UNIVERSITÀ DEGLI STUDI DI MILANO

Scuola di Dottorato in Fisica, Astrofisica e Fisica Applicata
Dipartimento di Fisica

Corso di Dottorato in Fisica, Astrofisica e Fisica Applicata
CICLO XXIV

Perturbative Approach to Integrability in Three-Dimensional Chern-Simons Theories

Settore Scientifico Disciplinare FIS/02

Supervisore: Dott. Alberto SANTAMBROGIO
Coordinatore: Prof. Marco BERSANELLI

Tesi di Dottorato di:
Stefano SCOLERI

Anno Accademico 2011-2012

A mia nonna Maria

Acknowledgements

This thesis work wouldn't have been possible without the help of many people. First of all, I would like to thank my supervisor Alberto Santambrogio, for his guidance and for having introduced me to such interesting areas of Physics.

The results presented here were also obtained with the fruitful collaboration of Andrea Mauri, whom I thank for his useful explanations and discussions.

I am also particularly grateful to Francesco Fiamberti and Christoph Sieg, for scientific discussions, and to Samuele Chimento, for fundamental informatic support.

Then, I would like to thank all the friends and colleagues who shared with me these exciting years. They include Fabrizio, Andrea, Chiara, Francesco, Emanuele, Giovanni, Matias, Fulvio, Stefano, Marta, Samuele and Laura.

Last, but not least, many heartfelt thanks to my family for their affection and continuous support.

Abstract

One of the most fascinating discoveries of contemporary Theoretical Physics is the AdS/CFT correspondence relating gauge theories to gravity theories. Soon after its formulation, tremendous developments allowed to obtain a deep comprehension of the four-dimensional $\mathcal{N} = 4$ SYM theory and, in particular, led to the discovery of integrable structures both in the gauge theory itself and in its string counterpart. In the last few years, much attention was devoted to the study of supersymmetric Chern-Simons-matter theories in three dimensions. In this class of theories a distinguished role is played by the $\mathcal{N} = 6$ ABJM model which is a $U(N)_k \times U(N)_{-k}$ superconformal gauge theory with Chern-Simons level k. Indeed, in the large N limit, the ABJM theory has been conjectured to be the AdS/CFT dual description of M-theory on an $AdS_4 \times S^7/\mathbb{Z}_k$ background and, for $k \ll N \ll k^5$, of a type IIA string theory on $AdS_4 \times \mathbb{CP}^3$. For this reason, soon after its discovery the ABJM model has quickly become the ideal three-dimensional playground to study AdS/CFT as much as $\mathcal{N} = 4$ SYM has been in the four-dimensional case. Quite surprisingly, the ABJM model seems to share a number of notable properties with $\mathcal{N} = 4$ SYM theory even though the two theories are *a priori* different in nature. One of the common features is provided by the fact that also in planar ABJM theory integrable structures naturally show up. In particular, a Bethe Ansatz approach to the computation of anomalous dimensions is possible and a set of all-loop Bethe equations was formulated. These equations are very similar to those of SYM theory, however, starting at eight loops, important new features are believed to appear, in particular in connection with the appearance of the dressing phase. With the aim to check such a picture, we analyze for the first time the form of the eight-loop dilatation operator in the $SU(2) \times SU(2)$ sector and, adopting a perturbative approach based on superspace techniques, we directly extract the value of the unknown leading order coefficient of the dressing phase from supergraphs involving maximal interactions.

Contents

Chapter 1.

Introduction

Since its first appearance, in the late sixties of the past century, String Theory turned out to be a useful tool in the analysis of strongly coupled systems: in those years, people were trying to understand the strong force and String Theory was formulated as a theory of hadrons. Later, QCD was discovered and String Theory was abandoned as a theory of strong interactions, becoming, on the other hand, the most promising candidate for a quantum theory of the gravitational interaction. Unlike other proposed theories of quantum gravity, it also offered a unified description of all fundamental interactions. To reach the goal of quantizing gravity, String Theory gives up the concept of point-like particles as the elementary constituents of nature and introduces extended objects, such as strings and branes. Moreover, it needs supersymmetry and requires the existence of extra spatial dimensions to the four known, in order to avoid inconsistencies such as the presence of tachyons and ghosts in the spectrum.

At first sight, if we believe in String Theory, the gravitational interaction seems to be described by a completely different theory, especially if we think that all the other fundamental interactions are consistently described in terms of *gauge theories*. Nevertheless, after the discovery of D-branes and, subsequently, of the AdS/CFT correspondence, in the nineties, it became clear that String theory and gauge theories are deeply related with each other.

The original formulation [1] of the AdS/CFT correspondence given by Maldacena states the equivalence of type IIB superstring theory on $AdS_5 \times S^5$ background and $\mathcal{N} = 4$ Super Yang-Mills (SYM) theory, which is a four-dimensional superconformal gauge theory living on the boundary of AdS_5 space. It is remarkable the fact that two

theories living in space-times of different dimensions are related: this is a consequence of holography, which has always been a guiding principle when dealing with gravity. All dynamics in AdS_5 can be reformulated as a boundary effect and it is captured by a four-dimensional local field theory. A very important point is that the AdS/CFT correspondence relates opposite regimes of the two theories: it is indeed a weak-strong duality, in the sense that the strong-coupling regime of one of the two theories is related to the weak-coupling regime of the other. In particular, the strong-coupling regime of the gauge theory corresponds to the supergravity limit of the string theory. So, we can see that String Theory is still useful to study the strong-coupling behaviour of a field theory. On the other hand, weakly coupled gauge theory corresponds to the full interacting string theory.

The most interesting fact about AdS/CFT correspondence is, perhaps, that it allows to investigate the non-perturbative regime of a theory by means of perturbative computations made on a different theory. Unfortunately, the strong/weak nature of the duality has also some disadvantages: in general, explicit computations are manageable only in the perturbative regime, *i.e.* at weak coupling. This makes the AdS/CFT correspondence difficult to prove and it remains substantially a conjecture, even though several strong tests have supported its validity. First of all, the symmetries of the two theories match. Moreover, the spectra of the two theories should coincide. A conformal field theory doesn't have asymptotic states or an S-matrix, so the natural objects to consider are operators. In particular, anomalous dimensions of composite operators on the field theory side are claimed to be equal to energies of string states. Right after the formulation of the correspondence, the connections between supergravity modes and chiral primary operators in the gauge theory had been investigated. Furthermore, a certain class of excited string states on the pp wave background, which is the Penrose limit of $AdS_5 \times S^5$, was shown to be dual to a class of non-BPS operators with large R-charge, the so called BMN operators. Also, some correlation functions are protected from quantum corrections and can be easily compared on both sides of the correspondence, being independent on the coupling constant. Further, stronger, tests of the correspondence require a deeper knowledge of some aspects of the gauge theory and of string theory, concerning quantities which depend on the coupling and their non-perturbative behavior.

This is a very hard task and seemed out of reach at first sight, but such a belief changed when, after the work of Minahan and Zarembo [7], there appeared hints sug-

gesting that both the involved theories may be integrable in the planar limit. This is a very important discovery, since integrability provides powerful tools, such as the Bethe ansatz, for the determination of large parts of the spectra of the two theories. In the most optimistic point of view, in the future, integrability can allow to compute the full spectra of the theories, at any value of the coupling constant: in particular, it can be possible to take into account also finite-size effects, which are not included in the standard asymptotic Bethe ansatz, such as wrapping interactions. Techniques based on the Thermodynamic Behte Ansatz [8, 9] go in this direction. In particular, a Y-system was introduced which could reproduce wrapping corrections by solving a set of functional equations which capture the integrability properties of the discrete classical Hirota dynamics.

The fundamental quantity in an integrable model is the scattering matrix of excitations. This isn't the S-matrix of asymptotic particles, which cannot be defined in a conformal theory as we said, but rather an asymptotic quantity which may be defined, in the planar limit, in an internal space. It is constrained by the symmetries of the theory and it appears to be two-particle factorized, because of integrability. It should be now clear why integrability in gauge and string theories plays a prominent role in testing the AdS/CFT correspondence. The idea behind integrability is that composite operators of the gauge theory are mapped to states of a closed spin chain. Operators whose dimensions are not protected by supersymmetry from receiving quantum corrections can be viewed as magnon states propagating on the spin chain, while the dilatation operator of the gauge theory corresponds to the hamiltonian of the integrable model. Minahan and Zarembo showed that the one-loop dilatation operator of $\mathcal{N} = 4$ SYM theory in the $SU(2)$ sector is equivalent to the hamiltonian of the Heisenberg model of ferromagnetism, which is an integrable model. Further developments suggested that integrability survives to all orders and to all sectors and an all loop Behte ansatz was formulated. Similarly, on the String Theory side, classical integrability of the world-sheet sigma model was proved and arguments were proposed, suggesting that integrability persists in the quantum theory. Intriguingly, the spectrum of this quantum string theory can be described by Bethe equations as well, whose form is surprisingly similar to that of gauge theory. However, the spectra of gauge and string theory match only if the S-matrices of the two theories are related by a global dressing factor, which is constrained by symmetry to be reduced to a pure phase.

The original formulation of the AdS/CFT correspondence was later extended to other theories with less symmetries[1] and to theories living in a different number of spacetime dimensions. In the present work we will focus on a particular version of the correspondence introduced in 2008 by Aharony, Bergman, Jafferis and Maldacena [10], where a three-dimensional $\mathcal{N} = 6$ superconformal Chern-Simons theory (in the following called ABJM theory) was conjectured to be equivalent to type IIA string theory on $AdS_4 \times \mathbb{CP}^3$. Also within this AdS_4/CFT_3 correspondence, one can define a coupling parameter λ which interpolates between gauge theory regime at $\lambda \ll 1$ and string theory regime at $\lambda \gg 1$.

Quite surprisingly, the ABJM model seems to share a number of notable properties with $\mathcal{N} = 4$ SYM theory even though the two theories are *a priori* different in nature. One of the common features is provided by the fact that also in the ABJM case, in the planar limit, integrable structures naturally show up. At first it was found in [12] that, at the two-loop order and in the $SU(4)$ flavour sector, the anomalous dimensions of composite operators could be mapped to the energy spectrum of an integrable Hamiltonian acting on an alternating fundamental-antifundamental spin-chain.

The two-loop analysis was then extended to the full theory in [14, 15] by the introduction of an $OSp(2, 2|6)$ chain. A generalization to the parity breaking ABJ model [16] was also studied at two loops in [15, 17], where it was found to be integrable at the given order.

Soon afterwards, paralleling the progresses done in the four-dimensional case, a set of an all-loop Bethe equations for the asymptotic spectrum of the full ABJM theory was proposed by Gromov and Vieira [18] . The Bethe equations nicely interpolated between the weak coupling results and the coset string construction at strong coupling [19, 20], together with the algebraic curve approach developed in [21]. These all-loop Bethe equations are very similar to those of SYM theory and a dressing factor of the same form is present.

Anyway, beside the many analogies between the two versions of the correspondence, also some differences can be observed. For instance, ABJM theory isn't maximally supersymmetric, as is the case of $\mathcal{N} = 4$ SYM: among the consequences of this fact, we

[1] Remarkable progress has been done, for example, towards an holographic description of non supersymmetric theories such as QCD and condensed matter systems.

mention that a non trivial function $h(\lambda)$ appears in the all-loop Bethe Ansatz. It interpolates between weak coupling behaviour $h(\lambda) \simeq \lambda$ and and strong coupling behaviour $h(\lambda) \simeq \sqrt{\lambda/2}$, but its functional form, as well as its ultimate meaning, is not completely known. This function enters the anomalous dimensions of composite operators and the dilatation operator through its coefficients in the weak-coupling expansion. Such coefficients cannot be fixed with the help of integrability alone.

One of the salient features of the all-loop Bethe equations is that they are strongly constrained by the symmetries of the theory. In fact only a pair of undetermined functions of the coupling λ are left open in the description of the spectrum. One is the aforementioned interpolation function $h(\lambda)$ and the other is the dressing function $\theta(\lambda)$, which, as we have seen, had to be introduced also in the $\mathcal{N} = 4$ SYM case. In the ABJM case, the dressing phase plays a fundamental role since its presence has also been conjectured to give rise to the coupling between even- and odd-site excitations on the spin-chain at high-loop orders. In particular, the coefficients of the dressing phase are conjectured to affect anomalous dimensions starting at eight loops: nevertheless, even though their values at weak coupling can be guessed on the ground of general considerations and a conjecture on the form of the phase factor at finite coupling can be made, a rigorous proof of its validity is still missing. It is, therefore, important to test such conjectures against direct field theory computations.

The description of the asymptotic spectrum depicted in [18] has been subsequently checked at the perturbative level beyond two-loops by direct Feynman diagrammatic computations in [22–25] . The four-loop dilatation operator has been fully computed for both the ABJM and ABJ models by using the component formulation in [22, 23] and the $\mathcal{N} = 2$ superspace formalism in [24]. As a result, the form of dilatation operator was found to be compatible with the spectrum predicted by the Bethe equations and moreover it was possible to fix the next-to-leading order coefficient in the weak coupling expansion of the function $h(\lambda)$.

In [26] the analysis was pushed up to six loops. At this order, a full Feynman diagrammatic analysis looks very complicated, even using superspace techniques. Nevertheless, in [26] the expression of the dilatation operator could be derived by computing a suitable set of Feynman diagrams and assuming the form of the Bethe equations of [18]. Moreover, the results of [26] imply that the dressing phase doesn't appear at six-loop order. This latter fact isn't so trivial and has the consequence that the order λ^4 coefficient of

the dressing factor which can be a priori present at six loops, actually vanishes, therefore confirming that the first non-trivial effects of the presence of the dressing phase are to be found at eight-loop order. Further perturbative checks on the spectrum were exploited in [27, 28].

Meanwhile the internal S-matrix approach to integrability has also been introduced in [29–32] and [33]). The all loop S-matrix has been found to be compatible with the all-loop Bethe equations of [18]. Moreover, the Y-system equations were extended to the ABJM case.

In this thesis we present the results of the perturbative analysis of the eight-loop dilatation operator of ABJM theory, which led to the computation of the leading order coefficient of the dressing phase [34]. The work is organized as follows: in Chapter 2 we introduce ABJM theory in the context of AdS_4/CFT_3 correspondence; in Chapter 3 we review the basic results of integrability and we formulate the Bethe ansatz for ABJM theory; in Chapter 4 we consider the dilatation operator of the theory, which is the basic tool for the computation of anomalous dimensions of composite operators. The Bethe ansatz provides useful techniques to construct the asymptotic dilatation operator in the $SU(2) \times SU(2)$ sector: such techniques are applied up to eight loops; Chapter 5 contains the perturbative computations which allow the extraction of the leading order coefficient of the dressing phase from the eight loop dilatation operator. In particular, this is achieved investigating the Feynman diagrams which contribute to maximal interactions in the dilatation operator. To this end, $\mathcal{N} = 2$ superspace techniques are used. Finally, Chapter 6 contains our conclusions. We have added some appendices which collect the more technical aspects of the subjects: Appendix A contains the superspace formulation of ABJM theory; Appendix B gives a general introduction to some techniques which can be adopted to deal with multiloop Feynman integrals: in particular, GPXT and Mellin-Barnes techniques are presented since they have been used in the computation of the integrals needed in Chapter 5; Appendix C collects some `Mathematica` routines which were written for the computations of Chapter 4.

Chapter 2.

The AdS_4/CFT_3 Correspondence

Superconformal Chern-Simons theories in $D = 3$ spacetime dimensions naturally arise in the study of AdS_4/CFT_3 correspondence. We give here a general introduction to this correspondence, in order to contextualize the gauge theory we'll focus on in the subsequent chapters, which is the so called ABJM theory: it is an $\mathcal{N} = 6$ superconformal Chern-Simons-Matter theory with gauge group $U(N) \times U(N)$. We outline here the main properties of such theory. Some technical details are collected in Appendix A.

2.1. General Motivation

According to the AdS/CFT correspondence[1] [1–3] some conformal gauge theories are related to a string theory on a curved background including anti-de-Sitter (AdS) space. The first, and most well understood, example of such a correspondence related the maximally supersymmetric $\mathcal{N} = 4$ SYM theory, with gauge group $SU(N)$, to type IIB superstring theory on $AdS_5 \times S^5$. The four-dimensional gauge theory lives on the boundary of AdS_5 space.

Gauge theories in three dimensions were less studied until the last decade. It is interesting to investigate whether an AdS_4/CFT_3 version of the correspondence can be formulated, relating three-dimensional theories to gravity theories. From this perspective, a three-dimensional gauge theory should live on the boundary of AdS_4 space. M-theory admits compactifications involving AdS_4 space: the most symmetrical one

[1] See also [4–6] for useful reviews.

is $AdS_4 \times S^7$. M-theory contains M2-branes and M5-branes: the worldvolume of M2-branes is three-dimensional and its low energy dynamics should be a conformal field theory: such a theory could be considered as a candidate to be the gauge theory dual to M-theory. Many efforts were made in order to find the right gauge theory describing the worldvolume of N coincident M2-branes: first, it was thought [36] that it could be the IR fixed-point of $D = 3, \mathcal{N} = 8$ $U(N)$ SYM theory. This is a maximally supersymmetric theory but it has the disadvantage of being strongly coupled, making it very hard to find an explicit lagrangian description.

In [37], for the first time, supersymmetric Chern-Simons theories, which are classically conformal invariant, were analyzed for the same purpose: these are topological field theories, but can be coupled to matter fields carrying physical degrees of freedom. Moreover, since the Chern-Simons level is not renormalized up to a possible one-loop shift, these theories are exactly conformal at the quantum level. Chern-Simons theories with $\mathcal{N} = 1$, $\mathcal{N} = 2$ and $\mathcal{N} = 3$ supersymmetries were constructed in [37, 38]. However, some difficulties were encountered to find an $\mathcal{N} = 8$ $U(N)$ gauge theory of this type. In [37] it was also argued that a lagrangian description cannot be found for such a theory. Subsequently, Bagger and Lambert proposed a three-dimensional field theory as a worldvolume description of multiple M2-branes in M-theory, which has $\mathcal{N} = 8$ supersymmetry with manifest $SO(8)$ R-symmetry, should be superconformal and has an explicit lagrangian description [39–41]: such lagrangian was constructed on the basis of a new algebraic structure called "three-algebra" (see also [42]). In [43] it was shown that, for a specific three-algebra, the Bagger-Lambert theory can be rewritten as an ordinary gauge theory with gauge group $SU(2) \times SU(2)$ and matter in the bifundamental.

Finally, in 2008, Aharony, Bergman, Jafferis and Maldacena succeeded in finding a three-dimensional $U(N) \times U(N)$ Chern-Simons-Matter theory [10] describing the low energy limit of N M2-branes: we will call this gauge theory ABJM theory. The price to pay is to give up maximal supersymmetry: in fact their theory is only $\mathcal{N} = 6$ supersymmetric. The coupling constant of this theory is $1/k$, where k is the Chern-Simons level, so that the theory becomes weakly coupled at large k. ABJM theory can be obtained as the IR limit of a particular brane construction in type IIB string theory which has $\mathcal{N} = 3$ supersymmetry. The gauge theory realizing such brane construction, which can be chosen to have $U(N) \times U(N)$ gauge group with Chern-Simons terms at opposite levels and matter fields in the bifundamental, flows in the IR to the $\mathcal{N} = 6$

ABJM theory. Thanks to T-duality, this brane construction can be lifted to M-theory, where it corresponds to M2-branes probing a $\mathbb{C}^4/\mathbb{Z}_k$ singularity. It is important to stress that, performing a $\mathbb{Z}_k$ orbifold, it was possible to find an explicit lagrangian description for the conformal field theory of N M2-branes. Another interesting feature of ABJM theory is that it has, at large N, a gravity dual description in terms of M-theory on the $AdS_4 \times S^7/\mathbb{Z}_k$ background with N units of four-form flux through AdS_4. Moreover, it has a 't Hooft limit if we send N and k to infinity keeping $\lambda = N/k$ fixed: in this case, the gravity dual becomes type IIA string theory in the $AdS_4 \times \mathbb{CP}^3$ background with N units of four-form flux through AdS_4 and k units of two-form flux through a $\mathbb{CP}^1 \subset \mathbb{CP}^3$. We will call λ the "'t Hooft coupling". When $N = 2$, *i.e.* the gauge group is $SU(2) \times SU(2)$, extra symmetries appear and supersymmetry is enlarged to $\mathcal{N} = 8$: in this case ABJM theory becomes equivalent to Bagger-Lambert theory. The parameters of gauge and string theories are related by the following identifications:

$$g_s \sim \frac{\lambda^{5/4}}{N}\,, \qquad R^2 = 4\pi\alpha'\sqrt{2\lambda}\,, \tag{2.1}$$

where g_s is the string coupling constant, α' the effective tension and R the $\mathbb{CP}^3$ radius, which is also twice the AdS_4 radius. We thus see that $N \gg k^5$ implies large values of g_s: this means that string theory is strongly coupled and, in fact, becomes M-theory in this case. When $k^5 \gg N$, on the contrary, g_s is small and strings do not interact. In this case, at large 't Hooft coupling, the background is weakly curved and the supergravity approximation is valid: this regime is dual to strongly coupled gauge theory. At small 't Hooft coupling, the background is highly curved and strings are subject to large quantum fluctuations: this regime is dual to perturbative gauge theory and will be explored in this work.

2.2. ABJM Theory

ABJM theory [10] is a three-dimensional $\mathcal{N} = 6$ superconformal Chern-Simons theory with gauge group $U(N) \times U(N)$ coupled to matter. We give here the $\mathcal{N} = 2$ superspace formulation of such theory. This was first given in [44], but we use a slightly different notation as in [24,47], which is adapted from [87]. A superspace formulation of supersymmetric gauge theories is preferable with respect to the component approach for several

reasons: first of all, it is particularly suitable for perturbative computations, especially at high loop orders. The main reason is the following: all the ordinary component fields of a supermultiplet are combined into a single superfield, which is function of bosonic and fermionic superspace coordinates, and Feynman supergraphs can be drawn directly in superspace. A supergraph involves such superfields and usually encodes the information on a large number of standard diagrams involving ordinary fields. Moreover, since supersymmetry is manifest from the beginning, many simplifications and cancellations directly related to supersymmetry are automatically implemented with supergraphs. Finally, fermionic interactions don't appear explicitly in supergraphs, being hidden in the superfield formalism: this fact further simplifies the calculations. A brief introduction to the superspace formalism is given in Appendix A.

ABJM theory has two $\mathcal{N}=2$ vector supermultiplets, V and $\hat{V}$, with V transforming in the adjoint representation of the first $U(N)$ and $\hat{V}$ in the adjoint representation of the second $U(N)$. Supersymmetry is extended to $\mathcal{N}=6$ if we add two sets of chiral matter superfields Z^A and W_A, $A=1,2$, with an appropriate superpotential. Z^A and W_A transform in the bifundamental representations $(\mathbf{N},\bar{\mathbf{N}})$ and $(\bar{\mathbf{N}},\mathbf{N})$ of the $U(N)\times U(N)$ gauge group. Moreover, the theory has a manifest $SU(2)\times SU(2)$ flavour symmetry: the scalars transform in the $(\mathbf{2},\mathbf{1})$ and $(\mathbf{1},\mathbf{2})$ of this global flavour symmetry respectively. The R-symmetry group is enhanced to $SO(6)\simeq SU(4)$ due to contributions from Chern-Simons terms. This group combines with the three-dimensional conformal group $SO(2,3)$ and with the 24 fermionic supercharges Q_α and S_α to give the full superconformal group $OSp(2,2|6)$. After gauge-fixing, this symmetry group reduces to $SU(2|2)$. The eight elementary excitations transform in the $(\mathbf{2}|\mathbf{2})_Z\oplus(\mathbf{2}|\mathbf{2})_W$ of the residual symmetry group[2]. This fact should be compared with the AdS_5/CFT_4 case, where the full superconformal group $PSU(2,2|4)$ reduces to two copies of $SU(2|2)$ and the sixteen elementary excitations transform in its $(\mathbf{2}|\mathbf{2})_L\otimes(\mathbf{2}|\mathbf{2})_R$.

[2]The subscripts Z and W stem for the Z- and W-particles, which will be later introduced.

The gauge fixed ABJM action in $\mathcal{N}=2$ superspace reads

$$\begin{aligned} S =& \frac{k}{4\pi}\ \Big\{ \int \mathrm{d}^3x\, \mathrm{d}^4\theta \int_0^1 \mathrm{d}t\ \mathrm{Tr}\left(V\, \bar{\mathrm{D}}^\alpha e^{-tV} \mathrm{D}_\alpha e^{tV} - \hat{V}\, \bar{\mathrm{D}}^\alpha e^{-t\hat{V}} \mathrm{D}_\alpha e^{t\hat{V}} \right) \\ &+ \int \mathrm{d}^3x\, \mathrm{d}^4\theta\, \mathrm{Tr}\left(\bar{Z}_A e^V Z^A e^{-\hat{V}} + \bar{W}^B e^{\hat{V}} W_B e^{-V} \right) \\ &+ \frac{i}{2}\left[\int \mathrm{d}^3x\, \mathrm{d}^2\theta\, \epsilon_{AC}\epsilon^{BD}\, \mathrm{Tr} Z^A W_B Z^C W_D + \int \mathrm{d}^3x\, \mathrm{d}^2\bar{\theta}\, \epsilon^{AC}\epsilon_{BD}\, \mathrm{Tr} \bar{Z}_A \bar{W}^B \bar{Z}_C \bar{W}^D \right] \\ &+ \text{ gauge fixing and ghost terms} \Big\}. \end{aligned} \tag{2.2}$$

The first line contains the non abelian Chern-Simons action, the second line contains the kinetic term of the matter superfields and their coupling with gauge superfields, while the third line is the superpotential. The terms containing exponentials can be expanded and just few terms are necessary for our perturbative computations. The expansion of the Chern-Simons action gives the kinetic term of the gauge superfields and their self-interactions:

$$\int_0^1 \mathrm{d}t\, \mathrm{Tr} V \bar{\mathrm{D}}^\alpha e^{-tV} \mathrm{D}_\alpha e^{tV} = \frac{1}{2} \mathrm{Tr}\, V \bar{\mathrm{D}}^\alpha \mathrm{D}_\alpha V - \frac{1}{6} \mathrm{Tr} V \bar{\mathrm{D}}^\alpha [V, \mathrm{D}_\alpha V] + \cdots \tag{2.3}$$

and similar for $\hat{V}$. The gauge-matter action expands as

$$\mathrm{Tr}\, \bar{Z}_A e^V Z^A e^{-\hat{V}} = \mathrm{Tr}\, \bar{Z}_A \Big(Z^A + V Z^A - Z^A \hat{V} + \frac{1}{2}(V^2 Z^A + Z^A \hat{V}^2) - V Z^A \hat{V} \Big) + \dots \tag{2.4}$$

and similar for W_A. In terms of gauge group indices, the gauge superfields are written as:

$$V = V^a T^a\,, \qquad \hat{V} = \hat{V}^a T^a\,, \tag{2.5}$$

where $a = 0, 1, \dots, N^2-1$ and the matrices T^a are the generators of the $\mathfrak{u}(N)$ gauge algebra. They are $N \times N$ hermitian matrices such that $T^0 = \frac{1}{\sqrt{N}}$ is proportional to the identity and, for $a = 1, \dots, N^2-1$, T^a are traceless, being the generators of the $\mathfrak{su}(N)$

subalgebra. The matrices T^a are normalized as

$$\mathrm{Tr}(T^a T^b) = \delta^{ab} \,. \tag{2.6}$$

The matter superfields can be written as:

$$Z^A = (Z^A)^i_{\hat{i}} \,, \qquad \bar{Z}_A = (\bar{Z}_A)^{\hat{i}}_i \,, \qquad W_A = (W_A)^{\hat{i}}_i \,, \qquad \bar{W}^A = (\bar{W}^A)^i_{\hat{i}} \,, \tag{2.7}$$

with fundamental indices $i, \hat{i} = 1, \ldots, N$.

In the action (2.2) also the three-dimensional, $\mathcal{N} = 2$ superspace spinor covariant derivatives $\mathrm{D}_\alpha, \bar{\mathrm{D}}_\alpha$ appear: they satisfy the algebra

$$\{\mathrm{D}_\alpha, \mathrm{D}_\beta\} = \{\bar{\mathrm{D}}_\alpha, \bar{\mathrm{D}}_\beta\} = 0 \,, \quad \{\mathrm{D}_\alpha, \bar{\mathrm{D}}_\beta\} = p_{\alpha\beta} \,. \tag{2.8}$$

The metric ϵ_{AB} for the SU(2) flavour indices is given by

$$\epsilon_{12} = 1 \,, \qquad \epsilon^{12} = 1 \,, \qquad \epsilon^{AB}\epsilon_{CD} = \delta^A_C \delta^B_D - \delta^A_D \delta^B_C \,. \tag{2.9}$$

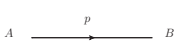
The Chern-Simons level, k, is fixed to be an integer by gauge invariance of the effective action. Its inverse, $1/k$, plays the role of the gauge coupling constant and perturbation theory is valid for large values of k. Moreover, ABJM theory admits a large N expansion, such that only planar supergraphs contribute. We can define the 't Hooft coupling constant

$$\lambda = \frac{N}{k} \,, \tag{2.10}$$

which can be kept fixed while taking N and k large. In fact, still remaining in the planar limit, we will be interested in the perturbative expansion in terms of the λ parameter. In the following, we will refer to λ as the coupling constant of the theory. This is justified by the fact that the AdS/CFT correspondence relates weak and strong coupling regimes of the gauge and string theories in terms of λ.

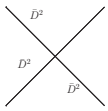
We now give the Euclidean Feynman rules of the theory. After the Wick-rotation $e^{-iS} \to e^S$ in the path integral, superspace Feynman rules are obtained in standard way. Here we give only the rules that will be needed for our computations in Chapter 5. We refer to Appendix A for a wider explanation. The chiral superfield propagators are given

by

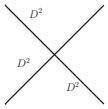


$$= \langle Z^B(p)\bar{Z}_A(-p)\rangle = \langle \bar{W}^B(p)W_A(-p)\rangle = \frac{4\pi}{k}\frac{\delta^B_A}{p^2}\delta^4(\theta_1-\theta_2)\,, \quad (2.11)$$

where θ_i are fermionic superspace coordinates and diagonality in the gauge indices has been suppressed. The vertices are obtained by taking the functional derivatives of the Wick rotated action w.r.t. the corresponding superfields. When a functional derivative w.r.t. the (anti)-chiral superfields is taken, factors of (D^2) $\bar{\mathrm{D}}^2$ are generated in the vertices. For the quartic superpotential vertices, we have:



$$= i\epsilon^{AC}\epsilon_{BD}\,\frac{k}{4\pi}\left(\delta^{\hat{i}}_{\hat{l}}\,\delta^{\hat{k}}_{\hat{j}}\,\delta^l_k\,\delta^j_i - \delta^{\hat{k}}_{\hat{l}}\,\delta^{\hat{i}}_{\hat{j}}\,\delta^l_i\,\delta^j_k\right) \quad (2.12)$$



$$= i\epsilon_{AC}\epsilon^{BD}\,\frac{k}{4\pi}\left(\delta^{\hat{l}}_{\hat{k}}\,\delta^{\hat{j}}_{\hat{i}}\,\delta^i_l\,\delta^k_j\; - \delta^{\hat{l}}_{\hat{i}}\,\delta^{\hat{j}}_{\hat{k}}\,\delta^k_l\,\delta^i_j\right), \quad (2.13)$$

Note that, in a standard way, one of the (D^2) $\bar{\mathrm{D}}^2$ factors has been absorbed into the (anti)chiral integration such that the integration measure of the (anti)chiral vertex is promoted to the full superspace measure. The surviving spinor covariant derivatives are written directly on the graphs. Indeed, manipulations of the D and $\bar{\mathrm{D}}$ can be made on the graphs and constitute the so called D-algebra procedure [87]: this essentially corresponds to repeated integrations by parts in θ-space which allow to reduce the full superspace integral to a standard momentum-space Feynman integral.

2.3. Anomalous Dimensions of Operators

As described in the Introduction, the AdS/CFT correspondence conjectures that the spectrum of energy states of string theory coincides with the spectrum of anomalous dimensions of the gauge theory operators. We thus introduce here the concept of gauge theory operators and of anomalous dimensions in ABJM theory.

Gauge invariant local operators are defined as traces of products of elementary fields or their covariant derivatives, all evaluated at the same spacetime point. In the planar

limit, it is sufficient to restrict to single-trace operators. In a conformal field theory, these composite operators can be thought of as the states of the theory. Given a set $\{O_a\}$ of operators, conformal symmetry constrains the form of their two-point correlation functions to be

$$\langle \mathcal{O}_a(x)\mathcal{O}_b(y)\rangle = \frac{\delta_{ab}}{|x-y|^{2\Delta_a}}\,, \tag{2.14}$$

where Δ_a is the scaling dimension of the operator $\mathcal{O}_a$. This quantity doesn't coincide with the classical scaling dimension $\Delta_a^{(0)}$, since quantum corrections in general appear: more precisely, a composite operator acquires an anomalous dimension γ as a consequence of renormalization, which has to be implemented in order to cancel divergences in correlation functions, so that $\Delta_a = \Delta_a^{(0)} + \gamma_a$. Some operators are protected by supersymmetry from receiving an anomalous dimensions: they are called *chiral primary* operators. In the other cases we have to renormalize them and, in this process, we usually face the problem of operator mixing. Let's take a basis of bare operators with the same quantum numbers, suche as the classical dimension and other charges. Transition to the basis where renormalization is multiplicative gives:

$$\mathcal{O}_a^{ren} = \mathcal{Z}_a{}^b(\Lambda)\mathcal{O}_b^{bare}\,, \tag{2.15}$$

where Λ is an UV cutoff and the matrix $\mathcal{Z}_a{}^b$ is the renormalization factor which subtracts the UV divergences from the correlation functions. In a perturbative approach, it can be expanded in powers of λ and gets contributions from all the Feynman diagrams with a single insertion of one of the composite operators $\mathcal{O}_a$. The renormalization factor $\mathcal{Z}$ can be related to the mixing matrix

$$\Gamma = \Lambda\frac{\mathrm{d}}{\mathrm{d}\Lambda}\,\ln\mathcal{Z}(\Lambda) \tag{2.16}$$

which, in a conformal field theory, coincides with the quantum part of the dilatation operator of, *i.e.* the generator of scaling transformations:

$$\mathcal{D} = \mathcal{D}_0 + \Gamma\,, \tag{2.17}$$

where $\mathcal{D}_0$ gives the classical dimensions. A basis of operators with well-defined anomalous dimensions is then found by diagonalizing the mixing matrix Γ:

$$\Gamma \mathcal{O}_a = \gamma_a \mathcal{O}_a \,. \tag{2.18}$$

The anomalous dimensions are the eigenvalues of the mixing matrix and the multiplicatively renormalized operators are the corresponding eigenvectors. In terms of the dilatation operator, we can write

$$\mathcal{D}\mathcal{O}_a = \Delta_a \mathcal{O}_a \,. \tag{2.19}$$

As $\mathcal{N} = 4$ SYM, also ABJM theory has some sectors which are closed under renormalization, in the sense that they contain a restricted number of operators which mix only among themselves, thus allowing simplified calculations. We will consider, in this work, only the simplest case, which is the $SU(2) \times SU(2)$ sector. It contains operators made out only of chiral superfields Z^A, W_B:

$$\mathcal{O}^{A_1 \cdots A_L}_{B_1 \cdots B_L} = \mathrm{Tr}[Z^{A_1} W_{B_1} \cdots Z^{A_L} W_{B_L}] \,. \tag{2.20}$$

From the above considerations it is clear that the dilatation operator is the basic object for the computation of anomalous dimensions. Nevertheless, relying solely on field theory perturbative techniques, it can be very hard to compute anomalous dimensions: this is due to the fact that, especially at high loop orders, one has usually to deal with a huge number of Feynman diagrams, even in the superspace approach. In the next chapter we will see that the integrability tools allow us to extremely simply this problem.

2.4. The Gravity Dual of ABJM Theory

For completeness, in this section we give a couple of comments on the gravity theory dual to ABJM theory. For a wider exposition we refer to [48] and references therein. As we said, the gravity dual of ABJM theory is type IIA superstring theory on $AdS_4 \times \mathbb{CP}^3$ background: this geometry is a solution of type IIA supergravity with two- and four-form fluxes turned on. It preserves 24 out of 32 supersymmetries and, therefore, is not maximally supersymmetric. A superstring sigma-model can be conveniently defined on

the coset

$$\frac{OSp(2,2|6)}{SO(1,3)\times U(3)} \tag{2.21}$$

with a κ-symmetry gauged-fixed action, in order to make it equivalent to the Green-Schwarz action, at least in non singular backgrounds. This sigma-model can be semi-classically quantized. One method for dealing with a curved background at the quantum level is to take a Penrose limit of the geometry which leads to a solvable plane-wave background and then to include curvature corrections perturbatively.

The metric on $AdS_4 \times \mathbb{CP}^3$ can be written as

$$ds^2 = R^2 \left(\frac{1}{4} ds^2_{AdS_4} + ds^2_{\mathbb{CP}^3} \right) , \tag{2.22}$$

where the AdS_4 metric in global coordinates reads

$$ds^2_{AdS_4} = -\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, (d\theta^2 + \sin^2\theta\, d\varphi^2) , \tag{2.23}$$

with $\rho \geq 0$, $t \in \mathbb{R}$, $\theta \in [0,\pi]$ and $\varphi \in [0,2\pi]$. The $\mathbb{CP}^3$ metric is given by:

$$\begin{aligned} ds^2_{\mathbb{CP}^3} =& d\xi^2 + \cos^2\xi \sin^2\xi \left(d\psi + \frac{1}{2}\cos\theta_1 d\varphi_1 - \frac{1}{2}\cos\theta_2 d\varphi_2 \right)^2 \\ &+ \frac{1}{4}\cos^2\xi \left(d\theta_1^2 + \sin^2\theta_1 d\varphi_1^2 \right) + \frac{1}{4}\sin^2\xi \left(d\theta_2^2 + \sin^2\theta_2 d\varphi_2^2 \right) , \end{aligned} \tag{2.24}$$

with $\xi \in [0,\pi/2]$, $\psi \in [0,2\pi]$ and $(\theta_{1,2}, \varphi_{1,2})$ parameterizing two 2-spheres. The background admits five Killing vectors leading to the five conserved charges: the worldsheet energy, the AdS-spin and the three $\mathbb{CP}^3$ angular momenta. Note that there is less symmetry than in the $AdS_5 \times S^5$ case, where there are two AdS-spins. The angular momenta are related to the charges of ABJM fields so that a dictionary between classical strings and gauge theory operators can be formulated. The classical string solution that corresponds to the gauge theory operator $\mathrm{Tr}(Z^1 W_1)^L$ (with L large so that the string becomes classical), is a point-like string that moves along a particular geodesic. Semi-classical quantization of strings is achieved by expanding the action about a classical solution and quantizing the fluctuations. In this way we find the worldsheet spectrum of string excitations. In the light-cone gauge, the massless fluctuations can be gauged

away and we are left with 4 light excitations $(\theta_{1,2}, \varphi_{1,2})$ from $\mathbb{CP}^3$ and 4 heavy excitations of which one (ξ) comes from $\mathbb{CP}^3$ and the other three $(x_{1,2,3})$ from AdS_4. For the physical fermions similar results hold. The light modes can be organized into two (2|2)-dimensional supermultiplets which can be mapped precisely to gauge theory operators with non vanishing anomalous dimensions. The heavy modes are an artifact of the above analysis which is done at infinite coupling λ. When going to finite coupling they dissolve into two light particles. At the technical level this is seen by looking at which particle poles appear in Green's functions at not strictly infinite coupling. The first observation is that in the free theory the pole for the heavy particles with mass κ coincides with the branch point of the branch cut that accounts for the pair production of two light modes with mass $\kappa/2$ each. When interactions are turned on, *i.e.* when $1/\sqrt{\lambda}$ corrections are considered, the pole moves into the branch cut, and the statement is that the exact propagator has a branch cut only.

