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Transfer matrix for Kogut-Susskind fermions in the spin basis

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In the absence of interaction, it is well known that the Kogut-Susskind regularizations of fermions in the spin and flavor basis are equivalent to each other. In this paper, we clarify the difference between the two formulations in the presence of interaction with gauge fields. We then derive an explicit expression of the transfer matrix in the spin basis by a unitary transformation on that one in the flavor basis which is known. The essential key ingredient is the explicit construction of the fermion Fock space for variables which live on blocks. Therefore, the transfer matrix generates time translations of two lattice units.

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I. INTRODUCTION

The naive discretization of the Dirac equation on the lattice [1] leads to a replication of the fermionic states, known as lattice fermion *doubling* [2]. The doublers appear as spurious poles in the fermion propagator at the nonzero corners of the Brillouin zone. The Wilson way of removing the doublers is to give them a mass which becomes infinite in the continuum limit, at the cost of an explicit breaking of chiral invariance on the lattice [2].

In the Kogut-Susskind [3–6] lattice formulation for relativistic fermions ([7], Chap. 4) the doublers are instead interpreted as physical fields by the introduction of additional quantum numbers. This has been done in two ways. In the former approach, first the fermion field is reduced to a single component per site by a procedure called *spin diagonalization*, and, for this reason, this method is referred to as the one in the *spin basis*. Afterward, spin and flavor degrees of freedom are associated to different corners of an elementary hypercube on the lattice [8–10], and, therefore, sometimes fermions in this formulation are said to be *staggered*. In the latter approach [11–13], said in the *flavor basis*, the additional quantum numbers, called *taste*, are associated, together with the spin, with blocks corresponding to the hypercubes of the spin basis of size twice the lattice spacing.

In the absence of coupling with gauge fields, these forms are changed into one another by a linear transformation on the fermion fields, but in the presence of gauge fields, they are not equivalent, as we shall make clear in the following. Their difference is of consequence in the construction of the corresponding transfer matrices.

For Kogut-Susskind fermions in the flavor basis, a simple operator realization of the transfer matrix is known [14]. It has been built in close analogy with the case of Wilson fermions [15–19] (see also Ref. [20]), the only difference being that it performs time translations by one block instead of one lattice spacing.

The situation is more complex for Kogut-Susskind fermions in the spin basis [11,12,21] because all attempts at constructing a positive definite transfer matrix that performs time translations by a single lattice spacing failed. The difficulty was circumvented by looking at time translations by two lattice spacings. Here, we meet with a subtlety. We must distinguish whether the transfer matrix acts on a Fock space built on one or two time slices. In the first case, we can get the operator which translates by one lattice spacing by taking the square root of the transfer matrix which translates by two lattice spacings. In the second case, instead, translations by one lattice spacing are not defined at all. This seems to be the case with Kogut-Susskind fermions, but the necessary construction of the Fock space on blocks, in the spin basis, has not been explicitly performed.

We became interested in a formulation of the transfer matrix in the spin basis in the framework of relativistic field theories of fermions whose partition function is dominated by bosonic composites [22]. We deem that this question is of more general interest. First, a positive transfer matrix means that unitarity is guaranteed also at finite lattice spacing. We will show that this requires that the gauge fields be defined on blocks, not only in the flavor basis but in the spin basis as well. Hence, fermion fields should transform, under gauge transformations, accordingly. As a practical consequence, correlation functions will not show oscillations. Moreover, a positive transfer matrix provides a Hamiltonian formulation directly on the lattice. Therefore, for example, it can be adopted as the starting point for the study of the spectrum. Or, as we shall see, it can be used to derive approximations based on a variational principle.

This subject became for us more relevant in the development of an approach to QCD hadronization (meant as the

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replacement of the QCD degrees of freedom by hadronic ones), called the nilpotency expansion, that makes use of the operator form of the transfer matrix [23–26]. It is based on the use of a Bogoliubov transformation on the fermion operators whose parameters depend on time and gauge fields in such a way that gauge invariance and other symmetries are trivially preserved. These parameters then become dynamical bosonic fields. At the lowest order in an expansion in the inverse of the number of fermion degrees of freedom entering in the bosonic composites (nilpotency index), interestingly enough, we got, after the integration of the fermion degrees of freedom, a statistical weight on gauge configurations, which does not suffer from the sign problem, also with a finite chemical potential [26].

We remark that, in spite of the mentioned approximations, our formulation at the lowest order in the nilpotency parameter has the features of a variational calculation, thus providing an upper bound for the ground-state energy.

Using Kogut-Susskind fermions, because of the lack of a convenient formulation of the transfer matrix, we were able to express our results only in the flavor basis. Numerical simulations are, instead, usually performed in the spin basis because they are much faster. We were thus motivated to find an operator form of the transfer matrix in this latter basis as well. Since, apparently, in any case, we should resign to time translations by one block, we decided to get an expression of the transfer matrix in the spin basis by a linear transformation from the flavor basis.

The presentation of our results is organized as follows. In Sec. II, we remind, for the convenience of the reader and in order to establish the notation, what is relevant to the present issue about the Kogut-Susskind regularization. We adopt the notations of Montvay and Münster [7] with some minor changes that will be specified. In Sec. III, we perform the transformation of the action from the flavor to the spin basis. Most of the results, with some qualification, are well known, but we think this section is a necessary preparation for Sec. IV, in which we perform the transformation of the transfer matrix and of the coherent states.

