EXTENSION OF CONTINUOUS CONVEX FUNCTIONS FROM SUBSPACES I

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Abstract. Let X be a topological vector space, $Y \subset X$ a subspace, and $A \subset X$ an open convex set containing 0. We are interested in extendability of a continuous convex function $f\colon A \cap Y \to \mathbb{R}$ to a continuous convex function $F\colon A \to \mathbb{R}$. We characterize such extendability being valid: (a) for a given f; (b) for every f. The case (b) for $A = X$ generalizes results from a paper by J. Borwein, V. Montesinos and J. Vanderwerff, and from another one by L. Zajíček and the second-named author. We also show that if X is locally convex and X/Y is "conditionally separable" then the couple (X, Y) satisfies the CE-property, saying that the above extendability holds for $A = X$ and every f. It follows that every couple (X, Y) has the CE-property for the weak topology.

We consider also a stronger SCE-property saying that the above extendability is true for every A and every f. A deeper study of the SCE-property will appear in a subsequent paper.

Introduction

Let Y be a subspace of a real topological vector space X. We say that the couple (X, Y) has the CE-*property* ("convex extension property") if each continuous convex (real-valued) function on Y admits a continuous convex extension to the whole X. It is natural to ask which couples (X, Y) have the CE-property.

The story started in [3] by Borwein and Vanderwerff, whose Theorem 2.3 implies that, for instance, the couple (ℓ_∞, c_0) fails the CE-property. In the same paper, the authors write: "However, even the case of extending continuous convex functions from separable [Banach] spaces to separable [Banach] superspaces is not clear to us" (p. 1802). This was answered in a paper by Borwein, Montesinos and Vanderwerff [2], where the CE-property is characterized in terms of extendability of certain weak*-null nets in Y^* to weak*-null

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nets in X^*; this characterization implies, via Rosenthal's extension theorem, that (X, Y) has the CE-property whenever X is a Banach space and $Y \subset X$ is a closed subspace such that X/Y is separable (see [2, Corollary 4.11]). An alternative proof, without using Rosenthal's theorem, appeared in [9] by Zajíček and the second-named author, and works also for incomplete normed spaces; this proof uses a new characterization of the CE-property, roughly speaking, in terms of "extendability", from Y to X, of certain increasing sequences of open convex sets (see [9, Theorem 4.3]).

In the present paper, we extend the above mentioned results to topological vector spaces, providing simpler proofs, and give some more results on the subject. In our characterizations of extendability, in part we further on develop techniques from [9], and in part we add a new dual equivalent condition which enables us both to get the net characterization of the CE-property from [2] in our general context, and to quickly prove that a kind of separability (called by us "conditional separability") of X/Y is sufficient for (X, Y) to have the CE-property whenever X is a locally convex topological vector space and Y its subspace. It turns out that our proofs are also suitable for characterizing the following (at least formally) stronger property.

We say that the couple (X, Y) has the SCE-*property* ("strong convex extension property") if, for each open convex set $A \subset X$ that intersects Y, every continuous convex function on $Y \cap A$ admits a continuous convex extension defined on A.

The paper is organized as follows. Section 1 contains preliminaries, a large part of which more or less belong to the mathematical folklore. In Section 2, we present characterizations of extendability, firstly for a single continuous convex function (Theorem 2.1), then for all continuous convex functions (Theorem 2.6, which is the core of the present paper). Section 3 collects some examples and basic facts about the CE- and the SCE-properties, either previously known or easy consequences of results in Section 2. In Section 4, we present the second main result of this paper, namely the above-mentioned sufficient condition for the CE-property, and its corollary saying that every couple (X, Y) has the CE-property in the weak topology.

The SCE-property is going to be studied more deeply in a subsequent paper [4].

1. Preliminaries

We consider topological vector spaces (t.v.s.) over $\mathbb{R}$ that are not necessarily Hausdorff. Recall that in such spaces the Hahn-Banach theorem still works for separating two convex sets such that one of them is disjoint from the nonempty interior of the other one. Moreover, each such space admits a base of neighborhoods of the origin made of circled (i.e., symmetric and starshaped with respect to 0) open sets. All convex functions we consider are real-valued.

If X is a real topological vector space, we denote by X^* the topological dual of X, usually endowed with the w^*-topology $\sigma(X^*, X)$. If Y is a subspace of X then we consider the linear w^*-w^*-continuous map

$$q\colon X^* \to Y^*, \qquad q(x^*) = x^*|_Y\,.$$

Notice that if X is locally convex then Y^* can be algebraically identified with the quotient space $X^*/Y^\perp$, and then q becomes the quotient map (see [5, 14.5]). Moreover, if Y is also closed in X then Y^* is topologically isomorphic to $(X^*, w^*)/Y^\perp$ (see [5, 17.14]). However, we shall not need this fact.

Given $A \subset X$, we denote by $A^\circ \subset X^*$ the polar set of A, i.e. the set

$$A^\circ = \{x^* \in X^*;\ x^*x \le 1\ \forall x \in A\}.$$

Analogously, for $B \subset X^*$, we denote by ${}^\circ B \subset X$ the prepolar set of B, i.e. the set

$${}^\circ B = \{x \in X;\ x^*x \le 1\ \forall x^* \in B\}.$$

Let us recall (cf. [6]) that if A is a convex subset of X, $x \in \operatorname{int} A$ and $y \in A$ then $[x, y) \subset \operatorname{int} A$; hence if A has nonempty interior then $\overline{\operatorname{int} A} = \overline{A}$ and $\operatorname{int} \overline{A} = \operatorname{int} A$.

Let us recall the following easy known fact (see, e.g., [8, Theorem 1.25]): if A, B are convex sets in a vector space then

$$\text{(1)} \qquad \operatorname{conv}(A \cup B) = \bigcup_{0 \le t \le 1} [(1-t)A + tB] = \bigcup_{a \in A,\, b \in B} [a, b]\,.$$

Let us start with a simple lemma.

Lemma 1.1. *Let Y be a subspace of a t.v.s. X, $C \subset Y$ and $A \subset X$ convex sets. Then:*

(a) $\operatorname{conv}(A \cup C) \cap Y = \operatorname{conv}[(Y \cap A) \cup C]$*;*

(b) *if C is open in Y, A is open in X and $A \cap C \ne \emptyset$, then* $\operatorname{conv}(A \cup C)$ *is open in X.*

Proof. (a) The inclusion "$\supset$" is obvious. To prove the other inclusion, consider an arbitrary $y \in Y \cap \operatorname{conv}(A \cup C)$. Then $y \in [a, c]$ for some $a \in A$, $c \in C$. If $y \ne c$ then necessarily $a \in Y$ (since $y, c \in Y$) and hence $y \in \operatorname{conv}[(Y \cap A) \cup C]$; and the last formula is trivial for $y = c$.

(b) Fix an arbitrary $a_0 \in A \cap C$. For each $x \in C$, there obviously exists $y \in C \setminus \{a_0\}$ such that $x \in (y, a_0]$; consequently, there exists $t \in (0, 1]$ with $x \in (1-t)C + tA$. Now we are done, since

$$\operatorname{conv}(A \cup C) = C \cup \bigcup_{0 < t \le 1} [(1-t)C + tA] = \bigcup_{0 < t \le 1} [(1-t)C + tA]$$

and the members of the last union are open.

□

The following two facts are well known. The first one is an easy consequence of the Hahn-Banach separation theorem, and a relative of [2, Lemma 4.5].

Fact 1.2. *Let Z be a t.v.s., $A \subset Z$ an open convex set, $L \subset Z$ an affine set, $A \cap L \neq \emptyset$. Let $h\colon A \to \mathbb{R}$ and $\varphi\colon L \to \mathbb{R}$ be continuous functions such that h is convex, φ is affine and $\varphi \leq h$ on $A \cap L$. Then φ can be extended to a continuous affine function $\hat{\varphi}\colon Z \to \mathbb{R}$ such that $\hat{\varphi} \leq h$ on A.*

Fact 1.3. *Let X be a topological vector space. Let $C \subset X$ be a nonempty open convex set containing the origin. Then its Minkowski gauge p_C is a nonnegative continuous sublinear (i.e., subadditive and positively homogeneous) function on X. Moreover,*

$$C = \{x \in X;\ p_C(x) < 1\}\,, \quad \overline{C} = \{x \in X;\ p_C(x) \leq 1\}\,,$$

and $|p_C(x) - p_C(y)| \leq \max\{p_C(x-y), p_C(y-x)\}$ $(x, y \in X)$. In particular, p_C is uniformly continuous.