Chapter 3.

Integrability

We have seen in the last chapter that the AdS/CFT correspondence states that the spectra of gauge and string theories should agree. As we explained, by *spectrum* of a conformal gauge theory such as ABJM theory, we mean the set of all possible anomalous dimensions of composite operators. Such quantities are in general functions of the coupling λ and their computation can be very hard if performed with perturbative QFT methods only, where a full-fledged computation of many multiloop Feynman diagrams is required to evaluate the mixing matrix. Remarkably, in the last decade, new methods emerged that reduce the computation of anomalous dimensions, as well as other observables, to the solution of a set of algebraic equations, called Bethe equations[1]. This is due to *integrability*: there are many hints that ABJM theory may be integrable in the planar limit, as its four-dimensional cousin $\mathcal{N} = 4$ SYM. If such a belief turns out to be true, the gauge theory might even be solved exactly. Similar integrable structures appear on the string theory side of the correspondence, where a two-dimensional nonlinear sigma model is defined on the string worldsheet. In this chapter we introduce the basic ingredients of integrability in gauge theories and, in particular, related results in ABJM theory. Some of these results are purely conjectures and need to be tested by means of independent techniques: this motivates the perturbative computations that we will perform in the following chapters of this work. A recent review of integrability in the context of AdS/CFT correspondence can be found in [11].

[1] In more complicated situations, we are left with a set of integral equations instead: this is the case of the Thermodynamic Bethe Ansatz and related methods (such as the Y-system), which we won't describe in this work.

3.1. The Algebraic Bethe Ansatz

While classical integrability can be rigorously formulated in the Hamiltonian formalism, the issue of quantum integrability requires more attention to be understood. Roughly speaking, an integrable system can be thought of as solvable. We will define a quantum system to be integrable if it admits an infinite number of commuting charges $\mathcal{Q}_r$

$$[\mathcal{Q}_r, \mathcal{Q}_s] = 0\,, \qquad r, s = 1, \ldots, \infty \tag{3.1}$$

where $\mathcal{Q}_2$ is taken to be the hamiltonian (so they are conserved charges). By "solving a theory" we mean to compute the eigenvalues of physical observables. Even though finding a complete solution might be very hard in many cases, integrability provides a set of tools which extremely simplify these computations. Such techniques culminate in the Bethe equations, a set of algebraic equations named after Hans Bethe, who found a method to solve the Heisenberg model of ferromagnetism in 1931 [49]. Quantum integrability plays a fundamental role in many areas of Physics, from condensed matter physics to mathematical physics, and, remarkably, appears also in the context of AdS/CFT correspondence, where it proves very useful *e.g.* in the problem of renormalization of gauge theory operators. Here, we don't follow the chronological flow, which would demand to start with Bethe's original method, but rather outline first the more general method originated from the work of Baxter in the early seventies and developed by the "Leningrad school" around Faddeev. This modern approach is usually known as the *algebraic Bethe ansatz* [50, 51] (see also [52–54, 56, 79] for some reviews). In the next section we will outline Bethe's original method, which usually goes under the name of *coordinate Bethe ansatz*.

For the sake of simplicity, we discuss in this section the $\mathrm{XXX}_{\frac{1}{2}}$ Heisenberg model[2] as a concrete example. However, our results are very general in nature. The Hilbert space of such a system is the tensor product of L copies of Hilbert spaces $\mathbb{C}^2$ for individual spins, where L is the total number of sites in the spin-chain.

[2] Such model was introduced in 1928 to dscribe a closed ferromagnetic one-dimensional spin-chain with spin $\frac{1}{2}$ degrees of freedom at each site and with identical coupling constants for spin-spin interactions in all spatial directions. As we will see, the integrable models appearing in the context of AdS/CFT are more general and will be introduced in the subsequent sections. Nevertheless, the Heisenberg model is a good playground to understand how the Bethe ansatz works.

The starting point of the algebraic Bethe ansatz approach is a generating object called *Lax operator*, rather than the hamiltonian. For the $\mathrm{XXX}_{\frac{1}{2}}$ spin chain it is given by

$$\mathcal{L}_{na}(u) = \left(u - \frac{i}{2}\right) \mathbb{1}_{na} + iP_{na}\,, \tag{3.2}$$

where $P_{j,k}$ is a permutation operator exchanging the spins at sites j and k, u is a spectral parameter and a labels an auxiliary space $\mathbb{C}^2$. In this auxiliary space, $\mathcal{L}_{na}$ is then a 2×2 matrix. Let $\mathcal{L}_{na}(u)$ and $\mathcal{L}_{nb}(v)$ be two Lax operators which act at the same quantum space (labelled by n) but at different auxiliary spaces (labelled by a and b): then, it can be shown that there exists an operator $R_{ab}(u-v)$ acting in the tensor product of the two auxiliary spaces, such that the following *fundamental commutation relation* holds on the triple tensor product of $\mathbb{C}^2$ spaces:

$$R_{ab}(u-v)\mathcal{L}_{na}(u)\mathcal{L}_{nb}(v) = \mathcal{L}_{nb}(v)\mathcal{L}_{na}(u)R_{ab}(u-v)\,. \tag{3.3}$$

The operator R_{ab} is called *R-matrix* and, in the case of the Heisenberg model, it is given by

$$R_{ab}(u) = u\mathbb{1}_{ab} + iP_{ab}\,. \tag{3.4}$$

We see that the R-matrix is essentially the same thing as the Lax operator. The fundamental commutation relation implies that the R-matrix satisfies the *Yang-Baxter* equation:

$$R_{ab}(u-v)R_{na}(u)R_{nb}(v) = R_{nb}(v)R_{na}(u)R_{ab}(u-v)\,. \tag{3.5}$$

This equation is of primary importance in integrable theories and has a simple interpretation in terms of factorized scattering: it says that multiparticle[3] scattering factorizes into pairwise scattering processes, in which the order of the pairwise scattering is of no importance.

[3] We don't further characterize, for the moment, such "particles". In condensed matter systems, as the Heisenberg model, they are usually magnons, while in our applications they will be certain gauge theory operators, as we will explain in a moment.

From the Lax operator one can build the *monodromy matrix* $T_a(u)$ which describes the transport of spins around the chain:

$$T_a(u) = \mathcal{L}_{La}(u) \cdots \mathcal{L}_{1a}(u) \,. \tag{3.6}$$

It also fulfills a fundamental commutation relation of the form

$$R_{ab}(u-v) T_a(u) T_b(v) = T_b(v) T_a(u) R_{ab}(u-v) \,. \tag{3.7}$$

Taking the trace in the auxiliary space, we then introduce the *transfer matrix*

$$T(u) = \mathrm{Tr}_a T_a(u) \,. \tag{3.8}$$

It is an operator acting on the quantum space. From (3.7) it follows that transfer matrices at different values of the spectral parameter commute:

$$[T(u), T(v)] = 0 \,. \tag{3.9}$$

Thanks to this property, the transfer matrix is used to generate the conserved charges of the integrable system: expanding $T(u)$ in power series around a point u_0 in complex plane, we obtain a set of linearly independent commuting operators $\mathcal{Q}_n$ acting on the quantum space:

$$T(u) = \sum_{n=0}^{\infty} (u-u_0)^n \mathcal{Q}_n \,, \qquad \mathcal{Q}_n = \frac{1}{n!} \frac{\mathrm{d}^n}{\mathrm{d}u^n} T(u) \bigg|_{u=u_0} \,, \tag{3.10}$$

This resembles the formulation of classical integrability, involving integrals of motion in involution, in the isospectral deformation approach of Lax. On the ground of this method there is the existence of the R-matrix (or, equivalently, of the Lax operator) satisfying the Yang-Baxter equation. The Heisenberg model has L degrees of freedom and the expansion of the transfer matrix, conveniently made around $u_0 = i/2$, yields a polynomial of degree L in the spectral parameter, $T(u) = \sum_{n=0}^{L} (u - i/2)^n \mathcal{Q}_n$. Since $\mathcal{Q}_{L-1} = 0$, we have exactly L commuting charges, as required by integrability. Apart from the trivial $\mathcal{Q}_L = 2\mathbb{1}$, the first contribution in (3.10) gives

$$T(i/2) = i^L U \,, \qquad U = P_{1,2} \cdots P_{L-1,L} \,, \tag{3.11}$$

where U simultaneously shifts all the spins by one site and is related to the momentum operator by $e^{iP} = U$, while the second contribution provides

$$H = \frac{1}{2}\left(i \left.\frac{\mathrm{d}}{\mathrm{d}u} \log T(u)\right|_{u=\frac{i}{2}} - L \right) = \frac{1}{2}\sum_{n=1}^{L} (P_{n,n+1} - \mathbb{1}) \,, \tag{3.12}$$

which is the well known Heisenberg hamiltonian and can be equivalently written in terms of the spin operator $\vec{S}_n = \frac{1}{2}\vec{\sigma}_n$, where $\vec{\sigma}_n$ are Pauli matrices acting on site n, as

$$H = \sum_{n=1}^{L} \left(\frac{1}{4} - \vec{S}_n \cdot \vec{S}_{n+1} \right) . \tag{3.13}$$

We have thus formally proven integrability of the $\mathrm{XXX}_{\frac{1}{2}}$ spin-chain. Moreover, our formalism allows us to solve the eigenvalue problem for H in a very elegant way, even though L is very large, as we now explain. Let's write the monodromy matrix in auxiliary space as:

$$T_a(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} , \tag{3.14}$$

where A, B, C, D are operators acting on the quantum space. We will proceed to the diagonalization of $T(u)$, which therefore will enable us to diagonalize all the charges $\mathcal{Q}_n$ at the same time. Since $T(u) = A(u) + D(u)$, the off-diagonal components $C(u)$ and $B(u)$ may be ignored. They are however very useful for the construction of the eigenvectors: let's observe that the operator C annihilates the ferromagnetic vacuum[4], *i.e.* the state with all "spin up":

$$C(u)|0\rangle = 0 \,. \tag{3.15}$$

We now make the following ansatz for the eigenvectors of $T(u)$ at level M:

$$|u_1 \cdots u_M\rangle = B(u_1) \cdots B(u_M)|0\rangle \,. \tag{3.16}$$

[4]By symmetry one can exchange simultaneously $A \leftrightarrow D$ and $B \leftrightarrow C$: this corresponds to exchange up and down spins. In this case the vacuum is the state with all "spin down".

The state $|u_1 \cdots u_M\rangle$ is called *Bethe state*. Using the commutation relations (3.7) one can show that it is an eigenstate of the transfer matrix with eigenvalues

$$\Lambda(u, u_1, \ldots, u_M) = \left(u + \frac{i}{2}\right)^L \prod_{j=1}^{M} \frac{u - u_j - i}{u - u_j} + \left(u - \frac{i}{2}\right)^L \prod_{j=1}^{M} \frac{u - u_j + i}{u - u_j} \tag{3.17}$$

if the set $\{u_1, \ldots, u_M\}$ satisfies the following *Bethe equations*:

$$\left(\frac{u_j + i/2}{u_j - i/2}\right)^L = \prod_{k \neq j}^{M} \frac{u_j - u_k + i}{u_j - u_k - i}\,. \tag{3.18}$$

These equations are necessary in order to cancel out the poles in Λ so that it is a polynomial of degree L in u, as it should. The solutions of the Bethe equations, called *Bethe roots*, then, allow us to find the eigenvalues $\Lambda(u, u_1, \ldots, u_M)$ of $T(u)$. From (3.10) we can then immediately find the eigenvalues of the conserved charges. In particular, we have for the momentum operator:

$$P|u_1 \cdots u_M\rangle = \sum_{j=1}^{M} p(u_j)|u_1 \cdots u_M\rangle\,, \qquad p(u) = -i \log \frac{u + i/2}{u - i/2} \tag{3.19}$$

and for the hamiltonian

$$H|u_1 \cdots u_M\rangle = \sum_{j=1}^{M} E(u_j)|u_1 \cdots u_M\rangle\,, \qquad E(u) = -\frac{1}{2}\frac{1}{u^2 + 1/4}\,. \tag{3.20}$$

We see that momentum and energy are additive. From (3.19) we also see that the Bethe equations can be cast into the following form:

$$e^{iLp(u_j)} = \prod_{k \neq j}^{M} S(u_j - u_k)\,, \tag{3.21}$$

where the S-matrix is given by

$$S(u_j - u_k) = \frac{u_j - u_k + i}{u_j - u_k - i}\,. \tag{3.22}$$

The eigenstate $|u\rangle$ represents a quasi-particle, called *magnon*, with definite momentum $p(u)$, propagating along the spin-chain. The quantity u is also called magnon rapid-

ity. A state $|u_1 \cdots u_M\rangle$ represents M magnons propagating on the spin chain: Bethe equations tell that the scattering process factorizes into $M-1$ scattering processes of the j^{th} magnon with the remaining magnons. The S-matrix describes the scattering of two magnons and the phase-shift for the j^{th} magnon is given by the exponential of its momentum. It is possible to express the individual magnon energy as a function of the individual magnon momentum: it is called *dispersion relation* and is given by:

$$E(p) = \cos p - 1\,. \tag{3.23}$$

We have presented the Bethe Ansatz technique for the $\text{XXX}_{\frac{1}{2}}$ Heisenberg spin-chain: this means that the spins are in the fundamental representation of $SU(2)$. The Bethe Ansatz can be extended to spin-chains with more general symmetry groups. The Bethe equations become more involved since one has to deal with the nested Bethe Ansatz: for example, for $SU(n)$ symmetry, one needs $n-1$ Bethe Ansatz levels.

3.2. The Coordinate Bethe Ansatz

The most important advantage of the algebraic Bethe Ansatz is that it allows to find a rather systematic way to generate large classes of integrable models. Once we have defined an R-matrix, it automatically gives an integrable hamiltonian. We now describe the more "physical" coordinate Bethe Ansatz. Useful introductions can be found in [54, 55, 59]. We still descibe the example of the Heisenberg model.

Let's start with the Heisenberg hamiltonian (3.13). The ferromagnetic vacuum state is taken to be

$$|0\rangle = |\uparrow\uparrow\cdots\uparrow\rangle \tag{3.24}$$

and has zero energy. One constructs excited states by flipping one ore more spins at selected sites:

$$|n_1 \cdots n_M\rangle := |\uparrow\uparrow\cdots\downarrow\cdots\downarrow\cdots\uparrow\rangle = S^-_{n_1}\cdots S^-_{n_M}|0\rangle\,, \tag{3.25}$$

where $S_n^{\pm} = S_n^1 \pm iS_n^2$ are the usual ladder operators. Our aim is to diagonalize the hamiltonian. The states (3.25) are not eigenstates of H. Consider one excitation first: we find that the superposition

$$|p\rangle = \sum_{n=1}^{L} e^{ipn} |n\rangle \tag{3.26}$$

is indeed an eigenstate of H with eigenvalue

$$E(p) = -2\sin^2\frac{p}{2}\,, \tag{3.27}$$

which coincides with (3.23). Such a state is called "magnon". Two-magnon states are defined by

$$|p_1p_2\rangle = \sum_{n_1<n_2}^{L} \psi_{n_1n_2}|n_1n_2\rangle \tag{3.28}$$

where ψ is the wave function in the discrete position space of the spin-chain. Schrödinger equation $H\,|p_1p_2\rangle = E(p_1,p_2)\,|p_1p_2\rangle$ can be solved by the following Bethe ansatz:

$$\psi_{n_1n_2} = e^{i(p_1n_1+p_2n_2)} + S(p_1,p_2)e^{i(p_1n_2+p_2n_1)}\,. \tag{3.29}$$

It means that the magnons freely move around the chain with momenta p_1, p_2 until they hit each other when $n_2 = n_1+1$. If the system is integrable, they either pass through each other or exchange momenta, with an amplitude $S(p_1,p_2)$, which is then the S-matrix of the process. Because of total momentum conservation, momenta are individually conserved. Integrability implies that it is true also for the M-body scattering problem. Plugging (3.29) into the Schrödinger equation we find that it is satisfied if

$$S(p_1,p_2) = -\frac{e^{ip_1+ip_2} - 2\,e^{ip_1} + 1}{e^{ip_1+ip_2} - 2\,e^{ip_2} + 1}\,, \quad E(p_1,p_2) = E(p_1) + E(p_2)\,, \tag{3.30}$$

where $E(p)$ is given by (3.27). Imposing the boundary conditions

$$\psi_{n_1n_2} = \psi_{n_2n_1+L} \tag{3.31}$$

on the wave function, we obtain the Bethe equations:

$$e^{ip_1 L} = S(p_1, p_2)\,, \qquad e^{ip_2 L} = S(p_2, p_1)\,. \tag{3.32}$$

Integrability allows us to immediately generalize them to the problem of M-magnon scattering: factorized scattering demands that the total phase shift acquired by a magnon circling around the chain is the products of phase factors due to individual collisions with the other $M-1$ magnons

$$e^{ip_k L} = \prod_{j=1, j\neq k}^{M} S(p_k, p_j)\,, \quad k = 1, \ldots, L\,, \tag{3.33}$$

with the same S-matrix (3.30). The total energy is given by

$$E(p_1, \ldots, p_M) = \sum_{k=1}^{M} E(p_k)\,. \tag{3.34}$$

The eigenstates are given by

$$|p_1 p_2 \cdots p_M\rangle = \sum_{n_1<n_2<\cdots<n_M} \sum_{\{\tau\}} A(\tau) e^{ip_{\tau_1} n_1 + \cdots + ip_{\tau_M} n_M} |n_1 \cdots n_M\rangle\,, \tag{3.35}$$

where $\{\tau\}$ is the set of all $M!$ permutations τ of the spin down. The amplitudes are given by

$$A(\tau) = \text{sign}(\tau) \prod_{j<k} (e^{ip_k + ip_j} - 2e^{ip_k} + 1)\,, \tag{3.36}$$

where $\text{sign}(\tau)$ is the signature of the permutation. Changing variables to $u_k = \frac{1}{2}\cot(\frac{p_k}{2})$ we recover (3.18),(3.20), (3.22).

3.3. The ABJM Spin-Chain

We are now ready to describe how integrability shows up in ABJM theory. Consider single trace operators, made up only of chiral superfields, of the form:

$$\mathcal{O}^{A_1\cdots A_L}_{B_1\cdots B_L} = \mathrm{Tr}(Y^{A_1}Y^\dagger_{B_1}\cdots Y^{A_L}Y^\dagger_{B_L}) \tag{3.37}$$

where $A_i, B_i = 1,\ldots,4$ and

$$Y^A = (Z^1, Z^2, W^{\dagger 1}, W^{\dagger 2})\,, \qquad Y^\dagger_A = (Z^\dagger_1, Z^\dagger_2, W_1, W_2) \tag{3.38}$$

combine the $SU(2)$ superfields Z^A and W_A into fundamental and anti-fundamental representations of the R-symmetry group $SU(4)$. It is very convenient to represent the operators (3.37) as states of a circular *spin-chain* with $2L$ sites. The spins are alternating between the fundamental representation of $SU(4)$ on odd sites and the anti-fundamental representation on the even sites:

$$\mathrm{Tr}(Y^1Y^\dagger_3Y^2Y^\dagger_1Y^4Y^\dagger_2\cdots) \;\leftrightarrow\; |Y^1Y^\dagger_3Y^2Y^\dagger_1Y^4Y^\dagger_2\cdots\rangle \tag{3.39}$$

The spin-chain is closed because of the presence of the trace. In this spin-chain analogy, the dilatation operator of the gauge theory can then be regarded as the hamiltonian of the model acting in the Hilbert space $(V\otimes\bar{V})^{\otimes L}$, where V and $\bar{V}$ are the $\mathbf{4}$ and $\bar{\mathbf{4}}$ of $SU(4)$. It was shown that the planar two-loop dilatation operator of ABJM theory, in the spin-chain picture, is mapped to an integrable hamiltonian [12, 13]. This hamiltonian can be written in terms of permutation operators $P : V\otimes V \to V\otimes V$ (or $P : \bar{V}\otimes\bar{V}\to\bar{V}\otimes\bar{V}$) and trace operators $K : V\otimes\bar{V}\to V\otimes\bar{V}$ (or $K : \bar{V}\otimes V\to\bar{V}\otimes V$) acting on flavour space in the following way

$$P^{AB}{}_{A'B'} = \delta^A{}_{B'}\delta^B{}_{A'}\,, \qquad K^{AB'}_{BA'} = \delta^A{}_{A'}\delta_B{}^{B'}\,, \tag{3.40}$$

and reads

$$\mathcal{D}_2 = \frac{1}{2}\,\lambda^2\sum_{j=1}^{2L}(2\,\mathbb{1} - 2P_{j,j+2} + P_{j,j+2}K_{j,j+1} + K_{j,j+1}P_{j,j+2})\,. \tag{3.41}$$

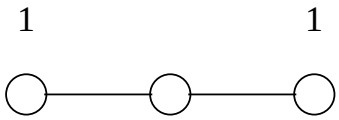


Figure 3.1.: The $SU(4)$ Dynkin diagram. The numbers indicate the Dynkin labels of the representation. The outer roots are u_j and v_j while r_j is the middle root.

The ground states of the Hamiltonian are chiral primary operators. Their spectrum can be directly compared with the supergravity harmonics on $AdS_4 \times \mathbb{CP}^3$. It can be checked that the chiral primaries are in one-to-one correspondence with the spherical functions on $\mathbb{CP}^3$. Integrability of the hamiltonian (3.41) can be shown in the following way (see [12,13] for details): generalizing the formalism explained in the previous section, one builds a general alternating integrable spin-chain hamiltonian in any representations starting with appropriate R-matrices [57]. The resulting hamiltonian involves nearest-neighbor and three-site interactions, but in general breaks charge conjugation symmetry $\mathbf{4} \leftrightarrow \bar{\mathbf{4}}$. It turns out for $SL(n)$ groups that the nearest neighbor interactions always cancel out. If one further makes a special choice of parameters then the conjugation symmetry is preserved and the spin-chain hamiltonian exactly coincides with the dilatation operator (3.41). Therefore, (3.41) is integrable by construction. It proves two-loop integrability of ABJM theory. The R-matrix formalism also provides a set of nested Bethe equations:

$$\begin{aligned}
\left(\frac{u_j + i/2}{u_j - i/2}\right)^L &= \prod_{k\neq j}^{M_u} \frac{u_j - u_k + i}{u_j - u_k - i} \prod_{k=1}^{M_r} \frac{u_j - r_k - i/2}{u_j - r_k + i/2}\,, \\
1 &= \prod_{k\neq j}^{M_r} \frac{r_j - r_k + i}{r_j - r_k - i} \prod_{k=1}^{M_u} \frac{r_j - u_k - i/2}{r_j - u_k + i/2} \prod_{k=1}^{M_v} \frac{r_j - v_k - i/2}{r_j - v_k + i/2}\,, \\
\left(\frac{v_j + i/2}{v_j - i/2}\right)^L &= \prod_{k\neq j}^{M_v} \frac{v_j - v_k + i}{v_j - v_k - i} \prod_{k=1}^{M_r} \frac{v_j - r_k - i/2}{v_j - r_k + i/2}\,,
\end{aligned} \tag{3.42}$$

where u_j, v_j and r_j are a set of Bethe roots associated with the $SU(4)$ Dynkin diagram (see Figure 3.1). The outer roots u_j and v_j are momentum carrying roots:

$$e^{2iP} = \prod_{j=1}^{M_u} \frac{u_j + i/2}{u_j - i/2} \prod_{j=1}^{M_v} \frac{v_j + i/2}{v_j - i/2}\,. \tag{3.43}$$

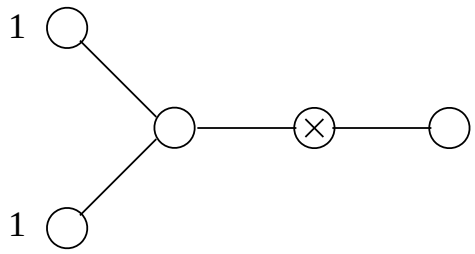


Figure 3.2.: One choice for the $OSp(2,2|6)$ Dynkin diagram. The Dynkin labels are taken from the $SU(4)$ spin chain. Some subsectors of the theory are visible from this diagram: two $SU(2|2)$ subsectors are obtained by exciting only momentum carrying roots of one of the wings (u and v) plus an arbitrary amount of auxiliary roots in the $SU(2|2)$ tail (r, s, w). The $SU(2) \times SU(2)$ subsector is obtained by only exciting the momentum carrying roots.

Since the operators (3.37) contain the trace, Behte equations must be supplemented by the condition of vanishing momentum:

$$\prod_{j=1}^{M_u} \frac{u_j + i/2}{u_j - i/2} \prod_{j=1}^{M_v} \frac{v_j + i/2}{v_j - i/2} = 1\,. \tag{3.44}$$

It follows from the cyclicity of the trace and is hence called also "trace condition". Once we have solved the Bethe equations (3.42) we can find the energy eigenvalues, which correspond to anomalous dimensions:

$$E = \lambda^2 \left(\sum_{j=1}^{M_u} \frac{1}{u_j^2 + 1/4} + \sum_{j=1}^{M_v} \frac{1}{v_j^2 + 1/4} \right)\,. \tag{3.45}$$

Let's observe that, for the $SU(2) \times SU(2)$ subsector, the trace operators K vanish and we are left with two decoupled Heisenberg $SU(2)$ spin-chains of L sites each, one living at odd sites and the other at even sites. This corresponds to the absence of the middle roots r_j. The trace condition couples the two chains through the relation $P = P_1 + P_2 = 0 \pmod{2\pi}$, where $P_{1,2}$ are the momenta for the first and second spin-chain. In [12] the Bethe Ansatz was extended to the full $OSp(2,2|6)$ superconformal group and integrability for the full planar dilatation operator was proved in [15]. The coupling between the Bethe roots is encoded in the Dynkin diagram of Figure 3.2. The $OSp(2,2|6)$ Bethe Ansatz was tested against explicit diagonalization of the spin-chain hamiltonian in [27].

3.4. The All-Loop Bethe Ansatz

In the previous section we have presented the two-loop Bethe Ansatz for ABJM theory. At strong coupling we have seen that the theory can be described by a coset sigma model: this sigma model was shown to be classically integrable [19,20] and the algebraic curve was constructed in [21]. It is tempting to try and extend the Bethe Ansatz beyond the two-loop level and connect the resulting Bethe equations to the string sigma model. Dealing with all-orders systems leads to long-range spin-chains: remarkably, even though we neither know the exact dilatation operator nor the generating Lax operator, some progress can be still made. In the context of AdS_5/CFT_4, thanks to symmetry considerations, an all-loop *Asymptotic Bethe Ansatz* was derived in [58,59,78]. The word "asymptotic" can be understood in the context of the coordinate Behte Ansatz: because of long-range interactions, there must be an asymptotic region where two magnons are far enough from each other so that they cannot interact directly: in this case the wave function can be put in the form of the Bethe Ansatz. The proposed Bethe equations contain a scalar dressing factor, which was later conjectured in [75]. Being asymptotic, these equations don't take into account *wrapping interactions*[5] and are, therefore, valid only for sufficiently long operators. Put in other words, the length of the spin-chain has to exceed the range of the interactions.

Inspired by these considerations, Gromov and Vieira proposed an analogous all-loop Bethe Ansatz for the AdS_4/CFT_3 case. The reason of this similarity is that both versions of the correspondence share an $SU(2|2)$ symmetry which highly constrains the form of the Bethe equations. Introducing five Behte roots u, v, r, s, w, as required by the structure of the Dynkin diagram associated with the $OSp(2,2|6)$ symmetry group, the

[5] Such interactions have a range greater than the length of the operator and, thus, wrap around it.

all-loop Bethe equations read [18]:

$$
\begin{aligned}
1 &= \prod_{k=1}^{M_w} \frac{s_j - w_k + i/2}{s_j - w_k - i/2} \prod_{k=1}^{M_u} \frac{1 - 1/x(s_k)x(u_j + 1/2)}{1 - 1/x(s_k)x(u_j - 1/2)} \\
&\quad \times \prod_{k=1}^{M_v} \frac{1 - 1/x(s_k)x(v_j + 1/2)}{1 - 1/x(s_k)x(v_j - 1/2)}\,, \\
1 &= \prod_{k\neq j}^{M_w} \frac{w_j - w_k - i}{w_j - w_k + i} \prod_{k=1}^{M_s} \frac{w_j - s_k + i/2}{w_j - s_k - i/2} \prod_{k=1}^{M_r} \frac{s_j - r_k + i/2}{s_j - r_k - i/2}\,, \\
1 &= \prod_{k=1}^{M_w} \frac{r_j - w_k + i/2}{r_j - w_k - i/2} \prod_{k=1}^{M_u} \frac{x(r_j) - x(u_k + i/2)}{x(r_j) - x(u_k - i/2)} \prod_{k=1}^{M_v} \frac{x(r_j) - x(v_k + i/2)}{x(r_j) - x(v_k - i/2)}\,, \\
\left(\frac{x(u_j + i/2)}{x(u_j - i/2)}\right)^L &= \prod_{k\neq j}^{M_u} \frac{u_j - u_k + i}{u_j - u_k - i} \prod_{k=1}^{M_s} \frac{1 - 1/x(u_j - i/2)x(u_k)}{1 - 1/x(u_j + i/2)x(u_k)} \prod_{k=1}^{M_r} \frac{x(u_j - i/2) - x(r_k)}{x(u_j + i/2) - x(r_k)} \\
&\quad \times \prod_{k=1}^{M_u} e^{i\theta(u_j,u_k)} \prod_{k=1}^{M_v} e^{i\theta(v_k,u_j)}\,, \\
\left(\frac{x(v_j + i/2)}{x(v_j - i/2)}\right)^L &= \prod_{k\neq j}^{M_v} \frac{v_j - v_k + i}{v_j - v_k - i} \prod_{k=1}^{M_s} \frac{1 - 1/x(v_j - i/2)x(v_k)}{1 - 1/x(v_j + i/2)x(v_k)} \prod_{k=1}^{M_r} \frac{x(vu_j - i/2) - x(r_k)}{x(v_j + i/2) - x(r_k)} \\
&\quad \times \prod_{k=1}^{M_v} e^{i\theta(v_j,v_k)} \prod_{k=1}^{M_u} e^{i\theta(v_k,u_j)}
\end{aligned}
\tag{3.46}
$$

and must be supplemented by the zero-momentum condition:

$$
1 = \prod_{k=1}^{M_u} \frac{x(u_k + i/2)}{x(u_k - i/2)} \prod_{k=1}^{M_v} \frac{x(v_k + i/2)}{x(v_k - i/2)}\,. \tag{3.47}
$$

We see that only the u and v roots carry momentum. However, they are coupled to the auxiliary s, w, r roots through Bethe equations. The variables x are related to the Bethe roots through the relation

$$
x + \frac{h^2(\lambda)}{x} = u \quad \Rightarrow \quad x(u) = \frac{u}{2} + \frac{1}{2}\sqrt{u^2 - 4h^2(\lambda)}\,, \tag{3.48}
$$

where $h(\lambda)$ is an unfixed function of the coupling constant which appears in the magnon dispersion relation [60–62] as well as in many other quantities. Indeed, the form of the

exact all-order magnon dispersion relation of ABJM theory is fixed by the underlying $SU(2|2)$ symmetry to be [63]

$$E(p) = \sqrt{Q^2 + 4h^2(\lambda)\sin^2\frac{p}{2}} - Q\,, \tag{3.49}$$

where Q is the R-charge of the fundamental magnon. It, thus, has the same form as in $\mathcal{N}=4$ SYM, where $Q=1$ and $h(\lambda)$ turns out to be simply $\sqrt{\lambda/4\pi}$. In ABJM theory, $Q=1/2$ and $h(\lambda)$ appears to be highly non-trivial. Some guesses for the functional form have been made, *e.g.* in [18, 24, 60], but none of them is fully acceptable. Any proposal for $h(\lambda)$ should interpolate between the known weak- and strong-coupling asymptotics, which are given by

$$h^2(\lambda) = \lambda^2 - 4\zeta(2)\lambda^4 + \mathcal{O}(\lambda^6)\,, \quad \lambda \ll 1 \tag{3.50}$$

and

$$h(\lambda) = \sqrt{\frac{\lambda}{2}} + a_1 + \mathcal{O}(1/\sqrt{\lambda})\,, \quad \lambda \gg 1\,. \tag{3.51}$$

The two terms displayed in (3.50) were computed in [12, 22, 23], while higher-order terms are still unknown. We will make a conjecture on the transcendentality of the coefficients in the weak-coupling expansion of $h^2(\lambda)$ in the next sections. The leading term in (3.51) was computed in [61, 62], while the value of the constant term is still debated. A worldsheet computation for the one-loop correction to the energy of a folded string yields a non-vanishing value for a_1 [64], while the algebraic curve computation gives $a_1 = 0$ [65], which was also assumed in [18] in deriving (3.46). This disagreement can be traced back to different prescriptions used for adding up fluctuation frequencies in string computations. It, thus, seems that the subleading terms in (3.51) are scheme dependent. In order to unambiguously compare different approaches, one should therefore express all results in terms of a physical reference observable, and neither in terms of λ nor $h(\lambda)$.

The Bethe equations (3.46) also contain the scalar *dressing factor* $e^{i\theta}$, which introduces extra self-interactions for the momentum-carrying roots, as well as a new interaction between them, at higher orders in perturbation theory. It is a very important quantity since it interpolates between weak and strong coupling behavior. Its form is

also conjectured in [18] to be essentially the same as the BES phase [75] and will be discussed in the next section.

Once the system (3.46) is solved, the spectrum of all conserved charges $\mathcal{Q}_r$ is given by the momentum-carrying roots alone:

$$\mathcal{Q}_r = \sum_{k=1}^{M_u} q_r(u_k) + \sum_{k=1}^{M_v} q_r(v_k)\,, \tag{3.52}$$

where

$$q_r(u) = \frac{1}{r-1}\left(\frac{i}{x(u+i/2)^{r-1}} - \frac{i}{x(u-i/2)^{r-1}}\right)\,. \tag{3.53}$$

In particular, the spectrum of anomalous dimensions (or energies of string states) is given by the eigenvalues of the second charge:

$$E = h^2(\lambda)\mathcal{Q}_2\,. \tag{3.54}$$

For our purposes, this is the best method available for the computation of anomalous dimensions of long operators in ABJM theory. We recall that the all-loop Bethe equations (3.46) are not proven, but just conjectured. However they enjoy many nice properties that allow to trust their correctness. First of all they exhibit an $OSp(2,2|6)$ global symmetry, which is to be expected from AdS_4/CFT_3 correspondence, besides other interesting dualities (see [18]). Moreover, they reduce to the two-loop Bethe equations of [12, 13] discussed in the previous section, for $\lambda \to 0$, and to the algebraic curve of [21] in the continuum limit, for $\lambda \to \infty$. The all-loop Bethe Ansatz has passed some non trivial tests [29–32, 74]. In particular, it has been shown that it can be derived from an exact two-particle S-matrix[6]. Nevertheless, it is useful to look for further independent checks of its validity. In this work, we perturbatively compute the leading-order weak-coupling correction to the dressing phase $\theta(\lambda)$ appearing in the scalar factor inside (3.46)

[6]The staggered nature of the spin-chain naturally breaks the S-matrix up into pieces: interactions between two Z-particles, between two W-particles, and between one of each kind, where each piece is proportional to the S-matrix derived in the context of the AdS_5/CFT_4 [63]. The scalar factor cannot be fixed by symmetry and, consistently with [18], has been conjectured to be related to the BES dressing phase [29]. We will return on these results in section 3.5

and which is predicted to affect anomalous dimensions starting at eight-loop order [18]. We find an agreement with the value[7] expected from the Asymptotic Bethe Ansatz.

3.5. The Dressing Phase

We now describe the dressing factor $e^{i\theta}$ appearing in the all-loop Bethe equations (3.46). The phase $\theta(\lambda)$ is called *dressing phase* and is conjectured to coincide with the BES phase [75] introduced in the context of the AdS_5/CFT_4 correspondence, upon the replacement $g \to h(\lambda)$, where g is the SYM coupling constant given by $\sqrt{\lambda/4\pi}$. The form $e^{i\theta}$ of the ABJM dressing factor is a consequence of the $SU(2|2)_Z \oplus SU(2|2)_W$ symmetry, whereas in the AdS_5/CFT_4 case the analogous factor $e^{2i\theta}$ appears squared because of the $SU(2|2) \otimes SU(2|2)$. Such a factor is necessary to interpolate between the weak-coupling regime described by gauge theory and the strong-coupling regime described by string theory. In fact, the S-matrices[8] of the respective theories are equal up to the dressing factor [59, 68].

The two-body S-matrix is the main object in the solution of integrable theories. Symmetry considerations, together with the requirement of factorized scattering, usually fix the S-matrix up to an overall scalar factor: this is precisely the dressing factor. It encodes much more dynamical information and is harder to derive. In [66] it is shown how the integral form of Dorey-Hofman-Maldacena [67] for the dressing phase can be derived from the assumptions of unitarity, crossing symmetry and some physical constraints on its analytical structure. This DHM representation is given by

$$\theta(u,v) = \chi(x^+,y^-) - \chi(x^-,y^-) - \chi(x^+,y^+) + \chi(x^-,y^+)\,, \tag{3.55}$$

where $\chi(x,y) = -\chi(y,x)$, $x^\pm = x(u \pm i/2)$, $y^\pm = x(v \pm i/2)$ and

$$\chi(x,y) = -i \oint_{|z_1|=1} \frac{\mathrm{d}z_1}{2\pi i} \oint_{|z_2|=1} \frac{\mathrm{d}z_2}{2\pi i} \frac{1}{(x-z_1)(y-z_2)} \log \frac{\Gamma[1 + ih(\lambda)(z_1 + \frac{1}{z_1} - z_2 - \frac{1}{z_2})]}{\Gamma[1 - ih(\lambda)(z_1 + \frac{1}{z_1} - z_2 - \frac{1}{z_2})]}\,. \tag{3.56}$$

[7] See section 3.5 for details.

[8] We're referring, here, to the internal S-matrix describing the scattering of excitations on the lattice hidden inside the trace of composite operators.

