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"BCVA and Funding Costs under Different Models and Different Credit Contagion Hypotheses"

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To my Grandmother because she came to my dreams and encouraged me.

To my Family because they are always there.

To Roberto because he supported me through all these years.

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Preface

Counterparty Credit Risk and Funding Costs have recently become very urgent topics in derivatives pricing, stimulating new research fields within banks and corporates.

On one side, in fact, the risk of default has reached financial institutions that were previously thought to be default immune, channeling enormous stress on credit spreads.

On the other hand, and to a certain extent consequently, the access to liquidity has become more difficult and quite expensive for all market participants, especially for those with a lower credit quality. The fundamental reason for rising cost of liquidity along with the increase in credit spreads, lies in the fact that funding provisions are not performed at the risk-free rate, but they are linked to the creditworthiness of the party itself.

The financial framework for Counterparty Credit Risk is that of a bilateral contract, where one or both parties may have the right to receive a payment, or conversely may have the obligation to make a payment, during the life of the transaction or at maturity. In IRS pricing, where it is not possible to know in advance the sign of the mark-to-market of the derivative during the life of the transaction, both parties may eventually have the right or the obligation to make a payment. When dealing with bilateral contracts such as swaps, it is therefore advisable to consider the risk of default of both parties, as it is not possible to know in advance which one will be the surviving or defaulting party, and for whom the mark-to-market shall be negative or positive at time of default.

In order to assess Counterparty Credit Risk it is necessary to assume the point of view of one of the two parties, and consequently make all evaluations from the reference party's perspective.

Counterparty Credit Risk therefore represents the risk for one party that the counterparty in a OTC transaction defaults prior to maturity, thus not fulfilling all its payment obligations.

The relevance of this risk has increased after 2007 credit crisis, which showed that counterparties could indeed default, especially big financial institutions that were previously considered almost risk-free.

Market players converged in the search for an indicator that could include in the pricing of a transaction the relevant counterparty risk they were implicitly bearing when entering a deal. This search led to the identification of the so called ”CVA”, i.e. Credit Value Adjustment.

This risk factor indicator must be therefore intended as the adjustment that has to be included when pricing OTC derivatives, in order to properly account for the probability of default of one’s counterparty. Counterparty Credit Risk is therefore relevant when the mark-to-market of the derivative is negative for the defaulting party. In this case, in fact, the surviving party will only receive a portion of its positive mark-to-market and will consequently incur a loss, that has to be priced in advance and charged accordingly.

When dealing with bilateral contracts, though, where the mark-to-market may be positive or negative to both parties, also the counterparty may be exposed to the risk of default of the reference party, and in order to proper account for this eventuality, also the ”DVA”, i.e. Debt Value Adjustment, is introduced. For this reason, Counterparty Credit Risk will be addressed in the form of Bilateral Credit Value Adjustment (BCVA), in order to properly account for the risk of default of both parties.

Relevant literature concerning pricing techniques for Counterparty Credit Risk includes, among all, Burgar and Kjaer [20], Morini and Prampolini [44], Gregory [33] and [34], Brigo [8], Brigo et. al. [10], Brigo and Capponi [11], Brigo and Morini [16] and [17].

The increasing importance of risk of default of a party has also led to a significant impact on the cost of funding for any liquidity disbursement. In fact, as the cost of funding on the bond primary market depends on the credit spread of the party itself, the higher the credit riskiness of a party perceived by the market, the more expensive the cost of funding the market would require to that party. As a consequence, the cost for funding any cashflow expected to be paid during the life of the underlying transaction, has recently started to be relevant and worth be priced in the overall value of the underlying contract in the context of credit risk. Literature is only at dawn in developing a coherent pricing framework for Funding Costs to be charged in a OTC derivative, see on this regard Brigo et. al. [19] and Crepey [24]. This is the reason why a computation methodology for Funding Costs pricing and accountancy is developed and proposed in the current work. Again Funding Costs are priced from the point of view of a reference party, as the funding spread to be added to the risk-free rate depends on the credit quality of the reference party. Different assumptions may be performed on the proper funding spread to apply in the funding transaction, either simply considering the credit spread observable in the market or adding a liquidity premium to it in order to account for a friction between primary and secondary market. The concept of ”credit riskiness” has overflowed the computation of Funding Costs too, as the borrowing party in the funding transaction may not be able to fulfill its obligation in case of own default. When pricing Funding Costs, a sort of cost reduction may therefore be conceived in order to properly account for the possibility of own risk of default.

Another important topic to consider when addressing Counterparty Credit Risk is the correlation between default risks of the two parties. When computing Credit Value Adjustment, the approach of the current work was to introduce correlation between default risks of two parties through common jumps in the intensity process for default intensities. This choice was quite innovative with respect to previous literature. Another approach, in fact, may have been that of considering copulas on default triggers. We decided not to follow this path for two main reasons. First of all, because the classical copula approach would lead to an unrealistic behavior of default events. Secondly, correlation between default triggers is not observable in the market, while instead correlation between the intensity process and jumps is observable in the CDS behavior.

We may summarize the objectives of the current work in the following points.

1. *Bilateral Credit Value Adjustment.* Study the importance and size the impact of Counterparty Credit Risk in derivatives pricing, accounting not only for risk of default of the counterparty but also of own risk of default.
2. *Funding Costs in the contest of a credit risky funding.* Introduce Funding Costs when liquidity disbursements are due to the counterparty during the life of the transaction, and the funding spread is different from the risk-free rate, because of own risk of default to be priced in. Find an appropriate setup to model and quantify the concept of a ”credit-risky” funding, and to account for own risk of default when computing Funding Costs. In order to fulfill this objective, ”Funding Cost Adjustment” and ”Adjustment for Default Risk” within ”Funding Cost Adjustment” are introduced.
3. *Correlation in default events through common jumps.* Verify the impact of correlation between default risks of two parties when computing Credit Value Adjustment.

We will here briefly present the structure of the current work.

In Chapter 1 an introduction to the Counterparty Credit Risk setting is given, with a description of all relevant measures, including above all Credit Value Adjustment (CVA), Debt Value Adjustment (DVA), Bilateral Credit Value Adjustment (BCVA), Expected Exposure (EE) and Negative Expected Exposure (NEE). An overview of market practice is provided, concerning risk-free or replacement closeout, collateral, netting and re-hypothecation. To conclude, the newly born topic of Funding Costs is introduced.

In Chapter 2 and Chapter 3 a model comparison for Bilateral Conterparty Credit Risk and Funding Costs is provided.

In particular, Chapter 2 examines the Black&Scholes approach followed by Burgard and Kjaer (2010), with detailed mathematical passages followed by the authors in order to obtain their pricing formulas. Chapter 3, instead, investigates the approach proposed by Morini and Prampolini (2010), where a pure liquidity basis is introduced when pricing Funding Costs, in the framework of Bilateral Counterparty Credit Risk. A compared analysis of both models is proposed.

Chapter 4 can be regarded as the most important one, as it presents the approach developed in the current work, and a comprehensive formula for derivatives pricing is given, in order to account for Bilateral Credit Value Adjustment (BCVA) and for Funding Cost Adjustment (FCA).

Chapter 5 describes the pricing setup followed, and it can be considered a technical support to understand the stochastic intensity modeling approach adopted in order to perform numerical tests.

Chapter 6 presents all numerical tests performed and it provides an overview of results obtained. This chapter can be intended as a comprehensive journey through the overall work, as the complexity and the meaning of tests evolve along with the presentation of the chapter itself.

Finally, in Chapter 7 we find the conclusions to the current work.

Chapter 1
Counterparty Credit Risk: introducing Credit and Debt Value Adjustments

Abstract We derive a pricing formula for a derivative with Bilateral Counterparty Credit Risk and Funding Costs. In particular, we compute the relevant pricing adjustments to the risk-free value that need to be considered when including the risk of default of both parties in a bilateral contract, and the cost for the access to liquidity in a credit risky funding environment. Bilateral Credit Value Adjustment is built through the concepts of Adjusted CVA and Adjusted DVA, meaning that the probability of default of one party at a certain point in time is always weighted by the survival probability of the other party up to that moment. A fundamental assumption is that, at time of default, the credit riskless value of the derivative is considered, in case with an haircut applied in order to account for a proper recovery rate. This work innovates on the existing literature in two directions. First of all, a computation methodology for Funding Costs is provided, given that the search for a comprehensive pricing formula is still at dawn either within practitioners and academics. Secondly, correlation between default risks of the two parties is included in bilateral counterparty risk pricing not through the imposition of a gaussian copula but through the introduction of common jumps in the process for default intensities.

1.1 Introducing Counterparty Credit Risk

The specification of CVA as an instrument for pricing, and not for capital requirements (see for instance Credit VaR to this purpose), brings CVA in the risk-neutral pricing world under the risk-neutral pricing measure Q, as opposed to the real world probability measure P. Risk-neutral probabilities can be extrapolated from the market, via calibration of the pricing model to market observable CDS quotes.

Setting the scene, we may usefully resort to the distinction between a ”borrower” and a ”lender”.

In literature, the borrower is assumed to be the party with lower credit quality, as it is the party that is asking to be financed at inception and is expected to repay its own debt at the final date, for example in a zero-coupon bond. Eventually payments may be planned to be paid also during the life of the transaction, if for instance we think of periodical coupon payments for a coupon-bearing bond, or periodical instalments for loans and mortgages.

The lender, instead, is assumed to be the party with higher credit quality, as it has the possibilty to fund in cash the other counterparty at start date. This is of course a simplistic approach for explanatory purpose.

As a matter of fact, the lender is the one conceptually bearing the Counterparty Credit Risk, as it may not receive back its initial disbursement, partially or totally, subject to any default occurrences of the borrower. It flows almost automatic that the lender should ask for a credit premium over risk-free interest rate for the anticipated amount, in order to properly account for CVA.

Analogously, in an option framework, the option buyer would be the Counterparty Credit Risk bearer, as cashing a premium at inception and buying the right to receive a certain payoff at the option expiry. As a consequence, the option buyer should deduce from the risk-free premium a CVA adjustment for the option seller's credit quality risk.

A further step was to introduce the risk of default for both parties, assigning a risk of default also to the lending counterparty, leading to the introduction of the "DVA", i.e. Debt Value Adjustment. Equivalently, in the option context, a default risk was assigned also to the option seller.

This passage is linked to the idea that both parties may be supposed to exchange payments and naturally converges in the concept of Bilateral Counterparty Value Adjustment, referred to as "BCVA".

A more detailed investigation of these concepts and their implications will be the object of next chapters, where we will review some relevant literature, see [20] and [44].

In this chapter, instead, we will further explore all other relevant aspects connected with Counterparty Ccredit Risk, as some mechanisms have been disciplined in order to minimize this risk factor. CSA agreements between market counterparties, in fact, allow for collateral exchange and margining procedures.

1.2 Relevant risk measures for Counterparty Credit Risk

In order to deal with Counterparty Credit Risk we shall resort to a set of relevant risk metrics, that will guide us through our calculations.

As also illustrated in [34] and [8], these measures refer to the concept of future exposure for counterparty risk, intended as the present expectation, under a certain probability measure, of future exposure to a counterparty, if any positive is expected and zero otherwise. The probability measure shall be the real world one P, for risk management purposes, and the risk-neutral one Q, for pricing purposes.

In particular, finance literature presents us with Expected Exposure (EE), Potential Future Exposure (PFE) and Expected Positive Exposure (EPE).

What is common to all these measures is that default is given for granted, and they attempt to quantify the loss expected to be suffered in this event. These metrics do not include any assumption or pricing of default probability, contrary to Credit VaR.

PFE and *EE* refer to a single point in time, whereas *EPE* characterizes an evolution in time.

- *PFE*

Potential Future Exposure brings along the concept of confidence level, quite similarly to VaR.

PFE_β at a certain confidence level β, in fact, represents the exposure that will only be exceeded with a probability smaller than $(1-\beta)$. So one can be $\beta\%$ confident that an exposure of amount bigger than PFE_β will not be exceeded.

As an example, if the mark-to-market of a derivative X is distributed according to a normal distribution $X \sim N(\mu,\sigma^2)$, then the PFE_β will be given by the following:

$$PFE_\beta = \mu + \sigma\Phi^{-1}(\beta)$$

In fact, if $X \sim N(\mu,\sigma^2)$, one can standardize the random variable X and get $Z \sim N(0,1)$:

$$Z = \frac{X-\mu}{\sigma}$$

One can then assign a value β to the probability of X being equal or smaller than a certain initially unknown amount PFE_β, and obtain the value of PFE_β associated to β.

$$\begin{aligned} P\left(X \leq PFE_\beta\right) &= \beta \\ P\left(\frac{X-\mu}{\sigma} \leq \frac{PFE_\beta - \mu}{\sigma}\right) &= \beta \\ P\left(Z \leq \frac{PFE_\beta - \mu}{\sigma}\right) &= \beta \end{aligned}$$

from which, through the inverse of the cumulative distribution function of a standard normal distribution calculated in β, one gets:

$$PFE_\beta = \mu + \sigma\Phi^{-1}(\beta)$$

Risk management would normally set $\beta = 99\%$.

One can remark that, unlike VaR that is usually referred to as a measure of a loss, PFE_β indeed deals with the concept of a gain, in that it represents the potential future exposure, meaning a positive amount one counterparty will be expecting to receive from the other counterparty. It is nonetheless an amount at risk, because if the counterparty defaults, the amount will not be received, in whole or in part (if a recovery rate is applicable).

PFE for a derivative is usually calculated with reference to a given future time, and it does not describe the evolution of exposure through time.

If one wants to characterise the PFE through time, one can simulate the price of a derivative at each future time until a desired time horizon, and then take the $\beta-$percentile of the distribution of exposures as the PFE within the desired time horizon. This will be an approximation and a "representative" value of the exposure that will be exceeded only with a probability of maximum $\beta\%$, within the chosen time horizon.

- *EE* and *EPE*

Expected Exposure represents the average exposure at a future date, under the probability measure P, where the definition of EE as "exposure at a future date" embeds the concept of "expectation" and "positivity".

The exposure, in fact, must be intended as a positive quantity, given that we are interested in knowing our gain at risk in case of default of the counterparty.

As a result, EE shall be the average of only expected positive values at a given future date.

If the expected value of a derivative is given by its mark-to-market, the EE shall be the average of only positive mark-to-market at a future date.

The curve of EE in time, represents the expected exposure profile of a derivative.

This brings along the metric of EPE, i.e. Expected Positive Exposure, which is the average of positive expected values up to a certain time.

This definition means that one has to integrate the EE over time in order to obtain the EPE.

One shall resort to compute the EE as the average of only positive discounted expected values under the risk-neutral measure Q, instead of the real probability measure P, when dealing with pricing tasks rather than risk management problems. It is in fact the no-arbitrage pricing that brings the requirement of a risk-neutral probability measure.

As an example for the EE, if the mark-to-market of a derivative X is distributed according to a normal distribution $X \sim N(\mu, \sigma^2)$, we have that the Exposure E is given by:

$$\begin{aligned} E &= X^+ \\ &= \max(X;0) \\ &= \max(\mu + \sigma Z;0) \end{aligned}$$

with $Z = X - \mu\sigma$ and $Z \sim N(0,1)$.

The expected value of a continuous random variable X with probability density function $f(x)$ is given by:

$$E(X) = \int_{-\infty}^{+\infty} x f(x) dx$$

and $X \geq 0$ implies $\mu + \sigma Z \geq 0$ and therefore $Z \geq -\mu/\sigma$.

The Expected Exposure EE being the average of only positive mark-to-market values at a future date, will be given through the following [34]:

$$\begin{aligned} EE &= \int_{-\frac{\mu}{\sigma}}^{+\infty} (\mu + \sigma x)\, \phi(x) dx \\ &= \mu \Phi(\frac{\mu}{\sigma}) + \sigma \phi(\frac{\mu}{\sigma}) \end{aligned}$$

where $\phi(x)$ is the probability density function of a standard normal distribution and $\Phi(x)$ is the cumulative distribution function of a standard normal distribution.

As anticipated, EPE is defined as the average of the EE profile through time up to a given point in the future.

Important useful approximations for the coming chapters are the following:

$$\int_t^T EE(u,T) du \approx \frac{T-t}{n} \sum_{i=1}^{n} EE(t_i,T) \tag{1.1}$$

if the time interval $[t,T]$ is divided in n time intervals with $i = 0,1,..,n$ and $t_0 = t$ and $t_n = T$.

$$EPE = \frac{\int_t^T EE(u,T) du}{T-t} \approx \frac{1}{n} \sum_{i=1}^{n} EE(t,t_i) \tag{1.2}$$

- NEE and ENE

For future use, we shall introduce corresponding measures for the negative case, when we may be interested in considering only negative expected values of mark-to-market. In particular we have Negative Exposure NE:

$$\begin{aligned} NE &= X^{-} \\ &= \min(X;0) \end{aligned}$$

with resulting Negative Expected Exposure NEE, opposite to Expected Exposure EE, as the average of expected values, only if negative.

NEE can be approximated as:

$$\int_t^T NEE(u,T) du \approx \frac{T-t}{n} \sum_{i=1}^{n} NEE(t_i,T) \tag{1.3}$$

Correspondently, Expected Negative Exposure ENE, is equal to the average of the NEE profile through time up to a given point in the future.

$$ENE = \frac{\int_t^T NEE(u,T)du}{T-t} \approx \frac{1}{n}\sum_{i=1}^{n} NEE(t,t_i) \tag{1.4}$$

1.3 Introduction to pricing formulas for CVA, DVA and BCVA

- *Unilateral CVA*

CVA refers to the case where there is one defaultable counterparty and a risk-free counterparty, or to the case when obligations are structured as one-sided, and the party that has the obligation to pay may default.

As CVA deals with the concept of exposure, the formula for CVA is intuitively given by the expected amount that is at risk in case of default of the counterparty (EE), multiplied by the probability of default of the defaultable counterparty, and adjusted by the recovery rate the surviving party is likely to obtain in the unfavourable event of default of the counterparty. As explained in [8], CVA is basically an option on the residual value of a portfolio at default of the counterparty.

The CVA being an option to be priced, the expectation has to be taken under the risk-neutral measure Q, rather than P.

As a matter of fact, the recourse to the probability measure Q for pricing purposes derives from an hedging argument, and the choice of Q may thus be arguable in case CVA was not indeed hedged [34].

Leaving aside this consideration, and pricing CVA as an option under Q, the price for CVA is calculated in t for a derivative X with final maturity T:

$$CVA(t,T) = E^Q\left[(1-\delta)I(\tau \leq T)X(\tau,T)^+\right] \tag{1.5}$$

where δ is the recovery rate the surviving party will receive in case of default of the obligor.

The recovery fraction δ will be applied to the residual value of the derivative from the moment of default τ until maturity T, and only in case this residual value is positive: $X(\tau,T)^+$. Approximating the above formula we get the following more practical one (details of the derivation will be given in coming chapters):

$$CVA(t,T) \approx (1-\delta)\sum_{i=1}^{n} DF(t,t_i)EE(t,t_i)q(t_{i-1},t_i) \tag{1.6}$$

where $DF(t,t_i)$ is the discount factor for time t_i calculated in t, $EE(t,t_i)$ is the exposure at time t_i, and $q(t_{i-1},t_i)$ is the marginal default probability of the counterparty in the time interval $(t_{i-1},t_i]$.

This formulation of CVA is unilateral.

- *Adjusted CVA*

In literature we also find a formulation for the *Adjusted* CVA, which is equal to the unilateral CVA, multiplied by the survival probability of the receiving party, the one previously assumed to be risk-free. The *Adjusted* CVA will of course be smaller than unilateral CVA.

$$AdjCVA(t,T) \approx (1-\delta)\sum_{i=1}^{n} DF(t,t_i)EE(t,t_i)S_A(t_i)q_B(t_{i-1},t_i) \tag{1.7}$$

Here $S_A(t_i)$ represents the survival probability of the creditor, named A, up to time t_i, and $q_B(t_{i-1},t_i)$ is the marginal default probability of the obligor, here named B, in the time interval $(t_{i-1},t_i]$.

- *DVA* and *Bilateral CVA*

Bilateral CVA considers the possibility that both counterparties can default, unlike unilateral CVA, and the case where the exposure for the original creditor, may also be negative, meaning that payment obligations may be expected on both sides of the deal. This may seem a more simmetric situation, if it were not for the fact that, in general, creditor and obligor do not bear the same credit risk. As a consequence, survival and default probabilities are not simmetric.

A first formula for Bilateral CVA is given by:

$$\begin{aligned} BCVA(t,T) \approx{}& (1-\delta)\sum_{i=1}^{n} DF(t,t_i)EE(t,t_i)S_A(t_i)q_B(t_{i-1},t_i) \\ &+(1-\delta_A)\sum_{i=1}^{n} DF(t,t_i)NEE(t,t_i)S_B(t_i)q_A(t_{i-1},t_i) \end{aligned} \tag{1.8}$$

where δ_A is the recovery rate for counterparty A. The portion $(1-\delta_A)$ of the NEE represents the obligations party A was expected to meet before his own default, but that will not be respected because of own default. In this setting, the second term on the r.h.s. of the expression is the adjusted Debt Value Adjustment (DVA), which reduces the amount of Counterparty Credit Risk when including a counterparty's own credit risk.

A counterparty, in fact, at his own default, will honor only a recovery fraction of his debt, but will receive the whole positive credit.

As we can see, if the adjusted CVA term of BCVA is positive, and it is a cost that will be subtracted from the risk-free value of a derivative, the adjusted DVA term will be a negative component, thus a gain that will decrease the amount of counterparty credit risk to charge for a transaction.

A more detailed derivation of these formula is given in following chapters.

1.4 Risk-free or Replacement Closeout

Closeout defines the valuation procedure for the residual of a deal, in case one party exercises the right to terminate the transaction, upon default of the other, or upon other specific agreed events. This termination right must be intended as unilateral, and enables the surviving party to immediately come out of a transaction freezing his position. The advantage of the closeout right lies in the possibility to fully re-hedge with another counterparty just upon default (or specific event), without further exposure to market movements.

In case the surviving party has a net creditor position, the exposure loss incurred will be claimed for, but the position will be hedged straight after default and the mark-to-market of the original transaction frozen at default time.

There are two possible alternatives in the valuation approach. One may choose to calculate the residual value of the contract according to its risk-free value, or still considering the risk-adjusted value.

In particular, considering the case of a one-sided transaction, where only one party bears the obligation to pay, at default of the creditor, the obligor may choose to evaluate its liability as risk-free ("risk free closeout"), or still considering his own default risk ("replacement closeout").

The problem with the choice of a risk-free valuation approach upon creditor's default, lies in the sudden value increase of the liability just after creditor's default.

On the contrary, a "replacement closeout", based on the potential charge of unilateral DVA in case the defaulted counterpart was to be replaced, ensures continuity and consistency in the valuation approach to one's own liabilities.

On the other hand, a "risk free closeout" may be preferred by a creditor position upon default of the debtor, especially in case of high correlation among market players.

Contagion effect due to high correlation, may result in a decrease in value of defaulted party's assets, due to a deterioration in the credit quality of other market participants (assets supposed to be on correlated names). In this sense, a creditor would rather choose a risk-free closeout approach, in order not to see his assets dramatically diminished in value because of a contagion effect. See on the topic [16].

1.5 Collateral, Netting and Re-hypothecation

Collateral has been introduced as a guarantee to limit Counterparty Credit Risk, and, as a consequence, to reduce CVA charges and facilitate market transactions.

Collateral can be designed as a one-way or as a two-way clause, that basically imposes to the counterparty for which the mark-to-market of the transaction is negative, to provide a guarantee in cash or liquid securities, such as bonds, to the other counterparty to reduce his exposure. The "posting" of cash, or other eligible securities, ensures protection to the surviving party from the default of his obligor with respect to the outstanding exposure. In case of a two-way collateral agreement, both parties may in turn have the obligation to post collateral, depending on the sign of the mark-to-market.

As the mark-to-market of a deal fluctuates in time according to market movements of the underlying risk factors, a crucial point is that of monitoring it as often as possible, and exchanging collateral accordingly. The purpose, of course, is to minimize any potential mismatch between the underlying asset value and the corresponding collateral position.

Netting is a practice that allows for compensation between positive and negative values of the portfolio of contracts in place with a defaultable counterparty.

It is possible to agree on the posting of collateral on the netted portfolio position with respect to a counterparty.

In case of a counterparty's default that had previously posted collateral against his negative net position, the surviving party, with positive net mark-to-market, will not be required to return the collateral. As a matter of fact, collateral will serve as an exposure compensation. Considering that, in most cases, it is under direct control of the collateral taker, in case of default of the collateral provider, collateral may be liquidated immediately to the collateral taker, without requiring all legal actions needed for other creditors.

Collateral posting may indeed not sufficiently mitigate Counterparty Credit Risk, and it may be affected by two major drawbacks: i) the frequency of margining and ii) the practice of re-hypothecation.

The first point i) was already anticipated, and it mainly substantiates in the fact that, in order to have collateral always in line with underlying mark-to-market, one should recur to collateral checking with a high frequency, with resulting high operational costs and risks. Moreover, even accepting the costs of a high frequency of collateral exchange, full exposure protection is not ensured. The bias of time discretization for observation dates can not be totally eliminated.

The second point ii), instead, refers to the practice of re-investing the collateral received, and gain interests on it. This process is stimulated by the necessity to remunerate the collateral provider. When a collateral taker re-invests it, collateral itself is at risk and the probability of easily mobilizing it, in the event of a switch in the mark-to-market of the underlying derivative, decreases. This is the reason why re-hypotecation may be not favoured or allowed by regulators.

On the other hand, without re-hypotecation, collateral posting may be too expensive and therefore avoided by market participants. As a matter of fact, whenever possible, counterparties are likely to prefer collateralized trades rather than uncollateralized ones. The cost of uncollateralization is essentially represented by the charge of CVA or BCVA, depending on the approach. It is possible to infer that, in order to decide whether to prefer collateralized over uncollateralized trades, one should confront remuneration required on collateral (case of collateralization) against CVA charges (case of uncollateralization).

Of course this is a simplistic approach, and reality may offer a combination of solutions, such as posting of collateral, according to predefined rules, in order to diminish CVA charges.

1.6 Introducing Funding Costs

The liquidity crisis following the credit crisis induced by major defaults of 2008 (Lehman Brothers, Freddie Mac, Fannie Mae, etc), brought to the attention of market players the importance of liquidity, and the impact of debtors' creditworthiness in the definition of a price, if any, for giving liquidity. The cost of liquidity raised continuously along with the credit crisis.

This led to a significant introduction of funding costs in financial contracts pricing. Funding costs in pricing represent the necessity to fund any liquidity esboursement that a party has to face. A liquidity esboursement can be identified as the net negative cash flow at a certain point in time. The assumption is to have any liquidity esboursement financed , from the moment it occurs until the end of the corresponding transaction. The interest rate a counterparty has to pay is usually referred to as "funding rate", which of course depends on the credit quality of the counterparty itself. One may also assume net positive cash flows at certain points in time. For these occurrences, we shall introduce interest rates to be earned, from the moment positive cash flows happen until maturity of the related contract.

Funding costs consistent inclusion in the framework of Counterparty Credit Risk is still at dawn.

This is the reason why funding costs pricing will be analysed in this work. In existing literature the topic is addressed in [20], [44], [19], [24].

Chapter 2
The Black & Scholes approach to Bilateral Counterparty Credit Risk modeling with Funding Costs

Abstract This chapter will try to illustrate previous literature findings for continuous time modeling of Bilateral Counterparty Credit Risk together with Funding Costs. Important reference on this regard is represented by Burgard K. and Kjaer M. in [20]. An overview of the main model assumptions and financial results will be given, together with detailed mathematical passages leading from initial model setting and problem proposition to final formulas suggested by the authors.

2.1 Describing a possible Black & Scholes model setting

In the Black & Scholes setting of [20], an economy consisting of two parties B and C and four traded assets is assumed, where these are one default-free zero-coupon bond P_R, two risky zero-coupon bonds respectively bearing the risk of the two parties in the market P_B and P_C, and an asset with no default risk S.