The following last fact is well known for normed spaces. This general version is a part of the mathematical folklore.

Fact 1.4. *Let A be an open convex subset of a t.v.s. X, $f\colon A \to \mathbb{R}$ a convex function. Then f is continuous on A if and only if f is locally bounded on A.*

In the next lemma, we collect some known or easy-to-prove results about polar and prepolar sets.

Lemma 1.5. *Let Y be a subspace of a t.v.s. X. Let $0 \in A \subset X$ and $0 \in B \subset X^*$. Then:*

(i) *A° is convex and w^*-closed, moreover if $0 \in \operatorname{int} A$ then A° is w^*-compact;*
(ii) *$({}^\circ B)^\circ = \overline{\operatorname{conv}}^{\,w^*} B$;*
(iii) *if A is open and convex and $B = A^\circ$, then*

$$\operatorname{int} {}^\circ B = A = \{x \in X; \sup x(B) < 1\};$$

(iv) *if $A = \operatorname{int} {}^\circ B$, then $A^\circ = ({}^\circ B)^\circ$;*
(v) *if A is open and convex and $B = A^\circ$, then*

$$q(B) = ({}^\circ B \cap Y)^\circ = (A \cap Y)^\circ.$$

Proof. The first part of (i) follows by the definition of A°, the second part of (i) is the Banach-Alaoglu theorem [5, 17.4]; (ii) follows easily by the Hahn-Banach separation theorem and the fact that $(X^*, w^*)^* = X$.

Let us prove (iii). Since A is open $A \subset \operatorname{int}{}^\circ B$. Suppose that $A \neq \operatorname{int}{}^\circ B$, then, by the Hahn-Banach theorem and since $\operatorname{int} {}^\circ B$ is open, there exist $x \in \operatorname{int} {}^\circ B$

and $f \in X^*$ such that $\sup f(A) = 1 < f(x)$, hence $f \in A^\circ = B$ and $x \notin {}^\circ B$, which is a contradiction. This proves that $A = \operatorname{int} {}^\circ B$. Now, if $x \in A$ it is clear that $\sup x(B) \neq 1$, indeed, if this is not the case, there exists $\varepsilon > 0$ such that $y = (1+\varepsilon)x \in A$ and $\sup y(B) = 1+\varepsilon$, a contradiction since $B = A^\circ$. This proves that $A \subset \{x \in X; \sup x(B) < 1\}$. To prove the other inclusion, fix $x \in X$ such that $\sup x(B) < 1$, then $x \in {}^\circ B$ and there exists $\varepsilon > 0$ such that $y = (1+\varepsilon)x \in {}^\circ B$. Since $0 \in A = \operatorname{int} {}^\circ B$, we get $[0, y) \subset \operatorname{int} {}^\circ B$ and, in particular, $x \in \operatorname{int} {}^\circ B$.

To prove (iv), let us observe that $A \subset {}^\circ B$ and then $A^\circ \supset ({}^\circ B)^\circ$. For the other inclusion, fix $a^* \in A^\circ$ and $b \in {}^\circ B$. Since $0 \in A = \operatorname{int} {}^\circ B$, $[0, b) \subset \operatorname{int} {}^\circ B = A$. Since $a^* a \leq 1$, for each $a \in [0, b)$, we get $a^* b \leq 1$.

Now, to prove (v), let us observe that, by the definition of q and by (iii), it is easy to see that $q(B) \subset ({}^\circ B \cap Y)^\circ \subset (A \cap Y)^\circ$. Suppose that $y^* \in Y^*$ is such that $\sup y^*(A \cap Y) \leq 1$, i.e. $y^* y \leq p_A(y)$, for each $y \in Y$ (recall that p_A denotes the Minkowski gauge of A). By the Hahn-Banach extension theorem, there exists $x^* \in X^*$ such that $x^*|_Y = y^*$ and such that, for every $x \in X$, $x^* x \leq p_A(x)$, i.e. $\sup x^*(A) \leq 1$. This proves that $(A \cap Y)^\circ \subset q(B)$ and concludes the proof. $\square$

Let A be a nonempty convex open subset of a t.v.s. X, and $f \colon A \to \mathbb{R}$ a continuous convex function. Let us recall that, for $x_0 \in A$ and $\varepsilon \geq 0$, the *ε-subdifferential* of f at x_0 is the set

$$\partial_\varepsilon f(x_0) = \{x^* \in X^*;\ f(x) \geq f(x_0) + x^*(x - x_0) - \varepsilon,\ x \in A\}.$$

Moreover, $\partial f(x_0) := \partial_0 f(x_0)$ is the *subdifferential* of f at x_0. It is easy to prove that $\partial_\varepsilon f(x_0)$ is a convex w^*-compact set in X^* and that if $\{\varepsilon_n\}$ is a null sequence of positive reals then $\bigcap_n \partial_{\varepsilon_n} f(x_0) = \partial f(x_0)$.

Lemma 1.6. *Let X be a t.v.s., $A \subset X$ an open convex set, $Y \subset X$ a subspace, $0 \in A$. Let $f : A \cap Y \to \mathbb{R}$ and $G : A \to \mathbb{R}$ be convex continuous functions such that $f(0) = G(0)$, $f \leq G|_{A \cap Y}$ and $q\big(\partial G(0)\big) = \partial f(0)$. Then f admits a continuous convex extension $F : A \to \mathbb{R}$ such that $\partial F(0) = \partial G(0)$.*

Proof. For every $x \in X$, put

$$F(x) = \sup\{f(y) + x^*(x - y);\ y \in A \cap Y, x^* \in \Omega(y)\},$$

where

$$\Omega(y) = \{x^* \in X^*;\ q(x^*) \in \partial f(y),\ f(y) + x^*(z - y) \leq G(z)\ \forall z \in A\}.$$

Let us observe that F is a convex function with values in $(-\infty, \infty]$. Using the Hahn-Banach theorem it is easy to see that $F|_{A \cap Y} = f$. By the definition of F, we have $F \leq G$ on A and hence F is real valued and continuous on A. Moreover, since $F \leq G$ on A and $F(0) = G(0)$, we have $\partial F(0) \subset \partial G(0)$. Now, let $x^* \in \partial G(0)$, i.e., $x^* \in X^*$ and $G(x) \geq x^* x$ for each $x \in A$. Then $F(x) \geq$

$f(0) + x^*x = x^*x$, for each $x \in A$, i.e., $x^* \in \partial F(0)$. Hence $\partial F(0) = \partial G(0)$ and the proof is complete. □

2. Characterizations of extendability

Given sets $E, E_1, E_2 \dots$, the notation $E_n \nearrow E$ means that the sequence $\{E_n\}$ is nondecreasing and its union is the set E. The notation $E_n \searrow E$ means that the sequence $\{E_n\}$ is nonincreasing and its intersection is the set E.

Our first theorem characterizes extendability of a single convex continuous function. The case when X is a normed space and $A = X$ was proved in [9, Theorem 4.1]. The present general proof is even simpler that the proof therein.

Theorem 2.1. *Let X be a t.v.s., $A \subset X$ an open convex set, $Y \subset X$ a subspace, $A \cap Y \neq \emptyset$. Let $f \colon A \cap Y \to \mathbb{R}$ be a continuous convex function. Then the following assertions are equivalent.*

(i) *There exists a continuous convex $F \colon A \to \mathbb{R}$ such that $f = F|_{A\cap Y}$.*
(ii) *There exists a continuous convex $G \colon A \to \mathbb{R}$ such that $f \leq G|_{A\cap Y}$.*
(iii) *There exist open convex sets $A_n \subset A$ ($n \in \mathbb{N}$) such that $A_n \nearrow A$ and f is bounded above on each $A_n \cap Y$.*

Proof.

(ii) $\Rightarrow$ *(i)*. For each fixed $y \in A \cap Y$, apply Fact 1.2 with $Z = Y$, $L = \{y\}$ and $h = f$ to find a continuous affine function $\varphi_y \colon Y \to \mathbb{R}$ that supports f at y, that is, $\varphi_y \leq f$ and $\varphi_y(y) = f(y)$. Apply Fact 1.2 again, this time with $Z = X$, $L = Y$ and $h = G$, to get a continuous affine extension $\hat{\varphi}_y \colon X \to \mathbb{R}$ of φ_y such that $\hat{\varphi}_y \leq G$. Then the function $F := \sup_{y\in A\cap Y} \hat{\varphi}_y$ on A is convex, continuous (by Fact 1.4, since $\hat{\varphi}_y \leq F \leq G$) and extends f.

