Identification in Structural Vector Autoregressive Models with Structural Changes*

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Abstract

An increasing strand of the literature uses structural changes and different heteroskedasticity regimes found in the data constructively to improve the identification of structural parameters in Structural Vector Autoregressions (SVAR). A standard assumption in this literature is that the reduced form unconditional covariance matrix of the system varies while the structural parameters remain constant. Under this condition it is possible to identify the SVAR without the need to resort to theory-driven restrictions. With macroeconomic data, the hypothesis that the structural parameters are invariant to breaks is untenable. This paper investigates the identification issues that arise in SVARs when structural breaks occurring at known dates affect both the reduced form covariance matrix and the structural parameters. The knowledge that different heteroskedasticity regimes characterize the data is combined with theory-driven restrictions giving rise to new necessary and sufficient local identification rank conditions which generalize the ones which apply for SVARs with constant parameters. This approach opens interesting possibilities for practitioners. An empirical illustration shows the usefulness of the suggested identification strategy by focusing on a small monetary policy SVAR of the U.S. economy. Two heteroskedasticity regimes are found to characterize the data before and after the 1980s and this information is combined with economic reasoning to identify the effect of monetary policy shocks on output and inflation.

Keywords: Heteroskedasticity, Identification, Monetary policy, Structural VAR.
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1 Introduction

Structural Vector Autoregressions (SVAR) are widely used for policy analysis and to provide stylized facts about business cycle. As is known, one needs to identify the structural shocks to run policy simulations. The recent literature seems to suggest that structural changes, in particular changes in the value of the VAR reduced form unconditional covariance matrix, can be used constructively to improve the identification of structural parameters that are assumed to be stable over time and across volatility regimes, see Rigobon (2003), Lanne and Lütkepohl (2008, 2010), Lanne *et al.* (2010) and Ehrmann *et al.* (2011).[1] Since the seminal paper of Rigobon (2003), a common assumption in this literature is that structural breaks affect the unconditional covariance matrix of the system disturbances but not the structural parameters. In a similar setup, the information that there exist different volatility regimes in the data represents an 'additional' identification source that can be exploited to identify the shocks without the need to resort to theory-driven identification restrictions, see also Lanne and Lütkepohl (2008, 2010) and Ehrmann *et al.* (2011).

However, if the hypothesis that the data generating process is a VAR with constant parameters apart from changes in the volatility of the disturbances appears reasonable in certain applications, it is questionable with macroeconomic data since it is well recognized that structural breaks have marked consequences on both the transmission and propagation mechanisms of the shocks. In general, there is no compelling reason to believe that structural breaks that affect the covariance matrix of VAR disturbances have no impact on the structural parameters.

In this paper, we investigate the identification issues that arise in SVARs when structural breaks occurring at known dates affect both the reduced form unconditional covariance matrix of the reduced form VAR disturbances and the structural parameters. In principle, there are two identification strategies which can be followed when the reduced form unconditional VAR covariance matrix changes along with the structural parameters at known dates. The obvious strategy (henceforth Strategy A) is the one that treats the SVARs before and after the break as independent models characterized by independent identification rules. In this case, one simply applies the 'standard' identification rules that hold for SVARs within each heteroskedasticity regime. The alternative strategy (henceforth Strategy B) is the one that treats the SVARs before and after the break as a unique model whose shocks can be identified in one solution. Strategy B combines the statistical

[1] In the recent literature, Sentana (1992) and Fiorentini and Sentana (2001) have introduced similar ideas in the context of factor models, Klein and Vella (2010) and Lewbel (2010) in the context of simultaneous systems of equations and Mavroeidis amd Magnusson (2010) in the context of dynamic models featuring forward-looking behavior. See Kilian (2012) for a review.

information provided by the data with theory-driven structural information. Surprisingly, no contribution about the identification of SVARs has been provided in this case.

This paper shows that the Strategy B leads to substantial gains in the identification analysis of SVARs. Our main result is that the combination of heteroskedasticity and theory-driven restrictions gives rise to necessary and sufficient local identification conditions that generalize the 'standard' ones that apply for SVARs with constant parameters and open interesting possibilities for practitioners. The necessary and sufficient local identification rank condition derived in this paper extends the ones discussed in e.g. Giannini (1992), Hamilton (1994) and Amisano and Giannini (1997) to a more general framework.

Rubio-Ramìrez *et al.* (2010) have established novel sufficient conditions for global identification in SVARs and necessary and sufficient conditions for exactly identified systems. Although one could potentially generalize their global identification results to our setup, we confine our attention to identification conditions which are necessary other than sufficient also in overidentified models. Hence the rank identification conditions discussed in this paper are 'local' in the sense of Rothemberg (1971).

Our analysis is in the spirit of e.g. Rigobon (2003), Lanne and Lütkepohl (2008) and Ehrmann *et al.* (2011), but represents one step further in the identification analysis of SVARs for two reasons: (i) our parameterization is designed such that when the VAR covariance matrix changes, the structural parameters automatically change; (ii) the heteroskedasticity found in the data is combined with the information provided by economic theory to identify the structural shocks.

In our framework, estimation and inference is of standard type, hence no new result is needed.

To show the usefulness of our approach we identify and estimate a small monetary policy SVAR for the U.S. economy based on quarterly data. In particular, we exploit the change in volatility occurred in key U.S. variables in the move from the 'Great Inflation' to the 'Great Moderation' period to identify the effects of monetary policy shocks on output and inflation.

The paper is organized as follows. In Section 2 we introduce the reference SVAR considering the case of constant parameters and some definitions. In Section 3 we deal with the case of a single break and provide the main results of the paper: Sub-section 3.1 summarizes the representation and assumptions, Sub-section 3.2 discusses the identification strategies and Sub-section 3.3 derives the formal necessary and sufficient conditions for identification. In Section 4 we review the existing related literature and compare our approach to existing work. In Section 5 we summarize the empirical analysis based on a small monetary policy SVAR estimated on U.S. quarterly data. Section 6 contains some concluding remarks.

In Appendix A we extend the analysis to the case of nonstationary cointegrated variables and in Appendix B we briefly summarize some estimation and testing issues. In Appendix C we consider the case of multiple (more than one) breaks. Proofs are in Appendix D.