The credit risky zero-coupon bonds P_B and P_C are supposed to pay 1 at maturity T if the respective issuing party does not prematurely default.

The processes for the four traded assets have the following dynamics under the hystorical probability measure:

$$\begin{cases} \frac{dP_R}{P_R} = r(t)dt \\ \frac{dP_B}{P_B} = r_B(t)dt - dJ_B \\ \frac{dP_C}{P_C} = r_C(t)dt - dJ_C \\ \frac{dS}{S} = \mu(t)dt + \sigma(t)dW \end{cases}$$

The two parties B and C are then assumed to enter a derivative on the asset S, with $S \geq 0$, where B pays a given payoff $H(S)$ to C at maturity T, where $H(S) \in R$.

As a matter of fact, party C buys an option from party B. The value of this option, from the point of view of C, is denoted by $\widehat{V}(t,S,J_B,J_C)$, which is therefore the value of a risky derivative that depends on time t, on the underlying S, and on the jump states J_B and J_C of the two parties B and C.

At the same time, the value of a derivative on the same underlying, but between default-free parties, is indicated by $V(t,S)$.

In both cases the value of the derivative is given by its mark-to-market M.

Mark-to-market M can be either positive or negative, and it can be represented alternatively by M^+ or M^-.

Embracing a shared assumption in financial literature, in case of default of one of the two parties, the surving party always receives the recovery rate of a positive mark-to-market, and pays the full amount of a negative mark-to-market, at time of default of the other party.

This approach is represented in [20] through M^+, i.e. the positive mark-to-market for the surviving party, and M^-, i.e. the negative mark-to-market for the surviving party.

Assuming as said that $\widehat{V}$ is the value of the derivative from the point of view of C, meaning the party who has to receive $H(S)$ in T, with $\widehat{V} = M$, we would have the following.

If party B defaults first, denoted by $J_B = 1$ and $J_C = 0$:

$$\widehat{V}(t,S,1,0) = M^-(t,S) + R_B M^+(t,S) \tag{2.1}$$

and instead if party C defaults first, denoted by $J_B = 0$ and $J_C = 1$:

$$\widehat{V}(t,S,0,1) = R_C M^-(t,S) + M^+(t,S) \tag{2.2}$$

R_B and R_C are the recovery rates of party B and C respectively, and they represent the percentage of mark-to-market that would be recovered by the other party in case of default of party B and C respectively.

If B defaults first, as in $\widehat{V}(t,S,1,0)$ with $J_B = 1$, as we are considering the value of the risky derivative from the point of view of C:

- if the mark-to-market is negative for C, C will pay to B the full negative mark-to-market $M^-(t,S)$
- if the mark-to-market is positive for C, C will receive only party B's recovery rate R_B times the mark-to-market itself, $R_B M^+(t,S)$.

If C defaults first, as in $\widehat{V}(t,S,0,1)$ with $J_C = 1$, instead:

- if the mark-to-market is negative for C, C will only pay its recovery rate R_C times the mark-to-market, $R_C M^-(t,S)$
- if the mark-to-market is positive for C, C will receive the full mark-to-market from B given by $M^+(t,S)$.

2.2 Find $d\widehat{V}$ as replicating self-financing portfolio

The classic idea is to build a portfolio that replicates the derivative by reproducing all its risk factors, so that the replicating portfolio mirrors any infinitesimal change in the value of the derivative by an identical change in its value.

A portfolio $\Pi(t)$ is built and it is made up of convenient amounts of S, P_B,P_C and $\beta(t)$, where β is the cash amount that will be specified here below. So we have:

$$\widehat{V}(t) = \Pi(t) = \delta(t)S(t) + \alpha_B(t)P_B(t) + \alpha_C(t)P_C(t) + \beta(t) \tag{2.3}$$

The strategy $\Pi(t)$ is put in place by party B, as it is the party expected to pay the payoff $H(S)$ to party C in T.

In this context, a funding rate $r_F(t)$ is associated to party B, to represent the costs that party B may pay on the negative cash positions deriving from the replicating strategy. Party B is therefore referred to as the issuer as well.

The self-financing property is then imposed to the replicating portfolio:

$$d\widehat{V}(t) = d\Pi(t) = \delta(t)dS(t) + \alpha_B(t)dP_B(t) + \alpha_C(t)dP_C(t) + d\beta(t) \tag{2.4}$$

Always with reference to [20], the change in cash is described as it follows:

$$d\beta(t) = d\beta_S(t) + d\beta_F(t) + d\beta_C(t) \tag{2.5}$$

- $d\beta_S(t)$:

$$d\beta_S(t) = \delta(t)\left(\gamma_S(t) - q_S(t)\right)S(t)dt \tag{2.6}$$

The share position earns a dividend income of $\gamma_S(t)$ and has a financing cost of $q_S(t)$.

- $d\beta_F(t)$:

$$\begin{aligned} d\beta_F(t) &= \left\{ r(t)(\widehat{V} - \alpha_B P_B)^+ + r_F(t)(\widehat{V} - \alpha_B P_B)^- \right\} dt \\ &= \left\{ r(t)(\widehat{V} - \alpha_B P_B)^+ + (r(t) + s_F)(\widehat{V} - \alpha_B P_B)^- \right\} dt \\ &= \left\{ r(t)(\widehat{V} - \alpha_B P_B) + s_F(\widehat{V} - \alpha_B P_B)^- \right\} dt \end{aligned} \tag{2.7}$$

where $r_F(t) = r(t) + s_F$.

Here, the extra positive positive cash balance remaining after own bonds have been purchased is represented by $(\widehat{V} - \alpha_B P_B)^+$, and it must earn the risk-free rate $r(t)$.

On negative cash balances, represented by $(\widehat{V} - \alpha_B P_B)^-$, the issuer has to pay $r_F(t)$, which is equal to the risk-free rate $r(t)$ plus the funding spread s_F.

The funding spread s_F is equal to zero if $\widehat{V}$ can be used as collateral, while it is equal to $(1 - R_B)\lambda_B$ if $\widehat{V}$ cannot be used as collateral.

As funding spread for the case of uncollateralization, we use $(1 - R_B)\lambda_B$ which is the yield on unsecured issuer bonds with recovery rate R_B.

- $d\beta_C(t)$:

$$d\beta_C(t) = -\alpha_C(t)r(t)P_C(t)dt \tag{2.8}$$

The issuer will short a portion $\alpha_C(t)$ of counterparty bonds through a repurchase agreement and have financing cost as in $d\beta_C(t)$.

Therefore we have the following expression for $d\beta(t)$:

$$d\beta(t) = \delta(t)\left(\gamma_S(t) - q_S(t)\right)S(t)dt + \left\{r(t)(\widehat{V} - \alpha_B P_B) + s_F(\widehat{V} - \alpha_B P_B)^-\right\}dt - \alpha_C(t)r(t)P_C(t)dt$$

which, by omitting the time indicator, brings to:

$$\begin{aligned} d\widehat{V} &= \delta dS + \alpha_B dP_B + \alpha_C dP_C + d\beta \\ &= \delta dS + \alpha_B P_B(r_B dt - dJ_B) + \alpha_C P_C(r_C dt - dJ_C) \\ &\quad + \left\{r(\widehat{V} - \alpha_B P_B) + s_F(\widehat{V} - \alpha_B P_B)^- - \alpha_C r P_C - \delta\left(q_S - \gamma_S\right)S\right\}dt \\ &= \left\{r\widehat{V} + s_F(\widehat{V} - \alpha_B P_B)^- + \delta\left(\gamma_S - q_S\right)S + \alpha_B P_B(r_B - r) + \alpha_C P_C(r_C - r)\right\}dt \\ &\quad + \delta dS - \alpha_B P_B dJ_B - \alpha_C P_C dJ_C \end{aligned} \tag{2.9}$$

It is here important to recall that:

$$\begin{aligned} \delta dS &= \delta(\mu S dt + \sigma S dW) \\ &= \delta\mu S dt + \delta\sigma S dW \end{aligned}$$

2.3 Finding $d\widehat{V}$ through Ito's Lemma

$$d\widehat{V} = \frac{\partial}{\partial t}\widehat{V}dt + \frac{\partial}{\partial S}\widehat{V}dS + \frac{1}{2}\frac{\partial^2}{\partial S^2}\widehat{V}\sigma^2 S^2 dt + \Delta\widehat{V}_B dJ_B + \Delta\widehat{V}_C dJ_C \tag{2.10}$$

with:

$$\Delta\widehat{V}_B = \widehat{V}(t,S,1,0) - \widehat{V}(t,S,0,0) \tag{2.11}$$

and

$$\Delta\widehat{V}_C = \widehat{V}(t,S,0,1) - \widehat{V}(t,S,0,0) \tag{2.12}$$

It is here important to recall that:

$$\frac{\partial}{\partial S}\widehat{V}dS = \frac{\partial}{\partial S}\widehat{V}(\mu S dt + \sigma S dW)$$

$$= \frac{\partial}{\partial S}\widehat{V}\mu S dt + \frac{\partial}{\partial S}\widehat{V}\sigma S dW$$

2.4 Eliminating all risk sources

Now we eliminate all risk sources: dW, dJ_B and dJ_C.

- dW :

The Brownian motion dW is present in the process for the asset S, in both representation of $d\widehat{V}$, both in the one deriving from the self-financing replicating strategy and in the one coming from the derivation of Itô's Lemma.

We must therefore equate both terms in dW and get:

$$\delta\sigma S = \frac{\partial}{\partial S}\widehat{V}\sigma S$$

$$\delta = \frac{\partial}{\partial S}\widehat{V} \tag{2.13}$$

- dJ_B :

We equate the terms containing the jump state associated with the default of party B, dJ_B, which are present in both derivations of $d\widehat{V}$:

$$-\alpha_B P_B dJ_B = \Delta\widehat{V}_B dJ_B$$

$$\begin{aligned}\alpha_B &= -\frac{\Delta\widehat{V}_B}{P_B} \\ &= \frac{\widehat{V} - (M^- + R_B M^+)}{P_B}\end{aligned} \tag{2.14}$$

recalling that:

$$\begin{aligned}\Delta\widehat{V}_B &= \widehat{V}(t,S,1,0) - \widehat{V}(t,S,0,0) \\ &= (M^- + R_B M^+) - \widehat{V}\end{aligned}$$

- dJ_C :

We now equate the terms containing the jump state associated with the default of party C, dJ_C, which are present in both derivations of $d\widehat{V}$:

$$-\alpha_C P_C dJ_C = \Delta \widehat{V}_C dJ_C$$

$$\begin{aligned}\alpha_C &= -\frac{\Delta \widehat{V}_C}{P_C} \\ &= \frac{\widehat{V} - (M^+ + R_C M^-)}{P_C}\end{aligned} \tag{2.15}$$

recalling that:

$$\begin{aligned}\Delta \widehat{V}_C &= \widehat{V}(t,S,0,1) - \widehat{V}(t,S,0,0) \\ &= (M^+ + R_C M^-) - \widehat{V}\end{aligned}$$

Then we see that from the expression for $d\widehat{V}$ as a self-financing replicating strategy, after eliminating the risk factors, we have:

$$d\widehat{V} = \left\{ r\widehat{V} + s_F(\widehat{V} - \alpha_B P_B)^- + \delta\,(\gamma_S - q_S)\,S + \alpha_B P_B(r_B - r) + \alpha_C P_C(r_C - r) + \delta S \mu \right\} dt$$

where we substitute the recent findings for α_B, α_C and δ, and we recall that:

$$r_B - r = \lambda_B$$

$$r_C - r = \lambda_C$$

therefore obtaining:

$$d\widehat{V} = \left\{ r\widehat{V} + s_F(\widehat{V} + \Delta \widehat{V}_B)^- + (\gamma_S - q_S)\,\frac{\partial}{\partial S}\widehat{V} S - \lambda_B \Delta \widehat{V}_B - \lambda_C \Delta \widehat{V}_C + \frac{\partial}{\partial S}\widehat{V} S \mu \right\} dt \tag{2.16}$$

From the expression for $d\widehat{V}$ derived through Itô's Lemma, after eliminating all risk factors, instead, we get:

$$d\widehat{V} = \left\{ \frac{\partial}{\partial t}\widehat{V} + \frac{\partial}{\partial S}\widehat{V} \mu S + \frac{1}{2}\frac{\partial^2}{\partial S^2}\widehat{V}\sigma^2 S^2 \right\} dt \tag{2.17}$$

2.5 Finding the general solution

Equating the two findings we obtain:

$$r\widehat{V} + s_F(\widehat{V} + \Delta \widehat{V}_B)^- + (\gamma_S - q_S)\,\frac{\partial}{\partial S}\widehat{V} S - \lambda_B \Delta \widehat{V}_B - \lambda_C \Delta \widehat{V}_C = \frac{\partial}{\partial t}\widehat{V} + \frac{1}{2}\frac{\partial^2}{\partial S^2}\widehat{V}\sigma^2 S^2$$

Now we introduce the elliptic differential operator A_t defined as:

$$A_t V \equiv \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} V + (q_S - \gamma_S) S \frac{\partial}{\partial S} V$$

we recognize it in the equating PDE and obtain:

$$\begin{cases} \frac{\partial}{\partial t}\widehat{V} + A_t \widehat{V} - r\widehat{V} = s_F(\widehat{V} + \Delta\widehat{V}_B)^- - \lambda_B \Delta\widehat{V}_B - \lambda_C \Delta\widehat{V}_C \\ \widehat{V}(T,S) = H(S) \end{cases} \tag{2.18}$$

We now insert the boundary condition for $\Delta\widehat{V}_B$:

$$\begin{aligned} \Delta\widehat{V}_B &= \widehat{V}(t,S,1,0) - \widehat{V}(t,S,0,0) \\ &= (M^- + R_B M^+) - \widehat{V} \end{aligned}$$

and $\Delta\widehat{V}_C$:

$$\begin{aligned} \Delta\widehat{V}_C &= \widehat{V}(t,S,0,1) - \widehat{V}(t,S,0,0) \\ &= (M^+ + R_C M^-) - \widehat{V} \end{aligned}$$

and we see that:

$$\begin{cases} \frac{\partial}{\partial t}\widehat{V} + A_t \widehat{V} - r\widehat{V} = s_F(\widehat{V} + M^- + R_B M^+ - \widehat{V})^- - \lambda_B(M^- + R_B M^+ - \widehat{V}) - \lambda_C(M^+ + R_C M^- - \widehat{V}) \\ \widehat{V}(T,S) = H(S) \end{cases}$$

where the negative part of the term multiplying s_F is simply M^-, therefore resulting in:

$$\begin{cases} \frac{\partial}{\partial t}\widehat{V} + A_t \widehat{V} - r\widehat{V} = s_F M^- - \lambda_B(M^- + R_B M^+) + \lambda_B \widehat{V} - \lambda_C(M^+ + R_C M^-) + \lambda_C \widehat{V} \\ \widehat{V}(T,S) = H(S) \end{cases}$$

$$\begin{cases} \frac{\partial}{\partial t}\widehat{V} + A_t \widehat{V} - r\widehat{V} = (\lambda_B + \lambda_C)\widehat{V} + s_F M^- - \lambda_B(M^- + R_B M^+) - \lambda_C(M^+ + R_C M^-) \\ \widehat{V}(T,S) = H(S) \end{cases}$$

It can be recognized that, in case of the non-risky derivative V, the regular B&S PDE would be satisfied:

$$\begin{cases} \frac{\partial}{\partial t} V + A_t V - rV = 0 \\ V(T,S) = H(S) \end{cases} \tag{2.19}$$

2.6 The solution when $M(t,S) = \widehat{V}(t,S)$

The authors in [20] then proceed with two different assumptions on the value of the mark-to-market M, which can be either equal to V or $\widehat{V}$. As introduced in Chapter 1, in fact, it is possible to adopt either a risk-free or a replacement closeout procedure, for the valuation of the residual of a deal in case of early termination of a transaction.

For the purpose of this work we are interested in the analysis of the case when $M(t,S) = \widehat{V}(t,S)$, which leads to:

$$\begin{cases} \frac{\partial}{\partial t}\widehat{V} + A_t\widehat{V} - r\widehat{V} = (\lambda_B + \lambda_C)\widehat{V} + s_F(\widehat{V})^- - \lambda_B((\widehat{V})^- + R_B(\widehat{V})^+) - \lambda_C((\widehat{V})^+ + R_C(\widehat{V})^-) \\ \widehat{V}(T,S) = H(S) \end{cases}$$

In order to solve this PDE we split the case where $\widehat{V} \geq 0$ and $\widehat{V} \leq 0$. The two cases describe respectively the purchase and the sale of an option.

More specifically, in our setting where $\widehat{V}$ is the value of the derivative from the point of view of C, when $\widehat{V} \geq 0$ party C buys an option from party B, while when $\widehat{V} \leq 0$ party C sells an option to party B. The replicating strategy is put in place by the party that is expected to fulfill the payment at expiry.

- $\widehat{V} \geq 0$

Where $\widehat{V} \geq 0$ we can assume that we have only the positive component of the mark-to-market of the derivative, and therefore we can eliminate all the terms where the assumed value of $\widehat{V}$ is negative.

$$\begin{cases} \frac{\partial}{\partial t}\widehat{V} + A_t\widehat{V} - r\widehat{V} = \lambda_B((\widehat{V})^+ - R_B(\widehat{V})^+) + \lambda_C((\widehat{V})^+ - (\widehat{V})^+) \\ \widehat{V}(T,S) = H(S) \end{cases}$$

$$\begin{cases} \frac{\partial}{\partial t}\widehat{V} + A_t\widehat{V} - r\widehat{V} = \lambda_B(1 - R_B)(\widehat{V})^+ \\ \widehat{V}(T,S) = H(S) \end{cases} \tag{2.20}$$

- $\widehat{V} \leq 0$:

Where $\widehat{V} \leq 0$ we can eliminate the terms where the mark-to-market would be positive.

$$\begin{cases} \frac{\partial}{\partial t}\widehat{V} + A_t\widehat{V} - r\widehat{V} = \lambda_B((\widehat{V})^- - (\widehat{V})^-) - \lambda_C((\widehat{V})^- - R_C(\widehat{V})^-) + s_F(\widehat{V})^- \\ \widehat{V}(T,S) = H(S) \end{cases}$$

$$\begin{cases} \frac{\partial}{\partial t}\widehat{V} + A_t\widehat{V} - r\widehat{V} = \lambda_C(1 - R_C)(\widehat{V})^- + s_F(\widehat{V})^- \\ \widehat{V}(T,S) = H(S) \end{cases} \tag{2.21}$$

and if we merge the two solutions we found, we get:

$$\begin{cases} \frac{\partial}{\partial t}\widehat{V} + A_t\widehat{V} - r\widehat{V} = \lambda_B(1 - R_B)(\widehat{V})^+ + \lambda_C(1 - R_C)(\widehat{V})^- + s_F(\widehat{V})^- \\ \widehat{V}(T,S) = H(S) \end{cases} \tag{2.22}$$

We can now introduce the so called Credit Value Adjustment, CVA= $\widehat{U}$, which allows us to decompose $\widehat{V}$ in the following form:

$$\widehat{V} = V + \widehat{U} \tag{2.23}$$

The meaning of this decomposition is to see the value of the credit risky derivative $\widehat{V}$ as the sum of the credit risk-free value of the derivative V and an adjustment for the credit risk $\widehat{U}$, which can be separately calculated.

In this way we can re-write the PDE as it follows:

$$\begin{cases} \frac{\partial}{\partial t}(V+\widehat{U}) + A_t(V+\widehat{U}) - r(V+\widehat{U}) = \lambda_B(1-R_B)(V+\widehat{U})^+ + \lambda_C(1-R_C)(V+\widehat{U})^- + s_F(V+\widehat{U})^- \\ (V+\widehat{U})(T,S) = H(S) \end{cases}$$

We can here remember that V satisfies the regular B&S PDE, so that the terms in V sum up to zero in the left hand side and the same V has a terminal value equal to $H(S)$, which means that, at maturity date$T$, $\widehat{V}$ and V must converge to the same value of $H(S)$. Given V as known, we can see that the remaining terms in $\widehat{U}$ must satisfy the following:

$$\begin{cases} \frac{\partial}{\partial t}\widehat{U} + A_t\widehat{U} - r\widehat{U} = \lambda_B(1-R_B)(V+\widehat{U})^+ + \lambda_C(1-R_C)(V+\widehat{U})^- + s_F(V+\widehat{U})^- \\ U(T,S) = 0 \end{cases}$$

The solution of this general case when $M = \widehat{V}$ can be found by either applying the Feyman-Kač theorem (and the Fubini theorem) and solving the resulting integral equation, or by solving the above non-linear PDE.

First we adopt the formal solution provided by the authors in [20], where the Feyman-Kač theorem is applied. We will then analyse the two separate cases when:

- $M = \widehat{V}$ and $r_F = r$
- $M = \widehat{V}$ and $r_F = r + s_F$

The calculation of the value of $\widehat{V}(t,s)$, of $\widehat{U}_0(t,s)$ (if $s_F = 0$) and of $\widehat{U}(t,s)$ (if $s_F \neq 0$) will necessarily be done by separating the two cases when $\widehat{V} \geq 0$ and $\widehat{V} \leq 0$.

$\widehat{U}_0(t,s)$ is the value of $\widehat{U}(t,s)$ when $s_F = 0$.

To sum up, we will find results for $\widehat{U}_0(t,s)$ or $\widehat{U}(t,s)$ in four different cases, and we will then try to see how they can be summarized in the formal general solution provided by the authors.

The formal solution provided by the authors in [20] is the following:

$$\begin{aligned} \widehat{U}(t,s) = &-(1-R_B)\int_t^T \lambda_B(u)D_r(t,u)E_t\left[(V(u,S(u))+\widehat{U}(u,S(u)))^+\right]du \\ &-(1-R_C)\int_t^T \lambda_C(u)D_r(t,u)E_t\left[(V(u,S(u))+\widehat{U}(u,S(u)))^-\right]du \\ &-\int_t^T s_F(u)D_r(t,u)E_t\left[(V(u,S(u))+\widehat{U}(u,S(u)))^-\right]du \end{aligned} \tag{2.24}$$

2.6.1 *The solution when* $M = \widehat{V}$ and $r_F = r$

When $r_F = r$ it is because the funding spread s_F is equal to zero. In this case the PDE reduces to:

$$\begin{cases} \frac{\partial}{\partial t}\widehat{V} + A_t\widehat{V} - r\widehat{V} = \lambda_B(1-R_B)(\widehat{V})^+ + \lambda_C(1-R_C)(\widehat{V})^- \\ \widehat{V}(T,S) = H(S) \end{cases}$$

This non-linear PDE has to be solved numerically, unless we distinguish the two cases where $\widehat{V} \geq 0$ and $\widehat{V} \leq 0$, which allow us to apply the Feyman-Kač.

- $\widehat{V} \geq 0$:

  When $\widehat{V} \geq 0$, we have:

$$\begin{cases} \frac{\partial}{\partial t}\widehat{V} + A_t\widehat{V} - r\widehat{V} - \lambda_B(1-R_B)\widehat{V} = 0 \\ \widehat{V}(T,S) = H(S) \end{cases} \tag{2.25}$$

  and the Feyman-Kač representation gives:

$$\widehat{V}(t,s) = E_t\left[D_{r+(1-R_B)\lambda_B}(t,T)H(S(T))\right] \tag{2.26}$$

  where:

$$D_y(t,T) = \exp\left(-\int_t^T y(s)ds\right)$$

  We can work in order to write the Feyman-Kač representation as:

$$\begin{aligned} \widehat{V}(t,s) &= E_t\left[D_{r+(1-R_B)\lambda_B}(t,T)H(S(T))\right] \\ &= D_{(1-R_B)\lambda_B}(t,T)E_t\left[D_r(t,T)H(S(T))\right] \\ &= D_{(1-R_B)\lambda_B}(t,T)V(t,s) \end{aligned}$$

  given that $V(t,s) = E_t\left[D_r(t,T)H(S(T))\right]$.
  If we now consider the specification that $\widehat{V} = V + U$, we can also see that:

$$\begin{aligned} U(t,s) &= \widehat{V}(t,s) - V(t,s) \\ &= D_{(1-R_B)\lambda_B}(t,T)V(t,s) - V(t,s) \\ &= V(t,s)[D_{(1-R_B)\lambda_B}(t,T) - 1] \\ &= V(t,s)[D_x(t,T) - 1] \end{aligned}$$

  if we call $(1-R_B)\lambda_B(t) = x(t)$, with:

$$D_x(t,T) - 1 = \exp\left(-\int_t^T x(s)ds\right) - 1 := \exp(-f(T)) - 1$$

  We can now apply the Fundamental Theorem of Integral Calculus:

$$f(x) = \int_a^x \frac{\partial f(x)}{\partial x} dx$$

to $\exp(-f(T))$ so that:

$$\begin{aligned}
\exp(-f(T)) - 1 &= \int_t^T \frac{\partial(\exp(-f(u)) - 1)}{\partial u} du \\
&= -\int_t^T \exp(-f(u)) \frac{\partial f(u)}{\partial u} du \\
&= -\int_t^T \exp\left(-\int_t^u x(s)ds\right) \frac{\partial\left(\int_t^u x(s)ds\right)}{\partial u} du \\
&= -\int_t^T \exp\left(-\int_t^u x(s)ds\right) x(u)du \\
&= -\int_t^T D_x(t,u)x(u)du \\
&= -\int_t^T D_{(1-R_B)\lambda_B}(t,u)(1-R_B)\lambda_B(u)du
\end{aligned}$$

so that:

$$U_0(t,s) = -V(t,s) \int_t^T D_{(1-R_B)\lambda_B}(t,u)(1-R_B)\lambda_B(u)du \tag{2.27}$$

- $\widehat{V} \leq 0$:

  If $\widehat{V} \leq 0$, instead, we have:

$$\begin{cases} \frac{\partial}{\partial t}\widehat{V} + A_t\widehat{V} - r\widehat{V} - \lambda_C(1-R_C)\widehat{V} = 0 \\ \widehat{V}(T,S) = H(S) \end{cases} \tag{2.28}$$

  and the Feyman-Kač representation is:

$$\begin{aligned}
\widehat{V}(t,s) &= E_t\left[D_{r+(1-R_C)\lambda_C}(t,T)H(S(T))\right] \\
&= D_{(1-R_C)\lambda_C}(t,T)E_t\left[D_r(t,T)H(S(T))\right] \\
&= D_{(1-R_C)\lambda_C}(t,T)V(t,s)
\end{aligned}$$

  If we now consider the specification that $\widehat{V} = V + U$, we can also see that:

$$\begin{aligned}
U(t,s) &= \widehat{V}(t,s) - V(t,s) \\
&= D_{(1-R_C)\lambda_C}(t,T)V(t,s) - V(t,s) \\
&= V(t,s)[D_{(1-R_C)\lambda_C}(t,T) - 1]
\end{aligned}$$

$$= V(t,s)[D_x(t,T)-1]$$

and, if we call $(1-R_C)\lambda_C(t) = x(t)$, again with:

$$D_x(t,T)-1 = \exp\left(-\int_t^T x(s)ds\right) - 1 := \exp\left(-f(T)\right) - 1$$

we can obtain, through the same process as in the case of $V \geq 0$, the following result for $U_0(t,s)$, when $V \leq 0$:

$$U_0(t,s) = -V(t,s)\int_t^T D_{(1-R_C)\lambda_C}(t,u)(1-R_C)\lambda_C(u)du \qquad (2.29)$$

2.6.2 *The solution when* $M = \widehat{V}$ *and* $r_F = r + (1-R_B)\lambda_B$

We can here assume that the funding spread s_F is not equal to zero, but to $(1-R_B)\lambda_B$.