(i) $\Rightarrow$ *(iii)* follows immediately by putting $A_n := \{x \in A : F(x) < n\}$.

(iii) $\Rightarrow$ *(ii)*. Obviously, we can suppose that $0 \in A_1$ and $f(0) = 0$. We claim that we can also suppose that, for each n,

$$A_n \subset \lambda_{n+1} A_{n+1} \qquad \text{with } \lambda_{n+1} \in (0, 1). \tag{2}$$

This can be done by substituting each A_n with $\widetilde{A}_n := \alpha_n A_n$, where $\alpha_n \in (0, 1), \alpha_n \nearrow 1$. Indeed, $\widetilde{A}_n = \alpha_n A_n \subset \alpha_n A_{n+1} = \frac{\alpha_n}{\alpha_{n+1}} \widetilde{A}_{n+1}$; and, for $x \in A$, fix an index n such that $\frac{x}{\alpha_n} \in A$, and then an index $k \geq n$ with $\frac{x}{\alpha_n} \in A_k$, to see that $x \in \alpha_n A_k \subset \alpha_k A_k = \widetilde{A}_k$. So, let us assume (2).

We have $s_n := \sup_{A_n\cap Y} f \geq f(0) = 0$. Let p_n denote the Minkowski gauge of A_n. Define $G \colon A \to [0, +\infty]$ by

$$G := s_1 + s_2 p_1 + \sum_{n=2}^{\infty} \frac{s_{n+1}}{1 - \lambda_n} (p_n - \lambda_n)^+ ,$$

where t^+ denotes the positive part of t, that is, $t^+ = \max\{t, 0\}$. Obviously, G is a convex function on A. If $k \geq 2$ and $x \in A_k$, then, for each $n \geq k+1$, we have $x \in A_{n-1} \subset \lambda_n A_n$ which implies $p_n(x) < \lambda_n$. This shows that, for each $k \geq 2$,

$$G = s_1 + s_2 p_1 + \sum_{n=2}^{k} \frac{s_{n+1}}{1-\lambda_n} (p_n - \lambda_n)^+ \qquad \text{on the set } A_k.$$

Consequently, F is finite-valued and continuous on A. It remains to show that $G|_Y \geq f$. To see this, consider an arbitrary $y \in Y$. Then: if $y \in A_1$ then $G(y) \geq s_1 \geq f(y)$; if $y \in A_2 \setminus A_1$ then $G(y) \geq s_2 p_1(y) \geq s_2 \geq f(y)$; if $y \in A_{k+1} \setminus A_k$ for some $k \geq 2$ then $p_k(y) \geq 1 > \lambda_k$ and hence $G(y) \geq \frac{s_{k+1}}{1-\lambda_k}(p_k(y) - \lambda_k) \geq s_{k+1} \geq f(y)$. The proof is complete. □

Corollary 2.2. *Let X be a t.v.s., $A \subset X$ an open convex set containing* 0, *and $Y \subset X$ a subspace. Let $f\colon A \cap Y \to \mathbb{R}$ be a continuous convex function. Then f can be extended to a continuous convex function on A, provided any of the following two conditions is satisfied.*

(a) *f is bounded above on $tA \cap Y$ for each $t \in (0, 1)$.*

(b) *X is a normed space and f is bounded above on each bounded set $E \subset A \cap Y$ such that $\sup_E p_A < 1$ (where p_A is the Minkowski gauge of A).*

Proof. Let us prove (b). Let U denote the open unit ball of X. Then the bounded open convex sets $A_n = \frac{n}{n+1} A \cap nU$ $(n \in \mathbb{N})$ satisfy $A_n \nearrow A$. Moreover, by our assumptions, f is bounded above on each $A_n \cap Y$. Apply Theorem 2.1. The proof (a) goes in the same way by considering $A_n = \frac{n}{n+1} A$. □

Remark 2.3. The above Corollary 2.2(b) for $A = X$ immediately gives the following fact from [3, p.1801]: *each continuous convex function f on a subspace of a normed space X, such that f is bounded on bounded subsets of the subspace, admits a continous convex extension to the whole X.*

Lemma 2.4. *Let A be an open convex set in a t.v.s X. Suppose that $\{A_n\}_{n \in \mathbb{N}}$ is a sequence of open convex sets in X such that $0 \in A_1$. Then $A_n \nearrow A$ iff $A_n^\circ \searrow A^\circ$.*

Proof. If $A_n \nearrow A$ then clearly $\{A_n^\circ\}$ is a decreasing sequence of sets in X^* containing A°. We claim that $A_n^\circ \searrow A^\circ$. Indeed, if this is not the case, by the Hahn-Banach theorem, there exists $x \in X$ such that $\sup x(A^\circ) < 1 < \sup x(\bigcap_n A_n^\circ)$. Hence $x \in A$ by Lemma 1.5(iii), and $x \notin \bigcup_n A_n$, which is a contradiction.

Now, suppose that $A_n^\circ \searrow A^\circ$. By Lemma 1.5(iii), $A_n = \operatorname{int} {}^\circ(A_n^\circ)$ for each $n \in \mathbb{N}$, and hence $\{A_n\}$ is an increasing sequence of sets in A. We claim that $A_n \nearrow A$. Indeed, if this is not the case, there exists $x \in A \setminus \bigcup_n A_n$. By the

Hahn-Banach theorem, there exists $x^* \in X^*$ such that $1 = \sup x^*(\bigcup_n A_n) \leq x^*(x) < \sup x^*(A)$, where the last inequality is strict since A is open in X. Hence we have $x^* \notin A^\circ$ and $x^* \in \bigcap_n A_n^\circ$, a contradiction. $\square$

Definition 2.5.

(a) In the sequel, all nets of the form $\{t_{n,\gamma}\}_{n\in\mathbb{N},\gamma\in\Gamma}$, where Γ is a nonempty set, are directed by the following relation:

$$(n_1, \gamma_1) \leq (n_2, \gamma_2) \ \text{ iff } \ n_1 \leq n_2.$$

(b) Given an equicontinuous net $\{x^*_{n,\gamma}\}$ in X^*, we denote by $(\{x^*_{n,\gamma}\})'$ the set of its w^*-limit points (that is, the set of limits of all w^*-converging subnets).

The following theorem is the first main result of the present paper.

Theorem 2.6. *Let X be a t.v.s., $A \subset X$ an open convex set, $Y \subset X$ a subspace, $0 \in A$. Then the following assertions are equivalent.*

(i) *Every continuous convex function on $A\cap Y$ admits a continuous convex extension to A.*

(ii) *For every sequence $\{C_n\}$ of open convex sets in Y such that $C_n \nearrow A \cap Y$, there exists a sequence $\{D_n\}$ of open convex sets in X such that $D_n \nearrow A$ and $D_n \cap Y = C_n$, for each $n \in \mathbb{N}$.*

(ii') *For every sequence $\{C_n\}$ of open convex sets in Y such that $C_n \nearrow A \cap Y$, there exists a sequence $\{D_n\}$ of open convex sets in X such that $D_n \nearrow A$ and $D_n \cap Y \subset C_n$, for each $n \in \mathbb{N}$.*

(iii) *For every sequence $\{B_n\}$ of w^*-compact convex sets in Y^* such that $B_n \searrow q(A^\circ)$ and such that ${}^\circ B_1$ is a neighborhood of the origin in Y, there exists a sequence $\{E_n\}$ of w^*-compact convex sets in X^* such that $E_n \searrow A^\circ$, such that ${}^\circ E_1$ is a neighborhood of the origin in X and such that $q(E_n) = B_n$, for each $n \in \mathbb{N}$.*

(iv) *Every continuous convex function f on $A\cap Y$ such that $\partial f(0) = q(A^\circ)$ admits a continuous convex extension F to A such that $\partial F(0) = A^\circ$.*

(v) *For every equicontinuous net $\{y^*_{n,\gamma}\} \subset Y^*$ such that $(\{y^*_{n,\gamma}\})' \subset q(A^\circ)$, there exists an equicontinuous net $\{x^*_{n,\gamma}\} \subset X^*$ such that $(\{x^*_{n,\gamma}\})' \subset A^\circ$ and $x^*_{n,\gamma}|_Y = y^*_{n,\gamma}$ for each $(n,\gamma) \in \mathbb{N}\times\Gamma$.*

Since the proof of Theorem 2.6 is quite long, we shall divide it in several steps for convenience of the reader.

Proof of $(i) \Leftrightarrow (ii) \Leftrightarrow (ii')$**.**

$(ii') \Rightarrow (i)$. Let f be a continuous convex function on $A \cap Y$. For the sets $C_n := \{y \in A\cap Y : f(y) < n\}$ find the corresponding sets D_n from (ii'). Apply Theorem 2.1 to get (i).