Throughout the paper we use the following notation, matrices and conventions. K_n is the $n^2 \times n^2$ *commutation matrix*, i.e. the matrix such that $K_n vec(M) = vec(M')$ where M is $n \times n$, and D_n is the *duplication matrix,* i.e. the $n^2 \times \frac{1}{2}n(n+1)$ full column-rank matrix such that $D_n vech(M) = vec(M)$, where $vech(M)$ is the column obtained from $vec(M)$ by eliminating all supradiagonal elements. Given K_n and D_n, $N_n{:=}\frac{1}{2}(I_{n^2} + K_n)$ is a $n^2 \times n^2$ matrix such that rank$[N_n]{=}\frac{1}{2}n(n+1)$ and $D_n^{+}{:=}(D_n' D_n)^{-1} D_n'$ is the Moore-Penrose inverse of D_n. Finally, when we say that the matrix $M{:=}M(v)$, whose elements depend (possibly nonlinearly) on the elements of the vector v, 'has rank r evaluated at v_0', we implicitly mean that v_0 is a 'regular point', i.e. that rank$(M){:=}r$ does not change within a neighborhood of v_0.

2 Reference model: constant parameters case

In this section we introduce the main concepts used in the paper and fix some notation by using a SVAR with time-invariant parameters as reference model. The analysis will be extended to the case of structural breaks in the next sections.

Let Z_t be the $n \times 1$ vector of observable variables. The reference reduced form model is given by the constant parameters VAR:

$$Z_t = A_1 Z_{t-1} + \ldots + A_k Z_{t-k} + \Psi D_t + \varepsilon_t \quad , \quad t = 1, ..., T \tag{1}$$

where ε_t is a n-dimensional White Noise process with positive definite time-invariant covariance matrix Σ_ε, A_j, $j = 1, ..., k$ are $n \times n$ matrices of time-invariant coefficients, k is the VAR lag order, D_t is an m-dimensional vector containing deterministic components (constant and dummies), Ψ is the $m \times m$ matrix of associated coefficients, and T is the sample length.

Let

$$\mathring{A}{:=}\begin{bmatrix} A_1 & \ldots & A_{k-1} & A_k \\ I_n & \ldots & 0_{n\times n} & 0_{n\times n} \\ \vdots & \ddots & \vdots & \vdots \\ 0_{n\times n} & \cdots & I_n & 0_{n\times n} \end{bmatrix} := \begin{bmatrix} & A & \\ I_{n(k-1)} & , & 0_{n(k-1)\times n} \end{bmatrix} \tag{2}$$

be the $nk{\times}nk$ companion matrix associated with the VAR system (1), where $A{:=}[A_1, ..., A_k]$.

As is known, the stationarity/nonstationarity of the VAR depends on the location of the eigenvalues of the matrix $\mathring{A}$ on the unit disk. Let $\lambda_{\max}(\cdot)$ be the function that delivers the modulus of the largest eigenvalue of the matrix in the argument. We consider two possibilities: $\lambda_{\max}(\mathring{A}) < 1$, which corresponds to the case of asymptotically stationary VARs, and $\lambda_{\max}(\mathring{A})$=1, which corresponds, under additional restrictions reported in Appendix A, to the case of nonstationary cointegrated VARs, see Johansen (1996).

In the asymptotically stationary case, $\lambda_{\max}(\mathring{A}) < 1$, the coefficients of the vector moving average (VMA) representation associated with the VAR system (1) are given by $\Phi_h{:=}J'\mathring{A}^h J$, $h = 0, 1, 2, ...$, where $J'{:=} \left[I_n, 0_{n\times(nk-n)}\right]$ is a selection matrix. In this case, the maximum likelihood (ML) estimators of A, Σ_ε and Φ_h $(h = 1, 2, ...)$ are asymptotically Gaussian, and the impulse response analysis can be carried out in the 'conventional' way.

In the nonstationary case, $\lambda_{\max}(\mathring{A})$=1, treated in detail in Appendix A, the inference and impulse response analysis can be treated in the 'conventional' way if the sub-matrices in $A{:=}[A_1, ..., A_k]$ (and hence the companion matrix $\mathring{A}$) are suitably restricted to account for the number of unit roots in the system, see Lütkepohl and Reimers (1992), Amisano and Giannini (1997, Ch. 6) and Phillips (1998).

For future reference, we compact the VAR system (1) in the expression

$$Z_t = \Pi W_t + \varepsilon_t \quad , \quad E(\varepsilon_t\varepsilon_t'){:=}\Sigma_\varepsilon \quad , \quad t = 1, ..., T \tag{3}$$

where $W_t{:=}(Z_{t-1}', ..., Z_{t-k}', D_t')$ and $\Pi{:=}(A, \Psi)$. If the VAR in levels is asymptotically stationary ($\lambda_{\max}(\mathring{A}) < 1$) the A matrix in Π is left unrestricted, while if the VAR features unit roots ($\lambda_{\max}(\mathring{A})$=1) the A matrix in Π is assumed to fulfill the three restrictions summarized in Appendix A. The matrix Π is $n \times f$, $f{:=}\dim(W_t){:=}nk + m$, and the VAR reduced form parameters are collected in the p-dimensional vector $\theta{:=}(\pi', \sigma_+')'$, where $\pi{:=}vec(\Pi)$ and $\sigma_+{:=}vech(\Sigma)$, $p{:=}nf + \frac{1}{2}n(n+1)$.

The SVAR system associated with the reduced form in Eq. (3) is given by

$$\varepsilon_t{:=}Ce_t \quad , \quad E(e_te_t'){:=}I_n \tag{4}$$

where C is a non-singular $n \times n$ matrix of structural parameters and e_t is a n-dimensional i.i.d. vector with covariance matrix I_n which collects the structural shocks. Using the terminology in Amisano and Giannini (1997), the SVAR in Eq. (4) defines the 'C-model'. In the C-model, the unexpected movements in the variables, $\varepsilon_t{:=}Z_t - E(Z_t \mid Z_{t-1}, ..., Z_1)$, are directly linked to the structural shocks e_t by the C matrix.

We consider the formulation in Eq. (4) of the SVAR because it is largely used in empirical analysis, although our approach is consistent with the alternative specification

$$K\varepsilon_t := e_t \quad , \quad E(e_t e_t') := I_n$$

(termed 'K-model' in Amisano and Giannini, 1997) where $K := C^{-1}$.