II. KOGUT-SUSSKIND FERMIONS

Let x_μ be the coordinates of hypercubic lattice sites, $0 \le x_\mu \le L_\mu - 1$, $0 \le \mu \le 3$ (Montvay and Münster in Ref. [7] use indices from 1 to 4), and y_μ the coordinates of hypercubic blocks. They are related by

$$x_\mu = 2y_\mu + \eta_\mu, \tag{1}$$

with $0 \le y_\mu \le L'_\mu - 1$, $L_\mu = 2L'_\mu$, and $\eta_\mu = 0,\ 1$ the position vectors within the block. The sum over lattice points can be split into the sum over the blocks and the sum over the sites within a block, that is

$$\sum_x = \sum_y \sum_\eta. \tag{2}$$

We denote by ψ_x the fermionic fields on the lattice sites and by $q_y^{\alpha a}$ the fields on the blocks. The latter have Dirac spinor indices $1 \le \alpha \le 4$, in Greek letters, and taste indices $1 \le a \le 4$, in Latin letters.

It is important to remark that the gauge transformations in the first case act at the sites of the basic lattice and in the second at the coordinates of the blocks

$$\psi_x \to g_x \psi_x, \qquad q_y^{\alpha a} \to g_y q_y^{\alpha a}. \tag{3}$$

While g_y is the same transformation for all x in a given block with coordinate y, g_x will, in general, change also within the same block.

A. The flavor basis

The gauge link variables on the blocks are denoted by $U_\mu(y)$. Under gauge transformations, they change according to the rule

$$U_\mu(y) \to g_y U_\mu(y) g^\dagger_{y+\hat\mu}. \tag{4}$$

The action of the fermion fields in the flavor basis can be written as

$$S(U) = 2^4 \sum_y \mathcal{L}_q(U), \tag{5}$$

where the factor 16 keeps into account the volume of the elementary cell when using variables defined on the blocks, and the Lagrangian in the flavor basis is

$$\mathcal{L}_q(U) := m\bar q_y(\mathbb{1}\otimes\mathbb{1})q_y + \sum_{\mu=0}^{3} \bar q_y \Big\{ \Big[(\gamma_\mu \otimes \mathbb{1}) \frac{1}{2}(\nabla_\mu^{(+)} + \nabla_\mu^{(-)}) - (\gamma_5 \otimes t_5 t_\mu)\Delta_\mu \Big] q \Big\}_y. \tag{6}$$

The flavour matrices t_μ are defined for $\mu = 0, \ldots, 3$ and $\mu = 5$ by

$$t_\mu := \gamma_\mu^T = t_\mu^\dagger, \tag{7}$$

and the other operators are defined in terms of translations on the blocks

$$[T_\mu^{(\pm)} f]_y := 2^4 \sum_{y'} \frac{1}{2^4} \delta_{y', y\pm\hat\mu} f(y') = f(y \pm \hat\mu) \tag{8}$$

and the identity on the blocks

$$[\mathbb{1} f]_y := 2^4 \sum_{y'} \frac{1}{2^4} \delta_{y',y} f(y) = f(y) \tag{9}$$

according to

$$\nabla_\mu^{(+)} := \frac{1}{2}(U_\mu T_\mu^{(+)} - \mathbb{1}), \qquad \nabla_\mu^{(-)} := \frac{1}{2}(\mathbb{1} - T_\mu^{(-)} U_\mu^\dagger) \tag{10}$$

$$\Delta_\mu := \frac{1}{2}(\nabla_\mu^{(+)} - \nabla_\mu^{(-)}) = \frac{1}{4}(U_\mu T_\mu^{(+)} + T_\mu^{(-)} U_\mu^\dagger - 2\mathbb{1}). \tag{11}$$

We can recognize that the projections of the fermionic field

$$q_+ = P_+ q, \qquad q_-^\dagger = P_- q, \tag{12}$$

where

$$P_\pm = \frac{1}{2}(\mathbb{1} \otimes \mathbb{1} \mp \gamma_0\gamma_5 \otimes t_5 t_0), \tag{13}$$

propagate forward/backward in time and, therefore, describe particles/antiparticles, respectively. Accordingly, we introduce creation and annihilation operators $\hat{q}_\pm^\dagger$, $\hat{q}_\pm$. They are defined at one and the same time, so that in addition to spin and flavor, they depend on the spatial position only, denoted by boldface letters. They satisfy canonical anticommutation relations

$$\begin{aligned} \{(\hat{q}_\pm^\dagger)_{\mathbf{y}_1}^{a\alpha}, (\hat{q}_\pm)_{\mathbf{y}_2}^{\beta b}\} &= \frac{1}{8}\delta_{\mathbf{y}_1\mathbf{y}_2} P_\pm^{\beta b,\alpha a}, \\ \{(\hat{q}_\pm^\dagger)_{\mathbf{y}_1}^{a\alpha}, (\hat{q}_\mp)_{\mathbf{y}_2}^{\beta b}\} &= 0. \end{aligned} \tag{14}$$

As the factor $\frac{1}{8}$ accounts for the spatial volume of the blocks, the above anticommutation relations become canonical in the basis in which $P_\pm$ are diagonal.

The transfer matrix corresponding to the flavor-Lagrangian (6) in the gauge $U_0 = \mathbb{1}$ is [14,27]

$$\mathcal{T}_{t,t+1} = \exp(\hat{q}_- N_t \hat{q}_+)^\dagger \exp(2\mu \hat{n}_B) \exp(\hat{q}_- N_{t+1} \hat{q}_+). \tag{15}$$