(i) ⇒ *(ii′)*. As in the last part of the proof of Theorem 2.1, we can suppose that $0 \in C_1$ and $C_n \subset \lambda_{n+1} C_{n+1}$ $(n \in \mathbb{N})$ with $\lambda_{n+1} \in (0,1)$. Denote by p_n be the Minkowski functional of C_n, where the sets C_n are as in (ii'), and put

$$f := 1 + \sum_{n=2}^{\infty} \frac{1}{1-\lambda_n} (p_n - \lambda_n)^+.$$

As in Theorem 2.1, f is a continuous convex function on $A \cap Y$. Let $F\colon A \to \mathbb{R}$ be a continuous convex extension of f. We claim that the sets

$$D_n := \{x \in A : F(x) < n\}$$

have the desidered properties. Note that $D_1 = \emptyset$. To see that $D_k \cap Y \subset C_k$ for $k \geq 2$, fix an arbitrary $y \in Y \setminus C_k$. Then, for $n \leq k$, $y \notin C_n$ and hence $p_n(y) \geq 1$. Consequently, $F(y) = f(y) \geq 1 + \sum_{n=2}^{k} \frac{1}{1-\lambda_n}(p_n(y) - \lambda_n) \geq k$, that is, $y \notin D_k$.

(ii) ⇒ *(ii′)*. This implication is obvious.

(ii′) ⇒ *(ii)*. Let $\{C_n\}$ be as in *(ii)*. Without any loss of generality, we can suppose that $0 \in C_1$. Find the corresponding sets D_n from (ii'). We claim that we can suppose that $0 \in D_1$. To see this, let $n_0 \in \mathbb{N}$ be the smallest index such that $0 \in D_{n_0}$, and assume that $n_0 > 1$. By (i) (which has been already proved to be equivalent to (ii')), there exists a continuous convex function $F\colon A \to \mathbb{R}$ such that $F(y) = p_{C_1}(y)$ whenever $y \in A \cap Y$. Then the properties of $\{D_n\}$ remain satisfied if we change the definition of D_i $(i < n_0)$ in the following way:

$$D_i := \{x \in D_{n_0};\ F(x) < 1\} \quad \text{for each } i < n_0.$$

Our claim is proved. Now, by Lemma 1.1, the sets $\widetilde{D}_n := \operatorname{conv}(D_n \cup C_n)$ are open and convex, $\widetilde{D}_n \cap Y = C_n$ for each n, and $\widetilde{D}_n \nearrow A$. □

Proof of *(ii)* ⇔ *(iii)***.**

(ii) ⇒ *(iii)*. Let $\{B_n\}$ be a sequence of w^*-compact convex sets in Y^* such that $B_n \searrow q(A^\circ)$ and such that ${}^\circ B_1$ is a neighborhood of the origin in Y. Put, for every $n \in \mathbb{N}$, $C_n = \operatorname{int}({}^\circ B_n)$. Then $\{C_n\}$ is a sequence of nonempty open convex sets in $A \cap Y$ such that $0 \in C_1$. Since $C_n^\circ = ({}^\circ B_n)^\circ = B_n$ and $(A \cap Y)^\circ = q(A^\circ)$ (Lemma 1.5), we have $C_n \nearrow A \cap Y$ by Lemma 2.4.

Now, we can find a sequence $\{D_n\}$ of open convex sets in X such that $D_n \cap Y = C_n$ and $D_n \nearrow A$. Put, for every $n \in \mathbb{N}$, $E_n = D_n^\circ$. Then $\{E_n\}$ is a sequence of w^*-compact convex sets in X^* such that ${}^\circ E_1$ is a neighborhood of the origin in X and, by Lemma 2.4, $E_n \searrow A^\circ$. Moreover, for every $n \in \mathbb{N}$, Lemma 1.5 implies that

$$q(E_n) = q(D_n^\circ) = (D_n \cap Y)^\circ = C_n^\circ = (\operatorname{int}{}^\circ B_n)^\circ = B_n.$$

(iii) ⇒ *(ii)*. Let $\{C_n\}$ be a sequence of open convex sets in Y such that $C_n \nearrow A \cap Y$. We can (and do) suppose that $0 \in C_1$. Put $B_n = C_n^\circ$ $(n \in \mathbb{N})$.

Then $\{B_n\}$ is a sequence of w^*-compact convex sets in Y^* such that ${}^\circ B_1$ is a neighborhood of the origin in Y. By Lemma 2.4, $B_n \searrow q(A^\circ)$.

Now, we can find a sequence $\{E_n\}$ of w^*-compact convex sets in X^* such that $E_n \searrow A^\circ$, such that ${}^\circ E_1$ is a neighborhood of the origin in X and such that $q(E_n) = B_n$, for each $n \in \mathbb{N}$. Put, for every $n \in \mathbb{N}$, $D_n = \operatorname{int}({}^\circ E_n)$. Then $\{D_n\}$ is a sequence of open convex subsets of X contained in A such that $D_n \cap Y = C_n$, for each $n \in \mathbb{N}$; indeed, since $D_n = \operatorname{int}{}^\circ E_n$, we have

$$C_n{}^\circ = B_n = q(E_n) = (D_n \cap Y)^\circ$$

and hence

$$D_n \cap Y = \operatorname{int}{}^\circ B_n = \operatorname{int}{}^\circ(C_n{}^\circ) = C_n.$$

Again by Lemma 2.4, $D_n \nearrow A$. □

Proof of $(iii) \Leftrightarrow (iv)$**.**

$(iii) \Rightarrow (iv)$. Let $f : A \cap Y \to \mathbb{R}$ be a continuous convex function such that $\partial f(0) = q(A^\circ)$. Without any loss of generality, we can suppose that $f(0) = 0$. Put, for every $n \in \mathbb{N}$, $B_n = \partial_{1/n} f(0)$. Then $\{B_n\}$ is a of w^*-compact convex sets in Y^*, and $B_n \searrow \partial f(0) = q(A^\circ)$. Moreover, if $V = f^{-1}\big((-\infty, 1)\big)$, we have that, for every $v \in V$ and $x^* \in B_1 = \partial_1 f(0)$, $x^* v \le f(v) - f(0) + 1 < 2$; hence $V/2 \subset {}^\circ B_1$. By (iii), we can find a sequence $\{E_n\}$ of w^*-compact convex sets in X^* such that $E_n \searrow A^\circ$, $q(E_n) = B_n$, for each $n \in \mathbb{N}$, and such that ${}^\circ E_1$ is a neighborhood of the origin in X.

For $u \in A \cap Y$ and $y^* \in \partial f(u)$, let us put $a(y^*, u) = y^*(u) - f(u)$ and observe that $a(y^*, u) \ge 0$. Let us consider the function $\varphi : A \cap Y \to \mathbb{R}$, defined by

$$\varphi(y) = \sup\big(\{f(u) + y^*(y - u);\, u \in A \cap Y, y^* \in \partial f(u), a(y^*, u) \ge 1\} \cup \{-1\}\big).$$

Let us observe that, since $-1 \le \varphi \le f$, φ is a continuous convex function and then, since we already proved that $(i) \Leftrightarrow (iii)$, it admits a continuous convex extension $\Phi : A \to \mathbb{R}$.

Now, for $x \in A$, put

$$G(x) = \max\{\sup_{n\in\mathbb{N}}(\sigma_{E_n}(x) - \tfrac{1}{n+1}), \Phi(x)\},$$

where σ_E denotes the support function of a nonempty set $E \subset X^*$, i.e. the convex function with values in $(-\infty, \infty]$, defined by $\sigma_E(x) = \sup_{x^* \in E} x^* x$. Then G is a convex function and, since ${}^\circ E_1$ is a neighborhood of the origin in X, G is also continuous on A. Notice that $\Phi(0) = \varphi(0) = -1$, hence

$$G(x) = \sup_{n\in\mathbb{N}}(\sigma_{E_n}(x) - \tfrac{1}{n+1}) \quad \text{for every } x \text{ in a neighborhood of } 0. \tag{3}$$

We claim that $f \le G|_{A \cap Y}$. To see this, fix $y \in A \cap Y$ and take any $y^* \in \partial f(y)$. We consider three different cases:

- $a(y^*, y) \ge 1$. In this case we have:

$$G(y) \ge \Phi(y) = \varphi(y) \ge f(y) + y^*(y - y) = f(y).$$

- $0 < a(y^*, y) < 1$. In this case there exists a unique $n \in \mathbb{N}$ such that $1/(n+1) < a(y^*, y) \le 1/n$ and we have:

$$G(y) \ge \sigma_{B_n}(y) - \tfrac{1}{n+1} = \sigma_{\partial_{1/n} f(0)}(y) - \tfrac{1}{n+1} \ge y^*y - a(y^*, y) = f(y).$$

- $a(y^*, y) = 0$. In this case $y^* \in \partial f(0) = q(A^\circ)$ and $f(y) = y^*y$. Then, for every $n \in \mathbb{N}$, we have

$$G(y) \ge \sigma_{E_n}(y) - \tfrac{1}{n+1} \ge \sigma_{q(A^\circ)}(y) - \tfrac{1}{n+1} \ge y^*y - \tfrac{1}{n+1}.$$

  Then $G(y) \ge y^*y = f(y)$.