We state that the points vec(C^{-1}, Γ) and vec(C^{*-1}, Γ^*), where $\Gamma := C^{-1}\Pi$, $\Gamma^* := C^{*-1}\Pi$ and C^* is any non-singular matrix, are observationally equivalent if and only if they imply the same distribution of Z_t for $t = 1, ..., T$. Identification requires the imposition of a proper set of restrictions on the matrix C. From Eq. (4) one obtains the relationships

$$\Sigma_\varepsilon := CC' \tag{5}$$

which place $n(n+1)/2$ restrictions on the elements of C, leaving $n^2 - n(n+1)/2 :=$ $n(n-1)/2$ unidentified parameters. Thus a necessary identification order condition is that at least $n(n-1)/2$ restrictions are imposed on C. One way to achieve identification is to complement Eq. (5) with a set of linear restrictions that we write in 'explicit form'

$$vec(C) := S_C \ \gamma + s_C \tag{6}$$

where S_C is a $n^2 \times a_C$ selection matrix, γ is the $a_C \times 1$ vector containing the free elements of C, and s_C is a $n^2 \times 1$ vector. The information required to specify the matrix S_C and the vector r_C usually comes from economic theory or from structural and institutional knowledge related to the problem under study; the condition a_C:=dim(γ)$\leq n^2 - n(n-1)/2 \leq n(n+1)/2$ is necessary for identification.

Throughout the paper θ_0 denotes the 'true' vector value of the reduced form parameters and γ_0 the 'true' vector value of the structural parameters. The matrix C_0 denotes the counterpart of C that fulfills the restriction $vec(C_0) := S_C\gamma_0 + s_C$.

The identification problem of the SVAR in Eq. (4) amounts to the issue of recovering γ_0 (C_0) uniquely from Eq. (5) and Eq. (6). In addition to the necessary order condition, $a_C \leq n(n+1)/2$, necessary and sufficient condition for identification is that the $n(n+1)/2 \times a_C$ matrix

$$2D_n^+(C \otimes I_n)S_C \tag{7}$$

has full column rank evaluated at C_0.[2]

[2]Equivalently, Giannini (1992) and Amisano and Giannini (1997, Ch. 3) derive the necessary and sufficient identification rank condition by referring to linear restrictions in 'implicit form' $R_C vec(C) := r_C$, where R_C is a $b_C \times n^2$ known selection matrix, $b_C \geq n(n-1)/2$, and r_C is $b_C \times 1$ known vector; in their setup identification requires the full-column rank condition of the $b_C \times n(n-1)/2$ matrix

$$R_C(I_n \otimes C)\tilde{D}_n$$

If the rank condition in Eq. (7) holds, the orthogonalized impulse responses are defined by $\Xi_h := [\psi_{lm,h}] := \Phi_h C := (J'\mathring{A}^h J)C$, $h = 0, 1, 2, ...$, where $\psi_{lm,h}$ is the response of variable l to a one-time impulse in variable m, h periods before.[3]

The necessary and sufficient rank condition in Eq. (7) can be checked ex-post at the ML estimate but also prior to estimation at random points drawn uniformly from the parameter space, see e.g. Giannini (1992).[4] Lucchetti (2006) has shown that Eq. (7) can be replaced with a 'structure condition' which is independent on the knowledge of the structural parameters but is still confined to the local identification case. Rubio-Ramìrez *et al.* (2010) have established novel sufficient conditions for global identification that circumvent the knowledge of Eq. (7) and novel necessary and sufficient conditions for global identification that hold for exactly identified SVARs.

In the next sections we generalize the rank condition in Eq. (7) to the case in which there are structural breaks at known dates in the matrices Σ_ε and C. We shall use the concepts of 'identified version of C' and 'similar/different identification structure' with the following meanings.

Definition 1 [Identified version of C] Given the SVAR model in Eq.s (3)-(4) and the restrictions in Eq. (6), the matrix $C := \breve{C}$ denotes an 'identified version' of C if $\det(\breve{C}) \neq 0$ and $\breve{C}$ satisfies the conditions

$$vec(\breve{C}) := S_C\, \gamma + s_C \quad , \; \Sigma_\varepsilon := \breve{C}\, \breve{C}',$$

$$D_n^+(\breve{C} \otimes I_n)S_C \text{ has full column-rank } a_C \leq n(n+1)/2.$$

Definition 2 [Similar/different identification structures] The $n \times n$ matrices C_i associated with the covariance matrices Ω_i, $i = 1, ..., s$, such that each $C_i := \breve{C}_i$ is an identified version of C and S_{C_i} is $n^2 \times a_{C_i}$, $a_{C_i} \leq \frac{n(n+1)}{2}$, have *similar identification*

evaluated at C_0, where $\tilde{D}_n$ denotes the $n^2 \times n(n-1)/2$ full column-rank matrix such that the general solution of the system $N_n y = 0_{n^2 \times 1}$, where y is a non null n^2-dimensional vector, is $y := \tilde{D}_n v$, and v is $\frac{n(n-1)}{2} \times 1$; (in other words, each column of $\tilde{D}_n$ belongs to the null space of N_n).

[3] The identification of C can also be achieved by complementing the symmetry restrictions in Eq. (5) with a set of constraints on the matrix

$$\Xi_\infty := (I_n - A_1 - \cdots - A_k)^{-1} C := \sum_{h=0}^{\infty} \Phi_h C := J'[I_{nk} - \mathring{A}]^{-1} JC$$

which measures the long run impact of the structural shocks on the variables (Blanchard and Quah,1989); the constraints on Ξ can be used in place of, or in conjunction with, the 'short run' restrictions. In this paper we focus, without loss of generality, on the 'short run' restrictions alone. Moreover, the identification analysis presented in this paper does not rely on the recently developed methodologies based on sign restrictions, but can potentially be extended to that case; we refer to Fry and Pagan (2011) for some notes of cautions about the use of sign restrictions in the identification analysis of SVARs, see also Kilian (2012).

[4] Iskrev (2010) applies the same idea to check the identification of DSGE models.

structures if $S_{C_i} = S_{C_j}$, $i \neq j$, $i, j = 1, ..., s$. They have *different identification structures* if for a given $i{:=}\bar{\imath}$, $1 \leq \bar{\imath} \leq s$ and $j{:=}\bar{j}$, $1 \leq \bar{j} \leq s$, such that $\bar{\imath} \neq \bar{j}$, $S_{C_{\bar{\imath}}} \neq S_{C_{\bar{j}}}$.