In the above equation, N_t is a matrix which depends on the time of the blocks only because it depends on the gauge link variables

$$N_t := N[U(t)], \tag{16}$$

and μ is the chemical potential

$$\hat{n}_B = 2^3 \sum_{\mathbf{y}} (\hat{q}_+^\dagger \hat{q}_+ - \hat{q}_-^\dagger \hat{q}_-)_{\mathbf{y}} \tag{17}$$

that we omitted for simplicity in the Lagrangian. By keeping into account the spatial volume factors,

$$\hat{q}_- N_t \hat{q}_+ = 64 \sum_{\mathbf{y}',\mathbf{y}} (\hat{q}_-)_{\mathbf{y}'} (N_t)_{\mathbf{y}'\mathbf{y}} (\hat{q}_+)_{\mathbf{y}} \tag{18}$$

$$\begin{aligned} N_{\mathbf{y}'\mathbf{y}} = -2\Big\{ & m(\gamma_0 \otimes \mathbb{1})\mathbb{1}_{\mathbf{y}'\mathbf{y}} + \sum_{k=1}^{3} (\gamma_0\gamma_k \otimes \mathbb{1}) \\ & \times [P_k^{(-)}\nabla_k^{(+)} + P_k^{(+)}\nabla_k^{(-)}]_{\mathbf{y}'\mathbf{y}} \Big\}, \end{aligned} \tag{19}$$

where

$$P_k^{(\pm)} = \frac{1}{2}(\mathbb{1} \otimes \mathbb{1} \pm \gamma_k\gamma_5 \otimes t_5 t_k), \tag{20}$$

and

$$\mathbb{1}_{\mathbf{y}'\mathbf{y}} := \frac{1}{8}\delta_{\mathbf{y}'\mathbf{y}}, \qquad (T_k^{(\pm)})_{\mathbf{y}'\mathbf{y}} := \frac{1}{8}\delta_{\mathbf{y}'\pm\hat{k},\mathbf{y}} \tag{21}$$

enter in the definitions of $(\nabla_k^{(\pm)})_{\mathbf{y}'\mathbf{y}}$.

Notice that

$$q_\pm^\dagger N q_\pm = 0. \tag{22}$$

B. The spin basis

For the sake of later comparison, we report the regularization of a Lagrangian in the spin basis. The gauge fields on the hypercubic lattice are denoted by $u_\mu(x)$ and transform according to

$$u_\mu(x) \to g_x u_\mu(x) g_{x+\hat{\mu}}^\dagger. \tag{23}$$

The Lagrangian in the spin basis is

$$\begin{aligned} \mathcal{L}_\psi(u) := & m\bar{\psi}_x\psi_x \\ & + \frac{1}{2}\sum_{\mu=0}^{3} \alpha_{x\mu}[\bar{\psi}_x u_\mu(x)\psi_{x+\hat{\mu}} - \bar{\psi}_{x+\mu} u_\mu^\dagger(x)\psi_x], \end{aligned} \tag{24}$$

where the signs $\alpha_{x\mu}$ are defined for $\mu = 0, \ldots, 3$ by

$$\alpha_{x\mu} := (-1)^{x_0+\cdots+x_{\mu-1}}. \tag{25}$$

There is no direct way of identifying forward and backward movers. This is the difficulty encountered in the construction of a transfer matrix in operator form for this Lagrangian. Indeed, as far as we know, such a construction has been achieved only after a reduction of the Lagrangian itself, in which the fermion fields and their conjugates live on odd and, respectively, even sites [11].

At the classical level, however, the fields in the spin and flavor basis are related according to

$$q_y^{\alpha a} = \frac{1}{8}\sum_\eta \Gamma_{\eta;\alpha a}\psi_{2y+\eta} \tag{26}$$

$$\bar{q}_y^{a\alpha} = \frac{1}{8}\sum_\eta \bar{\psi}_{2y+\eta}\Gamma_{\eta;a\alpha}^\dagger, \tag{27}$$

where

$$\Gamma_\eta := \gamma_0^{\eta_0}\gamma_1^{\eta_1}\gamma_2^{\eta_2}\gamma_3^{\eta_3}. \tag{28}$$

The matrices Γ satisfy the relations

$$\frac{1}{4}\operatorname{tr}(\Gamma_\eta^\dagger\Gamma_{\eta'}) = \delta_{\eta\eta'} \tag{29}$$

$$\frac{1}{4}\sum_\eta \Gamma_{\eta;b\beta}^\dagger\Gamma_{\eta;\alpha a} = \delta_{ba}\delta_{\beta\alpha} \tag{30}$$

that allow us to invert Eqs. (27),

$$\psi_{2y+\eta} = 2\,\mathrm{tr}(\Gamma_\eta^\dagger q_y) \tag{31}$$

$$\bar{\psi}_{2y+\eta} = 2\,\mathrm{tr}(\bar{q}_y \Gamma_\eta). \tag{32}$$

We will use these relationships in order to derive an action and a transfer matrix in the spin basis from those in the flavor basis.

III. TRANSFORMATION OF THE LAGRANGIAN

In this section, we express the Lagrangian (6) in the spin basis using the transformations (27)

$$\sum_x \mathcal{L}'_\psi(U) := 2^4 \sum_y \mathcal{L}_q(U). \tag{33}$$

While in the absence of gauge interaction $\mathcal{L}'_\psi$ coincides with $\mathcal{L}_\psi$, reported in Eq. (24), we shall see that this does not occur, in general, in the presence of gauge fields.

The mass term of the action is proportional to

$$\begin{aligned} 2^4 \sum_y \bar{q}_y q_y &= \frac{1}{4} \sum_y \sum_\eta \sum_{\eta'} \bar{\psi}_{2y+\eta}\, \mathrm{tr}(\Gamma_{\eta'}^\dagger \Gamma_\eta) \psi_{2y+\eta'} \\ &= \sum_x \bar{\psi}_x \psi_x. \end{aligned} \tag{34}$$