In this hypothesis, the PDE stays:

$$\begin{cases} \frac{\partial}{\partial t}\widehat{V} + A_t\widehat{V} - r\widehat{V} = \lambda_B(1-R_B)(\widehat{V})^+ + \lambda_C(1-R_C)(\widehat{V})^- + s_F(\widehat{V})^- \\ \widehat{V}(T,S) = H(S) \end{cases} \qquad (2.30)$$

Again we have to split the two cases where $\widehat{V} \geq 0$ and $\widehat{V} \leq 0$.We can easily see that the first case when $\widehat{V} \geq 0$ simplifies to the same PDE with $s_F = 0$, and therefore we do not need to deduce again what the Feyman-Kač representation is and what value of $U_0(t,s)$ we obtain.

- $\widehat{V} \leq 0$:

  When $\widehat{V} \leq 0$, the PDE reduces to:

$$\begin{cases} \frac{\partial}{\partial t}\widehat{V} + A_t\widehat{V} - r\widehat{V} - \lambda_C(1-R_C)\widehat{V} - s_F\widehat{V} = 0 \\ \widehat{V}(T,S) = H(S) \end{cases}$$

  and since $s_F = (1-R_B)\lambda_B$:

$$\begin{cases} \frac{\partial}{\partial t}\widehat{V} + A_t\widehat{V} - r\widehat{V} - \lambda_C(1-R_C)\widehat{V} - \lambda_B(1-R_B)\widehat{V} = 0 \\ \widehat{V}(T,S) = H(S) \end{cases} \qquad (2.31)$$

  which leads to:

$$\begin{aligned}\widehat{V}(t,s) &= E_t\left[D_{r+\lambda_C(1-R_C)+\lambda_B(1-R_B)}(t,T)H(S(T))\right]\\ &= D_{\lambda_C(1-R_C)+\lambda_B(1-R_B)}(t,T)E_t\left[D_r(t,T)H(S(T))\right]\\ &= D_k(t,T)V(t,s)\end{aligned}$$

with $k = \lambda_C(1-R_C)+\lambda_B(1-R_B)$.
In this case, when looking for $U(t,s)$, we find:

$$\begin{aligned}U(t,s) &= \widehat{V}(t,s) - V(t,s)\\ &= D_k(t,T)V(t,s) - V(t,s)\\ &= V(t,s)\left[D_k(t,T)-1\right]\end{aligned}$$

We now assume $k(t) = x(t)$, with $D_k(t,T) = D_x(t,T)$

$$D_x(t,T) - 1 = \exp\left(-\int_t^T x(s)ds\right) - 1 := \exp\left(-f(T)\right) - 1$$

and again:

$$\begin{aligned}\exp\left(-f(T)\right) - 1 &= \int_t^T \frac{\partial(\exp\left(-f(u)\right)-1)}{\partial u}du\\ &= -\int_t^T \exp(-f(u))\frac{\partial f(u)}{\partial u}du\\ &= -\int_t^T \exp\left(-\int_t^u x(s)ds\right)\frac{\partial\left(\int_t^u x(s)ds\right)}{\partial u}du\\ &= -\int_t^T \exp\left(-\int_t^u x(s)ds\right)x(u)du\\ &= -\int_t^T D_x(t,u)x(u)du\\ &= -\int_t^T D_k(t,u)k(u)du\\ &= -\int_t^T D_{\lambda_C(1-R_C)+\lambda_B(1-R_B)}(t,u)\left[\lambda_C(u)(1-R_C)+\lambda_B(u)(1-R_B)\right]du\end{aligned}$$

so that:

$$\begin{aligned}U(t,s) &= -V(t,s)\int_t^T D_{\lambda_C(1-R_C)+\lambda_B(1-R_B)}(t,u)\left[\lambda_C(u)(1-R_C)+\lambda_B(u)(1-R_B)\right]du\\ &= -V(t,s)\int_t^T D_k(t,u)k(u)du \qquad (2.32)\end{aligned}$$

2.7 Verifying the general formal PDE integral solution

The general PDE for $\widehat{V}$ found by Burgar and Kjaer cannot be solved analitically as it is non-linear. We saw that it is possible to transform the PDE in $\widehat{V}$ in a PDE in $\widehat{U}$ by assuming that V satisfies the regular B&S PDE and therefore acting as a known parameter.

Analogously, the PDE in $\widehat{U}$ cannot be solved analitically but only numerically, unless we assume and separately analyse the two different scenarios of $\widehat{V} \geq 0$ and $\widehat{V} \leq 0$.

This is the reason why we applied the Feyman-Kač theorem to $\widehat{V}$ either for $\widehat{V} \geq 0$ and for $\widehat{V} \leq 0$.

Indeed, we applied the Feyman-Kač theorem four times, as we specified the outcomes for $\widehat{V}(t,S)$, $\widehat{U}_0(t,S)$ (if $s_F = 0$) and $\widehat{U}(t,S)$ (if $s_F \neq 0$), also on the basis of the presence of the funding spread. Therefore we investigated the following four cases:

- $\widehat{V} \geq 0$ and $s_F = 0$
- $\widehat{V} \leq 0$ and $s_F = 0$
- $\widehat{V} \geq 0$ and $s_F \neq 0$
- $\widehat{V} \leq 0$ and $s_F \neq 0$

As we anticipated in the previous section, we want to show how the different results we have just found can be summarized in the general formal solution provided by the authors.

More specifically we will first separately investigate the cases where $\widehat{V} \geq 0$ and $\widehat{V} \leq 0$, and we will then merge the findings.

- $\widehat{V} \geq 0$

Given the PDE we found in the previous section:

$$\begin{cases} \frac{\partial}{\partial t}\widehat{V} + A_t\widehat{V} - r\widehat{V} = \lambda_B(1-R_B)(\widehat{V})^+ + \lambda_C(1-R_C)(\widehat{V})^- + s_F(\widehat{V})^- \\ \widehat{V}(T,S) = H(S(T)) \end{cases}$$

we saw that, the in case of $\widehat{V} \geq 0$, it reduces to:

$$\begin{cases} \frac{\partial}{\partial t}\widehat{V} + A_t\widehat{V} - r\widehat{V} = \lambda_B(1-R_B)\widehat{V} \\ \widehat{V}(T,S) = H(S(T)) \end{cases}$$

and the Feyman-Kač representation of $\widehat{V}(t,s)$ is:

$$\begin{aligned} \widehat{V}(t,s) &= E_t\left[D_{r+(1-R_B)\lambda_B}(t,T)H(S(T))\right] \\ &= D_{(1-R_B)\lambda_B}(t,T)V(t,s) \end{aligned}$$

We also saw that the CVA, here indicated by $\widehat{U}(t,s)$, is in this case given by:

$$
\begin{aligned}
\widehat{U}(t,s) &= \widehat{V}(t,s) - V(t,s) \\
&= D_{(1-R_B)\lambda_B}(t,T)V(t,s) - V(t,s) \\
&= V(t,s)\left[D_{(1-R_B)\lambda_B}(t,T) - 1\right]
\end{aligned}
$$

We already proved either for $r_F = r$ and for $r_F = r + s_F$ that when $\widehat{V} \geq 0$ we have:

$$
\widehat{U}(t,s) = -V(t,s)\int_t^T (1-R_B)\lambda_B(u)D_{(1-R_B)\lambda_B}(t,u)du
$$

At this point we can further elaborate this integral solution for $\widehat{U}(t,s)$ and see that, through $V(t,s) = E_t\left[D_r(t,T)V(T,S)\right]$:

$$
\begin{aligned}
\widehat{U}(t,s) &= -\int_t^T (1-R_B)\lambda_B(u)D_{(1-R_B)\lambda_B}(t,u)E_t\left[D_r(t,u)V(u,S(u))\right]du \\
&= -(1-R_B)\int_t^T \lambda_B(u)D_r(t,u)E_t\left[D_{(1-R_B)\lambda_B}(t,u)V(u,S(u))\right]du \\
&= -(1-R_B)\int_t^T \lambda_B(u)D_r(t,u)E_t\left[(V(u,S(u)) + \widehat{U}(u,S(u)))^+\right]du
\end{aligned}
$$

where we recognized that $D_{(1-R_B)\lambda_B}(t,u)V(u,S(u)) = \widehat{V}(u,S(u))$ when $\widehat{V} \geq 0$, and where we applied the positive sign to indicate we are in the case when $\widehat{V} \geq 0$.

- $\widehat{V} \leq 0$

  Given the same PDE we found in the previous section:

$$
\begin{cases}
\frac{\partial}{\partial t}\widehat{V} + A_t\widehat{V} - r\widehat{V} = \lambda_B(1-R_B)(\widehat{V})^+ + \lambda_C(1-R_C)(\widehat{V})^- + s_F(\widehat{V})^- \\
\widehat{V}(T,S) = H(S(T))
\end{cases}
$$

  we saw that, the in case of $\widehat{V} \leq 0$, it reduces to:

$$
\begin{cases}
\frac{\partial}{\partial t}\widehat{V} + A_t\widehat{V} - r\widehat{V} = \lambda_C(1-R_C)\widehat{V} + s_F\widehat{V} \\
\widehat{V}(T,S) = H(S(T))
\end{cases}
$$

  and the Feyman-Kač representation of $\widehat{V}(t,s)$ is:

$$
\begin{aligned}
\widehat{V}(t,s) &= E_t\left[D_{r+(1-R_B)\lambda_B+(1-R_C)\lambda_C}(t,T)H(S(T))\right] \\
&= D_{(1-R_B)\lambda_B+(1-R_C)\lambda_C}(t,T)V(t,s)
\end{aligned}
$$

  We also saw that the CVA, here indicated by $\widehat{U}(t,s)$, is in this case given by:

$$
\widehat{U}(t,s) = \widehat{V}(t,s) - V(t,s)
$$

$$
\begin{aligned}
&= D_{(1-R_B)\lambda_B+(1-R_C)\lambda_C}(t,T)V(t,s) - V(t,s) \\
&= V(t,s)\left[D_{(1-R_B)\lambda_B+(1-R_C)\lambda_C}(t,T) - 1\right]
\end{aligned}
$$

As we are managing the general case for $\widehat{V} \leq 0$, without specifying if $s_F \neq 0$, the rate s_F actually appears in our representation of $\widehat{U}(t,s)$, and we can therefore use the solution we found in the case when $s_F \neq 0$, which is:

$$
\begin{aligned}
\widehat{U}(t,s) &= -V(t,s)\int_t^T D_{\lambda_C(1-R_C)+\lambda_B(1-R_B)}(t,u)\left[\lambda_C(u)(1-R_C)+\lambda_B(u)(1-R_B)\right]du \\
&= -V(t,s)\int_t^T (1-R_C)\lambda_C(u)D_{\lambda_C(1-R_C)+\lambda_B(1-R_B)}(t,u)du \\
&\quad -V(t,s)\int_t^T (1-R_B)\lambda_B(u)D_{\lambda_C(1-R_C)+\lambda_B(1-R_B)}(t,u)du \\
&= -(1-R_C)\int_t^T \lambda_C(u)D_{\lambda_C(1-R_C)+\lambda_B(1-R_B)}(t,u)E_t\left[D_r(t,u)V(u,S(u))\right]du \\
&\quad -\int_t^T (1-R_B)\lambda_B(u)D_{\lambda_C(1-R_C)+\lambda_B(1-R_B)}(t,u)E_t\left[D_r(t,u)V(u,S(u))\right]du \\
&= -(1-R_C)\int_t^T \lambda_C(u)D_r(t,u)E_t\left[D_{\lambda_C(1-R_C)+\lambda_B(1-R_B)}(t,u)V(u,S(u))\right]du \\
&\quad -\int_t^T s_F(u)D_r(t,u)E_t\left[D_{\lambda_C(1-R_C)+\lambda_B(1-R_B)}(t,u)V(u,S(u))\right]du \\
&= -(1-R_C)\int_t^T \lambda_C(u)D_r(t,u)E_t\left[\left(V(u,S(u))+\widehat{U}(u,S(u))\right)^-\right]du \\
&\quad -\int_t^T s_F(u)D_r(t,u)E_t\left[\left(V(u,S(u))+\widehat{U}(u,S(u))\right)^-\right]du
\end{aligned}
$$

by recognizing that $(1-R_B)\lambda_B(u) = s_F(u)$ and $D_{\lambda_C(1-R_C)+\lambda_B(1-R_B)}(t,u)V(u,S(u)) = \widehat{V}(u,S(u))$ when $\widehat{V} \leq 0$. Moreover we added the negative sign to indicate we are indeed in the case when $\widehat{V} \leq 0$.

Basically we can see that this solution summarizes the two cases when $\widehat{V} \leq 0$, when $s_F = 0$ and when $s_F = (1-R_B)\lambda_B$.

If we put together the solution we found for $\widehat{V} \geq 0$ and the solution for $\widehat{V} \leq 0$, we can see that the general formal solution provided by the authors is verified.

$$
\begin{aligned}
\widehat{U}(t,s) &= -(1-R_B)\int_t^T \lambda_B(u)D_r(t,u)E_t\left[\left(V(u,S(u))+\widehat{U}(u,S(u))\right)^+\right]du \\
&\quad -(1-R_C)\int_t^T \lambda_C(u)D_r(t,u)E_t\left[\left(V(u,S(u))+\widehat{U}(u,S(u))\right)^-\right]du \\
&\quad -\int_t^T s_F(u)D_r(t,u)E_t\left[\left(V(u,S(u))+\widehat{U}(u,S(u))\right)^-\right]du
\end{aligned}
$$

Chapter 3
Introducing a pure Liquidity Cost in the framework of Bilateral Counterparty Credit Risk with Funding Costs

Abstract In this chapter we describe the approach followed by Morini and Prampolini in [44], for the calculation of Bilateral Counterparty Credit Risk charges in association with Funding Costs. A comparison between formulas obtained in [44] and the Black & Scholes results of [20] is then proposed. We show that the two approaches rejoin in the same result, when considering a vanilla payoff, where only one of the two parties commits itself to pay a predetermined amount of money to the other party, at a fixed date in the future. Nevertheless, both parties are yet subject to risk of default.

3.1 CVA and DVA

The approach followed in [44] consists in the calculation of the expected value of a transaction in an economy where a party B commits to pay a fixed amount K to party C, at a given time T in the future. This is a single period model with a given payoff at expiry T and with deterministic interest rate r. The time interval is defined by t, with $t \in [0;T]$.

The only source of uncertainty is introduced in the model through the probability of default of the two parties. The risk of default is developed through the intensity model approach that will be deeply investigated also in the following chapters. Of course the most relevant impact of a default event would verify in case of default of the borrowing party B, the one expected to pay the agreed amount K at time T. This is the reason why the amount K is weighted by the probability of default of B, also referred to as the ”borrower” or the”issuer”.

As in [20], also in [44] the probability of default of the two parties is introduced along with the concept of CVA and DVA, which are respectively ”Credit Value Adjustment” and ”Debt Value Adjustment”, as explained in Chapter 1.

In [44] the value V_B for the borrower B of this transaction is given by:

$$V_B = P + DVA_B - Ke^{-rT} \tag{3.1}$$

where DVA_B is the "Debt Value Adjustment" deriving from the fact that B will pay the amount K at expiry date T only upon its survival until that moment. The amount P is the premium that party B receives at start date.

Specifically, considering τ_B the time of default of party B with $\tau_B \in [0;T]$, we have that the formula for it is given by:

$$\begin{aligned} DVA_B &= E\left[e^{-rT} K \mathbf{1}\{\tau_B \leq T\}\right] \\ &= Ke^{-rT} \Pr\{\tau_B \leq T\} \end{aligned} \tag{3.2}$$

If we look at DVA_B, it is the expected value of the amount K that B may not pay in T, weighted by its own probability of default. In fact, B will only pay K at maturity if it does not default in advance.

Substituting the expression for DVA_B in the formula for V_B in $t = 0$ we obtain the following:

$$\begin{aligned} V_B &= P + DVA_B - Ke^{-rT} \\ &= P + Ke^{-rT} \Pr\{\tau_B \leq T\} - Ke^{-rT} \\ &= P - Ke^{-rT}(1 - \Pr\{\tau_B \leq T\}) \\ &= P - Ke^{-rT} \Pr\{\tau_B > T\} \\ &= P - Ke^{-rT} e^{-\pi_B T} \end{aligned}$$

where the intensity model approach gives us:

$$\Pr\{\tau_B > T\} = e^{-\pi_B T}$$

with π_B being the deterministic and instantaneous default probability of party B. In credit structuring and pricing, π_B is set equal to the CDS spread of party B, which is observable in the market. The spread π_B is intended to be $\pi_B = \lambda_B LGD_B = \lambda_B(1 - R_B)$, where λ_B is the instantaneous default intensity of B, LGD_B is the "loss given default" of B, and R_B is the recovery rate of B.

In [44] π_B is therefore correctly addressed to as the "risk-adjusted instantaneous default probability".

Going back to the formula for V_B we just found in [44], it is very intuitive to describe the flows for the borrower as the cashing of the premium at inception minus the discounted value of the capital reimbursement weighted by its own probability of survival until maturity. This effect is indeed obtained through the DVA_B.

Solving for the value of P that gives a breakeven level for B we obtain, by putting $V_B = 0$:

$$P = Ke^{-rT} e^{-\pi_B T} \tag{3.3}$$

The borrower will enter this transaction if the premium received at inception will be at least equal to the net present value of the amount K weighted by its own probability of survival.

Analogously, in [44] Morini and Prampolini compute the value V_L of the same transaction from the point of view of L, the ”lender”, who has to receive the amount K at time T, upon survival of the borrower until that moment:

$$V_L = -P - CVA_L + Ke^{-rT} \tag{3.4}$$

The component CVA_L is the ”Credit Value Adjustment” that the lender L is bearing for the fact that he will receive the amount K at maturity T only if the borrower does not default before that moment. It is therefore equal to the same DVA_B:

$$\begin{aligned} CVA_L &= E\left[e^{-rT} K \mathbf{1}\{\tau_B \leq T\}\right] \\ &= Ke^{-rT}\,\mathrm{Pr}\{\tau_B \leq T\} \end{aligned} \tag{3.5}$$

Inserting it in the expression for V_L it gives:

$$\begin{aligned} V_L &= -P - CVA_L + Ke^{-rT} \\ &= -P - Ke^{-rT}\,\mathrm{Pr}\{\tau_B \leq T\} + Ke^{-rT} \\ &= -P + Ke^{-rT}(1 - \mathrm{Pr}\{\tau_B \leq T\}) \\ &= -P + Ke^{-rT}\,\mathrm{Pr}\{\tau_B > T\} \\ &= -P + Ke^{-rT} e^{-\pi_B T} \end{aligned}$$

and solving for the breakeven value for the lender through $V_L = 0$:

$$P = Ke^{-rT} e^{-\pi_B T}$$

Therefore the lender L should enter this transaction by paying a premium P equal to or smaller than the discounted value of K, times the survival probability of the borrower.

Indeed we can see that the equilibrium price P for both the borrower B and the lender L coincides.

3.2 Introducing Funding Cost and Funding Benefit

The authors in [44] try to understand if they should also make some further considerations regarding the premium P paid at the start date $t = 0$.

On the lender side, they ask themselves if a funding cost for financing the premium should be accounted for. Whereas, on the borrower side, they investigate if a funding benefit from receiving the same premium should be considered.

They verify that introducing these elements is not indeed necessary because the introduction of DVA already allows for a proper comprehension of these funding effects.

3.2.1 *Does the Borrower have to consider a Funding Benefit by receiving P in $t = 0$?*

From the point of view of the borrower B, one can argue that funding an amount P, if not provided by the lender L through this transaction, would have anyway generated a negative cashflow at maturity T equal to $-Pe^{rT}e^{s_B T}\mathbf{1}\{\tau_B > T\}$, where s_B is here introduced and it is the funding spread for the borrower B.

The funding spread s_B in [44] is assumed to be equal to $s_B = \pi_B + \gamma_B$, where π_B is the CDS spread for B, and γ_B is a pure "liquidity basis".

In the context of our deal, it is therefore possible to say that receiving the premium P in $t = 0$ is equivalent to receiving $Pe^{rT}e^{s_B T}\mathbf{1}\{\tau_B > T\}$ in T, to be added to what B has to pay in T, that is $K\mathbf{1}\{\tau_B > T\}$. We therefore investigate if a "Funding Benefit" should be accounted for, when computing the "fair" premium P from the point of view of the borrower B.

We indicate with $\widetilde{V}_B$ the "new" total payoff for B at time $t = T$, and we see that it becomes consequently equal to:

$$\widetilde{V}_B = Pe^{rT}e^{s_B T}\mathbf{1}\{\tau_B > T\} - K\mathbf{1}\{\tau_B > T\} \tag{3.6}$$

and, discounting it to $t = 0$, the payoff $\widetilde{V}_B$ transforms in the following "new" V_B:

$$\begin{aligned} V_B &= Pe^{s_B T}\mathbf{1}\{\tau_B > T\} - Ke^{-rT}\mathbf{1}\{\tau_B > T\} \\ &= Pe^{\pi_B T}e^{\gamma_B T}\mathbf{1}\{\tau_B > T\} - Ke^{-rT}\mathbf{1}\{\tau_B > T\} \\ &= Pe^{\pi_B T}e^{\gamma_B T}e^{-\pi_B T} - Ke^{-rT}e^{-\pi_B T} \\ &= Pe^{\gamma_B T} - Ke^{-rT}e^{-\pi_B T} \end{aligned}$$

As a matter of fact, apart from a pure liquidity basis γ_B, the price P for the borrower B would be equal to the one previously found when no Funding Benefit was accounted for:

$$P = Ke^{-rT}e^{-\pi_B T}e^{-\gamma_B T}$$

and by putting $\gamma_B = 0$:

$$P = Ke^{-rT}e^{-\pi_B T}$$

So, considering the own probability of default eliminates any Funding Benefit, apart from a pure liquidity basis if $\gamma_B > 0$.

3.2.2 Does the Lender have to consider a Funding Cost by anticipating P in $t = 0$?

From the lender's perspective, instead, one can point out that cashing the premium P at inception in $t = 0$ may generate a "Funding Cost", arising when the lender L has to be financed for the amount of the premium P until maturity $t = T$.

Anticipating the amount P until time T can generate in $t = T$ a negative cashflow equal to $-Pe^{rT}e^{s_L T}\mathbf{1}\{\tau_L > T\}$, which is paid only if the lender L survives until expiry date $t = T$. Here s_L is the funding spread of the lender L, and it is given by $s_L = \pi_L + \gamma_L$, where π_L is the CDS spread of party L observable in the market, and γ_L is the liquidity basis for party L.

This negative cashflow, deriving from the funding deal, has to be added to the amount that L receives at time T if the borrower does not default until that moment, equal to $K\mathbf{1}\{\tau_B > T\}$, resulting in the following "new" payoff $\widetilde{V}_L$ for L in $t = T$:

$$\widetilde{V}_L = -Pe^{rT}e^{s_L T}\mathbf{1}\{\tau_L > T\} + K\mathbf{1}\{\tau_B > T\} \tag{3.7}$$

and, discounting it to $t = 0$, $\widetilde{V}_L$ results in the following "new" V_L:

$$\begin{aligned} V_L &= -Pe^{s_L T}\mathbf{1}\{\tau_L > T\} + Ke^{-rT}\mathbf{1}\{\tau_B > T\} \\ &= -Pe^{\pi_L T}e^{\gamma_L T}e^{-\pi_L T} + Ke^{-rT}e^{-\pi_B T} \\ &= -Pe^{\gamma_L T} + Ke^{-rT}e^{-\pi_B T} \end{aligned}$$

with the following "new" value for the equilibrium P for L:

$$P = Ke^{-rT}e^{-\pi_B T}e^{-\gamma_L T}$$

and by putting $\gamma_L = 0$:

$$P = Ke^{-rT}e^{-\pi_B T}$$

Apart from a pure liquidity basis γ_L, the price P for the lender L is again equal to the one already calculated when no Funding Cost was accounted for.

In fact, including the lender's own probability of default, eliminates the necessity to include any Funding Cost deriving from financing the premium, apart from a pure liquidity basis.

As a general finding, we may say that considering a "risky" funding, where one also takes into account its own probability of not surviving, compensates any further effect of Funding Cost or Funding Benefit, apart from a pure liquidity basis.

3.3 Assimilating different approaches

It is possible to see that the approach developed by Morini and Prampolini in [44] can be assimilated to the one presented by Burgard and Kjaer in [20], through the following adjustments.

- Funding rate: $r_F = r + s_F$ in Burgard and Kjaer (see [20]) equal to $r_B = r + s_B$ in Morini and Prampolini (see [44]).

Morini and Prampolini assume that the funding rate is always for a "risky" funding, and therefore they add the funding spread of the borrowing party to the risk-free rate. Their model can therefore be assimilated to the hypothesis of Burgar and Kjaer where they assume that $r_F = r + s_F$.

- Funding spread: $s_F = (1 - R_B)\lambda_B$ in Burgard and Kjaer (see [20]) equal to $s_B = \pi_B = LGD_B\lambda_B = (1 - R_B)\lambda_B$, with $\gamma_B = 0$ in Morini and Prampolini (see [20]).

Morini and Prampolini assume the funding spread is equal to the CDS spread from intensity models literature plus a pure liquidity basis spread, resulting in $s_F = \pi_B + \gamma_B$. If we assume in the Morini and Prampolini model that $\gamma_B = 0$, and therefore the funding spread for the borrower is only given by its CDS spread, then we are working under the same assumption in both models.

- The payoff at maturity: $H(S(T)) = K$

In order to assimilate the two approaches, the Burgar and Kjaer's payoff that the borrowing party has to pay at maturity, has to be assumed to be strictly positive and equal to K, as in the Morini and Prampolini's model.

This last assumption brings the Burgar and Kjaer model in the eventuality of always generating a positive value of the derivative itself, thus eliminating the case when $\widehat{V} \leq 0$.