3 Single break

Consider the SVAR summarized in Eq.s (3)-(4) and assume that it is known that at time T_B, where $1 < T_B < T$, the unconditional reduced form covariance matrix Σ_ε changes. We denote with $\Sigma_{\varepsilon,1}$ and $\Sigma_{\varepsilon,2}$ the VAR covariance matrix before and after the break, respectively, where $\Sigma_{\varepsilon,1} \neq \Sigma_{\varepsilon,2}$. In this section, the case of a single break is discussed for ease of exposition; all technical results derived in this section will be extended to the case of a finite number $s \geq 2$ of breaks in Appendix C.

Inspired by the seminal work of Rigobon (2003), an hypothesis that is receiving increasing attention in the recent literature is based on the idea that the change in the variance of the disturbances Σ_ε does not affect the structural parameters in C. The intuition of Rigobon (2003), extended to the case of SVARs in Lanne and Lütkepohl (2008), is that identification can be achieved by exploiting the algebraic result in Horn and Johnson (1985, Corollary 7.6.5), according to which the condition $\Sigma_{\varepsilon,1} \neq \Sigma_{\varepsilon,2}$ guarantees the simultaneous factorization

$$\Sigma_{\varepsilon,1}{:=}PP' \quad , \quad \Sigma_{\varepsilon,2}{:=}PVP' \tag{8}$$

where P is a $n \times n$ non-singular matrix and $V{:=}\text{diag}(v_1, ..., v_n) \neq I_n$ is a diagonal matrix with $v_i > 0$, $i = 1, ..., n$. Given Eq. (4), identification can be achieved by setting $C{:=}P$, where the choice $C{:=}P$ is unique except for sign changes if all v_i's are distinct, see Lanne and Lütkepohl (2008, 2010).

Intuitively, the result follows from the observation that given $C{:=}P$, Eq. (8) doubles the number of restrictions implied by symmetry and these restrictions are enough to identify the elements of C and $v_1, ..., v_n$. Such a 'purely statistical' approach to the identification of SVAR maximizes the role attached to the data in the identification of the shocks: no theory-driven restriction on C is needed. However, this approach leads to misleading results when C changes when Σ_ε changes. With macroeconomic data it is reasonable to expect that a break affects both the reduced form and structural parameters.

We discuss the identification analysis of the SVAR in the presence of a single break that affects both Σ_ε and C. In Sub-section 3.1 we summarize the representation of the model and introduce the main assumptions; in Sub-section 3.2 we discuss and compare two identification strategies that can be followed when it is known that a change occurs at time T_B and, finally, in Sub-section 3.3 we provide the formal necessary and sufficient conditions for identification.

3.1 Representation and assumptions

A structural change is introduced in the VAR in Eq. (3) by letting the reduced form parameters change at time T_B. We consider the following specification:

$$Z_t = \Pi(t)W_t + \varepsilon_t \quad , \quad E(\varepsilon_t\varepsilon_t'):=\Sigma_\varepsilon(t) \quad , \quad t = 1, ..., T \tag{9}$$

where

$$\Pi(t):=\Pi_1 \times 1\,(t \leq T_B) + \Pi_2 \times 1\,(t > T_B) \quad , \quad t = 1, ..., T \tag{10}$$

$$\Sigma_\varepsilon(t):=\Sigma_{\varepsilon,1} \times 1\,(t \leq T_B) + \Sigma_{\varepsilon,2} \times 1\,(t > T_B) \quad , \quad t = 1, ..., T, \tag{11}$$

$1\,(\cdot)$ is the indicator function, $\Pi_1:=[A_1, \Psi_1]$ and $\Pi_2:=[A_2, \Psi_2]$ are the $n \times f$ matrices containing the autoregressive coefficients before and after the break, respectively. The covariance matrices $\Sigma_{\varepsilon,1}$ and $\Sigma_{\varepsilon,2}$ have been defined above.

The model in Eq.s (9)-(11) covers the case in which the structural break affects both the autoregressive and covariance matrix coefficients. Moreover, as shown in Appendix A, in the case of nonstationary variables the setup is consistent with Hansen's (2002) cointegrated VAR model with structural breaks.

Our analysis requires that $\Sigma_{\varepsilon,1} \neq \Sigma_{\varepsilon,2}$, and that T_B is known to the econometrician. We then formalize the following assumptions.

Assumption 1 [Change of variance at (known) T_B**]** Given the VAR system in Eq.s (9)-(11), T_B is known, $T_B \geq f$ and $T - (T_B + 1) \geq f$, and given Eq. (11) it holds

$$\sigma_{+1}:=vech(\Sigma_{\varepsilon,1}) \neq \sigma_{+2}:=vech(\Sigma_{\varepsilon,2}).$$

Assumption 2 [Invariance of persistence] If in Eq. (10) $A_1 \neq A_2$, the companion matrices

$$\mathring{A}_1^*:=\begin{bmatrix} & A_1 & \\ I_{n(k-1)} & & 0_{n(k-1)\times n} \end{bmatrix} , \; \mathring{A}_2^*:=\begin{bmatrix} & A_2 & \\ I_{n(k-1)} & & 0_{n(k-1)\times n} \end{bmatrix}$$

are such that $\lambda_{\max}(\mathring{A}_1^*) < 1$ and $\lambda_{\max}(\mathring{A}_2^*) < 1$, or, alternatively, are such that $\lambda_{\max}(\mathring{A}_1^*)=1=\lambda_{\max}(\mathring{A}_2^*)$, where $\mathring{A}_1^*$ and $\mathring{A}_2^*$ fulfill the restrictions summarized in Appendix A.

Assumption 1 states that the break date is known and that the unconditional VAR covariance matrix changes. Its main implication is that the two sub-samples of observations

$$Z_1, ..., Z_{T_B} \tag{12}$$

$$Z_{T_B+1}, ..., Z_T \tag{13}$$

represent two distinct regimes and there are sufficient observations to estimate the VAR in each regime. Hereafter the sub-sample in Eq. (12) will be denoted as the 'pre-change' regime and the sub-sample in Eq. (13) as the 'post-change' regime. Since estimation is typically carried out under the assumption of Gaussian disturbances, we interpret the ML estimators of $(\Pi_1, \Sigma_{\varepsilon,1})$ and $(\Pi_2, \Sigma_{\varepsilon,2})$ as quasi-ML (Q)ML estimators. Asymptotic inference is of standard type under Assumptions 1-2, see Appendix . Only Assumption 1 will be crucial to the derivation of our necessary and sufficient identification conditions.