Let us show that $\partial G(0) = A^\circ$. If $x^* \notin A^\circ$, then there exists $n_0 \in \mathbb{N}$ such that $x^* \notin E_n$ $\forall n \ge n_0$. By the Hahn-Banach theorem, there exists $x_0 \in X$ such that $x^*x_0 > \sup x_0(E_{n_0})$. By homogeneity and (3), we can (and do) suppose that $x_0 \in A$, $\sup x_0(E_n) < \frac{1}{n_0+1}$ for each $n < n_0$, and $G(x_0) = \sup_{n\in\mathbb{N}}(\sigma_{E_n}(x_0) - \frac{1}{n+1})$. Now, if $n \ge n_0$ then

$$\sigma_{E_n}(x_0) - \tfrac{1}{n+1} \le \sigma_{E_{n_0}}(x_0) - \tfrac{1}{n+1} < \sigma_{E_{n_0}}(x_0) < x^*x_0\,;$$

and if $n < n_0$ then

$$\sigma_{E_n}(x_0) - \tfrac{1}{n+1} < \sigma_{E_n}(x_0) - \tfrac{1}{n_0+1} < 0 \le \sigma_{E_{n_0}}(x_0) < x^*x_0\,.$$

Hence $x^*x_0 > \sup_{n\in\mathbb{N}}(\sigma_{E_n}(x_0) - \frac{1}{n+1}) = G(x_0) = G(x_0) - G(0)$, which shows that $x^* \notin \partial G(0)$. Thus $\partial G(0) \subset A^\circ$. On the other hand, if $x^* \in A^\circ$ and $x \in A$, we have

$$G(x) \ge \sigma_{E_n}(x) - \tfrac{1}{n+1} \ge x^*x - \tfrac{1}{n+1} \qquad (n \in \mathbb{N}).$$

Consequently, $G(x) \ge x^*x$, which proves that $\partial G(0) \supset A^\circ$.

Finally, Lemma 1.6 concludes the proof.

(iv) $\Rightarrow$ *(iii)* Let $\{B_n\}$ be a sequence of w^*-compact convex sets in Y^* such that $B_n \searrow q(A^\circ)$ and such that ${}^\circ B_1$ is a neighborhood of the origin in Y. For every $y \in A \cap Y$, put $f(y) = \sup_n\big(\sigma_{B_n}(y) - \frac{1}{n}\big)$. Then $f : Y \cap A \to \mathbb{R}$ is a convex nonnegative function such that $f(0) = 0$. Moreover, since ${}^\circ B_1$ is a neighborhood of the origin in Y, the function f is continuous.

Observe that $B_n \subset \partial_{1/n} f(0)$; indeed, if $y^* \in B_n$ and $y \in A \cap Y$, then

$$f(y) \ge \sigma_{B_n}(y) - \tfrac{1}{n} \ge y^*y - \tfrac{1}{n}.$$

Consequently,

$$q(A^\circ) = \bigcap_{n\in\mathbb{N}} B_n \subset \bigcap_{n\in\mathbb{N}} \partial_{1/n} f(0) = \partial f(0).$$

Now, suppose that $y^* \notin q(A^\circ)$, that is, there exists $n \in \mathbb{N}$ such that $y^* \in Y^* \setminus B_n$. By the Hahn-Banach theorem and by the w^*-compactness of the sets $B_1, \ldots, B_{n-1}$, there exists $y \in A \cap Y$ such that $y^*y > \sup y(B_n) = \sigma_{B_n}(y)$ and such that $y^*y > \sigma_{B_k}(y) - \frac{1}{k}$ whenever $k < n$. Then

$$y^*y > \max\{\max_{k<n}(\sigma_{B_k}(y) - \tfrac{1}{k}), \sup_{k\ge n} \sigma_{B_k}(y)\} \ge f(y)\,;$$

hence $y^* \notin \partial f(0)$. This proves that $q(A^\circ) = \partial f(0)$.

By (iv), there exists a continuous convex extension $F\colon A \to \mathbb{R}$ of f such that $\partial F(0) = A^\circ$. Put $E_n = \partial_{1/n} F(0) \cap q^{-1}(B_n)$ for $n \in \mathbb{N}$. Then $\{E_n\}$ is a sequence of w^*-compact convex sets in X^* such that $E_n \searrow \partial F(0) = A^\circ$. By continuity of F, we have that ${}^\circ E_1$ is a neighborhood of the origin in X. Observe that clearly $q(\partial_{1/n} F(0)) = \partial_{1/n} f(0)$. Thus

$$q(E_n) \subset q(\partial_{1/n} F(0)) \cap B_n = \partial_{1/n} f(0) \cap B_n = B_n.$$

On the other hand, if $y^* \in B_n$ then $y^* \in \partial_{1/n} f(0) = q(\partial_{1/n} F(0))$. Hence $y^* = q(x^*)$ for some $x^* \in \partial_{1/n} F(0)$. Since $x^* \in q^{-1}(B_n)$, we have $x^* \in E_n$, and hence $y^* \in q(E_n)$. This proves that $q(E_n) = B_n$. $\square$

Proof of *(iii)* $\Leftrightarrow$ *(v)***.**

(v) $\Rightarrow$ *(iii)* Let $\{B_n\}$ be a sequence of w^*-compact convex sets in Y^* such that ${}^\circ B_1$ is a neighborhood of the origin in Y, and $B_n \searrow q(A^\circ)$. Put $\Gamma = B_1$ and, for each $(n,\gamma) \in \mathbb{N} \times \Gamma$, put $y^*_{n,\gamma} = \gamma$ if $\gamma \in B_n$ and $y^*_{n,\gamma} = 0$ otherwise. Since ${}^\circ B_1$ is a neighborhood of the origin in Y, $\{y^*_{n,\gamma}\}$ is an equicontinuous net. Moreover, since $B_n \searrow q(A^\circ)$, we have

$$(\{y^*_{n,\gamma}\})' \subset \bigcap_{n\in\mathbb{N}} (\{y^*_{k,\gamma}\}_{k\ge n})' \subset \bigcap_{n\in\mathbb{N}} B_n = q(A^\circ).$$

By (v), there exists an equicontinuous net $\{x^*_{n,\gamma}\} \subset X^*$ such that $(\{x^*_{n,\gamma}\})' \subset A^\circ$ and $x^*_{n,\gamma}|_Y = y^*_{n,\gamma}$ for each $(n,\gamma) \in \mathbb{N} \times \Gamma$. For each $n \in \mathbb{N}$, put $E_n = \overline{\mathrm{conv}}^{\,w^*}(\{x^*_{k,\gamma}\}_{k\ge n} \cup A^\circ)$. Then $\{E_n\}$ is a decreasing sequence of w^*-compact convex sets in X^*. Moreover, ${}^\circ E_1$ is a neighborhood of the origin in X since E_1 is equicontinuous (because both $\{x_{k,\gamma}\}_{k\ge n}$ and A° are). Let us observe that

$$B_n = \{q(x^*_{k,\gamma})\}_{k\ge n} \subset q(E_n) \subset \overline{\mathrm{conv}}^{\,w^*}(\{q(x^*_{k,\gamma})\}_{k\ge n} \cup q(A^\circ)) = B_n,$$

and hence $q(E_n) = B_n$. Moreover, by the Hahn-Banach theorem, it is easy to see that

$$\bigcap_{n\in\mathbb{N}} E_n \subset \overline{\mathrm{conv}}^{\,w^*}\big((\{x^*_{n,\gamma}\})' \cup A^\circ\big) \subset A^\circ.$$

Consequently, $\bigcap_n E_n = A^\circ$.