Given this adaptation, we can see how the Burgar and Kjaer's case of $M = \widehat{V}$, $\widehat{V} \geq 0$ and $r_F = r + s_F$ with $s_F = (1 - R_B)\lambda_B$, by putting $H(S(T)) = K$, becomes:

$$
\begin{aligned}
\widehat{V}(t,s) &= E_t\left[D_{r+(1-R_B)\lambda_B}(t,T)H(S(T))\right] \\
&= E_t\left[D_{r+(1-R_B)\lambda_B}(t,T)K\right] \\
&= \exp(-\int_t^T (r(s) + (1-R_B)\lambda_B(s))ds)K \\
&= \exp(-\int_t^T r(s)ds)\exp(-\int_t^T (1-R_B)\lambda_B(s)ds)K
\end{aligned}
$$

and it is equal to the continuous time version of the following Morini and Prampolini case:

$$
\begin{aligned}
P &= e^{-rT} e^{-\pi_B T} K \\
&= e^{-rT} e^{-(1-R_B)\lambda_B T} K
\end{aligned}
$$

Here below you can find a more appropriate proof of this.
We shall remember findings from Chapter 2.
If B defaults first:

$$\widehat{V}(t,s,1,0) = M^- + R_B M^+$$

If C defaults first:

$$\widehat{V}(t,s,0,1) = R_C M^- + M^+$$

where M^- is the negative mtm for the surviving party, and M^+ is the positive mtm for the surving party. In this case the negative part of the mark to market does not exist, and therefore we obtain the following modified conditions.
If B defaults first:

$$\widehat{V}(t,s,1,0) = R_B M^+$$

If C defaults first:

$$\widehat{V}(t,s,0,1) = M^+$$

Recalling that:

$$
\begin{cases}
\frac{\partial}{\partial t}\widehat{V} + A_t\widehat{V} - r\widehat{V} = s_F(\widehat{V} + \Delta\widehat{V}_B)^- - \lambda_B \Delta\widehat{V}_B - \lambda_C \Delta\widehat{V}_C \\
\widehat{V}(T,S) = H(S)
\end{cases}
$$

with boundary condition for $\Delta\widehat{V}_B$:

$$
\begin{aligned}
\Delta\widehat{V}_B &= \widehat{V}(t,S,1,0) - \widehat{V}(t,S,0,0) \\
&= (M^- + R_B M^+) - \widehat{V}
\end{aligned}
$$

and $\Delta\widehat{V}_C$:

$$
\begin{aligned}
\Delta\widehat{V}_C &= \widehat{V}(t,S,0,1) - \widehat{V}(t,S,0,0) \\
&= (M^+ + R_C M^-) - \widehat{V}
\end{aligned}
$$

we see that in our case the boundary conditions modify to:

$$
\begin{aligned}
\Delta\widehat{V}_B &= \widehat{V}(t,S,1,0) - \widehat{V}(t,S,0,0) \\
&= R_B M^+ - \widehat{V}
\end{aligned}
$$

and

$$\begin{aligned}\Delta\widehat{V}_C &= \widehat{V}(t,S,0,1) - \widehat{V}(t,S,0,0)\\ &= M^+ - \widehat{V}\end{aligned}$$

Plugging them into the PDE:

$$\begin{cases}\frac{\partial}{\partial t}\widehat{V} + A_t\widehat{V} - r\widehat{V} = s_F(\widehat{V} + R_B M^+ - \widehat{V})^- - \lambda_B(R_B M^+ - \widehat{V}) - \lambda_C(M^+ - \widehat{V})\\ \widehat{V}(T,S) = K\end{cases}$$

$$\begin{cases}\frac{\partial}{\partial t}\widehat{V} + A_t\widehat{V} - r\widehat{V} = -\lambda_B R_B M^+ + \lambda_B\widehat{V} - \lambda_C M^+ + \lambda_C\widehat{V}\\ \widehat{V}(T,S) = K\end{cases}$$

$$\begin{cases}\frac{\partial}{\partial t}\widehat{V} + A_t\widehat{V} - r\widehat{V} = (\lambda_B + \lambda_C)\widehat{V} - \lambda_B R_B M^+ - \lambda_C M^+\\ \widehat{V}(T,S) = K\end{cases}$$

Substituting $M^+ = (\widehat{V})^+$ and considering that we are in the case where $\widehat{V} > 0$ always:

$$\begin{cases}\frac{\partial}{\partial t}\widehat{V} + A_t\widehat{V} - r\widehat{V} = (\lambda_B + \lambda_C)\widehat{V} - \lambda_B R_B\widehat{V} - \lambda_C\widehat{V}\\ \widehat{V}(T,S) = K\end{cases}$$

$$\begin{cases}\frac{\partial}{\partial t}\widehat{V} + A_t\widehat{V} - r\widehat{V} = \lambda_B\widehat{V} - \lambda_B R_B\widehat{V}\\ \widehat{V}(T,S) = K\end{cases}$$

$$\begin{cases}\frac{\partial}{\partial t}\widehat{V} + A_t\widehat{V} - r\widehat{V} = (1 - R_B)\lambda_B\widehat{V}\\ \widehat{V}(T,S) = K\end{cases}$$

$$\widehat{V}(t,s) = E_t\left[D_{r+(1-R_B)\lambda_B}(t,T)K\right]$$

What can be seen in both the approaches we examined is that, in the hypothesis of $s_B = \pi_B$ with no liquidity basis, when there is only one defaultable party that has the obligation to pay at expiry, a ”Unilateral Credit Counterparty Risk” has been considered.

To give a better explanation of the point, we can identify two aspects of the overall financial operation, being the ”transaction deal”, where one of the two parties commits to pay a payoff at expiry, and the ”funding deal”, where the other party may enter a financing transaction to cash the premium at inception.

In both models, in the ”transaction deal” it is only the probability of default of the party obliged to pay at expiry that is taken into account. In other terms, the ”debtor-risky” payoff is not weighted by the survival probability of the creditor, until we introduce Funding Costs and a pure liquidity basis component is then included. In fact, it is only through the introduction of Funding Costs for the ”funding deal” that we introduce also the probability of default of the ”creditor” party in the ”transaction deal”, i.e. the lender L, otherwise in the ”transaction deal” we would only see the default risk of the party expected to pay at expiry.

In the approach that will be developed in the following chapters, instead, both positive and negative payoffs will be weighted by the creditworthiness of both parties at the same time.

Chapter 4
Bilateral Counterparty Risk with Funding Costs: a discretized approach

Abstract In this chapter we will try to gather all the information learnt so far and develop a systematic approach to price a two-sided payoff, considering both Bilateral Counterparty Credit Risk and Funding Costs. In order to implement the outcoming approach, the theoretical model will be discretized, which will also allow us to make numerical tests. Basic assumption of no wrong-way risk is in place, meaning that there is not a direct link between the default risk of the counterparty and a possible increase in the exposure of the other party in the underlying transaction. Moreover, default probabilities of the two parties are not correlated.

4.1 Funding Cost Adjustment in the framework of Bilateral Credit Value Adjustment

We will assume that we have two parties, namely party A and party B, where party A may be thought of as being the ”financial institution”, while party B may be regarded as the ”counterparty”. As far as now, no consideration on the credit quality of either parties has been done.

As also discussed in previous chapters, we will assume that if one of the two parties defaults, then the other party will receive only a portion of the present value of its positive exposure towards the defaulting party, namely the recovery rate times the positive exposure. On the other hand, if at default the surviving party owes any amount to the defaulting party, then the surviving party will pay the full negative exposure to its counterparty. This is standard market practice.

We will study the value of a ”credit risky” derivative from the point of view of party A. From a funding perspective, we will assume that positive exposures do not generate any specific revenue, but, on the other side, we will consider that negative cash positions need to be funded from the moment they take place until maturity of the transaction. Positive or negative positions must be intended as ”net” negative or positive positions at a certain moment in time. This is the point where Funding Costs are indeed introduced in our model. As we are analyzing the value of the derivative from the point of view of party A, we will of course consider Funding Costs from the point of view of A. The inclusion of Funding Costs may be explained as the necessity to fund any liquidity disbursement that a party will have to face. Assuming that a party has no liquidity reserves implies recognizing that the party will have to go into the market, in case through the Treasury department, and fund itself at its current funding spread for the relevant maturity. As proposed also in [44], the funding spread can be set to be equal to the credit spread, plus a liquidity basis which can be explained as a friction between the primary and the secondary market. On the other hand, if the party had the liquidity to face the negative future cashflows, one should consider the opportunity cost of not investing that liquidity in the market rather than facing directly the cash outflows resulting from the derivative position. If we assume that the party could go at least in the market and buy back its own bonds with the excess liquidity, we may again come to the conclusion that the cost of a negative exposure is equal to its cost of funding, because this is what a party is giving up when not buying back its own bonds. As the buyback of own bonds in the secondary market would be done at a liquidity premium with respect to the funding spread used in the primary market, we can say that Funding Costs to fund negative positions must be at least equal to the opportunity cost of not extinguishing outstanding debt, i.e. the sum of the credit spread - where the credit spread is assumed to be the cost for issuances in the primary market - plus a liquidity premium specific of the secondary market.

We assume the time interval to be $[0,T]$, with $t_{i=0} = 0 < t_i < ... < t_{i=n} = T$ and $i \in N$.

Moreover we assume a possible default time at τ^1, with $\tau^1 \in (0,T)$, where it may be either $\tau^1 = \tau_A$ if party A defaults first or $\tau^1 = \tau_B$ if party B defaults first instead.

$\widetilde{X}(0,T)$ is the payoff valued at time $t_{i=0} = 0$ with maturity T. The value $\widetilde{X}(0,T)$ of the derivative must be intended as $\widetilde{X}(0,T) = \widetilde{X}^+(0,T) + \widetilde{X}^-(0,T)$, as we are thinking of a derivative where it is not possible to know in advance whether the mark-to-market $\widetilde{X}(0,T)$ will be positive or negative at any time t_i for our reference party. This would be for instance the case of an interest rate swap prior to any fixing date, where one of the two parties is paying a floating rate, like Euribor.

In order to proceed with the setup of a model able to properly account for Funding Costs, we will introduce the concept of net negative expected cash flows.

Net negative expected cash flows, in fact, will be the quantities to be funded from the moment they are actually due in the underlying transaction, until its maturity.

Each single net negative expected cash flow known in t_{i-1} - applicable for the period $(t_{i-1},t_i]$ and occurring in t_i - will be funded from t_i until final maturity T.

It is not the whole negative mark-to-market to be funded at each period until maturity date, but only the single net cash flow if negative.

For this reason we introduce Negative Expected Cash Flow quantity $ECF^-(t_{i-1},t_i)$.

Basically we have that:

- t_{i-1} is the point in time where party A knows the net negative cash flow $ECF^-(t_{i-1},t_i)$ (e.g. when there is the fixing of relevant Euribor);
- t_i is the point in time where party A actually pays the cash flow $ECF^-(t_{i-1},t_i)$;

- Party A will pay funding on $ECF^{-}(t_{i-1},t_i)$ from t_{i+1} until T, so total funding on it will be $ECF^{-}(t_{i-1},t_i)F_A(t_i,t_n)$, with $F_A(t_i,t_n) = Floating\ Leg(t_i,t_n)$ + $Spread\ Leg(t_i,t_n)$
- $Floating\ Leg(t_i,t_n) = B(0,0) - B(t_i,t_n)$
- $Spread\ Leg(t_i,t_n) = Funding\ Spread * \sum_{j=i}^{n} B(t_i,t_j)$
- $B(t_i,t_j)$ is the discount factor valued in t_i with maturity t_j with CIR bonds formula
- Introduce proper risk-free discount factors.

We will here analyze separate cases according to different possible times of default of both parties.

In our approach we separate the derivation of BCVA and FCA, and we start from BCVA.

We consider the risk-free discouted payoff $\widetilde{X}(0,T)$.

- If neither party A nor party B defaults before maturity T, the discounted payoff in $t_{i=0} = 0$ shall be:

  $1\{\tau^1 > T\}\left[\widetilde{X}(0,T)\right]$

- If both party A and party B default before T, the discounted payoff before τ^1 is given by:

  $1\{\tau^1 \le T\}\left[\widetilde{X}(0,\tau^1)\right]$

- If party A defaults before T, with δ_A being the "recovery rate" of party A, the discounted payoff at time τ^1 is:

  $1\{\tau^1 \le T, \tau^1 = \tau_A\}\left[\delta_A\widetilde{X}^{-}(\tau^1,T) + \widetilde{X}^{+}(\tau^1,T)\right]$

- If party B defaults before T, with δ_B being the "recovery rate" of party B, the discounted payoff at time τ^1 is:

  $1\{\tau^1 \le T, \tau^1 = \tau_B\}\left[\widetilde{X}^{-}(\tau^1,T) + \delta_B\widetilde{X}^{+}(\tau^1,T)\right]$

Merging all the components, we obtain the following formula for the discounted risky derivative under the risk neutral probability measure Q:

$$\widehat{X}(0,T) = E_Q\begin{bmatrix} \mathbf{1}\{\tau^1 > T\}\left[\widetilde{X}(0,T)\right] + \\ \mathbf{1}\{\tau^1 \le T\}\left[\widetilde{X}(0,\tau^1)\right] + \\ \mathbf{1}\{\tau^1 \le T, \tau^1 = \tau_A\}\left[\delta_A\widetilde{X}^{-}(\tau^1,T) + \widetilde{X}^{+}(\tau^1,T)\right] + \\ \mathbf{1}\{\tau^1 \le T, \tau^1 = \tau_B\}\left[\widetilde{X}^{-}(\tau^1,T) + \delta_B\widetilde{X}^{+}(\tau^1,T)\right] \end{bmatrix}$$

and since $\widetilde{X}^{+}(\tau^1,T) = \widetilde{X}(\tau^1,T) - \widetilde{X}^{-}(\tau^1,T)$ on the third line, and $\widetilde{X}^{-}(\tau^1,T) = \widetilde{X}(\tau^1,T) - \widetilde{X}^{+}(\tau^1,T)$ on the fourth line, we get:

$$\widehat{X}(0,T) = E_Q \begin{bmatrix} \mathbf{1}\left\{\tau^1 > T\right\}\left[\widetilde{X}(0,T)\right] + \\ \mathbf{1}\left\{\tau^1 \leq T\right\}\left[\widetilde{X}(0,\tau^1)\right] + \\ \mathbf{1}\left\{\tau^1 \leq T, \tau^1 = \tau_A\right\}\left[\delta_A \widetilde{X}^-(\tau^1,T) + \widetilde{X}(\tau^1,T) - \widetilde{X}^-(\tau^1,T)\right] + \\ \mathbf{1}\left\{\tau^1 \leq T, \tau^1 = \tau_B\right\}\left[\widetilde{X}(\tau^1,T) - \widetilde{X}^+(\tau^1,T) + \delta_B \widetilde{X}^+(\tau^1,T)\right] \end{bmatrix}$$

At this point we recognize the risk-free value of the derivative $\widetilde{X}(0,T)$:

$$\begin{aligned} &\mathbf{1}\left\{\tau^1 > T\right\}\widetilde{X}(0,T) + \mathbf{1}\left\{\tau^1 \leq T\right\}\widetilde{X}(0,\tau^1) + \\ &\mathbf{1}\left\{\tau^1 \leq T, \tau^1 = \tau_A\right\}\widetilde{X}(\tau^1,T) + \mathbf{1}\left\{\tau^1 \leq T, \tau^1 = \tau_B\right\}\widetilde{X}(\tau^1,T) \\ = &\widetilde{X}(0,T) \end{aligned}$$

which leads to:

$$\widehat{X}(0,T) = \widetilde{X}(0,T) + E_Q \begin{bmatrix} \mathbf{1}\left\{\tau^1 \leq T, \tau^1 = \tau_A\right\}\left[\delta_A \widetilde{X}^-(\tau^1,T) - \widetilde{X}^-(\tau^1,T)\right] + \\ \mathbf{1}\left\{\tau^1 \leq T, \tau^1 = \tau_B\right\}\left[-\widetilde{X}^+(\tau^1,T) + \delta_B \widetilde{X}^+(\tau^1,T)\right] \end{bmatrix}$$

Here we recognize the pricing adjustment that we were looking for, namely the Bilateral Credit Value Adjustment (BCVA). Specifically, rearranging terms we can find a new formula for the "credit-risky" value of the derivative $\widehat{X}(0,T)$:

$$\widehat{X}(0,T) = \widetilde{X}(0,T) - E_Q \begin{bmatrix} \mathbf{1}\left\{\tau^1 \leq T, \tau^1 = \tau_B\right\}(1-\delta_B)\widetilde{X}^+(\tau^1,T) + \\ \mathbf{1}\left\{\tau^1 \leq T, \tau^1 = \tau_A\right\}(1-\delta_A)\widetilde{X}^-(\tau^1,T) \end{bmatrix}$$

We can in fact extrapolate the Bilateral Credit Value Adjustment (BCVA), given by:

$$\text{BCVA} = \text{E}_Q\left[\mathbf{1}\left\{\tau^1 \leq T, \tau^1 = \tau_B\right\}(1-\delta_B)\widetilde{X}^+(\tau^1,T) + \mathbf{1}\left\{\tau^1 \leq T, \tau^1 = \tau_A\right\}(1-\delta_A)\widetilde{X}^-(\tau^1,T)\right]$$

We now turn to the derivation of a "Funding Cost Adjustment" (FCA) following the same procedure.

We consider funding for party A. Funding spread is equal to party A's credit spread, as we refer to party A as the "pricing party".

- If neither party A nor party B defaults before maturity T, funding in $t_{i=0} = 0$ shall be:

$$1\left\{\tau^1 > T\right\}\left[\sum_{i=1}^{n-1}\left[D(t,t_i)ECF^-(t_{i-1},t_i)F_A(t_i,t_n)\right]\right]$$

- If both party A and party B default before T, funding before τ^1 is given by:

$$1\left\{\tau^1 \leq T\right\}\left[\sum_{i=1}^{\tau^1-1}\left[D(t,t_i)ECF^-(t_{i-1},t_i)F_A(t_i,t_{\tau^1})\right]\right]$$

- If party A defaults before T, funding at time τ^1 is:

$$1\left\{\tau^1 \leq T, \tau^1 = \tau_A\right\}\left[\delta_A \sum_{i=\tau^1}^{n-1}\left[D(t,t_i)ECF^-(t_{i-1},t_i)F_A(t_i,t_n)\right]\right]$$

- If party B defaults before T, funding at time τ^1 is:

$$1\left\{\tau^1 \leq T, \tau^1 = \tau_B\right\}\left[\sum_{i=\tau^1}^{n-1}\left[D(t,t_i)ECF^-(t_{i-1},t_i)F_A(t_i,t_n)\right]\right]$$

Merging all the components, we get for FCA under the risk neutral probability measure Q:

$$\text{FCA=E}_Q\left[\begin{array}{l}\mathbf{1}\left\{\tau^1 > T\right\}\left[\sum_{i=1}^{n-1}\left[D(t,t_i)ECF^-(t_{i-1},t_i)F_A(t_i,t_n)\right]\right]+\\ \mathbf{1}\left\{\tau^1 \leq T\right\}\left[\sum_{i=1}^{\tau^1-1}\left[D(t,t_i)ECF^-(t_{i-1},t_i)F_A(t_i,t_{\tau^1})\right]\right]+\\ \mathbf{1}\left\{\tau^1 \leq T, \tau^1 = \tau_A\right\}\left[\delta_A\sum_{i=\tau^1}^{n-1}\left[D(t,t_i)ECF^-(t_{i-1},t_i)F_A(t_i,t_n)\right]\right]+\\ \mathbf{1}\left\{\tau^1 \leq T, \tau^1 = \tau_B\right\}\left[\sum_{i=\tau^1}^{n-1}\left[D(t,t_i)ECF^-(t_{i-1},t_i)F_A(t_i,t_n)\right]\right]\end{array}\right]$$

which simplifies to:

$$\begin{aligned}FCA = &\sum_{i=1}^{n-1}\left[D(t,t_i)ECF^-(t_{i-1},t_i)F_A(t_i,t_n)\right] - \\ &E_Q\left[\mathbf{1}\left\{\tau^1 \leq T, \tau^1 = \tau_A\right\}(1-\delta_A)\left[\sum_{i=\tau^1}^{n-1}\left[D(t,t_i)ECF^-(t_{i-1},t_i)F_A(t_i,t_n)\right]\right]\right]\end{aligned}$$

We see that BCVA and FCA can both be regarded as net costs, in that:

- BCVA is a cost equal to Adjusted CVA diminuished by Adjusted DVA, where a party reduces charges for Counterparty Credit Risk by a component related to its own default probability
- FCA is a cost equal to the funding of the entire profile of net Negative Expected Cash Flows, reduced by the amount of Funding Costs that will not be due in case of own default.

4.2 Bilateral Credit Value Adjustment: a discretized approach

We now want to discretize the expressions for BCVA and FCA. We assume the time interval to be $[0,T]$, with $t_{i=0} = 0 < t_{i=1} < ... < t_{i=n} = T$.

Specifically, from the general definition for default and survival probabilities built through the expected value of the Indicator function:

$$\text{E}_Q\left[\mathbf{1}\left\{\tau^1 \leq T\right\}\right] = 1 - E_Q\left[\mathbf{1}\left\{\tau^1 > T\right\}\right] \Rightarrow \Pr\left\{\tau^1 \leq T\right\} = 1 - \Pr\left\{\tau^1 > T\right\}$$

considering the infinitesimal change of the above expression, and then integrating over the time interval $[0,T]$:

$$\mathrm{dPr}\left\{\tau^1 \leq T\right\} = -d\,\mathrm{Pr}\left\{\tau^1 > T\right\} \Rightarrow \int_0^T d\,\mathrm{Pr}\left\{\tau^1 \leq T\right\} = -\int_0^T d\,\mathrm{Pr}\left\{\tau^1 > T\right\}$$

and by calling $S(0,y)$ the survival probability in the time interval $[0,y]$, and thus having $-\int_0^T d\,\mathrm{Pr}\left\{\tau^1 > T\right\} = -\int_0^T dS(0,y)$, we have that:

$$\int_0^T d\,\mathrm{Pr}\left\{\tau^1 \leq T\right\} = -\int_0^T dS(0,y)$$

From the formula we found for BCVA:

$$\text{BCVA=E}_Q\begin{bmatrix} \mathbf{1}\left\{\tau^1 \leq T, \tau^1 = \tau_B\right\}(1-\delta_B)\widetilde{X}^+(\tau^1,T) \\ +\mathbf{1}\left\{\tau^1 \leq T, \tau^1 = \tau_A\right\}(1-\delta_A)\widetilde{X}^-(\tau^1,T) \end{bmatrix}$$

we can obtain the following discounted and continuous version of the formula for BCVA, with $\widetilde{X}(0,T) = D(0,T)X(0,T)$ and $D(0,T)$ being the risk-free discount factor in $t_{i=0} = 0$ for maturity T.

$$\begin{aligned} BCVA = &-(1-\delta_B)E_Q\left[\int_0^T D(0,y)X^+(y,T)S_A(0,y)dS_B(t,y)\right] \\ &-(1-\delta_A)E_Q\left[\int_0^T D(0,y)X^-(y,T)S_B(0,y)dS_A(t,y)\right] \end{aligned}$$

with $S_A(0,y)$ equal to the cumulative probability distribution of party A, $S_B(0,y)$ the cumulative probability distribution of party B and $D(0,y)$ a risk-free discount factor.

As the expected value of X^+ under the risk neutral measure Q is equal to the Expected Exposure EE , i.e. the average of only expected positive values at a certain point in time:

$$\mathrm{E}_Q\left[X^+(y,T)\right] = E_Q\left[\max(0,X(y,T))\right] = EE(y,T)$$

whereas the expected value of X^- under the risk neutral measure Q is equal to the Negative Expected Exposure NEE, i.e. the average of only expected negative values at a certain point in time:

$$\mathrm{E}_Q\left[X^-(y,T)\right] = E_Q\left[\min(0,X(y,T)]\right) = NEE(y,T)$$

discretizing the expression for BCVA we have:

$$BCVA \approx (1-\delta_B)\sum_{i=1}^{n} D(t,t_i)EE(t_i)S_A(t_i)\left[S_B(t_{i-1})-S_B(t_i)\right]$$
$$+(1-\delta_A)\sum_{i=1}^{n} D(t,t_i)NEE(t_i)S_B(t_i)\left[S_A(t_{i-1})-S_A(t_i)\right]$$

where each $EE(t_i)$ is the Expected Exposure that would occur in t_i in case party A was to unwind the entire position in the derivative. This means that $D(t,t_i)EE(t_i)$ does not represent the discounted value for the single period $[t_{i-1},t_i]$ but the entire positive exposure deriving from the position in t_i, considering all future cash flows discounted up to t. The same applies to $NEE(t_i)$. $S_A(t_i)$ is the survival probability of party A up to time t_i, while $[S_B(t_{i-1})-S_B(t_i)]$ is the default probability of party B in the time interval $(t_{i-1},t_i]$. At the same time $S_B(t_i)$ is the survival probability of party B up to time t_i, while $[S_A(t_{i-1})-S_A(t_i)]$ is the default probability of party A in the time interval $(t_{i-1},t_i]$.

4.3 Adjustment for Default Risk in FCA

For what is concerned with FCA, instead, we can rearrange the formula as indicated here below.

From the above formula:

$$FCA = \sum_{i=1}^{n-1}\left[D(t,t_i)ECF^{-}(t_{i-1},t_i)F_A(t_i,t_n)\right] -$$
$$E_Q\left[\mathbf{1}\left\{\tau^1 \leq T, \tau^1 = \tau_A\right\}(1-\delta_A)\left[\sum_{i=\tau^1}^{n-1}\left[D(t,t_i)ECF^{-}(t_{i-1},t_i)F_A(t_i,t_n)\right]\right]\right]$$

specifying $E_Q\left[\mathbf{1}\left\{\tau^1 \leq T, \tau^1 = \tau_A\right\}\right]$, we obtain:

$$FCA = \sum_{i=1}^{n-1}\left[D(t,t_i)ECF^{-}(t_{i-1},t_i)F_A(t_i,t_n)\right] -$$
$$(1-\delta_A)\sum_{i=1}^{n-1}\left[\Pr(\tau_B > t_i)\Pr(\tau_A \leq t_i)D(t,t_i)ECF^{-}(t_{i-1},t_i)F_A(t_i,t_n)\right]$$

The value of the "credit-risky" and "funding-adjusted" value of the derivative $\widehat{X}(0,T)$ shall then be:

$$\widehat{X}(0,T) = \widetilde{X}(0,T) - BCVA + FCA$$
$$\approx \widetilde{X}(0,T) +$$

$$(1-\delta_B)\sum_{i=1}^{n} D(t,t_i)EE(t_i)S_A(t_i)\left[S_B(t_{i-1})-S_B(t_i)\right]+$$
$$(1-\delta_A)\sum_{i=1}^{n} D(t,t_i)NEE(t_i)S_B(t_i)\left[S_A(t_{i-1})-S_A(t_i)\right]+$$
$$\sum_{i=1}^{n-1}\left[D(t,t_i)ECF^{-}(t_{i-1},t_i)F_A(t_i,t_n)\right]-$$
$$(1-\delta_A)\sum_{i=1}^{n-1}\left[\Pr(\tau_B>t_i)\Pr(\tau_A\le t_i)D(t,t_i)ECF^{-}(t_{i-1},t_i)F_A(t_i,t_n)\right]$$

For the purpose of further numerical tests, we notice that also for FCA we may adopt the following notations for $\Pr(\tau_B > t_i)$ and for $\Pr(\tau_A \le t_i)$ respectively.

In particular, we can indicate $\Pr(\tau_B > t_i) = S_B(t_i)$, where this is the survival probability of party B from $t_{i=0} = 0$ to t_i.

We can notice two important differences between BCVA and FCA.

First of all, while for BCVA what matters is always the whole Expected Exposure, or Negative Expected Exposure, of the derivative, in case with some haircuts applied, for FCA, instead, one has to consider a single net cash flow at a time.

In fact, in case of default, the surviving party will suffer a loss - in case partially recovered - on the entire position, from the moment of default of its counterparty until maturity.

On the other hand, Funding Costs are applied to any single net Negative Expected Cash Flow due at each payment date, not to the whole mark-to-market of the derivative.

It would not make sense indeed to fund the entire mark-to-market of the derivative from start date until maturity. It is instead more meaningful to fund any net negative cash flow from the moment it has to be paid until maturity of the transaction.

As a second point, it is important to note that, while for BCVA we refer to differential default probability from t_i to t_{i-1}, to be applied to the exposure at t_i, for FCA, instead, we need to consider the entire default probability from $t_{i=0} = 0$ to t_i to be applied to the net cash flow $ECF^{-}(t_{i-1},t_i)$.

Chapter 5
Stochastic intensity modeling

Abstract When dealing with Credit Counterparty Risk and Funding Costs it is necessary to model both interest rates and default intensities. Both variables can be modeled either constant, deterministic and stochastic. In finance literature the most sophisticated approach is that of modeling stochastic interest rates and this is the path we shall follow in the remaining of this work. Interest Rates will be modeled according to a Cox-Ingersoll-Ross process, from now on CIR process, and relevant properties and simulation techniques will be investigated. For more details on CIR term structure model see [23]. Also for default intensities, their modeling shall represent a stochastic behaviour in time, rather than assuming default intensities as some constants or deterministic functions. To this purpose we shall see that the CIR process can be adopted as a suitable process also for default intensities. As a consequence, we shall have both interest rates and default intensities evolving stochastically according to their respective CIR process.

5.1 A setting for deterministic intensity model: a Poisson distribution for default events

Before moving to the stochastic setting, we shall go through a review of the determistic model for default intensities.

As illustrated also in [39], when modeling default intensities, it is possible to define a random variable Y_t representing the number of "arrivals" in the time interval, distributed according to a Poisson distribution. "Arrivals" shall be "defaults" in our environment. Specifically:

- Y_t: "number of arrivals in t" is a discrete random variable
- $Y_t \sim P(\gamma t)$ where γ is the average number of arrivals in the time unit

The probability function of Y_t will be given by the following:

$$\Pr(Y_t = y) = \begin{cases} \frac{(\gamma t)^y \exp(-\gamma t)}{y!} & if \quad y = 0, 1, ..., N \\ 0 & otherwise \end{cases}$$

with $t \in (0, +\infty)$ continuous values, and therefore the probability function is itself a continuous function of time.