3.2 Identification strategies

Given Assumption 1, we replace the C matrix in Eq. (4) with the specification

$$C(t){:=}C_1 \times 1\left(t \leq T_B\right) + C_2 \times 1\left(t > T_B\right) \quad , \quad t = 1, ..., T \tag{14}$$

where C_1 and C_2 are two $n \times n$ matrices that refer to the pre- and post-change regimes, respectively. C_1 and C_2 need to be identified for the SVAR to be identified. The appealing feature of Eq. (14) is that the number of free elements that enter the C_1 matrix may potentially differ from the number of free elements that enter the C_2 matrix, meaning that the identification structure of the SVAR may change radically from the pre- to the post-change regime.

We discuss two identification strategies that can be followed when Assumption 1 and the specification in Eq. (14) are taken into account. The first, Strategy A, is the one VAR practitioners typically follow when it is known that the reduced form parameters change at time T_B. The second, Strategy B, is our suggested approach.

Strategy A

An obvious identification strategy can be implemented by replacing the structural specification in Eq. (4) with the systems

$$\varepsilon_t{:=}\breve{C}_1 e_t \quad , \quad E(e_t e_t'){:=}I_n \quad , \quad t = 1, ..., T_B$$

$$\varepsilon_t{:=}\breve{C}_2 e_t \quad , \quad E(e_t e_t'){:=}I_n \quad , \quad t = T_B + 1, ..., T$$

where $\breve{C}_1$ and $\breve{C}_2$ are two identified versions of C and may have similar or different identification structures.

With this strategy, the change in the variance of the reduced form disturbances is solely used to separate the two regimes because one deals with two independent SVARs, one relative to the pre-change period, and the other relative to the post-change period. The identification analysis is treated separately in the two regimes.

An example in which the Strategy A is followed in practice is e.g. Boivin and Giannoni (2006), see Section 5.

Strategy B

Assumption 1 can be fully exploited in the identification analysis if the matrices C_1 and C_2 are identified simultaneously. More precisely, consider the following counterpart of the structural specification in Eq. (4):

$$\varepsilon_t{:=}C(t)e_t \quad , \quad E(e_t e_t'){:=}I_n \quad , \quad t = 1, ..., T \tag{15}$$

$$C(t){:=}C + Q \times 1\,(t > T_B) \quad , \quad t = 1, ..., T \tag{16}$$

where Q is a $n \times n$ matrix whose free (non-zero) elements are collected in the vector q. The correspondence between Eq. (14) and Eq. (16) is obtained with $C{:=}C_1$ and $Q{:=}C_2 - C_1$, hence the matrix Q captures the changes, if any, of the structural parameters in the switch from the pre- to the post-change regime.

In general, C and Q must be restricted for the SVAR defined by the Eq.s (15)-(16) to be identified. More specifically, Eq.s (15)-(16) give rise to the set of restrictions

$$\Sigma_{\varepsilon,1}{:=}CC' \tag{17}$$

$$\Sigma_{\varepsilon,2}{:=}(C + Q)(C + Q)' \tag{18}$$

so that the condition $\Sigma_{\varepsilon,1} \neq \Sigma_{\varepsilon,2}$ automatically implies $Q \neq 0_{n \times n}$.[5] The $n(n+1)$ symmetry restrictions provided by Eq.s (17)-(18) are not sufficient alone to identify the $2n^2$ elements of C and Q, suggesting that it is necessary to add at least $2n^2 - (n^2 + n) = n^2 - n$ restrictions on these matrices. Hence the setup described by Eq.s (17)-(18) is sharply different from that in Lanne and Lütkepohl (2008, 2010) based on the factorization in Eq. (8).

[5]The converse, instaed, is not generally true because it is possible to find examples in which $Q \neq 0_{n \times n}$ but $\Sigma_{\varepsilon,1}{:=}C_1C_1'{=}\Sigma_{\varepsilon,2}{:=}C_2C_2'$.This fact has no consequence on the identification results derived in Proposition 1, which require the validity of Assumption 1.

Linear restrictions on C and Q are given by

$$\begin{pmatrix} vec(C) \\ vec(Q) \end{pmatrix} := \begin{bmatrix} S_C & S_I \\ 0_{n^2 \times a_C} & S_Q \end{bmatrix} \begin{pmatrix} \gamma \\ q \end{pmatrix} + \begin{pmatrix} s_C \\ s_Q \end{pmatrix} \tag{19}$$

where q is the vector containing the free elements of Q, S_Q and s_Q are $n^2 \times a_Q$ and $n^2 \times 1$, respectively, and S_I is a possibly non-zero selection matrix which imposes cross-restrictions on the elements of C and Q; in general, there is no reason for C and Q being specified with similar identification structure.

The appealing feature of the framework described by Eq.s (17)-(19) is that the identification of the SVAR can be obtained by exploiting the heteroskedasticity found in the data in conjunction with theory-driven information. It will be shown that the combination of theory-driven and statistically-driven information delivers more 'flexible' identification conditions compared to the cases in which only one of the two sources of identification is used.

Assume temporarily that Eq.s (17)-(19) are specified such that to ensure the identifiability of the SVAR. The next example illustrates the benefits of this approach compared to the Strategy A.

Example 1 [Strategy B: similar identification structures across regimes] Consider the three-variable SVAR (n:=3) adapted from Rubio-Ramìrez *et al.* (2010, p. 677):

$$\underset{\varepsilon_t}{\begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{pmatrix}} = \underset{C}{\begin{bmatrix} c_{11} & 0 & 0 \\ c_{21} & c_{22} & 0 \\ 0 & c_{32} & c_{33} \end{bmatrix}} \underset{e_t}{\begin{pmatrix} e_{1t} \\ e_{2t} \\ e_{3t} \end{pmatrix}} \tag{20}$$

where ε_t:=(ε_{1t}, ε_{2t}, ε_{3t})$'$ is the vector of VAR reduced form disturbances and e_t:=(e_{1t}, e_{2t}, e_{3t})$'$ is the vector of structural shocks, which is i.i.d. with covariance matrix I_3. It is assumed that $c_{ii} \neq 0$, $i = 1, 2, 3$. If the matrix C:=$\breve{C}$ fulfills the conditions of Definition 1, the SVAR based on Eq. (20) is identified. Suppose that it is known that at time $1 < T_B < T$ the VAR covariance matrix changes as in Assumption 1. Suppose further that the change in Σ_ε is associated with a corresponding change in the magnitude of the structural parameters. Then the specification in Eq. (20) is replaced with