With the representation (3.55)-(3.56) the dressing factor reduces to unit when $\lambda \to 0$ and to the AFS phase [68] when $\lambda \to \infty$, thus reproducing the correct dynamics in the asymptotic regimes.

A useful rewriting of the dressing phase is given in terms of the conserved charges as [68, 80]

$$\theta(u,v) = \sum_{r=2}^{\infty} \sum_{s=r+1}^{\infty} \beta_{r,s}(\lambda)[q_r(u)q_s(v) - q_s(u)q_r(v)]\,, \tag{3.57}$$

where the coefficient functions $\beta_{r,s}(\lambda)$ should have the following strong coupling behavior [18]

$$\beta_{r,s}(\lambda) = h(\lambda)^{r+s-1}\delta_{r+1,s} + h(\lambda)^{r+s-2}\frac{-1+(-1)^{r+s}}{\pi}\,\frac{(r-1)(s-1)}{(r+s-2)(s-r)} + \mathcal{O}(1/\sqrt{\lambda}) \tag{3.58}$$

yielding the AFS phase at leading order and the HL phase [84] at next-to leading order[9].

In the context of AdS_5/CFT_4 it was argued [75] that the weak-coupling expansion of $\beta_{r,s}(\lambda)$ can be extrapolated by analytical continuation of the all-loop strong-coupling coefficient functions, conjectured in [85] to be given by $\beta_{r,s}(\lambda) = \sum_{n=0}^{\infty} c_{r,s}^{(n)} g^{r+s-1-n}$ with

$$c_{r,s}^{(n)} = \frac{(1-(-1)^{r+s})\zeta(n)}{2(-2\pi)^n\Gamma(n-1)}(r-1)(s-1)\frac{\Gamma[(\frac{1}{2}(s+r+n-3)]\Gamma[(\frac{1}{2}(s-r+n-1)]}{\Gamma[(\frac{1}{2}(s+r-n+1)]\Gamma[(\frac{1}{2}(s-r-n+3)]}\,, \tag{3.59}$$

to negative values of n:

$$\beta_{r,s}(g) = -\sum_{n=1}^{\infty} c_{r,s}^{(-n)}\, g^{r+s+n-1}\,, \tag{3.60}$$

where $g = \frac{\sqrt{\lambda}}{4\pi}$ as usual. So, the dressing phase coefficients were conjectured to be

$$\beta_{r,s}^{(\ell)} = -c_{r,s}^{(r+s-2\ell-1)}\,. \tag{3.61}$$

[9] This assumes that $a_1 = 0$ in (3.51). However see [64] for the case where $a_1 = -\frac{\ln 2}{2\pi}$.

In particular, $c_{2,3}^{(-2)} = -4\zeta(3)$. Moreover, there were provided arguments to say that the appearance of the dressing phase coefficient functions $\beta_{r,s}(\lambda)$ is postponed to order λ^{r+s-2} instead of λ^{s-1} [80] so that the leading order coefficient of the dressing phase was predicted to be

$$\beta_{2,3}^{(3)} = 4\zeta(3)\,. \tag{3.62}$$

By means of field theory computations, it was proved that (3.62) is indeed the right value [82].

Since in the Bethe Ansatz proposal of [18] the strong coupling limit of the dressing phase is the same of the AdS_5/CFT_4 case, we can suppose, by analogy of the $\mathcal{N}=4$ SYM case and with the usual replacement $g \to h(\lambda)$, that in ABJM theory $\beta_{r,s}(\lambda) = \mathcal{O}(\lambda^{2(r+s-2)})$ and that:

$$\beta_{r,s}(\lambda) = -\sum_{n=1}^{\infty} c_{r,s}^{(-n)}\, h(\lambda)^{r+s+n-1}\,. \tag{3.63}$$

For $r=2$, $s=3$, we see that we should obtain

$$\beta_{2,3}(\lambda) = -c_{2,3}^{(-2)}\,\lambda^6 - \left(c_{2,3}^{(-4)} + 3h_4 c_{2,3}^{(-2)}\right)\lambda^8 + \cdots \tag{3.64}$$

We observe that, since the first weak coupling coefficient of $h(\lambda)$ is $h_2 = 1$ (3.50), the leading order contribution to ABJM dressing phase is expected to be

$$\beta_{2,3}^{(6)} = 4\zeta(3)\,, \tag{3.65}$$

which is precisely the same value obtained in $\mathcal{N}=4$ SYM. At higher orders, the $c_{r,s}^{(-n)}$ coefficients mix with the non trivial coefficients of $h(\lambda)$ yielding dressing phase coefficients which differ from the $\mathcal{N}=4$ SYM ones. We stress that, in our context, this is merely a conjecture: one of the goals of the present work is the computation of the coefficient $\beta_{2,3}^{(6)}$ by independent field theory techniques. We will find in Chapter 5 that the prediction (3.65) is indeed correct.

3.6. Transcendentality principle

Let's now make a general discussion on the dressing phase coefficients of ABJM theory. In the case of $\mathcal{N}=4$ SYM theory, it was shown [75] that the coefficients of the dressing phase should have a well defined degree of transcendentality in order to preserve the Kotikov-Lipatov transcendentality principle on the scaling function of the theory [76]. The scaling function $f_{\rm SYM}(g)$ (also known as the cusp anomalous dimension) is a universal function of the coupling constant g that appears, in a generic planar gauge theory, in many situations: it is appears, *e.g.*, in on-shell gluon scattering amplitudes, in the anomalous dimensions of large-spin twist operators and so on. Its weak coupling expansion in g fulfills the Kotikov-Lipatov transcendentality principle [92] up to four loops. It is widely believed that such principle is valid at any loop-order. Since the scaling function can be derived from a Bethe Ansatz approach, it puts constraints on the transcendentality of the dressing phase. In particular, the leading order coefficient of the dressing phase directly computed in [82] precisely satisfies such constraints. Here, we want to generalize these results to the ABJM case: the scaling function of ABJM theory is prescribed to be [18]

$$f_{\rm ABJM}(\lambda)=\frac{1}{2}\,f_{\rm SYM}(g)|_{g\to h(\lambda)}\,. \tag{3.66}$$

Taking the weak coupling expansion for $f_{\rm SYM}(g)$ obtained in [75] and inserting the expansion of $h(\lambda)$, we thus get, up to eight loops,

$$\begin{aligned} f_{\rm ABJM}(\lambda) =& \,4\,\lambda^2-\left(\frac{4}{3}\pi^2-4h_4\right)\lambda^4+\left(\frac{44}{45}\pi^4-\frac{8}{3}h_4\pi^2+4h_6\right)\lambda^6 \\ &-4\left(\frac{73}{315}\pi^6-8\zeta(3)^2-\frac{11}{15}h_4\pi^4+\frac{1}{3}h_4^2\pi^2+\frac{2}{3}h_6\pi^2-h_8+4\beta_{2,3}^{(6)}\zeta(3)\right)\lambda^8 \\ &+\mathcal{O}(\lambda^{10})\,. \end{aligned} \tag{3.67}$$

The transcendentality principle, extended to ABJM theory, states that:

Assigning degree of transcendentality k to constants π^k as well as $\zeta(k)$, the ℓ-loop contribution to the cusp anomalous dimension f_{ABJM} has uniform degree of transcendentality $\ell-2$.

Recalling that $h_4 = -4\zeta(2)$, we see that this principle is satisfied up to four loops. Moreover, from the six-loop contribution, we immediately see that h_6 should have degree four. Starting from eight loops, also the unknown dressing phase starts contributing to the scaling function. One very natural way[10] to preserve the transcendentality principle is to conjecture that:

- h_{2k} has degree of transcendentality $2k-2$,
- $\beta_{r,s}^{(2k)}$ has degree of transcendentality $2k+2-r-s$.

Therefore, the transcendentality principle implies that the leading order coefficient of the dressing phase should be of the form

$$\beta_{2,3}^{(6)} = a\,\pi^3 + b\,\zeta(3)\,, \tag{3.68}$$

where a and b are some rational numbers. One of the main goals of this work is to compute such constants. It will be done in Chapter 5.

[10] We're neglecting the possibility that h_8 and $\beta_{2,3}^{(6)}$ can have higher (or lower) degree of transcendentality which cancel out between them.

Chapter 4.

The Dilatation Operator

In this chapter we compute the planar asymptotic dilatation operator of ABJM theory, up to eight loops, in the $SU(2) \times SU(2)$ sector. Its eigenvalues are the anomalous dimensions of long operators, *i.e.* operators that aren't affected by wrapping interactions at the considered perturbative order. In this sector, operators are made out only of the chiral superfields Z^A, W_B:

$$\mathcal{O}^{A_1 \cdots A_L}_{B_1 \cdots B_L} = \mathrm{Tr}[Z^{A_1} W_{B_1} \cdots Z^{A_L} W_{B_L}] \,, \tag{4.1}$$

where $A_j, B_j = 1, 2$. This sector is closed under renormalization, *i.e.* the operators (4.1) mix only among themselves. The restriction to the planar asymptotic case is motivated by the fact that, in this case, a large information on the dilatation operator can be obtained relying only on integrability of the theory. This chapter covers the first part of [34] and works out the details of the computations required to obtain the results contained there.

4.1. The Bethe Ansatz in $SU(2) \times SU(2)$ sector

Let's first collect the integrability tools we need for the construction of the dilatation operator. We just have to specialize the main results of the previous section to the $SU(2) \times SU(2)$ sector.

In the spin chain picture, the operators (4.1) are mapped to states of a circular alternating $\mathrm{XXX}_{\frac{1}{2}}$ Heisenberg spin chain, with the fields Z^A being interpreted as spins lying on odd sites and the W_A as lying on even sites. We will refer to fields Z^2 and W_2 as impurities and they correspond to "spin down" states. The ground state of length $2L$ is chosen to be

$$|0\rangle = \mathrm{Tr}[Z^1 W_1 \cdots Z^1 W_1]\,, \tag{4.2}$$

while the operators

$$\begin{aligned} |2k\rangle &= \mathrm{Tr}[Z^1 W_1 \cdots Z^1 W_2 \cdots Z^1 W_1]\,, \\ |2k+1\rangle &= \mathrm{Tr}[Z^1 W_1 \cdots Z^2 W_1 \cdots Z^1 W_1] \end{aligned} \tag{4.3}$$

represent states with a single excitation on the site $2k$ and $2k+1$ respectively. There are two kinds of one-magnon states, depending on their momentum p being excited on even or odd sites[1]:

$$\begin{aligned} |p\rangle_e &= \sum_{k=1}^{L} e^{ipk} |2k\rangle \\ |p\rangle_o &= \sum_{k=1}^{L} e^{ipk} |2k+1\rangle\,. \end{aligned} \tag{4.4}$$

Magnon dispersion relation is

$$E(p) = \frac{1}{2}\left(\sqrt{1 + 16h^2(\lambda)\sin^2\frac{p}{2}} - 1\right)\,, \tag{4.5}$$

where $h(\lambda)$ is the interpolating function, which has the weak-coupling expansion

$$h^2(\lambda) = \sum_{k=1}^{\infty} h_{2k}\lambda^{2k} \tag{4.6}$$

with $h_2 = 1$ and $h_4 = -4\zeta(2)$ the only known coefficients [22–24].

[1] In the sum, the identification $2L+1 \sim 1$ is understood.

We will need the expansion $E(p) = \sum_{k=1}^{\infty} E_{2k}(p)\lambda^{2k}$ of (4.5) up to eight loops. We get:

$$\begin{aligned}
E_2(p) &= 2 - 2\,\cos p\,,\\
E_4(p) &= -2\,(3+h_4) + 2\,(4-h_4)\cos p - 2\,\cos(2p)\,,\\
E_6(p) &= 40 - 12h_4 + 2h_6\\
&\quad + 2\,(8h_4 - h_6 - 30)\,\cos p\\
&\quad + 4\,(6-h_4)\,\cos(2p)\\
&\quad - 4\,\cos(3p)\,,\\
E_8(p) &= 2\,(-175 + 60h_4 - 3h_4^2 - 6h_6 + h_8)\\
&\quad + 2\,(280 - 90h_4 + 4h_4^2 + 8h_6 - h_8)\cos(p)\\
&\quad + 2\,(-140 + 36h_4 - h_4^2 - 2h_6)\cos(2p)\\
&\quad + 2\,(40 - 6h_4)\cos(3p)\\
&\quad - 10\,\cos(4p)\,.
\end{aligned} \tag{4.7}$$

The all-loop asymptotic Bethe equations for the alternating spin chain of length $2L$ can be derived from [18] and read

$$\begin{aligned}
\left[\frac{x(u_j+i/2)}{x(u_j-i/2)}\right]^L &= \prod_{k=1,k\neq j}^{M_u} \frac{u_j-u_k+i}{u_j-u_k-i}\, e^{i\theta(u_j,u_k)} \prod_{k=1}^{M_v} e^{i\theta(u_j,v_k)}\\
\left[\frac{x(v_j+i/2)}{x(v_j-i/2)}\right]^L &= \prod_{k=1,k\neq j}^{M_v} \frac{v_j-v_k+i}{v_j-v_k-i}\, e^{i\theta(v_j,v_k)} \prod_{k=1}^{M_u} e^{i\theta(v_j,u_k)}
\end{aligned} \tag{4.8}$$

while the momentum constraint is

$$\prod_{j=1}^{M_u} \frac{x(u_j+i/2)}{x(u_j-i/2)} \prod_{j=1}^{M_v} \frac{x(v_j+i/2)}{x(v_j-i/2)} = 1\,. \tag{4.9}$$

We have called u_i and v_i the Bethe roots of each $SU(2)$ factor. We also denoted by M_u and M_v the number of u and v roots. Moreover, we introduced the function

$$x(w) = \frac{w}{2}\left(1 + \sqrt{1 - 4\frac{h^2(\lambda)}{w^2}}\right)\,, \tag{4.10}$$

where $w = u, v$. The dressing phase is [18]

$$\theta(w_i, w_j) = \sum_{r=2}^{\infty} \sum_{s=r+1}^{\infty} \beta_{r,s}(\lambda)[q_r(w_i)q_s(w_j) - q_s(w_i)q_r(w_j)]\,, \tag{4.11}$$

where the coefficient functions $\beta_{r,s}(\lambda)$ can be expanded in λ as

$$\beta_{r,s}(\lambda) = \sum_{k=s-1}^{\infty} \beta_{r,s}^{(2k)} \lambda^{2k}\,. \tag{4.12}$$

It is given in terms of the quantities

$$q_r(w) = \frac{1}{r-1}\left(\frac{i}{x(w+i/2)^{r-1}} - \frac{i}{x(w-i/2)^{r-1}}\right)\,, \tag{4.13}$$

which are related to the eigenvalues of the conserved charges $\mathcal{Q}_r$, whose existence is ensured by integrability. In particular, $\mathcal{Q}_2$ is the hamiltonian of the integrable model and is identified with the dilatation operator of the gauge theory. The energy eigenvalues are given by

$$E = h^2(\lambda)\left(\sum_{j=1}^{M_u} q_2(u_j) + \sum_{j=1}^{M_v} q_2(v_j)\right) \tag{4.14}$$

Let's observe that, for two-impurity states, the Bethe Ansatz is simplified. The momentum constraint requires

$$w_1 = -w_2 \equiv w\,, \tag{4.15}$$

while the Bethe equations reduce to

$$\left(\frac{w+i/2}{w-i/2}\right)^L = \left(\frac{1+\sqrt{1-\frac{4h^2(\lambda)}{(w-i/2)^2}}}{1+\sqrt{1-\frac{4h^2(\lambda)}{(w+i/2)^2}}}\right)^L e^{i\theta(w,-w)} \tag{4.16}$$

for $M_u = M_v = 1$, and

$$\left(\frac{w+i/2}{w-i/2}\right)^{L-1} = \left(\frac{1+\sqrt{1-\frac{4h^2(\lambda)}{(w-i/2)^2}}}{1+\sqrt{1-\frac{4h^2(\lambda)}{(w+i/2)^2}}}\right)^{L} e^{i\theta(w,-w)} \tag{4.17}$$

for $M_u = 2$, $M_v = 0$ or $M_u = 0$, $M_v = 2$. For any L fixed, these equations can be solved order by order in perturbation theory. This is the basic tool for the computation of anomalous dimensions of long operators.

As we see from (4.8), the dressing factor introduces extra self-interactions for the roots and also couples the two $SU(2)$ spin chains at higher loops. In the proposed all-loop Bethe Ansatz [18] it is mentioned that the dressing phase starts contributing at eight loops, leading to the result that the two $SU(2)$ spin chains are decoupled up to six loops: this means that the two types of magnons, associated to the two $SU(2)$ factors, propagating on the spin chain, don't interact at this order. It was shown in [22] and [24], by direct computations, that this is true at four-loop order: indeed, the contributions to the dilatation operator of the respective diagrams that could lead to interactions between the two types of magnons cancel. The results of [26] imply that this is true also at six-loop order. This latter fact isn't so trivial and has the consequence that the coefficient $\beta_{2,3}^{(4)}$, which can be a priori present at six loops, actually vanishes. A similar result was valid in $\mathcal{N} = 4$ SYM in the $SU(2)$ secotr, where the coefficient of the dressing phase $\beta_{2,3}^{(2)}$ was equal to zero at order λ^2. In fact, the form of the dressing phase for the ABJM case is conjectured to be the same as for $\mathcal{N} = 4$ SYM [18]. From this fact we conjectured in Section 3.5 that the leading-order coefficient of the dressing phase should be $\beta_{2,3}^{(6)} = 4\zeta(3)$. It therefore should appear at eight loops in the dilatation operator. The next Chapter of this work is aimed to provide a perturbative test of this conjecture.

4.2. General procedure

Here we discuss the general procedure we used to construct the asymptotic dilatation operator of ABJM theory. It is similar to the procedure used to construct the dilatation operator of $\mathcal{N} = 4$ SYM, described in [79–81].

We have seen that the two-loop dilatation operator is integrable and that in the $SU(2) \times SU(2)$ sector is described by an alternating Heisenberg spin chain with next-to-nearest-neighbor interactions. We then assume integrability for the whole theory, as it is widely believed to be true. For the construction of the dilatation operator up to a given order λ^ℓ, it suffices to require *perturbative integrability*. This means that the integrable structure closes up to the considered perturbative order. More precisely, if

$$\mathcal{Q}_r(\lambda) = \sum_{k=0}^{\infty} \mathcal{Q}_r^{(2k)} \lambda^{2k} \tag{4.18}$$

is the perturbative expansion of the charges, we ask that the maximal range of $\mathcal{Q}_r^{(2k)}$ is $r + 2k - 1$, *i.e.* $\mathcal{Q}_r^{(2k)}$ acts on $r + 2k - 1$ adjacent spins, and that at ℓ loops

$$[\mathcal{Q}_r(\lambda), \mathcal{Q}_s(\lambda)] = \mathcal{O}(\lambda^{\ell+1}) \,. \tag{4.19}$$

The physical interpretation is the following: starting from an exactly integrable hamiltonian (which is next-to-nearest neighbor for ABJM theory), we deform it with local interactions of range linearly increasing with the perturbative order. Hence, at finite λ, one would obtain a long-ranged spin chain. In the same way, the charges commute exactly only if all the terms of the series (4.19) are taken into account. Let's note that, when the range $r + 2k - 1$ exceeds the length $2L$ of the spin chain, the notion of locality of an interaction breaks down and *wrapping interactions* typically appear. So, we work with *asymptotic states*, *i.e.* states of length $2L$ satisfying

$$L > \frac{\ell + r - 2}{2} \,, \tag{4.20}$$

where ℓ is the loop-order.

Let's now turn to the problem of the construction of the asymptotic ℓ-loop dilatation operator. Since the dilatation operator is identified with the charge $\mathcal{Q}_2$, on states of length $2L > \ell$ its perturbative expansion is

$$\mathcal{D}(\lambda) = L + \sum_{k=1}^{\infty} \mathcal{D}_{2k} \, \lambda^{2k} \,, \tag{4.21}$$

where the 0-loop term $2L$ is proportional to the identity and yields the classical dimension of an operator made up of $2L$ chiral superfields, while the second term is the quantum contribution and yields its anomalous dimension. The ℓ-loop dilatation operator D_ℓ has maximal range $\ell + 1$. Let's introduce a basis of operators for $\mathcal{D}$: the local interactions among spins are represented by products of permutations of next-to-neighbor spins, so we define the *permutation structures*

$$\{a_1, \dots, a_n\} = \sum_{j=1}^{L} P_{2j+a_1,2j+a_1+2} \cdots P_{2j+a_n,2j+a_n+2} \,, \tag{4.22}$$

where the permutation operators $P_{i,j}$ exchange spins at sites i and j. Indices are understood modulo $2L$. If $n = 0$, we denote $\{\,\} = L$. The range $\mathcal{R}$ of a permutation structure is

$$\mathcal{R} = \max(a_1, \dots, a_n) - \min(a_1, \dots, a_n) + 3 \,. \tag{4.23}$$

The integer n is called *length* of the permutation structure and, in terms of the associated Feynman diagrams, it coincides with the number of chiral and antichiral vertices. The permutation structures satisfy the following relations:

$$\begin{aligned}
\{\dots, a, a, \dots\} &= \{\dots, \dots\} \,; \\
\{\dots, a, b, \dots\} &= \{\dots, b, a, \dots\} \,, \qquad \text{if } |b - a| \neq 2 \,; \\
\{a, b, \dots\} &= \{a + 2m, b + 2m, \dots\} \,, \qquad m = 0, 1, 2, \dots \\
\{\dots, a, a + 2, a, \dots\} &= \{\dots, a, a + 2, \dots\} + \{\dots, a + 2, a, \dots\} - \{\dots, a, \dots\} \\
&\quad - \{\dots, a + 2, \dots\} + \{\dots, \dots\} \,.
\end{aligned} \tag{4.24}$$

Under hermitian conjugation and parity, they transform as

$$\begin{aligned}
\{a_1, \dots, a_n\}^\dagger &= \{a_n, \dots, a_1\} \,; \\
\mathcal{P}\{a_1, \dots, a_n\}\mathcal{P}^{-1} &= \{-a_1, \dots, -a_n\} \,.
\end{aligned} \tag{4.25}$$

If $\{a, b, c, \ldots\}$ contains only even/odd entries, it will act, regarding single-impurity states, on states with impurity on even/odd sites only:

$$\begin{aligned}\{\ldots, a, \ldots\}|2k\rangle &= \sum_{j=1}^{L} \cdots P_{2j+a,2j+a+2} \cdots |2k\rangle \\ &= \cdots P_{a,a+2} \cdots |\uparrow\uparrow\uparrow \cdots \downarrow \cdots \uparrow\uparrow\rangle \\ &\quad + \cdots P_{a+2,a+4} \cdots |\uparrow\uparrow\uparrow \cdots \downarrow \cdots \uparrow\uparrow\rangle\end{aligned} \tag{4.26}$$

where the impurity (here written as a "spin-down") is at $2k^{th}$ position. Because of the sum, the permutation operator $P_{a,a+2}$ "goes" through the entire chain and permutes the same spin twice. This applies for every permutation operator present in the permutation structure.

In Table 4.1 we list all the independent permutation structures that appear in the dilatation operator up to eight loops. In order to read a basis for the dilatation operator at ℓ loops, one has to include all the permutation structures up to range $\mathcal{R} = \ell + 1$, neglecting the ones in parenthesis, because they appear at the next loop order.

$\mathcal{R}$	**Basis of permutation structures**
1	{ }
2	-
3	{0}
4	{0,1}
5	{0,2}, {2,0} ({0,1,2}, {2,1,0})
6	{0,3} {0,1,3}, {0,3,1} {0,2,3}, {2,0,3} ({0,1,2,3}, {2,1,0,3}, {0,3,2,1}, {2,3,0,1})
7	{0,4} {0,2,4}, {4,2,0}, {0,4,2}, {2,0,4} ({0,1,4}, {0,3,4}) ({2,0,4,2}) ({0,1,2,4}, {2,1,0,4}, {4,1,2,0}, {4,1,0,2}) ({0,1,3,4}, {0,3,1,4}) ({0,2,3,4}, {2,0,3,4}, {0,4,3,2}, {4,3,2,0})
8	{0,5}, {0,1,5}, {0,4,5} {0,2,5}, {2,0,5}, {0,3,5}, {0,5,3} {0,1,3,5}, {0,3,1,5}, {0,1,5,3}, {5,3,1,0} {0,2,3,5}, {2,0,3,5}, {0,2,5,3}, {2,0,5,3} {0,2,4,5}, {2,0,4,5}, {0,4,2,5}, {4,2,0,5}
9	{0,6} {0,2,6}, {2,0,6}, {0,4,6}, {0,6,4} {0,3,6} {0,2,4,6}, {0,2,6,4}, {0,4,2,6}, {2,0,4,6} {2,0,6,4}, {0,6,4,2}, {4,2,0,6}, {6,4,2,0}

Table 4.1.: Permutation structures needed up to eight loops, grouped according to their range $\mathcal{R}$. Permutation structures of odd (even) range $\mathcal{R}$ appear at $\mathcal{R}-1$ ($\mathcal{R}$) loops, apart from the ones in parenthesis, which appear at the next loop order. Permutation structures contributing at ten loops or beyond are not written. Note that, in order to find the complete basis, one has to add $\{\ldots, a+1, \ldots\}$ to each permutation structure $\{\ldots, a, \ldots\}$ listed here.

We will need the action of permutation structures on one-particle states:

$$
\begin{aligned}
\{\,\}|2k\rangle &= L|2k\rangle \\
\{0\}|2k\rangle &= |2k-2\rangle + |2k+2\rangle + (L-2)|2k\rangle \\
\{0,2\}|2k\rangle &= |2k-4\rangle + 2\,|2k+2\rangle + (L-3)|2k\rangle \\
\{2,0\}|2k\rangle &= 2\,|2k-2\rangle + |2k+4\rangle + (L-3)|2k\rangle \\
\{0,4\}|2k\rangle &= 2\,|2k-2\rangle + 2\,|2k+2\rangle + (L-4)|2k\rangle \\
\{0,2,4\}|2k\rangle &= |2k-6\rangle + 3\,|2k+2\rangle + (L-4)|2k\rangle \\
\{4,2,0\}|2k\rangle &= 3\,|2k-2\rangle + |2k+6\rangle + (L-4)|2k\rangle \\
\{2,0,4\}|2k\rangle &= |2k-4\rangle + |2k+2\rangle + |2k-2\rangle + |2k+4\rangle + (L-4)|2k\rangle \\
\{0,4,2\}|2k\rangle &= |2k-2\rangle + |2k-4\rangle + |2k+4\rangle + |2k+2\rangle + (L-4)|2k\rangle \\
\{2,0,4,2\}|2k\rangle &= 2\,|2k-4\rangle + 2\,|2k+4\rangle + (L-4)|2k\rangle \\
\{0,6\}|2k\rangle &= 2\,|2k-2\rangle + 2\,|2k+2\rangle + (L-4)|2k\rangle \\
\{0,2,6\}|2k\rangle &= |2k-2\rangle + 3\,|2k+2\rangle + |2k-4\rangle + (L-5)|2k\rangle \\
\{2,0,6\}|2k\rangle &= 3\,|2k-2\rangle + |2k+2\rangle + |2k+4\rangle + (L-5)|2k\rangle \\
\{0,4,6\}|2k\rangle &= |2k-4\rangle + 3\,|2k+2\rangle + |2k-2\rangle + (L-5)|2k\rangle \\
\{0,6,4\}|2k\rangle &= 3\,|2k-2\rangle + |2k+4\rangle + |2k+2\rangle + (L-5)|2k\rangle \\
\{0,2,4,6\}|2k\rangle &= |2k-8\rangle + 4\,|2k+2\rangle + (L-5)|2k\rangle \\
\{6,4,2,0\}|2k\rangle &= 4\,|2k-2\rangle + |2k+8\rangle + (L-5)|2k\rangle \\
\{0,2,6,4\}|2k\rangle &= |2k-2\rangle + |2k-6\rangle + |2k+4\rangle + 2\,|2k+2\rangle + (L-5)|2k\rangle \\
\{2,0,4,6\}|2k\rangle &= |2k-6\rangle + 2\,|2k+2\rangle + |2k-2\rangle + |2k+4\rangle + (L-5)|2k\rangle \\
\{0,6,4,2\}|2k\rangle &= 2\,|2k-2\rangle + |2k-4\rangle + |2k+6\rangle + |2k+2\rangle + (L-5)|2k\rangle \\
\{4,2,0,6\}|2k\rangle &= 2\,|2k-2\rangle + |2k+2\rangle + |2k+6\rangle + |2k-4\rangle + (L-5)|2k\rangle \\
\{0,4,2,6\}|2k\rangle &= 2\,|2k-4\rangle + 2\,|2k+2\rangle + |2k+4\rangle + (L-5)|2k\rangle \\
\{2,0,6,4\}|2k\rangle &= |2k-4\rangle + 2\,|2k+4\rangle + 2\,|2k-2\rangle + (L-5)|2k\rangle\,.
\end{aligned}
\tag{4.27}
$$

Similar results hold for the action of the permutation structures on states with one impurity at odd sites $|2k+1\rangle$. Obviously, the action of mixed permutation structures

on one-particle states can be reduced to the ones above:

$$\begin{aligned} \{0,1\}|2k\rangle &= \{0\}|2k\rangle \\ \{0,1\}|2k+1\rangle &= \{1\}|2k+1\rangle \\ \{0,1,2\}|2k\rangle &= \{0,2\}|2k\rangle \\ \{0,1,2\}|2k+1\rangle &= \{1\}|2k+1\rangle \\ &\dots \end{aligned} \tag{4.28}$$

In order to construct the ℓ-loop dilatation operator, we first write the most general combination of permutation structures up to range $\ell+1$. Then, we fix the unknown coefficients imposing the following conditions:

1. Impose hermiticity and parity invariance:

$$\mathcal{D}_\ell^\dagger = \mathcal{D}_\ell\,, \qquad\qquad \mathcal{P}\mathcal{D}_\ell\mathcal{P}^{-1} = \mathcal{D}_\ell\,; \tag{4.29}$$

2. Require that the vacuum state is protected (*i.e.*, it has zero energy):

$$\mathcal{D}_\ell\,|0\,\rangle = 0\,; \tag{4.30}$$

3. Impose the magnon dispersion relation on one-impurity states on odd and even sites (let E_ℓ be the ℓ-loop coefficient of (4.5) in the small λ expansion):

$$\mathcal{D}_\ell\,|p\rangle_{o,e} = E_\ell(p)\,|p\rangle_{o,e}\,; \tag{4.31}$$

4. Use the asymptotic Bethe ansatz on two-impurity states in order to fix the remaining parameters.

Point 4 works in the following manner:

- Fix a length of the spin chain, with $L > \ell/2$, so that wrapping interactions are absent;

- Find a basis of two-impurity operators of length $2L$. Note that there are two types of such bases, as the impurities can be both on even (or odd) sites or one on even and the other on odd sites[2];
- Explicitly diagonalize the matrix representation of $\mathcal{D}(\lambda) = \sum_{k=1}^{\ell/2} \mathcal{D}_{2k}\, \lambda^{2k}$ on the basis of length $2L$ states;
- Compute the eigenvalues from the asymptotic Bethe equations (4.8) and compare them with those found in the previous step, containing the unknown parameters.

The explicit diagonalization of the dilatation operator and the solution of the Behte equations can be performed with `Mathematica`. We collect in Appendix C the relevant routines.

At the end of the procedure, some parameters may remain unfixed: some of them are related to *similarity transformations* of the dilatation operator:

$$\mathcal{D}' = e^{-i\chi} \mathcal{D} e^{i\chi} \,. \tag{4.32}$$

They don't appear in the spectrum but only affect the eigenvectors. Therefore, they can be regarded as unphysical and their values depend on the renormalization scheme.

Some other parameters may depend on some coefficients of the dressing phase and cannot be fixed through the Bethe Ansatz.

Other possible unknown parameters can be fixed through conservation of the first higher charge $\mathcal{Q}_3$: since the charge $\mathcal{Q}_3$ should be conserved up to order ℓ, this means that we have to impose

$$\sum_{k=1}^{\ell/2} [\mathcal{D}_{2k}, \mathcal{Q}_3^{(\ell+2-2k)}] = 0 \,. \tag{4.33}$$

This requires the construction of the $\mathcal{Q}_3$ up to order ℓ with a procedure similar to that described for the dilatation operator, with the difference that $\mathcal{Q}_3^{(\ell)}$ has odd parity and is anti-hermitian. Since $\mathcal{Q}_3$ has higher maximal range, $\ell + 2$, at high loops it can be quite complicated to compute.

[2] for the same reason, we have seen in section 4.1 that there are two types of Bethe equations for two-impurity states

4.3. Construction of the dilatation operator

We are now in a position to compute the perturbative asymptotic dilatation operator of ABJM theory up to eight loops.

First of all we compute the two-loop dilatation operator. A basis of permutation structures with range up to $\mathcal{R} = 3$ is

$$\{\}\,, \qquad \{0\}\,, \qquad \{1\}\,, \tag{4.34}$$

so the two-loop dilatation operator will be of the form

$$\mathcal{D}_2 = a\,\{\} + b\,\{0\} + c\,\{1\}\,. \tag{4.35}$$

Hermiticity implies that the coefficients a, b, c are real. We now use the relations (4.27) to fix these coefficients. Zero-energy of the vacuum gives the equation

$$a + b + c = 0\,. \tag{4.36}$$

Let's now compute the action of the four-loop dilatation operator on one-magnon states (4.4). Using the relations (4.27), for $|p\rangle_e$ we obtain

$$\begin{aligned}
\mathcal{D}_2|p\rangle_e =& aL\,|p\rangle_e + c\,L|p\rangle_e + b\sum_{k=1}^{L} e^{ipk}\{0\}|2k\rangle \\
=& a\,L|p\rangle_e + c\,L|p\rangle_e + b\sum_{k=1}^{L} e^{ipk}\left(|2k+2\rangle + |2k-2\rangle + (L-2)|2k\rangle\right) \\
=& (a+b+c)L|p\rangle_e + b\sum_{k=1}^{L} e^{ip(k-1)}e^{ip}|2(k-1)\rangle \\
&+ b\sum_{k=1}^{L} e^{ip(k+1)}e^{-ip}|2(k+1)\rangle - 2b|p\rangle_e \\
=& b\,e^{ip}|p\rangle_e + b\,e^{-ip}|p\rangle_e - 2b|p\rangle_e \\
=& 2b\,(\cos p - 1)|p\rangle_e\,.
\end{aligned} \tag{4.37}$$

A similar result holds for $|p\rangle_o$. So we have:

$$\begin{aligned}\mathcal{D}_2\,|p\rangle_e &= 2b\,(\cos p - 1)\,|p\rangle_e\,,\\ \mathcal{D}_2\,|p\rangle_o &= 2c\,(\cos p - 1)\,|p\rangle_e\,.\end{aligned} \tag{4.38}$$

Comparing these eigenvalues with (4.7) we obtain $b = c = -1$, so that:

$$\mathcal{D}_2 = 2\,\{\,\} - \{0\} - \{1\}\,, \tag{4.39}$$

which is exactly the same result of Minahan and Zarembo [12] and Bak and Rey [13].

Let's now compute the four-loop dilatation operator: this was done in [22] and, independently, in [25]. The authors of [22] used a procedure similar to ours, up to point 3. This fixes all but one coefficient in the dilatation operator, namely the coefficient of $\{0,1\}+\{1,2\}$. We simply use point 4 of our procedure on length six two-impurity states to prove that it vanishes, without computing any Feynman diagram! A basis of permutation structures up to range $\mathcal{R}=5$ is[3]

$$\{\,\}\,,\quad \{0\}\,,\quad \{1\}\,,\quad \{0,1\}\,,\quad \{1,2\}\,,\quad \{0,2\}\,,\quad \{2,0\}\,,\quad \{1,3\}\,,\quad \{3,1\}\,. \tag{4.40}$$

We can conveniently write the dilatation operator as

$$\mathcal{D}_4 = \; a\,\{\,\} + \mathcal{D}_{4,even} + \mathcal{D}_{4,odd} + \mathcal{D}_{4,mixed}\,, \tag{4.41}$$

where $\mathcal{D}_{4,even}$ and $\mathcal{D}_{4,odd}$ contain permutation structures with only even/odd entries, thus acting non trivially only on states with impurities on even/odd sites respectively, while $\mathcal{D}_{4,mixed}$ contains permutation structures with both even and odd entries:

$$\begin{aligned}\mathcal{D}_{4,even} &= b\,\{0\} + g_1\,\{0,2\} + g_2\,\{2,0\}\,,\\ \mathcal{D}_{4,odd} &= c\,\{1\} + f_1\,\{1,3\} + f_2\,\{3,1\}\,,\\ \mathcal{D}_{4,mixed} &= d_1\,\{0,1\} + d_2\,\{1,2\}\,,\end{aligned} \tag{4.42}$$

[3] All other permutation structures of range five, such as $\{0,1,2\}$, don't contribute at four loops. It is easily seen from the chiral graphs, which are the graphs displaying the minimum number of loops, associated to those permutation structures.

Hermiticity implies that the coefficients a, b, c, d_1, d_2 are real and that $f_1^* = f_2$, $g_1^* = g_2$. Furthermore, parity invariance implies that $d_1 = d_2$, $f_1 = f_2$ and $g_1 = g_2$. Setting $d = d_1 = d_2$, $f = f_1 = f_2$ and $g = g_1 = g_2$, we can, then, write the dilatation operator as

$$\mathcal{D}_4 = a\{\,\} + b\{0\} + c\{1\} + d(\{0,1\} + \{1,2\}) + g(\{0,2\} + \{2,0\}) + f(\{1,3\} + \{3,1\})\,, \tag{4.43}$$

where all the coefficients are now real. Zero-energy of the vacuum implies the following relation:

$$a + b + c + 2d + 2f + 2g = 0\,. \tag{4.44}$$

The action of $\mathcal{D}_4$ on the even one-magnon state is

$$\mathcal{D}_4|p\rangle_e = [(-2b - 4d - 6g) + (2b + 4d + 4g)\,\cos p + 2g\,\cos(2p)]\,|p\rangle_e\,. \tag{4.45}$$

A similar result holds for $|p\rangle_o$. Imposing the validity of the four-loop dispersion relation on one-magnon states we get the following equations[4]:

$$\begin{aligned} c + 2d + 3f &= 3 - h_4\,, \\ b + 2d + 3g &= 3 - h_4\,, \\ f = g &= -1\,. \end{aligned} \tag{4.46}$$

This fixes all but one coefficient, which will be found as described in point 4 of the procedure described in section 4.2. In order to avoid wrapping interactions that would break the Bethe ansatz procedure, we have to focus on length six two-impurity states. The matrix representation of the dilatation operator on a basis of states with two impurities of different types has the following eigenvalues:

$$\gamma_0 = 0\,, \quad \gamma_1 = 6\lambda^2 + (-18 + 3d + 6h_4)\lambda^4\,, \quad \gamma_2 = 6\lambda^2 + (-18 + 9d + 6h_4)\lambda^4\,, \tag{4.47}$$

while the Bethe Ansatz applied to a spin chain of length six gives the energy eigenvalue

$$E = 6\lambda^2 + (-18 + 6h_4)\lambda^4\,. \tag{4.48}$$

[4]We list here only the independent equations.