The behaviour of Y_t evolves according to the following three hypotesis:

- *Uniformity.*

$\Pr(Y_t = y)$ is constant, with y number of arrivals in a time interval of lenght t.

The probability of arrivals $\Pr(Y_t = y)$ in a time interval t is always constant, regardless of the specific time interval of lenght t.

This implies that $\Pr(Y_t = y)\,[t_i;t_{i+1}] = \Pr(Y_t = y)\,[t_{i+1};t_{i+2}]$ with $t_{i+1} - t_i = t_{i+2} - t_{i+1} = t$.

- *Absense of memory.*

The number of arrivals in a time interval of lenght t does not depend on the number of arrivals in preceding disjointed time intervals.

- *Impossibility of simultaneous arrivals.*

In a small time interval Δt, either no arrivals take place ($Y_{\Delta t} = 0$) or only one arrival verifies ($Y_{\Delta t} = 1$).

One can summarize the above hypotesis as following:

$\Longrightarrow \Pr_1(\Delta t) = \gamma \Delta t$. (The probability of one arrival in Δt is proportional to γ arrivals in the time unit multiplied by the time fraction Δt);

$\Longrightarrow \Pr_0(\Delta t) = 1 - \lambda \Delta t$. (The probability of zero arrivals in Δt is complementary to the probability of one arrival in Δt);

$\Longrightarrow \Pr_{>1}(\Delta t) = 0$. (In Δt the probability of more than one arrival is zero);

and show that the only random variable satisfying all three conditions is a random variable distributed according to the Poisson distribution.

5.1.1 First passage time in a Poisson process for the first default

A Poisson process is a family of Poisson random variables $Y_t \sim P(\gamma t)$ with $t \in R^+$, satisfying the following properties [21]:

1. $Y_0 = 0$

2. $\forall\ t_1 < t_2 < t_3 < t_4 \in R^+ : Y(t_4) - Y(t_3)$ is independent of $Y(t_2) - Y(t_1) \Longrightarrow Y_t$ has independent increments

3. $\forall\ t_1 < t_2 < t_3 < t_4 \in R^+ : Y(t_4) - Y(t_3) \sim Y(t_2) - Y(t_1) \Longrightarrow Y_t$ has a distribution with stationary increments

4.$\forall\ t_1 < t_2 \in R^+ : Y(t_2) - Y(t_1) \sim P(\gamma(t_2 - t_1)) \Longrightarrow$ Increments of Y_t distribute according to a Poisson distribution

In order to assess the waiting time for the first ”default” in a Poisson process, we can recur to a new continuous random variable τ_1 and study its behaviour.

The continuous random variable τ_1 shall therefore be the ”waiting time for the first default” in the Poisson process Y_t.

Its cumulative distribution function $F_{\tau_1}(t)$, representing the time we have to wait before we see the first ”passage” or ”arrival” or ”default”, will be:

$$\begin{aligned} F_{\tau_1}(t) &= \Pr(\tau_1 \leq t) \\ &= 1 - \Pr(\tau_1 > t) \\ &= 1 - \Pr(Y_t = 0) \\ &= 1 - \exp(-\gamma t) \end{aligned}$$

As a matter of fact, the probability that the first default event happens in τ_1 with $\tau_1 \leq t$, is equal to the probability of zero ”defaults” for the random variable Y_t until t, distributed according to the Poisson distribtion $Y_t \sim P(\gamma t)$. As a consequence, the probability of the first default event happening in τ_1 prior to t, is the complement to 1 of the probability of zero ”defaults” for the random variable Y_t until t.

The density function $f_{\tau_1}(t)$of τ_1 will be given by $f_{\tau_1}(t) = F'_{\tau_1}(t)$:

$$f_{\tau_1}(t) = \begin{cases} \gamma \exp(-\gamma t) & if \quad t > 0 \\ 0 & otherwise \end{cases}$$

which is indeed the probability density function of an exponential random variable. Therefore we have the following important result for the distribution of the first default time:

$$\tau_1 \sim \exp(\gamma)$$

To recap, the default time τ_1 is the first jump-time of a time-inhomogeneous Poisson process with strictly increasing, continuous, invertible hazard function Γ and hazard rate (deterministic intensity) γ, with $\int_0^t \gamma(t)dt = \Gamma(T)$, as mentioned also by Brigo and Alfonsi [9].

For the purpose of our work, we shall place ourselves under the risk-neutral measure Q, so that all expected values and probabilities shall be calculated accordingly.

Therefore, as found before, the risk-neutral probability of the first default time occurring before a certain moment t is:

$$Q(\tau_1 \leq t) = 1 - \exp(-\gamma t) \tag{5.1}$$

Default intensities may also be assumed to evolve according to a stochastic model, and this shall indeed be the purpose of the remaining of this work.

Furthermore, when in this work interest rates are introduced, we shall model a stochastic behaviour for this variable as well, eventually according to the well known CIR processes.

5.2 A CIR model for the Stochastic intensity

In this section we will investigate how to model and simulate stochastic default intensity.

In general, stochastic intensity is referred to as λ and the respective intensity process as $\Lambda(T) = \int_0^T \lambda(t)dt$.

When modeling stochastic λ , it is nonetheless necessary to grant that intensity stay always positive over time. From a financial point of view, in fact, negative intensities do not make any sense. On the contrary, allowing for negative interest rates, may still find a financial justification.

Moreover we may want to incorporate a mean reversion behaviour.

Summarizing all these features:

- Stochastic behaviour in time
- Independency of interest rates
- Mean reverting process
- Always non-negative values

a suitable model for stochastic default intensities has been identified in the CIR model. Specifically, the behaviour of stochastic default intensity λ_t may be described by the following SDE:

$$d\lambda(t) = \theta(\eta - \lambda(t))dt + \sigma_\lambda \sqrt{\lambda(t)}dW(t) \tag{5.2}$$

where θ and η are two mean reversion parameters, both positive constants, respectively θ is the speed of adjustment and η is the mean reversion level.

The volatility of the process for λ is here assumed to be a constant parameter σ_λ as well, but it may eventually be transformed in a function of time $\sigma(t)$, or be modeled as stochastic variable.

$W(t)$ is a Brownian motion and $dW(t) \sim N(0,1)\sqrt{dt}$.

In this setting we shall further investigate the case when this process for $\lambda(t)$ is actually ensuring to obtain positive values. To this purpose we will make reference to the so called *Feller condition*.

As suggested by J. Gregory [34], another interesting feature of default intensities behaviour in time may be that of representing sudden and discrete jumps, as a result of an unexpected shock in the credit quality of a counterparty, that either being an upgrade or a downgrade. This could be done by the introduction of some Poisson jumps, thus modifying the above SDE as it follows:

$$d\lambda(t) = \theta(\eta - \lambda(t))dt + \sigma_\lambda \sqrt{\lambda(t)}dW(t) + jdN$$

where N is a Poisson process with jump size j.

5.3 The conditional probability distribution of the CIR model

The properties of the CIR model as a mean-reverting square-root process have long been investigated in [28] and as referenced by Andersen, Jackel and Kahl (2009) [1] and by Lord (2008) [40]. The same approach was followed also in [48] when analyzing different simulation approaches for the CIR process.

It was studied that, if the random variable $\lambda(t)$ follows a CIR process, the process for $\lambda(t)$ can then be simulated exactly from the conditional probability distribution of $\lambda(T)\,|\lambda(t)$, with $\Delta t = T - t$ and $T > t$.

Specifically, it was studied [28] that the conditional probability distribution of $\lambda(t+\Delta t)\,|\lambda(t)$ follows a non-central chi-squared distribution, times some certain parameter that will be discussed shortly.

Starting from the CIR process for $\lambda(t)$ given by:

$$d\lambda(t) = \theta(\eta - \lambda(t))dt + \sigma_\lambda \sqrt{\lambda(t)}dW(t)$$

we shall set the following constant parameter d, being the degree of freedom of a non-central chi-square distribution:

$$d = \frac{4\theta\eta}{\sigma_\lambda^2} \tag{5.3}$$

and the parameter $ncp(t, t+\Delta t)$, being the non-central parameter of a non-central chi-square distribution:

$$ncp(t, t+\Delta t) = \frac{4\theta e^{-\theta(\Delta t)}}{\sigma_\lambda^2(1 - e^{-\theta\Delta t})} \tag{5.4}$$

Conditional on $\lambda(t)$, $\lambda(T)$ has a non-central chi-square distribution with degree of freedom d and non-central parameter $\lambda(t)ncp(t,t+\Delta t)$, times $e^{-\theta(\Delta t)}/ncp(t,t+\Delta t)$.

Given a certain value λ, the above results in:

$$\Pr[\lambda(T) < \lambda\,|\lambda(t)] = F^{\chi^2}_{d,\lambda(t)ncp(t,t+\Delta t)}\left(\lambda \frac{ncp(t,t+\Delta t)}{e^{-\theta(\Delta t)}}\right) \tag{5.5}$$

where, as said, $F^{\chi^2}_{d,\lambda(t)ncp(t,t+\Delta t)}$ is the cumulative distribution function of a non-central chi-square distribution $\chi^2(d, \lambda(t)ncp(t,t+\Delta t))$, with degree of freedom d and non-central parameter $\lambda(t)ncp(t,t+\Delta t)$. The above can be further rearranged substituting $ncp(t,t+\Delta t)$:

$$\Pr[\lambda(T) < \lambda\,|\lambda(t)] = F^{\chi^2}_{d,\lambda(t)ncp(t,t+\Delta t)}\left(\lambda \frac{4\theta e^{-\theta(\Delta t)}}{\sigma_\lambda^2(1 - e^{-\theta\Delta t})e^{-\theta(\Delta t)}}\right)$$

$$= F^{\chi^2}_{d,\lambda(t)ncp(t,t+\Delta t)}\left(\lambda \frac{4\theta}{\sigma_\lambda^2(1-e^{-\theta\Delta t})}\right)$$

For ease of reading, in the following we shall set a new parameter k, equal to the multiplicative factor of the distribution:

$$\begin{aligned} k &= \frac{e^{-\theta(\Delta t)}}{ncp(t,t+\Delta t)} \\ &= \frac{e^{-\theta(\Delta t)}\sigma_\lambda^2(1-e^{-\theta\Delta t})}{4\theta e^{-\theta(\Delta t)}} \\ &= \frac{\sigma_\lambda^2(1-e^{-\theta\Delta t})}{4\theta} \end{aligned}$$

Thus eventually leading to:

$$\Pr[\lambda(T) < \lambda\,|\lambda(t)] = F^{\chi^2}_{d,\lambda(t)ncp(t,t+\Delta t)}\left(\frac{\lambda}{k}\right)$$

5.3.1 The non-central Chi-square distribution $\chi^2(d,ncp)$

The Gamma Family

It is known that the Non-central Chi-square distribution $\chi^2(d,ncp)$ belongs to the Gamma family of probability distribution functions.

In particular a random variable X distributes following a Gamma distribution when

$$X \sim \Gamma\left(\alpha;\frac{1}{\lambda}\right)$$

with $\alpha,\lambda > 0$.

The probability density function is then given by:

$$f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)}e^{-\lambda x}\lambda^\alpha x^{\alpha-1} & if \quad x > 0 \\ 0 & if \quad x \leq 0 \end{cases}$$

The cumulative distribution function is instead given by:

$$\begin{aligned} F_X(x) &= \int_{-\infty}^{+\infty} f(x)dx \\ &= \int_0^{+\infty} f(x)dx \end{aligned}$$

$$= \int_0^{+\infty} \frac{1}{\Gamma(\alpha)} e^{-\lambda x} \lambda^{\alpha} x^{\alpha-1} dx$$

and $\Gamma(\alpha)$ is such that $\int_0^{+\infty} f(x)dx = 1$ and therefore:

$$\int_0^{+\infty} \frac{1}{\Gamma(\alpha)} e^{-\lambda x} \lambda^{\alpha} x^{\alpha-1}\, dx = 1$$

resulting in:

$$\Gamma(\alpha) = \int_0^{+\infty} e^{-\lambda x} \lambda^{\alpha} x^{\alpha-1}\, dx$$

The Chi-square as a Gamma distribution

A Chi-square distribution $\chi^2(v)$ is a sum of v squared standard normal distributions $X_i \sim N(0,1)$, specifically:

$$\chi^2(v) \sim \sum_{i=1}^{v} (X_i)^2 = X_1^2 + X_2^2 + ... + X_v^2$$

where v is called the degree of freedom of $\chi^2(v)$.

A random variable following a Chi-square distribution with v degrees of freedom, can be also recognized as being distributed according to a Gamma distribution, with parameters $\alpha = v/2$ and $\lambda = 1/2$, so that if $X \sim \chi^2(v)$ then it is also true that:

$$X \sim \Gamma\left(\alpha = \frac{v}{2}; \frac{1}{\lambda} = 2\right)$$

As a consequence, the probability density function and the cumulative distribution functions can be derived accordingly.

The probability density function of $X \sim \chi^2(v)$ for $x > 0$, recalling that for a generic $X \sim \Gamma\left(\alpha; \frac{1}{\lambda}\right)$ is

$$f(x) = \frac{1}{\Gamma(\alpha)} e^{-\lambda x} \lambda^{\alpha} x^{\alpha-1}$$

with $\alpha = v/2$ and $1/\lambda = 2$,or equivalently $\lambda = 1/2$, will be given by:

$$\begin{aligned} f_{\chi_v^2}(x) &= \frac{1}{\Gamma\left(\frac{v}{2}\right)} e^{-x/2} \left(\frac{1}{2}\right)^{v/2} x^{(v/2-1)} \\ &= \frac{1}{2^{v/2}\Gamma\left(\frac{v}{2}\right)} e^{-x/2} x^{(v/2-1)} \end{aligned}$$

where we have the known Gamma solution for positive half-integers:

$$\Gamma\left(\frac{v}{2}\right) = \sqrt{\pi}\frac{(v-2)!!}{2^{(v-1)/2}}$$

The cumulative distribution function for $X \sim \chi^2(v)$, will be given by:

$$\begin{aligned} F_{\chi^2_v}(x) &= \int_0^{+\infty} f_{\chi^2_v}(x)dx \\ &= \int_0^{+\infty} \frac{1}{2^{v/2}\Gamma\left(\frac{v}{2}\right)} e^{-x/2} x^{(v/2-1)} dx \end{aligned}$$

or if we only want to integrate up to a certain value of z and get:

$$\Pr(X < z)\,\big|X \sim \chi^2(v)$$

$$F_{\chi^2_v}(x;z) = \int_0^{z} \frac{1}{2^{v/2}\Gamma\left(\frac{v}{2}\right)} e^{-x/2} x^{(v/2-1)} dx \tag{5.6}$$

The Non-central Chi-square distribution derived $\chi^2(d,ncp)$

The Non-central Chi-square distribution $\chi^2(d,ncp)$ can be derived as a sum of d squared non standard normal distributions $X_i \sim N(\mu_{i,}\sigma_i)$, where σ_i may also be equal to some constant parameter σ_i or to 1. In particular we have that:

$$\chi^2(d,ncp) \sim \sum_{i=1}^{d} (X_i)^2 = X_1^2 + X_2^2 + \ldots + X_d^2$$

with $ncp = \sum_{i=1}^{d} \left(\frac{\mu_i}{\sigma_i}\right)^2$, being the non-central parameter of the $\chi^2(d,ncp)$ distribution. The distribution depends only on the ncp and not on the single $\left(\frac{\mu_i}{\sigma_i}\right)^2$.

A random variable distributed according to a Non-central Chi-square distribution is called a mixture of distributions, in that it is a random variable whose probability density function is a weighted average of others random variables density functions.

5.3.2 *The probability density function of a non-central chi-square distribution* $\chi^2(d,ncp)$

Specifically, the probability density function $f_{\chi^2}(x;d,ncp)$ of a Non-central Chi-square distribution $\chi^2(d,ncp)$ is a weighted average of probability density functions

$f_{\chi^2}(x; d+2j)$ of Chi-square distributions $\chi_j^2(x; d+2j)$, where the weights are given by Poisson distributed degrees of freedom $J \sim P(\frac{ncp}{2})$.

The degree of freedom of the each central Chi-square distribution $\chi_j^2(x; d+2j)$ is indeed $d+2j$, with $(j_1, j_2, ..., j_N) = J$ being a random variable with a Poisson distribution $J \sim P(\frac{ncp}{2})$ of parameter $(\frac{ncp}{2})$.

The probability density function $f_{\chi^2}(x; d, ncp)$ of the Non-central Chi-square distribution $\chi^2(d, ncp)$, with degree of freedom d and non-central parameter ncp, is therefore given by:

$$f_{\chi^2}(x; d, ncp) = \sum_{j=0}^{\infty} \frac{e^{-ncp/2} (\frac{ncp}{2})^j}{j!} f_{\chi^2}(x; d+2j)$$

5.3.3 *The cumulative distribution function of a non-central chi-square distribution* $\chi^2(d, ncp)$

The cumulative distribution function $F_{\chi^2}(z; d, ncp)$ up to a certain value z,of a Non-central Chi-square distribution $\chi^2(d, ncp)$ shall be equal to the weighted average of the cumulative distribution functions of $F_{\chi^2}(z; d+2j)$ of Chi-square distributions $\chi_j^2(z; d+2j)$, where the weights are given by Poisson distributed degrees of freedom $J \sim P(\frac{ncp}{2})$.

$$\begin{aligned} F_{\chi^2}(z; d, ncp) &= \sum_{j=0}^{\infty} \frac{e^{-ncp/2} (\frac{ncp}{2})^j}{j!} F_{\chi^2}(z; d+2j) \qquad (5.7) \\ &= e^{-ncp/2} \sum_{j=0}^{\infty} \frac{(\frac{ncp}{2})^j}{j! 2^{(d/2+j)} \Gamma\left(\frac{d}{2}+j\right)} \int_0^z e^{-x/2} x^{(d/2-1+j)} dx \end{aligned}$$

5.3.4 *The Non-central Chi-square distribution* $\chi^2(d, ncp)$ *when* $d > 1$

Recalling that the Non-central Chi-square distribution $\chi^2(d, ncp)$ can be derived as a sum of d squared non standard normal distributions $X_i \sim N(\mu_i, \sigma_i)$,

$$\chi^2(d, ncp) \sim \sum_{i=1}^{d} (X_i)^2 = X_1^2 + X_2^2 + ... + X_d^2$$

when $d > 1$, it is useful to see the distribution χ^2 as a sum of a Central Chi-square distribution with (d-1) degrees of freedom $\chi^2(d-1)$, and a Non-central Chi-

Square distribution with one degree of freedom and non-centrality parameter $ncp = \sum_{i=1}^{d} \left(\frac{\mu_i}{\sigma_i}\right)^2$.

$$\chi^2(d,ncp) \sim \chi^2(d-1) + \chi^2(1,ncp)$$

with, given $Z \sim N(0,1)$:

$$\chi^2(1,ncp) \sim (Z+\sqrt{ncp})^2$$

As a consequence, when $d > 1$, the $\chi^2(d,ncp)$ can be seen as:

$$\chi^2(d,ncp) \sim \chi^2(d-1) + (Z+\sqrt{ncp})^2 \tag{5.8}$$

where $\chi^2(d-1)$ is a central Chi-square distribution with $(d-1)$ degrees of freedom, $Z \sim N(0,1)$ is a standard normal distribution, and ncp is the non-centrality parameter of the original $\chi^2(d,ncp)$ distribution.

5.3.5 The Non-central Chi-square distribution $\chi^2(d,ncp)$ when $d \leq 1$

On the other hand, when $d \leq 1$, we shall recall that $\chi^2(d,ncp)$ is equal to a central Chi-square distribution $\chi^2(d+2J)$ with Poisson distributed degrees of freedom $J \sim P(\frac{ncp}{2})$.

$$\chi^2(d,ncp) \sim \chi^2(d+2J) \tag{5.9}$$

5.4 The Non-central Chi-square results for the CIR process

We can now apply the results for the Non-central Chi-square distribution to the CIR process, as it has been proved that if a random variable $\lambda(t)$ evolves according to a CIR process SDE:

$$d\lambda(t) = \theta(\eta - \lambda(t))dt + \sigma_\lambda \sqrt{\lambda(t)}dW(t)$$

then the distribution of $\lambda(t)$ is a Non-central Chi-square one.

In particular, recalling that $\lambda(T)\,|\lambda(t) \sim \chi^2(d, \lambda(t)ncp(t,t+\Delta t))$ times $e^{-\theta(\Delta t)}/ncp(t,t+\Delta t)$, with degree of freedom d :

$$d = \frac{4\theta\eta}{\sigma_\lambda^2}$$

and non-centrality parameter $ncp(t, t+\Delta t)$:

$$ncp(t, t+\Delta t) = \frac{4\theta e^{-\theta(\Delta t)}}{\sigma_\lambda^2(1-e^{-\theta\Delta t})}$$

The cumulative distribution function being represented by:

$$\Pr[\lambda(T) < \lambda\,|\lambda(t)] = F^{\chi^2}_{d,\lambda(t)ncp(t,t+\Delta t)}\left(\frac{\lambda}{k}\right)$$

with

$$\begin{aligned} k &= \frac{e^{-\theta(\Delta t)}}{ncp(t, t+\Delta t)} \\ &= \frac{\sigma_\lambda^2(1-e^{-\theta\Delta t})}{4\theta} \end{aligned}$$

for $d > 1$, we shall have that:

$$\lambda(t+\Delta t) = k\left(\chi^2(d-1) + (Z + \sqrt{ncp(t+\Delta t)})\right) \tag{5.10}$$

and for $d \leq 1$, instead:

$$\lambda(t+\Delta t) = k\left(\chi^2(d+2J)\right) \tag{5.11}$$

with $J \sim P\left(\frac{ncp(t+\Delta t)}{2}\right)$.

The above specification means that one should proceed as following:

1. Generate J as $J \sim P\left(\frac{ncp(t+\Delta t)}{2}\right)$ and find the outcome $J = j \in N$
2. Generate a Central Chi-Square $\chi^2(d+2j)$, with degrees of freedom $(d+2j)$ where j is the outcome of the previously generated Poisson distribution.
3. Simulate the Central Chi-Square distribution χ^2 as a Gamma distribution Γ, so that now d must not be necessarily an integer.

$$\chi^2(d+2j) \sim \Gamma\left(\frac{d+2j}{2};2\right)$$

5.5 The CIR process: a discretized approach

As we saw in the previous section, the CIR process has a known distribution and, as a consequence, it can be simulated exactly. One can argue, though, that exact simulation may be slow and therefore may be willing to recur to some approximating techniques, as it was reported also in Andersen, Jackel and Kahl [1], in Lord, Koekkoek and Van Dijk [40], and in Webber [48].

One possible alternative that we have is to recur to approximating schemes based on Itô-Taylor expansions. References for Itô-Taylor based techniques are Kloeden and Platen (1995), Glasserman (2004), Jäckel (2002) and Gatheral (2006).

In particular, we will investigate Itô-Taylor approximating schemes in 1-dimension.

As it is known, for a 1-dimensional process with SDE:

$$dX_t = \alpha X_t dt + \beta X_t dW_t$$

the Euler discretization scheme is given by:

$$\Delta_E \widetilde{X}_t = \alpha \widetilde{X}_t \Delta t + \beta \widetilde{X}_t \Delta W_t$$

whereas the Milstein discretization scheme is given by:

$$\Delta_M \widetilde{X}_t = \alpha \widetilde{X}_t \Delta t + \beta \widetilde{X}_t \sqrt{\Delta t}\varepsilon_t + \frac{1}{2}\beta \widetilde{X}_t \beta' \widetilde{X}_t (\Delta W_t^2 - \Delta t)$$

where $\beta' X_t = \partial\beta/\partial X_t$, $\Delta W_t \sim \sqrt{\Delta t}\varepsilon_t$ with $\varepsilon_t \sim N(0,1)$. As a consequence $\Delta W_t \sim N(0,\Delta t)$.

For the CIR process with the following SDE:

$$dX_t = \alpha(\mu - X_t)dt + \eta\sqrt{X_t}dW_t$$

the Euler discretization scheme for $\widetilde{X}_t$ results in:

$$\Delta_E \widetilde{X}_t = \alpha(\mu - \widetilde{X}_t)\Delta t + \eta\sqrt{\widetilde{X}_t}\Delta W_t \tag{5.12}$$

and the Milstein discretization scheme in:

$$\Delta_M \widetilde{X}_t = \left(\alpha(\mu - \widetilde{X}_t) - \frac{1}{4}\eta^2\right)\Delta t + \eta\sqrt{\widetilde{X}_t}\Delta W_t + \frac{1}{4}\eta^2 \Delta W_t^2 \tag{5.13}$$

One can notice that the Euler approximation is of order Δt only in the drift term, and $\sqrt{\Delta t}$ in the volatility term.

The Milstein approximation is, instead, of order Δt in both drift and volatility term.

As a matter of fact, both these discretization schemes, if directly applied to a CIR process, may return negative values for discretized $\widetilde{X}_t$, leading to impossibility of tractability of the discretized term for $\sqrt{\widetilde{X}_t}$.

In order to prevent negative values for discretized $\widetilde{X}_t$, one may resort to imposing boundary conditions, and, for example, using a rectification function, or, as an alternative, one may think of using a transformed variable.

As for the Euler scheme, what can be found in literature, in order to prevent discretized $\widetilde{X}_t$ to assume negative values, is the recourse to a rectification function of the type $\widetilde{X}_t^+ = \max(\widetilde{X}_t;0)$, as reported also in [1]. This was indeed the work of Lord, Koekkoek and Van Dijk [40] who referred to this method as to the ”full truncation” scheme.

A previous similar approach had been the one developped by Kloeden and Platen [38], who suggested to replace the $\sqrt{\widetilde{X}_t}$ term with the $\sqrt{\left|\widetilde{X}_t\right|}$ term instead.

The ”corrected” version of the Euler discretization scheme for the CIR process thus becomes:

$$\Delta_E \widetilde{X}_t = \alpha(\mu - \widetilde{X}_t^+)\Delta t + \eta\sqrt{\widetilde{X}_t^+}\Delta W_t \tag{5.14}$$

where $\widetilde{X}_t^+ = \max(\widetilde{X}_t;0)$. The result is that the process for $\widetilde{X}_t$ is allowed to become negative, and, whenever this happens, the process becomes deterministic with drift $\alpha\mu$.

As we will show afterwards, the Euler scheme has first-order weak convergence, meaning that expectations of functions of $\widetilde{X}_t$ will approach their true values as $O(\Delta t)$.

The above recourse to a rectification function of the type $\widetilde{X}_t^+ = \max(\widetilde{X}_t;0)$, may also be pursued for the Milstein scheme, leading to:

$$\Delta_M \widetilde{X}_t = \left(\alpha(\mu - \widetilde{X}_t^+) - \frac{1}{4}\eta^2\right)\Delta t + \eta\sqrt{\widetilde{X}_t^+}\Delta W_t + \frac{1}{4}\eta^2\Delta W_t^2 \tag{5.15}$$

The Milstein scheme should be caracterized by second-order convergence, though the outcome strictly depends on the choice of parameters.

Other positivity preserving techniques suggest the use of a Euler implicit scheme, see Brigo and Alfonsi [9], or of a Mistein implicit scheme, see Andersen, Jäckel and Kahl [1].

In literature it is possible to find other techniques to discretize the CIR process $\widetilde{X}_t$, such as moment freezing, log-euler approximation, log-normal approximation, normal approximation, etc.

5.6 Stochastic Interest Rates: a CIR model for the Interest Rates

As anticipated in the introductory section to this chapter, in order to model Credit Counterparty Risk and Funding Costs, we are both required to model interest rates and default intensities. As we already investigated the modeling and simulation techniques for stochastic intensities, we shall now focus our attention on interest rates.