$$\underbrace{\begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{pmatrix}}_{\varepsilon_t} := \left\{ \underbrace{\begin{bmatrix} c_{11} & 0 & 0 \\ c_{21} & c_{22} & 0 \\ 0 & c_{32} & c_{33} \end{bmatrix}}_{C} + \begin{bmatrix} q_{11} & 0 & 0 \\ q_{21} & q_{22} & 0 \\ 0 & q_{32} & q_{33} \end{bmatrix} \times 1\,(t > T_B) \right\} \underbrace{\begin{pmatrix} e_{1t} \\ e_{2t} \\ e_{3t} \end{pmatrix}}_{e_t} \tag{21}$$

where it is seen that for $t \leq T_B$ (pre-change regime), the response to the shocks on impact is given by c_{ij}, while for $t > T_B$ (post-change regime) the response to the shocks on impact is $c_{ij}+q_{ij}$. However, the structural break might involve only a subset of the structural parameters, not all of them. For instance, the Q matrix in Eq. (21) might be specified with $q_{21}{:=}0{=:}q_{32}$, meaning that for $j = 1, 2, 3$, only the response on impact of $e_{j,t}$ on $\varepsilon_{j,t}{:=}Z_{j,t}{-}E(Z_{j,t} \mid Z_{t-1}, ..., Z_1)$ changes after the break. The constraints $q_{21}{:=}0{=:}q_{32}$ give rise to two testable overidentifying restrictions which can be tested only within the Strategy B.

In principle, there is no specific reason to start from an identification structure in which C is kept fixed at $C{:=}\breve{C}$ as in Example 1. If such a requirement is relaxed, the SVAR can be identified by using Eq.s (17)-(18) placing simultaneous restrictions on C and Q, see the next example.

Example 2 [Strategy B: different identification structure across regimes] Consider the same three-variable SVAR of Example 1 and the structural specification

$$\underbrace{\begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{pmatrix}}_{\varepsilon_t} := \left\{ \underbrace{\begin{bmatrix} c_{11} & c_{12} & 0 \\ c_{21} & c_{22} & 0 \\ c_{31} & c_{32} & c_{33} \end{bmatrix}}_{C} + \begin{bmatrix} q_{11} & -c_{12} & 0 \\ q_{21} & q_{22} & 0 \\ 0 & q_{32} & q_{33} \end{bmatrix} \times\ 1\,(t \geq T_B) \right\} \underbrace{\begin{pmatrix} e_{1t} \\ e_{2t} \\ e_{3t} \end{pmatrix}}_{e_t} \tag{22}$$

in which the element q_{12} of the Q matrix is restricted such that $q_{12}{:=}{-}c_{12}$. In this case, there are 5 zero restrictions on C and Q plus a single cross-restrictions which implies that the response on impact of ε_{1t} to e_{2t} is effective in the pre-change period but not in the post-change period. Obviously, a researcher who follows the Strategy A can not identify the shocks implied by the specification in Eq. (22).

3.3 Identification analysis

Consider the SVAR with structural break at time T_B introduced in Sub-section 3.1, Assumption 1 and the set of restrictions in Eq.s (17)-(19), which we re-write here for completeness:

$$\Sigma_{\varepsilon,1} := CC' \tag{23}$$

$$\Sigma_{\varepsilon,2} := (C+Q)(C+Q)' \tag{24}$$

$$vec(C) := S_C\gamma + S_I q + s_C \tag{25}$$

$$vec(Q) := S_Q q + s_Q. \tag{26}$$

We denote with γ_0 and q_0 the vectors containing the 'true' values of γ and q, respectively, and with C_0 and Q_0 the matrices obtained from Eq.s (25)-(26) by replacing γ and q with γ_0 and q_0. Our main result is summarized in the next proposition.

Proposition 1 [Identification of C and Q] Assume that the data generating process belongs to the class of SVARs in Eq.s (9)-(11) and (15)-(16), that the C matrix is non-singular and that C and Q are subject to the restrictions in Eq.s (23)-(26). Under Assumption 1, the following statements hold.

(a) Necessary and sufficient condition for the SVAR to be identified is that the $n(n+1)\times(a_C+a_Q)$ matrix

$$(I_2 \otimes D_n^+)\begin{bmatrix} (C\otimes I) & 0_{n^2\times n^2} \\ (C+Q)\otimes I & (C+Q)\otimes I \end{bmatrix}\begin{bmatrix} S_C & S_I \\ 0_{n^2\times a_C} & S_Q \end{bmatrix} \tag{27}$$

has full-column rank evaluated at $C := C_0$ and $Q := Q_0$; necessary order condition is

$$(a_C + a_Q) \leq n(n+1) \tag{28}$$

(b) If $C := \breve{C}$ is an identified version of C, necessary and sufficient condition for the SVAR to be identified is that the $\frac{1}{2}n(n+1)\times a_Q$ matrix

$$D_n^+\left[\left(\tilde{C}+Q\right)\otimes I_n\right]S_Q \tag{29}$$

has full-column rank evaluated at $Q := Q_0$; necessary order condition is $a_Q \leq \frac{1}{2}n(n+1)$.

Proof: Appendix D.

Some remarks are in order.

Proposition 1 will be generalized to the case of multiple breaks in Appendix C, see Proposition 2. An example reported in Appendix C will also show that the identification of the

SVAR requires less and less theory-driven restrictions as the number of heteroskedasticity regimes (breaks) increases.

Point (a) of Proposition 1 deals with the most general case, i.e. the situation in which the identifying restrictions are placed simultaneously on C and Q, including the cross-restrictions governed by the matrix S_I.

The best way to understand the identification condition in point (b) is to think about a researcher who follows a two-step approach: in the first step, the matrix C is identified by applying the 'standard' identification rules that hold in the absence of breaks, obtaining $\tilde{C}$; in the second step, attention is focused on the identification of the Q matrix (post-change regime) since $C{:=}\tilde{C}$ is kept fixed. In this situation, the rank condition is enormously simplified and, as expected, the identification of the SVAR depends only on the restrictions characterizing Q.

