A priori bounds for superlinear problems involving the N-Laplacian *

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Abstract

In this paper we establish a priori bounds for positive solution of the equation

$$-\Delta_N u = f(u) \ , \ \ u \in H_0^1(\Omega)$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, and the nonlinearity f has at most exponential growth. The techniques used in the proofs are a generalization of the methods of Brezis-Merle to the N-Laplacian, in combination with the Trudinger-Moser inequality, the Moving Planes method and a Comparison Principle for the N-Laplacian.

Keywords and phrases: a priori bounds; moving planes; Trudinger-Moser inequality.

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1 Introduction

This paper is concerned with a priori bounds for positive solutions of equations involving the N-Laplacian and superlinear nonlinearities in bounded domains in $\mathbb{R}^N$. More precisely, we consider

$$\begin{cases} -\Delta_N u &= f(u) & \text{in } \Omega \\ u &> 0 & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega \end{cases} , \tag{1.1}$$

where Ω is a strictly convex, bounded and smooth domain in $\mathbb{R}^N$, and $\Delta_N u = div(|\nabla u|^{N-2}\nabla u)$ is the N-Laplacian operator. On the function $f : \mathbb{R}^+ \to \mathbb{R}^+$ we assume that it is a locally Lipschitz function satisfying the following hypotheses:

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(f_1) *$f(s) \geq 0$, for all $s \geq 0$,*

and either

(f_2) *there exists a positive constant d such that*

$$\liminf_{s \to +\infty} \frac{f(s)}{s^{N-1+d}} > 0$$

and

(f_3) *there exist constants c, $s_0 \geq 0$ and $0 < \alpha < 1$ such that*

$$f(s) \leq c\, e^{s^\alpha} \ , \ \text{for all } s \geq s_0 \, ,$$

or

(f_4) *there exist constants c_1, $c_2 > 0$ and $s_0 > 0$ such that*

$$c_1\, e^s \leq f(s) \leq c_2\, e^s \ , \ \text{for all } s \geq s_0 \, .$$

The main result is the following

Theorem 1.1 *(A priori bound). Under the assumptions (f_1) and either (f_2) and (f_3) (subcritical case) or (f_4) (critical case) there exists a constant $C > 0$ such that every weak solution $u \in W_0^{1,N}(\Omega) \cap C^1(\Omega)$ of Equation (1.1) satisfies*

$$\|u\|_{L^\infty(\Omega)} \leq C. \tag{1.2}$$

A priori bounds for superlinear elliptic equations have been a focus of research in nonlinear analysis in recent years. On the one hand, such results give interesting qualitative information on the positive solutions of such equations; on the other hand they are also useful to obtain existence results via degree theory.

It seems that the first general result for a priori bounds for superlinear elliptic equations is due to Brezis-Turner [5], 1977. They considered the equation

$$\begin{cases} -\Delta\, u \ = g(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \ , \end{cases} \tag{1.3}$$

and proved an a priori bound under the (main) hypothesis

$$0 \leq g(x,s) \leq c\, s^p \ , \ p < \frac{N+1}{N-1} \, .$$

Their method is based on the *Hardy-Sobolev inequality.*

In 1981, Gidas and Spruck [8] considered Equation (1.3) under the assumption

$$\lim_{s \to \infty} \frac{g(x,s)}{s^p} = a(x) > 0 \quad \text{in } \overline{\Omega} \ ,$$

and proved a priori estimates under the condition

$$1 < p < \frac{N+2}{N-2} = 2^* - 1 \ ,$$

using *blow-up techniques* and *Liouville theorems* on $\mathbb{R}^N$.

In 1982, De Figueiredo - P.L. Lions - Nussbaum [9] obtained a priori estimates under the assumptions that Ω is convex, and $g(s)$ is superlinear at infinity and satisfies

$$g(s) \le cs^p \ , \ 1 < p < \frac{N+2}{N-2} \ , \quad \text{(and some technical conditions)} \ .$$

Their method relied on the *moving planes technique*, see [7], to obtain estimates near the boundary, and on Pohozaev-type identities.

Due to the results by Gidas-Spruck and De Figueiredo-Lions-Nussbaum it was generally believed that the result of Brezis-Turner was not optimal. But surprisingly, Quittner-Souplet [14] showed in 2004 that under the general hypotheses of Brezis-Turner their result is optimal; in fact, they give a counterexample with a $g(x, s)$ with strong x-dependence.

Concerning to the $m-$Laplace case, Azizieh-Clément [3] studied the problem

$$\begin{cases} -\Delta_m u \ = \ g(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \ . \end{cases} \tag{1.4}$$

They obtain a priori estimates for the particular case $1 < m < 2$, assuming $g(x, u) = g(u)$, with $C_1 u^p \le g(u) \le C_2 u^p$, where $1 < p < N(m-1)/(N-m)$ and Ω is bounded and convex.

The more general case $1 < m \le 2$ was considered by Ruiz [16]; he studied problem (1.4) where g is as in Azizieh-Clement but depends on x; also, he does not need Ω convex. In these two works, a blow-up argument together with a non existence result of positive super solutions, due to Mitidieri-Pohozaev [13], are used.