As already suggested, we shall model interest rates thorugh a stochastic process as well, evolving according to the following SDE of the CIR model, which is the same we introduced for default intensities, except for the specific parametes:

$$dr(t) = \alpha(\mu - r(t))dt + \sigma_r\sqrt{r(t)}dZ(t) \tag{5.16}$$

where α, μ and σ_r are positive constants and, specifically, α is the mean reversion rate, μ is the equilibrium level, σ_r is the volatility of the short rate and $dZ(t)$ is a Brownian motion $dZ_t \sim N(0,1)\sqrt{dt}$.

As it was already stated when analysing the CIR process for stochastic default intensities, also for interest rates the condition:

$$2\alpha\mu > \sigma_r^2$$

guarantees that the process $r(t)$ remains positive *(Feller condition)*.

As it is well known in literature, see for instance Björk [4], the CIR model for interest rates belongs to the family of term structure affine models for short rates, and the term structure for pure disount bonds is given by:

$$B_t(T\,|r_t\,) = \exp(A(T-t) - B(T-t)r_t) \tag{5.17}$$

with:

$$B(x) = \frac{2(\exp(\gamma x)-1)}{(\gamma+a)(\exp(\gamma x)-1)+2\gamma}$$

$$A(x) = \left[\frac{2\gamma\exp((a+\gamma)(x/2))}{(\gamma+a)(\exp(\gamma x)-1)+2\gamma}\right]^{2ab/\sigma^2}$$

and:

$$\gamma = \sqrt{a^2+2\sigma^2}$$

5.7 Correlation between interest rates and default intensities

So far we have assumed independency between interest rates and default intensity stochastic processes.

In this section we will release this assumption, and assume the two Brownian motions dW_t and dZ_t are instantaneously correlated by means of a correlation parameter ρ.

$$dW_t dZ_t = \rho dt$$

This topic is approached also in Lando [39], Brigo and Alfonsi (2004) [9], Brigo and Pallavicini (2006) [18].

Recalling our SDE for interest rates:

$$dr(t) = \alpha(\mu - r(t))dt + \sigma_r\sqrt{r(t)}dZ(t)$$

and our SDE for default intensities:

$$d\lambda(t) = \theta(\eta - \lambda(t))dt + \sigma_\lambda \sqrt{\lambda(t)}dW(t)$$

we shall set correlation between the two Brownian motions as following:

$$W(t) = \rho Z(t) + \sqrt{1-\rho^2}Z^{'}(t)$$

where $Z(t) \sim N(0,1)\sqrt{dt}$ and $Z^{'}(t) \sim N(0,1)\sqrt{dt}$ are two independent Brownian motions (*Cholesky decomposition*).

As a consequence:

$$dW(t) = \rho dZ(t) + \sqrt{1-\rho^2}dZ^{'}(t) \tag{5.18}$$

5.7.1 Discretizing correlated interest rate and default intensity CIR processes

In the previous sections we approached the discretization of CIR processes for the interest rate and default intensity through Euler and Milstein techniques.

We can therefore try to discretize the new set of SDEs through the Euler scheme again.

In particular, we shall discretize the time interval $[0,T]$, with $t_0 = 0 < t_1 < ... < t_n = T$ and simulate discrete increments $\Delta Z(t) = Z(t_{i+1} - t_i)$ and $\Delta W(t) = W\ (t_{i+1} - t_i)$ for the two Brownian motions $Z(t)$ and $W(t)$.

In order to simulate all increments $\Delta W(t)$ for $W(t)$, we shall of course simulate increments $\Delta Z(t)$ and $\Delta Z^{'}(t)$ for $Z(t)$ and $Z^{'}(t)$, so as to get:

$$\Delta W(t) = \rho \Delta Z(t) + \sqrt{1-\rho^2}\Delta Z^{'}(t)$$

Once the time interval has been discretized and the Brownian motions increments generated as above, one can apply the following Euler discretization scheme to the CIR SDEs for the interest rate and default intensity respectively.

Specifically, the increment $\Delta_E \widetilde{r}_t$ for the Euler discretized interest rate shall be given by the following, as above:

$$\Delta_E \widetilde{r}_t = \alpha(\mu - \widetilde{r}_t)\Delta t + \sigma_r \sqrt{\widetilde{r}_t}\Delta Z(t)$$

and, with $\Delta t = t_{i+1} - t_i$, and $\Delta Z(t) = Z(t_{i+1}) - Z(t_i)$:

$$\begin{aligned}\widetilde{r}(t_{i+1}) &= \widetilde{r}(t_i) + \Delta_E \widetilde{r}(t_i, t_{i+1}) \\ &= \widetilde{r}(t_i) + \alpha(\mu - \widetilde{r}(t_i))(t_{i+1} - t_i) + \sigma_r \sqrt{\widetilde{r}(t_i)}(Z(t_{i+1}) - Z(t_i))\end{aligned}$$

The Euler discretized increment $\Delta_E \widetilde{\lambda}_t$ for the default intensity, correlated to interest rate, shall instead be given by:

$$\Delta_E \widetilde{\lambda}_t = \theta(\eta - \widetilde{\lambda}_t)\Delta t + \sigma_\lambda \sqrt{\widetilde{\lambda}_t}\Delta W(t)$$

and, with $\Delta t = t_{i+1} - t_i$, and $\Delta W(t) = W(t_{i+1}) - W(t_i)$:

$$\begin{aligned}\widetilde{\lambda}(t_{i+1}) &= \widetilde{\lambda}(t_i) + \Delta_E \widetilde{\lambda}(t_i, t_{i+1}) \\ &= \widetilde{\lambda}(t_i) + \theta(\eta - \widetilde{\lambda}(t_i))(t_{i+1} - t_i) + \sigma_\lambda \sqrt{\widetilde{\lambda}(t_i)}\,(W(t_{i+1}) - W(t_i))\end{aligned}$$

where in this case we remember:

$$W(t_{i+1}) - W(t_i) = \rho\,(Z(t_{i+1}) - Z(t_i)) + \sqrt{1-\rho^2}\left(Z^{'}(t_{i+1}) - Z^{'}(t_i)\right)$$

As in the previous case when no correlation was assumed, in order to prevent negative interest rates and negative default intensities, we shall recur to a rectification function of the type $\widetilde{r}_t^+ = \max(\widetilde{r}_t; 0)$ and $\widetilde{\lambda}_t^+ = \max(\widetilde{\lambda}_t; 0)$, thus our discretized SDEs become:

$$\widetilde{r}(t_{i+1}) = \widetilde{r}(t_i) + \alpha(\mu - \widetilde{r}^+(t_i))(t_{i+1} - t_i) + \sigma_r \sqrt{\widetilde{r}^+}\,(Z(t_{i+1}) - Z(t_i))$$

and

$$\widetilde{\lambda}(t_{i+1}) = \widetilde{\lambda}(t_i) + \theta(\eta - \widetilde{\lambda}^+(t_i))(t_{i+1} - t_i) + \sigma_\lambda \sqrt{\widetilde{\lambda}^+(t_i)}\,(W(t_{i+1}) - W(t_i))$$

This discretization scheme is applicable for the case of only one defaultable counterparty, with intensity correlated to payment obligations underlying risk factors (i.e. interest rates). In case of two defaultable counterparties, one should most likely assume correlation of both default intensities with interest rates, and eventually correlation between both default intensities.

5.8 Correlation between default intensities for bilateral contracts

In this section we will release the assumption of correlation between interest rates and default intensity of a counterparty, and focus the attention on correlation between default intensities of two counterparties A and B. This will be applicable for the case of a bilateral contract with both defaultable parties.

As a matter of fact we may reproduce the discretization scheme introduced in the above section, for correlated interest rates and default intensity.

To this purpose, we will assume two Brownian motions $dW_A(t)$ and $dW_B(t)$ are instantaneously correlated by means of a correlation parameter $\rho_{A,B}$.

$$dW_A(t)dW_B(t) = \rho_{A,B}dt \tag{5.19}$$

The SDE for default intensity of party A shall be:

$$d\lambda_A(t) = \theta_A(\eta_A - \lambda_A(t))dt + \sigma_{\lambda_A}\sqrt{\lambda_A(t)}dW_A(t) \tag{5.20}$$

the SDE for default intensity of party B shall be:

$$d\lambda_B(t) = \theta_B(\eta_B - \lambda_B(t))dt + \sigma_{\lambda_B}\sqrt{\lambda_B(t)}dW_B(t) \tag{5.21}$$

and we shall set correlation between the two Brownian motions as following:

$$W_B(t) = \rho_{A,B}W_A(t) + \sqrt{1-\rho_{A,B}^2}W_A^{'}(t) \tag{5.22}$$

where $W_A(t) \sim N(0,1)\sqrt{dt}$ and $W_A^{'}(t) \sim N(0,1)\sqrt{dt}$ are two independent Brownian motions (*Cholesky decomposition*).

As a consequence the noise term correlation shall be defined by:

$$dW_B(t) = \rho_{A,B}dW_A(t) + \sqrt{1-\rho_{A,B}^2}dW_A^{'}(t) \tag{5.23}$$

5.8.1 Discretizing correlated CIR processes of default intensities

As already suggested in the previous case, we can discretize default intensities SDEs through the Euler scheme.

In particular, we shall discretize the time interval $[0,T]$, with $t_0 = 0 < t_1 < ... < t_n = T$ and simulate discrete increments $\Delta W_A(t) = W_A(t_{i+1} - t_i)$ and $\Delta W_B(t) = W_B$ $(t_{i+1} - t_i)$ for the two Brownian motions $W_A(t)$ and $W_B(t)$.

To simulate all increments $\Delta W_B(t)$ for $W_B(t)$, we shall of course simulate increments $\Delta W_A(t)$ and $\Delta W_A^{'}(t)$ for $W_A(t)$ and $W_A^{'}(t)$, so as to get:

$$\Delta W_B(t) = \rho_{A,B}\Delta W_A(t) + \sqrt{1-\rho_{A,B}^2}\Delta W_A^{'}(t) \tag{5.24}$$

Again, once the time interval has been discretized and the Brownian motions increments generated as above, one can apply the Euler discretization scheme to the CIR SDEs for both default intensities.

Specifically, the increment $\Delta_E\widetilde{\lambda}_A(t_i,t_{i+1})$ for the Euler discretized default intensity for party A shall be given by the following:

$$\Delta_E\widetilde{\lambda}_A(t_i,t_{i+1}) = \theta_A(\eta_A - \widetilde{\lambda}_A(t_i))\Delta t + \sigma_{\lambda_A}\sqrt{\widetilde{\lambda}_A(t_i)}\Delta W_A(t) \tag{5.25}$$

and, with $\Delta t = t_{i+1} - t_i$, and $\Delta W_A(t) = W_A(t_{i+1} - t_i)$:

$$\widetilde{\lambda}_A(t_{i+1}) = \widetilde{\lambda}_A(t_i) + \Delta_E\widetilde{\lambda}_A(t_i,t_{i+1})$$

$$= \widetilde{\lambda}_A(t_i) + \theta_A(\eta_A - \widetilde{\lambda}_A(t_i))\,(t_{i+1} - t_i) + \sigma_{\lambda_A}\sqrt{\widetilde{\lambda}_A(t_i)}\,(W_A(t_{i+1}) - W_A(t_i))$$

The Euler discretized increment $\Delta_E\widetilde{\lambda}_B(t_i, t_{i+1})$ for the default intensity of party B, correlated to default intensity of party A, shall instead be given by:

$$\Delta_E\widetilde{\lambda}_B(t_i, t_{i+1}) = \theta_B(\eta_B - \widetilde{\lambda}_B(t_i))\Delta t + \sigma_{\lambda_B}\sqrt{\widetilde{\lambda}_B(t_i)}\Delta W_B(t) \tag{5.26}$$

and, with $\Delta t = t_{i+1} - t_i$, and $\Delta W_B(t) = W_B(t_{i+1}) - W_B(t_i)$:

$$\begin{aligned}\widetilde{\lambda}_B(t_{i+1}) &= \widetilde{\lambda}_B(t_i) + \Delta_E\widetilde{\lambda}_B(t_i, t_{i+1}) \\ &= \widetilde{\lambda}_B(t_i) + \theta_B(\eta_B - \widetilde{\lambda}_B(t_i))\,(t_{i+1} - t_i) + \sigma_{\lambda_B}\sqrt{\widetilde{\lambda}_B(t_i)}\,(W_B(t_{i+1}) - W_B(t_i))\end{aligned}$$

where in this case we remember:

$$W_B(t_{i+1}) - W_B(t_i) = \rho_{A,B}\,(W_A(t_{i+1}) - W_A(t_i)) + \sqrt{1 - \rho_{A,B}^2}\left(W_A'(t_{i+1}) - W_A'(t_i)\right)$$

As in the previous case when no correlation was assumed, in order to prevent negative default intensities, we shall recur to a rectification function of the type $\widetilde{\lambda}_A^+(t) = \max(\widetilde{\lambda}_A(t); 0)$ and $\widetilde{\lambda}_B^+(t) = \max(\widetilde{\lambda}_B(t); 0)$, thus discretized SDEs become:

$$\widetilde{\lambda}_A(t_{i+1}) = \widetilde{\lambda}_A(t_i) + \theta_A(\eta_A - \widetilde{\lambda}_A^+(t_i))\,(t_{i+1} - t_i) + \sigma_{\lambda_A}\sqrt{\widetilde{\lambda}_A^+(t_i)}\,(W_A(t_{i+1}) - W_A(t_i))$$

and

$$\widetilde{\lambda}_B(t_{i+1}) = \widetilde{\lambda}_B(t_i) + \theta_B(\eta_B - \widetilde{\lambda}_B^+(t_i))\,(t_{i+1} - t_i) + \sigma_{\lambda_B}\sqrt{\widetilde{\lambda}_B^+(t_i)}\,(W_B(t_{i+1}) - W_B(t_i))$$

5.9 Introducing jump defaults for stochastic default intensities

As anticipated in preceding sections, it is possible to introduce jump defaults in the CIR process for stochastic intensity.

Considering intensities $\lambda_A(t)$ and $\lambda_B(t)$, we obtain:

$$d\lambda_A(t) = \theta_A(\eta_A - \lambda_A(t))dt + \sigma_{\lambda_A}\sqrt{\lambda_A(t)}dW_A(t) + dJ_A(\alpha_1, \gamma_1) \tag{5.27}$$

see [13], [26] for references, and:

$$d\lambda_B(t) = \theta_B(\eta_B - \lambda_B(t))dt + \sigma_{\lambda_B}\sqrt{\lambda_B(t)}dW_B(t) + dJ_B(\alpha_2, \gamma_2) \tag{5.28}$$

where the jump component for $\lambda_A(t)$ is given by:

$$J_A(\alpha_1, \gamma_1) = \sum_{i=1}^{N_t} Y_i \tag{5.29}$$

with N_t a pure jump process with mean jump arrival rate (or intensity) α_1, and $Y_i \sim Exp(1/\gamma_1)$, where γ_1 is the expected (mean) jump size.

Jump times are therefore independent and distributed according to a pure Poisson process.

At the same time, jump sizes Y_i are independent and exponentially distributed.

The increment $dJ_A(\alpha_1, \gamma_1)$ is the jump, if any, that occurs at time t according to the pure jump process N_t.

We shall therefore simulate a pure jump process N_t, with intensity α_1, which returns the number of random variables, exponentially distributed with parameter γ_1 to add up, in order to define the jump component for $\lambda_A(t)$.

At the same time, the jump component for $\lambda_B(t)$ is given by:

$$J_B(\alpha_2, \gamma_2) = \sum_{i=1}^{M_t} X_i \tag{5.30}$$

with M_t a pure jump process with mean jump arrival rate α_2, and $X_i \sim Exp(1/\gamma_2)$, where γ_2 is the mean jump size.

Parameters $\alpha_1, \gamma_1, \alpha_2, \gamma_2$ shall all be positive.

Jump processes $J_A(\alpha_1, \gamma_1)$ and $J_B(\alpha_2, \gamma_2)$ are independent of Brownian motions $W_A(t)$ and $W_B(t)$.

5.10 Survival probabilities for default intensities

In the beginning of this chapter, we introduced that survival probabilities can be modeled through the exponential distribution of the default intensity γ, if deterministic, or λ if stochastic. As we are dealing with pricing, we placed ourselves in the risk-neutral world, and all assets are priced as the expected discounted value under the risk-neutral measure Q.

$$Q(\tau > t) = \exp(-\gamma t) \tag{5.31}$$

We will give here a more formal definition of survival probabilities. To this purpose we consider the probability space (Ω, F_t, Q).

All the information available is represented by the filtration F_t.

The intensity $\lambda(X_t)$ is built as a function λ of a state variable X_t, and $\lambda(X_t)$ is the intensity of a Poisson process Y_t. The filtration for Y_t is H_t.

The filtration generated by X_t is G_t. We then consider an exponential random variable with parameter 1, ε_1, independent of G_t.

So we have information on the state variable:

$$G_t = \sigma\{X_s; 0 \leq s \leq t\}$$

and on the jump process:

$$H_t = \sigma\{Y_s; 0 \leq s \leq t\}$$

and the filtration $\mathrm{F}_t = G_t \vee H_t$ collects both information sets.

The filtration $G_t = \sigma\{X_s; 0 \leq s \leq t\}$ carries information on default time occurrences. It says, in fact, if default has occurred up to time t, and if yes, it specifies default time τ.

The filtration $H_t = \sigma\{Y_s; 0 \leq s \leq t\}$, instead, relates to information on the magnitude of the jump process itself. This is an attempt to separate information on default-free quantities, and information on default itself.

In the remaing of this work, the stochastic intensity process will simply be referred to as $\lambda(t)$, instead of $\lambda(X_t)$.

In this setting, default time τ is usually defined through an exponential random variable ε_1, with mean 1.

The first jump is, in fact, built as the first time the integral of the stochastic intensity is equal to, or bigger than, an exponential random variable with mean 1.

$$\tau = \inf\left\{t : \int_0^t \lambda(s)ds \geq \varepsilon_1\right\} \tag{5.32}$$

Conditional survival probability after time t, given filtration G_t, is the probability that the integral of the default intensity is smaller than the exponential random variable.

When G_t is known, also $\int_0^t \lambda(s)ds$ is known, so that, given G_t, one can compute the integral of the path for $\lambda(s)$ up to t, and separately get the exponential distribution ε_1 in order to confront both the integral and ε_1.

$$\begin{aligned} Q(\tau > t|G_t) &= Q\left(\int_0^t \lambda(s)ds < \varepsilon_1 |G_t\right) \\ &= Q\left(\varepsilon_1 \geq \int_0^t \lambda(s)ds|G_t\right) \\ &= 1 - F^{Exp}_{\mu=1}\left(\int_0^t \lambda(s)ds\right) \\ &= 1 - \left[1 - \exp\left(-\int_0^t \lambda(s)ds\right)\right] \\ &= \exp\left(-\int_0^t \lambda(s)ds\right) \end{aligned} \tag{5.33}$$

the $F^{Exp}_{\mu=1}\left(\int_0^t \lambda(s)ds\right)$ being the C.D.F.of the exponential distribution with mean 1, calculated for the realization $\int_0^t \lambda(s)ds$.

We know that if $X \sim Exp(\mu)$, then C.D.F. $F^{Exp}_{\mu} = P(x \leq X) = 1 - e^{-\mu X}$ for $x \geq 0$.

In static models, in case we take the expectation, the survival probability is no longer conditional on the specific filtration and we have:

$$Q(\tau > t) = E^Q\left[\exp\left(-\int_0^t \lambda(s)ds\right)\right] \tag{5.34}$$

As seen also in [39] and [26], when dealing with dynamic survival probabilities, instead, we have to consider the following result:

$$Q(\tau > T\,|F_t) = \mathbf{1}\{\tau > t\}\,E^Q\left[\exp\left(-\int_t^T \lambda(s)ds\right)|G_t\right] \tag{5.35}$$

In case of dynamic models, all the information up to present time, represented by the filtration F_t, must be considered. Survival probability for the following time step will depend upon survival up to present time.

- *Case of no Jump defaults*

The solution to the above expression, when $\lambda(s)$ evolves according to a CIR process, is the pure discount bonds formula for the CIR model. Considering CIR SDE for intensity $\lambda_A(t)$:

$$d\lambda_A(t) = \theta_A(\eta_A - \lambda_A(t))dt + \sigma_{\lambda_A}\sqrt{\lambda_A(t)}dW_A(t)$$

survival probability from t until T, given information up to t, i.e. conditional on G_t, is $Q[(t,T)\,|G_t]$ for $(t,T]$(see [26]):

$$Q[(t,T)\,|G_t] = E^Q\left[\exp\left(-\int_t^T \lambda_A(s)ds\right)|G_t\right] = \exp\left[A(T-t) - B(T-t)\lambda(t)\right] \tag{5.36}$$

where:

$$B(x) = \frac{2(\exp(\gamma_A x)-1)}{(\gamma_A+\theta_A)(\exp(\gamma_A x)-1)+2\gamma_A}$$

$$A(x) = \left[\frac{2\gamma_A\exp((\theta_A+\gamma_A)(x/2))}{(\gamma_A+\theta_A)(\exp(\gamma_A x)-1)+2\gamma_A}\right]^{2\theta_A\eta_A/\sigma^2_{\lambda_A}}$$

$$\gamma_A = \sqrt{\theta_A^2 + 2\sigma^2_{\lambda_A}}$$

Survival probability $Q[(t,T)\,|F_t]$ for $(t,T]$, conditional on F_t,instead, will be:

$$\begin{aligned} Q(\tau > T\,|F_t) &= \mathbf{1}\{\tau > t\}\,Q[(t,T)\,|G_t] \\ &= \mathbf{1}\{\tau > t\}\,E^Q\left[\exp\left(-\int_t^T \lambda(s)ds\right)|G_t\right] \end{aligned} \tag{5.37}$$

As a matter of fact, $Q\left[(t,T)\,|G_t\,\right]$ gives the survival probability in the interval $(t,T]$ knowing we are past time t. $Q\left[(t,T)\,|F_t\,\right]$ gives the survival probability conditional on no default up to time t.

Same formulas can be obtained for survival probability of party B, with stochastic intensity $\lambda_B(t)$.

- *Case of Jump defaults*

We can obtain a survival probability formula also for the case of a CIR process with the jump component $J_A(\alpha_1,\gamma_1) = \sum_{i=1}^{N_t} Y_i$. The SDE being:

$$d\lambda_A(t) = \theta_A(\eta_A - \lambda_A(t))dt + \sigma_{\lambda_A}\sqrt{\lambda_A(t)}dW_A(t) + dJ_A(\alpha_1,\gamma_1)$$

the jump-adjusted survival probability formula is, see [13] and [26]:

$$Q\left[(t,T)\,|G_t\,\right] = E^Q\left[\exp\left(-\int_t^T \lambda_A(s)ds\right)|G_t\right] = \exp\left[\widetilde{A}(T-t) - \widetilde{B}(T-t)\lambda(t)\right] \tag{5.38}$$

with:

$$\widetilde{A}(x) = A(x)\left[\frac{2\gamma_A \exp\left(\frac{\theta_A+\gamma_A+2\gamma_1}{2}(T-t)\right)}{2\gamma_A + (\theta_A+\gamma_A+2\gamma_1)\exp\left[\gamma_A(T-t)-1\right]}\right]^{\frac{2\alpha_1\gamma_1}{\sigma_{\lambda_A}^2 - 2\theta_A\gamma_1 - 2\gamma_1^2}}$$

$$\widetilde{B}(x) = B(x)$$

$$\gamma_A = \sqrt{\theta_A^2 + 2\sigma_{\lambda_A}^2}$$

Same formulas can be obtained for survival probability of party B, with stochastic intensity $\lambda_B(t)$.

Chapter 6
Numerical Tests

Abstract In this chapter we will expose a comprehensive set of numerical examples, regarding the calculation of BCVA and Funding Costs for the case of an Interest Rate Swap. Relevant underlying variables to analyze will be interest rates and default intensities. The set of examples will depend on different criteria, specifically:

- model for stochastic interest rates and default intensities: CIR for stochastic interest rates, CIR for stochastic default intensities
- generation path approach for interest rates: exact method or approximated method, through Euler discretization approach
- choice of default intensities variable type: constant and stochastic default intensities
- allowance of jump defaults in the stochastic process for default intensities
- correlation between default intensities
- model parameters values for interest rates and default intensities

The combination of different choices on the above criteria, will lead to a set of numerical results that will be discussed in the course of the present chapter. The derivative instrument priced will be an Interest Rate Swap, where we have two defaultable parties, with different associated credit risk, measured by the relative CDS spread. Party A will be the one with higher credit quality, implied by a lower CDS spread, and party B will be the one with lower credit quality, and a corresponding higher CDS spread.

6.1 Case of stochastic interest rates and constant default intensities

In this section we give numerical example for the case of stochastic interest rates and constant default intensities, calculated on the basis of CDS spread quotes, assumed to be constant over time. No correlation and no jump defaults are allowed.

- *Interest Rates and Discount Factors*

We make the hypothesis of a 5 year IRS, where party A receives a fixed rate from party B, and party B receives a floating rate from party A, with 3 months tenor. Cash flows are exchanged quarterly.

We perform several simulations for the path of interest rate in order to build the Expected Exposure EE and the Negative Expected Exposure NEE profiles, as described in Chapter 1. We perform simulations because EE and NEE are built as options on mark-to-market $X(t_i)$, namely:

$$EE(t_i) = \frac{1}{m}\sum_{k=1}^{m}[\max(X(t_i);0)]$$

and

$$NEE(t_i) = \frac{1}{m}\sum_{k=1}^{m}[\min(X(t_i);0)]$$

where k, with $k = [1,...,m]$ is the number of simulations we perform and over which we want to compute the average of results for each observation date t_i, and where $X(t_i)$ is the mark-to-market of the derivative at the observation date t_i in which we are calculating the $EE(t_i)$ and $NEE(t_i)$ respectively. The time interval is $[0,T]$, with $t_{i=0} = 0 < t_{i=1} < ... < t_{i=n} = T$.

We then have that overall results are $EE = \sum_{i=1}^{n} EE(t_i)$ and $NEE = \sum_{i=1}^{n} NEE(t_i)$.

Formulas for EE and NEE at each observation period (we choose 3 months lenght for each observation period) are built as max and min functions in turn, with respect to the mark-to-market of an interest rate swap.

We have that for each simulation k, with $k = [1,...,m]$, the market value of fixed leg payments $X_k(fix,t)$ of an IRS is given by:

$$X_k(fix,t) = y\sum_{i=1}^{n} B_k(t,t_i)$$

where y is the fixed rate paid in the swap, $B_k(t,t_i)$ is the market value at time t of discount bond with maturity t_i, which means all $B_k(t,t_i)$ are discount factors from date t of valuation to the various maturities t_i, for $i = [1,...,n]$, of cashflows.

The market value of floating leg payments $X_k(flo,t)$ of an IRS, instead, is given by:

$$X_k(flo,t) = B_k(0,0) - B_k(t,n)$$

As a consequence, total market value of a payer swap contract will be given by:

$$X_k(t)_{payer} = X_k(flo,t) - X_k(fix,t)$$

and the total market value of a receiver swap contract by:

$$X_k(t)_{receiver} = X_k(fix,t) - X_k(flo,t)$$

As said, EE and NEE are built as options on mark-to-market, and, in order to calculate the values of these options on the mark-to-market we perform simulations of the path for the interest rate.

Each path for the spot interest rate is obtained according to the exact method for the CIR process, as described in Chapter 5.

In fact, as said in Chapter 5, when a random variable $r(t)$ evolves according to the CIR SDE:

$$dr(t) = \alpha(\mu - r(t))dt + \sigma_r \sqrt{r(t)}dZ(t)$$

then the distribution of $r(t)$ is a Non-central Chi-square one.