When in Eq. (28) $(a_C + a_Q){:=}\ n(n+1)$, the SVAR is 'exactly identified' and is over-identified when $(a_C + a_Q) < n(n+1)$; in the latter case, it is possible to compute a (quasi-)LR test to validate the $n(n+1)-\ (a_C + a_Q)$ overidentifying restrictions. The necessary order condition of point (a) can be re-stated by observing that it is necessary to place at least $n(n-1)$ joint restrictions on C and Q. If this result seems obvious when referred to the identification strategy in which $C{:=}\breve{C}$ is kept fixed (Example 1), it is less obvious and leads to beneficial effects when this restriction is relaxed (Example 2). The next two examples show that one appealing possibility is to concentrate all necessary identifying restrictions on the Q matrix alone, leaving the C matrix unrestricted (except from non-singularity): it is the heteroskedasticity stemming from the data that helps relaxing the 'standard' identification conditions.

Example 3a [Strategy B: consistency with DSGE modeling] Consider the same three-variable SVAR of the previous examples, the break at time T_B and the structural specification

$$\underset{\varepsilon_t}{\underbrace{\begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{pmatrix}}} := \left\{ \underset{C}{\underbrace{\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}}} + \begin{bmatrix} q_{11} & 0 & 0 \\ 0 & q_{22} & 0 \\ 0 & 0 & q_{33} \end{bmatrix} \times 1\left(t > T_B\right) \right\} \underset{e_t}{\underbrace{\begin{pmatrix} e_{1t} \\ e_{2t} \\ e_{3t} \end{pmatrix}}}. \tag{30}$$

In this case, the necessary order condition (Proposition 1, part (c)) is satisfied. Apparently the specification in Eq. (30) is 'close' to the one based on the factorization in Eq. (8), however, despite the C matrix is 'full' in both specifications, in our setup the diagonal elements of C are allowed to change in the two heteroskedasticity regimes while changes in the structural parameters are rule out in Eq. (8). Interestingly, Eq.

(30) can be related to the debate about the consistency of SVAR analysis and DSGE modeling. Imagine that the SVAR features output, inflation and a measure of the monetary policy instrument. As is known, small-scale new-Keynesian DSGE models of the type discussed in e.g. Lubik and Schorfheide (2004), Boivin and Giannoni (2006) and Carlstrom *et al.* (2009) typically admit an immediate reaction of output and inflation to monetary policy impulses, while 'conventional' SVARs feature a lag in such reactions. Thus, under the null that a solution of the structural model belongs to the data generating process, 'conventional' SVARs offer a misspecified representation of monetary policy shocks and their propagation. In other words, given the specification $\varepsilon_t{:=}Cu_t$, the C matrix should be 'full' with highly restricted coefficients and no zero restrictions in order to account for the cross-equation restrictions under rational expectations. Accordingly, the standard Cholesky assumption based on a triangular C might severely distort the impulse response functions, producing price puzzles and muted responses of inflation and the output gap to monetary shocks, see, *inter alia*, Carlstrom *et al.* (2009), Castelnuovo and Surico (2010) and Castelnuovo (2011). Eq. (30) suggests that an identified SVAR with a 'full' C matrix is potentially consistent with the solution of a DSGE system under the condition that a break in the underlying structural parameters occurs.

Example 3b [Strategy B: identification of fiscal shocks] Blanchard and Perotti (2002) is a seminal contribution in which a SVAR is used to identify the effects of fiscal shocks on output by exploiting the following specification:

$$\begin{bmatrix} 1 & 0 & -a_1 \\ 0 & 1 & -b_1 \\ -d_1 & -d_2 & 1 \end{bmatrix} \begin{pmatrix} tt_t \\ gg_t \\ x_t \end{pmatrix} := \begin{bmatrix} 1 & a_2 & 0 \\ b_2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} e_t^{tt} \\ e_t^{gg} \\ e_t^{x} \end{pmatrix} \tag{31}$$

where we have adopted a slight change of notation compared to their Eq.s (2)-(4). In Eq. (31), $\varepsilon_t{:=}(tt_t, g_t, x_t)'$ is the vector of reduced form disturbances of a three-dimensional VAR for $Z_t{:=}(T_t, GG_t, X_t)'$, where T_t is the logarithm of quarterly taxes, GG_t spending and X_t is the GDP in real per capita terms, and e_t^{tt}, e_t^{gg} and e_t^{x} are mutually uncorrelated structural shocks; we refer to the original article for the interpretation of the structural parameters a_1, a_2, b_1, b_2, d_1 and d_2. Using the terminology in Amisano and Giannini (1997), the SVAR defined by Eq. (31) is an

'AB-model'.[6] By re-writing Eq. (31) as a 'C-model' yields

$$\begin{pmatrix} tt_t \\ gg_t \\ x_t \end{pmatrix} := \begin{bmatrix} 1 & 0 & -a_1 \\ 0 & 1 & -b_1 \\ -d_1 & -d_2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & a_2 & 0 \\ b_2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} e_t^{tt} \\ e_t^{gg} \\ e_t^{x} \end{pmatrix} := \underbrace{\begin{bmatrix} c_{11}^* & c_{12}^* & c_{13}^* \\ c_{21}^* & c_{22}^* & c_{23}^* \\ c_{31}^* & c_{32}^* & c_{33}^* \end{bmatrix}}_{C^*} \begin{pmatrix} e_t^{tt} \\ e_t^{gg} \\ e_t^{x} \end{pmatrix} \tag{32}$$

where

$$C^* := \frac{1}{(a_1c_1 + b_1d_2 - 1)} \begin{bmatrix} -a_1b_2d_2 + b_1d_2 - 1 & -a_1d_2 + a_2(b_1d_2 - 1) & -a_1 \\ -b_1d_1 + b_2(a_1d_1 - 1) & -a_2b_1d_1 + (a_1d_1 - 1) & -b_1 \\ -d_1 - b_2d_2 & -c_{d2} - a_2d_1 & -1 \end{bmatrix}.$$

It is immediately seen that the system defined by Eq. (32) is not identified and is necessary to impose at least 3 'additional' restrictions to identify the structural shocks. However, if it is known that at time T_B the covariance matrix of the reduced form disturbances $\varepsilon_t := (tt_t, gg_t, x_t)'$ changes and that at least one among a_1, a_2, b_1, b_2, d_1 and d_2 also changes, the specification in Eq. (32) can be replaced with e.g.

$$\begin{pmatrix} tt_t \\ g_t \\ x_t \end{pmatrix} = \left\{ \begin{bmatrix} c_{11}^* & c_{12}^* & c_{13}^* \\ c_{21}^* & c_{22}^* & c_{23}^* \\ c_{31}^* & c_{32}^* & c_{33}^* \end{bmatrix} + \begin{bmatrix} q_{11} & 0 & 0 \\ 0 & q_{22} & 0 \\ 0 & 0 & q_{33} \end{bmatrix} \times 1\,(t > T_B) \right\} \begin{pmatrix} e_t^{tt} \\ e_t^{g} \\ e_t^{x} \end{pmatrix}.$$

Proposition 1 suggests that in this case the parameters a_1, a_2, b_1, b_2, d_1, d_2, q_{11}, q_{22} and q_{33} can potentially be recovered from the data.