In order to derive the kinetic term, we shall use the relations

$$\begin{aligned} \sum_\alpha \gamma_\mu^{\alpha'\alpha} \Gamma_{\eta:\alpha a} = {}& \delta_{0\eta_\mu} \alpha_{\eta\mu} \Gamma_{\eta+\hat{\mu}:\alpha' a} \\ &+ \delta_{1\eta_\mu} \alpha_{\eta\mu} \Gamma_{\eta-\hat{\mu}:\alpha' a} \end{aligned} \tag{35}$$

$$\begin{aligned} \sum_{\alpha,a} \gamma_5^{\alpha'\alpha} (t_5 t_\mu)^{a'a} \Gamma_{\eta:\alpha a} = {}& -\delta_{0\eta_\mu} \alpha_{\eta\mu} \Gamma_{\eta+\hat{\mu}:\alpha' a'} \\ &+ \delta_{1\eta_\mu} \alpha_{\eta\mu} \Gamma_{\eta-\hat{\mu}:\alpha' a'}. \end{aligned} \tag{36}$$

From the definition, Eq. (28), the relation (35) soon follows, and

$$\Gamma_\eta \gamma_\mu = (-1)^{\eta_0+\eta_1+\eta_2+\eta_3} (-1)^{\eta_\mu} \gamma_\mu \Gamma_\eta \tag{37}$$

so that

$$\Gamma_\eta \gamma_5 = (-1)^{\eta_0+\eta_1+\eta_2+\eta_3} \gamma_5 \Gamma_\eta. \tag{38}$$

Hence

$$\begin{aligned} \sum_{\alpha,a} \gamma_5^{\alpha'\alpha} (t_5 t_\mu)^{a'a} \Gamma_{\eta:\alpha a} &= (\gamma_5 \Gamma_\eta \gamma_\mu \gamma_5)_{\alpha' a'} \\ &= -(\gamma_5 \Gamma_\eta \gamma_5 \gamma_\mu)_{\alpha' a'} \\ &= -(-1)^{\eta_\mu} (\gamma_\mu \Gamma_\eta)_{\alpha' a'}, \end{aligned} \tag{39}$$

which, together with Eq. (35), implies the relation (36).

The kinetic term is proportional to

$$\frac{16}{4} \sum_y \sum_\mu \{\bar{q}_y (\gamma_\mu \otimes 1)[U_\mu(y) q_{y+\hat{\mu}} - U_\mu^\dagger(y-\hat{\mu}) q_{y-\hat{\mu}}] - \bar{q}_y (\gamma_5 \otimes t_5) t_\mu [U_\mu(y) q_{y+\hat{\mu}} + U_\mu^\dagger(y-\hat{\mu}) q_{y-\hat{\mu}} - 2q_y]\}; \tag{40}$$

that is

$$\begin{aligned} \frac{1}{16} \sum_y \sum_\mu \sum_{\eta,\eta'} \sum_{\alpha,\alpha',a,a'} & \bar{\psi}_{2y+\eta'} \Gamma^\dagger_{\eta':a'\alpha'} [U_\mu(y) (\gamma_\mu^{\alpha'\alpha} \delta^{a'a} - \gamma_5^{\alpha'\alpha} (t_5 t_\mu)^{a'a}) \Gamma_{\eta:\alpha a} \psi_{2y+2\hat{\mu}+\eta} \\ & - U_\mu^\dagger(y-\hat{\mu}) (\gamma_\mu^{\alpha'\alpha} \delta^{a'a} + \gamma_5^{\alpha'\alpha} (t_5 t_\mu)^{a'a}) \Gamma_{\eta:\alpha a} \psi_{2y-2\hat{\mu}+\eta} + 2\gamma_5^{\alpha'\alpha} (t_5 t_\mu)^{a'a} \Gamma_{\eta:\alpha a} \psi_{2y+\eta}], \end{aligned} \tag{41}$$

which because of Eqs. (35) and (36) becomes

$$\begin{aligned} \frac{1}{8} \sum_y \sum_\mu \sum_{\eta,\eta'} \sum_{\alpha,\alpha',a,a'} & \bar{\psi}_{2y+\eta'} \Gamma^\dagger_{\eta':a'\alpha'} \alpha_{\eta\mu} [U_\mu(y) \delta_{0\eta_\mu} \Gamma_{\eta+\hat{\mu}:\alpha' a'} \psi_{2y+2\hat{\mu}+\eta} \\ & - U_\mu^\dagger(y-\hat{\mu}) \delta_{1\eta_\mu} \Gamma_{\eta-\hat{\mu}:\alpha' a'} \psi_{2y-2\hat{\mu}+\eta} + (-\delta_{0\eta_\mu} \Gamma_{\eta+\hat{\mu}:\alpha' a'} + \delta_{1\eta_\mu} \Gamma_{\eta-\hat{\mu}':\alpha' a'}) \psi_{2y+\eta}], \end{aligned} \tag{42}$$

and, performing the trace on spinor and flavor indices (30),

$$\begin{aligned} \frac{1}{2} \sum_y \sum_\mu \sum_{\eta,\eta'} & \bar{\psi}_{2y+\eta'} \alpha_{\eta\mu} [U_\mu(y) \delta_{0\eta_\mu} \delta_{\eta',\eta+\hat{\mu}} \psi_{2y+2\hat{\mu}+\eta} - U_\mu^\dagger(y-\hat{\mu}) \delta_{1\eta_\mu} \delta_{\eta',\eta-\hat{\mu}} \psi_{2y-2\hat{\mu}+\eta} \\ & + (-\delta_{0\eta_\mu} \delta_{\eta',\eta+\hat{\mu}} + \delta_{1\eta_\mu} \delta_{\eta',\eta-\hat{\mu}}) \psi_{2y+\eta}], \end{aligned} \tag{43}$$

and, performing the sum over η',

$$\begin{aligned} \frac{1}{2} \sum_y \sum_\eta \sum_\mu \alpha_{\eta\mu} & [\delta_{0\eta_\mu} \bar{\psi}_{2y+\eta+\hat{\mu}} U_\mu(y) \psi_{2(y+\hat{\mu})+\eta} + \delta_{1\eta_\mu} \bar{\psi}_{2y+\eta-\hat{\mu}} \psi_{2y+\eta} \\ & - \delta_{1\eta_\mu} \bar{\psi}_{2y+\eta-\hat{\mu}} U_\mu^\dagger(y-\hat{\mu}) \psi_{2(y-\hat{\mu})+\eta} - \delta_{0\eta_\mu} \bar{\psi}_{2y+\eta+\hat{\mu}} \psi_{2y+\eta}]. \end{aligned} \tag{44}$$