(iii) $\Rightarrow$ *(v)* Let $\{y^*_{n,\gamma}\} \subset Y^*$ be an equicontinuous net such that $(\{y^*_{n,\gamma}\})' \subset q(A^\circ)$. For each $n \in \mathbb{N}$, put $B_n = \overline{\mathrm{conv}}^{\,w^*}\big(\{y^*_{k,\gamma}\}_{k\ge n} \cup q(A^\circ)\big)\big)$. Proceeding as above, it is easy to see that the sequence $\{B_n\}$ is a decreasing sequence of w^*-compact convex sets with $B_n \searrow q(A^\circ)$ and $0 \in \mathrm{int}({}^\circ B_1)$. Let $\{E_n\}$ be the sequence from *(iii)*. In particular, for each $(n,\gamma) \in \mathbb{N} \times \Gamma$, we can choose an element $x^*_{n,\gamma} \in E_n$ such that $x^*_{n,\gamma}|_Y = y^*_{n,\gamma}$. Then $\{x^*_{n,\gamma}\} \subset X^*$ is an equicontinuous net such that $(\{x^*_{n,\gamma}\})' \subset A^\circ$. Indeed,

$$(\{x^*_{n,\gamma}\})' \subset \bigcap_{n\in\mathbb{N}} (\{x^*_{k,\gamma}\}_{k\ge n})' \subset \bigcap_{n\in\mathbb{N}} E_n = A^\circ.$$

□

Applying Theorem 2.6 to $A = X$, we immediately get the following corollary. (Notice that it is easy to see that a set $B \subset X^*$ is equicontinuous if and only if $^\circ B$ is a neighborhood of 0 in X.)

Corollary 2.7. *Let Y be a subspace of a t.v.s. X. Then the following assertions are equivalent.*

(i) *Every continuous convex function on Y admits a continuous convex extension to X.*

(ii) *For every sequence $\{C_n\}$ of open convex sets in Y such that $C_n \nearrow Y$, there exists a sequence $\{D_n\}$ of open convex sets in X such that $D_n \cap Y = C_n$ and $D_n \nearrow X$.*

(iii) *For every sequence $\{B_n\}$ of w^*-compact convex equicontinuous sets in Y^* such that $B_n \searrow \{0\}$, there exists a sequence $\{E_n\}$ of w^*-compact convex equicontinuous sets in X^* such that $E_n \searrow \{0\}$ and such that $q(E_n) = B_n$, for each $n \in \mathbb{N}$.*

(iv) *Every continuous convex function f on Y that is Gâteaux differentiable at some $y_0 \in Y$ admits a continuous convex extension F to X that is Gâteaux differentiable at y_0.*

(v) *For every equicontinuous w^*-null net $\{y^*_{n,\gamma}\} \subset Y^*$, there exists an equicontinuous w^*-null net $\{x^*_{n,\gamma}\} \subset X^*$ such that $x^*_{n,\gamma}|_Y = y^*_{n,\gamma}$, for each $(n,\gamma) \in \mathbb{N} \times \Gamma$.*

Remark 2.8. The above Corollary 2.7 extends some results from [9] and [2] to the ambit of topological vector spaces. In particular,

(a) the equivalence $(i) \Leftrightarrow (ii)$ appears in [9, Theorem 4.3] for normed spaces;

(b) the equivalences $(i) \Leftrightarrow (iv) \Leftrightarrow (v)$ appear in [2, Theorems 4.8 and 4.15] for Banach spaces (and condition (iv) therein is formulated with f and F Lipschitz functions).

The new condition Corollary 2.7(iii) will be used in the proof of Theorem 4.5. The corresponding condition (iii) from Theorem 2.6 will be used in the forthcoming paper [4].

The following a bit technical observation reveals to be quite useful in some proofs.

Observation 2.9. *The (equivalent) conditions (ii') and (iii) in Theorem 2.6 are respectively equivalent to the following, formally weaker, conditions.*

(ii") *For every sequence $\{C_n\}$ of open convex sets in Y such that $C_n \nearrow A \cap Y$, there exists a sequence $\{D_k\}$ of open convex sets in X such that $D_k \nearrow A$ and for each $k \in \mathbb{N}$ there exists $n(k) \in \mathbb{N}$ for which $D_k \cap Y \subset C_{n(k)}$.*

(iii') *For every sequence* $\{B_n\}$ *of* w^**-compact convex sets in* Y^* *such that* $B_n \searrow q(A^\circ)$ *and such that* ${}^\circ B_1$ *is a neighborhood of the origin in* Y*, there exists a sequence* $\{E_k\}$ *of* w^**-compact convex sets in* X^* *such that* $E_k \searrow A^\circ$*, such that* ${}^\circ E_1$ *is a neighborhood of the origin in* X *and such that for each* $k \in \mathbb{N}$ *there exists* $n(k) \in \mathbb{N}$ *for which* $q(E_k) \supset B_{n(k)}$*.*

Proof. Let us prove that (ii") implies (ii') (the inverse implication is obvious). Assume (ii"). We can (and do) suppose that $\{n(k)\}$ is increasing. Let us define a sequence $\{\widetilde{D}_n\}$ of open convex sets in X. For $n < n(1)$, put $\widetilde{D}_n = \emptyset$. Given $n \geq n(1)$, take the unique $k \in \mathbb{N}$ such that $n(k) \leq n < n(k+1)$, and define $\widetilde{D}_n = D_k$. Observe that, for $n \geq n(1)$, we have $\widetilde{D}_n \cap Y = D_k \cap Y \subset C_{n(k)} \subset C_n$. Hence the condition (ii') in Theorem 2.6 holds for $\{\widetilde{D}_n\}$ in place of $\{D_n\}$.

Proceeding as in the proof of $(ii) \Leftrightarrow (iii)$ in Theorem 2.6, we easily get the equivalence $(ii'') \Leftrightarrow (iii')$. □

Let us conclude this section with an easy corollary concerning extensions from dense subspaces.

Corollary 2.10. *Let* X *be a t.v.s.,* $Y \subset X$ *a subspace,* $A \subset X$ *an open convex set,* $A \cap Y \neq \emptyset$*. Then each continuous convex function on* $A \cap Y$ *can be uniquely extended to a continuous convex function on* $A \cap \overline{Y}$*.*

Proof. We can suppose that $X = \overline{Y}$, i.e. Y is dense in X. Then $A \cap Y$ is dense in A. Now, it is clear that f has at most one continuous convex extension to A. To prove that such an extension exists, let us show that (iii) in Theorem 2.6 holds with $E_n = B_n$ (notice that $Y^* = X^*$ in our case). It is easy to see that a set $B \subset X^*$ is equicontinuous on Y if and only if it is equicontinuous on X. Now, it suffices to show that the topologies $\sigma(X^*, X)$ and $\sigma(X^*, Y)$ coincide on any such set B. By equicontinuity of B, there exists a symmetric neighborhood $V \subset X$ of the origin such that $|b^* v| \leq 1$ whenever $b^* \in B, v \in V$. Given $x \in X$ and $\varepsilon > 0$, choose $y \in Y \cap (x + \frac{\varepsilon}{2} V)$. Then we easily get $\{b^* \in B : |b^* y| < \frac{\varepsilon}{2}\} \subset \{b^* \in B : |b^* x| < \varepsilon\}$, and this completes the proof. □

3. The two extension properties – first facts

Definition 3.1. Let Y be a subspace of a t.v.s. X. We shall say that the couple (X, Y) has:

(a) the CE-*property* ("convex extension property") if each continuous convex function on Y can be extended to a continuous convex function on X;

(b) the SCE-*property* ("strong convex extension property") if, for every open convex set $A \subset X$ that intersects Y, each continuous convex function on $Y \cap A$ can be extended to a continuous convex function on A.

Clearly, the SCE-property implies the CE-property. Notice that Corollary 2.7 provides characterizations of the CE-property, and Theorem 2.6 implies characterizations of the SCE-property.

Lemma 3.2. *Let X be a t.v.s., $Y \subset Z$ two subspaces of X such that Z is dense in X. Let $\mathcal{P}$ be one of the properties* CE *and* SCE*. Then the following assertions are equivalent:*

(i) *(X, Y) has the property $\mathcal{P}$;*
(ii) *$(X, \overline{Y})$ has the property $\mathcal{P}$;*
(iii) *(Z, Y) has the property $\mathcal{P}$.*

Proof. We shall prove only the case when $\mathcal{P}$ is the SCE-property. (The other case is much simpler.)