Recently, Lorca-Ubilla [12] obtained a priori estimates for solutions of (1.4) for more general nonlinearities g. They only require $0 \le g(x, u) < Cu^p \ , \ 1 < p < N(m-1)/(N-m)$, together with a superlinearity assumption at infinity. In this case the blow-arguments used by Azizieh-Clément and by Ruiz are not sufficient to obtain a contradiction. However using an adaptation of Ruiz's argument, which consists in a combination of Harnack inequalities and local L^q estimates, it is possible to get the a priori estimate.

The above mentioned results are for $N > 2$; for $N = 2$ one has the embedding $H_0^1(\Omega) \subset L^p$, for all $p > 1$, but easy examples show that $H_0^1(\Omega) \not\subseteq L^\infty(\Omega)$. Thus, one may ask for the maximal growth function $g(s)$ such that $\int_\Omega g(u) < \infty$ for $u \in H_0^1(\Omega)$. This maximal possible growth was determined independently by Yudovich, Pohozaev and Trudinger, leading to what is now called the *Trudinger inequality*: it says that for $u \in H_0^1(\Omega)$ one has $\int_\Omega e^{u^2} dx < +\infty$.

So, one can ask whether in dimension $N = 2$ one can prove a priori estimates for nonlinearities with growth up to the Trudinger-Moser growth. This is not the case, however some interesting result for equations with exponential growth have been proved in recent years. First, we mention the result of Brezis-Merle [4] who proved in 1991 that under the growth restriction

$$c_1 e^s \leq g(x,s) \leq c_2 e^s$$

one has: if $\int_\Omega g(x,u)dx \leq c$, for all $u > 0$ solution of Equation (1.1), then there exists $C > 0$ such that

$$\|u\|_\infty \leq C$$

for all positive solutions.

This is not quite an a priori result yet; however, from the *boundary estimates* of De Figueiredo - Lions - Nussbaum one obtains, assuming that Ω is convex (and adding some technical assumptions) that the condition $\int_\Omega g(x,u) \leq c$ of Brezis-Merle is satisfied. Hence, on convex domains the Brezis-Merle result yields indeed the desired a priori bounds. We note also that Brezis-Merle give examples of nonlinearities $g(x,s) = h(x)e^{s^\alpha}$ with $\alpha > 1$ for which there exists a sequence of unbounded solutions.

Our Theorem 1.1 is motivated by the result of Brezis-Merle. We recall that in dimension N the Trudinger inequality gives as maximal growth $g(s) \leq e^{|s|^{N/(N-1)}}$, while our result shows that for a priori bounds it is again the exponential growth $g(s) \sim e^s$ which is the limiting growth to obtain a priori bounds.

The paper is organized as follows: in section 2 we obtain uniform bounds near the boundary $\partial\Omega$, using results of Damascelli-Sciunzi [6]. In section 3 we show that the boundary estimates yield easily a uniform bound on $\int_\Omega g(x,u)$. In section 4 we discuss the "subcritical case", i.e. when assumptions (f_2) and (f_3) hold, while in section 5 we prove the a priori bounds in the "critical case", i.e. under assumption (f_4).

2 The boundary estimate

In this section we obtain a priori estimates on a portion of Ω including the boundary.

Proposition 2.1 *Assume (f_2) or the left inequality in (f_4). Then there exist positive constants r, C such that every weak solution $u \in W_0^{1,N}(\Omega) \cap C^1(\Omega)$ of Equation (1.1) verifies*

$$u(x) \leq C \text{ and } |\nabla u(x)| \leq C, x \in \Omega_r \ ,$$

where $\Omega_r = \{x \in \Omega : d(x, \partial\Omega) \leq r\}$.

Proof. For $x \in \partial\Omega$, let $\eta(x)$ denote the outward normal vector to $\partial\Omega$ in x. By Damascelli-Sciunzi [6], Theorem 1.5, there exists $t_0 > 0$ such that $u(x - t\eta(x))$ is nondecreasing for $t \in [0, t_0]$ and for $x \in \partial\Omega$. Note that t_0 depends only on the geometry of Ω. Following the ideas of de Figueiredo, Lions and Nussbaum's paper [9] one now shows that there exists $\alpha > 0$, depending only on Ω, such that

$$\begin{aligned}&u(z-t\sigma) \text{ is nondecreasing for all } t \in [0,t_1],\\ &\text{where } \ |\sigma| = 1,\ \sigma \in \mathbb{R}^N \text{ verifies } \sigma\cdot\eta(z) \geq \alpha,\ z \in \partial\Omega\ ,\end{aligned}$$

and $t_1 > 0$ depends only on Ω.

Since $u(z-t\sigma)$ is nondecreasing in t for z and σ as above, for all $x \in \Omega_\epsilon$ we find a measure set I_x, and positive numbers γ and ϵ (depending only on Ω) such that

(i) $|I_x| \geq \gamma$

(ii) $I_x \subset \{x \in \Omega :\ d(x,\partial\Omega) \geq \frac{\varepsilon}{2}\}$

(iii) $u(y) \geq u(x)$, for all $y \in I_x$.