We therefore conclude that $d = 0$; this implies that $\mathcal{D}_{4,mixed} = 0$, as is consistent with the fact that the two types of magnons don't interact at 4 loops. The four-loop dilatation operator, then, reads

$$\mathcal{D}_4 = 2\,(-4 + h_4)\,\{\,\} + (6 - h_4)\,(\{0\} + \{1\}) - (\{0,2\} + \{2,0\} + \{1,3\} + \{3,1\}). \tag{4.49}$$

The six-loop dilatation operator was computed in [26] through Feynman diagrammatics. Here, we apply our technique to rederive it: the procedure of section 4.2 turns out to be very simple also in the six-loop case, and, assuming that the dressing phase doesn't appear at six loops as mentioned in section 4.1, it allows to completely fix the dilatation operator without computing any diagram. However, here, we don't assume this fact, but directly verify that $\beta^{(4)}_{2,3} = 0$, making use of a single input from the results of [26].

Let's start by writing the six-loop dilatation operator as the most general linear combination of the permutation structures contributing at six loops. We have

$$\mathcal{D}_6 = a\,\{\,\} + \mathcal{D}_{6,even} + \mathcal{D}_{6,odd} + \mathcal{D}_{6,mixed}\,, \tag{4.50}$$

where

$$\begin{aligned}
\mathcal{D}_{6,even} &= b\,\{0\} + d_1\,\{0,2\} + d_2\,\{2,0\} + m\,\{0,4\} \\
&\quad + l_1\,\{0,2,4\} + l_2\,\{4,2,0\} + l_3\,\{2,0,4\} + l_4\,\{0,4,2\}\,, \\
\mathcal{D}_{6,odd} &= \mathcal{D}_{6,even}(\{\ldots,a,\ldots\} \leftrightarrow \{\ldots,a+1,\ldots\})\,, \\
\mathcal{D}_{6,mixed} &= c\,\{0,1\} + c'\,\{1,2\} + e\,\{0,3\} + e'\,\{1,4\} \\
&\quad + f_1\,\{0,1,2\} + f_2\,\{2,1,0\} + f_1'\,\{1,2,3\} + f_2'\,\{3,2,1\} \\
&\quad + g_1\,\{0,1,3\} + g_2\,\{0,3,1\} + g_1'\,\{1,2,4\} + g_2'\,\{1,4,2\} \\
&\quad + g_3\,\{0,2,3\} + g_4\,\{2,0,3\} + g_3'\,\{1,3,4\} + g_4'\,\{3,1,4\}\,.
\end{aligned} \tag{4.51}$$

Imposing hermiticity on $\mathcal{D}_6$, we find that the coefficients a, b, c, e, m are real, while the others are restricted by the following relations:

$$\begin{aligned}
&d_1 = d_2^* \quad l_1 = l_2^* \quad l_3 = l_4^* \quad f_1 = f_2^* \quad g_1 = g_2^* \quad g_3 = g_4^* \\
&f_1' = f_2'^* \quad g_1' = g_2'^* \quad g_3' = g_4'^*\,.
\end{aligned} \tag{4.52}$$

Parity invariance, then, gives the following relations:

$$\begin{aligned} &d_1 = d_2 \equiv d \quad f_1 = f_2 \equiv f \quad l_1 = l_2 \equiv l \quad c = c' \quad e = e' \\ &f_1' = f_2' \equiv f' \quad g_1 = g_4' \quad g_2 = g_3' \quad g_3 = g_2' \quad g_4 = g_1' \quad f_1' = f_2 \quad f_2' = f_1 \,. \end{aligned} \tag{4.53}$$

Combining hermiticity and parity invariance, we have

$$g_1 = g + i\,\epsilon_1 \quad g_3 = \tilde{g} + i\,\epsilon_2 \quad l_3 = \tilde{l} + i\,\epsilon_3 \tag{4.54}$$

We can rewrite the dilatation operator as

$$\begin{aligned} \mathcal{D}_{6,even} = {} & b\,\{0\} + d\,(\{0,2\} + \{2,0\}) + m\,\{0,4\} \\ & + l\,(\{0,2,4\} + \{4,2,0\}) + \tilde{l}\,(\{2,0,4\} + \{0,4,2\}) \\ & + i\,\epsilon_3\,(\{2,0,4\} - \{0,4,2\}) \end{aligned} \tag{4.55}$$

$$\begin{aligned} \mathcal{D}_{6,mixed} = {} & c\,(\{0,1\} + \{1,2\}) + e\,(\{0,3\} + \{1,4\}) \\ & + f\,(\{0,1,2\} + \{2,1,0\} + \{1,2,3\} + \{3,2,1\}) \\ & + g\,(\{0,1,3\} + \{0,3,1\} + \{1,3,4\} + \{3,1,4\}) \\ & + i\,\epsilon_1\,(\{0,1,3\} - \{0,3,1\} - \{1,3,4\} + \{3,1,4\}) \\ & + \tilde{g}\,(\{0,2,3\} + \{2,0,3\} + \{1,4,2\} + \{1,2,4\}) \\ & + i\,\epsilon_2\,(\{0,2,3\} - \{2,0,3\} + \{1,4,2\} - \{1,2,4\}) \end{aligned} \tag{4.56}$$

Zero-energy of the vacuum gives:

$$a + 2b + 4d + 4l + 4\tilde{l} + 2m + 2c + 2e + 4f + 4g + 4\tilde{g} = 0\,. \tag{4.57}$$

The action of $\mathcal{D}_6$ on $|p\rangle_e$ is

$$\begin{aligned}\mathcal{D}_6|p\rangle_e = [&(-2b-4c-6d-4e-10f-8g-12\tilde{g}\\ &-8l-8\tilde{l}-4m)\\ &+2\,(b+2c+2d+2e+4f+4g+4\tilde{g}\\ &+3l+2\tilde{l}+2m)\cos(p)\\ &+2\,(d+f+2\tilde{g}+2\tilde{l})\cos(2p)\\ &+2l\cos(3p)\,]\,|p\rangle_e\end{aligned} \tag{4.58}$$

Repeating the analogous computation with $|p\rangle_o$ and comparing these eigenvalues with E_6 given in (4.7) we find the following equations:

$$\begin{aligned}&l=-2\,;\\ &d+f+2\tilde{g}+2\tilde{l}=2\,(6-h_4)\,;\\ &d+f+2g+2\tilde{l}=2\,(6-h_4)\,;\\ &b+2c+2d+2e+4f+4g+4\tilde{g}\\ &\qquad+3l+2\tilde{l}+2m=-30+8h_4-h_6\,;\end{aligned} \tag{4.59}$$

from second and third equation of (4.59) it follows that $\tilde{g}=g$. Solving (4.59) for b,d we find

$$\begin{aligned}b&=-48-2c-2e-2f-4g+12h_4-h_6+2\tilde{l}-2m\,,\\ d&=12-f-2g-2h_4-2\tilde{l}\,.\end{aligned} \tag{4.60}$$

Now we consider two-impurity states. In order to avoid wrapping, we have to start from $L=4$. For $L=4$ (length 8 states), comparing the eigenvalues of the dilatation operator with the solutions of the Bethe equations we get

$$\tilde{l}=0\,,\qquad 2m=4-3\,\beta^{(4)}_{2,3}\,,\qquad f=0\,,\qquad g=0\,,\qquad c+e=0\,. \tag{4.61}$$

For $L=5$ (length 10 states) we get

$$c=0\,,\qquad e=0\,. \tag{4.62}$$

We therefore have explicitly shown that $\mathcal{D}_{6,mixed} = 0$. It means that the two types of magnons don't interact at six loops, *i.e.* the two $SU(2)$ factors are decoupled, as claimed in [18]. The results of [26] (obtained with an independent approach) imply, in our notations, that $m = 2$. From (4.61) we thus see that $\beta_{2,3}^{(4)}$ must vanish, as conjectured in Section 3.5.

The six-loop dilatation operator is then given by:

$$\begin{aligned}\mathcal{D}_6 = &\, 2\,(30 - 8h_4 + h_6)\,\{\,\} + (-52 + 12h_4 - h_6)\,(\{0\} + \{1\}) \\ &+ 2\,(\{0,4\} + \{1,5\}) + (12 - 2h_4)\,(\{0,2\} + \{2,0\} + \{1,3\} + \{3,1\}) \\ &- 2\,(\{0,2,4\} + \{4,2,0\} + \{1,3,5\} + \{5,3,1\}) \\ &+ i\,\epsilon_3\,(\{2,0,4\} - \{0,4,2\} - \{1,5,3\} + \{3,1,5\})\end{aligned} \tag{4.63}$$

and coincides with that presented in [26] up to non physical parameters. Since the coefficient $\beta_{2,3}^{(4)}$ of the dressing phase, which could be a priori present according to 4.12, actually vanishes, the first contribution to the dressing phase should be searched at eight loops. We are thus led to consider the eight-loop dilatation operator: since it isn't known at present, we now use our procedure to constrain it as much as possible.

Its general representation in the basis of permutation structures is:

$$\mathcal{D}_8 = a\,\{\,\} + \mathcal{D}_{8,even} + \mathcal{D}_{8,odd} + \mathcal{D}_{8,mixed}\,, \tag{4.64}$$

where

$$\begin{aligned}\mathcal{D}_{8,even} = &\, b\,\{0\} + c_1\,\{0,2\} + c_2\,\{2,0\} + d\,\{0,4\} \\ &+ e_1\,\{0,2,4\} + e_2\,\{0,4,2\} + e_3\,\{2,0,4\} + e_4\,\{4,2,0\} + f\,\{0,6\} \\ &+ g_1\,\{0,2,6\} + g_2\,\{2,0,6\} + g_3\,\{0,4,6\} + g_4\,\{0,6,4\} \\ &+ l_1\,\{0,2,4,6\} + l_2\,\{0,2,6,4\} + l_3\,\{0,4,2,6\} + l_4\,\{2,0,4,6\} \\ &+ l_5\,\{2,0,6,4\} + l_6\,\{0,6,4,2\} + l_7\,\{4,2,0,6\} + l_8\,\{6,4,2,0\} \\ &+ m\,\{2,0,4,2\}\end{aligned} \tag{4.65}$$

$$\mathcal{D}_{8,odd} = \mathcal{D}_{8,even}(\{\ldots,a,\ldots\} \leftrightarrow \{\ldots,a+1,\ldots\}) \tag{4.66}$$

$$\begin{aligned}\mathcal{D}_{8,mixed} =& \alpha_1 \{0,1\} + \alpha_2 \{0,1,2\} + \alpha_3 \{2,1,0\} + \alpha_4 \{0,3\} \\ &+ \alpha_5 \{0,1,3\} + \alpha_6 \{0,3,1\} + \alpha_7 \{0,2,3\} + \alpha_8 \{2,0,3\} \\ &+ \alpha_9 \{0,1,2,3\} + \alpha_{10} \{2,1,0,3\} + \alpha_{11} \{0,3,2,1\} + \alpha_{12} \{2,3,0,1\} \\ &+ \alpha_{13} \{0,1,4\} + \alpha_{14} \{0,3,4\} + \alpha_{15} \{0,1,2,4\} + \alpha_{16} \{2,1,0,4\} \\ &+ \alpha_{17} \{4,1,2,0\} + \alpha_{18} \{4,1,0,2\} + \alpha_{19} \{0,1,3,4\} + \alpha_{20} \{0,3,1,4\} \\ &+ \alpha_{21} \{0,2,3,4\} + \alpha_{22} \{2,0,3,4\} + \alpha_{23} \{0,4,3,2\} + \alpha_{24} \{4,3,2,0\} \\ &+ \alpha_{25} \{0,5\} + \alpha_{26} \{0,1,5\} + \alpha_{27} \{0,2,5\} + \alpha_{28} \{2,0,5\} + \alpha_{29} \{0,3,5\} \\ &+ \alpha_{30} \{0,5,3\} + \alpha_{31} \{0,4,5\} + \alpha_{32} \{0,1,3,5\} + \alpha_{33} \{0,3,1,5\} \\ &+ \alpha_{34} \{0,1,5,3\} + \alpha_{35} \{5,3,1,0\} + \alpha_{36} \{0,2,3,5\} + \alpha_{37} \{2,0,3,5\} \\ &+ \alpha_{38} \{0,2,5,3\} + \alpha_{39} \{2,0,5,3\} + \alpha_{40} \{0,2,4,5\} + \alpha_{41} \{2,0,4,5\} \\ &+ \alpha_{42} \{0,4,2,5\} + \alpha_{43} \{4,2,0,5\} + \alpha_{44} \{0,3,6\} + (\{\ldots, a \leftrightarrow a+1, \ldots\})\end{aligned} \tag{4.67}$$

From now on we will focus only on $\mathcal{D}_{8,even}$ and $\mathcal{D}_{8,mixed}$, since the coefficients of $\mathcal{D}_{8,odd}$ are the same of $\mathcal{D}_{8,even}$. The hermitian conjugate of $\mathcal{D}_{8,even}$ is

$$\begin{aligned}\mathcal{D}^{\dagger}_{8,even} =& b^* \{0\} + c_2^* \{0,2\} + c_1^* \{2,0\} + d^* \{0,4\} \\ &+ e_4^* \{0,2,4\} + e_3^* \{0,4,2\} + e_2^* \{2,0,4\} + e_1^* \{4,2,0\} + f^* \{0,6\} \\ &+ g_2^* \{0,2,6\} + g_1^* \{2,0,6\} + g_4^* \{0,4,6\} + g_3^* \{0,6,4\} \\ &+ l_8^* \{0,2,4,6\} + l_7^* \{0,2,6,4\} + l_5^* \{0,4,2,6\} + l_6^* \{2,0,4,6\} \\ &+ l_3^* \{2,0,6,4\} + l_4^* \{0,6,4,2\} + l_2^* \{4,2,0,6\} + l_1^* \{6,4,2,0\} \\ &+ m^* \{2,0,4,2\}\end{aligned} \tag{4.68}$$

Imposing hermiticity $\mathcal{D}_{8,even} = \mathcal{D}^{\dagger}_{8,even}$ we obtain the first restrictions on the coefficients:

$$\begin{aligned} a =& a^* \quad b = b^* \quad c_1 = c_2^* \quad d = d^* \quad e_1 = e_4^* \quad e_2 = e_3^* \quad f = f^* \\ g_1 =& g_2^* \quad g_3 = g_4^* \quad l_1 = l_8^* \quad l_2 = l_7^* \quad l_3 = l_5^* \quad l_4 = l_6^* \quad m = m^* \,. \end{aligned} \tag{4.69}$$

Similar computations for $\mathcal{D}_{8,mixed}$ give

$$\begin{aligned}
&\alpha_2^* = \alpha_3 \quad \alpha_5^* = \alpha_6 \quad \alpha_7^* = \alpha_8 \quad \alpha_9^* = \alpha_{12} \quad \alpha_{10}^* = \alpha_{11} \quad \alpha_{15}^* = \alpha_{17} \quad \alpha_{16}^* = \alpha_{18} \\
&\alpha_{19}^* = \alpha_{20} \quad \alpha_{21}^* = \alpha_{24} \quad \alpha_{22}^* = \alpha_{23} \quad \alpha_{27}^* = \alpha_{28} \quad \alpha_{29}^* = \alpha_{30} \quad \alpha_{32}^* = \alpha_{35} \quad \alpha_{33}^* = \alpha_{34} \\
&\alpha_{36}^* = \alpha_{39} \quad \alpha_{37}^* = \alpha_{38} \quad \alpha_{40}^* = \alpha_{43} \quad \alpha_{41}^* = \alpha_{42} \quad \alpha_{26}^* = \alpha_{26} \quad \alpha_{31}^* = \alpha_{31} \ \ldots
\end{aligned} \tag{4.70}$$

where the ellipses denote real coefficients. Let's now compute now the parity conjugate of $\mathcal{D}_{8,even}$:

$$\begin{aligned}
\mathcal{P}\mathcal{D}_{8,even}\mathcal{P}^{-1} = {} & b\,\{0\} + c_2\,\{0,2\} + c_1\,\{2,0\} + d\,\{0,4\} \\
& + e_4\,\{0,2,4\} + e_2\,\{0,4,2\} + e_3\,\{2,0,4\} + e_1\,\{4,2,0\} + f\,\{0,6\} \\
& + g_4\,\{0,2,6\} + g_3\,\{2,0,6\} + g_2\,\{0,4,6\} + g_1\,\{0,6,4\} \\
& + l_8\,\{0,2,4,6\} + l_6\,\{0,2,6,4\} + l_5\,\{0,4,2,6\} + l_7\,\{2,0,4,6\} \\
& + l_3\,\{2,0,6,4\} + l_2\,\{0,6,4,2\} + l_4\,\{4,2,0,6\} + l_1\,\{6,4,2,0\} \\
& + m\,\{2,0,4,2\}
\end{aligned} \tag{4.71}$$

Imposing parity invariance $\mathcal{D}_{8,even} = \mathcal{P}\mathcal{D}_{8,even}\mathcal{P}^{-1}$ we obtain other restrictions:

$$\begin{aligned}
&c_1 = c_2 \equiv c \quad e_1 = e_4 \quad g_1 = g_4 \quad g_2 = g_3 \\
&l_1 = l_8 \quad l_2 = l_6 \quad l_3 = l_5 \quad l_4 = l_7\,.
\end{aligned} \tag{4.72}$$

Similarly for $\mathcal{D}_{8,mixed}$ we get

$$\begin{aligned}
&\alpha_2 = \alpha_3 \quad \alpha_5 = \alpha_8 \quad \alpha_6 = \alpha_7 \quad \alpha_9 = \alpha_{12} \quad \alpha_{13} = \alpha_{14} \quad \alpha_{15} = \alpha_{24} \\
&\alpha_{16} = \alpha_{22} \quad \alpha_{17} = \alpha_{21} \quad \alpha_{18} = \alpha_{23} \quad \alpha_{19} = \alpha_{20} \quad \alpha_{26} = \alpha_{31} \quad \alpha_{27} = \alpha_{30} \\
&\alpha_{28} = \alpha_{29} \quad \alpha_{32} = \alpha_{43} \quad \alpha_{33} = \alpha_{41} \quad \alpha_{34} = \alpha_{42} \quad \alpha_{35} = \alpha_{40} \quad \alpha_{36} = \alpha_{39}
\end{aligned} \tag{4.73}$$

In $\mathcal{D}_{8,mixed}$, besides $\{\ldots, a, \ldots\}$, one has to consider $\{\ldots, a+1, \ldots\}$, but this doesn't add new coefficients, because under parity we have

$$\begin{aligned}
\mathcal{P}\{1,2\}\mathcal{P}^{-1} &= \{0,1\} \\
\mathcal{P}\{1,2,3\}\mathcal{P}^{-1} &= \{2,1,0\} \\
&\ldots
\end{aligned} \tag{4.74}$$

Combining hermiticity and parity invariance, we have

$$\begin{aligned} g_1 =& g_4 \equiv g + i\,\epsilon_1 \quad g_2 = g_3 \equiv g - i\,\epsilon_1 \quad e_2 \equiv e_2 + i\,\epsilon_3 \quad e_3 \equiv e_3 - i\,\epsilon_3 \\ l_2 =& l_6 \equiv l_2 + i\,\epsilon_2 \quad l_7 = l_4 \equiv l_2 - i\,\epsilon_2 \quad l_3 = l_5 \equiv l_3 - i\,\epsilon_4 \end{aligned} \tag{4.75}$$

$$\begin{aligned} \alpha_6 =& \alpha_8 \equiv \alpha_6 + i\,\epsilon_{m6} \quad \alpha_{11} = \alpha_{11} + i\,\epsilon_{m11} \quad \alpha_{17} = \alpha_{17} + i\,\epsilon_{m17} \quad \alpha_{18} = \alpha_{18} + i\,\epsilon_{m18} \\ \alpha_{24} =& \alpha_{24} + i\,\epsilon_{m24} \quad \alpha_{23} = \alpha_{23} + i\,\epsilon_{m23} \quad \alpha_{28} = \alpha_{28} + i\,\epsilon_{m28} \quad \alpha_{30} = \alpha_{30} + i\,\epsilon_{m30} \\ \alpha_{35} =& \alpha_{35} + i\,\epsilon_{m35} \quad \alpha_{34} = \alpha_{34} + i\,\epsilon_{m34} \quad \alpha_{38} = \alpha_{38} + i\,\epsilon_{m38} \quad \alpha_{43} = \alpha_{43} + i\,\epsilon_{m43} \\ \alpha_{42} =& \alpha_{42} + i\,\epsilon_{m42} \end{aligned} \tag{4.76}$$

We can rewrite the dilatation operator as

$$\begin{aligned} \mathcal{D}_{8,even} =& \, b\,\{0\} + c\,(\{0,2\} + \{2,0\}) + d\,\{0,4\} \\ &+ e_1\,(\{0,2,4\} + \{4,2,0\}) + e_2\,(\{0,4,2\} + \{2,0,4\}) \\ &+ i\,\epsilon_3\,(\{0,4,2\} - \{2,0,4\}) + f\,\{0,6\} \\ &+ g\,(\{0,2,6\} + \{2,0,6\} + \{0,4,6\} + \{0,6,4\}) \\ &+ i\epsilon_1\,(\{0,2,6\} - \{2,0,6\} + \{0,4,6\} + \{0,6,4\}) \\ &+ l_1\,(\{0,2,4,6\} + \{6,4,2,0\}) \\ &+ l_2\,(\{0,2,6,4\} + \{2,0,4,6\} + \{0,6,4,2\} + \{4,2,0,6\}) \\ &+ i\,\epsilon_2\,(\{0,2,6,4\} - \{2,0,4,6\} + \{0,6,4,2\} - \{4,2,0,6\}) \\ &+ l_3\,(\{0,4,2,6\} + \{2,0,6,4\}) + m\,\{2,0,4,2\} \end{aligned} \tag{4.77}$$

and for the mixed part[5] we get

$$\begin{aligned}
\mathcal{D}_{8,mixed} = & \alpha_1 \{0,1\} + \alpha_2 (\{0,1,2\} + \{2,1,0\}) + \alpha_3 \{0,3\} \\
& + \alpha_4 (\{0,1,3\} + \{0,3,1\} + \{0,2,3\} + \{2,0,3\}) \\
& + i\,\epsilon_{m1} (\{0,1,3\} - \{0,3,1\} - \{0,2,3\} + \{2,0,3\}) \\
& + \alpha_5 (\{0,1,2,3\} + \{3,2,1,0\}) + \alpha_6 (\{2,1,0,3\} + \{0,3,2,1\}) \\
& + i\,\epsilon_{m2} (\{2,1,0,3\} - \{0,3,2,1\}) + \alpha_7 (\{0,1,4\} + \{0,3,4\}) \\
& + \alpha_8 (\{0,1,2,4\} + \{4,1,2,0\} + \{0,2,3,4\} + \{4,3,2,0\}) \\
& + i\,\epsilon_{m3} (\{0,1,2,4\} - \{4,1,2,0\} - \{0,2,3,4\} + \{4,3,2,0\}) \\
& + \alpha_9 (\{2,1,0,4\} + \{4,1,0,2\} + \{2,0,3,4\} + \{0,4,3,2\}) \\
& + i\,\epsilon_{m4} (\{2,1,0,4\} - \{4,1,0,2\} + \{2,0,3,4\} - \{0,4,3,2\}) \\
& + \alpha_{10} (\{0,1,3,4\} + \{0,3,1,4\}) + \alpha_{11} \{0,5\} \\
& + \alpha_{12} (\{0,1,5\} + \{0,4,5\}) \\
& + \alpha_{13} (\{0,2,5\} + \{2,0,5\} + \{0,3,5\} + \{0,5,3\}) \\
& + i\,\epsilon_{m5} (\{0,2,5\} - \{2,0,5\} - \{0,3,5\} + \{0,5,3\}) \\
& + \alpha_{14} (\{0,1,3,5\} + \{4,2,0,5\} + \{5,3,1,0\} + \{0,2,4,5\}) \\
& + i\,\epsilon_{m6} (\{0,1,3,5\} + \{4,2,0,5\} - \{5,3,1,0\} - \{0,2,4,5\}) \\
& + \alpha_{15} (\{0,3,1,5\} + \{0,1,5,3\} + \{2,0,4,5\} + \{0,4,2,5\}) \\
& + i\,\epsilon_{m7} (\{0,3,1,5\} - \{0,1,5,3\} + \{2,0,4,5\} - \{0,4,2,5\}) \\
& + \alpha_{16} (\{0,2,3,5\} + \{2,0,5,3\}) + \alpha_{17} (\{2,0,3,5\} + \{0,2,5,3\}) \\
& + i\,\epsilon_{m8} (\{2,0,3,5\} - \{0,2,5,3\}) \\
& + \alpha_{18} \{0,3,6\} + (\{\ldots, a \leftrightarrow a+1, \ldots\})\,.
\end{aligned} \tag{4.78}$$

The coefficients are now all real. Requiring that the vacuum state is protected we have

$$\begin{aligned}
& a + 2b + 4c + 2d + 4e_1 + 4e_2 + 2f + 8g + 4l_1 + 8l_2 + 4l_3 + 2m + 2\alpha_1 \\
& + 4\alpha_2 + 2\alpha_3 + 8\alpha_4 + 4\alpha_5 + 4\alpha_6 + 4\alpha_7 + 8\alpha_8 + 8\alpha_9 + 4\alpha_{10} + 2\alpha_{11} \\
& + 4\alpha_{12} + 8\alpha_{13} + 8\alpha_{14} + 8\alpha_{15} + 4\alpha_{16} + 4\alpha_{17} + 2\alpha_{18} = 0
\end{aligned} \tag{4.79}$$

[5] We have relabelled the coefficients in a more "natural" way.

We now impose the eight-loop dispersion relation on one-magnon states:

$$\begin{aligned}\mathcal{D}_8\,|p\rangle_e &= E_8\,|p\rangle_e\,,\\ \mathcal{D}_8\,|p\rangle_o &= E_8\,|p\rangle_o\,.\end{aligned} \tag{4.80}$$

The action of $\mathcal{D}_8$ on $|p\rangle_e$ is

$$\begin{aligned}(a\,\{\,\}+\mathcal{D}_{8,even})|p\rangle_e = [&(-2b-6c-4d-8e_1-8e_2-4f-20g-10l_1\\ &-20l_2-10l_3-4m)\\ &+2\,(b+2c+2d+3e_1+2e_2+2f+8g+4l_1\\ &+6l_2+2l_3)\cos(p)\\ &+2\,(c+2e_2+2g+2l_2+3l_3+2m)\cos(2p)\\ &+2\,(e_1+2l_2)\cos(3p)\\ &+2l_1\cos(4p)\,]\,|p\rangle_e\end{aligned} \tag{4.81}$$

and

$$\begin{aligned}\mathcal{D}_{8,mixed}|p\rangle_e = [&-(4\alpha_1+10\alpha_2+4\alpha_3+20\alpha_4+12\alpha_5+12\alpha_6+12\alpha_7\\ &+24\alpha_8+24\alpha_9+14\alpha_{10}+4\alpha_{11}+12\alpha_{12}+20\alpha_{13})\\ &+24\alpha_{14}+24\alpha_{15}+12\alpha_{16}+12\alpha_{17}+6\alpha_{18})\\ &+(4\alpha_1+8\alpha_2+4\alpha_3+16\alpha_4+8\alpha_5+8\alpha_6+12\alpha_7\\ &+20\alpha_8+16\alpha_9+12\alpha_{10}+4\alpha_{11}+12\alpha_{12}+16\alpha_{13})\\ &+20\alpha_{14}+16\alpha_{15}+8\alpha_{16}+8\alpha_{17}+6\alpha_{18})\cos(p)\\ &+(2\alpha_2+4\alpha_4+4\alpha_5+4\alpha_6+8\alpha_9+2\alpha_{10}+4\alpha_{13})\\ &+8\alpha_{15}+4\alpha_{16}+4\alpha_{17})\cos(2p)\\ &+4\,(\alpha_8+\alpha_{14})\cos(3p)\,]\,|p\rangle_e\,.\end{aligned} \tag{4.82}$$

Similar results are true for $|p\rangle_o$. Comparing these eigenvalues with E_8 given in (4.7) we find the following equations:

$$\begin{aligned}
&l_1 = -5\,;\\
&e_1 + 2l_2 + 2\alpha_8 + 2\alpha_{14} = 40 - 6h_4\,;\\
&c + 2e_2 + 2g + 2l_2 + 3l_3 + 2m + \alpha_2 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6\\
&+ 4\alpha_9 + \alpha_{10} + 2\alpha_{13} + 4\alpha_{15} + 2\alpha_{16} + 2\alpha_{17} = -140 + 36h_4 - h_4^2 - 2h_6\,;\\
&b + 2c + 2d + 3e_1 + 2e_2 + 2f + 8g + 4l_1 + 6l_2 + 2l_3 + 2\alpha_1 + 4\alpha_2\\
&+ 2\alpha_3 + 8\alpha_4 + 4\alpha_5 + 4\alpha_6 + 6\alpha_7 + 10\alpha_8 + 8\alpha_9 + 6\alpha_{10} + 2\alpha_{11}\\
&+ 6\alpha_{12} + 8\alpha_{13} + 10\alpha_{14} + 8\alpha 15 + 4\alpha_{16} + 4\alpha_{17} + 3\alpha_{18} =\\
&\qquad = 280 - 90h_4 + 4h_4^2 + 8h_6 - h_8\,;\\
&b + 3c + 2d + 4e_1 + 4e_2 + 2f + 10g + 5l_1 + 10l_2 + 5l_3 + 2m\\
&+ 2\alpha_1 + 5\alpha_2 + 2\alpha_3 + 10\alpha_4 + 6\alpha_5 + 6\alpha_6 + 6\alpha_7 + 12\alpha_8 + 12\alpha_9\\
&+ 7\alpha_{10} + 2\alpha_{11} + 6\alpha_{12} + 10\alpha_{13} + 12\alpha_{14} + 12\alpha_{15} + 6\alpha_{16} + 6\alpha_{17} + 3\alpha_{18} =\\
&\qquad = 175 - 60h_4 + 3h_4^2 + 6h_6 - h_8\,.
\end{aligned} \tag{4.83}$$

The last equation isn't independent. We finally get:

$$\begin{aligned}
l_1 &= -5\,;\\
e_1 &= 2\,(20 - l_2 - \alpha_8 - \alpha_{14} - 3h_4)\,;\\
c &= -140 - 2e_2 - 2g - 2l_2 - 3l_3 - 2m - \alpha_2 - 2\alpha_4 - 2\alpha_5 - 2\alpha_6\\
&\quad - 4\alpha_9 - \alpha_{10} - 2\alpha_{13} - 4\alpha_{15} - 2\alpha_{16} - 2\alpha_{17} + 36h_4 - h_4^2 - 2h_6\,;\\
b &= 460 - 2d - 2f - 4g + 4l_2 + 4l_3 + 4m + 2e_2\\
&\quad - 2\alpha_1 - 2\alpha_2 - 2\alpha_3 - 4\alpha_4 - 6\alpha_7 - 4\alpha_8 - 4\alpha_{10} - 2\alpha_{11} - 6\alpha_{12}\\
&\quad - 4\alpha_{13} - 4\alpha_{14} - 3\alpha_{18} - 144h_4 + 6h_4^2 + 12h_6 - h_8\,.
\end{aligned} \tag{4.84}$$

In order to fix other parameters we consider two-impurity states: for length beyond wrapping order, with the help of `Mathematica`, we explicitly diagonalize the dilatation operator and compare its eigenvalues with those obtained from the perturbative solutions of the asymptotic Behte equations.

For $L = 5$ (length 10 states) we get:

$$\begin{aligned}
d =& 10 - 4l_3 - m + 2e_2 - f - 12g + 6h_4 - 2\alpha_{10} \\
& - 2\alpha_{12} + 4\alpha_{15} - \alpha_{18} - 2\alpha_7 + 4\alpha_9 - 4\beta\,; \\
\alpha_1 =& \alpha_{11} + 4\alpha_{13} + 2\alpha_{14} + 4\alpha_{15} + 4\alpha_{16} + 3\alpha_{17} + 4\alpha_5 \\
& + 3\alpha_6 - 2(\alpha_7 + \alpha_8 - 2\alpha_9) + \beta \\
\alpha_2 =& \frac{1}{2}\,(-\alpha_{11} - 2\alpha_{12} - 4\alpha_{13} - 2\alpha_{14} - 4\alpha_{15} - 2\alpha_{16} \\
& - \alpha_{17} - 4\alpha_5 - 3\alpha_6 - 2\alpha_8 - 4\alpha_9 - \beta) \\
\alpha_3 =& -2(\alpha_{11} + 2\alpha_{12} + 2\alpha_{13} + 2\alpha_{14} - \alpha_{16} - \alpha_{17} \\
& + \alpha_{18} - \alpha_5 - \alpha_6 + \alpha_7) \\
\alpha_4 =& \frac{1}{4}\,(-4\alpha_{10} + \alpha_{11} + 2\alpha_{12} - 2\alpha_{14} - 4\alpha_{15} - 6\alpha_{16} \\
& - 5\alpha_{17} - 4\alpha_5 - 3\alpha_6 - 2\alpha_8 - 4\alpha_9 - \beta)
\end{aligned} \tag{4.85}$$

where $\beta = \beta_{2,3}^{(6)}$ is the leading coefficient of the dressing phase.
For $L = 6$ (length 12 states) we get

$$\begin{aligned}
e_2 =& \frac{2}{3}\,(-26 + 2l_3 + m + 5g - l_2 - 3\alpha_{15} - 3\alpha_9 + 2\beta) \\
f =& \frac{1}{3}\,(20 - 2l_3 - 4m - 8g + 4l_2 - 3\alpha_{18} - 2\beta) \\
\alpha_{13} =& \frac{1}{4}\,(-\alpha_{11} - 2(\alpha_{12} + 2\alpha_{15} + \alpha_{16} - \alpha_6 - 2\alpha_8 + \beta)) \\
\alpha_{14} =& \frac{1}{4}\,(-\alpha_{11} - 2(\alpha_{12} - \alpha_{16} + \alpha_6 + 2\alpha_8 - \beta)) \\
\alpha_{17} =& \frac{1}{2}\,\alpha_{11} + \alpha_{12} - \alpha_{16}
\end{aligned} \tag{4.86}$$

For $L = 7$ (length 14) states we get

$$\begin{aligned}
g =& \frac{1}{2}\,(10 - 2l_3 - 2m - \beta) \\
l_2 =& \frac{1}{4}\,(-4 + 2m + \beta)
\end{aligned} \tag{4.87}$$

For higher L, no more parameters enter the spectrum. So, point 4 of the procedure described in section 4.2 isn't sufficient to completely fix the dilatation operator. Nevertheless, we stress that an important result already follows from equations (4.85)-(4.86):

a non vanishing[6] coefficient β for the dressing phase would imply that the α_i coefficients, appearing in $\mathcal{D}_{8,mixed}$, cannot be all zero. This proves that $\mathcal{D}_{8,mixed} \neq 0$.

In principle, the remaining coefficients could be fixed imposing the commutation of $\mathcal{D}$ with the third charge $\mathcal{Q}_3$: this is laborious and we won't do it, since it isn't necessary for the computations of the dressing phase, as we will see in the next chapter. The part of the dilatation operator which acts on even sites only reads:

$$\begin{aligned}
\mathcal{D}_{8,even} =&(536 - 4l_3 - 6m - 4\alpha_5 - 6\alpha_6 + 6\alpha_7 - 4\alpha_9 + 4\alpha_{10} \\
&- 3\alpha_{11} - 4\alpha_{15} + 2\alpha_{16} + 3\alpha_{18} - 156h_4 + 6h_4^2 \\
&+ 12h_6 - h_8 + 2\beta)\,\{0\} \\
&+ (-148 + 3l_3 + 5m + 2\alpha_5 + 4\alpha_9 + \alpha_{10} + 4\alpha_{15} + 2\alpha_{16} \\
&+ 36h_4 - h_4^2 - 2h_6 + \frac{7}{2}\beta)\,(\{0,2\} + \{2,0\}) \\
&+ (-42 + 2l_3 + 3m - 2\alpha_7 - 2\alpha_{10} - 2\alpha_{12} + 6h_4)\,\{0,4\} \\
&+ (-8 + 2l_3 + 2m - \alpha_{18} + \beta)\,\{0,6\} \\
&+ (-2l_3 - 3m - 2\alpha_9 - 2\alpha_{15} + 2i\,\epsilon_{3a} - \beta)\,\{0,4,2\} \\
&+ (-2l_3 - 3m - 2\alpha_9 - 2\alpha_{15} - 2i\,\epsilon_{3a} - \beta)\,\{2,0,4\} \\
&+ (5 - l_3 - m + i\,\epsilon_{1a} - \frac{1}{2}\beta)\,(\{0,2,6\} + \{0,6,4\}) \\
&+ (5 - l_3 - m - i\,\epsilon_{1a} - \frac{1}{2}\beta)\,(\{0,4,6\} + \{2,0,6\}) \\
&+ (42 - m + \alpha_6 + \frac{1}{2}\alpha_{11} + \alpha_{12} - 6h_4 - \frac{3}{2}\beta)\,(\{0,2,4\} + \{4,2,0\}) \\
&+ m\,\{2,0,4,2\} \\
&+ l_3\,(\{0,4,2,6\} + \{2,0,6,4\}) \\
&+ (-1 + \frac{1}{2}m + i\,\epsilon_{2a} + \frac{1}{4}\beta)\,(\{0,2,6,4\} + \{0,6,4,2\}) \\
&+ (-1 + \frac{1}{2}m - i\,\epsilon_{2a} + \frac{1}{4}\beta)\,(\{2,0,4,6\} + \{4,2,0,6\}) \\
&- 5\,(\{0,2,4,6\} + \{6,4,2,0\})
\end{aligned} \tag{4.88}$$

where the coefficients ϵ_{ia} are combinations of the ϵ_i: they correspond to similarity transformations and can be fixed only selecting a particular renormalization scheme. The form of (4.88) is quite similar to the four-loop dilatation operator of $\mathcal{N} = 4$ SYM [82].

[6] We will see in Chapter 5 that this is, indeed, the case.

The presence of the "mixed" hamiltonian, coupling the two types of magnons, is a completely new feature of ABJM theory which appears at eight loops, making this theory substantially different from the N = 4 SYM theory at four loops. Its coefficients α_i affect also the even and odd parts, as can be seen in (4.88). However, they don't enter the coefficients of the last five terms in (4.88). The same is true for the coefficients of $h(\lambda)$ (recall that h_6 and h_8 are not known at present). This will allow us to compute the coefficients m and l_3, as well as β, in the next chapter via direct field theory methods.

Chapter 5.

The Leading Order Dressing Phase

In this chapter we proceed to the computation of the leading-order coefficient of the dressing phase, *i.e.* the parameter $\beta_{2,3}^{(6)}$ which appears in the eight-loop dilatation operator. We determine it by direct field-theory calculations, in a similar fashion to that used in [82] for the $\mathcal{N}=4$ SYM case. Here, we use $\mathcal{N}=2$ superspace methods for the evaluation of Feynman diagrams. We refer to Appendix A for a very brief review of such methods. The results of the following analysis are presented in [34].