Remark that if we increase the component x_μ of a site x, we jump on a block different from that of x if x_μ is odd. This is the case when $x = 2y + \eta + \hat{\mu}$ and $\eta_\mu = 0$, but not when $x = 2y + \eta - \hat{\mu}$ and $\eta_\mu = 1$. Similarly, if we decrease x_μ, we jump on a different block only when x_μ is even. This is the case when $x = 2y + \eta - \hat{\mu}$ and $\eta_\mu = 1$, but not when $x = 2y + \eta + \hat{\mu}$ and $\eta_\mu = 0$. If $x = 2y + \eta$ then

$$\alpha_{\eta\mu} = \alpha_{x\mu}. \tag{45}$$

Then the kinetic term has the same form as that of $\mathcal{L}_\psi(u')$, where

$$u'_\mu(x) = \begin{cases} U_\mu(y) & \text{for } x = 2y + \eta \quad \text{and} \quad \eta_\mu = 1 \\ \mathbb{1} & \text{elsewhere} \end{cases} \tag{46}$$

that is, the gauge field couples only sites which belong to different blocks.

In conclusion,

$$\mathcal{L}'_\psi(u') = m\bar{\psi}_x\psi_x + \frac{1}{2}\sum_{\mu=0}^{3}\alpha_{x\mu}[\bar{\psi}_x u'_\mu(x)\psi_{x+\hat{\mu}} - \bar{\psi}_{x+\mu}u'^{\dagger}_\mu(x)\psi_x]. \tag{47}$$

We have the constraint, however, that the fermion fields within a block should all transform in the same way under gauge transformations. One might think that we could relax this constraint by a different transformation from the spin to the flavor basis

$$\begin{aligned} q_y^{\alpha a} &= \frac{1}{8}\sum_\eta \Gamma_{\eta;\alpha a}\mathcal{C}_{2y+\eta}\psi_{2y+\eta} \\ \bar{q}_y^{a\alpha} &= \frac{1}{8}\sum_\eta \bar{\psi}_{2y+\eta}\mathcal{C}^\dagger_{2y+\eta}\Gamma^\dagger_{\eta;\alpha a}. \end{aligned} \tag{48}$$

Such a generalization, however, is only apparent because the curvature for the plaquettes with all the vertices within one and the same block vanishes. Indeed, such a generalization, as the particular ones chosen, for example, in Refs. [9], Eq. (35), [27], Eq. (56), are pure-gauge transformations of Eq. (27).

We conclude that, in the presence of a *generic* gauge-field configuration, the Lagrangian in the spin basis $\mathcal{L}'_\psi(u)$ and that in the flavor basis $\mathcal{L}_q(U)$ are not equivalent.

The transformed Lagrangian $\mathcal{L}'_\psi(u')$ could also be regarded, in the spirit of the previous quoted attempt [11], as a modification of $\mathcal{L}_\psi(u)$, defined in Eq. (24), for which a transfer matrix can be constructed.

The above construction refers to the case of vanishing chemical potential. Its inclusion is, however, straightforward [27]. We only note that, at variance with respect to the coupling with gauge fields, the chemical potential can be attached to all links in the transformed Lagrangian $\mathcal{L}'_\psi(u')$, provided its value be half the one in the flavor basis.

IV. TRANSFORMATION OF TRANSFER MATRIX AND COHERENT STATES

As a first step, we must transform creation-annihilation operators from the flavor to the spin basis. To this end, we must determine the expressions of the fields $q_\pm$ in the spin basis

$$\begin{aligned} (q_+)_y &= P_+\frac{1}{8}\sum_\eta \Gamma_\eta\psi_{2y+\eta} \\ (q_-^\dagger)_y &= P_-\frac{1}{8}\sum_\eta \Gamma_\eta\psi_{2y+\eta}. \end{aligned} \tag{49}$$

Using the relation (39), we find

$$P_+\Gamma_\eta = \delta_{0\eta_0}\Gamma_\eta, \qquad P_-\Gamma_\eta = \delta_{1\eta_0}\Gamma_\eta, \tag{50}$$

and similar relations hold for $\Gamma^\dagger$.

We, therefore, have

$$\begin{aligned} (q_+)_y &= \frac{1}{8}\sum_\eta \delta_{0\eta_0}\Gamma_\eta\psi_{2y+\eta}, \\ (q_-^\dagger)_y &= \frac{1}{8}\sum_\eta \delta_{1\eta_0}\Gamma_\eta\psi_{2y+\eta}. \end{aligned} \tag{51}$$

Next, we define the operators corresponding to the ψ fields according to

$$\begin{aligned} (\hat{q}_+)_y &= \frac{1}{8}\sum_{\boldsymbol{\eta}} \delta_{0\eta_0}\Gamma_\eta\hat{\psi}_{2y+\eta}, \\ (\hat{q}_-^\dagger)_y &= \frac{1}{8}\sum_\eta \delta_{1\eta_0}\Gamma_\eta\hat{\psi}_{2y+\eta} \end{aligned} \tag{52}$$

and assume that

$$\left\{\hat{\psi}^\dagger_{2\mathbf{y}'+\eta'}, \hat{\psi}_{2\mathbf{y}+\eta}\right\} = 2\delta_{\mathbf{y}'\mathbf{y}}\delta_{\eta'\eta}. \tag{53}$$