We now use Piccone's identity (see [2]), which says that if v and u are C^1 functions with $v \geq 0$ and $u > 0$ in Ω, then

$$|\nabla v|^N \geq |\nabla u|^{N-2}\nabla\Big(\frac{v^N}{u^{N-1}}\Big)\,\nabla u\ .$$

We apply this inequality with $v = e_1$, the first (positive) eigenfunction of the N-Laplacian on Ω, and $u > 0$ a (weak) solution of $-\Delta_N u = f(u)$. We assume that e_1 is normalized, i.e. $\int_\Omega e_1^N = 1$. Then we have (observe that $\frac{e_1^N}{u^{N-1}}$ belongs to $W_0^{1,N}(\Omega)$ since u is positive in Ω and has nonzero outward derivative on the boundary because of Hopf's lemma, see [17])

$$c \geq \int_\Omega |\nabla e_1|^N dx \geq \int_\Omega |\nabla u|^{N-2}\nabla u\ \nabla\frac{e_1^N}{u^{N-1}} = \int_\Omega \frac{f(u)\,e_1^N}{u^{N-1}}$$

Thus condition (f_2) (or condition (f_4)) implies $\displaystyle\int_\Omega u^d e_1^N \leq \widetilde{C}$, and so

$$\eta^N \int_{\Omega\setminus\Omega_{\frac{\varepsilon}{2}}} u^d \leq \widetilde{C}$$

where $e_1(z) \geq \eta > 0, z \in \Omega\setminus\Omega_{\frac{\varepsilon}{2}}$. By (ii), given $x \in \Omega_\epsilon$, we have

$$\eta^N \int_{I_x} u^d \leq \widetilde{C}\,.$$

Now since $u^d(x)|I_x| \leq \int_{I_x} u^d$ by (i) and (ii), we have $u^d(x) \leq \dfrac{\tilde{C}}{\gamma\eta^N}$, and so $u(x) \leq C'$, for all $x \in \Omega_\epsilon$. Finally by Lieberman [11] (see also Azizieh and Clément [3]) we have

$$u \in C^{1,\alpha}(\Omega_{\frac{\varepsilon}{2}}) \ \text{ with } \ \|u\|_{C^{1,\alpha}(\Omega_{\frac{\varepsilon}{2}})} \leq C\ . \tag{2.1}$$

■

3 Uniform bound on $\int_\Omega f(u)$

In this section we show that the boundary estimates yield easily a bound on the term $\int_\Omega f(u)dx$, for all positive solutions of Equation (1.1).

Proposition 3.1 *Suppose estimate (2.1) holds. Then there exists a positive constant C such that for every weak solution of Equation (1.1) we have*

$$\int_\Omega f(u) \le C\,. \tag{3.1}$$

Proof. Let $\psi \in C_0^\infty(\Omega)$ such that $\psi \equiv 1$ on $\Omega \setminus \Omega_{\frac{\varepsilon}{2}}$. We have

$$\int_\Omega |\nabla u|^{N-2}\nabla u \nabla\psi = \int_\Omega f(u)\psi \tag{3.2}$$

Using

$$\int_{\Omega\setminus\Omega_{\frac{\varepsilon}{2}}} f(u) \le \int_\Omega f(u)\psi$$

and the a priori estimates in $\Omega_{\frac{\varepsilon}{2}}$, see (2.1), we get

$$\int_{\Omega\setminus\Omega_{\frac{\varepsilon}{2}}} f(u) \le \int_\Omega |\nabla u|^{N-2}\nabla u\nabla\psi = \int_{\Omega_{\frac{\varepsilon}{2}}} |\nabla u|^{N-2}\nabla u\nabla\psi \le C.$$

Hence the estimate (3.1) is proved. ■

We also state here an adaptation of Theorems 2 and 6 in [15] to the N-Laplace operator Δ_N which will be useful in the sequel.

Lemma 3.2 *Let $u \in W^{1,N}_{loc}(\Omega)$ be a solution of*

$$-\Delta_N u \;=\; h(x) \quad in \quad \Omega\;.$$

where $h \in L^p(\Omega),\ p > 1$. Let $B_{2R} \subset \Omega$. Then

$$\|u\|_{L^\infty(B_R)} \le CR^{-1}(\|u\|_{L^N(B_{2R})} + RK)$$

where $C = C(N,p)$ and $K = \left(R^{N(p-1)/p}\|h\|_{L^p(\Omega)}\right)^{1/(N-1)}$.

4 Subcritical Case

In this section, we prove Theorem 1.1 under the assumptions (f_1), (f_2) and (f_3), i.e. in the *subcritical case.*

The proof will be based on Hölder's inequality in Orlicz spaces (cf. [1]): Let ψ and $\widetilde{\psi}$ be two complementary N-functions. Then

$$\Big|\int_\Omega h\,g\,\Big| \le 2\|h\|_\psi\|g\|_{\widetilde{\psi}}\;, \tag{4.1}$$

where $\|h\|_\psi$ and $\|g\|_{\widetilde{\psi}}$ denote the *Luxemburg* (or *gauge*) norms.