(i) $\Rightarrow$ *(ii)*. If $A \subset X$ is a open convex set intersecting $\overline{Y}$, then A intersects Y, as well. Clearly, $A \cap Y$ is dense in $A \cap \overline{Y}$. Given a continuous convex function f on $A \cap \overline{Y}$, every continuous convex extension of $f|_{A\cap Y}$ to X is also an extension of f.

(ii) $\Rightarrow$ *(i)*. Given an open convex set $A \subset X$ that intersects Y, and a continuous convex function f on $A \cap Y$, apply Corollary 2.10 to extend f to $A \cap \overline{Y}$, then apply (ii) to extend it to X.

(i) $\Rightarrow$ *(iii)*. Let $A \subset Z$ be an open (in Z) convex set intersecting Y. We can (and do) suppose that $0 \in A$. Since the Minkowski gauge p_A of A is uniformly continuous (Fact 1.3), it admits a (unique) continuous (necessarily convex) extension $f\colon X \to \mathbb{R}$. Then the set $\widetilde{A} = \{x \in X : f(x) < 1\}$ is open (in X) and convex, $\widetilde{A} \supset A$, and $\widetilde{A} \cap Z = A$. By (i), every continuous convex function on $A \cap Y = \widetilde{A} \cap Y$ can be extended to a continuous convex function on $\widetilde{A}$.

(iii) $\Rightarrow$ *(i)*. Let $A \subset X$ be an open convex set, f a continuous convex function on $A \cap Y$. By (iii), f can be extended to a continuous convex function on $A \cap Z$; by Corollary 2.10, this extension can be extended to a continuous convex function on A. □

The following proposition collects some easy or known sufficient conditions for the CE-property.

Proposition 3.3. *Let Y be a subspace of a t.v.s. X. Then (X, Y) has the* CE*-property provided* at least one *of the following conditions is satisfied.*

(a) *Y is complemented in X (i.e., there exists a continuous linear projection P of X onto Y).*
(b) *X is a normed linear space, and X/Y is separable.*
(c) *X is locally convex, and Y is isomorphic to some $\ell_\infty(\Gamma)$ space.*

(d) *X is a Banach space, and Y is isomorphic to a $C(K)$ Grothendieck space.*

(e) *Y is separable, and X is a Banach space having the separable complementation property (saying that every separable subspace of X is contained in a complemented separable subspace); for instance, this is satisfied in any of the following cases:*
   (i) *X is weakly compactly generated;*
   (ii) *X is a dual space with the Radon-Nikodým property;*
   (iii) *X has a countably norming M-basis;*
   (iv) *X is a Banach lattice not containing c_0.*

Proof. (a) This is quite easy: if $f\colon Y \to \mathbb{R}$ is a continuous convex function, then $f \circ P$ is a continuous convex extension of f.
(b) By Lemma 3.2, we may assume that X is a Banach space and Y is closed. Apply [2, Corollary 4.10] or [9, Theorem 4.5].
(c) follows from (a) since $\ell_\infty(\Gamma)$ is complemented in any locally convex topological vector superspace X. Indeed, let $V \subset X$ be an open symmetric convex set such that $V \cap \ell_\infty(\Gamma)$ is contained in the open unit ball of $\ell_\infty(\Gamma)$. By the Hahn–Banach theorem, each coordinate-functional e_γ ($\gamma \in \Gamma$) on $\ell_\infty(\Gamma)$ admits a continuous linear extension $f_\gamma \in X^*$ such that $\sup_V f_\gamma \le 1$. Then $Px = \big(f_\gamma(x)\big)_{\gamma\in\Gamma}$ defines a continuous linear projection of X onto $\ell_\infty(\Gamma)$.
For (d), see [2, Proposition 4.12].
(e) appears in the proof of [2, Corollary 4.11] (which corresponds to our particular case (iii)). The idea is simple: given a continuous convex function f on Y, take a complemented separable subspace Z containing Y, extend f first to Z using (b), and then to X by (a). References for sufficient conditions (i)-(iv) can be found in [7], pp. 481–482. □

In the next section, we shall extend the above case (b) to locally convex spaces X (Theorem 4.5). The following proposition, providing a few examples of couples (X, Y) failing the CE-property, is an immediate consequence of results in [3]. See also [2, Example 4.2]. Other examples can be found using [3, Theorem 2.3].

Proposition 3.4. *Let Y be an infinite dimensional closed subspace of $\ell_\infty(\Gamma)$. Then the couple $\big(\ell_\infty(\Gamma), Y\big)$ fails the* CE-*property in any of the following cases.*

(a) *Y is isomorphic to $c_0(\Lambda)$ or some $\ell_p(\Lambda)$ with $1 < p < \infty$.*
(b) *Y does not contain any isomorphic copy of ℓ_1.*

In particular, (ℓ_∞, c_0) fails the CE-*property.*

Proof. (a) follows from (b), while (b) follows from [3, Theorem 2.5] and the fact that there exists a continuous convex function f on Y which is unbounded on some bounded set (see [1, Theorem 2.2] or [2, Theorem 3.1]). □

Proposition 3.5. *Let Y be a subspace of a Banach space X. Suppose that X is a Grothendieck space and that the couple (X, Y) has the* CE*-property. Then Y is a Grothendieck space.*

Proof. Fix any w^*-null sequence $\{y_n^*\} \subset Y^*$. Since the couple (X, Y) has the CE-property, by Corollary 2.7, there exists a w^*-null sequence $\{x_n^*\} \subset X^*$ such that $x_n^*|_Y = y_n^*$, for each $n \in \mathbb{N}$. Since X is a Grothendieck space, $\{x_n^*\}$ is w-null in X^*, and hence also $\{y_n^*\}$ is w-null in Y^*. This proves that Y is a Grothendieck space. □

Corollary 3.6. *Let Y be a separable nonreflexive Banach space, considered as a subspace of ℓ_∞. Then (ℓ_∞, Y) fails the* CE*-property.*

Proof. It is well known that ℓ_∞ is a Grothendieck space. On the other hand, Y is not, since every separable Grothendieck space is reflexive. Apply Proposition 3.5. □

We know much less about the SCE-property. In particular, *we do not have any example of a couple (X, Y) satisfying the* CE*-property, but not the* SCE*-property.* The SCE-property will be studied in a subsequent paper [4]. Here we only state the following two easy sufficient conditions.

Proposition 3.7. *Let Y be a subspace of a t.v.s. X. Then (X, Y) has the* SCE*-property provided* at least one *of the following conditions is satisfied.*

(a) *Y is dense in X.*
(b) *X is locally convex, and Y is finite-dimensional.*

Proof. (a) follows from Corollary 2.10.

(b) By [5, Sec. 7, Prob. A], Y can be written as a (topological) direct sum of two subspaces $Y = Y_1 \oplus Y_2$, where Y_1 is topologically trivial and Y_2 is Hausdorff. It is well-known that Y_2 is isomorphic to some $\mathbb{R}^d$ and there exists a continuous linear projection P of X onto Y_2. Fix a bounded convex symmetric open set $B \subset Y_2$. Let $0 \in A \subset X$ be an open convex set. Define

$$D_k = (1 - \tfrac{1}{k+1})A \cap P^{-1}(kB) \qquad (k \in \mathbb{N}).$$

Then $\overline{D}_k \subset (1 - \frac{1}{k+1})\overline{A} \cap P^{-1}(k\overline{B}) \subset D_{k+1} \subset A$, and $D_k \nearrow A$. Let $\{C_n\}$ be any sequence of open convex sets in Y such that $C_n \nearrow A \cap Y$. For each k, the set $\overline{D}_k \cap Y_2$ is closed and bounded in Y_2, and hence compact. Consequently, the set $\overline{D}_k \cap Y = Y_1 + (\overline{D}_k \cap Y_2)$ is compact in Y. It follows that there exists $n(k) \in \mathbb{N}$ with $D_k \cap Y \subset \overline{D}_k \cap Y \subset C_{n(k)}$. Apply Observation 2.9. □

4. Conditional separability of X/Y and the CE-property

The main result of the present section is Theorem 4.5 saying, roughly speaking, that if X is locally convex and X/Y is "conditionally separable" (see below) then (X, Y) has the CE-property. This theorem generalizes Proposition 3.3(b) from normed spaces to locally convex spaces.

Let us start with a brief discussion of "conditional separability". Likely, this notion (maybe under another terminology) has already been considered in the literature, but we have not found any reference.

Definition 4.1. Let X be a t.v.s., $A \subset X$. We say that A is *conditionally separable* if for each neighborhood V of 0 there exists a set $Q \subset X$ such that Q is at most countable and $A \subset Q + V$.

Remark 4.2. The following properties are quite easy.