This is obviously consistent with the second set of equations in Eqs. (14). Consistency with the first set requires that

$$\frac{1}{64}\sum_{\eta,\eta'}\delta_{\sigma\eta_0}\delta_{\tau\eta'_0}\Gamma^\dagger_{\eta':b\beta}\Gamma_{\eta:\alpha a}\left\{\hat{\psi}^\dagger_{2\mathbf{y}'+\eta'}, \hat{\psi}_{2\mathbf{y}+\eta}\right\} = \frac{1}{32}\delta_{\mathbf{y}'\mathbf{y}}\delta_{\sigma\tau}\sum_\eta \delta\sigma\eta_0\Gamma^\dagger_{\eta:b\beta}\Gamma_{\eta:\alpha a} \tag{54}$$

$$= \frac{1}{8}\delta_{\mathbf{y}'\mathbf{y}}\delta_{\sigma\tau}P_\pm^{\alpha a,\beta b}, \tag{55}$$

where $\sigma = 0, 1$, respectively, when the index of the projector is $+$ or $-$. The second equality follows from the equations

$$\sum_{a',\alpha'} P_{\pm}^{\alpha a,\alpha' a'} \frac{1}{4}\sum_{\eta}\Gamma^{\dagger}_{\eta:b\beta}\Gamma_{\eta:\alpha' a'} = \frac{1}{4}\sum_{\eta}\delta_{\sigma\eta_0}\Gamma^{\dagger}_{\eta:b\beta}\Gamma_{\eta:\alpha a} = P_{\pm}^{\alpha a,\beta b} \tag{56}$$

that can be proven using Eqs. (30) and (50).

Some comments about our results are in order. We see that the temporal component η_0 of the fields in the spinor basis corresponds to the $\pm$ projection of the field in the flavor basis. The 8 Dirac-taste degrees of freedom of particles/antiparticles are spread on the 8 sites of the even/odd time slice in the corresponding block. In this connection, looking at Eq. (53), η_0 can be regarded as a quantum number. But this quantum number changes when time increases by one unit in the original lattice, so that, unlike the $q_{\pm}$ projections, the fields $\psi_{2\mathbf{y}+\eta}$ with η_0, respectively, 1 or 0 cannot be identified as forward/backward movers. Changing time, we change a particle into the hole of an antiparticle.

A. Transfer matrix

We first transform the baryon number

$$\hat{n}_B = 2^3\sum_{\mathbf{y}}(\hat{q}_+^{\dagger}\hat{q}_+ - \hat{q}_-^{\dagger}\hat{q}_-)_{\mathbf{y}} = \frac{1}{2}\sum_{\mathbf{y},\eta}[(\hat{\psi}^{\dagger}\hat{\psi})_{2\mathbf{y}+\eta}\delta_{0\eta_0} - (\hat{\psi}\hat{\psi}^{\dagger})_{2\mathbf{y}+\eta}\delta_{1\eta_0})] \tag{57}$$

$$= \frac{1}{2}\sum_{\mathbf{x}}[(\hat{\psi}^{\dagger}\hat{\psi})_{\mathbf{x}0} - (\hat{\psi}^{\dagger}\hat{\psi})_{\mathbf{x}1}], \tag{58}$$

where we relabeled the operators $\hat{\psi}$ with the spatial coordinates

$$\mathbf{x} = 2\mathbf{y} + \boldsymbol{\eta} \tag{59}$$

and η_0 and made the identifications

$$\hat{\psi}_{\mathbf{x}0} := \hat{\psi}_{2\mathbf{y}+(0,\boldsymbol{\eta})}, \qquad \hat{\psi}_{\mathbf{x}1} := \hat{\psi}^{\dagger}_{2\mathbf{y}+(1,\boldsymbol{\eta})} \tag{60}$$

in agreement with the relations (52) which show that when $\eta_0 = 1$, the operator $\hat{\psi}_{2\mathbf{y}+\eta}$ is a creation operator.

In this notation, the commutation relations (53) become

$$\{\hat{\psi}^{\dagger}_{\mathbf{x}'\eta_0'}, \hat{\psi}_{\mathbf{x}\eta_0}\} = 2\delta_{\mathbf{x}'\mathbf{x}}\delta_{\eta_0'\eta_0}. \tag{61}$$

Next, we must determine a matrix N_t' such that

$$64\sum_{\mathbf{y}',\mathbf{y}}(\hat{q}_-)_{\mathbf{y}'}(N_t)_{\mathbf{y}'\mathbf{y}}(\hat{q}_+)_{\mathbf{y}} = \sum_{\mathbf{y}',\mathbf{y}}\sum_{\eta',\eta}\hat{\psi}^{\dagger}_{2\mathbf{y}'+\eta'}\,\mathrm{tr}(\Gamma^{\dagger}_{\eta'}(N_t)_{\mathbf{y}'\mathbf{y}}P_+\Gamma_{\eta})\hat{\psi}_{2\mathbf{y}+\eta} \tag{62}$$

$$= \sum_{\mathbf{y}',\mathbf{y}}\sum_{\eta',\eta}\hat{\psi}^{\dagger}_{2\mathbf{y}'+\eta'}(N_t')_{\mathbf{y}'\eta',\mathbf{y}\eta}\hat{\psi}_{2\mathbf{y}+\eta}. \tag{63}$$