We first prove the following inequality:

Lemma 4.1 *Let $\gamma > 0$; then*

$$s\,t \le s(\log(s+1))^{1/\gamma} \; + \; t\,(e^{t^\gamma} - 1)\ , \quad \textit{for all} \;\; s,t \ge 0$$

Proof. Consider for fixed $t > 0$

$$\max_{s\ge 0}\{st - s(\log(s+1))^{1/\gamma}\}$$

In the maximum point s_t we have

$$t = (\log(s_t+1))^{1/\gamma} + \frac{s_t}{\gamma(s_t+1)}(\log(s_t+1))^{\frac{1}{\gamma}-1} \ge (\log(s_t+1))^{1/\gamma}$$

and hence $e^{t^\gamma} \ge s_t + 1$. Thus

$$\begin{aligned} \max_{s\ge 0}\ \{st - s(\log(s+1))^{1/\gamma}\} \;&=\; s_t t - s_t(\log(s_t+1))^{1/\gamma} \\ &\le\; s_t t \;\le\; t\,(e^{t^\gamma} - 1)\ . \end{aligned}$$

■

Note that for the N-function $\psi(s) = s(\log(s+1))^{1/\gamma}$, the *complementary N-function* $\widetilde{\psi}(t)$ is by definition given by

$$\widetilde{\psi}(t) = \max_{s\ge 0}\{st - s(\log(s+1))^{1/\gamma}\}\ .$$

The above Lemma shows that $\varphi(t) := t\,(e^{t^\gamma} - 1) \ge \widetilde{\psi}(t)$, for all $t \ge 0$, and hence $\|g\|_{\widetilde{\psi}} \le \|g\|_\varphi$, and so the Hölder inequality (4.1) is valid also for the gauge norm φ in place of $\widetilde{\psi}$:

$$\Big|\int_\Omega h\,g\,\Big| \le 2\|h\|_\psi \|g\|_\varphi\ , \tag{4.2}$$

Let now $u \in W_0^{1,N}(\Omega)$ be a weak solution of (1.1), denote

$$\gamma = \frac{N}{N-1} - \alpha\ , \quad \text{and} \quad \beta = \frac{\alpha}{\gamma}\ ,$$

and consider

$$\int_\Omega |\nabla u|^N = \int_\Omega f(u)u \;=\; \int_\Omega \frac{f(u)}{u^\beta}\; u^{1+\beta} \le \int_\Omega \frac{f(u)}{u_\beta}\chi_u\; u^{1+\beta} + c\ , \tag{4.3}$$

where χ_u is the characteristic function of the set $\{x \in \Omega\ :\ u(x) \ge 1\}$. By (4.2) we conclude that

$$\int_\Omega |\nabla u|^N \le 2\big\|u^{1+\beta}\big\|_\varphi\ \big\|\frac{f(u)}{u^\beta}\chi_u\big\|_\psi + c\ . \tag{4.4}$$

We now estimate the two Orlicz-norms in (4.4):

First note that there exists $d_\gamma > 0$ such that $\varphi(t) = t\,(e^{t^\gamma} - 1) \le e^{d_\gamma t^\gamma} - 1$, and hence

$$\begin{aligned} \|u^{1+\beta}\|_\varphi &= \inf\ \left\{k > 0\ :\ \textstyle\int_\Omega \varphi(\frac{u^{1+\beta}}{k}) \le 1\right\} \\ &\le \inf\ \left\{k > 0 : \int_\Omega \left(e^{d_\gamma \left(\frac{u^{1+\beta}}{k}\right)^\gamma} - 1\right) \le 1\right\} \\ &= \inf\ \left\{k > 0 : \int_\Omega \left(e^{d_\gamma \frac{u^{\frac{N}{N-1}}}{k^\gamma}} - 1\right) \le 1\right\}\ , \end{aligned} \tag{4.5}$$

since $(1+\beta)\gamma = \gamma + \alpha = N/(N-1)$. Now recall the Trudinger-Moser inequality which says that

$$\sup_{\|u\|_{W_0^{1,N}} \le 1} \int_\Omega e^{\alpha |u|^{N/(N-1)}} dx \quad < +\infty\ ,\quad \text{if}\ \ \alpha \le \alpha_N\ , \tag{4.6}$$

where $\alpha_N = N\omega_N^{1/(N-1)}$, and ω_N is the measure of the unit sphere in $\mathbb{R}^N$. Thus, if we take $k^\gamma = \frac{d_\gamma}{\alpha_N} \|\nabla u\|_{L^N(\Omega)}^{N/(N-1)}$ in (4.5), we see that the last integral in (4.5) is finite, and it becomes smaller than 1 if we choose $k^\gamma = c\,\|\nabla u\|_{L^N(\Omega)}^{N/(N-1)}$, for $c > 0$ suitably large, since φ is a convex function. Thus, we get

$$\|u^{1+\beta}\|_\varphi \le c\,\|\nabla u\|_{L^N(\Omega)}^{\frac{N}{N-1}\frac{1}{\gamma}}\ .$$