- (i) Conditional separability is stable under countable unions, under taking closures, and under passing to subsets.
- (ii) One can write equivalently "$Q \subset A$" instead of "$Q \subset X$" in Definition 4.1.
- (iii) A locally convex t.v.s. X is conditionally separable iff for each continuous seminorm ν on X the seminormed space (X, ν) is separable.
- (iv) All separable sets, all Lindelöf sets and all totally bounded (in particular, compact) sets are conditionally separable.
- (v) Assume that a t.v.s. X has a countable base of neighborhoods of 0 (equivalently, X is metrizable [5]). A set $A \subset X$ is conditionally separable iff A is separable (iff A is Lindelöf).
- (vi) Let X and Z be topological vector spaces, let $T : X \to Z$ be a continuous linear operator and let A be a conditionally separable subset of X, then $T(A)$ is conditionally separable. In particular, if Y is a subspace of a conditionally separable t.v.s. X, then X/Y is conditionally separable.

Lemma 4.3. *Let X be a t.v.s. If X is conditionally separable, then*

- $(*)$ *for every neighborhood $V \subset X$ of 0 there exists a countable set $Q \subset X$ that separates the points of V°.*

Moreover, if X is locally convex, also the converse is true.

Proof. First notice that, both in Definition 4.1 and in $(*)$, we can equivalently consider only neighborhoods V belonging to an appropriate local base.

Let X be conditionally separable, and $V \subset X$ a symmetric neighborhood of 0. Then $X = Q + V$ with $Q \subset X$ countable. Since V° is symmetric, it suffices to show that Q separates every $x^* \in V^\circ \setminus \{0\}$ from the origin. To this end, consider $x \in X$ with $x^*x > 1$, and write it in the form $x = q + v$ where $q \in Q$, $v \in V$. Then $x^*q = x^*x - x^*v > 1 - 1 = 0$.

Now, let X be a locally convex space satisfying $(*)$, and $V \subset X$ an open convex neighborhood of 0. Fix a countable set $Q \subset X$ separating the points of V°, and define a countable set Q_0 as the set of all finite rational linear combinations of elements of Q. It is easy to see that $Q_0 + V$ is a nonempty open convex set. If $Q_0 + V \neq X$, choose an arbitrary $x \in X \setminus (Q_0 + V)$ and a nonzero $x^* \in X^*$ such that

$$1 = x^*x > \sup_{Q_0+V} x^* = \sup_{Q_0} x^* + \sup_{V} x^* \geq \max\bigl\{\sup_{Q_0} x^*, \sup_{V} x^*\bigr\}.$$

But this implies that $x^* \in V^\circ \setminus \{0\}$ and $x^*|_Q \equiv 0$. This contradiction shows that $Q_0 + V = X$, and we are done. □

Observation 4.4. *Let L be a nonempty subset of the algebraic dual of a vector space X. Then X is conditionally separable in the weak topology $\sigma(X, L)$.*

Proof. Each neighborhood V of 0 contains a neighborhood of the form

$$W = \{x \in X : |f_i(x)| < \varepsilon \text{ for } i = 1, \dots, n\}$$

where the functionals $f_1, \dots, f_n$ belong to L and are linearly independent. We claim that for each $q = (q_1, \dots, q_n) \in \mathbb{Q}^n$ there exists $x_q \in X$ such that $f_i(x_q) = q_i$ $(i = 1, \dots, n)$. (Indeed, since f_i's are linearly independent, there exist $u_1, \dots, u_n \in X$ such that $f_i(u_j) = \delta_{ij}$ (Kronecker's delta); then $x_q := \sum_{j=1}^n q_j u_j$ works.) Then the set $Q = \{x_q : q \in \mathbb{Q}^n\}$ clearly satisfies $Q + V \supset Q + W = X$. □

The above observation implies that, in general, *conditional separability is not equivalent to separability.* Indeed, if X is a normed space then X is weakly separable if and only if it is separable. On the other hand, X is always weakly conditionally separable by Observation 4.4. However, for the norm topology the two notions coincide by Remark 4.2(v).

Now, we are ready for the second main result of the present paper.

Theorem 4.5. *Let X be a locally convex t.v.s., $Y \subset X$ a subspace for which the quotient space X/Y is conditionally separable. Then the couple (X, Y) has the* CE*-property.*

Proof. It suffices to verify the condition (iii) in Corollary 2.7. Let $\{B_n\}$ be a sequence of w^*-compact convex sets in Y^* such that $B_n \searrow \{0\}$ and such that ${}^\circ B_1$ is a neighborhood of the origin in Y. Since X is locally convex, there exists V a convex symmetric neighborhood of the origin such that $V \cap Y \subset {}^\circ B_1$ and hence such that $q(V^\circ) = (V \cap Y)^\circ \supset B_1$. Let us observe that $W = \{v + Y; v \in V\}$ is a neighborhood of the origin in X/Y and then, by Lemma 4.3, there exists a countable set $Q = \{x_n + Y\} \subset X/Y$, with $\{x_n\} \subset X$, such that Q separates the points of $W^\circ = V^\circ \cap Y^\perp$. For every $n \in \mathbb{N}$, put

$$M_n = \{x^* \in X^*;\ |x^*x_i| \leq 1/n,\ i = 1, \dots, n\}, \quad E_n = (2V^\circ) \cap M_n \cap q^{-1}(B_n).$$

Then $\{E_n\}$ is a decreasing sequence of w^*-compact convex sets in X^* such that ${}^\circ E_1$ is a neighborhood of the origin in X (indeed, $E_1 \subset 2V^\circ$ implies ${}^\circ E_1 \supset {}^\circ(2V^\circ) = \frac{1}{2}V$).

We claim that $E_n \searrow \{0\}$. To see this, suppose that $x^* \in \bigcap_n E_n$, then $q(x^*) \in \bigcap_n B_n$, i.e. $x^* \in Y^\perp$. Moreover, $x^* \in \bigcap_n M_n$ and hence $x^*|_Q \equiv 0$. It follows that $x^* = 0$ since Q separates the points of $V^\circ \cap Y^\perp$.

By Observation 2.9, it remains to prove that each $q(E_n)$ contains some B_k. By definition of E_n, it suffices to show that $q\big((2V^\circ) \cap M_n\big)$ contains some B_k. Suppose this is not the case; then, for every $k \in \mathbb{N}$, there exists $z_k^* \in B_k \setminus q\big((2V^\circ) \cap M_n\big)$. Since $B_k \searrow \{0\}$, we have $z_k^* \to 0$ in the w^*-topology of Y^*. Since $z_k^* \in B_1 \subset q(V^\circ)$, we have $z_k^* = q(x_k^*)$ for some $x_k^* \in V^\circ$. Let $\{x_\alpha^*\}$ be a w^*-convergent subnet of $\{x_k^*\}$. Then $x_\alpha^* \to x^* \in V^\circ$. Let us observe that $x_\alpha^* - x^* \in 2V^\circ$ and $q(x^*) = 0$. Moreover, since $x_\alpha^* - x^* \to 0$ in the w^*-topology, eventually $x_\alpha^* - x^* \in M_n$. Then eventually $z_\alpha^* = q(x_\alpha^* - x^*) \in q\big((2V^\circ) \cap M_n\big)$, which is a contradiction with the choice of z_k^*. □

Remark 4.6. If $X = L^p([0,1])$ with $0 < p < 1$ and if Y is a finite dimensional subspace of X, then the couple (X, Y) does not have the CE-property. Indeed, since $X^* = \{0\}$, every continuous convex function on X is constant. This example shows that in the above theorem we cannot omit the assumption that X is locally convex.

Let us conclude with the case of the weak-type topologies. Theorem 4.5 and Observation 4.4 immediately imply the following corollary.

Corollary 4.7. *Let X be a vector space, $Y \subset X$ a subspace, L a subset of the algebraic dual of X. Then the couple (X, Y) has the* CE*-property for the $\sigma(X, L)$-topology.*

Proof. By Observation 4.4, $\big(X, \sigma(X, L)\big)$ is conditionally separable and hence, by Remark 4.2, X/Y is conditionally separable. Theorem 4.5 completes the proof. □

Corollary 4.8. *Let X be a normed linear space.*

(a) *For every subspace $Y \subset X$, the couple (X, Y) has the* CE*-property for the weak topology.*
(b) *For every subspace $Y \subset X^*$, the couple (X^*, Y) has the* CE*-property for the weak* topology. (That is, every $\sigma(Y, X)$-continuous convex function on Y can be extended to a weak*-continuous convex function on X^*.)*

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