In the above equation, color taste and Dirac indices have been omitted. We observe that

$$(\gamma_0\gamma_k\otimes\mathbb{1})P_k^{(\pm)}P_+\Gamma_{\eta} = \frac{1}{2}\delta_{0\eta_0}[(\gamma_0\gamma_k\otimes\mathbb{1}) \pm (\gamma_0\gamma_5\otimes t_5t_k)]\Gamma_{\eta} \tag{64}$$

$$= \delta_{0\eta_0}\alpha_{\eta k}\frac{1\mp(-1)^{\eta_k}}{2}(\delta_{0\eta_k}\Gamma_{\eta+\hat{0}+\hat{k}} + \delta_{1\eta_k}\Gamma_{\eta+\hat{0}-\hat{k}}) \tag{65}$$

and

$$\Gamma^{\dagger}_{\eta'}(\gamma_0\otimes\mathbb{1})P_+\Gamma_{\eta} = \delta_{0\eta_0}\delta_{1\eta_0'}\Gamma^{\dagger}_{\eta'}\Gamma_{\eta}, \tag{66}$$

so that

$$\mathrm{tr}[\Gamma^{\dagger}_{\eta'}(\gamma_0\gamma_k\otimes\mathbb{1})P_k^{(\pm)}P_+\Gamma_{\eta}] = 4\delta_{0\eta_0}\delta_{1\eta_0'}\alpha_{\eta k}\frac{1\mp(-1)^{\eta_k}}{2}(\delta_{0\eta_k}\delta_{\boldsymbol{\eta}',\boldsymbol{\eta}+\hat{k}} + \delta_{1\eta_k}\delta_{\boldsymbol{\eta}',\boldsymbol{\eta}-\hat{k}}) \tag{67}$$

and

$$\mathrm{tr}[\Gamma^{\dagger}_{\eta'}(\gamma_0\otimes\mathbb{1})P_+\Gamma_{\eta}] = 4\delta_{0\eta_0}\delta_{1\eta_0'}\delta_{\boldsymbol{\eta}'\boldsymbol{\eta}}. \tag{68}$$

Finally, we get the transformed N matrix

$$(N')_{\mathbf{y}'\eta',\mathbf{y}\eta} = -8\delta_{0\eta_0}\delta_{1\eta_0'}\Big[m\delta_{\boldsymbol{\eta}'\boldsymbol{\eta}}\mathbb{1}_{\mathbf{y}'\mathbf{y}} + \sum_{\mu=1}^{3}\alpha_{\eta\mu}(\delta_{0\eta_\mu}\delta_{\boldsymbol{\eta}',\boldsymbol{\eta}+\hat{\mu}}\nabla_{\mu}^{(+)} + \delta_{1\eta_\mu}\delta_{\boldsymbol{\eta}',\boldsymbol{\eta}-\hat{\mu}}\nabla_{\mu}^{(-)})_{\mathbf{y}'\mathbf{y}}\Big]$$
$$= -\delta_{0\eta_0}\delta_{1\eta_0'}\Big\{m\delta_{\boldsymbol{\eta}',\boldsymbol{\eta}}\delta_{\mathbf{y}',\mathbf{y}} + \frac{1}{2}\sum_{\mu=1}^{3}\alpha_{\eta\mu}[(-\delta_{0\eta_\mu}\delta_{\boldsymbol{\eta}',\boldsymbol{\eta}+\hat{\mu}} + \delta_{1\eta_\mu}\delta_{\boldsymbol{\eta}',\boldsymbol{\eta}-\hat{\mu}})\delta_{\mathbf{y}'\mathbf{y}}$$
$$+ \delta_{0\eta_\mu}\delta_{\boldsymbol{\eta}',\boldsymbol{\eta}+\hat{\mu}}U_{\mu}(\mathbf{y}')\delta_{\mathbf{y},\mathbf{y}'+\hat{\mu}} - \delta_{1\eta_\mu}\delta_{\boldsymbol{\eta}',\boldsymbol{\eta}-\hat{\mu}}U_{\mu}^{\dagger}(\mathbf{y})\delta_{\mathbf{y},\mathbf{y}'-\hat{\mu}}]\Big\}. \tag{69}$$

Notice that the terms that involve the gauge variables refer to sites belonging to different blocks, while in the other terms, the sites belong to the same blocks. The same operator can be relabeled by using the coordinates $\mathbf{x}$ and η_0; then,

$$(N')_{\mathbf{x}'\eta_0',\mathbf{x}\eta_0} = -\delta_{0\eta_0}\delta_{1\eta_0'}\Big\{m\delta_{\mathbf{x}'\mathbf{x}} + \frac{1}{2}\sum_{\mu=1}^{3}\alpha_{\mathbf{x}\mu}[\delta_{\mathbf{x}',\mathbf{x}-\hat{\mu}}u_{\mu}'(\mathbf{x}') - \delta_{\mathbf{x}',\mathbf{x}+\hat{\mu}}u_{\mu}'^{\dagger}(\mathbf{x})]\Big\}, \tag{70}$$

where the values $\eta_\mu = 0, 1$ simply control the presence of the gauge field according to the definition of u' given in Eq. (46).

In conclusion,

$$\hat{q}_{-} N_t \hat{q}_{+} = \hat{\psi}_1 N_t' \hat{\psi}_0. \tag{71}$$

It should not be necessary to repeat that the expression of the transfer matrix so obtained is positive definite and performs time translations by two lattice spacings.

B. Coherent states

In order to complete our analysis, we perform the transformation also on the coherent states. This will enable us to make, as a cross-check, the derivation of the Lagrangian (47) starting from the transfer matrix.