Next, we show that $\frac{\alpha}{\gamma} = \beta$ and (3.1) imply

$$\Big\|\frac{f(u)}{u^\beta}\chi_u\Big\|_\psi \le \int_\Omega d\,f(u) \le C\,.$$

Indeed, assumption (f_3) implies

$$\begin{aligned} \Big\|\frac{f(u)}{u^\beta}\chi_u\Big\|_\psi &= \inf\left\{k > 0 : \int_\Omega \frac{f(u)}{ku^\beta}\chi_u\Big(\log\big(1 + \frac{f(u)}{k\,u^\beta}\chi_u\big)\Big)^{\frac{1}{\gamma}} \le 1\right\} \\ &\le \inf\left\{k > 1 : \int_\Omega \frac{f(u)}{ku^\beta}\chi_u\Big(\log\big(1 + f(u)\big)\Big)^{\frac{1}{\gamma}} \le 1\right\} \\ &\le \inf\left\{k > 1 : \int_\Omega \frac{f(u)}{k\,u^\beta}\chi_u\big(\log(c\,e^{u^\alpha})\big)^{\frac{1}{\gamma}} \le 1\right\} \\ &\le \inf\left\{k > 1 : \int_\Omega \frac{f(u)}{k}\,d\,u^{\frac{\alpha}{\gamma}-\beta} \le 1\right\} \\ &\le \int_\Omega d\,f(u) \le C\,. \end{aligned}$$

Hence, joining these estimates, we conclude by (4.4) that

$$\|\nabla u\|_{L^N(\Omega)}^N \le C\,\|\nabla u\|_{L^N(\Omega)}^{\frac{N}{N-1}\frac{1}{\gamma}} + c\ .$$

Finally, note that $\alpha < 1$ implies that $\dfrac{N}{N-1}\,\dfrac{1}{\gamma} < N$, and so

$$\|\nabla u\|_{L^N(\Omega)} \leq C_N\,, \tag{4.7}$$

for any solution positive $u \in W^{1,N}(\Omega)$, with C_N depending only on N and Ω.

To obtain also a uniform L^∞-bound, we proceed as follows: Let $p > 1$, then given $\varepsilon > 0$ there exists $C(\varepsilon)$ such that

$$p\, s^{\alpha} \leq \varepsilon s^{\frac{N}{N-1}} + C(\varepsilon)\,.$$

Thus we can estimate

$$\int_\Omega |f(u)|^p \;\leq\; C_1(\varepsilon) \int_\Omega e^{\varepsilon |u|^{\frac{N}{N-1}}}\;.$$

Now, choosing $\epsilon > 0$ such that $\varepsilon\, C_N^{N/(N-1)} \leq \alpha_N$, the estimate (4.7) and the Trudinger–Moser inequality imply

$$\int_\Omega |f(u)|^p \quad\leq\quad C_1\,(\varepsilon) \int_\Omega e^{\varepsilon C_N^{\frac{N}{N-1}} \left|\frac{u}{\|\nabla u\|_{L^N(\Omega)}}\right|^{\frac{N}{N-1}}} \leq C\;.$$

And so, since $\displaystyle\int_\Omega |f(u)|^p \leq C$, we have by Lemma 3.2 that $\|u\|_{L^\infty(K)} \leq C = C(K)$ for every compact $K \subset\subset \Omega$. We are finished, since in Section 3 we have proved a priori estimates near the boundary.

5 Critical Case

In this section, we will prove Theorem 1.1 under assumptions (f_1) and (f_4). It is convenient to introduce the following number

$$d_N = \inf_{X \neq Y} \frac{\langle |X|^{N-2}X - |Y|^{N-2}Y, X - Y\rangle}{|X - Y|^N}\,. \tag{5.1}$$

By Proposition 4.6 of [10] we know that $d_N \geq \left(\dfrac{2}{N}\right)\left(\dfrac{1}{2}\right)^{N-2}$. Also, by taking $Y = 0$ we see that $d_N \leq 1$.

We will use the following standard comparison result

Lemma 5.1 *Suppose that u, $v \in W^{1,N}(\Omega) \cap C(\overline{\Omega})$ verify $-\Delta_N\, u \leq -\Delta_N\, v$ weakly in Ω, that is*

$$\int_\Omega \langle |\nabla u|^{N-2}\nabla u - |\nabla v|^{N-2}\nabla v, \nabla\phi\rangle \leq 0\;,$$

for all $\phi \in W_0^{1,N}$ such that $\phi \geq 0$ in Ω. If $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω.

Proof.
By taking $\phi = (u - v)^+$ we get

$$d_N \int_{\{u \geq v\}} |\nabla(u - v)|^N \leq \int_{\{u \geq v\}} \langle |\nabla u|^{N-2}\nabla u - |\nabla v|^{N-2}\nabla v, \nabla(u - v)\rangle \leq 0\;,$$

where d_N is given by (5.1). This inequality implies $u \le v$ in Ω. ■

We also need the following results by Ren and Wei [15] (Lemmas 4.1 and 4.3), which generalize the corresponding inequality for $N = 2$ of Brezis-Merle.

Lemma 5.2 *Let $u \in W^{1,N}(\Omega)$ verifying $-\Delta_N u = h$ in Ω and $u = 0$ on $\partial\Omega$, where $h \in L^1(\Omega) \cap C^0(\Omega)$ is nonnegative. Then, for every δ with $0 < \delta < N\omega_N^{\frac{1}{N-1}}$*

$$\int_\Omega e^{\frac{(N\omega_N^{\frac{1}{N-1}} - \delta)}{\|h\|_{L^1(\Omega)}^{\frac{1}{N-1}}}|u|} \le \frac{N\omega_N^{\frac{1}{N-1}}|\Omega|}{\delta} ,$$

where ω_N denotes the surface measure of the unit sphere in $\mathbb{R}^N$.