5.1. The maximal reshuffling hamiltonian

In the last chapter we constructed the eight-loop dilatation operator $\mathcal{D}_8$. We have seen that a large number of permutation structures contributes to $\mathcal{D}_8$ and, correspondingly, many parameters have to be determined in order to completely fix it. Our procedure based on the Bethe Ansatz could fix only few of them. In particular, as can be seen from (4.88), lower range interactions depend on the unknown coefficients of the weak-coupling expansion of $h(\lambda)$ as well as on some coefficients α_i of the mixed hamiltonian, which couples the two $SU(2)$ sectors. Remarkably, the last five terms in (4.88) don't contain such coefficients. These terms have length four. At ℓ loops, interactions of length $\ell/2$ are called *maximal*, since they lead to maximal reshuffling of spins inside the operator. It turns out that their coefficients are, in principle, the easiest to compute through Feynman diagrams, since vector interactions are absent, so that they get contribution from a single diagram. Let's describe now how to construct such a diagram. To this

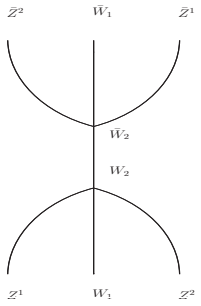


Figure 5.1.: Building block for chiral structures

purpose it is useful to rewrite the dilatation operator in a different basis, the basis of *chiral functions*, which is defined in terms of the basis of permutation structures as [24, 81, 88, 89]

$$\begin{aligned}
\chi() &= \{\,\}\,, \\
\chi(a) &= \{a\} - \{\,\}\,, \\
\chi(a,b) &= \{a,b\} - \{a\} - \{b\} + \{\,\}\,, \\
\chi(a,b,c) &= \{a,b,c\} - \{a,b\} - \{a,c\} - \{b,c\} + \{a\} + \{b\} + \{c\} - \{\,\}\,, \\
\chi(a,b,c,d) &= \{a,b,c,d\} - \{a,b,c\} - \{a,b,d\} - \{a,c,d\} - \{b,c,d\} \\
&\quad + \{a,b\} + \{a,c\} + \{a,d\} + \{b,c\} + \{b,d\} + \{c,d\} \\
&\quad - \{a\} - \{b\} - \{c\} - \{d\} + \{\,\}\,.
\end{aligned} \tag{5.1}$$

These functions are directly related to the chiral structure of a supergraph and precisely describe the flavor flow of fields inside (4.1) under the action of the interaction. All diagrams contributing to the same chiral function share the same chiral structure. For maximal interactions, no vector interactions can be present and the chiral structure alone determines the unique diagram which contributes. The chiral structure of the function $\chi(a_1, \ldots, a_n)$ contains n chiral vertices and n antichiral vertices, connected by $\langle Z\bar{Z}\rangle$ or $\langle W\bar{W}\rangle$ propagators. Each pair of chiral and antichiral vertices (plus the propagator connecting them) describes an elementary permutation of next-to-nearest neighbors and constitutes the building block of any supergraph. This is represented in Figure 5.1. The whole chiral structure of a supergraph is recovered by assembling n building blocks in the order indicated in the chiral function (arguments should be read from right to left).

Let's now consider the eight-loop dilatation operator written in the basis of chiral functions. The part of the hamiltonian leading to maximal reshuffling of spins[1] preserves the form given in terms of the permutation structures:

$$\begin{aligned}\mathcal{D}_{8,mr} = & \, m\,\chi(2,0,4,2) \\ & + l_3\,[\chi(0,4,2,6)+\chi(2,0,6,4)] \\ & + (-1+\frac{1}{2}m + i\,\epsilon_{2a}+\frac{1}{4}\beta)\,[\chi(0,2,6,4)+\chi(0,6,4,2)] \\ & + (-1+\frac{1}{2}m - i\,\epsilon_{2a}+\frac{1}{4}\beta)\,[\chi(2,0,4,6)+\chi(4,2,0,6)] \\ & - 5\,(\chi(0,2,4,6)+\chi(6,4,2,0)]\,.\end{aligned} \tag{5.2}$$

We note that the form of (5.2) is just the same as the maximal reshuffling hamiltonian of $\mathcal{N}=4$ SYM [82]. The coefficients of the five terms in (5.2) can be computed from the supergraphs contributing to $\chi(2,0,4,2)$, $\chi(0,4,2,6)$, $\chi(0,2,6,4)$, $\chi(2,0,4,6)$, $\chi(0,2,4,6)$. As we explained before, only one supergraph contributes to each term and it contains only scalar interactions. This is a great simplification. The coefficient of the last term was already fixed by imposing the dispersion relation on one-magnon states. The first four terms contain precisely four undetermined parameters, namely m, l_3, ϵ_{2a} and β. Therefore, the maximal reshuffling hamiltonian (5.2) can be directly determined by the computation of a very small number of Feynman diagrams. In particular, this will provide us with the leading order coefficient of the dressing phase $\beta=\beta^{(6)}_{2,3}$.

5.2. Computation of maximal diagrams

In order to determine the coefficients of the dilatation operator from Feynman diagrams, we consider the renormalization of composite operators [91] and the subtraction of subdivergences, which proceeds by introducing renormalization factors and counterterm diagrams. This technique was successfully applied to the dilatation operator of $\mathcal{N}=4$ SYM [77,82,88–90] and to the dilatation operator of ABJM theory up to six loops [22–26]. We will use $\mathcal{N}=2$ superspace methods.

[1] We give here only the terms which act non-trivially on even sites. Besides them there are terms acting on odd sites, which are obtained by the shifting $\chi(\ldots,a\to a+1,\ldots)$. They have the same coefficients as (5.2)

We therefore have to deal with Feynman diagrams with one vertex being the composite operator and additional vertices which, for maximal reshuffling diagrams, are only chiral or antichiral vertices.

Let $\mathcal{O}_a$ be a vector of operators with the same quantum numbers. The dilatation operator acts on such operators as

$$\mathcal{D}\mathcal{O}_a = \Delta_a{}^b \mathcal{O}_b \,, \tag{5.3}$$

where $\Delta_a{}^b$ represents the matrix of scaling dimensions and in general isn't diagonal, due to operator mixing. The relation between bare and renormalized quantities is

$$\mathcal{O}_a^{ren} = \mathcal{Z}_a{}^b \mathcal{O}_b^{bare} \,. \tag{5.4}$$

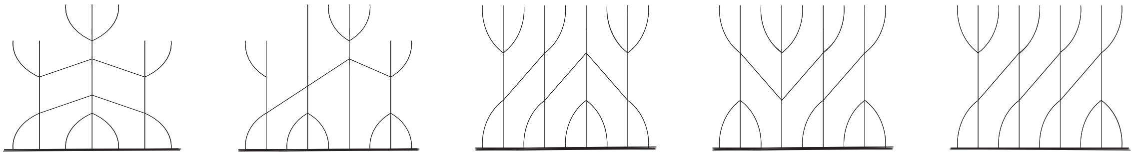
The matrix $\mathcal{Z}$ is the renormalization factor: in a perturbative approach, it can be expanded as

$$\mathcal{Z} = \mathbb{1} + \sum_{k=1}^{\infty} \mathcal{Z}_{2k}\, \lambda^{2k} \,, \tag{5.5}$$

and gets contributions from all the Feynman diagrams with a single insertion of one of the composite operators $\mathcal{O}_a$. The ℓ-loop contribution to $\mathcal{Z}$ is found from the overall divergence of ℓ-loop diagrams of that kind. We use dimensional regularization in $D = 3 - 2\varepsilon$ dimensions in order to regularize quantum divergences. In dimensional regularization, these divergences show up as poles in the dimensional regulator ε, in the limit $\varepsilon \to 0$. The relation between the renormalization factor and the dilatation operator is

$$\mathcal{D} = \mathcal{D}_0 + \lim_{\varepsilon \to 0} \left[2\varepsilon\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \ln \mathcal{Z}(\lambda, \varepsilon) \right] \,, \tag{5.6}$$

where $\mathcal{D}_0$ gives the classical dimension. We see that only the divergent part of $\mathcal{Z}$ contributes to anomalous dimensions. The ℓ-loop dilatation operator is then given by the $\lambda^{2\ell}$ coefficient of the $\frac{1}{\varepsilon}$ pole of $\ln \mathcal{Z}$ multiplied by -2ℓ. Higher order poles must be absent in $\ln \mathcal{Z}$.

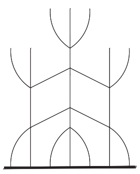


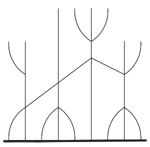
$\mathcal{S}_a$ $\mathcal{S}_b$ $\mathcal{S}_c$ $\mathcal{S}_d$ $\mathcal{S}_e$

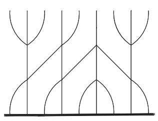
Figure 5.2.: Supergraphs corresponding to the maximal reshuffling terms of the eight-loop dilatation operator of ABJM theory. From left to right, and together with their reflected diagrams, they are associated to each line in (5.2). The horizontal bar represents the operator itself, or analogously, the spin chain.

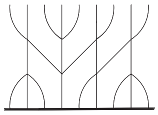
In order to compute the coefficients of (5.2) we need to compute the supergraphs in Figure 5.2 and isolate the overall UV divergence by subtracting all their UV subdivergences.

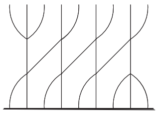
First of all, we perform standard D-algebra manipulations (see Appendix A) to reduce the supergraphs to ordinary momentum-space integrals I_{8x}. The Feynman rules imply that the color factor is always N^8, which combines with $\left(\frac{4\pi}{k}\right)^8$ to give the right power of the 't Hooft coupling, *i.e.* λ^8. The $(4\pi)^8$ factor will be simplified by the factor $\frac{1}{(4\pi)^8}$ present in each integral I_{8x}. The pole parts of the integrals with the subdivergenes subtracted are denoted by $\bar{I}_{8a}$. The maximal reshuffling supergraphs, thus, evaluate to:

$\mathcal{S}_a =$  $= \lambda^8\,(4\pi)^8\,\bar{I}_{8a}|_{\frac{1}{\varepsilon}}\;\chi(2,0,4,2)\,,$ (5.7)

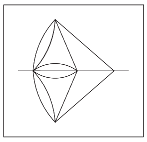
$\mathcal{S}_b =$  $= \lambda^8\,(4\pi)^8\,\bar{I}_{8b}|_{\frac{1}{\varepsilon}}\;\chi(0,4,2,6)\,,$ (5.8)

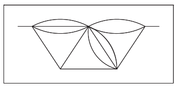
$\mathcal{S}_c =$  $= \lambda^8\,(4\pi)^8\,\bar{I}_{8c}|_{\frac{1}{\varepsilon}}\;\chi(0,2,6,4)\,,$ (5.9)

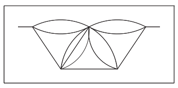
$$\mathcal{S}_d \;=$$

$$= \lambda^8\,(4\pi)^8\,\bar{I}_{8d}|_{\frac{1}{\varepsilon}}\;\chi(2,0,4,6)\,, \tag{5.10}$$

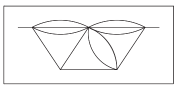
$$\mathcal{S}_e \;=$$

$$= \lambda^8\,(4\pi)^8\,\bar{I}_{8}|_{\frac{1}{\varepsilon}}\;\chi(0,2,4,6)\,. \tag{5.11}$$

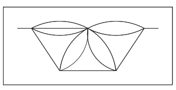
The subtracted integrals are computed in Appendix B and are given by:

$$\bar{I}_{8a} \;=$$

$$= \frac{1}{(4\pi)^8}\left(-\frac{1}{3072\,\varepsilon^4}+\frac{1}{192\,\varepsilon^3}-\frac{1}{48\,\varepsilon^2}+\frac{y}{\varepsilon}\right)\,, \tag{5.12}$$

$$\bar{I}_{8b} \;=$$

$$= \frac{1}{(4\pi)^8}\left(-\frac{5}{6144\,\varepsilon^4}+\frac{5}{768\,\varepsilon^3}-\frac{1}{384\,\varepsilon^2}-\frac{1}{32\,\varepsilon}\right)\,, \tag{5.13}$$

$$\bar{I}_{8c} \;=$$

$$= \frac{1}{(4\pi)^8}\left(-\frac{1}{2048\,\varepsilon^4}+\frac{1}{128\,\varepsilon^3}-\frac{3}{64\,\varepsilon^2}+\frac{5}{192\,\varepsilon}\right)\,, \tag{5.14}$$

$$\bar{I}_{8d} \;=$$

$$= \frac{1}{(4\pi)^8}\left(-\frac{1}{2048\,\varepsilon^4}+\frac{1}{192\,\varepsilon^3}-\frac{1}{64\,\varepsilon^2}-\frac{11}{192\,\varepsilon}\right)\,, \tag{5.15}$$

$$\bar{I}_{8e} \;=$$

$$= \frac{1}{(4\pi)^8}\left(-\frac{1}{6144\,\varepsilon^4}+\frac{1}{256\,\varepsilon^3}-\frac{19}{384\,\varepsilon^2}+\frac{5}{16\,\varepsilon}\right)\,. \tag{5.16}$$

For their evaluation, we used the *Gegenbauer polynomials x-space technique* (GPXT) and, in the simplest cases, the method of contracting bubbles. Nevertheless, these two methods seem to be quite inefficient for the evaluation of the $\frac{1}{\varepsilon}$ pole of $\bar{I}_{8a}$. In this case, we evaluated the integral numerically through Mellin-Barnes techniques. The result, together with its error, is obtained in Appendix B and reads

$$y \;=\; \frac{1}{(4\pi)^8}\,(-0.0059921 \pm 0.0000008)\,. \tag{5.17}$$

Remarkably, it is possible to extract the corresponding analytical result from this number, as we discuss in the next section. The idea is to fit the numerical value with a combination of some analytical constants via an integer relation detection algorithm. Fortunately, as we argued in Section 3.6, only few constants can appear in the $\frac{1}{\varepsilon}$ pole of $\bar{I}_{8a}$.

5.3. PSLQ Algorithm

In section 3.6 we argued that the parameter β of (5.2) is purely transcendental and that it can be a rational combination of just two transcendental constants, namely π^3 and $\zeta(3)$. Since the only transcendental contribution to β can come from the $\frac{1}{\varepsilon}$ pole of $\bar{I}_{8a}$, such pole should be a rational combination of the constants 1, π^3, $\zeta(3)$. In order to extract the coefficients of this generic linear combination from the numerical value (5.12) we use the `Mathematica` implementation [104] of the PSLQ algorithm [83] (see also [105]).

The PSLQ algorithm is, perhaps, the most powerful integer relation detection algorithm. A vector $x = (x_1, \ldots, x_n)$ of real or complex numbers is said to possess an integer relation $(a_1, \ldots, a_n)$, if there exist integers a_i, not all zero, such that

$$a_1 x_1 + \cdots + a_n x_n = 0\,. \tag{5.18}$$

An integer relation detection algorithm can recover the vector of integers a_i, if it exists. PSLQ takes as inputs the vector x and a given precision and constructs a sequence of integer-valued matrices B_n that reduces the vector $y = xB_n$, until either the relation is found (as one of the columns of B_n), or else precision is exhausted. At the same time, PSLQ generates a steadily growing bound on the size of any possible relation. When a relation is found, the size of smallest entry of the vector y abruptly drops to roughly 10^{-p}, where p is the number of digits of precision. The size of this drop can be viewed as a confidence level that the relation is real and not merely a numerical artifact. A drop of 20 or more orders of magnitude almost always indicates a real relation. PSLQ usually needs high precision in the numerical input data. If one wishes to recover a relation of length n, with coefficients of maximum d digits, then the input vector x must be specified to at least nd digits. In our case, the vector x contains the numerical value

in $\bar{I}_{8a}\,|_{1/\varepsilon}$ and the transcendental constants it should be fitted with:

$$x = \big(-0.0059921, 1, \pi^3, \zeta(3)\big)\,. \tag{5.19}$$

Application of PSLQ algorithm with precision 10^{-7}, which is the same within which the numerical value of $\bar{I}_{8a}\,|_{1/\varepsilon}$ has been computed, gives the solution

$$a = (-32, -5, 0, 4)\,. \tag{5.20}$$

It means that the exact value of the $\frac{1}{\varepsilon}$ pole of the $\bar{I}_{8a}$ integral should be

$$y \;=\; \frac{1}{(4\pi)^8}\left(-\frac{5}{32}+\frac{1}{8}\,\zeta(3)\right)\,. \tag{5.21}$$

We can verify *a posteriori* that the value (5.21) matches with (5.17) w.r.t. all the digits within the numerical precision.

5.4. Results

We now proceed to the computation of the coefficients of the hamiltonian (5.2) following the procedure we initiated in section 5.2.

The final results for the supergraphs contributing to maximal reshuffling terms of the dilatation operator are

$$\begin{aligned}
\mathcal{S} &= \lambda^8\,\frac{5}{16}\,\chi(0,2,4,6)\,,\\
\mathcal{S}_a &= \lambda^8\left(-\frac{5}{32}+\frac{1}{8}\,\zeta(3)\right)\,\chi(2,0,4,2)\,,\\
\mathcal{S}_b &= -\,\lambda^8\,\frac{1}{32}\,\chi(0,4,2,6)\,,\\
\mathcal{S}_c &= \lambda^8\,\frac{5}{192}\,\chi(0,2,6,4)\,,\\
\mathcal{S}_d &= -\,\lambda^8\,\frac{11}{192}\,\chi(2,0,4,6)\,.
\end{aligned} \tag{5.22}$$

The corresponding contributions to the renormalization constant are obtained multiplying these values by -16. At this point, we can find the unknown parameters appearing

in (5.2). Our results are:

$$m = \frac{5}{2} - 2\zeta(3)\,, \qquad l_3 = \frac{1}{2}\,, \qquad \beta = 4\,\zeta(3)\,, \qquad \epsilon_{2a} = \frac{2}{3}i\,. \tag{5.23}$$

We recall that the parameter ϵ_{2a} is unphysical: the value listed in (5.23) depends on the renormalization scheme we used. The presence of the factor i seems to spoil hermiticity of the dilatation operator, see [82] for further comments on this. Moreover, the coefficient of the chiral function $\chi(0,2,4,6)$ turns out to be -5, as it was already known from the previous chapter: it, thus, constitutes a non trivial check of the whole procedure. Finally, we have computed the value of the leading order coefficient of the dressing phase. Its value agrees with the prediction (3.65) and is the same as in $\mathcal{N} = 4$ SYM. In the Conclusions we further comment on this result.

Chapter 6.

Conclusions

In this work we have explored some aspects of integrability in three-dimensional Chern-Simons-matter theories, focusing in particular on ABJM theory. Integrability provides powerful tools to analyse the spectrum of the theory, as well as other observables. Nevertheless, integrability of the full theory is only conjectured and proposals such as the all-loop Bethe Ansatz [18] need to be tested via independent methods. We have performed some perturbative computations in order to check some ideas related to integrability. In particular, we have investigated the weak-coupling dressing phase of ABJM theory. It appears in the Bethe equations, as well as in the dilatation operator, starting at eight loops. A simple procedure based on the Bethe Ansatz allowed us to construct the asymptotic dilatation operator in the $SU(2) \times SU(2)$ sector up to six loops without the computations of Feynman diagrams. We also verified that the dressing phase is not present up to this order. Furthermore, we could largely constrain the eight-loop dilatation operator. In this case, we then focused on the terms corresponding to maximal interactions: they contain the unknown parameter $\beta_{2,3}^{(6)}$, which is the leading order coefficient of the dressing phase. Thanks to superspace techniques and with the help of the transcendentality principle, we could directly compute its value from Feynman supergraphs. A great simplification follows from the fact that we considered maximal interactions: we need to compute a very small number of Feynman diagrams (in fact, just three are necessary to completely fix $\beta_{2,3}^{(6)}$) which contain only scalar interactions. The result we found is $\beta_{2,3}^{(6)} = 4\zeta(3)$ and coincides with the value of the leading coefficient of the dressing phase of $\mathcal{N} = 4$ SYM. This value was expected on the basis of the results of [18] where a conjecture for the weak-coupling expansion of the coefficient functions

can be inferred:

$$\beta_{r,s}(\lambda) = -\sum_{n=1}^{\infty} c_{r,s}^{(-n)}\, h(\lambda)^{r+s+n-1}\,. \tag{6.1}$$

Here, $c_{r,s}^{(-n)}$ denote the analytical continuation [85] of the all-loop strong-coupling coefficient functions[1] proposed in [84]. For $r = 2$, $s = 3$, we see that we should obtain

$$\beta_{2,3}(\lambda) = -c_{2,3}^{(-2)}\,\lambda^6 - \left(c_{2,3}^{(-4)} + 3h_4 c_{2,3}^{(-2)}\right)\,\lambda^8 + \cdots \tag{6.2}$$

We observe that, since the first weak coupling coefficient of $h(\lambda)$ is $h_2 = 1$, the leading order contribution to ABJM dressing phase is just $\beta_{2,3}^{(6)} = 4\zeta(3)$, as in $\mathcal{N} = 4$ SYM. Our field theory computation is consistent with such predictions. At higher orders, however, the $c_{r,s}^{(-n)}$ coefficients mix with the non trivial coefficients of $h(\lambda)$ yielding dressing phase coefficients which differ from the $\mathcal{N} = 4$ SYM ones. We notice that, at least, the next-to-leading order of (6.2) has the right transcendentality. It would be interesting to further test the validity of (6.1) beyond the leading order, through direct computations: hopefully, restricting to the case of maximal interactions, the calculations will be simplified and have a chance of being performed with standard field theory techniques. It would be interesting also to extend our analysis to the parity breaking ABJ theory, to check whether the lack of parity symmetry influence the integrable structure of the theory.

[1] See Section 3.5.

Appendix A.

$\mathcal{N} = 2$ Superspace Formalism

In this Appendix we give a general overview of the superspace formalism, which was used in this work both for the formulation of ABJM theory in Chapter 2 and for our perturbative computations of Chapter 5. As we said in Chapter 2, this formalism is particularly suitable for perturbative computations in supersymmetric gauge theories, especially at high loop orders. It happens because supersymmetry is intrinsic to the formalism and not sovra-imposed. Here, we briefly review the basic facts about three-dimensional [1] superspace in the $\mathcal{N} = 2$ formalism, basically with the aim to fix the notations, relevant for the treatment of ABJM theory, used throughout the work. We follow the conventions of [87]. We don't pretend to be self-contained and refer to [45,46] for a wider introduction to the same subjects.

A.1. Superspace Conventions

The Lorentz group in three-dimensional Minkowski space is $SL(2,\mathbb{R})$, rather than the four-dimensional $SL(2,\mathbb{C})$, and the corresponding fundamental representation acts on two-component real spinors ψ^α. For this reason, in three dimensions there's no difference between dotted and undotted indices and new types of contractions are allowed. Spinor indices are raised and lowered with the "spinorial metric" $C^{\alpha\beta} = i\epsilon^{\alpha\beta}$, where $\epsilon^{\alpha\beta}$ is the

[1] We mean, here, that spacetime dimension is $D = 3$.

antysymmetric tensor with $\epsilon^{12} = 1$:

$$\psi^\alpha = C^{\alpha\beta}\psi_\beta\,, \quad \psi_\alpha = \psi^\beta C_{\beta\alpha}\,. \tag{A.1}$$

We follow the convention that, when contracting spinor indices, the upper one is always placed on the left:

$$\psi\chi = \psi^\alpha\chi_\alpha = \chi^\alpha\psi_\alpha = \chi\psi\,, \quad \psi^2 = \frac{1}{2}\psi^\alpha\psi_\alpha \tag{A.2}$$

We work directly in Wick-rotated Euclidean spacetime, with metric $g_{\mu\nu} = \mathrm{diag}(1,1,1)$. Gamma matrices satisfy the algebra

$$(\gamma^\mu)^\alpha{}_\gamma(\gamma^\nu)^\gamma{}_\beta = -g^{\mu\nu}\delta^\alpha{}_\beta - \epsilon^{\mu\nu\rho}(\gamma_\rho)^\alpha{}_\beta \tag{A.3}$$

where the Levi-Civita tensor is such that $\epsilon^{012} = 1$. Gamma matrices with both spinor indices raised (or lowered) are symmetric. We use the bi-spinor notation to represent vectors: our conventions for coordinates, momenta and fields are respectively

$$x^{\alpha\beta} = \frac{1}{2}(\gamma_\mu)^{\alpha\beta}x^\mu\,, \quad p_{\alpha\beta} = (\gamma^\mu)_{\alpha\beta}p_\mu\,, \quad A_{\alpha\beta} = \frac{1}{\sqrt{2}}(\gamma^\mu)_{\alpha\beta}A_\mu \tag{A.4}$$

and

$$x^\mu = x^{\alpha\beta}(\gamma^\mu)_{\alpha\beta}\,, \quad p_\mu = \frac{1}{2}(\gamma_\mu)^{\alpha\beta}p_{\alpha\beta}\,, \quad A_\mu = \frac{1}{\sqrt{2}}(\gamma_\mu)^{\alpha\beta}A_{\alpha\beta}\,. \tag{A.5}$$

It follows that the scalar product of two vectors ia given by

$$u\cdot v = \frac{1}{2}u^{\alpha\beta}v_{\alpha\beta}\,. \tag{A.6}$$

As usual, from the momentum $p_{\alpha\beta}$ one constructs standard derivatives as

$$p_{\alpha\beta} = i\partial_{\alpha\beta}\,. \tag{A.7}$$

Extended supersymmetry transformations are generated by the supercharges Q^I_α, $I = 1,\ldots,\mathcal{N}$, which satisfy the algebra

$$\{Q^I_\alpha, Q^J_\beta\} = 2p_{\alpha\beta}\delta^{IJ}\,. \tag{A.8}$$

the fundamental relation (A.8) is left invariant by the R-symmetry group $SO(\mathcal{N})$.

We describe $\mathcal{N}$-extended *superspace* with bosonic and (anticommuting) fermionic coordinates $(x^{\alpha\beta}, \theta^\alpha_I)$, $\alpha = 1, 2$, $I = 1, \ldots, \mathcal{N}$. From now on we restrict to the case $\mathcal{N} = 2$, where the R-symmetry group $SO(2) \simeq U(1)$ is the same for the $\mathcal{N} = 1$ four-dimensional formalism. It is convenient to change coordinates as

$$\theta^\alpha = \theta^\alpha_1 + i\theta^\alpha_2 \,, \quad \bar\theta^\alpha = \theta^\alpha_1 - i\theta^\alpha_2 \tag{A.9}$$

and to define spinor derivatives acting on these coordinates as

$$\partial_\alpha \theta^\beta = \delta^\beta_\alpha \,, \quad \bar\partial_\alpha \bar\theta^\beta = \delta^\beta_\alpha \,, \quad \partial_\alpha \bar\theta^\beta = \bar\partial_\alpha \theta^\beta = 0 \,. \tag{A.10}$$

Starting from spinor derivatives one constructs three-dimensional spinor covariant derivatives as

$$\mathrm{D}_\alpha = \partial_\alpha + \frac{i}{2}\bar\theta^\beta \partial_{\alpha\beta} \,, \quad \bar{\mathrm{D}}_\alpha = \bar\partial_\alpha + \frac{i}{2}\theta^\beta \partial_{\alpha\beta} \tag{A.11}$$

which satisfy the algebra[2]

$$\{\mathrm{D}_\alpha, \mathrm{D}_\beta\} = \{\bar{\mathrm{D}}_\alpha, \bar{\mathrm{D}}_\beta\} = 0 \,, \quad \{\mathrm{D}_\alpha, \bar{\mathrm{D}}_\beta\} = p_{\alpha\beta} \,. \tag{A.12}$$

We observe that $\mathcal{N} = 2$ three-dimensional superspace is essentially the same as $\mathcal{N} = 1$ four-dimensional superspace, with the only difference that there aren't undotted indices: this implies, for example, that contractions such as $\bar{\mathrm{D}}^\alpha \mathrm{D}_\alpha$ are now possible.

Quantities such as the action of supersymmetric field theories are very conveniently written in superspace, where supersymmetry transformations are realized in a natural way on the coordinates $(x^{\alpha\beta}, \theta^\alpha, \bar\theta^\beta)$. The basic objects are *superfields*, which are functions of the bosonic as well as of the fermionic coordinates and may carry external indices, according to representations of the symmetry algebra. If a superfield is expanded in Taylor series in θ (which is finite, due to the anticommuting nature of the fermionic coordinates) we get the corresponding component fields, functions only of the spacetime

[2] This is the same algebra of the supercharges in momentum space. If we define $Q_\alpha = \frac{1}{2}(Q^{(1)}_\alpha + iQ^{(2)}_\alpha)$ and $\bar Q_\alpha = \frac{1}{2}(Q^{(1)}_\alpha - iQ^{(2)}_\alpha)$ the non vanishing anticommutator reads $\{Q_\alpha, \bar Q_\beta\} = P_{\alpha\beta}$.

coordinates. Integration over fermionic coordinates is defined by

$$\int \mathrm{d}^2\theta = \frac{1}{2}\partial^\alpha\partial_\alpha\,, \quad \int \mathrm{d}^2\bar{\theta} = \frac{1}{2}\bar{\partial}^\alpha\bar{\partial}_\alpha\,, \quad \int \mathrm{d}^4\theta = \int \mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\,, \tag{A.13}$$

such that supersymmetric invariant actions can be easily written in superspace and can be related to ordinary actions through projection:

$$\begin{aligned} &\int \mathrm{d}^3x\mathrm{d}^2\theta = \int \mathrm{d}^3x\,\mathrm{D}^2|_{\theta=\bar{\theta}=0}\,, \quad \int \mathrm{d}^3x\mathrm{d}^2\bar{\theta} = \int \mathrm{d}^3x\,\bar{\mathrm{D}}^2|_{\theta=\bar{\theta}=0}\,, \\ &\int \mathrm{d}^3x\mathrm{d}^4\theta = \int \mathrm{d}^3x\,\mathrm{D}^2\bar{\mathrm{D}}^2|_{\theta=\bar{\theta}=0}\,. \end{aligned} \tag{A.14}$$

The θ-space δ-function is given by

$$\delta^4(\theta-\theta') = (\theta-\theta')^2(\bar{\theta}-\bar{\theta}')^2\,. \tag{A.15}$$

A.2. Supersymmetric Chern-Simons-Matter Theories

The class of supersymmetric gauge theories which is of interest for the present work is constructed starting from two basic types of superfields: real scalar superfields $V(x,\theta,\bar{\theta})$ and complex chiral superfields $\Phi(x,\theta,\bar{\theta})$.

Real unconstrained superfields can be used to describe gauge fields and their superpartners (plus possible auxiliary fields). The $\mathcal{N} = 2$ vector multiplet contains a real scalar, a complex fermion and the gauge field. In three dimensions, the complex fermion has only one physical helicity on-shell. The gauge field, on the other hand, will lose its longitudinal helicity by gauge invariance as usual, and only one of its two transverse helicities is physical. Together with the real scalar, these give two bosonic and two fermionic physical degrees of freedom. As we now see, these degrees of freedom are provided by the real superfield V.

In the abelian case, one can see that the following action

$$S_{CS,abelian} = \int \mathrm{d}^3x\mathrm{d}^4\theta\,\frac{1}{2}V\bar{\mathrm{D}}^\alpha\mathrm{D}_\alpha V \tag{A.16}$$

is invariant under the gauge transformations

$$V' = V + i(\bar{\Lambda} - \Lambda)\,, \tag{A.17}$$

where Λ is a chiral superfield, and yields the following equation of motion:

$$\bar{\mathrm{D}}^{\alpha}\mathrm{D}_{\alpha}V = 0\,. \tag{A.18}$$

The generalization to the non abelian case is a bit involved [35] and requires the addition of an extra coordinate t such that $t \in [0,1]$ and the generalized gauge superfield $\tilde{V}(x,\theta,\bar{\theta},t)$ depends on with the boundary conditions:

$$\tilde{V}(x,\theta,\bar{\theta},1) = V(x,\theta,\bar{\theta})\,, \quad \tilde{V}(x,\theta,\bar{\theta},0) = 0 \tag{A.19}$$

We refer to [45] for such a construction and we give here only the result, in terms of the simplest t-parametrization for the generalized gauge superfield, namely $\tilde{V} = tV$:

$$S_{CS} = \frac{k}{4\pi}\int \mathrm{d}^3x\mathrm{d}^4\theta \int_0^1 \mathrm{d}t\ \mathrm{Tr}\left(V\,\bar{\mathrm{D}}^{\alpha}(e^{-tV}\mathrm{D}_{\alpha}e^{tV})\right)\,, \tag{A.20}$$

which is invariant under the non abelian gauge transformations

$$e^{V'} = e^{i\bar{\Lambda}}e^{V}e^{-i\Lambda}\,. \tag{A.21}$$

The θ-expansion of the real superfield in the Wess-Zumino gauge leaves us with the right degrees of freedom of the vector multiplet[3]

$$\begin{aligned} V(x,\theta,\bar{\theta}) =& \theta^{\alpha}\bar{\theta}_{\alpha}\sigma(x) + \theta\gamma^{\mu}\bar{\theta}A_{\mu}(x) + \theta^2\bar{\theta}\bar{\chi}_{\alpha}(x) \\ &+ \bar{\theta}^2\theta^{\alpha}\chi_{\alpha}(x) + \theta^2\bar{\theta}^2 D(x)\,. \end{aligned} \tag{A.22}$$

Moreover, we can see that the action (A.20) projects to

$$S = -\frac{ik}{4\pi}\int \mathrm{d}^3x\epsilon^{\mu\nu\rho}\left(A_{\mu}\partial_{\nu}A_{\rho} + i\frac{2}{3}A_{\mu}A_{\nu}A_{\rho} - \chi\bar{\chi} - 2\sigma D\right) \tag{A.23}$$

[3] One scalar field is always auxiliary for general gauge theories. In particular it turns out, for pure Chern-Simons theories, that every field is auxiliary.

which is the Chern-Simons action for the vector field A_μ and the other members of the vector multiplet. None of these fields has physical, propagating degrees of freedom: they are all auxiliary fields. Even the gauge field A_μ is, in some sense, topological: since the action is linear in the derivative of the field, the equation of motion

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] = 0 \tag{A.24}$$

is a first order differential equation, which is an unusual situation for a bosonic field. Therefore, A_μ is a "pure gauge", in the sense that any solution of (A.24), by a suitable gauge transformation, can be made to vanish.

We now consider chiral superfields, suitable to describe matter fields, constrained by the condition[4]

$$\bar{\mathrm{D}}_\alpha \Phi(x, \theta, \bar{\theta}) = 0\,. \tag{A.25}$$

A scalar field with the superspace dependence $\Phi = \Phi(x_L, \theta)$, where

$$x_L^{\alpha\beta} = x^{\alpha\beta} + \frac{i}{4}(\theta^\alpha \bar{\theta}^\beta + \theta^\beta \bar{\theta}^\alpha)\,, \quad x_R^{\alpha\beta} = x^{\alpha\beta} - \frac{i}{4}(\theta^\alpha \bar{\theta}^\beta + \theta^\beta \bar{\theta}^\alpha)\,, \tag{A.26}$$

automatically satisfies the chiral constraint (and analogously an anti-chiral superfield with the dependence $\bar{\Phi} = \bar{\Phi}(x_R, \bar{\theta})$ satisfies the anti-chiral constraint $\mathrm{D}_\alpha \bar{\Phi}(x, \theta, \bar{\theta}) = 0$). The θ-expansion of such a superfield yields the $\mathcal{N} = 2$ scalar multiplet, containing a complex scalar boson ϕ, a two-component complex fermion ψ and a complex scalar F:

$$\Phi(x_L, \theta) = \phi(x_L) + \theta^\alpha \psi_\alpha(x_L) - \theta^2 F(x_L)\,. \tag{A.27}$$

The supersymmetric invariant action

$$S_0 = \int \mathrm{d}^3x \mathrm{d}^4\theta\, \bar{\Phi}\Phi \tag{A.28}$$

[4]Because of the presence of the spinor covariant derivative, this condition is invariant under supersymmetry transformations.

gives standard kinetic terms for the component fields (F isn't dynamical), while the superpotential

$$S_{pot} = \frac{\lambda}{4!}\int \mathrm{d}^3x\mathrm{d}^2\theta\,\Phi^4 + \frac{\lambda}{4!}\int \mathrm{d}^3x\mathrm{d}^2\bar{\theta}\,\bar{\Phi}^4 \tag{A.29}$$

gives[5] sextic self-interactions for the scalar field ϕ and quartic Yukawa couplings between ϕ and ψ (D-terms). In an on-shell analysis from the $\mathcal{N} = 1$ perspective, the scalar multiplet is formed by a real scalar and a real two-component fermion. The boson/fermion balance is satisfied on shell where the two-component fermion looses one of its helicities since in three dimensions, in contrast to four dimensions, there is a unique physical helicity. From this perspective, the $\mathcal{N} = 2$ scalar multiplet we have just constructed with the use of an $\mathcal{N} = 2$ chiral-constrained superfield is composed of a pair of $\mathcal{N} = 1$ scalar multiplets. Its physical on-shell degrees of freedom are a complex scalar and a complex two-component fermion, both with two physical degrees of freedom since, once again, the fermion has only one physical helicity. Notice that the complex nature of the fermion, as opposed to the four-dimensional case, is not related to it being in the fundamental representation of the Lorentz group but it is related to it being part of an $SO(2) = U(1)$ R-symmetry multiplet.

We now discuss the coupling of gauge and matter superfields. There are three ways in which we may couple chiral multiplets in a gauge invariant way with vector multiplets: adjoint, fundamental and bifundamental couplings. We restrict to bifundamental coupling, since it is the relevant coupling for ABJM theory. If V_1 and V_2 are two gauge superfields and Φ is a chiral superfield, the action

$$S = \int \mathrm{d}^3x\mathrm{d}^4\theta\,\mathrm{Tr}(e^{-V_1}\bar{\Phi}e^{V_2}\Phi) \tag{A.30}$$

is invariant under the gauge transformations

$$e^{V_1'} = e^{i\bar{\Lambda}}e^{V_1}e^{-i\Lambda}\,,\quad e^{V_2'} = e^{i\bar{\Lambda}}e^{V}e^{-i\Lambda}\,,\quad \Phi' = e^{i\Lambda_1}\Phi e^{-i\Lambda_2}\,,\quad \bar{\Phi}' = e^{i\bar{\Lambda}_2}\bar{\Phi}e^{-i\bar{\Lambda}_1}\,, \tag{A.31}$$

where $\Lambda_{1,2}$ are chiral superfields. By expanding this action in powers of the gauge fields, the first term of the expansion corresponds to the kinetic term of the chiral field (A.28). The rest of the terms determine the interaction between the scalar and gauge

[5]The auxiliary field F is integrated out by its equation of motion.

superfields. They give rise to sextic interactions for the scalars and Yukawa interactions between scalars and fermions (F-terms) in addition to the minimal coupling between gauge vectors and matter fields through the covariant derivative. We note that, even though pure Chern-Simons theory seemed trivially identical to pure bosonic CS theory since the superpartners of the gauge field were auxiliary fields, the lifting of the bosonic theory to an $\mathcal{N} = 2$ theory, when coupled to matter, has produced non trivial matter self-interactions.