In particular, $r(T)\,|r(t) \sim \chi^2(d, r(t)ncp(t,t+\Delta t))$ times $e^{-\mu(\Delta t)}/ncp(t,t+\Delta t)$, with degree of freedom d :

$$d = \frac{4\alpha\mu}{\sigma_r^2}$$

and non-centrality parameter $r(t)ncp(t,t+\Delta t)$:

$$ncp(t,t+\Delta t) = \frac{4\alpha e^{-\alpha(\Delta t)}}{\sigma_r^2(1-e^{-\alpha\Delta t})}$$

The cumulative distribution function being represented by:

$$\Pr[r(T) < r\,|r(t)\,] = F^{\chi^2}_{d,r(t)ncp(t,t+\Delta t)}\left(\frac{r}{k}\right)$$

with

$$\begin{aligned} k &= \frac{e^{-\alpha(\Delta t)}}{ncp(t,t+\Delta t)} \\ &= \frac{\sigma_r^2(1-e^{-\alpha\Delta t})}{4\alpha} \end{aligned}$$

It is possible to exactly obtain $r(t_{i+1})$, given $r(t_i)$, from the distribution of $r(t)$. Of course each path will be calculated according to discretization of the time grid, and the time increment shall be Δt.

For $d > 1$, we shall have that:

$$r(t+\Delta t) = k\left(\chi^2(d-1) + (Z + \sqrt{r(t)ncp(t+\Delta t)})\right)$$

and for $d \leq 1$, instead:

$$r(t+\Delta t) = k\left(\chi^2(d+2J)\right)$$

with $J \sim P\left(\frac{ncp(t+\Delta t)}{2}\right)$.

As explained in Chapter 5, one should proceed as following:

1. Generate J as $J \sim P\left(\frac{ncp(t+\Delta t)}{2}\right)$ and find the outcome $J = j \in N$
2. Generate a Central Chi-Square $\chi^2(d+2j)$, with degrees of freedom $(d+2j)$ where j is the outcome of the previously generated Poisson distribution.
3. Simulate the Central Chi-Square distribution χ^2 as a Gamma distribution Γ, so that now d must not be necessarily an integer.

$$\chi^2(d+2j) \sim \Gamma\left(\frac{d+2j}{2};2\right)$$

Given each path for the spot interest rate, the term structure of discount factors is calculated through the closed formula for CIR process, as seen in Chapter 5.

$$B_t(T\,|r_t\,) = \exp(A(T-t) - B(T-t)r_t)$$

where $B_t(T\,|r_t\,)$ is the discount factor for maturity T calculated in t, with:

$$B(x) = \frac{2(\exp(\gamma x)-1)}{(\gamma+\alpha)(\exp(\gamma x)-1)+2\gamma}$$

$$A(x) = \left[\frac{2\gamma\exp((\alpha+\gamma)(x/2))}{(\gamma+\alpha)(\exp(\gamma x)-1)+2\gamma}\right]^{2\alpha\mu/\sigma^2}$$

and:

$$\gamma = \sqrt{\alpha^2+2\sigma^2}$$

always with $2\alpha\mu > \sigma_r^2$.

- *Default Intensities*

In this case default intensities for party A e party B shall be considered as constants, γ_A and γ_B, calculated from CDS quote CDS_A and CDS_B, according to the formula:

$$\gamma_A = \frac{CDS_A}{(1-\delta_A)}$$

$$\gamma_B = \frac{CDS_B}{(1-\delta_B)}$$

Survival probabilities later used in BCVA and FCA formulas for constant default intensities are given by:

$$Q_A\left(\tau > t_i \left| F_{t_{i-1}} \right.\right) = S_A(t_i)\left| F_{t_{i-1}} \right. = \mathbf{1}\{\tau > t_{i-1}\}\exp - [\gamma_A(t_i - t_{i-1})]$$

Recovery rates δ_A and δ_B are assumed at a level of 40%.

- *Bilateral Credit Value Adjustment and Funding Cost Adjustment Formulas*

We assume to be valuing the IRS from the point of view of party A. As a consequence, Funding Costs will depend upon party A's funding rate and recovery rate. The terms $F_A(t_i, t_{i+1})$ will be used to build the FCA from t_i to t_{i+1}.

Formulas used for Bilateral Credit Value Adjustment and Funding Cost Adjustment are those of Chapter 4. Recalling them, we have BCVA:

$$\begin{aligned} BCVA \approx\ & (1-\delta_B)\sum_{i=1}^{n} D(t,t_i)EE(t_i)S_A(t_i)\left[S_B(t_{i-1}) - S_B(t_i)\right] \\ & +(1-\delta_A)\sum_{i=1}^{n} D(t,t_i)NEE(t_i)S_B(t_i)\left[S_A(t_{i-1}) - S_A(t_i)\right] \end{aligned}$$

and FCA:

$$\begin{aligned} FCA =\ & \sum_{i=1}^{n-1}\left[D(t,t_i)ECF^-(t_{i-1},t_i)F_A(t_i,t_n)\right] - \\ & (1-\delta_A)\sum_{i=1}^{n-1}\left[\Pr(\tau_B > t_i)\Pr(\tau_A \le t_i)D(t,t_i)ECF^-(t_{i-1},t_i)F_A(t_i,t_n)\right] \end{aligned}$$

- *Results*

We give here an overview of results obtained.

Parameters for interest rate are $\alpha = 0.2$, $\mu = 0.05$, $\sigma_r = 0.1$ with $r_{t=0} = 0.05$ and $T = 5$ years.

Fig. 6.1 represents the EE and NEE profile for a receiver IRS at 5% fixed rate against payer 3m-tenor floating rate, with 200 simulations for interest rates as described above.

In the first set of tables represented by Fig. 6.2 and Fig. 6.3, survival and default probabilities are calculated assuming constant default intensities with $CDS_A = 300bps$ and $CDS_B = 500bps$.

$$\gamma_A = \frac{CDS_A}{(1-\delta_A)} = \frac{0.03}{(1-0.4)} = 0.05$$

$$\gamma_B = \frac{CDS_B}{(1-\delta_B)} = \frac{0.05}{(1-0.4)} = 0.0833$$

BCVA and FCA Results

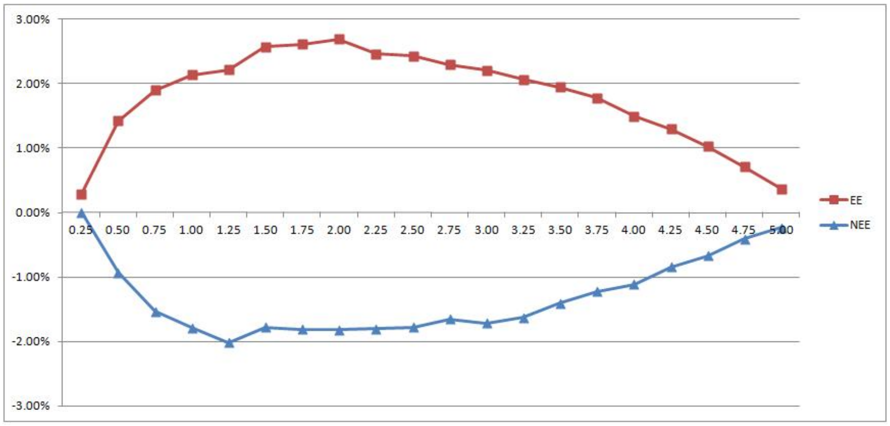

Fig. 6.1 EE and NEE profile for an IRS

FCA is calculated assuming a funding spread for party A equal to its $CDS_A = 300bps$. We are here saying that the funding spread is equal to the CDS spread, and assuming a liquidity premium equal to zero, as if there was no bid/offer spread between primary and secondary market. It is therefore a "credit-risky" funding.

As per the "Adjustment for Default Risk" in FCA, given:

$$FCA = \sum_{i=1}^{n-1} \left[D(t,t_i) ECF^{-}(t_{i-1},t_i) F_A(t_i,t_n) \right] -$$
$$(1-\delta_A) \sum_{i=1}^{n-1} \left[\Pr(\tau_B > t_i) \Pr(\tau_A \leq t_i) D(t,t_i) ECF^{-}(t_{i-1},t_i) F_A(t_i,t_n) \right]$$

we consider k, with $k = [1,...,m]$, the number of simulations used to build EE and NEE profiles, and time interval $[0,T]$, with $t_{i=0} = 0 < t_{i=1} < ... < t_{i=n} = T$, so that $t_{i,k}$ shall be time period t_i in simulation k.

6.2 Case of stochastic interest rates and stochastic default intensities

Interest rate modeling through the CIR process, discount factors, BCVA and FCA formulas are still as in the previous section case.

We now introduce stochastic default intensities, under the assumption of no correlation and no jump defaults allowed.

- *Default Intensities*

Adjusted CVA	0.2978%
Adjusted DVA	-0.1302%
BCVA	0.1676%

Time	DF	EE	NEE	B survives	A survives	B defaults	A defaults
0.00	1.0000			100.00%	100.00%		
0.25	0.9876	0.29%	0.00%	97.94%	98.76%	2.06%	1.24%
0.50	0.9753	1.42%	-0.93%	95.92%	97.53%	2.02%	1.23%
0.75	0.9632	1.90%	-1.53%	93.94%	96.32%	1.98%	1.21%
1.00	0.9512	2.14%	-1.79%	92.00%	95.12%	1.94%	1.20%
1.25	0.9394	2.22%	-2.02%	90.11%	93.94%	1.90%	1.18%
1.50	0.9277	2.57%	-1.78%	88.25%	92.77%	1.86%	1.17%
1.75	0.9162	2.61%	-1.81%	86.43%	91.62%	1.82%	1.15%
2.00	0.9048	2.69%	-1.82%	84.65%	90.48%	1.78%	1.14%
2.25	0.8936	2.46%	-1.81%	82.90%	89.36%	1.75%	1.12%
2.50	0.8825	2.43%	-1.78%	81.19%	88.25%	1.71%	1.11%
2.75	0.8715	2.30%	-1.65%	79.52%	87.15%	1.67%	1.10%
3.00	0.8607	2.21%	-1.72%	77.88%	86.07%	1.64%	1.08%
3.25	0.8500	2.07%	-1.63%	76.27%	85.00%	1.61%	1.07%
3.50	0.8395	1.95%	-1.40%	74.70%	83.95%	1.57%	1.06%
3.75	0.8290	1.78%	-1.22%	73.16%	82.90%	1.54%	1.04%
4.00	0.8187	1.49%	-1.11%	71.65%	81.87%	1.51%	1.03%
4.25	0.8086	1.29%	-0.84%	70.18%	80.86%	1.48%	1.02%
4.50	0.7985	1.03%	-0.67%	68.73%	79.85%	1.45%	1.00%
4.75	0.7886	0.71%	-0.41%	67.31%	78.86%	1.42%	0.99%
5.00	0.7788	0.37%	-0.23%	65.92%	77.88%	1.39%	0.98%

Fig. 6.2 BCVA for Constant Intensity

In this second set of tables, survival and default probabilities are calculated considering a stochastic intensity behaviour, evolving through a CIR process.

Given the CIR SDEs for stochastic default intensities of party A and party B, with independent Brownian Motions:

$$d\lambda_A(t) = \theta_A(\eta_A - \lambda_A(t))dt + \sigma_{\lambda_A}\sqrt{\lambda_A(t)}dW_A(t)$$

$$d\lambda_B(t) = \theta_B(\eta_B - \lambda_B(t))dt + \sigma_{\lambda_B}\sqrt{\lambda_B(t)}dW_B(t)$$

parameters for stochastic intensities are $\theta_A = 0.3$, $\eta_A = 0.02$, $\sigma_{\lambda_A} = 0.01$ and $\lambda_A(t=0) = 0.05$ for party A, and $\theta_B = 0.3$, $\eta_B = 0.06$, $\sigma_{\lambda_B} = 0.01$ and $\lambda_B(t = 0) = 0.0833$ for party B. Initial values are therefore equal to those of the constant intensity case, but evolve thereafter according to CIR parameters. The CIR process for λ_A and λ_B is here calculated just once through the exact method.

Funding with no Default Risk	0.8283%
Adjustment for Default Risk	-0.0420%
FCA	0.7862%

Time	Avg Funded Net CF	PV Funded Net CF	Adjustment for Default Risk	PV risky funding
0.00				
0.25	0.0000%	0.0000%	0.0000%	0.0000%
0.50	-0.0353%	-0.0345%	0.0005%	-0.0340%
0.75	-0.0575%	-0.0553%	0.0011%	-0.0542%
1.00	-0.0660%	-0.0627%	0.0017%	-0.0610%
1.25	-0.0745%	-0.0700%	0.0023%	-0.0677%
1.50	-0.0642%	-0.0596%	0.0023%	-0.0573%
1.75	-0.0661%	-0.0606%	0.0026%	-0.0579%
2.00	-0.0658%	-0.0595%	0.0029%	-0.0567%
2.25	-0.0641%	-0.0573%	0.0030%	-0.0542%
2.50	-0.0627%	-0.0553%	0.0032%	-0.0522%
2.75	-0.0573%	-0.0499%	0.0031%	-0.0469%
3.00	-0.0598%	-0.0514%	0.0033%	-0.0481%
3.25	-0.0575%	-0.0488%	0.0034%	-0.0455%
3.50	-0.0488%	-0.0409%	0.0029%	-0.0380%
3.75	-0.0418%	-0.0346%	0.0026%	-0.0320%
4.00	-0.0376%	-0.0308%	0.0024%	-0.0284%
4.25	-0.0282%	-0.0228%	0.0018%	-0.0209%
4.50	-0.0229%	-0.0182%	0.0015%	-0.0167%
4.75	-0.0129%	-0.0102%	0.0009%	-0.0093%
5.00	-0.0073%	-0.0057%	0.0005%	-0.0052%

Fig. 6.3 FCA for Constant Intensity

Survival probabilities are calculated as indicated in Chapter 5, using pure discount bond formulas for the CIR process, for the case of no jumps.

$$Q[(t,T)|G_t] = E^Q\left[\exp\left(-\int_t^T \lambda_A(s)ds\right)|G_t\right] = \exp[A(T-t) - B(T-t)\lambda(t)]$$

where:

$$B(x) = \frac{2(\exp(\gamma_A x) - 1)}{(\gamma_A + \theta_A)(\exp(\gamma_A x) - 1) + 2\gamma_A}$$

$$A(x) = \left[\frac{2\gamma_A \exp((\theta_A + \gamma_A)(x/2))}{(\gamma_A + \theta_A)(\exp(\gamma_A x) - 1) + 2\gamma_A}\right]^{2\theta_A \eta_A / \sigma_{\lambda_A}^2}$$

$$\gamma_A = \sqrt{\theta_A^2 + 2\sigma_{\lambda_A}^2}$$

The same applies for survival probabilities of party B.

The EE and NEE profiles are left unchanged with respect to the previous case, as the only varying component is represented by stochastic intensities.

Results are given below.

BCVA Results

Adjusted CVA	0.2710%
Adjusted DVA	-0.0998%
BCVA	0.1712%

Time	**DF**	**EE**	**NEE**	**B intensity**	**B survives**	**A intensity**	**A survives**	**B defaults**	**A defaults**
0.00	1.0000			8.33%	100.00%	5.00%	100.00%		
0.25	0.9876	0.29%	0.00%	8.22%	97.96%	4.53%	98.78%	2.04%	1.22%
0.50	0.9753	1.42%	-0.93%	7.91%	96.00%	4.40%	97.64%	1.96%	1.15%
0.75	0.9632	1.90%	-1.53%	7.71%	94.11%	4.21%	96.55%	1.89%	1.09%
1.00	0.9512	2.14%	-1.79%	7.62%	92.30%	3.87%	95.51%	1.82%	1.03%
1.25	0.9394	2.22%	-2.02%	7.58%	90.55%	3.83%	94.53%	1.75%	0.98%
1.50	0.9277	2.57%	-1.78%	7.54%	88.85%	3.77%	93.59%	1.69%	0.94%
1.75	0.9162	2.61%	-1.81%	7.28%	87.22%	3.69%	92.70%	1.64%	0.89%
2.00	0.9048	2.69%	-1.82%	7.32%	85.63%	3.62%	91.84%	1.58%	0.86%
2.25	0.8936	2.46%	-1.81%	7.14%	84.10%	3.43%	91.02%	1.53%	0.82%
2.50	0.8825	2.43%	-1.78%	7.05%	82.61%	3.27%	90.23%	1.49%	0.79%
2.75	0.8715	2.30%	-1.65%	6.88%	81.17%	3.22%	89.48%	1.45%	0.76%
3.00	0.8607	2.21%	-1.72%	6.81%	79.76%	3.11%	88.75%	1.40%	0.73%
3.25	0.8500	2.07%	-1.63%	6.66%	78.39%	3.04%	88.05%	1.37%	0.70%
3.50	0.8395	1.95%	-1.40%	6.74%	77.06%	2.95%	87.37%	1.33%	0.68%
3.75	0.8290	1.78%	-1.22%	6.66%	75.77%	2.90%	86.72%	1.30%	0.66%
4.00	0.8187	1.49%	-1.11%	6.53%	74.50%	2.86%	86.08%	1.26%	0.63%
4.25	0.8086	1.29%	-0.84%	6.49%	73.27%	2.81%	85.47%	1.23%	0.62%
4.50	0.7985	1.03%	-0.67%	6.61%	72.07%	2.83%	84.87%	1.20%	0.60%
4.75	0.7886	0.71%	-0.41%	6.68%	70.89%	2.75%	84.29%	1.18%	0.58%
5.00	0.7788	0.37%	-0.23%	6.71%	69.74%	2.73%	83.72%	1.15%	0.57%

Fig. 6.4 BCVA for Stochastic Intensity

We will refer to DVA and CVA for simplicity, but we shall remember that we are here indeed referring to ”Adjusted DVA” and ”Adjusted CVA”. With respect to the constant case, with $\gamma_A = 0.05$ and $\gamma_B = 0.0833$, in this stochastic case, the process for λ_A has a mean reversion trend towards $\eta_A = 0.02$, and therefore the adjustment for DVA shall be lower. In fact, credit quality of party A will be on average higher over the life of the transaction, with respect to the constant case, given that the process for λ_A will tend to a level of $\eta_A = 0.02$, against the constant value of $\gamma_A = 0.05$ for the case of constant intensity. We observe a DVA value of -0.098% instead of -0.1302%.

At the same time, CVA will be lower in the stochastic case, because a mean reversion trend to a lower long run average value is assumed also for party B. The mean reversion level for $\eta_B = 0.06$, compared to the constant case with $\gamma_B = 0.0833$, brings along a reduction of CVA contribution to a value of 0.2710% instead of 0.2978%.

When we input mean reversion levels equal to initial values in the processes Λ_A and Λ_B respectively, we obtain values for CVA and DVA equal to the constant intensity case.

As a consequence, a financial institution should introduce stochastic intensities predominantly when willing to account for a certain mean reversion trend in the credit quality of a counterparty, and/or its own creditworthiness. This trend will derive from credit model calibration to CDS curve observable in the market.

One of these cases may be, for example, if the actual level of a counterparty's CDS is due to an exceptional market situation, that brings along an increase in the CDS level that will not likely hold in the near future. This may happen if the CDS increase can be mostly ascribed to systemic risk the market is pricing in, but it is not supported by effectively deteriorated financial figures of the counterparty itself.

FCA Results

In the same way as for BCVA, we see that the introduction of $\eta_A = 0.02$, lower than $\lambda_A(t=0) = 0.05$, results in a decrease in the "Adjustment for Default Risk" in overall funding costs with respect to the constant case.

Moreover we should notice that the original component of "Funding with no Default Risk" is always calculated assuming a constant funding spread for party A equal to $CDS_A = 300bps$, to be applied to the floating rate. It is only in the "Adjustment for Default Risk" that we introduce the stochasticity of the intensity process for λ_A, in order to properly account for our assumptions on non deterministic dynamics for the creditworthiness for party A (i.e. mean reverting trend, volatility, etc). This choice can indeed be supported by a financial argument, provided that party A should enter in $t = 0$ in a financing contract in order to fund its expected negative cash flows, thus the cost of funding to be applied would be fixed in $t = 0$.

In the following table, instead, keeping previous intensity parameters fo default intensity of party B, we study the impact of a change in the equilibrium level of party A's intensity. The equilibrium level η_A is allowed to change from 0.02 to 0.07, given initial value $\lambda_A(t=0) = 0.05$.

With the equilibrium level moving upwards, we see that BCVA decreases, as the contribution of DVA increases, thus reducing the charge for Counterparty Credit Risk required. At the same time we see that FCA decreases, because as the equilibrium level increases there is a corresponding increase in the funding amount "at risk", meaning a higher "Adjustment for Default Risk".

Funding with no Default Risk	0.8283%
Adjustment for Default Risk	-0.0355%
FCA	0.7928%

Time	Avg Funded Net CF	PV Funded Net CF	Adjustment for Default Risk	PV risky funding
0.00				
0.25	0.0000%	0.0000%	0.0000%	0.0000%
0.50	-0.0353%	-0.0345%	0.0005%	-0.0340%
0.75	-0.0575%	-0.0553%	0.0011%	-0.0543%
1.00	-0.0660%	-0.0627%	0.0016%	-0.0612%
1.25	-0.0745%	-0.0700%	0.0021%	-0.0679%
1.50	-0.0642%	-0.0596%	0.0020%	-0.0575%
1.75	-0.0661%	-0.0606%	0.0023%	-0.0582%
2.00	-0.0658%	-0.0595%	0.0025%	-0.0570%
2.25	-0.0641%	-0.0573%	0.0026%	-0.0547%
2.50	-0.0627%	-0.0553%	0.0027%	-0.0526%
2.75	-0.0573%	-0.0499%	0.0026%	-0.0474%
3.00	-0.0598%	-0.0514%	0.0028%	-0.0487%
3.25	-0.0575%	-0.0488%	0.0027%	-0.0461%
3.50	-0.0488%	-0.0409%	0.0024%	-0.0385%
3.75	-0.0418%	-0.0346%	0.0021%	-0.0325%
4.00	-0.0376%	-0.0308%	0.0019%	-0.0289%
4.25	-0.0282%	-0.0228%	0.0015%	-0.0213%
4.50	-0.0229%	-0.0182%	0.0012%	-0.0171%
4.75	-0.0129%	-0.0102%	0.0007%	-0.0095%
5.00	-0.0073%	-0.0057%	0.0004%	-0.0053%

Fig. 6.5 FCA for Stochastic Intensity

As expected, the value for DVA with $\eta_A = \lambda_A(t = 0) = 0.05$ is almost exactly equal to the constant case, as we have DVA for the stochastic case equal to -0.1322% and DVA for the constant case equal to -0.1302%.

CVA for this stochastic case, instead, is not equal to the constant case because $\eta_B = 0.06$ with $\lambda_B(t = 0) = 0.0833$.

We see that, as the mean reversion level of the process for λ_A increases, there is also a slight effect on CVA which decreases, because we have to account for a lower survival probability of party A in the calculation of CVA. We are in fact talking about ”Adjusted CVA”.

Intuitively, if the credit quality of a party is deteriorating, this party should be less concerned about the CDS level of its counterparty and more about its own creditworthiness. We can say that, if the default intensity of one party increases in the long run, this party will see the relative importance of its own exposure towards

Stochastic Intensity Party A	Adjusted CVA	Adjusted DVA	BCVA	FCA
Initial Value=0.05 Equilibrium level=0.02 Volatility=0.01 Mean Reversion=0.3	0.2710%	-0.0998%	0.1712%	0.7928%
Initial Value=0.05 Equilibrium level=0.03 Volatility=0.01 Mean Reversion=0.3	0.2692%	-0.1108%	0.1584%	0.7902%
Initial Value=0.05 Equilibrium level=0.04 Volatility=0.01 Mean Reversion=0.3	0.2675%	-0.1216%	0.1459%	0.7878%
Initial Value=0.05 Equilibrium level=0.05 Volatility=0.01 Mean Reversion=0.3	0.2657%	-0.1322%	0.1335%	0.7853%
Initial Value=0.05 Equilibrium level=0.06 Volatility=0.01 Mean Reversion=0.3	0.2640%	-0.1426%	0.1214%	0.7829%
Initial Value=0.05 Equilibrium level=0.07 Volatility=0.01 Mean Reversion=0.3	0.2623%	-0.1529%	0.1094%	0.7804%

Fig. 6.6 Table Increasing Equilibrium Level

the other counterparty decreasing, because the probability of falling in own financial distress will be predominant.

Moreover, we recognize that the value for FCA with stochastic intensity better approximates the value for FCA with constant default intensities, when the equilibrium level is set equal to the initial value.

6.3 Correlation between stochastic default intensities

In this section we introduce correlation between stochastic default intensities, with no jump defaults allowed. We will study the impact of correlation between default intensities in the computation of BCVA.

In order to model correlation between default intensities, we shall resort to MonteCarlo simulations for default events, because no closed formula for survival probabilities is applicable for the case of correlation.

- *Expected Exposure and Negative Expected Exposure*

Also in the case of correlation between default intensities, values for EE and NEE are calculated as in the preceding set of numerical tests. This means that EE and NEE are obtained assuming stochastic interest rates, evolving through the above CIR SDEs, and are calculated through the exact method.

- *Default simulation*

When willing to simulate default through MC simulations, instead of directly computing survival and default probabilities with closed formulas, one needs to resort to the definition of "first default" itself. Given the definition of τ we provided in Chapter 5:

$$\tau = \inf\left\{ t : \int_0^t \lambda(s)ds \geq \varepsilon_1 \right\}$$

simulation should proceed as following:

1. Simulate an exponential distribution ε_1 with parameter 1
2. Simulate the path for stochastic intensity $\lambda_A(t)$, in this case through Euler discretization scheme
3. Integrate the path for stochastic intensity $\lambda_A(t)$ with respect to time
4. Confront the exponential distribution with the integral of the path, according to the following $Q(\tau > t \,|G_t) = Q\left(\int_0^t \lambda(s)ds < \varepsilon_1 \,|G_t\right)$
5. If ε_1 is smaller than, or equal to, the integral of the path before final point in time T, then first default τ^1 occurs at that moment, otherwise no default happens
6. Compute BCVA given τ^1
7. Repeat the above steps for stochastic intensity $\lambda_B(t)$, generating another exponential distribution ε_2 with parameter 1. In this case $\lambda_B(t)$ shall be correlated to $\lambda_A(t)$, through the diffusive noise term, as described in Chapter 5
8. Perform k times the above steps for $\lambda_A(t)$ and $\lambda_B(t)$, where k is the number of MC simulations, and then average results

In particular, in order to compute BCVA component, if any, we proceed as following:

1. At each simulation we see if default τ^1 happens, and, if yes, we see if it is party A or party B who defaults first, i.e. we see if $\tau^1 = \tau_A$ or $\tau^1 = \tau_B$.

2. If $\tau^1 = \tau_A$, we take the $NEE(\tau^1)$ component corresponding to the point in time τ^1, weighted for $(1-\delta_A)$, that would not be recovered by party B in case of default of party A. We then apply the appropriate discount factor up to present time and obtain $NEE(\tau^1)\ (1-\delta_A)D(t,\tau^1)$, where $D(t,\tau^1)$ is the risk-free discount factor calculated in $t=0$ for maturity τ^1. We are in fact intereste in the contribution to DVA in case of A defaulting first.
3. If $\tau^1 = \tau_B$, we take the $EE(\tau^1)$ component corresponding to the point in time τ^1, weighted for $(1-\delta_B)$, that would not be recovered by party A in case of default of party B. We then apply the appropriate discount factor up o present time and obtain $EE(\tau^1)\ (1-\delta_B)D(t,\tau^1)$, where $D(t,\tau^1)$ is the risk-free discount factor calculated in $t=0$ for maturity τ^1. We are in fact intereste in the contribution to CVA in case of B defaulting first.
4. We calculate the average of all CVA and DVA contributions, over all simulations.

6.3.1 BCVA for the case of no jumps

Here are results for BCVA with $k = 5000$ simulations for default events, with parameters for stochastic intensities respectively $\theta_A = 0.3$, $\eta_A = 0.02$, $\sigma_{\lambda_A} = 0.01$ and $\lambda_A(t=0) = 0.05$ for party A, and $\theta_B = 0.3$, $\eta_B = 0.06$, $\sigma_{\lambda_B} = 0.01$ and $\lambda_B(t=0) = 0.0833$ for party B. Recovery rates δ_A and δ_B are assumed at a level of 40%.