Lemma 5.3 *Let $u \in W^{1,N}(\Omega)$ verifying $-\Delta_N u = h$ in Ω and $u = g$ on $\partial\Omega$, where $h \in L^1(\Omega) \cap C^0(\Omega)$ and $g \in L^\infty(\Omega)$. Let $\phi \in W^{1,N}(\Omega)$ such that $\Delta_N \phi = 0$ in Ω and $\phi = g$ on $\partial\Omega$. Then, for every δ with $0 < \delta < N\omega_N^{\frac{1}{N-1}}$*

$$\int_\Omega e^{\frac{(N\omega_N^{\frac{1}{N-1}} - \delta) d_N^{\frac{1}{N-1}}}{\|h\|_{L^1(\Omega)}^{\frac{1}{N-1}}}|u-\phi|} \le \frac{N\omega_N^{\frac{1}{N-1}}|\Omega|}{\delta} .$$

Proof of Theorem 1.1 (*critical case*)
Suppose by contradiction that there is no a priori estimate, then there would exist a sequence $\{u_n\}_n \subset W^{1,N}(\Omega) \cap C^{1,\alpha}(\bar{\Omega})$ of weak solutions of (1.1) such that $\|u_n\|_{L^\infty(\Omega)} \to \infty$. Observe that by Proposition 3.1 there exists a constant C such that $\int_\Omega f(u_n) \le C$.

We may assume that $f(u_n)$ converges in the sense of measures on Ω to some nonnegative bounded measure μ, that is

$$\int_\Omega f(u_n)\,\psi \to \int_\Omega \psi\, d\mu, \text{ for all simple functions } \psi.$$

As in [4], let us introduce the concept of *regular point.* We say that $x_0 \in \Omega$ is a regular point with respect to μ if there exists an open neighborhood $V \subset \Omega$ of x_0 such that

$$\int_\Omega \chi_V\, d\mu < N^{N-1}\,\omega_N .$$

Next, we define the set A as follows: $x \in A$ if and only if there exists an open neighborhood $U \subset \Omega$ of x such that

$$\int_\Omega \chi_U\, d\mu < N^{N-1}\,\omega_N\, d_N ,$$

where d_N is the constant introduced in (5.1).

Because $d_N \le 1$, we have that the set A contains only regular points. Also, note that there is only a finite number of points $x \in \Omega \setminus A$; in fact, if $x \in \Omega \setminus A$ then

$$\int_{B_R(x)} d\mu \ge N^{N-1}\omega_N\, d_N, \text{ for all } R > 0 \text{ such that } B_R(x) \subset \Omega,$$

which implies $\mu(\{x\}) \geq N^{N-1}\omega_N\, d_N$. Hence, since

$$\sum_{x\in\Omega\setminus A} \mu(\{x\}) \leq \mu(\Omega) = \int_\Omega d\mu \leq C\ ,$$

the set of points in $\Omega \setminus A$ is finite.

Before finishing the proof we need two claims.

Claim 1. Let x_0 be a regular point, then there exist C and R such that for all $n \in \mathbb{N}$

$$\|u_n\|_{L^\infty(B_R(x_0))} \leq C$$

Proof of Claim 1. We divide the proof into two cases.

Case 1: $x_0 \in A$
By the definitions of the set A and the measure μ, there exist R, δ and $n_0 > 0$ such that for all $n > n_0$ we have

$$\left(\int_{B_R(x_0)} f(u_n)\right)^{\frac{1}{N-1}} < \left(N\omega_N^{\frac{1}{N-1}} - \delta\right) d_N^{\frac{1}{N-1}}\ . \tag{5.2}$$

Let ϕ_n be satisfying

$$\begin{cases} -\Delta_N \phi_n &= 0 \text{ in } B_R \\ \phi_n &= u_n \text{ on } \partial B_R\,. \end{cases}$$

Then $\phi_n \leq u_n$ in B_R by Lemma 5.1. Since $c \geq \int_\Omega f(u_n) \geq c_1 \int_\Omega e^{u_n}$ by (f_4), we have $\int_\Omega u_n^N < C'$ and thus $\int_\Omega \phi_n^N < C'$. Now, by using Lemma 3.2 we have

$$\|\phi_n\|_{L^\infty(B_{\frac{R}{2}})} \leq CR^{-1}(\|\phi_n\|_{L^N(B_R)} + c) \leq C''\ . \tag{5.3}$$

By applying Lemma 5.3, we get

$$\int_{B_R} e^{\frac{(N\omega_N^{\frac{1}{N-1}} - \delta')}{\|f(u_n)\|_{L^1(B_R)}^{\frac{1}{N-1}}} d_N^{\frac{1}{N-1}} |u_n - \phi_n|} < \frac{N\omega_N^{\frac{1}{N-1}} R^N C}{\delta'}$$

for any $\delta' \in (0, N\omega_N^{1/(N-1)})$. Taking δ' small enough we have by (5.2) that $q = \frac{(N\omega_N^{\frac{1}{N-1}} - \delta')}{\|f(u_n)\|_{L^1(B_R)}^{\frac{1}{N-1}}} d_N^{\frac{1}{N-1}} > 1$, and hence we get

$$\int_{B_{\frac{R}{2}}} e^{q|u_n - \phi_n|} \leq \int_{B_R} e^{q|u_n - \phi_n|} < K.$$