We have now introduced all the ingredients needed to write the action of the ABJM theory. It includes two $U(N) \times U(N)$ non abelian vector multiplets V and $\hat{V}$, with opposite Chern-Simons levels and transforming in the adjoint representation of the first and second $U(N)$ respectively, and two pairs of scalar multiplets $Z^{1,2}$ and $W_{1,2}$, transforming in the bifundamental and anti-bifundamelntal of the gauge group respectively. The action is given by

$$S = S_{CS} + S_{mat} + S_{pot}\,, \tag{A.32}$$

where

$$\begin{aligned}
S_{CS} =& \frac{k}{4\pi} \int \mathrm{d}^3x\, \mathrm{d}^4\theta \int_0^1 \mathrm{d}t\ \mathrm{Tr}\left(V\, \bar{\mathrm{D}}^\alpha e^{-tV} \mathrm{D}_\alpha e^{tV} - \hat{V}\, \bar{\mathrm{D}}^\alpha e^{-t\hat{V}} \mathrm{D}_\alpha e^{t\hat{V}}\right)\,, \\
S_{mat} =& \int \mathrm{d}^3x\, \mathrm{d}^4\theta\, \mathrm{Tr}\left(\bar{Z}_A e^V Z^A e^{-\hat{V}} + \bar{W}^B e^{\hat{V}} W_B e^{-V}\right)\,, \\
S_{pot} =& \frac{2\pi i}{k}\left[\int \mathrm{d}^3x\, \mathrm{d}^2\theta\, \epsilon_{AC}\epsilon^{BD}\, \mathrm{Tr} Z^A W_B Z^C W_D + \int \mathrm{d}^3x\, \mathrm{d}^2\bar{\theta}\, \epsilon^{AC}\epsilon_{BD}\, \mathrm{Tr}\bar{Z}_A \bar{W}^B \bar{Z}_C \bar{W}^D\right]\,,
\end{aligned} \tag{A.33}$$

where ϵ^{AB} is the two-dimensional Levi-Civita tensor. Taking a look to the superpotential S_{pot} a global $SU(2) \times SU(2)$ flavour symmetry, acting separately on Z and W fields, is manifest. Moreover, we can build doublets $(Z^1, \bar{W}^1)$ and $(Z^2, \bar{W}_2)$ which transform under another $SU(2)$ group of R-symmetry. Combining these two symmetries we have enhanced $SU(4)$ symmetry for the multiplet $(Z^1, Z^2, \bar{W}^1, \bar{W}^2)$ transforming in the **4** of the group. This $SU(4) \simeq SO(6)$ is the R-symmetry group of $\mathcal{N} = 6$ supersymmetry. In the $\mathcal{N} = 2$ formalism, only a $U(1) \times SU(2) \times SU(2)$ symmetry is manifest: the $U(1)$ factor is the R-symmetry group of $\mathcal{N} = 2$ supersymmetry.

A.3. Superspace Quantization

We now proceed to the quantization of ABJM theory. It is very useful to quantize the theory directly in $\mathcal{N} = 2$ superspace. There are may computational simplifications, when dealing with superspace Feynman diagrams (usually denoted as supergraphs) since supersymmetry is kept manifest at every stage of the computation.

First of all we rescale the chiral fields as

$$Z^A \to \frac{k}{4\pi} Z^A \,, \quad W_A \to \frac{k}{4\pi} W_A \,, \tag{A.34}$$

so that all the terms in (A.33) have the coupling $k/4\pi$ as in [24]. We use standard functional methods to construct the path integral $Z = \int e^S$ and apply covariant quantization introducing gauge-fixing terms, gauge averaging, and superfield Faddeev-Popov ghosts.

To quantize each of the two Chern-Simons terms, we have to choose gauge-fixing functions. Corresponding to the chiral gauge parameter Λ we need a chiral gauge-variant function which is made to vanish by a suitable gauge transformation. A suitable choice is $F = \bar{\mathrm{D}}^2 V$. For every chiral function f we can find gauge transformations such that $F = f$. Defining the functional determinant

$$\Delta(V) = \int \mathcal{D}\Lambda \mathcal{D}\bar{\Lambda}\, \delta(F(V,\Lambda,\bar{\Lambda}) - f)\delta(\bar{F}(V,\Lambda,\bar{\Lambda}) - \bar{f}) \tag{A.35}$$

we insert the factor

$$\int \mathcal{D}f \mathcal{D}\bar{f} \Delta(V)\Delta^{-1}(V) \exp\left(-\frac{k}{2\alpha}\int \mathrm{d}^3x \mathrm{d}^2\theta\, \mathrm{Tr}(ff) + \mathrm{h.c.}\right) , \tag{A.36}$$

where α is a dimensionless gauge parameter, into the functional integral $Z = \int \mathcal{D}V\, e^{S[V]}$ and we average Z with gaussian chiral weights[6] obtaining, after some standard manipulations,

$$Z = \int \mathcal{D}V \mathcal{D}c \mathcal{D}\bar{c} \mathcal{D}c' \mathcal{D}\bar{c}' e^{S_{CS}+S_{GF}+S_{FP}} \,. \tag{A.37}$$

[6]Note that in the four-dimensional case one uses non-chiral weights instead.

The gauge-fixed action is given by

$$S_{CS} + S_{GF} = \frac{k}{4\pi} \int \mathrm{d}^3x \mathrm{d}^4\theta \int_0^1 \mathrm{d}t \, \mathrm{Tr}\, V \left(\bar{\mathrm{D}}^\alpha e^{-tV} \mathrm{D}_\alpha e^{tV} + \left(\frac{1}{2\alpha} \mathrm{D}^2 + \frac{1}{2\bar{\alpha}} \bar{\mathrm{D}}^2 \right) V \right) . \tag{A.38}$$

A similar term, in Z, should be added with $\hat{V}$ and $\hat{\alpha}$ in place of V and α and with $k \to -k$. Moreover, we introduced Faddeev-Popov ghosts c, c', $\bar{c}$, $\bar{c}'$ in order to rewrite $\Delta^{-1}(V)$ in the path integral in terms of anticommuting chiral superfields instead of the parameters Λ, $\bar{\Lambda}$:

$$S_{FP} = \frac{k}{4\pi} \int \mathrm{d}^3x \mathrm{d}^4\theta \, \mathrm{Tr}(c' + \bar{c}') \mathrm{L}_{\frac{1}{2}V} [c + \bar{c} + \coth \mathrm{L}_{\frac{1}{2}V} (c - \bar{c})] \, . \tag{A.39}$$

where $\mathrm{L}_V X = [V, X]$. A similar term, in Z, should be added with $\hat{V}$, $\hat{c}$, $\hat{c}'$, $\bar{\hat{c}}$, $\bar{\hat{c}}'$ in place of V, c, c', $\bar{c}$, $\bar{c}'$ and with $k \to -k$. Quantization of the matter action is straightforward and won't be derived here.

In order to read propagators and vertices form the actions, it is useful to expand them. The first term in the gauge action can be expanded as

$$\int_0^1 \mathrm{d}t \, \mathrm{Tr} V \bar{\mathrm{D}}^\alpha e^{-tV} \mathrm{D}_\alpha e^{tV} = \frac{1}{2} \mathrm{Tr}\, V \bar{\mathrm{D}}^\alpha \mathrm{D}_\alpha V - \frac{1}{6} \mathrm{Tr} V \bar{\mathrm{D}}^\alpha [V, \mathrm{D}_\alpha V] + \cdots \tag{A.40}$$

and similar for $\hat{V}$. The ghost action expands as

$$\mathrm{Tr}(c' + \bar{c}') \mathrm{L}_{\frac{1}{2}V} [c + \bar{c} + \coth \mathrm{L}_{\frac{1}{2}V} (c - \bar{c})] = \mathrm{Tr} \left(\bar{c}' c - c' \bar{c} + \frac{1}{2} (c' + \bar{c}') [V, c + \bar{c}] \right) + \cdots \tag{A.41}$$

The gauge-matter action expands as

$$\mathrm{Tr}\, \bar{Z}_A e^V Z^A e^{-\hat{V}} = \mathrm{Tr}\, \bar{Z}_A \Big(Z^A + V Z^A - Z^A \hat{V} + \frac{1}{2} (V^2 Z^A + Z^A \hat{V}^2) - V Z^A \hat{V} \Big) \ + \ldots \tag{A.42}$$

and similar for W_A.

The quadratic piece in (A.40), together with the α- and $\hat{\alpha}$-dependent gauge fixing terms, determines the gauge superfield propagators by inversion. We obtain, in momen-

tum space:

$$\begin{aligned}\langle V^a(p)V^b(-p)\rangle &= \frac{4\pi}{k}\frac{1}{p^2}\left(\bar{\mathrm{D}}^\alpha \mathrm{D}_\alpha + 2\alpha \mathrm{D}^2 + 2\bar{\alpha}\bar{\mathrm{D}}^2\right)\delta^4(\theta_1-\theta_2)\delta^{ab}\,,\\ \langle \hat{V}^a(p)\hat{V}^b(-p)\rangle &= -\frac{4\pi}{k}\frac{1}{p^2}\left(\bar{\mathrm{D}}^\alpha \mathrm{D}_\alpha + 2\hat{\alpha}\mathrm{D}^2 + 2\bar{\hat{\alpha}}\bar{\mathrm{D}}^2\right)\delta^4(\theta_1-\theta_2)\delta^{ab}\end{aligned} \tag{A.43}$$

In order to simplify the computations, we will choose the Landau gauge where $\alpha = \hat{\alpha} = 0$. The ghost propagators read

$$\begin{aligned}\langle \bar{c}'^a(p)c^b(-p)\rangle = \langle \bar{c}^a(p)c'^b(-p)\rangle &= \frac{4\pi}{k}\frac{1}{p^2}\delta^4(\theta_1-\theta_2)\delta^{ab}\,,\\ \langle \bar{\hat{c}}'^a(p)\hat{c}^b(-p)\rangle = \langle \bar{\hat{c}}^a(p)\hat{c}'^b(-p)\rangle &= -\frac{4\pi}{k}\frac{1}{p^2}\delta^4(\theta_1-\theta_2)\delta^{ab}\end{aligned} \tag{A.44}$$

Finally, the chiral propagators read

$$\langle (Z^B)^i_{\hat{i}}(p)(\bar{Z}_A)^{\hat{j}}_j(-p)\rangle = \langle (\bar{W}^B)^i_{\hat{i}}(p)(W_A)^{\hat{j}}_j(-p)\rangle = \frac{4\pi}{k}\frac{\delta^B_A}{p^2}\delta^4(\theta_1-\theta_2)\delta^i_j\delta^{\hat{j}}_{\hat{i}}\,, \tag{A.45}$$

The vertices can be read directly from the lagrangian by making functional derivatives of the superfields. When a functional derivatives w.r.t. the (anti)-chiral superfields is taken, factors of (D^2) $\bar{\mathrm{D}}^2$ are generated in the vertices. In the case of quartic chiral (antichiral) vertices, one of the $\bar{\mathrm{D}}^2$ (D^2) factors can be absorbed into the (anti)chiral integration such that the integration measure of the (anti)chiral vertex is promoted to the full superspace measure.

A.4. D-algebra

Once Feynman rules have been given for the superfields, one can draw Feynman diagrams in superspace (supergraphs). For every vertex on the graph there is a $\int \mathrm{d}^4\theta$ integral and for every loop there is, as usual, a $\int \mathrm{d}^3p$ integral. The fundamental step in order to handle such supergraphs is the so called D-algebra, which is a technique aimed to perform on the graphs the integration by parts on θ coordinates allowing us to reduce to a standard momentum-space Feynman integral. We summarize here the steps that should be followed.

One may transfer spinor covariant derivatives along a line producing a minus sign for each D-operator transferred. Notice that this does not produce any ambiguity in the construction of the diagram since the propagators always come with an even number of derivatives such that their derivatives may be thought to be acting in any of the two vertices which the line connects. Choosing a given line and integrating by parts the operators acting on it, lines are contracted to points in θ-space and, at the end of the procedure, the whole diagram is contracted to a point with a single θ integral. While dealing with integrations by parts one often uses the following rules, which follow from the algebra of spinor covariant derivatives:

$$\begin{aligned}
[\mathrm{D}^\alpha, \bar{\mathrm{D}}^2] &= i\partial^{\alpha\beta}\bar{\mathrm{D}}_\beta\,, \qquad [\bar{\mathrm{D}}^\alpha, \mathrm{D}^2] = i\partial^{\alpha\beta}\mathrm{D}_\beta \\
\mathrm{D}^2\bar{\mathrm{D}}^2\mathrm{D}^2 &= \Box\mathrm{D}^2\,, \qquad \bar{\mathrm{D}}^2\mathrm{D}^2\bar{\mathrm{D}}^2 = \Box\bar{\mathrm{D}}^2 \\
\mathrm{D}_\alpha\mathrm{D}_\beta &= C_{\alpha\beta}\mathrm{D}^2\,, \qquad \bar{\mathrm{D}}_\alpha\bar{\mathrm{D}}_\beta = C_{\alpha\beta}\bar{\mathrm{D}}^2 \\
\{\mathrm{D}^2, \bar{\mathrm{D}}^2\} &= \Box + \mathrm{D}^\alpha\bar{\mathrm{D}}^2\mathrm{D}_\alpha = \Box + \bar{\mathrm{D}}^\alpha\mathrm{D}^2\bar{\mathrm{D}}_\alpha \\
[\mathrm{D}^2, \bar{\mathrm{D}}^2] &= \frac{i}{2}\partial^{\alpha\beta}[\bar{\mathrm{D}}_\beta, \mathrm{D}_\alpha]
\end{aligned} \tag{A.46}$$

where $\Box = \partial^\mu\partial_\mu$ is ordinary D'Alembert operator. Moreover, a product of three or more Ds or $\bar{\mathrm{D}}$s vanishes. Furthermore, when one of the possible paths of integration by parts produces products of the form $p_{\alpha\beta}C^{\alpha\beta}$, symmetry in the indices of p and antisymmetry in the indices of C imply a zero contribution to the supergraph: this is an intrinsic three-dimensional phenomenum. Another result specific of three dimensions is that $\bar{\mathrm{D}}^\alpha\mathrm{D} = \mathrm{D}^\alpha\bar{\mathrm{D}}_\alpha$. This permits to have the freedom of combining this operator with the contiguous ones acting on it and usually leads to simplifications. Finally one uses the fact that the product of two $\delta_4(\theta_1 - \theta_2)$ on lines connecting the same vertices is zero due to fermion anticommutation. At the end of the procedure, one has to remove all the derivatives from all the lines of each loop, but one, which must have precisely four spinor covariant derivatives in the form $\mathrm{D}^2\bar{\mathrm{D}}^2$ or $\bar{\mathrm{D}}^2\mathrm{D}^2$. Otherwise the result is zero.

When working on the graphs, one can ignore all the signs coming from transfers and integration by parts and at the end count the number of transpositions of spinorial indices that the D-algebra produced. An (odd) even number of transpositions produces a (minus) plus sign.

Among the advantages of using supergraphs instead of ordinary Feynman diagrams, with supersymmetric gauge theories, there is the fact that the number of diagrams is usually significantly smaller. Furthermore, one can often find cancellation patterns between different supergraphs or demonstrate finiteness theorems for classes of diagrams, which follow from power-counting arguments. Some of these results, in the case of three-dimensional theories, are described in [24].

Appendix B.

Techniques for the evaluation of multiloop Feynman integrals

In this appendix we review some methods for the computation of the UV divergent part of the Feynman integrals needed in the present work. All the integrals are computed using dimensional regularization in Euclidean space of dimension

$$D = 2(\lambda + 1)\,, \tag{B.1}$$

where

$$\lambda = \frac{1}{2} - \varepsilon \tag{B.2}$$

in order to get three-dimensional space in the $\varepsilon \to 0$ limit.
A generic Wick-rotated ℓ-loop integral without numerators, in momentum space, has always the form

$$I_\ell = \frac{1}{(2\pi)^{\ell D}} \int \frac{d^D k_1 \cdots d^D k_\ell}{\Pi_1 \cdots \Pi_P}\,, \tag{B.3}$$

where P is the number of propagators Π_i, which, in general, depend on the loop momenta $k_1, \ldots, k_\ell$ and the external momenta $p_1, \ldots, p_e$. We always consider massless propagators in this thesis.

A propagator with weight α in momentum space is Fourier-transformed to coordinate space according to the following formula

$$\frac{1}{k^{2\alpha}} = \frac{\Gamma(\lambda+1-\alpha)}{\Gamma(\alpha)\pi^{\lambda+1}} \int \frac{\mathrm{d}^D x\, e^{2ikx}}{x^{2(\lambda+1-\alpha)}}\,, \tag{B.4}$$

where $\Gamma(z)$ is the Euler gamma function. The weight α is a generic complex number. We stress the presence of the unconventional factor 2 in the exponential. According to this definition, we have

$$\int \mathrm{d}^D k\, e^{2ikx} = \pi^{2(\lambda+1)}\delta(x)\,. \tag{B.5}$$

We don't consider, here, integrals with non trivial numerators, since they never appear in the computations of maximal-reshuffling diagrams. Nevertheless, they can be treated via the same methods we're going to describe in the following sections: we refer to [81,94–96] for these subjects.

B.1. Contraction of bubbles

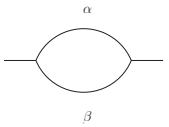
The first method we discuss here, for the computation of multiloop Feynman integrals, makes use of the so called G-functions. The one-loop master integral with general weights for the propagators is easily computed with the help of (B.4) and (B.5):

$$\begin{aligned} I_1(p;\alpha,\beta) &= \frac{1}{(2\pi)^D} \int \frac{d^D k}{k^{2\alpha}(k-p)^{2\beta}} \\ &= \frac{\Gamma(\lambda+1-\alpha)\Gamma(\lambda+1-\beta)}{(2\pi)^{2(\lambda+1)}\Gamma(\alpha)\Gamma(\beta)} \int \frac{\mathrm{d}^D x e^{2ipx}}{x^{2(2\lambda+2-\alpha-\beta)}} \\ &= \frac{\Gamma(\lambda+1-\alpha)\Gamma(\lambda+1-\beta)\Gamma(\alpha+\beta-\lambda-1)}{(4\pi)^{\lambda+1}\Gamma(\alpha)\Gamma(\beta)\Gamma(2\lambda+2-\alpha-\beta)} \frac{1}{p^{2(\alpha+\beta-\lambda-1)}} \\ &= G(\alpha,\beta)\,\frac{1}{p^{2(\alpha+\beta-\lambda-1)}}\,, \end{aligned} \tag{B.6}$$

where we have defined the G-function

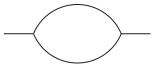
$$G(\alpha,\beta) = \frac{\Gamma(\lambda+1-\alpha)\Gamma(\lambda+1-\beta)\Gamma(\alpha+\beta-\lambda-1)}{(4\pi)^{\lambda+1}\Gamma(\alpha)\Gamma(\beta)\Gamma(2\lambda+2-\alpha-\beta)}\,. \tag{B.7}$$

We can represent graphically the integral $I_1(p;\alpha,\beta)$, indicating the weight of each propagator on the corresponding line, as



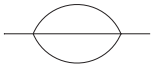
$$= G(\alpha,\beta) \quad \underline{\qquad}^{\alpha+\beta-1-\lambda} \,. \tag{B.8}$$

From power counting we immediately see that, in three dimensions, the one-loop bubble integral is finite[1]:



$$= G(1,1) \quad \underline{\qquad}^{\frac{1}{2}+\varepsilon} \,. \tag{B.9}$$

The two-loop bubble integral, on the contrary, is logarithmically UV divergent and, thanks to (B.8), is given by :



$$= G(1,1)G(1,1/2+\varepsilon) \quad \underline{\qquad}^{2\epsilon} \,. \tag{B.10}$$

The divergence of a diagram shows up as a pole, in ε, of the gamma function: the parameter ε thus works as a dimensional regulator. In order to obtain the Laurent expansions in ε of the integral, we make use of the following formula, valid for $|z|<1$:

$$\Gamma(1+z) = \exp\left(-\gamma z + \sum_{k=2}^{\infty}\frac{(-1)^k}{k}\zeta(k)z^k\right)\,, \tag{B.11}$$

where γ is the Euler-Mascheroni constant and

$$\zeta(k) = \sum_{n=1}^{\infty}\frac{1}{n^k} \tag{B.12}$$

[1]We recall that the corresponding four-dimensional integral is logarithmically divergent.

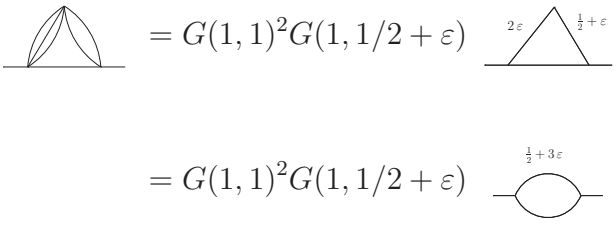
is the Riemann zeta function. The two-loop bubble integral under consideration, *e.g.*, has the following expansion[2]:

$$I_2 = \frac{1}{64\pi^2\varepsilon} + \frac{1}{32\pi^2}(3 - \gamma + \log 4\pi) + \mathcal{O}(\varepsilon)\,. \tag{B.13}$$

We now consider higher loop integrals. In the simplest cases, the loop integral can be computed by recursively contracting bubbles in momentum space, reducing the number of loops of the original diagram by repeated use of the (B.8). The following trick often simplifies the computations: put some external momentum to zero; if this doesn't introduce any IR divergence, then only the finite part of the result will be affected. So, if we are interested only in the UV divergent part of the integral (as always in this work, since the renormalization of operators is concerned) this procedure won't alter the result. Graphically, we eliminate the external line connecting to two propagators and draw a single propagator, whose weight is the sum of the original weights. The method of contracting bubbles is successful if we end up with a single bubble, which is resolved again by (B.8).

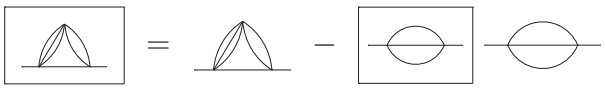
When dealing with multiloop integrals, the divergence usually shows itself as a higher-order pole in ε: this signals the presence of subdivergences in the integral which must be consistently subtracted in order to get the overall divergence (see [97]). We graphically denote by a box around a diagram the principal part of the diagram itself, after all the subdivergences have been subtracted.

We explicitly show how this method works with a simple four-loop example:



(B.14)

[2]From now on, we will omit the dependence of the result of a loop integral on the external momentum p.



$$= \frac{1}{(8\pi)^4}\left(-\frac{1}{2\varepsilon^2}+\frac{2}{\varepsilon}\right). \qquad \text{(B.15)}$$

In the following, we list the results of the integrals, relevant for the computation of maximal reshuffling diagrams of ABJM theory in the $SU(2)\times SU(2)$ sector, which can be effectively computed through the method of contracting bubbles described above. We denote by I_j the value of the integral[3] and by $\bar{I}_j$ the subtracted one. We conveniently factor out $1/(4\pi)^\ell$ from the results:

$$I_2 \; = \; = \; G(1,1)G(1,1/2+\varepsilon)$$

$$I_4 \; = \; = \; G(1,1)^2G(1,1/2+\varepsilon)G(1,1/2+3\,\varepsilon) \qquad \text{(B.16)}$$

$$I_{4a} \; = \; = \; G(1,1)^2G(1,1/2+\varepsilon)^2$$

$$I_6 \; = \; = \; G(1,1)^2G(1,1/2+\varepsilon)G(1,1+2\,\varepsilon)G(1,1/2+3\,\varepsilon) \times G(1+4\,\varepsilon,1/2+\varepsilon)$$

$$I_{6a} \; = \; = \; G(1,1)^2G(1,1/2+\varepsilon)^2G(1,1+2\,\varepsilon)G(1+2\,\varepsilon,1/2+3\,\varepsilon) \qquad \text{(B.17)}$$

$$I_{6b} \; = \; = \; G(1,1)^2G(1,1/2+\varepsilon)^2G(1,1+2\,\varepsilon)G(1,1/2+3\,\varepsilon)$$

[3]Recall that, because of the procedure described above, the value of I_j can differ from the true one in the finite parts.

$$I_{8d} = \quad = G(1,1)^2 G(1,1/2+\varepsilon)^2 G(1,1+2\,\varepsilon) G(1,1/2+3\,\varepsilon) \times G(1,1+4\,\varepsilon) G(1+2\,\varepsilon,1/2+5\,\varepsilon)$$

$$I_{8e} = \quad = G(1,1)^2 G(1,1/2+\varepsilon) G(1,1+2\,\varepsilon) G(1,1/2+3\,\varepsilon) \times G(1,1+4\,\varepsilon) G(1,1/2+5\,\varepsilon) G(1+6\,\varepsilon,1/2+\varepsilon) \tag{B.18}$$

$$\bar{I}_2 = \quad = \frac{1}{(4\pi)^2}\,\frac{1}{4\,\varepsilon}$$

$$\bar{I}_4 = \quad = \quad - \quad = \frac{1}{(4\pi)^4}\left(-\frac{1}{32\,\varepsilon^2}+\frac{1}{8\,\varepsilon}\right)$$

$$\bar{I}_{4a} = \quad = \quad - \; 2 \quad$$

$$= \frac{1}{(4\pi)^4}\left(-\frac{1}{16\,\varepsilon^2}\right)$$

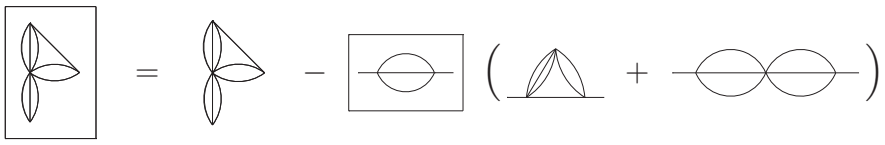
$$\bar{I}_6 = \quad = \quad - \quad - \quad \tag{B.19}$$

$$= \frac{1}{(4\pi)^6}\left(-\frac{1}{384\,\varepsilon^3}-\frac{1}{32\,\varepsilon^2}+\frac{1}{6\,\varepsilon}\right)$$

$$\bar{I}_{6a} = \quad = \quad - \; 2 \quad - \quad$$

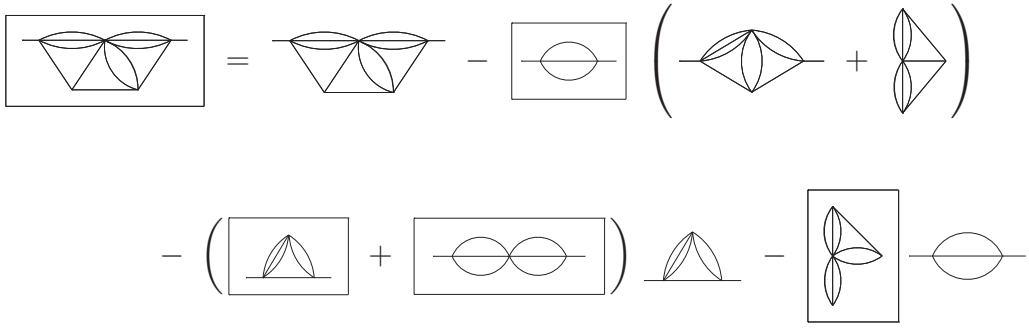
$$= \frac{1}{(4\pi)^6}\left(\frac{1}{192\,\varepsilon^3}-\frac{1}{48\,\varepsilon^2}-\frac{1}{24\,\varepsilon}\right)$$

$\bar{I}_{6b}$ =



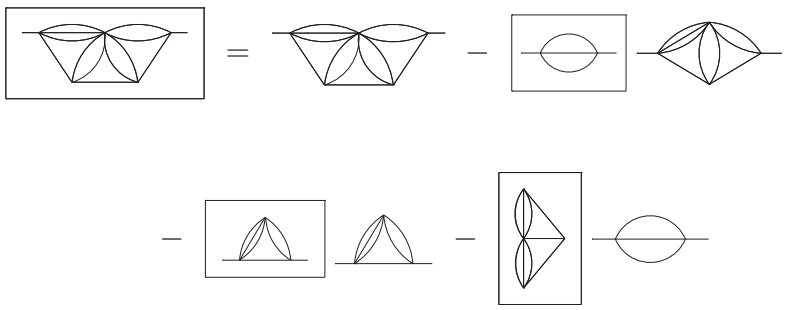
$$= \frac{1}{(4\pi)^6}\left(\frac{1}{128\,\varepsilon^3} - \frac{1}{32\,\varepsilon^2}\right)$$

$\bar{I}_{8d}$ =



$$= \frac{1}{(4\pi)^8}\left(-\frac{1}{2048\,\varepsilon^4} + \frac{1}{192\,\varepsilon^3} - \frac{1}{64\,\varepsilon^2} - \frac{11}{192\,\varepsilon}\right)$$

$\bar{I}_{8e}$ =



$$= \frac{1}{(4\pi)^8}\left(-\frac{1}{6144\,\varepsilon^4} + \frac{1}{256\,\varepsilon^3} - \frac{19}{384\,\varepsilon^2} + \frac{5}{16\,\varepsilon}\right)$$

(B.20)

B.2. The Gegenbauer polynomials x-space technique

We discuss here a technique which has proved very powerful for the computation of complicated multiloop Feynman integrals, especially if our goal is the renormalization of operators. This is the so called "*Gegenbauer polynomial x-space technique*" and was introduced in [93] and developed in [94, 95]. See also [81, 96] for useful reviews.

According to this technique, which we will call GPXT, the computations are made directly in coordinate space rather than in momentum space. The technique is grounded on the observation that, in x-space, the scalar propagator always depends on the difference of two points,

$$\Delta(x_i, x_j) = \frac{1}{(x_i - x_j)^{2\lambda}}\,, \tag{B.21}$$

and can thus be expanded in terms of the Gegenbauer polynomials, which form an orthogonal set on the unit sphere in $\mathbb{R}^D$. For the moment, D is an arbitrary integer dimension and $\lambda = D/2 - 1$. The analytic continuation to complex D will be done in a second step.

The Gegenbauer polynomials C_n^α, also known as ultraspherical polynomials, are the generalization of the Legendre polynomials of ordinary two-dimensional spherical harmonics to D-dimensional space. They can be defined in terms of a generating function,

$$\frac{1}{(1 - 2xt + t^2)^\alpha} = \sum_{n=0}^{\infty} = C_n^\alpha(x)\, t^n\,, \tag{B.22}$$

where $x \in [-1, 1]$. We refer to the quantity α as the weight of the polynomial, while n is its index. When $\alpha = \lambda$, the Gegenbauer polynomials are orthogonal with respect to the weight function $(1 - x^2)^{\lambda - 1/2}$:

$$\int_{-1}^{1} \mathrm{d}x\, (1 - x^2)^{\lambda - \frac{1}{2}} C_n^\lambda(x) C_m^\lambda(x) = \frac{\pi 2^{1-2\lambda}\Gamma(n + 2\lambda)}{n!(n + \lambda)\Gamma(\lambda)^2}\, \delta_{nm}\,. \tag{B.23}$$

The formulae

$$C_n^\alpha(x) = \sum_{p=0}^{[n/2]} \frac{(2x)^{n-2p}(-1)^p \Gamma(n - p + \alpha)}{(n - 2p)!p!\Gamma(\alpha)}\,, \tag{B.24}$$

$$\frac{(2x)^n}{n!} = \sum_{p=0}^{[n/2]} C^{\alpha}_{n-2p}(x)\, \frac{(n-2p+\alpha)\Gamma(\alpha)}{p!\Gamma(n-p+\alpha+1)} \tag{B.25}$$

allow us to express a polynomial with generic weight α as a combination of polynomials with weight λ:

$$C^{\alpha}_{n}(x) = \sum_{k=0}^{[n/2]} C^{\lambda}_{n-2k}(x)\, \frac{(n-2k+\lambda)\Gamma(\lambda)\Gamma(n-k+\alpha)\Gamma(k+\alpha-\lambda)}{k!\Gamma(\alpha)\Gamma(n+\lambda+1-k)\Gamma(\alpha-\lambda)}\,. \tag{B.26}$$

The value of a Gegenbauer polynomial at $x = 1$ is often needed:

$$C^{\alpha}_{n}(1) = \frac{\Gamma(n+2\alpha)}{n!\Gamma(2\alpha)}\,. \tag{B.27}$$

The Gegenbauer polynomials satisfy the recurrence relation:

$$\begin{aligned} C^{\alpha}_{0}(x) &= 1\,, \\ C^{\alpha}_{1}(x) &= 2\alpha x\,, \\ C^{\alpha}_{n}(x) &= \frac{1}{n}\left[2x(n+\alpha-1)C^{\alpha}_{n-1}(x) - (n+2\alpha-2)C^{\alpha}_{n-2}(x)\right]. \end{aligned} \tag{B.28}$$

The product of two polynomials with different indices, evaluate at the same point, can be converted into a sum

$$C^{\lambda}_{l}(x)C^{\lambda}_{m}(x) = \sum_{\substack{n=|l-m| \\ n+l+m\in 2\mathbb{N}}}^{l+m} D_{\lambda}(l,m;n)\, C^{\lambda}_{n}(x) \tag{B.29}$$

with coefficients given by

$$\begin{aligned} D_{\lambda}(l,m;n) = {} & \frac{n!(n+\lambda)\Gamma(\frac{n+m+l}{2}+2\lambda)}{\Gamma(\lambda)^2\Gamma(\frac{n+m+l}{2}+\lambda+1)\Gamma(n+2\lambda)} \\ & \times \frac{\Gamma(\frac{n+m+l}{2}-l+\lambda)\Gamma(\frac{n+m+l}{2}-m+\lambda)\Gamma(\frac{n+m+l}{2}-n+\lambda)}{\Gamma(\frac{n+m+l}{2}-l+1)\Gamma(\frac{n+m+l}{2}-m+1)\Gamma(\frac{n+m+l}{2}-n+1)}\,. \end{aligned} \tag{B.30}$$

Let's now turn to the description of the technique. We will consider here only diagrams with a single external momentum p, entering and leaving the graph at points $x_{\rm in}$ and

$x_{\rm out}$ in x-space. The generic ℓ-loop integral in (B.3) becomes, in x-space,

$$I_\ell = \frac{\Gamma(\lambda)^P}{(4^\ell \pi^P)^{\lambda+1}} \int \mathrm{d}^D x_1 \cdots \mathrm{d}^D x_{P-\ell} \frac{e^{2ip(x_{\rm out}-x_{\rm in})}}{\Delta_1 \cdots \Delta_P} \tag{B.31}$$

where Δ_i are the propagators in coordinate space.

It is now convenient to move to spherical coordinates in D dimensions: to this purpose we define

$$r = x^2\,, \qquad \hat{x} = \frac{x}{\sqrt{r}}\,. \tag{B.32}$$

So r is the (squared) radial coordinate and $\hat{x}$ is the unit vector pointing in the same direction as x. The integration measure changes to

$$\mathrm{d}^D x = \frac{1}{2} S_{D-1} r^\lambda \mathrm{d}r \mathrm{d}\hat{x}\,, \tag{B.33}$$

where

$$S_{D-1} = \frac{2\pi^{\lambda+1}}{\Gamma(\lambda+1)} \tag{B.34}$$

is the surface of the unit sphere in $\mathbb{R}^D$. The integral (B.31), therefore, transforms to

$$I_\ell = N_\lambda(\ell, P) \int \frac{\mathrm{d}r_1 \cdots \mathrm{d}r_{P-\ell} \mathrm{d}\hat{x}_1 \cdots \mathrm{d}\hat{x}_{P-\ell} r_1^\lambda \cdots r_{P-\ell}^\lambda \, e^{2ip(x_{\rm out}-x_{\rm in})}}{\Delta_1 \cdots \Delta_P} \tag{B.35}$$

with the normalization factor

$$N_\lambda(\ell, P) = \frac{\Gamma(\lambda+1)^\ell}{(4\pi)^{\ell(\lambda+1)} \lambda^P}\,. \tag{B.36}$$

At this point we can expand the propagators in terms of the Gegenbauer polynomials: from (B.22) we have

$$\Delta(x_i, x_j) = \frac{1}{(x_i - x_j)^{2\lambda}} = \frac{1}{M_{i,j}^\lambda} \sum_{n=0}^{\infty} C_n^\lambda(\hat{x}_i \cdot \hat{x}_j) \left(\frac{m_{i,j}}{M_{i,j}}\right)^{\frac{n}{2}}\,, \tag{B.37}$$

where we have introduced the notation

$$m_{i,j} = \min(r_i, r_j)\,, \qquad M_{i,j} = \max(r_i, r_j)\,. \tag{B.38}$$

The exponential too can be expanded in terms of the Gegenbauer polynomials:

$$e^{2ipx} = \Gamma(\lambda) \sum_{n=0}^{\infty} i^n (n+\lambda) C_n^{\lambda}(\hat{x}\cdot\hat{p})(p^2 r)^{\frac{n}{2}} j_{\lambda+n}(p^2 r)\,, \tag{B.39}$$

where

$$j_\alpha(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\alpha+1)}\, t^n = t^{-\alpha/2} J_\alpha(2\sqrt{t}) \tag{B.40}$$

is related to the Bessel function J_α and has the following property:

$$\int_0^\infty \mathrm{d}t\, t^a j_b(t) = \frac{\Gamma(b+1)}{\Gamma(a-b)}\,, \qquad \mathrm{Re}\, b > -1,\ \mathrm{Re}\, a > 2\mathrm{Re}\, b + \frac{1}{2}\,. \tag{B.41}$$

We see that the radial integration is somewhat complicated by the appearance of the Bessel functions and by an additional infinite sum. However, if we are interested only in the UV divergent part of the loop integral, as in all the cases in this work, a great simplification occurs: in x-space, the UV divergence is located where all coordinates are small, so we can approximate the exponential factor with unit, neglecting it. Since it is equivalent to set to zero the external momentum p, dropping the exponential will introduce IR divergences which mix to UV ones altering the final result. In x-space, IR divergences come from the region where some coordinates are large, so we can regulate them by introducing an infrared cutoff R as upper bound for the radial integrations. At the end of the computation, when all the subdivergences have been subtracted, the principal part of the integral must be independent of the regulator R, while the finite part in general depends on R and should be discarded from the result.

The expansion of the propagators in series of Gegenbauer polynomials allows us to separate the integral in radial and angular parts and introduces as many infinite sums as the number of the propagators themselves. However, we can minimize the number of series in this way: thanks to translational invariance of the integral (B.31), we can choose one of the vertices of the diagram as the origin of D-dimensional space. We call this

vertex the *root vertex*. All the propagators directly connected to the root vertex, which we call *root propagators*, are simply given by $1/r_i^\lambda$, where r_i is the radial coordinate of the vertex which connects the propagator to the root verex, and so they don't produce any series expansion in the Gegenbauer polynomials. Obviously, the best choice of the root vertex is usually such that the number of propagators attached to it is maximized. Therefore, in most cases, the root vertex will coincide with the composite operator.