These simulations, as a matter of facts, are done assuming low volatility ($\sigma_{\lambda_A} = \sigma_{\lambda_B} = 0.01$), high speed of mean reversion ($\theta_A = \theta_B = 0.3$), different levels of initial default intensities between party A and party B ($\lambda_A(t=0) = 0.05$ and $\lambda_B(t=0) = 0.0833$), and different levels for default intensities between initial and average levels ($\lambda_A(t=0) = 0.05$ with $\eta_A = 0.02$ and $\lambda_B(t=0) = 0.0833$ with $\eta_B = 0.06$).

Correlation is assumed to vary from -1 to +1.

When we perform BCVA calculations for $k = 5000$ simulations we see that the results tend to converge to the case of $\rho = 0$, where we had $BCVA = 0.1712\%$, for same stochastic parameters. The following table summarizes results for $k = 5000$.

BCVA Results

From this numerical test performed through MC simulations we can extrapolate two main findings.

The first result we should notice is that CVA and DVA levels we obtain through MC simulations are coherent with those obtained when applying closed formula solutions. This coherence result validates both methods applied.

The second result we should consider is that, over a sufficiently large number of simulations, CVA and DVA values do not depend upon the correlation parameter we apply between Brownian Motions of default intensities processes for λ_A and λ_B. This is not intuitively the result we would expect from a financial point of view at first. In fact, it can be argued that, if there is correlation between party A and party B,

Correlation	CVA	DVA	BCVA
-1	0.265%	-0.092%	0.173%
-0.1	0.270%	-0.096%	0.175%
0.1	0.262%	-0.100%	0.162%
1	0.257%	-0.089%	0.168%

Fig. 6.7 BCVA with Correlation between Stochastic Intensities

the former should reduce CVA charges for Counterparty Credit Risk, given that high default probability of party B should be associated with high default probability of party A.

Nonetheless, this expected result fails to realize when modeling correlation. The technical reason for this will be explored in the next section, when introducing common jumps.

From a financial point of view, though, one should further think and understand that default events are not necessarily anticipated by a deterioration in the credit quality of a counterparty, here represented through the dynamics for the stochastic intensity. In general, defaults are more frequently sudden and unpredictable events, that may not be associated with a long run upward trend of the CDS spread.

For $k = 5000$ in the following table, instead, parameters for party A's stochastic intensity are left unchanged, $\theta_A = 0.3$, $\eta_A = 0.02$, $\sigma_{\lambda_A} = 0.01$ and $\lambda_A(t = 0) = 0.05$ while for party B the equilibrium level is set higher so that λ_B has un upward trend, $\theta_B = 0.3$, $\eta_B = 0.10$, $\sigma_{\lambda_B} = 0.01$ and $\lambda_B(t = 0) = 0.0833$. Recovery rates δ_A and δ_B are assumed at a level of 40%. Basic assumptions on model parameters are the same as above, in particular low volatility and high speed of mean reversion for both processes.

We see that absolute level of BCVA increases, as the contribution of CVA increases. This is a result of the higher level for the average value of λ_B process, as we have $\eta_B = 0.10$. This, in fact, may represent a deterioration in the credit quality for party B, with respect to the previous numerical test with $\eta_B = 0.06$.

The level of DVA instead remains almost unchanged with respect to the previous case, because parameters for λ_A are not modified.

Correlation	CVA	DVA	BCVA
-1	0.321%	-0.093%	0.228%
-0.1	0.323%	-0.092%	0.231%
0.1	0.308%	-0.096%	0.213%
1	0.327%	-0.099%	0.227%

Fig. 6.8 BCVA with correlation between stochastic intensities and higher mean reversion level

6.3.2 BCVA for the case of jump defaults

Here below we give results for BCVA calculations for the case of correlation between default intensities, when also jump defaults are allowed in the CIR SDEs for λ_A and λ_B.

Simulation is done according to the procedure explained in the previous section, i.e. thorugh the simulation of default events, as we are in the case of correlation between default events. Parameters for party A's stochastic intensity are left unchanged, $\theta_A = 0.3$, $\eta_A = 0.02$, $\sigma_{\lambda_A} = 0.01$ and $\lambda_A(t=0) = 0.05$, and $\theta_B = 0.3$, $\eta_B = 0.06$, $\sigma_{\lambda_B} = 0.01$ and $\lambda_B(t=0) = 0.0833$. Recovery rates δ_A and δ_B are assumed at a level of 40%.

The difference lies in the introduction of jump defaults.

Jump processes $J_A(\alpha_1, \gamma_1)$ and $J_B(\alpha_2, \gamma_2)$ are independent of Brownian motions $W_A(t)$ and $W_B(t)$, while $W_A(t)$ and $W_B(t)$ are correlated between themselves as in the previuous numerical test. Parameters $\alpha_1, \gamma_1, \alpha_2, \gamma_2$ shall all be set positive, and specifically jump arrival rates are set to $\alpha_1 = 0.15$ and $\alpha_2 = 0.15$, while expected jump sizes are set to $\gamma_1 = 0.03$ and $\gamma_2 = 0.05$. We shall see that correlation between default intensities does not really affect the level of BCVA one party should charge, as what really matters is the equilibrium level for λ_A and λ_B.

BCVA Results

For $k = 5000$, we see that BCVA values stabilize and converge to one level, across different values for the correlation parameter.

It can be observed that the contribution of DVA increases significantly when jump defaults are included, both for λ_A and λ_B.

In particular, the following effects are tested:

- Increase of expected jump size for λ_B, as we set $\gamma_2 = 0.10$ (with $\gamma_1 = 0.03$ as in the original case)

Correlation	CVA	DVA	BCVA
-1	0.373%	-0.151%	0.222%
-0.5	0.377%	-0.152%	0.225%
-0.1	0.368%	-0.150%	0.218%
0.1	0.370%	-0.156%	0.214%
0.5	0.380%	-0.147%	0.233%
1	0.369%	-0.154%	0.215%

Fig. 6.9 BCVA with Correlation and Jumps in the Stochastic Intensity processes

- Increase of expected jump size for λ_A, as we set $\gamma_1 = 0.10$ (with $\gamma_2 = 0.03$ as in the original case)

Increase of expected jump size for λ_B

Here we see the impact of an increase of jump size for λ_B as we set $\gamma_2 = 0.10$, with $k = 5000$.

Other jump parameters remain unchanged from previuos test, $\gamma_1 = 0.03$, $\alpha_1 = 0.15$ and $\alpha_2 = 0.15$. Other stochastic parameters are set at $\theta_A = 0.3$, $\eta_A = 0.02$, $\sigma_{\lambda_A} = 0.01$ and $\lambda_A(t = 0) = 0.05$, and $\theta_B = 0.3$, $\eta_B = 0.06$, $\sigma_{\lambda_B} = 0.01$ and $\lambda_B(t = 0) = 0.0833$. Recovery rates δ_A and δ_B are assumed at a level of 40%.

Again final values for BCVA are not affected by correlation, but raising γ_2 to $\gamma_2 = 0.10$ results in higher BCVA level, as CVA increases along with γ_2 while DVA remains unchanged.

Increase of expected jump size for λ_A

Here we see the impact of an increase of jump size for λ_A as we set $\gamma_1 = 0.10$, with $k = 5000$.

Other jump parameters remain unchanged from original test, $\gamma_2 = 0.03$, $\alpha_1 = 0.15$ and $\alpha_2 = 0.15$. Other stochastic parameters are still set at $\theta_A = 0.3$, $\eta_A = 0.02$, $\sigma_{\lambda_A} = 0.01$ and $\lambda_A(t = 0) = 0.05$, and $\theta_B = 0.3$, $\eta_B = 0.06$, $\sigma_{\lambda_B} = 0.01$ and $\lambda_B(t = 0) = 0.0833$. Recovery rates δ_A and δ_B are assumed at a level of 40%.

Final values for BCVA do not change following the variation of correlation ρ. Absolute value for BCVA, though, reduces significantly, as DVA increases close to the level of CVA.

Correlation	CVA	DVA	BCVA
-1	0.467%	-0.152%	0.315%
-0.1	0.466%	-0.144%	0.322%
0.1	0.460%	-0.144%	0.316%
1	0.468%	-0.139%	0.329%

Fig. 6.10 BCVA with Correlation, Jumps and higher mean reversion level in the Stochastic Intensity process for B

Correlation	CVA	DVA	BCVA
-1	0.295%	-0.270%	0.025%
-0.1	0.311%	-0.261%	0.050%
0.1	0.290%	-0.261%	0.028%
1	0.293%	-0.272%	0.020%

Fig. 6.11 BCVA with Correlation, Jumps and higher mean reversion level in the Stochastic Intensity process for A

6.3.3 Common jumps

In all preceding numerical tests we experienced a low or null impact of correlation between λ_A and λ_B processes in the computation of BCVA.

We remember, from the beginning of this work, that the intensity process for stochastic intensity λ_A is referred to as $\Lambda_A(T) = \int_0^T \lambda_A(t)dt$, and that $\tau_A^1 = \Lambda_A^{-1}(\varepsilon_1)$. The same applies of course for λ_B.

If we want to include in our model correlation of default times between party A and party B, we need to understand the following.

As also explained in Morini (2011) [45], the stochasticity of τ_A^1 and τ_B^1 can either derive from the stochasticity of the the processes $\Lambda_A(T)$ and $\Lambda_B(T)$, or from the stochasticity of ε_1, ε_2. This means that, one can either choose to:

- simulate correlated processes for λ_A and λ_B, and keep ε_1, ε_2 independent respectively;
- simulate independent processes for λ_A and λ_B, and correlate ε_1, ε_2.

The choice of this work was to follow the first approach, introducing correlation in the stochastic processes for default intensities, leaving ε_1, ε_2 independent respectively. Unfortunately, as reported in Morini (2011),see [45], when simulating diffusive intensities, this approach is not able to provide sufficient effect of correlation in the results. The main reason for this failure appears to be the low dependance between the stochastic process for a generic λ_i and the respective first default time τ_i^1. As suggested in [45], in order to see a stronger correlation effect in our results, a possible solution should be that of increasing the dependency between the process for λ_i and τ_i^1.

This objective can be fulfilled by introducing jumps in the SDEs for λ_A and λ_B. In particular, we try to introduce a common jump in the dynamics for λ_A and λ_B, so that jumps happen at the same time in both processes.

Moreover, in these simulations, we decide to increase volatility level and decrease speed of mean reversion for both intensity processes.

Here below we provide some examples of paths and new results.

The following graph represents the dynamics of the CIR process for λ_A, where parameters are set as $\theta_A = 0.03$, $\eta_A = 0.03$, $\sigma_{\lambda_A} = 0.1$ and $\lambda_A(t = 0) = 0.03$. This implies low speed of mean reversion and relatively high volatility, with initial value equal to mean value. Parameters for jumps are set as jump arrival rate $\alpha_1 = 0.003$ and expected jump size $\gamma_1 = 0.02$.

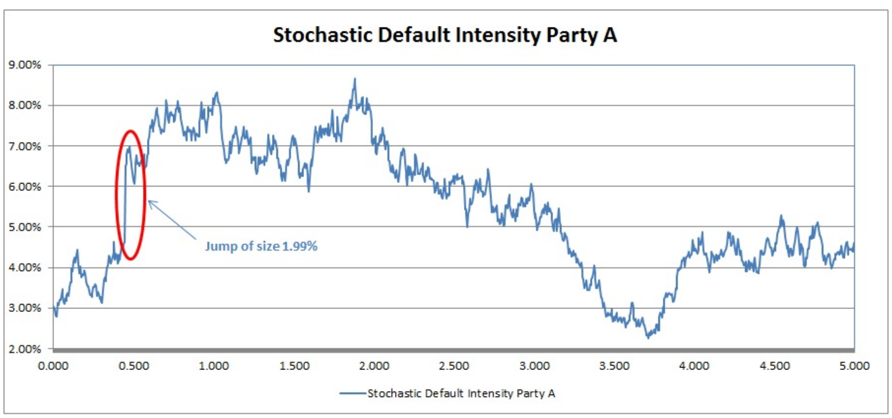

Fig. 6.12 Stochastic Intensity Jumps

Here below (see Fig. 6.13) we can see the same CIR process for λ_A, compared with the CIR process for λ_B, with $\theta_B = 0.03$, $\eta_B = 0.05$, $\sigma_{\lambda_B} = 0.1$ and $\lambda_B(t = 0) = 0.05$.

The speed of mean reversion of both paths is the same, low, as well as the volatility level, high. We show that when correlation is set at $\rho = 1$, stochastic intensities follow the same path, adjstusted for different parameter values. Moreover, we adopt here common jumps in both paths. Parameters for jumps are set as jump arrival rate $\alpha_1 = 0.005$ and expected jump size $\gamma_1 = 0.02$.

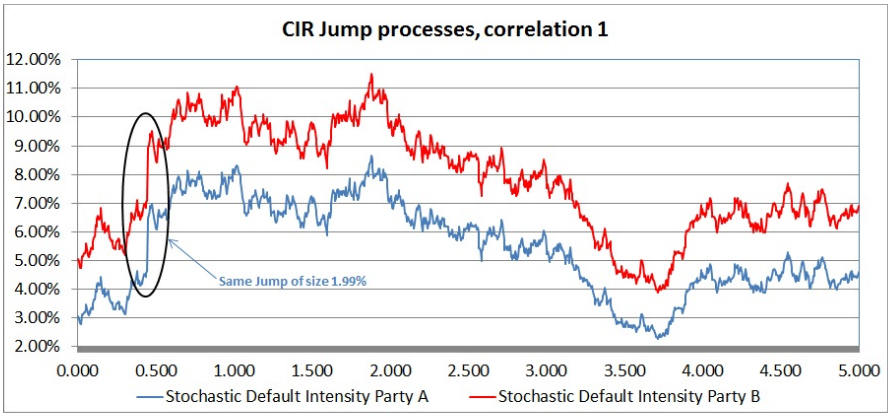

Fig. 6.13 CIR Jump Processes with Correlation 1

In Fig. 6.14 we present instead graphical results for correlation close to zero, with $\rho = 0.1$. Parameters for jumps are set as jump arrival rate $\alpha_1 = 0.005$ and expected jump size $\gamma_1 = 0.02$.

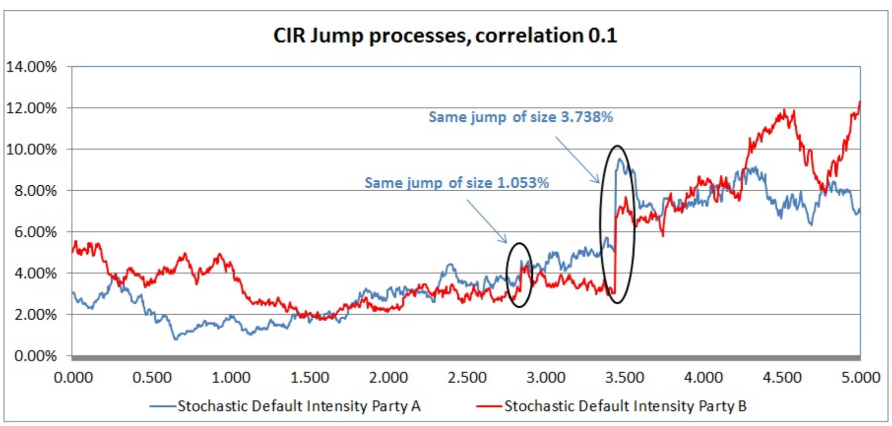

Fig. 6.14 CIR Jump Processes with Correlation 0.1

We see that, diminuishing the value of ρ to $\rho = 0.1$, paths tend to be non-correlated. Nonetheless, both paths are subject to the same jumps, arriving at the same time and of the same magnitude.

At last, in Fig. 6.15 we show the impact of $\rho = -1$. Dynamics of the two paths are almost exactly opposite to one another, though subject to the same jumps. Here again jump parameters are set as jump arrival rate $\alpha_1 = 0.005$ and expected jump size $\gamma_1 = 0.02$.

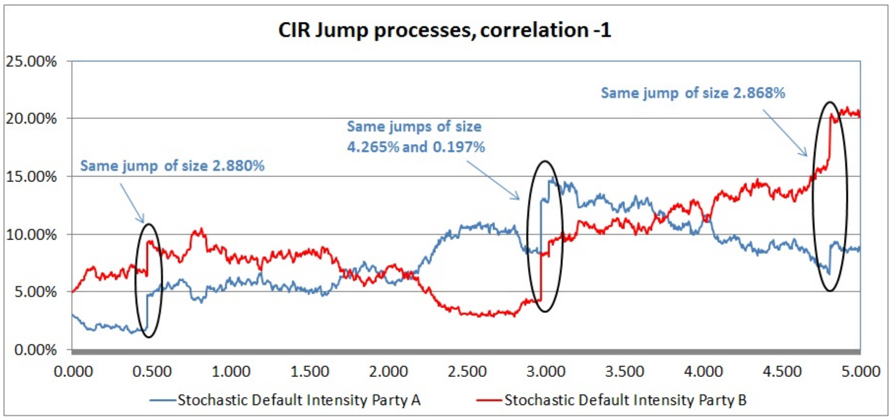

Fig. 6.15 CIR Jump Processes with Correlation -1

6.3.4 Common and independent jumps compared

Here below we present another set of results from numerical tests, where we compare results for CVA, DVA and BCVA in case of common or independent jumps, and in case of equal or different counterparties' names.

Case of different counterparties' names

In this case we assume that one party has a higher credit quality with respect to the other party, and we represent this feature through different CIR parameters. Specifically we have $\theta_A = 0.03$, $\eta_A = 0.03$, $\sigma_{\lambda_A} = 0.2$ and $\lambda_A(t=0) = 0.03$, and $\theta_B = 0.03$, $\eta_B = 0.05$, $\sigma_{\lambda_B} = 0.2$ and $\lambda_B(t=0) = 0.05$, so that party A has a higher credit quality compared to party B. BCVA is calculated from the point of view of party A.

For the case of different names, we then compare the case of common and independent jumps.

In Fig. 6.16 we first present numerical results for the case of independent jumps between party A and party B. Jumps are therefore simulated independently at each run for party A and party B, but with equal parameters, i.e. arrival rate $\alpha_1 = \alpha_2 = 0.005$ and expected jump size $\gamma_1 = \gamma_2 = 0.02$.

Correlation	CVA	DVA	BCVA
0	0.292%	-0.172%	0.119%
1	0.278%	-0.160%	0.118%

Fig. 6.16 Independent Jumps Different Names

In Fig. 6.17 instead we then present numerical results for the case of common jumps, meaning that jumps are simulated only once for both parties, leaving all parameters unchanged.

So we still have $\theta_A = 0.03$, $\eta_A = 0.03$, $\sigma_{\lambda_A} = 0.2$ and $\lambda_A(t = 0) = 0.03$, and $\theta_B = 0.03$, $\eta_B = 0.05$, $\sigma_{\lambda_B} = 0.2$ and $\lambda_B(t = 0) = 0.05$, with arrival rate $\alpha = 0.005$ and expected jump size $\gamma = 0.02$.

Correlation	CVA	DVA	BCVA
0	0.287%	-0.171%	0.117%
1	0.267%	-0.156%	0.111%

Fig. 6.17 Common Jumps Different Names

In this compared analysis we prefer to exclude the case of negative correlation, as we do not regard that eventuality as too meaningful from a financial point of view.

The case of maximum independence between default risks of the two parties can be identified in the case of independent jumps and zero correlation, where CVA $(\rho = 0) = 0.292\%$.

The case of maximum dependence, instead, can be retraced in the case of common jumps and correlation equal to 1, where CVA $(\rho = 1) = 0.267\%$.

As a matter of facts, we see that as we increase the links between default risks of the parties, either through correlation in the path dynamics and/or through the introduction of common jumps, the magnitude of CVA tends to reduce.

Case of equal counterparties' names

In this case we assume that both parties have the same credit quality, and we still represent this feature through equal CIR parameters. Specifically we have $\theta_A = 0.03$, $\eta_A = 0.03$, $\sigma_{\lambda_A} = 0.2$ and $\lambda_A(t=0) = 0.03$, and $\theta_B = 0.03$, $\eta_B = 0.03$, $\sigma_{\lambda_B} = 0.2$ and $\lambda_B(t=0) = 0.03$. BCVA is still calculated from the point of view of party A.

Also for the case of equal counterparties' names, we compare the case of common and independent jumps.

In Fig. 6.18 we first present numerical results for the case of independent jumps between party A and party B, simulated independently at each run for party A and party B, but with equal parameters, i.e. arrival rate $\alpha_1 = \alpha_2 = 0.005$ and expected jump size $\gamma_1 = \gamma_2 = 0.02$.

Correlation	CVA	DVA	BCVA
0	0.239%	-0.182%	0.057%
1	0.221%	-0.174%	0.047%

Fig. 6.18 Independent Jumps Same Names

In Fig. 6.19 we then present numerical results for the case of common jumps, meaning that jumps are simulated only once for both parties, leaving all parameters unchanged.

So we still have $\theta_A = 0.03$, $\eta_A = 0.03$, $\sigma_{\lambda_A} = 0.2$ and $\lambda_A(t=0) = 0.03$, and $\theta_B = 0.03$, $\eta_B = 0.03$, $\sigma_{\lambda_B} = 0.2$ and $\lambda_B(t=0) = 0.03$, with arrival rate $\alpha = 0.005$ and expected jump size $\gamma = 0.02$.

The case of maximum independence between default risks of the two parties can be identified in the case of independent jumps and zero correlation, where CVA $(\rho = 0) = 0.239\%$.

The case of maximum dependence, instead, can be recognized in the case of common jumps and correlation equal to 1, where CVA $(\rho = 1) = 0.229\%$. For this case of equal names, we still see that CVA charges have to be reduced in case of maximum dependence between the two parties' default risks. Nonetheless, we notice that the difference in CVA values between the case of maximum dependence and maximum independence is less sensible, with respect to the case of different names.

Correlation	CVA	DVA	BCVA
0	0.240%	-0.180%	0.060%
1	0.229%	-0.174%	0.055%

Fig. 6.19 Common Jumps Same Names

Chapter 7
Conclusions

The main results of the current work are synthesized in the following paragraphs.

Bilateral Credit Value Adjustment

The impact of Counterparty Credit Risk is not negligible and it must be accounted for when pricing financial derivatives, through the introduction of Adjusted CVA for Counterparty Credit Risk, and through Adjusted DVA for own risk of default, leading to the so called BCVA for bilateral contracts.

Numerical evidence of stochastic intensity pricing shows that Adjusted CVA varies as following:

- Adjusted CVA increases as we set higher levels of long-term average for the counterparty intensity process.
- Adjusted CVA that party A would charge to party B decreases as we set higher levels of mean-reversion of the intensity process for party A. In fact, with increasing long-run average values for party A's default intensity, one should account for a lower survival probability of party A in the calculation of Adjusted CVA. As a matter of facts, as the default intensity of a counterparty is increasing, this counterparty should consider the relative importance of its positive exposure towards the other counterparty decreasing, and concentrate on the eventuality of falling first into financial distress.
- Adjusted CVA increases as jump defaults are present in the intensity process for the counterparty.

When both parties may be subject to default, accountancy of (Adjusted) DVA should be pursued as well, reducing even significantly the impact of (Adjusted) CVA. The resulting BCVA level of credit charges will be positive or negative depending upon the relative levels of default intensities between the two parties.

Funding Costs in the contest of a credit risky funding

Funding Costs must be introduced when the cost of financing liquidity disbursements is significant. We assume that a party can finance itself at its funding spread to be added to the floating rate, entering into a funding transaction with maturity equivalent to that of the underlying contract, that is generating the net negative cash flows to be funded.

The funding spread of a party can be set to be equal to the credit spread, in case increased of a liquidity basis which can be explained as a friction between the primary and the secondary market for bonds.

If a party had the liquidity to face the negative future cashflows, one should consider the opportunity cost of not investing that liquidity in the market rather than facing directly the cash outflows resulting from the derivative position. If we assume that the party could go at least in the market and buy back its own bonds with the excess liquidity, we may again come to the conclusion that the cost of negative cashflows is equal to its cost of funding, i.e. spreads for bond issues, because this is what a party is giving up when not buying back its own bonds. As the buyback of own bonds in the secondary market would be done at a liquidity premium with respect to the funding spread used in the primary market, we can say that Funding Costs to fund negative positions must be at least equal to the opportunity cost of not extinguishing outstanding debt, i.e. the sum of the credit spread - where the credit spread is assumed to be the cost for issuances in the primary market - plus a liquidity premium specific of the secondary market.

When pricing Funding Costs, own risk of default has to be considered in order to reduce the overall amount of Funding Costs by a portion equivalent to the amount that would not be recovered by the counterparty in the funding deal in case of default.

In the approach of the current work, the computation of Funding Costs is fulfilled through the modelling of a "Funding Cost Adjustment" (FCA) for the risk-free value of the derivative. Moreover an "Adjustment for Default Risk" within "Funding Cost Adjustment" is introduced, in order to properly account for the necessity of a "risky" funding.

The calculation of "Funding Cost Adjustment" and of its "Adjustment for Default Risk" is based on the concept of Negative Expected Cash Flows, meaning the sequence of net cash flows that would translate into liquidity disbursements during the life of the transaction. In the setup of this work, we assume to fund these net Negative Expected Cash Flows from the moment they are due until maturity of the underlying transaction, at the relevant funding rate applicable to the debtor of the negative cashflows, to be added to the floating rate.

In the numerical tests that we performed, the funding spread was fixed as the relevant CDS spread at the start date of the contract, with a tenor equivalent to the maturity of the underlying transaction.

In order to calculate the "Adjustment for Default Risk", where the risk of default is accounted for, we assumed a default intensity calculated through the current level of CDS spread in case of constant intensity, or following the dynamics of the CIR SDE in case of stochastic intensity. This is financially consistent with the fact that a counterparty would enter into the funding transaction at the start date of the underlying contract, but the adjustment in Funding Costs for its own default risk would depend upon the dynamics of its survival probability over the life of the transaction.

Correlation in default events through common jumps

Under the hypothesis that links between default risks of the two parties are visible in the market, we investigate to what extent this dependency can impact Counterparty Credit Risk pricing. Dependency between default risks of two parties can be introduced setting a copula between the two default triggers, but this has some unpleasant implications. First, such dependency is not observable before a default event, since it has no effect on pre-default spread movements, so that the value for dependency parameters is undetermined. Second, the use of a copula like the gaussian one creates a predictable ordering of defaults that is unrealistic, see Morini (2011) [45] and Brigo and Chourdakis (2008) [12]. An alternative for creating dependency between defaults is assuming correlation between default intensities, and / or assuming common jumps in the path for default intensities. In such a case the default correlation is observable and we do not have predictable ordering of defaults.

We followed the latter path and verified in our numerical tests that, when pricing Counterparty Credit Risk, correlation between default intensities can have a limited impact, while a stronger impact is obtained when correlation is coupled with high levels of volatility and when we introduce common jumps in the paths for default intensities, as reported in our numerical tests.

This confirms numerically our analysis of the linkage between the default intensity representing the creditworthiness of a counterparty and the default event, which is not indeed so strong because the default trigger is not directly observable in the market, but it is rather unpredictable. This phenomenon leads to a situation where correlation between the credit quality of two counterparties, which is observable, may not indeed translate in correlated default events. This effect of weak correlation between default events of two counterparties when modeling diffusive intensities, is reflected in the fact that CVA charges do not vary upon the imposition of high or low correlation, until we introduce common jumps, strenghtening the dependency effect between default riskiness.

In order to amplify the impact of default risks dependency, in fact, we introduced common jumps in the process for default intensities. The situation of highest dependency in default risks can be identified when correlation is maximum and common jumps are introduced in the paths for default intensities. In the test we considered, we verified that in this case a minimum charge for Adjusted CVA has to be accounted for.

On the contrary, the situation of maximum independency between default risks verifies in case of null correlation and independent jumps in the paths for default intensities. As expected, in this case Adjusted CVA charges are maximum. This pattern for Counterparty Credit Risk charges is more visible for the case of two parties with different creditworthiness, with respect to the case of two equal names.

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