By (5.3) we conclude that $\int_{B_{\frac{R}{2}}} e^{qu_n} \leq K'$, and by (f_4) we get $\int_{B_{\frac{R}{2}}} f(u_n)^q < K$. Again by Lemma 3.2 we infer

$$\begin{aligned} \|u_n\|_{L^\infty(B_{\frac{R}{4}})} &\leq CR^{-1}\left(\|u_n\|_{L^N(B_{\frac{R}{2}})} + RK\right) \\ &\leq K_1\,, \end{aligned}$$

where $K_1 = K\Big(R, \|u_n\|_{L^N(B_{\frac{R}{2}})}, \|f(u_n)\|_{L^q(B_{\frac{R}{2}})}\Big)$

Case 2: $x_0 \notin A$

Since $\Omega \setminus A$ is finite we can choose $R > 0$ such that $\partial B_R(x_0) \subset A$. Taking $x \in \partial B_R(x_0)$, by case 1 there is $r = r(x)$ such that for all $n \in \mathbb{N}$

$$\|u_n\|_{L^\infty(B_{r(x)}(x))} \leq c(x).$$

This implies by compactness, for some $k \in \mathbb{N}$

$$\partial B_R \subseteq \bigcup_{i=1}^{k} B_{r(x_i)}(x_i).$$

Now, if $y \in \partial B_R$, then $y \in B_{r(x_{i_0})}(x_{i_0})$, for some $1 \leq i \leq k$. Hence

$$\|u_n\|_{L^\infty(\partial B_R)} \leq \max_{i=1,\cdots,k} C(x_i) =: K \text{ for all } n \in \mathbb{N}\,.$$

Let U_n be the solution of

$$\begin{cases} -\Delta_N U_n &= f(u_n) \text{ in } B_R \\ U_n &= K \text{ on } \partial B_R, \end{cases}$$

which is equivalent to

$$\begin{cases} -\Delta_N (U_n - K) &= f(u_n) \text{ in } B_R \\ U_n - K &= 0 \text{ on } \partial B_R\,. \end{cases}$$

Therefore

$$U_n \geq u_n \ , \quad \text{on} \quad B_R \ ,$$

by Lemma 5.1. Thus by applying Lemma 5.2 we have

$$\int_{B_R} e^{\frac{(N\omega_N^{\frac{1}{N-1}} - \delta')}{\|f(u_n)\|_{L^1}^{\frac{1}{N-1}}}|U_n - K|} \leq \frac{N\omega_N^{\frac{1}{N-1}} C R^N}{\delta'} \tag{5.4}$$

for any $\delta' \in (0, N\omega_N^{1/(N-1)})$.

Since x_0 is a regular point, there exist $R_1 < R$ and $n_0 \in \mathbb{N}$ such that for every $n > n_0$ we have for some $\delta > 0$

$$\Big(\int_{B_{R_1}(x_0)} f(u_n)\Big)^{\frac{1}{N-1}} < N\omega_N^{\frac{1}{N-1}} - \delta\,.$$

Taking $\delta' > 0$ sufficiently small, we have

$$1 < q = \frac{N\omega_N^{\frac{1}{N-1}} - \delta'}{N\omega_N^{\frac{1}{N-1}} - \delta} < \frac{N\omega_N^{\frac{1}{N-1}} - \delta'}{\|f(u_n)\|_{L^1}^{\frac{1}{N-1}}}\,,$$

and hence by (5.4)

$$\int_{B_{R_1}} e^{q|U_n - K|} < C \ , \quad \text{and then} \quad \int_{B_{R_1}} e^{qU_n} < K' \ ;$$

this implies

$$\int_{B_{R_1}} e^{qu_n} \leq K''' .$$

and therefore by (f_4)

$$\int_{B_{R_1}} f(u_n)^q \leq K(q) \ , \quad \text{and also} \quad \|u_n\|_{L^N(B_{R_1})} \leq C \ .$$

Hence, by Lemma 4.1

$$\begin{aligned} \|u_n\|_{L^\infty(B_{\frac{R_1}{2}})} &\leq C\, R_1^{-1}(\|u_n\|_{L^N(B_{R_1})} + C\|f(u_n)\|_{L^q(B_{R_1})}) \\ &< K''' . \end{aligned}$$

This finishes the proof of Claim 1.

Next, we define

$$\Sigma = \{x \in \Omega : x \text{ is not regular for } \ \mu\} .$$

We note that $\Sigma \subset \Omega \backslash A$ where A is defined in the proof of Theorem 1.1. Hence, also Σ has finitely many elements.

The second claim is

Claim 2. $\Sigma = \emptyset$.

