nabla(u - v)|^2 = \int_\Omega (g(u) - g(v))(u - v).$$

Because of Mean Value Theorem and (g_5), we have that

$$\int_\Omega |\nabla(u-v)|^2 \leq (\lambda_1 - \epsilon) \int_\Omega (u-v)^2,$$

for some small $\epsilon > 0$. From Poincarè's Inequality we conclude that $u \equiv v$.

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