Once we have chosen the root vertex and we have expanded the propagators depending on differences of coordinates in terms of Gegenbauer polynomials, the angular and radial integrations are performed separately.

The angular integration can be performed by repeated use of the orthogonality relation (B.23) of the Gegenbauer polynomials, which can be rewritten in terms of the angular variables as

$$\int \mathrm{d}\hat{x}\, C_n^\lambda(\hat{x}_i \cdot \hat{x}) C_m^\lambda(\hat{x} \cdot \hat{x}_j) = \frac{\lambda}{n+\lambda}\, \delta_{nm}\, C_n^\lambda(\hat{x}_i \cdot \hat{x}_j)\,. \tag{B.42}$$

The angular integration is normalized as $\int \mathrm{d}\hat{x} = 1$. In particular, since $C_0^\lambda(x) = 1$, we have

$$\int \mathrm{d}\hat{x}_i \mathrm{d}\hat{x}_j\, C_n^\lambda(\hat{x}_i \cdot \hat{x}_j) = \delta_{n0}\,. \tag{B.43}$$

If the same angular variables $\hat{x}_i$ appear inside more than two polynomials, we have to expand the product according to (B.29) and (B.30) and then use (B.42). Krönecker deltas can be used to decrease the number of summations. The angular integration is the main obstacle in multiloop computations: since complexity of the angular integrals crucially depends on the choice of the root vertex, one should place it so as to minimize the angular loop number and the number of infinite summations. In fact, GPXT is at its best when the number of infinite summations can be reduced at most to one.

Let's now turn to the radial integration. Having dropped the exponential and introduced the infrared cutoff, radial integrands consist of simple powers. The only difficulty is that, because of the presence of the min and max functions, the domain of integration (which is an hypercube of length R) has to be split into $(P-\ell)!$ subdomains, defined by the different orderings of the radial variables. This number can be large. Of course this

is not a problem if the procedure is automated with the help of a computer[4]. Anyway, it is useful to find all the possible symmetries of the integrand in order to reduce the independent domains of integrations.

At this point, when the angular and radial integrations have been performed, the next point is to promote the dimension D, or equivalently λ, to a complex parameter through the formula (B.2) and then perform the Laurent expansion of the result around $\varepsilon = 0$. If multiple poles are present, we proceed to the subtraction of subdivergences. These must be computed within the same renormalization scheme as the original integral, *i.e.* using GPXT and introducing the same cutoff procedure for infrared regularization.

The last step is to perform the summations that possibly survived after the angular integrations. As stated before, finding analytical results can be very hard, especially when multiple series are present, and sometimes only a numerical analysis is possible.

In order to illustrate the technique, we explicitly compute the following six-loop integral, which cannot be resolved through the method of contracting bubbles:

$$I_{6c} \;=\; \quad = \; N_\lambda(6,9)\int \frac{\mathrm{d}r_1\mathrm{d}r_2\mathrm{d}r_3\mathrm{d}\hat{x}_1\mathrm{d}\hat{x}_2\mathrm{d}\hat{x}_3\, r_1^\lambda r_2^\lambda r_3^\lambda}{r_1^{2\lambda}r_2^{3\lambda}r_3^{2\lambda}(x_1-x_2)^{2\lambda}(x_2-x_3)^{2\lambda}} \tag{B.44}$$

where the root vertex, denoted by 0, has been conveniently chosen as the upper one. Now we expand the two propagators depending on differences of coordinates in series of Gegenbauer polynomials and use the orthogonality relation:

$$\begin{aligned}
I_{6c} =& N_\lambda(6,9)\int \frac{\mathrm{d}r_1\mathrm{d}r_2\mathrm{d}r_3\mathrm{d}\hat{x}_1\mathrm{d}\hat{x}_2\mathrm{d}\hat{x}_3}{r_1^\lambda r_2^{2\lambda} r_3^\lambda M_{1,2}^\lambda M_{2,3}^\lambda}\sum_{n,m=0}^{\infty}\left(\frac{m_{1,2}}{M_{1,2}}\right)^{\frac{n}{2}}\left(\frac{m_{2,3}}{M_{2,3}}\right)^{\frac{m}{2}} C_n^\lambda(\hat{x}_1\cdot\hat{x}_2)C_m^\lambda(\hat{x}_2\cdot\hat{x}_3)\\
=& N_\lambda(6,9)\int \frac{\mathrm{d}r_1\mathrm{d}r_2\mathrm{d}r_3\mathrm{d}\hat{x}_1\mathrm{d}\hat{x}_3}{r_1^\lambda r_2^{2\lambda} r_3^\lambda M_{1,2}^\lambda M_{2,3}^\lambda}\sum_{n,m=0}^{\infty}\left(\frac{m_{1,2}}{M_{1,2}}\right)^{\frac{n}{2}}\left(\frac{m_{2,3}}{M_{2,3}}\right)^{\frac{m}{2}} \frac{\lambda}{n+\lambda}\delta_{nm}C_n^\lambda(\hat{x}_1\cdot\hat{x}_3)\\
=& N_\lambda(6,9)\int \frac{\mathrm{d}r_1\mathrm{d}r_2\mathrm{d}r_3}{r_1^\lambda r_2^{2\lambda} r_3^\lambda M_{1,2}^\lambda M_{2,3}^\lambda}\sum_{n,m=0}^{\infty}\left(\frac{m_{1,2}}{M_{1,2}}\right)^{\frac{n}{2}}\left(\frac{m_{2,3}}{M_{2,3}}\right)^{\frac{m}{2}} \frac{\lambda}{n+\lambda}\delta_{nm}\delta_{n0}\\
=& N_\lambda(6,9)\int \frac{\mathrm{d}r_1\mathrm{d}r_2\mathrm{d}r_3}{r_1^\lambda r_2^{2\lambda} r_3^\lambda M_{1,2}^\lambda M_{2,3}^\lambda}\,.
\end{aligned} \tag{B.45}$$

[4]In particular, we used the `Mathematica` routine for radial integrals described in [96].

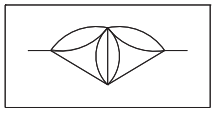
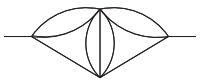
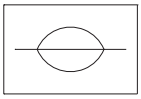
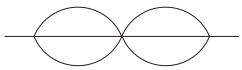
In this case, we could use the two Krönecker deltas coming from angular integrations to eliminate all the infinite summations, thus obtaining a great simplification. We are now left with the radial integral: introducing the cutoff R, the domain of integration is a cube, but, because of the presence of the two max functions, it should be split into 6! subdomains defined by the orderings of $0 \leq r_1 \leq r_2 \leq r_3 \leq R$ and permutations thereof. However, we note that the integrand is symmetric under exchange of the r_1 and r_3 variables, so only three independent integrations have to be done:

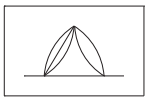
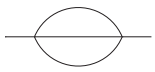
$$\begin{aligned} I_{6c} &= N_\lambda(6,9)\left(\int_0^R \frac{\mathrm{d}r_3}{r_3^{2\lambda}} \int_0^{r_3} \frac{\mathrm{d}r_2}{r_2^{3\lambda}} \int_0^{r_2} \frac{\mathrm{d}r_1}{r_1^{\lambda}} + \int_0^R \frac{\mathrm{d}r_3}{r_3^{2\lambda}} \int_0^{r_3} \frac{\mathrm{d}r_1}{r_1^{2\lambda}} \int_0^{r_1} \frac{\mathrm{d}r_2}{r_2^{\lambda}} \right. \\ &\qquad \left. + \int_0^R \frac{\mathrm{d}r_2}{r_2^{4\lambda}} \int_0^{r_2} \frac{\mathrm{d}r_3}{r_3^{\lambda}} \int_0^{r_3} \frac{\mathrm{d}r_1}{r_1^{\lambda}} \right) \\ &= N_\lambda(6,9)\, R^{3-6\lambda} \left(\frac{1}{3(1-\lambda)(1-2\lambda)^2} + \frac{1}{3(1-2\lambda^3)} + \frac{1}{3(1-\lambda)^2(1-2\lambda)} \right). \end{aligned} \tag{B.46}$$

Using (B.2), we can now make the ε-expansion of the integral. We have:

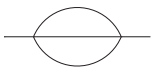
$$\frac{I_{6c}}{N_\lambda(6,9)} = \frac{1}{24\,\varepsilon^3} + \frac{1}{\varepsilon^2}\left(\frac{1}{6} + \frac{1}{4}\log R\right) + \frac{1}{\varepsilon}\left(\frac{1}{3} + \log R + \frac{3}{4}\log^2 R\right). \tag{B.47}$$

The R-dependence disappears if we subtract the subdivergences from this result:

$\bar{I}_{6c} =$  $=$  $-$  

$-\ 2$   . (B.48)

The pole parts that appear on the right hand side of the last equation obviously coincide with the ones computed in the previous section. However, the two integrals that multiply these pole parts have to be computed with GPXT in order to consistently regulate the infrared divergences hidden in R. Here, it is sufficient to compute the two loop integral:

 $= N_\lambda(2,3) \int_0^R \frac{\mathrm{d}r\, r^\lambda}{r^{3\lambda}} = N_\lambda(2,3)\, R^{1-2\lambda} \frac{1}{1-2\lambda}$. (B.49)

Expanding in ε and inserting the result in (B.48), we finally obtain

$$\bar{I}_{6c} = \frac{1}{(4\pi)^6}\left(\frac{1}{192\,\varepsilon^3} - \frac{1}{24\,\varepsilon^2} + \frac{1}{24\,\varepsilon}\right). \tag{B.50}$$

In order to perform calculations with GPXT more quickly, it is useful to work directly on the graphs. Since the expansion of propagators in terms of Gegenbauer polynomials allows us to separate the integral into its angular and radial part, one can draw separate graphs.

For the angular graph, we introduce the following rules:

1. Starting from the original graph, draw the same graph without root propagators, labelling the vertices $\hat{x}_1, \hat{x}_2, \ldots$ and the lines $n, m, \ldots$.
2. For each line introduce an infinite summation $\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\cdots$
3. Associate to each line a Gegenbauer polynomial:

$$\underset{\hat{x}_i \qquad n \qquad \hat{x}_j}{\bullet\!\!-\!\!-\!\!\bullet} = C_n^{\lambda}(\hat{x}_i\cdot\hat{x}_j)\,. \tag{B.51}$$

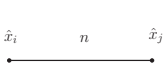
4. Impose orthogonality relation to reduce the number of lines:

$$\underset{\hat{x}_i \quad n \quad \hat{x}_k \quad m \quad \hat{x}_j}{\bullet\!\!-\!\!-\!\!\bullet\!\!-\!\!-\!\!\bullet} = \delta_{nm}\,\frac{\lambda}{\lambda+1}\;\underset{\hat{x}_i \qquad n \qquad \hat{x}_j}{\bullet\!\!-\!\!-\!\!\bullet}\,. \tag{B.52}$$

5. For each bubble, introduce a finite sum $\sum_{n=|l-m|,l+m+n\in 2\mathbb{N}}^{l+m}$ and contract it according to:

$$\underset{\hat{x}_i \qquad\qquad \hat{x}_j}{\overset{\overset{l}{\frown}}{\bullet\!\!\bullet}\!\!\underset{m}{\smile}} = D_\lambda(l,m;n)\;\underset{\hat{x}_i \qquad n \qquad \hat{x}_j}{\bullet\!\!-\!\!-\!\!\bullet}\,. \tag{B.53}$$

6. At the end of the procedure, we are left with one of the following situations:

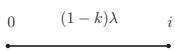


$$= \delta_{n0}\,, \qquad \hat{x} \;\; n \;\; = \frac{\Gamma(n+2\lambda)}{n!\Gamma(2\lambda)}\,. \tag{B.54}$$

For the radial graph, we introduce the following rules:

1. Draw the original graph, contracting possible bubbles made out of root propagators, and label the root vertex with 0 and the other vertices with positive integers. Moreover, label each non-root line with the index of the corresponding Gegenbauer polynomial.

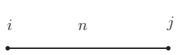
2. Introduce a factor $r_i^{-(k-1)\lambda}$ for each vertex connected to the root vertex (k is the number of root lines, -1 stands for the integration measure contribution):



$$= r_i^{(1-k)\lambda}\,. \tag{B.55}$$

3. Introduce a factor r_j^{λ} for each vertex which is not connected to the root vertex (contribution of the integration measure).

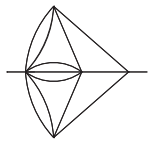
4. Associate to each non-root line the Max and min functions according to:



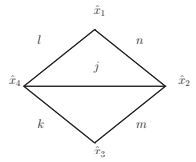
$$= M_{i,j}^{-\lambda-\frac{n}{2}}\, m_{i,j}^{\frac{n}{2}}\,. \tag{B.56}$$

Once we have written the angular and the radial graphs, we can implement the Krönecker deltas, perform the radial integrations, multiply by the normalization factor $N_\lambda(\ell, P)$, expand the result in ε and perform the remaining summations.

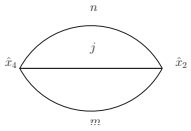
Consider, as an example, the following complicated eight-loop integral, which is required in the present thesis:

$$I_{8a} \;=$$  . (B.57)

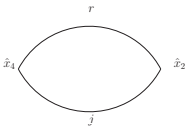
We choose to put the root vertex on the composite operator. The angular graph is quite complicated, since it turns out to be two-loop: this didn't happen in the $\mathcal{N}=4$ SYM counterpart, making the computation much harder in our case[5]. We obtain:



$$
\begin{aligned}
&= \delta_{nl}\,\delta_{mk}\,\frac{\lambda}{n+\lambda}\frac{\lambda}{m+\lambda}
\end{aligned}
$$

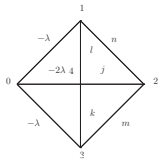


$$
\begin{aligned}
&= \frac{\lambda^2}{(n+\lambda)(m+\lambda)}\,\delta_{nl}\,\delta_{mk}\,D_\lambda(n,m;r)
\end{aligned}
$$



$$
\begin{aligned}
&= \frac{\lambda^2}{(n+\lambda)(m+\lambda)}\,\delta_{nl}\,\delta_{mk}\,\delta_{s0}\,D_\lambda(n,m;r)D_\lambda(r,j;s)\,.
\end{aligned}
\tag{B.58}
$$

The radial graph is given by



$$
\begin{aligned}
&= r_1^{-\lambda} r_4^{-2\lambda} r_3^{-\lambda}\, M_{1,2}^{-\lambda-\frac{n}{2}} M_{2,3}^{-\lambda-\frac{m}{2}} \\
&\quad \times M_{3,4}^{-\lambda-\frac{k}{2}} M_{1,4}^{-\lambda-\frac{l}{2}} M_{2,4}^{-\lambda-\frac{j}{2}} m_{1,2}^{\frac{n}{2}} m_{2,3}^{\frac{m}{2}} m_{3,4}^{\frac{k}{2}} m_{1,4}^{\frac{l}{2}} m_{2,4}^{\frac{j}{2}}\,.
\end{aligned}
\tag{B.59}
$$

Combining the two contributions, we can write

$$
I_{8a} \;=\; N_\lambda(8,12) \sum_{n,m,j=0}^{\infty} \mathcal{R}_{n,m,j}(\lambda)\,\mathcal{A}_{n,m,j}(\lambda)\,, \tag{B.60}
$$

[5]Note that, if we choose the central vertex as the root vertex, the angular graph becomes one-loop when the two bubbles have been contracted. One can think that this can be a better choice. Unfortunately, the contraction of bubbles introduces extra sums and doesn't lead to any simplification in the computation.

where

$$\mathcal{R}_{n,m,j}(\lambda) = \int \frac{\mathrm{d}r_1\mathrm{d}r_2\mathrm{d}r_3\mathrm{d}r_4\, r_2^\lambda}{r_1^\lambda r_4^{2\lambda} r_3^\lambda M_{1,2}^\lambda M_{2,3}^\lambda M_{3,4}^\lambda M_{4,1}^\lambda M_{2,4}^\lambda} \left(\frac{m_{1,2}m_{1,4}}{M_{1,2}M_{1,4}}\right)^{\frac{n}{2}} \left(\frac{m_{2,3}m_{3,4}}{M_{2,3}M_{3,4}}\right)^{\frac{m}{2}} \left(\frac{m_{2,4}}{M_{2,4}}\right)^{\frac{j}{2}}$$

$$\mathcal{A}_{n,m,j}(\lambda) = \sum_{\substack{r=|n-m| \\ n+m+r\in 2\mathbb{N}}}^{n+m} \sum_{\substack{s=|r-j| \\ r+s+j\in 2\mathbb{N}}}^{r+j} \frac{\lambda^2}{(n+\lambda)(m+\lambda)} D_\lambda(n,m;r) D_\lambda(r,j;s)\delta_{s0}\,. \tag{B.61}$$

Computing the integral in (B.61), one can see that, for $n = m = j = 0$, $\mathcal{R}_{n,m,j}$ has a fourth order pole in ε, otherwise it has a simple pole. On the contrary, $\mathcal{A}_{n,m,j}$ is always finite. Manipulations on Krönecker deltas allow us to write

$$\begin{aligned} I_{8a} &= N_\lambda(8,12) \sum_{n,m,j=0}^{\infty} \sum_{\substack{r=|n-m| \\ n+m+r\in 2\mathbb{N}}}^{n+m} \mathcal{R}_{n,m,j}(\lambda) \frac{\lambda^2}{(n+\lambda)(m+\lambda)} D_\lambda(n,m;r) D_\lambda(r,j;0)\delta_{rj} \\ &= N_\lambda(8,12) \sum_{n,m=0}^{\infty} \sum_{\substack{j=|n-m| \\ n+m+j\in 2\mathbb{N}}}^{n+m} \mathcal{R}_{n,m,j}(\lambda) \frac{\lambda^2}{(n+\lambda)(m+\lambda)} D_\lambda(n,m;j) D_\lambda(j,j;0) \\ &= N_\lambda(8,12) \sum_{n,m=0}^{\infty} \sum_{\substack{j=|n-m| \\ n+m+j\in 2\mathbb{N}}}^{n+m} T_{n,m,j}(\lambda)\,, \end{aligned} \tag{B.62}$$

where we have defined

$$T_{n,m,j}(\lambda) = \mathcal{R}_{n,m,j}(\lambda) \frac{\lambda^2}{(n+\lambda)(m+\lambda)} D_\lambda(n,m;j) D_\lambda(j,j;0)\,. \tag{B.63}$$

Since $\mathcal{A}_{000}(\lambda) = 1$, we can write

$$I_{8a} = N_\lambda(8,12) \left\{ \mathcal{R}_{000}(\lambda) + 2\sum_{n=1}^{\infty} T_{n0n}(\lambda) + \sum_{n,m=1}^{\infty} \sum_{\substack{j=|n-m| \\ n+m+j\in 2\mathbb{N}}}^{n+m} T_{n,m,j}(\lambda) \right\}, \tag{B.64}$$

which, after expanding around $\varepsilon = 0$, becomes[6]

$$I_{8a} = N_\lambda(8,12)\left\{\frac{1}{192\,\varepsilon^4} + \frac{1}{12\,\varepsilon^3}\Big(1 + \frac{1}{2}\log R\Big) + \frac{1}{3\,\varepsilon^2}\Big(\frac{1}{4} + 2\log R + \frac{1}{2}\log^2 R\Big)\right.$$
$$\left. + \frac{1}{3\,\varepsilon}\Big(-\frac{19}{18} + \frac{27}{4}\,\zeta(3) + 3a - 2\log R + 8\log^2 R + \frac{4}{3}\log^3 R\Big)\right\}, \tag{B.65}$$

where we have denoted by a the multiple series

$$a = \sum_{n,m=1}^{\infty} \sum_{\substack{j=|n-m| \\ n+m+j\in 2\mathbb{N}}}^{n+m} \mathrm{Res}\big(\mathcal{R}_{n,m,j}, \varepsilon = 0\big)\,\frac{1}{(2n+1)(2m+1)} D_{1/2}(n,m;j) D_{1/2}(j,j;0)\,. \tag{B.66}$$

It is very hard to find the sum of such a series analytically. One can try to rely on numerical methods, but the slow convergence of the series would make the result little accurate. For these reasons, it seems that GPXT isn't the best strategy to solve I_{8a} integral. In the next section we will describe a different approach to the same integral.

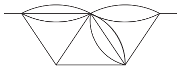
We computed all the integrals needed for the evaluation of maximal reshuffling diagrams of ABJM theory in the $SU(2) \times SU(2)$ with GPXT. The integrals $\bar{I}_2$, $\bar{I}_4$, $\bar{I}_{4a}$, $\bar{I}_6$, $\bar{I}_{6a}$, $\bar{I}_{6b}$, $\bar{I}_8$, $\bar{I}_{8d}$, consistently, give the answers we already obtained with G-functions. In addition, we could also compute the following ones[7]:

I_{6c} =  $= N_\lambda(6,9)\Big\{\frac{1}{24\,\varepsilon^3} + \frac{1}{\varepsilon^2}\Big(\frac{1}{6} + \frac{1}{4}\log R\Big)$

$$+ \frac{1}{\varepsilon}\Big(\frac{1}{3} + \log R + \frac{3}{4}\log^2 R\Big)\Big\} \tag{B.67}$$

[6] We see the appearance of the $\zeta(3)$ constant: it comes from the first series in (B.64), which can be computed analytically.

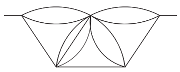
[7] For convenience, we leave here the normalization factor unexpanded: this will be easily reintroduced in the subtraction of subdivergences.

$I_{8b} =$ 

$$= N_\lambda(8,12)\Big\{\frac{5}{384\,\varepsilon^4} + \frac{5}{48\,\varepsilon^3}\Big(1+\log R\Big)$$

$$+\frac{1}{6\,\varepsilon^2}\Big(\frac{1}{2}+5\log R+\frac{5}{2}\log^2 R\Big)$$

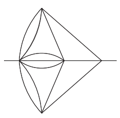
$$+\frac{1}{\varepsilon}\Big(-\frac{7}{4}+\frac{2}{3}\log R+\frac{10}{3}\log^2 R+\frac{10}{9}\log^3 R\Big)\Big\}$$

$I_{8c} =$ 

$$= N_\lambda(8,12)\Big\{\frac{1}{128\,\varepsilon^4} + \frac{1}{4\,\varepsilon^3}\Big(\frac{1}{3}+\frac{1}{4}\log R\Big)$$

$$+\frac{1}{\varepsilon^2}\Big(\frac{5}{24}+\frac{2}{3}\log R+\frac{1}{4}\log^2 R\Big)$$

$$+\frac{1}{3\,\varepsilon}\Big(\frac{1}{2}+5\log R+8\log^2 R+2\log^3 R\Big)\Big\}$$

$I_{8a} =$ 

$$= N_\lambda(8,12)\Big\{\frac{1}{192\,\varepsilon^4} + \frac{1}{12\,\varepsilon^3}\Big(1+\frac{1}{2}\log R\Big)$$

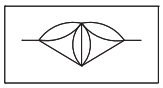
$$+\frac{1}{3\,\varepsilon^2}\Big(-\frac{1}{4}+2\log R+\frac{1}{2}\log^2 R\Big)$$

$$+\frac{1}{3\,\varepsilon}\Big(-\frac{19}{18}+\frac{27}{4}\zeta(3)+3a-2\log R+8\log^2 R+\frac{4}{3}\log^3 R\Big)\Big\}, \tag{B.68}$$

where a is the sum of the series defined in (B.66).

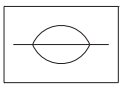
For the subtracted diagrams we, then, have

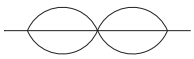
$\bar{I}_{6c}$ =



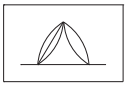
=

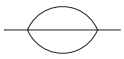
−





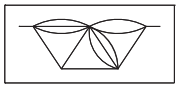
− 2



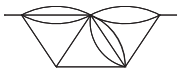


$$= \frac{1}{(4\pi)^6}\left(\frac{1}{192\,\varepsilon^3} - \frac{1}{24\,\varepsilon^2} + \frac{1}{24\,\varepsilon}\right)$$

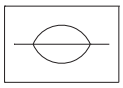
$\bar{I}_{8b}$ =

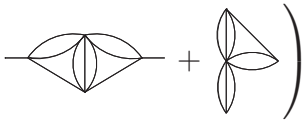


=



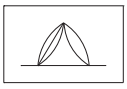
−



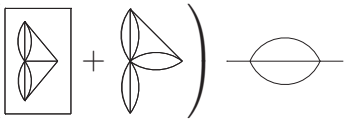


(B.69)

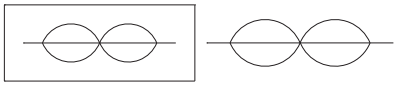
−



−

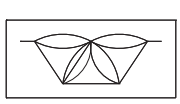


−

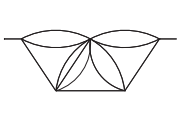


$$= \frac{1}{(4\pi)^8}\left(-\frac{5}{6144\,\varepsilon^4} + \frac{5}{768\,\varepsilon^3} - \frac{1}{384\,\varepsilon^2} - \frac{1}{32\,\varepsilon}\right)$$

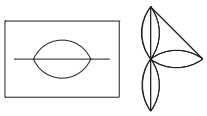
$\bar{I}_{8c}$ =



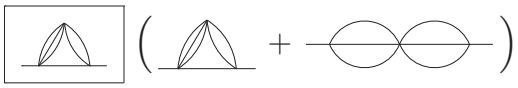
=



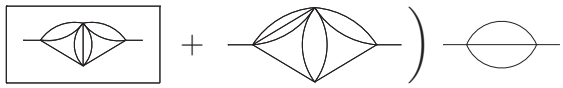
−



−

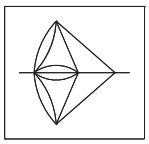


−

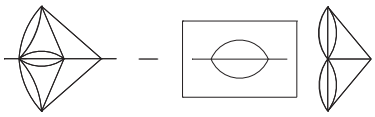


$$= \frac{1}{(4\pi)^8}\left(-\frac{1}{2048\,\varepsilon^4}+\frac{1}{128\,\varepsilon^3}-\frac{3}{64\,\varepsilon^2}+\frac{5}{192\,\varepsilon}\right) \tag{B.70}$$

$\bar{I}_{8a}$ =

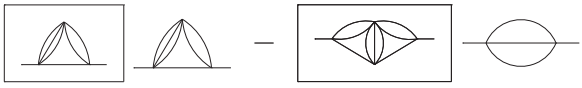


=



−

− 2



−

$$= \frac{1}{(4\pi)^8}\left\{-\frac{1}{3072\,\varepsilon^4}+\frac{1}{192\,\varepsilon^3}-\frac{1}{48\,\varepsilon^2}\right.$$

$$\left.+\frac{1}{\varepsilon}\left[-\frac{77}{432}+\frac{a}{16}+\frac{9}{64}\zeta(3)\right]\right\}.$$

Regarding $\bar{I}_{8a}$, we observe that GPXT allowed us to analytically compute the higher order poles in ε and to reduce the first order pole to a multiple series.

B.3. Mellin-Barnes representations

The last technique we discuss makes use of Mellin-Barnes representations of Feynman integrals [98], [99]. The idea is to replace a sum of two terms raised to some power by the product of these terms raised to some other powers and to perform integrations in terms of gamma functions, at the cost of introducing extra integrations over contours in complex plane along the imaginary axis (Mellin integral). Singularities in ε are, then, resolved by appropriate techniques and Mellin integrals are finally evaluated by means of analytical or numerical methods.

At the basis of the whole procedure there is the following formula:

$$\frac{1}{(A+B)^\alpha} = \frac{1}{\Gamma(\alpha)} \int_{-i\infty}^{i\infty} \frac{\mathrm{d}z}{2\pi i} \Gamma(\alpha+z)\Gamma(-z) \frac{B^z}{A^{\alpha+z}}\,, \tag{B.71}$$

which can be generalized to

$$\begin{aligned}\frac{1}{(A_1+\cdots+A_n)^\alpha} = \frac{1}{\Gamma(\alpha)} \int_{-i\infty}^{i\infty} \frac{\mathrm{d}z_2}{2\pi i} \cdots \int_{-i\infty}^{i\infty} \frac{\mathrm{d}z_n}{2\pi i} \prod_{i=2}^{n} A_i^{z_i} \\ \times A_1^{-\alpha-z_2-\cdots-z_n} \Gamma(\alpha+z_2+\cdots+z_n) \prod_{i=2}^{n} \Gamma(-z_i)\,.\end{aligned} \tag{B.72}$$

The decompositions (B.71) and (B.72) can be used in various situations, *e.g.* to turn massive propagators into massless ones. For our applications, they can be used in Feynman parametrization of loop integrals. Using the formula for Feynman parameters,

$$\frac{1}{\Pi_1^{\alpha_1}\cdots\Pi_n^{\alpha_n}} = \frac{\Gamma(\alpha_1+\cdots+\alpha_n)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_n)} \int_0^1 \mathrm{d}x_1 \cdots \int_0^1 \mathrm{d}x_n \frac{x_1^{\alpha_1-1}\cdots x_n^{\alpha_n-1}\delta(1-x_1-\cdots-x_n)}{(x_1\Pi_1+\cdots+x_n\Pi_n)^{\alpha_1+\alpha_n}}\,, \tag{B.73}$$

one can write an ℓ-loop scalar integral with generic powers of the propagators,

$$J_\ell = \frac{1}{(2\pi)^{\ell D}} \int \frac{d^D k_1 \cdots d^D k_\ell}{\Pi_1^{\alpha_1}\cdots\Pi_P^{\alpha_P}}\,, \tag{B.74}$$

in the following form:

$$J_\ell = \frac{1}{(4\pi)^{\lambda+1}} \frac{\Gamma(\alpha_1+\cdots+\alpha_P)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_P)} \int_0^1 \mathrm{d}x_1 \cdots \int_0^1 \mathrm{d}x_P \, x_1^{\alpha_1-1}\cdots x_P^{\alpha_P-1} \times \delta(1-x_1-\cdots-x_P) \frac{U(x)^{\alpha_1+\cdots+\alpha_P-(\ell+1)(\lambda+1)}}{F(x)^{\alpha_1+\cdots+\alpha_P-\ell(\lambda+1)}}, \tag{B.75}$$

where F and U are polynomials in $x_1, \cdots, x_P$ as well as in invariants of external momenta ($U(x) = 1$ for one-loop integrals). They are characteristics of the topology of the integral. We don't describe here the derivation of these polynomials (see, for example, [100]). Now, formula (B.72) can be used for $F(x)^{\alpha_1+\cdots+\alpha_P-\ell(\lambda+1)}$ and, then, the integral on Feynman parameters can be done by means of the formula

$$\int_0^1 \mathrm{d}x_1 \cdots \int_0^1 \mathrm{d}x_P \, x_1^{q_1-1}\cdots x_P^{q_P-1} \, \delta(1-x_1-\cdots-x_P) = \frac{\Gamma(q_1)\cdots\Gamma(q_P)}{\Gamma(q_1+\cdots+q_P)}. \tag{B.76}$$

Integration on Feynman parameters is possible because, for scalar integrals, one can rewrite $U(x)$ and $F(x)$ so that (B.76) becomes applicable[8]. As a result, any scalar Feynman integral may be represented by a single multi-dimensional Mellin-Barnes integral.

Finally, one cane use the following two Barnes' lemmas to try and reduce the dimensionality of the Mellin-Barnes representation obtained with the previous steps (lower dimension Mellin integrals are easier to handle):

$$\int_{-i\infty}^{i\infty} \mathrm{d}z \, \Gamma(a+z)\Gamma(b+z)\Gamma(c-z)\Gamma(d-z) = \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)},$$
$$\int_{-i\infty}^{i\infty} \mathrm{d}z \, \frac{\Gamma(a+z)\Gamma(b+z)\Gamma(c+z)\Gamma(d-z)\Gamma(e-z)}{\Gamma(a+b+c+d+e+z)} = \frac{\Gamma(a+d)\Gamma(a+e)\Gamma(b+d)\Gamma(b+e)\Gamma(c+d)\Gamma(c+e)}{\Gamma(a+b+d+e)\Gamma(a+c+d+e)\Gamma(b+c+d+e)}. \tag{B.77}$$

The construction of Mellin-Barnes representations of a given ℓ-loop Feynman integral, and in particular the manipulations on the F and U polynomials, has been automated in the `Mathematica` package `AMBRE`, written by Gluza, Riemann and Kajda [100]. It finds Mellin-Barnes representations for multiloop integrals by a loop-by-loop technique, which essentially allows to restrict the formalism described above to the one-loop case.

[8] This is also true for any integral at one loop. As stated before, in that case only $F(x)$ appears.

Up to this stage, our ℓ-loop integral has been transformed into a multidimensional Mellin integral of products and ratios of gamma functions which contain, among other objects, the dimensional regulator ε. If one chooses the integration contour in (B.71) such that the poles of the gamma function with $+z$ are separated from the poles with $-z$, the MB representation is well defined and corresponds to the original Feynman integral if the real parts of all of the gamma functions have positive arguments. If these conditions cannot be satisfied with $\varepsilon = 0$, then the integral may develop divergences and analytic continuation to 0 is necessary to make an expansion in ε.

In order to perform the analytical continuation in ε, we use the `Mathematica` package `MB` written by Czakon [101]. It assumes fixed contours parallel to the imaginary axis. If it is not possible for all integration contours, one introduces auxiliary analytic regularization to provide the existence of such straight contours. The analytic continuation consists in accounting for pole crossings past the contours, which are chosen such that no two contours can be crossed simultaneously: this assumption can always be satisfied by infinitesimal shifts of one of the concerned contours. Whenever a pole of some gamma function is crossed in the limit $\varepsilon \to 0$, one takes into account the corresponding residue (if the auxiliary analytic regularization was introduced, one first performs, in a similar way, the analytic continuation to zero values of the corresponding analytic parameters). For every resulting residue, which involves one integration less, a similar procedure is applied, and so on.

The `MB` package also contains a routine suitable for numerical integration of MB representations. Besides the built-in `Mathematica` function `NIntegrate`, it uses the `CUBA` library [102] of numerical integration routines and the CERN libraries [103] for the implementation of gamma and psi functions, in order to prepare `Fortran` programs, which are more efficient, in terms of computational time, for high dimension MB integrals.

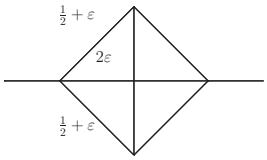
We now show how the Mellin-Barnes technique works, in the evaluation of the I_{8a} integral defined in the previous section.

First of all, we contract the bubbles to reduce to the evaluation of a four-loop integral:

$$I_{8a} \;=\; \quad = \; G(1,1)^3\, G(1,1/2+\varepsilon) \qquad . \tag{B.78}$$

Since the G-functions present a simple pole in ε, the four-loop integral

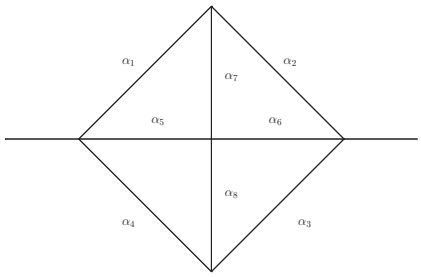
$$I_{4b} \quad = \quad$$



(B.79)

has to be computed up to order zero in ε. Let's introduce the corresponding four-loop master integral with generic powers of the propagators:

$$J_{4b}(\alpha_1,\ldots,\alpha_8) \quad = \quad$$



$$= \frac{1}{(2\pi)^{4D}} \int \mathrm{d}^D k_1 \mathrm{d}^D k_2 \mathrm{d}^D k_3 \mathrm{d}^D k_4 \, \frac{1}{(k_1^2)^{\alpha_1}(k_2^2)^{\alpha_2}(k_3^2)^{\alpha_3}(k_4^2)^{\alpha_4}}$$

$$\times \frac{1}{[(k_1-k_4-p)^2]^{\alpha_5}[(k_2-k_3-p)^2]^{\alpha_6}[(k_1-k_2)^2]^{\alpha_7}[(k_3-k_4)^2]^{\alpha_8}} \,. \tag{B.80}$$

With the help of AMBRE, we find the following six-fold MB representation for the integral:

$$
\begin{aligned}
J_{4b} =& \frac{1}{(4\pi)^{\lambda+1}} \int_{-i\infty}^{i\infty} \frac{\mathrm{d}z_1}{2\pi i} \cdots \int_{-i\infty}^{i\infty} \frac{\mathrm{d}z_6}{2\pi i} \, \frac{\Gamma(\lambda+1-\alpha_{17}-z_1)\Gamma(-z_1)\Gamma(\lambda+1-\alpha_{15}-z_2)}{\Gamma(\alpha_1)\Gamma(\alpha_3)\Gamma(\alpha_5)\Gamma(\alpha_6)} \\
& \times \frac{\Gamma(-z_2)\Gamma(\alpha_1+z_{12})\Gamma(\lambda+1-\alpha_{26}+z_1-z_3)\Gamma(2\lambda+2-\alpha_{1257}-z_{24})\Gamma(-z_4)}{\Gamma(2\lambda+2-\alpha_{157})\Gamma(\alpha_7)\Gamma(\alpha_2-z_1)\Gamma(3\lambda+3-\alpha_{12567}-z_2)} \\
& \times \frac{\Gamma(\alpha_2-z_1+z_{34})\Gamma(-2\lambda-2+\alpha_{12567}+z_{234})\Gamma(3\lambda+3-\alpha_{1235678}-z_{2345})}{\Gamma(4\lambda+4\alpha_{1235678}-z_{24})\Gamma(-2\lambda-2+\alpha_{125678}+z_{234})\Gamma(\alpha_4-z_5)} \\
& \times \frac{\Gamma(-z_5)\Gamma(\lambda+1-\alpha_4+z_5)\Gamma(\lambda+1-\alpha_3+z_3-z_6)\Gamma(4\lambda+4-\alpha_{1235678}-z_{56})}{\Gamma(5\lambda+5-\alpha_{12345678}-z_6)} \\
& \times \frac{\Gamma(-z_6)\Gamma(-4\lambda-4+\alpha_{12345678}+z_6)\Gamma(\alpha_3+z_{56})\Gamma(-3\lambda-3+\alpha_{1235678}+z_{2456})}{\Gamma(-3\lambda-3+\alpha_{1235678}+z_{56})} \,,
\end{aligned}
\tag{B.81}
$$

where we have denoted $\alpha_{ijk\cdots} = \alpha_i + \alpha_j + \alpha_k + \cdots$ and similarly for z_k.