Proof of Claim 2. Arguing by contradiction, let us assume that there exists $x_0 \in \Sigma$ and $R > 0$ such that

$$B_R(x_0) \cap \Sigma = \{x_0\}.$$

We recall that u_n verifies

$$\begin{cases} -\Delta_N u_n &= f(u_n) & \text{in } B_R(x_0) \\ \quad\ u_n &> 0 & \text{on } \partial B_R(x_0) . \end{cases}$$

By the previous claim and because all the points are regular in $B_R(x_0) \backslash \{x_0\}$, passing to a subsequence we can assume that $u_n \to u \ \ C^1-$uniformly on compact subsets of $B_R(x_0) \setminus \{x_0\}$. Consider the function $w(x) = N \log \frac{R}{|x-x_0|}$, which satisfies

$$\begin{cases} -\Delta_N w &= N^{N-1}\omega_N \delta_{x_0} & \text{in } B_R(x_0) \\ \quad\ w &= 0 & \text{on } \partial B_R(x_0) . \end{cases}$$

For $k > 0$, and define the functions

$$T_k(s) = \begin{cases} 0 & \text{if} \quad s < 0 \ , \\ s & \text{if} \quad 0 \leq s \leq k \ , \\ k & \text{if} \quad k < s \ . \end{cases}$$

Consider now the functions given by $z_n^{(k)} = T_k\left(w-u_n\right)$; because the functions u_n are positive we have that $z_n^{(k)} \in W_0^{1,N}\left(B_R\right)$, and $z_n^{(k)}\left(x_0\right) = k$, for all $n \in \mathbb{N}$. Also

$$z_n^{(k)} \to z^{(k)} = \begin{cases} T_k(w-u), & \text{if } x \neq x_0 \\ k\,, & \text{if } x = x_0. \end{cases}$$

Note that $z^{(k)}$ is a measurable function. We have

$$\int_{B_R} \left(|\nabla w|^{N-2} \nabla w - |\nabla u_n|^{N-2} \nabla u_n \right) \nabla z_n^{(k)} = N^{N-1}\omega_N k - \int_{B_R} f\left(u_n\right) z_n^{(k)}\ . \quad (5.5)$$

Now set $d\mu_n = f(u_n)dx$; then we may apply the following Proposition which is a generalization of Fatou's Lemma (see e.g. Royden, Real Analysis, Proposition 11.17):

Proposition: *Suppose that μ_n is a sequence of (positive) measures which converges to μ setwise, and g_n is a sequence of measurable, nonnegative functions that converge pointwise to g. Then*

$$\liminf_{n\to\infty} \int g_n\, d\mu_n \geq \int g\, d\mu$$

Hence, we can write

$$\int_{B_R} f(u_n) z_n^{(k)} dx = \int z_n^{(k)} d\mu_n$$

and conclude that

$$\begin{aligned} \liminf_{n\to\infty} \int_{B_R} f(u_n) z_n^{(k)} &= \liminf_{n\to\infty} \int z_n^{(k)} d\mu_n \\ &\geq \int z^{(k)} d\mu \\ &\geq \int_{\{x_0\}} z^{(k)} d\mu \\ &\geq N^{N-1}\omega_N\, k\,, \end{aligned}$$

where we have used that $z^{(k)}(x_0) = k$ and $\mu(x_0) \geq N^{N-1}\omega_N$, because $x_0 \in \Sigma$.

Thus we obtain from (5.5) that for all $k \in \mathbb{N}$

$$\int_{B_R} \left(|\nabla w|^{N-2} \nabla w - |\nabla u|^{N-2} \nabla u \right) \nabla z^{(k)} \leq 0\ ,$$

that is

$$\int_{B_R \cap \{0 \leq w-u \leq k\}} \left(|\nabla w|^{N-2} \nabla w - |\nabla u|^{N-2} \nabla u \right) \nabla (w-u) \leq 0\ ,\ k \in \mathbb{N}\ .$$

By inequality (5.1) we obtain

$$d_N \int_{B_R \cap \{0 \leq w-u \leq k\}} |\nabla (w-u)|^N \leq 0\ ,\ k \in \mathbb{N}\ .$$

Finally, letting $k \to \infty$, we conclude that

$$d_N \int_{B_R} \left|\nabla (w-u)^+\right|^N \le 0 \ .$$

Because we know that $(w-u) \le 0$ on ∂B_R, the above inequality implies that $w \le u$ in $W_0^{1,N}(B_R)$, and therefore we conclude that

$$\begin{aligned} \liminf_{n\to+\infty} \int_{B_R} f(u_n) &\ge \liminf_{n\to+\infty} \int_{B_R} c_1\, e^{u_n} \\ &\ge c_1 \int_{B_R} e^u \\ &\ge \int_{B_R} \frac{C}{|x-x_0|^N} = +\infty \end{aligned}$$

This is a contradiction and the proof of Claim 2 is complete.

To finish the proof of Theorem 1.1, we observe that there exists a sequence x_n of points in Ω such that $u_n(x_n) = \|u_n\|_{L^\infty(\Omega)}$ and we can assume that $x_n \to x_0$. Because we have an priori estimate near the boundary of Ω, we have $x_0 \in \Omega$. It is easy to see that for all $R > 0$ we have

$$\lim_{n\to+\infty} \|u_n\|_{L^\infty(B_R)} = +\infty.$$

By Claim 1, we conclude that x_0 is not a regular point, but this is impossible by Claim 2. Hence there are no blow-up points. ■

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