Let

$$|\alpha, \beta\rangle := \exp\Big\{-2^3 \sum_{\mathbf{y}} \sum_{\gamma,c} [\alpha_{\mathbf{y}}^{\gamma c} (\hat{q}_+^\dagger)_{\mathbf{y}}^{c\gamma} + \beta_{\mathbf{y}}^{c\gamma} (\hat{q}_-^\dagger)_{\mathbf{y}}^{\gamma c}]\Big\} |0\rangle \tag{72}$$

be a coherent state in the flavor basis, where $\alpha_{\mathbf{y}}^{\gamma c}$ and $\beta_{\mathbf{y}}^{\gamma c}$ are Grassmann variables, such that

$$\begin{aligned} (\hat{q}_+)_{\mathbf{y}}^{\gamma c} |\alpha, \beta\rangle &= \alpha_{\mathbf{y}}^{\gamma c} |\alpha, \beta\rangle, \\ (\hat{q}_-)_{\mathbf{y}}^{c\gamma} |\alpha, \beta\rangle &= \beta_{\mathbf{y}}^{c\gamma} |\alpha, \beta\rangle. \end{aligned} \tag{73}$$

Now,

$$2^3 \sum_{\mathbf{y}} \sum_{\gamma,c} \alpha_{\mathbf{y}}^{\gamma c} (\hat{q}_+^\dagger)_{\mathbf{y}}^{c\gamma} = \sum_{\mathbf{y},\eta} \mathrm{tr}(\Gamma_\eta^\dagger \alpha_{\mathbf{y}}) \delta_{0\eta_0} \hat{\psi}_{2\mathbf{y}+\eta}^\dagger \tag{74}$$

$$2^3 \sum_{\mathbf{y}} \sum_{\gamma,c} \beta_{\mathbf{y}}^{c\gamma} (\hat{q}_-^\dagger)_{\mathbf{y}}^{\gamma c} = \sum_{\mathbf{y},\eta} \mathrm{tr}(\beta_{\mathbf{y}} \Gamma_\eta) \delta_{1\eta_0} \hat{\psi}_{2\mathbf{y}+\eta}, \tag{75}$$

and, therefore, because of the anti-commutation relations (61),

$$\hat{\psi}_{\mathbf{x}0} |\alpha, \beta\rangle = \sum_{\eta 0} \hat{\psi}_{2\mathbf{y}+\eta} \delta_{0\eta_0} |\alpha, \beta\rangle = 2\,\mathrm{tr}(\Gamma_{(0,\boldsymbol{\eta})}^\dagger \alpha_{\mathbf{y}}) |\alpha, \beta\rangle \tag{76}$$

$$\hat{\psi}_{\mathbf{x}1} |\alpha, \beta\rangle = \sum_{\eta_0} \hat{\psi}_{2\mathbf{y}+\eta}^\dagger \delta_{1\eta_0} |\alpha, \beta\rangle = 2\,\mathrm{tr}(\beta_{\mathbf{y}} \Gamma_{(1,\boldsymbol{\eta})}) |\alpha, \beta\rangle. \tag{77}$$

This means that we can define

$$\alpha_{\mathbf{x}}' := 2\,\mathrm{tr}(\Gamma_{(0,\boldsymbol{\eta})}^\dagger \alpha_{\mathbf{y}}), \quad \beta_{\mathbf{x}}' := 2\,\mathrm{tr}(\beta_{\mathbf{y}} \Gamma_{(1,\boldsymbol{\eta})}) \tag{78}$$

and rewrite

$$|\alpha, \beta\rangle = \exp\Big[-\frac{1}{2} \sum_{\mathbf{x}} (\alpha_{\mathbf{x}}' \hat{\psi}_{\mathbf{x}0}^\dagger + \beta_{\mathbf{x}}' \hat{\psi}_{\mathbf{x}1}^\dagger)\Big] |0\rangle. \tag{79}$$

Notice that the Grassmann variables α, β, and α' as well are defined at even times. The variable β' instead, because of the matrix $\Gamma_{(1,\boldsymbol{\eta})}$ in its definition, must be considered attached at odd times. This is confirmed by the evaluation of the partition function using the transformed transfer matrix and coherent states. After the identifications

$$\begin{aligned} \bar{\psi}_{2x_0} &= (\alpha_{2x_0}')^*, & \psi_{2x_0} &= (\beta_{2x_0+1}')^* \\ \bar{\psi}_{2x_0+1} &= \beta_{2x_0+3}', & \psi_{2x_0+1} &= \alpha_{2x_0+2}', \end{aligned} \tag{80}$$

we get the Lagrangian (47).

V. CONCLUSION

Numerical simulations with Kogut-Susskind fermions are faster in the spin basis than in the flavor basis. Such calculations are usually performed in the Lagrangian formulation, but we are interested in numerical simulations in the framework of the nilpotency expansion, which makes use of the transfer matrix. So we need an expression of the transfer matrix in the spin basis. In any case, the knowledge of a positive definite transfer matrix in the spin basis is *per se* interesting being related to the unitarity of the theory.

We found in the literature essentially two formulations of the transfer matrix in the spin basis. In the first one, the Lagrangian is reduced by defining fermion fields and their conjugates at the odd, respectively, even sites, and a transfer matrix is constructed that performs time translations by 2 lattice spacings [11,12]. The fermion determinant, even at vanishing chemical potential, is, however, not positive definite, which makes this way less suitable to numerical simulations.

In the second formulation [11], a positive definite transfer matrix, called T^2, was defined that also performs time translations by 2 lattice spacings. As a consequence, the corresponding Fock space must be constructed on blocks. The explicit construction of such Fock space, however, is not given.

If the Fock space is associated to a block, we can get the transfer matrix in the spin basis by a unitary transformation from that in the flavor basis, whose expression, together with the construction of the Fock space, is known. The transfer matrix in the flavor basis is expressed in terms of a matrix N, and the transformed matrix is given in terms of the matrix N', given explicitly in Eq. (70). In order to do numerical simulations in the nilpotency expansion, all we need is to replace everywhere in the equations of the nilpotency expansion N by N' and remember that the gauge fields are now defined on blocks.

It would be now natural to compare our result with the expression of the previously derived transfer matrix [11]. One might expect that such a comparison should provide the definition of the Fock space in the latter. Unfortunately, this is not the case. The transfer matrix of Ref. [11] cannot be related to ours in a simple way, the most remarkable differences being that there is no requirement concerning the gauge variables which remain defined on the links of the original lattice, and creation and annihilation operators appear not only in exponential form, but also as powers.

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