We have to compute

$$
I_{4b} \;=\; J_{4b}(1/2+\varepsilon, 1, 1, 1/2+\varepsilon, 2\varepsilon, 1, 1, 1)\,. \tag{B.82}
$$

Putting this integral into the MB package we can make the ε-expansion and find the following numerical result:

$$
\begin{aligned}
I_{4b} =& \frac{2.116213934935895\,10^{-8}}{\varepsilon^3} + \frac{(8.26225 \pm 0.00003)\,10^{-7}}{\varepsilon^2} \\
& + \frac{(0.0000177432 \pm 0.0000000003)}{\varepsilon} + (0.0002705914 \pm 0.0000000003)\,.
\end{aligned}
\tag{B.83}
$$

We then insert this result in (B.78), make the ε-expansion and subtract the subdivergences, to obtain the following numerical result:

$$\bar{I}_{8a} = -\frac{5.2348\ldots\,10^{-13}}{\varepsilon^4} + \frac{8.37567\ldots\,10^{-12}}{\varepsilon^3} - \frac{3.35028\ldots\,10^{-11}}{\varepsilon^2} - \frac{9.63613\ldots\,10^{-12}}{\varepsilon}\,. \tag{B.84}$$

Comparing this result with the one obtained at the end of the previous section, we see that the poles of second, third and fourth order perfectly agree. We recall that these poles were computed analytically via GPXT. In addition, we now have the numerical result for the first order pole:

$$\bar{I}_{8a}\,|_{1/\varepsilon} = \frac{1}{(4\pi)^8}\,(-0.0059921 \pm 0.0000008)\,. \tag{B.85}$$

Appendix C.

Mathematica Routines

We collect in this Appendix the `Mathematica` codes of the routines used in Chapter 4 for the computation of the spectrum of two-impurity operators in ABJM theory.

C.1. Symbolic diagonalization

The first routine is aimed to the explicit diagonalization of the ℓ-loop dilatation operator in the $SU(2) \times SU(2)$ sector, written in terms of the basis of permutation structures. It is adapted from the routine available in [79] to the case of the ABJM alternating spin-chain, where two kinds of basis of operators have to be taken into account. The first contains two impurities of the same type (Z or W fields), while the second contains impurities of different kind.

The dilatation operator is entered in abstract notation by the user. The routine constructs a matrix representation on a basis of operators of specified length, via list manipulation. Finally, diagonalization is simply achieved by the built-in `Mathematica` function `Eigenvalues`.

Let's write a generic state, *i.e.* a single trace operator in the $SU(2) \times SU(2)$ sector, as an undefined function of this kind:

```
tr[W1, Z1, W2, Z1, W1, Z2, W1, Z1, W2, Z2];          (C.1)
```

and a generic permutation structure $\{a, b, c, d\}$ as the following undefined function:

```
Perm[a, b, c, d];
```
(C.2)

The function `Cyclic` implements cyclicity of the trace counting cyclic permutations of operators inside the trace just once: it sorts these cyclic permutations in a canonical order and takes a representative as the first operator:

```
Cyclic[x_] := x/.C_tr :> Module[{k},Sort[Table[RotateLeft[C, k],
                                        {k, Length[C]}]][[1]]];
```
(C.3)

For example, these two operators are equivalent, because of the cyclicity of the trace:

```
Cyclic[tr[W1, Z1, W2, Z1, W1, Z2, W1, Z1, W2, Z2]]
    tr[W1, Z1, W2, Z1, W1, Z2, W1, Z1, W2, Z2]

Cyclic[tr[W1, Z2, W1, Z1, W2, Z2, W1, Z1, W2, Z1]]
    tr[W1, Z1, W2, Z1, W1, Z2, W1, Z1, W2, Z2]
```
(C.4)

`PermuteList[C,P,s]` implements the permutation structure P on the state C at site s, working recursively from the right. It uses `PermuteElements` to perform pairwise permutations of next-to-nearest neighbor sites, mod L in order to consider closed chains. The permutation structure { } is treated separately:

```
PermuteList[C_tr, Perm[], s_] := C;
PermuteList[C_tr, P_Perm, s_] := PermuteList[PermuteElements[C, Last[P]
                                                + s], Drop[P, -1], s];
PermuteElements[C_tr, p_] := Module[{p0 = Mod[p, Length[C], 1], p1 =
          Mod[p + 2, Length[C], 1]},
          ReplacePart[ReplacePart[C, C[[p0]], p1], C[[p1]], p0]];
```
(C.5)

For example, on the state $|W_1Z^1W_2Z^1W_1Z^2W_1Z^1W_2Z^2\rangle$ these functions work in the following way:

```
PermuteElements[ tr[W1, Z1, W2, Z1, W1, Z2, W1, Z1, W2, Z2], 10]
    tr[W1, Z2, W2, Z1, W1, Z2, W1, Z1, W2, Z1]

PermuteList[tr[W1, Z1, W2, Z1, W1, Z2, W1, Z1, W2, Z2], Perm[1, 2], 2]
    tr[W1, Z1, W1, Z2, W2, Z1, W1, Z1, W2, Z2]

PermuteList[tr[W1, Z1, W2, Z1, W1, Z2, W1, Z1, W2, Z2], Perm[], 2]
    tr[W1, Z1, W2, Z1, W1, Z2, W1, Z1, W2, Z2]
```

(C.6)

`ActionPermPC[P,C]` gives the action of the permutation structure P on the state C, $\{\ldots\}\mathrm{Tr}[\ldots]$, summing over single permutations at each site s of the spin chain:

```
ActionPermPC[P_Perm, C_tr] := Module[{s}, Sum[PermuteList[C, P, 2 s],
                                        {s, Length[C]/2}]];
```

(C.7)

For example, $\{1\}|W_1Z^1W_2Z^1W_1Z^2\rangle$ is implemented with `ActionPermPC` as

```
ActionPermPC[Perm[1], tr[W1, Z1, W2, Z1, W1, Z2]]
    tr[W1, Z1, W1, Z1, W2, Z2] + tr[W1, Z1, W2, Z1, W1, Z2] +
    tr[W2, Z1, W1, Z1, W1, Z2]
```

(C.8)

```
Cyclic[%]
    tr[W1, Z1, W1, Z1, W2, Z2] + tr[W1, Z1, W1, Z2, W2, Z1] +
    tr[W1, Z1, W2, Z1, W1, Z2]
```

`ActionPerm`[`P`,`C`] gives the action of a linear combination of permutation structures, *e.g.* the dilatation operator, on a generic linear combination of states

```
ActionPerm[P_, C_]:= P /.P0_Perm:>(C/C0_tr -> ActionPermPC[P0, C0]);
```
(C.9)

Example of the use of `ActionPerm`:

```
ActionPerm[Perm[] - Perm[1], tr[W1, Z1, W2, Z2] - tr[W2, Z1, W1, Z2]]
    4 tr[W1, Z1, W2, Z2] - 4 tr[W2, Z1, W1, Z2]


Cyclic[%]
    4 tr[W1, Z1, W2, Z2] - 4 tr[W1, Z2, W2, Z1]


ActionPerm[ Perm[] - Perm[1], {tr[W1, Z1, W2, Z2] - tr[W2, Z1, W1, Z2],
            tr[W1, Z1, W2, Z1] - tr[W2, Z1, W1, Z1]}]
     {4 tr[W1, Z1, W2, Z2] - 4 tr[W2, Z1, W1, Z2],
      4 tr[W1, Z1, W2, Z1] - 4 tr[W2, Z1, W1, Z1]}


Simplify[Cyclic[%]]
    {4 (tr[W1, Z1, W2, Z2] - tr[W1, Z2, W2, Z1]), 0}
```
(C.10)

`Basis1`[`L`] generates a basis of operators of length $2L$ with 2 impurities of the same type; `Basis2`[`L`] generates a basis of operators of length $2L$ with 2 impurities of the two types. We create states with impurities from 0 to M on the odd-site and on the even-site spin-chain (`BasisW`[`L`,`M1`], `BasisZ`[`L`,`M2`]) and then fuse all their possible combinations, in an alternating way, through `FusePerm`[`L`,`M1`,`M2`]. The latter function uses `Fuse`[`A`,`B`,`L`] to fuse in an alternating way two vectors `A` and `B` of length `L` each:

```
BasisW[L_, M_] := Join[Array[W2 &,M], Array[W1 &,L - M]] // Permutations;
```
(C.11)

```
BasisZ[L_, M_] := Join[Array[Z2 &,M], Array[Z1 &,L - M]] // Permutations;
```
(C.12)

```
Fuse[A_, B_, L_] := Table[If[OddQ[k] == True, A[[1 + IntegerPart[k/2]]],
                B[[k/2]]], {k, 1, 2 L}];
```
(C.13)

```
Basis1[L_] := Flatten[Table[ tr @@ Fuse[BasisW[L, 2][[i]],
  BasisZ[L, 0][[1]], L], {i, 1, Binomial[L, 2]}], 1] // Cyclic // Union;
```
(C.14)

```
Basis2[L_] := Flatten[Table[ tr @@ Fuse[BasisW[L, 1][[i]],
  BasisZ[L, 1][[j]], L], {i, 1, Binomial[L, 1]}, {j, 1, Binomial[L, 1]}],
  1] // Cyclic // Union;
```
(C.15)

The following example shows how these functions work for two impurity states at length four:

```
Basis2[3]
    {tr[W1, Z1, W1, Z1, W2, Z2], tr[W1, Z1, W1, Z2, W2, Z1],
      tr[W1, Z1, W2, Z1, W1, Z2]}
```
(C.16)

`ActionMatrix[P,C]` gives the matrix representation of the linear operator `P`, *e.g.* the dilatation operator, on the basis `C`:

```
ActionMatrix[P_, C_] := CoeffList[ActionPerm[P, C] // Cyclic, C];
```
(C.17)

```
CoeffList[x_, L_] := Map[Coefficient[x, #] &, L];
```
(C.18)

`M[l,L]` is the matrix representation of the hamiltonian up to `l` loops on states of length $2L$

```
M1[l_,L_] := Sum[λ^(2 k) ActionMatrix[H[2 k], Basis1[L]], {k, 1, l/2}]
M2[l_,L_] := Sum[λ^(2 k) ActionMatrix[H[2 k], Basis2[L]], {k, 1, l/2}] .
```
(C.19)

As an example of application of this routine we present the diagonalization of the four-loop dilatation operator of ABJM theory. From Chapter 4, after having imposed hermiticity, parity invariance, zero energy of the vacuum and dispersion relation on one-magnon states we get

$$\begin{aligned}\mathcal{D}_4 =& 2(h_4-4+d)\,\{\,\}+(6-h_4-2d)\,\{0\}+(6-h_4-2d)\,\{1\}+d\,(\{0,1\}+\{1,2\})\\ &-\,(\{0,2\}+\{2,0\}+(\{1,3\}+\{3,1\})\,.\end{aligned} \tag{C.20}$$

If we consider the basis of length $2L=6$ states with two impurities of the same kind we find:

```
Bsais1[3]
    {tr[W1, Z1, W2, Z1, W2, Z1]} .
```
(C.21)

This state is protected. We find, indeed,

```
Normal[Series[Eigenvalues[M1[4, 3]],{λ, 0, 4}]]
    {0} .
```
(C.22)

This is coherent with the fact that the operator $\mathcal{O}=\mathrm{Tr}(Z^1,W_1,Z^1,W_2,Z^1,W_2)$ can be seen as a state with a single impurity on the even-site spin-chain with vacuum state $\mathrm{Tr}(W_2,W_2,W_2)$ and, because of the trace condition, has zero energy.

With the other type of basis we get instead

```
Basis2[3]
    {tr[W1, Z1, W1, Z1, W2, Z2], tr[W1, Z1, W1, Z2, W2, Z1],
      tr[W1, Z1, W2, Z1, W1, Z2]} .
```
(C.23)

In this case the diagonalization yields:

```
Normal[Series[Eigenvalues[M2[4, 3]],{λ, 0, 4}]]
```

$$\{0, 6\lambda^2 + (-18 + 3d + 6h4)\lambda^4, 6\lambda^2 + (-18 + 9d + 6h4)\lambda^4\}\,. \tag{C.24}$$

These eigenvalues have to be compared with the ones computed with Bethe Ansatz through the second routine we are going to describe.

C.2. Perturbative Bethe equations

The second routine is a `Mathematica` implementation of the all-loop Bethe equations (4.16) and (4.17), for two impurity states, and their perturbative solution.

First of all we define the functions appearing in such equations, needed for our eight-loop computation:

```
x[u_] := (u/2) (1 + Sqrt[1 - 4 h2/u^2]);
```
(C.25)

```
q[r_,u_] := (I/(r - 1)) (1/(x[u + I/2])^(r - 1) -1/(x[u - I/2])^(r - 1));
```
(C.26)

```
θ[u_] := c(q[2,u] q[3,-u]-q[3,u]q[2,-u]);
```
(C.27)

The perturbative expansion of the coefficients of the dressing phase $\beta_{2,3}^{(k)}$, relevant for the eight-loop dilatation operator, is

```
cpert[k_] := Sum[β[2 j]λ^(2 j), {j, 2, k/2 - 1}];
```
(C.28)

In particular, we get

```
cpert[8]
```

$$\lambda^4\beta[4] + \lambda^6\beta[6] \tag{C.29}$$

With `BAE1` we define the asymptotic Bethe Ansatz equations for two impurities with $M_u = M_v = 1$. We denote by $w = u, v$ the Bethe roots and by `h2` te interpolating function $h^2(\lambda)$. `DP` is for the moment undefined and denotes the dressing factor:

```
BAE1[L_]:=((w+I/2)/(w-I/2))^ L
    ((1+Sqrt[1-4 h2/(w+I/2)^ 2])/(1+Sqrt[1-4 h2/(w-I/2)^ 2]))^ L DP
```
(C.30)

In the same way, `BAE2` are the asymptotic Bethe Ansatz equations for two impurities with either $M_u = 2$ and $M_v = 0$ or $M_u = 0$ and $M_v = 2$:

```
BAE2[L_]:=((w+I/2)/(w-I/2))^ (L-1)
    ((1+Sqrt[1-4 h2/(w+I/2)^ 2])/(1+Sqrt[1-4 h2/(w-I/2)^2]))^ L DP
```
(C.31)

We need the perturbative expansion of the Bethe roots w for spin chains of length $2L$, up to k loops. To this end we define the function `BetheRoot[L,k]`: $w = \sum_{j=1}^{k/2} w_{2j} \lambda^{2(j-1)}$, with w_{2j} given by `RootCoeff[L,2j]`. The two-loop contribution is w_2. The maximum order coefficient is renamed u and will be computed as the solution of the Bethe Ansatz equations.

```
BetheRoot[L_,k_]:=(Sum[RootCoeff[L,2 j]λ^(2 (j-1)),
    {j,IntegerPart[k/2]}])/.{RootCoeff[L,k] -> u};
```
(C.32)

The perturbative expansion of $h^2(\lambda)$ up to k loops is called `hpert[k]`

```
hpert[k_]:=Sum[h[2 j]λ^(2 j),{j,1,k/2}]/.h[2] -> 1;
```
(C.33)

With `BetheAnsatz[L,k]` we solve the k-loop Bethe equations. The user has to solve such equations order by order in perturbation theory, filling the coefficients `RootCoeff[L,2j]`, until now not evaluated, by hand. `PertBAE[L,k]` is the perturbative expansion of the Bethe equations up to k loops. The solution for the Bethe root w is given by `sol[L,k]`. The perturbative expansion of the dressing factor is given by `DPpert`. We stress that is

suited for the eight-loop computations since it contains only the $\beta_{2,3}$ coefficient[1].

```
DPpert[k_,w_]:=Normal[Series[Exp[-Iθ[w]]/.
                {c -> cpert[k], h2 -> hpert[k]},{λ,0,k-2}]]
```
(C.34)

The expansion of the dressing factor up to order λ^6, relevant for our eight loop computations, reads

$$1+\frac{128iu\lambda^4\beta[4]}{(-i+2u)^3(i+2u)^3}+\frac{128i\lambda^6(-20u\beta[4]+112u^3\beta[4]+u\beta[6]+8u^3\beta[6]+16u^5\beta[6])}{(-i+2u)^5(i+2u)^5} \tag{C.35}$$

```
PertBAE1[L_,k_]:=Normal[Series[(BAE1[L] /.{DP -> DPpert[k, w]}) /.
    {w -> BetheRoot[L, k], h2 -> hpert[k]}, {λ,0,k-1}]]
```
(C.36)

```
BetheAnsatz1[L_,k_]:=u /.  Solve[PertBAE1[L,k] == 1, u];
```
(C.37)

```
PertBAE2[L_,k_]:=Normal[Series[(BAE2[L] /.  DP -> DPpert[k,w]) /.
    {w -> BetheRoot[L,k], h2 -> hpert[k]}, {λ,0,k-2}]]
```
(C.38)

```
BetheAnsatz2[L_,k_]:=u /.  Solve[PertBAE2[L,k] == 1, u];
```
(C.39)

```
sol[L_,k_]:= BetheRoot[L,k] /.  u -> RootCoeff[L,k];
```
(C.40)

The k-loop anomalous dimensions, *i.e.* the energy eigenvalues, are given by the function `AnomalousDimension[L,k]`. `q[2,u]` is the eigenvalue of the second conserved charge for a single magnon. The factor 2 corresponds to the fact that u-root and v-root have

[1] A generalization of this procedure for generic $\theta(\lambda)$ requires to modify the present routine.

identical solutions:

```
AnomalousDimension[L_,k_]:=Collect[Simplify[ Normal[Series[ hpert[k]
    Normal[ Series[(2 q[2,sol[L,k]]) /.  {h2 -> hpert[k]}, {λ,0,k-1}]],
     {λ,0,k}]]], λ];
```
(C.41)

In order to illustrate this procedure, we compute the anomalous dimension of the operator of length $2L = 6$, with two impurities $M_u = M_v = 1$. As described above, we run the following commands:

```
BetheAnsatz1[3, 2]
```
$$\{-\frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}\} \tag{C.42}$$

```
RootCoeff[3,2] = BetheAnsatz1[3,2][[2]];
```
(C.43)

```
sol[3,2]
```
$$\frac{1}{2\sqrt{3}} \tag{C.44}$$

```
BetheAnsatz1[3,4]
```
$$\{\sqrt{3}\} \tag{C.45}$$

```
RootCoeff[3,4] = BetheAnsatz1[3,4][[1]];
```
(C.46)

```
sol[3,4]
```
$$\frac{1}{2\sqrt{3}} + \sqrt{3}\lambda^2 \tag{C.47}$$

```
AnomalousDimension[3,4]
```

$$6\lambda^2 + 6\lambda^4(-3 + h[4]) \tag{C.48}$$

We note that, comparing this result with the one obtained in the previous section, we can obtain $d = 0$. Other anomalous dimensions are computed in a similar fashion.

Colophon

This thesis was typeset with LaTeX class "hepthesis", by Andy Buckley [106].

Feynman diagrams and figures were created with JaxoDraw [107].

Bibliography

[1] J. M. Maldacena, "*The Large N limit of superconformal field theories and supergravity*," Adv. Theor. Math. Phys. **2** (1998) 231 [hep-th/9711200].

[2] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, "*Gauge theory correlators from noncritical string theory*," Phys. Lett. B **428** (1998) 105 [hep-th/9802109].

[3] E. Witten, "*Anti-de Sitter space and holography*," Adv. Theor. Math. Phys. **2** (1998) 253 [hep-th/9802150].

[4] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, "*Large N field theories, string theory and gravity*," Phys. Rept. **323** (2000) 183 [hep-th/9905111].

[5] E. D'Hoker and D. Z. Freedman, "*Supersymmetric gauge theories and the AdS / CFT correspondence*," [hep-th/0201253].

[6] A. Zaffaroni, "*Introduction to the AdS/CFT correspondence*," Lecture notes for the "Troisième cycle de la physique en Suisse romande" held in Lausanne at EPFL laces.web.cern.ch/Laces/LACES09/notes/dbranes/lezioniLosanna.pdf

[7] J. A. Minahan and K. Zarembo, "*The Bethe ansatz for N=4 superYang-Mills*," JHEP **0303** (2003) 013 [hep-th/0212208].

[8] G. Arutyunov and S. Frolov, "*On String S-matrix, Bound States and TBA*," JHEP **0712** (2007) 024 [arXiv:0710.1568 [hep-th]].

[9] N. Gromov, V. Kazakov and P. Vieira, "*Exact Spectrum of Anomalous Dimensions of Planar N=4 Supersymmetric Yang-Mills Theory*," Phys. Rev. Lett. **103** (2009) 131601 [arXiv:0901.3753 [hep-th]].

[10] O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, "*N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals*," JHEP **0810**

(2008) 091 [arXiv:0806.1218 [hep-th]].

[11] N. Beisert, C. Ahn, L. F. Alday, Z. Bajnok, J. M. Drummond, L. Freyhult, N. Gromov and R. A. Janik *et al.*, "*Review of AdS/CFT Integrability: An Overview*," Lett. Math. Phys. **99** (2012) 3 [arXiv:1012.3982 [hep-th]].

[12] J. A. Minahan and K. Zarembo, "*The Bethe ansatz for superconformal Chern-Simons*," JHEP **0809** (2008) 040 [arXiv:0806.3951 [hep-th]].

[13] D. Bak and S. -J. Rey, JHEP **0810** (2008) 053 [arXiv:0807.2063 [hep-th]].

[14] J. A. Minahan, W. Schulgin and K. Zarembo, "*Two loop integrability for Chern-Simons theories with N=6 supersymmetry*," JHEP **0903** (2009) 057 [arXiv:0901.1142 [hep-th]].

[15] B. I. Zwiebel, "*Two-loop Integrability of Planar N=6 Superconformal Chern-Simons Theory*," J. Phys. A **42** (2009) 495402 [arXiv:0901.0411 [hep-th]].

[16] O. Aharony, O. Bergman and D. L. Jafferis, "*Fractional M2-branes*," JHEP **0811** (2008) 043 [arXiv:0807.4924 [hep-th]].

[17] D. Bak, D. Gang and S. -J. Rey, "*Integrable Spin Chain of Superconformal U(M) x anti-U(N) Chern-Simons Theory*," JHEP **0810** (2008) 038 [arXiv:0808.0170 [hep-th]].

[18] N. Gromov and P. Vieira, "*The all loop AdS4/CFT3 Bethe ansatz*," JHEP **0901** (2009) 016 [arXiv:0807.0777 [hep-th]].

[19] G. Arutyunov and S. Frolov, "*Superstrings on AdS(4) x CP**3 as a Coset Sigma-model*," JHEP **0809** (2008) 129 [arXiv:0806.4940 [hep-th]].

[20] B. Stefanski, jr, "*Green-Schwarz action for Type IIA strings on AdS(4) x CP**3*," Nucl. Phys. B **808** (2009) 80 [arXiv:0806.4948 [hep-th]].

[21] N. Gromov and P. Vieira, "*The AdS(4) / CFT(3) algebraic curve*," JHEP **0902** (2009) 040 [arXiv:0807.0437 [hep-th]].

[22] J. A. Minahan, O. Ohlsson Sax and C. Sieg, "*Magnon dispersion to four loops in the ABJM and ABJ models*," J. Phys. A **43** (2010) 275402 [arXiv:0908.2463 [hep-th]].

[23] J. A. Minahan, O. Ohlsson Sax and C. Sieg, "*Anomalous dimensions at four loops in N=6 superconformal Chern-Simons theories*," Nucl. Phys. B **846** (2011) 542 [arXiv:0912.3460 [hep-th]].

[24] M. Leoni, A. Mauri, J. A. Minahan, O. Ohlsson Sax, A. Santambrogio, C. Sieg and G. Tartaglino-Mazzucchelli, "*Superspace calculation of the four-loop spectrum in N=6 supersymmetric Chern-Simons theories*," JHEP **1012** (2010) 074 [arXiv:1010.1756 [hep-th]].

[25] D. Bak, H. Min and S. -J. Rey, "*Generalized Dynamical Spin Chain and 4-Loop Integrability in N=6 Superconformal Chern-Simons Theory*," Nucl. Phys. B **827** (2010) 381 [arXiv:0904.4677 [hep-th]].

[26] D. Bak, H. Min and S. -J. Rey, "*Integrability of N=6 Chern-Simons Theory at Six Loops and Beyond*," Phys. Rev. D **81** (2010) 126004 [arXiv:0911.0689 [hep-th]].

[27] G. Papathanasiou and M. Spradlin, "*Two-Loop Spectroscopy of Short ABJM Operators*," JHEP **1002** (2010) 072 [arXiv:0911.2220 [hep-th]].

[28] F. Levkovich-Maslyuk, "*Numerical results for the exact spectrum of planar AdS4/CFT3*," JHEP **1205** (2012) 142 [arXiv:1110.5869 [hep-th]].

[29] C. Ahn and R. I. Nepomechie, "*N=6 super Chern-Simons theory S-matrix and all-loop Bethe ansatz equations*," JHEP **0809** (2008) 010 [arXiv:0807.1924 [hep-th]].

[30] C. Ahn and R. I. Nepomechie, "*An Alternative S-matrix for N=6 Chern-Simons theory?*," JHEP **0903** (2009) 068 [arXiv:0810.1915 [hep-th]].

[31] C. Ahn and R. I. Nepomechie, *"Two-loop test of the N=6 Chern-Simons theory S-matrix*," JHEP **0903** (2009) 144 [arXiv:0901.3334 [hep-th]].

[32] C. Ahn, P. Dorey and R. I. Nepomechie, *"On the absence of reflection in AdS(4) / CFT(3)*," JHEP **1001** (2010) 129 [arXiv:0910.5584 [hep-th]].

[33] C. Ahn, M. Kim and B. -H. Lee, "*Quantum finite-size effects for dyonic magnons in the AdS_4xCP^3*," JHEP **1009** (2010) 062 [arXiv:1007.1598 [hep-th]].

[34] A. Mauri, A. Santambrogio and S. Scoleri, "*The Leading Order Dressing Phase in ABJM Theory*," to appear in JHEP arXiv:1301.7732 [hep-th].

[35] B.M. Zupnik, D.G. Pak "*Superfield formulation of the simplest three-dimensional gauge theories and conformal supergravities*," Teoret. Mat. Fiz. **77:1** (1988) 97106

[36] N. Itzhaki, J. M. Maldacena, J. Sonnenschein and S. Yankielowicz, "*Supergravity and the large N limit of theories with sixteen supercharges*," Phys. Rev. D **58** (1998) 046004 [hep-th/9802042].

[37] J. H. Schwarz, "*Superconformal Chern-Simons theories*," JHEP **0411** (2004) 078 [hep-th/0411077].

[38] D. Gaiotto and X. Yin, "*Notes on superconformal Chern-Simons-Matter theories*," JHEP **0708** (2007) 056 [arXiv:0704.3740 [hep-th]].

[39] J. Bagger and N. Lambert, "*Modeling Multiple M2's*," Phys. Rev. D **75** (2007) 045020 [hep-th/0611108].

[40] J. Bagger and N. Lambert, "*Gauge symmetry and supersymmetry of multiple M2-branes*," Phys. Rev. D **77** (2008) 065008 [arXiv:0711.0955 [hep-th]].

[41] J. Bagger and N. Lambert, "*Comments on multiple M2-branes*," JHEP **0802** (2008) 105 [arXiv:0712.3738 [hep-th]].

[42] A. Gustavsson, "*Algebraic structures on parallel M2-branes*," Nucl. Phys. B **811** (2009) 66 [arXiv:0709.1260 [hep-th]].

[43] M. Van Raamsdonk, "*Comments on the Bagger-Lambert theory and multiple M2-branes*," JHEP **0805** (2008) 105 [arXiv:0803.3803 [hep-th]].

[44] M. Benna, I. Klebanov, T. Klose and M. Smedback, "*Superconformal Chern-Simons Theories and AdS(4)/CFT(3) Correspondence*," JHEP **0809** (2008) 072 [arXiv:0806.1519 [hep-th]].

[45] M. Leoni, "*Applied N=2 Superspace Formalism in Three Dimensions*," Ph.D. thesis, 2011.

[46] M. Bianchi, "*Superspace Computation 3D.*," Ph.D. thesis, 2012

[47] M. Leoni and A. Mauri, "*On the infrared behaviour of 3d Chern-Simons theories in N=2 superspace*," JHEP **1011** (2010) 128 [arXiv:1006.2341 [hep-th]].

[48] T. Klose, "*Review of AdS/CFT Integrability, Chapter IV.3: N=6 Chern-Simons*

and Strings on AdS4xCP3," Lett. Math. Phys. **99** (2012) 401 [arXiv:1012.3999 [hep-th]].

[49] H. Bethe, "*On the theory of metals. 1. Eigenvalues and eigenfunctions for the linear atomic chain*," Z. Phys. **71** (1931) 205.

[50] L. D. Faddeev, "*How algebraic Bethe ansatz works for integrable model*," hep-th/9605187.

[51] L. D. Faddeev, "*Algebraic aspects of Bethe Ansatz*," Int. J. Mod. Phys. A **10** (1995) 1845 [hep-th/9404013].

[52] R. I. Nepomechie, "*A Spin chain primer*," Int. J. Mod. Phys. B **13** (1999) 2973 [hep-th/9810032].

[53] C. Sieg, "*Lecture notes on classical integrability and on the algebraic Bethe ansatz for the XXX1/2 Heisenberg spin chain*" Lecture notes prepared for the Course "Integrable systems in string theory and condensed matter" held in Milano, May 2009.

[54] M. Staudacher, "Review of AdS/CFT Integrability, Chapter III.1: Bethe Ansátze and the R-Matrix Formalism," Lett. Math. Phys. **99** (2012) 191 [arXiv:1012.3990 [hep-th]].

[55] J. A. Minahan, "*A brief introduction to the Bethe ansatz in N=4 super-Yang-Mills*," J. Phys. A **39** (2006) 12657.

[56] A. Doikou, S. Evangelisti, G. Feverati, and N. Karaiskos, "*Introduction to Quantum Integrability*," [arXiv:0912.3350 [math-ph]].

[57] H. J. de Vega and F. Woynarovich, "*New integrable quantum chains combining different kinds of spins*," J. Phys. A **25** (1992) 4499.

[58] N. Beisert and M. Staudacher, "*Long-range psu(2,2—4) Bethe Ansatze for gauge theory and strings*," Nucl. Phys. B **727** (2005) 1 [hep-th/0504190].

[59] M. Staudacher, "*The Factorized S-matrix of CFT/AdS*," JHEP **0505** (2005) 054 [hep-th/0412188].

[60] T. Nishioka and T. Takayanagi, "*On Type IIA Penrose Limit and N=6 Chern-*

Simons Theories," JHEP **0808** (2008) 001 [arXiv:0806.3391 [hep-th]].

[61] D. Gaiotto, S. Giombi and X. Yin, "*Spin Chains in N=6 Superconformal Chern-Simons-Matter Theory*," JHEP **0904** (2009) 066 [arXiv:0806.4589 [hep-th]].

[62] G. Grignani, T. Harmark and M. Orselli, *"The SU(2) x SU(2) sector in the string dual of N=6 superconformal Chern-Simons theory"*, Nucl. Phys. B **810** (2009) 115 [arXiv:0806.4959 [hep-th]].

[63] N. Beisert, "*The su(2—3) dynamic spin chain*," Nucl. Phys. B **682** (2004) 487 [hep-th/0310252].

[64] T. McLoughlin, R. Roiban, A. A. Tseytlin, *"Quantum spinning strings in AdS(4) x CP**3: Testing the Bethe Ansatz proposal"*, JHEP **0811** (2008) 069 [arXiv:0809.4038 [hep-th]].

[65] I. Shenderovich, "*Giant magnons in AdS(4) / CFT(3): Dispersion, quantization and finite-size corrections*," arXiv:0807.2861 [hep-th].

[66] P. Vieira and D. Volin, "*Review of AdS/CFT Integrability, Chapter III.3: The Dressing factor*," Lett. Math. Phys. **99** (2012) 231 [arXiv:1012.3992 [hep-th]].

[67] N. Dorey, D. M. Hofman and J. M. Maldacena, "*On the Singularities of the Magnon S-matrix*," Phys. Rev. D **76** (2007) 025011 [hep-th/0703104 [HEP-TH]].

[68] G. Arutyunov, S. Frolov and M. Staudacher, "*Bethe ansatz for quantum strings*," JHEP **0410** (2004) 016 [hep-th/0406256].

[69] N. Gromov, V. Mikhaylov, *"Comment on the Scaling Function in AdS(4) x CP**3"*, JHEP **0904** (2009) 083 [arXiv:0807.4897 [hep-th]].

[70] C. Krishnan, *"AdS(4)/CFT(3) at One Loop"*, JHEP **0809** (2008) 092 [arXiv:0807.4561 [hep-th]].

[71] L. F. Alday, G. Arutyunov, D. Bykov, *"Semiclassical Quantization of Spinning Strings in AdS(4) x CP**3"*, JHEP **0811** (2008) 089 [arXiv:0807.4400 [hep-th]].

[72] V. Mikhaylov, *"On the Fermionic Frequencies of Circular Strings"*, J. Phys. A **A43** (2010) 335401 [arXiv:1002.1831 [hep-th]].

[73] M. C. Abbott, I. Aniceto, D. Bombardelli, *"Quantum Strings and the AdS_4/CFT_3*

Interpolating Function", JHEP **1012** (2010) 040 [arXiv:1006.2174 [hep-th]].

[74] D. Astolfi, V. G. M. Puletti, G. Grignani, T. Harmark and M. Orselli, *"Finite-size corrections for quantum strings on AdS_4xCP^3"*, JHEP **1105** (2011) 128 [arXiv:1101.0004 [hep-th]].

[75] N. Beisert, B. Eden and M. Staudacher, "*Transcendentality and Crossing*," J. Stat. Mech. **0701** (2007) P01021 [hep-th/0610251].

[76] A. V. Kotikov and L. N. Lipatov, "*DGLAP and BFKL equations in the N=4 supersymmetric gauge theory*," Nucl. Phys. B **661** (2003) 19 [Erratum-ibid. B **685** (2004) 405] [hep-ph/0208220].

[77] N. Beisert, C. Kristjansen and M. Staudacher, "*The Dilatation operator of conformal N=4 superYang-Mills theory*," Nucl. Phys. B **664** (2003) 131 [hep-th/0303060].

[78] N. Beisert, V. Dippel and M. Staudacher, "*A Novel long range spin chain and planar N=4 super Yang-Mills*," JHEP **0407** (2004) 075 [hep-th/0405001].

[79] N. Beisert, "*The Dilatation operator of N=4 super Yang-Mills theory and integrability*," Phys. Rept. **405** (2005) 1 [hep-th/0407277].

[80] N. Beisert and T. Klose, "*Long-range gl(n) integrable spin chains and plane-wave matrix theory*," J. Stat. Mech. **0607** (2006) P07006 [hep-th/0510124].

[81] F. Fiamberti, "*Wrapping Effects in Supersymmetric Gauge Theories*," Fortsch. Phys. **58** (2010) 1023 [arXiv:1003.3118 [hep-th]].

[82] N. Beisert, T. McLoughlin and R. Roiban, "*The Four-loop dressing phase of N=4 SYM*," Phys. Rev. D **76** (2007) 046002 [arXiv:0705.0321 [hep-th]].

[83] H.R.P. Ferguson, D.H. Bailey and S. Arno, "*Analysis of PSLQ, An Integer Relation Finding Algorithm*," Math. Comput. **68**, 351 (1999)

[84] R. Hernandez and E. Lopez, "*Quantum corrections to the string Bethe ansatz*," JHEP **0607** (2006) 004 [hep-th/0603204].

[85] N. Beisert, R. Hernandez and E. Lopez, "*A Crossing-symmetric phase for AdS(5) x S**5 strings*," JHEP **0611** (2006) 070 [hep-th/0609044].

[86] T. McLoughlin and R. Roiban, "*Spinning strings at one-loop in AdS(4) x P**3*,"

JHEP **0812** (2008) 101 [arXiv:0807.3965 [hep-th]].

[87] S. J. Gates, M. T. Grisaru, M. Rocek and W. Siegel, "*Superspace Or One Thousand and One Lessons in Supersymmetry*," Front. Phys. **58** (1983) 1 [hep-th/0108200].

[88] F. Fiamberti, A. Santambrogio, C. Sieg and D. Zanon, "*Wrapping at four loops in N=4 SYM*," Phys. Lett. B **666** (2008) 100 [arXiv:0712.3522 [hep-th]].

[89] F. Fiamberti, A. Santambrogio, C. Sieg and D. Zanon, "*Anomalous dimension with wrapping at four loops in N=4 SYM*," Nucl. Phys. B **805** (2008) 231 [arXiv:0806.2095 [hep-th]].

[90] F. Fiamberti, A. Santambrogio and C. Sieg, "*Five-loop anomalous dimension at critical wrapping order in N=4 SYM*," JHEP **1003** (2010) 103 [arXiv:0908.0234 [hep-th]].

[91] D. J. Gross, A. Mikhailov and R. Roiban, "*Operators with large R charge in N=4 Yang-Mills theory*," Annals Phys. **301** (2002) 31 [hep-th/0205066].

[92] Z. Bern, M. Czakon, L. J. Dixon, D. A. Kosower and V. A. Smirnov, "*The Four-Loop Planar Amplitude and Cusp Anomalous Dimension in Maximally Supersymmetric Yang-Mills Theory*," Phys. Rev. D **75** (2007) 085010 [hep-th/0610248].

[93] K. G. Chetyrkin, A. L. Kataev and F. V. Tkachov, "*New Approach to Evaluation of Multiloop Feynman Integrals: The Gegenbauer Polynomial x Space Technique*," Nucl. Phys. B **174** (1980) 345.

[94] A. V. Kotikov, "*The Gegenbauer polynomial technique: The Evaluation of a class of Feynman diagrams*," Phys. Lett. B **375** (1996) 240 [hep-ph/9512270].

[95] A. V. Kotikov, "*Some methods for the evaluation of complicated Feynman integrals*," hep-ph/0112347.

[96] C. Sieg, "*Notes on the calculation of loop integrals*," http://msstp.org/?q=node/9

[97] J. Collins, "*Renormalization*," Cambridge University Press (2008), 392p, Cambridge Monographs on Mathematical Physics

[98] V.A. Smirnov, "*Evaluating Feynman Integrals*," Springer (2005), 247p, Springer Tracts in Modern Physics.

[99] V.A. Smirnov, "*Feynman Integral Calculus*," Springer (2006), Berlin, Germany, 283p.

[100] J. Gluza, K. Kajda and T. Riemann, "*AMBRE: A Mathematica package for the construction of Mellin-Barnes representations for Feynman integrals*," Comput. Phys. Commun. **177** (2007) 879 [arXiv:0704.2423 [hep-ph]].

[101] M. Czakon, "*Automatized analytic continuation of Mellin-Barnes integrals*," Comput. Phys. Commun. **175** (2006) 559 [hep-ph/0511200].

[102] T. Hahn, "*CUBA: A Library for multidimensional numerical integration*," Comput. Phys. Commun. **168** (2005) 78 [hep-ph/0404043].

[103] http://cernlib.web.cern.ch/cernlib/

[104] P. Bertok, PSLQ Integer Relation Algorithm Implementation, http://library.wolfram.com/infocenter/MathSource/4263

[105] D.H. Bailey and J.M. Borwein, "*PSLQ: An Algorithm to Discover Integer Relations*," Computeralgebra Rundbrief, **45** (2009) 8 LBNL-2144E.

[106] A. Buckley, The "hepthesis" LaTeX class. http://www.ctan.org/pkg/hepthesis

[107] D. Binosi and L. Theussl, "*JaxoDraw: A graphical user interface for drawing Feynman diagrams*," Comput. Phys. Commun. **161** (2004) 76