If the specified matrices C and Q meet the requirements of Proposition 1, the (population) orthogonalized impulse responses implied by the SVAR are given by

$$\Xi_{1,h} := \left[\psi_{1,lm,h}\right] := J'(\mathring{A}_1)^h JC\ ,\ \ h = 0, 1, 2, ...\ \text{'pre-change' regime} \tag{33}$$

$$\Xi_{2,h} := \left[\psi_{2,lm,h}\right] := J'(\mathring{A}_2)^h J(C + Q)\ ,\ \ h = 0, 1, 2, ...\ \text{'post-change' regime} \tag{34}$$

where

$$\mathring{A}_i := \begin{bmatrix} & A_i & \\ I_{n(k-1)} & , & 0_{n(k-1)\times n} \end{bmatrix}$$

and the matrices A_is, $i = 1, 2$ have to be restricted as explained in Appendix A in case of

[6] Since there are 5 restrictions on the matrix on the left-hand-side and 7 restrictions on the matrix on the right-hand side of Eq. (31), the total number of restrictions (including symmetry) is 12+6=18, and the necessary condition for (exact) identification is met. Actually, Blanchard and Perotti (2002) resort to 'external' information to identify a subset of the structural parameters.

nonstationary cointegrated variables. Note that A_1 and A_2 may be equal or differ under Assumptions 1-2.

The coefficient $\psi_{i,lm,h}$ captures the response of variable l to a one-time impulse in variable m, h periods before, in the heteroskedasticity regime i.Consistent estimates of impulse response functions are obtained from Eq.s (33)-(34) by replacing $\mathring{A}_1$, $\mathring{A}_2$ and the identified C and Q with their (Q)ML estimated counterparts.

The necessary and sufficient rank condition of Proposition 1 can be checked numerically ex-post at the (Q)ML estimate or prior to estimation by using algorithms no more complicated than the one originally proposed by Giannini (1992) for SVARs and successively suggested by Iskrev (2010) for DSGE models. The following algorithm is a possible example but it is clear that any algorithm of this type can not be used to derive any indication as to whether or not the SVAR is identified globally.

Algorithm [Numerical check of the rank condition prior to estimation]

1. Given the restrictions on C and Q, define the matrices S_C, s_C, S_Q, s_Q and S_I and the matrices K_n, N_n;
2. check the validity of the necessary order condition in Eq. (28); if the order condition is not satisfied stop the algorithm otherwise consider the next step;
3. for each element γ_j, $j = 1, ..., a_C$ of γ consider a grid $\mathcal{U}_{\gamma_j}$:=$[\gamma_j^{\min}, \gamma_j^{\max}]$ of economically plausible values, and for each element q_l, $l = 1, ..., a_Q$ of the vector q consider a grid $\mathcal{U}_{q_l}$:=$[q_l^{\min}, q_l^{\max}]$ of economically plausible values;
4. for $j = 1, ..., a_C$ and $l = 1, ..., a_Q$, draw γ_j from $\mathcal{U}_{\gamma_j}$ using the uniform distribution obtaining the point γ_0:=$(\gamma_1, ..., \gamma_{a_C})$; then draw q_l from $\mathcal{U}_{q_l}$ using the uniform distribution obtaining the point q_0:=$(q_1, ..., q_{a_Q})$; then construct the matrices C_0 and Q_0, where $vec(C_0)$:=$S_C\gamma_0 + s_C$ and $vec(Q_0)$:=$S_Q q_0 + s_Q$;
5. check whether the matrix defined in Eq. (27) has rank $(a_C + a_Q)$; [7] if full column-rank is not obtained, the algorithm is stopped, otherwise consider the next step;
6. repeat steps 4-5 M times;
7. If no stop has occurred, the SVAR is identified locally.

[7] In GAUSS, which is used in our paper, the rank of a matrix is determined as the number of singular values that exceed the tolerance value $10e^{-13}$.

4 Related literature

In this section we briefly review the contributions to the literature in which the heteroskedasticity found in the data is used to identify SVARs in macroeconomic analysis (see also Kilian, 2012) and compare those works with our approach; we refer to Klein and Vella (2010) and Lewbel (2012) and references therein for the use of heteroskedasticity in the identification analysis of endogenous regressor models and simultaneous systems of equations, and to Mavroeidis and Magnusson (2010) for dynamic macroeconomic models based on forward-looking behavior.[8] Moreover, it is worth emphasizing that Sentana and Fiorentini (2001) provide identification conditions, in the context of conditionally heteroskedastic factor models, that can be applied in a large number of cases, including GARCH-type residuals (e.g. Caporale, Cipollini and Demetriades, 2005; Dungey and Martin, 2001; King *et al.*, 1994), regime switching processes (e.g. Caporale, Cipollini and Spagnolo, 2005; Rigobon and Sack, 2003, 2004), and structural VAR models (e.g. Normandin and Phaneuf, 2004).

Focusing on the case of time-varying unconditional second-order moments, in his seminal contribution, Rigobon (2003) formalizes the intuition that when the data exhibit volatility clusters and the structural parameters remain constant, the number of equations connecting the parameters of the reduced form with those of the structural form can be enriched in order to solve the identification problem. In his baseline model, Rigobon assumes that the heteroskedasticity in the data can be described as a two-regime process and shows that in his setup the structural parameters of the system are just identified. More precisely, in his Proposition 1, Rigobon (2003) shows that in a bivariate system of linear equations with uncorrelated structural shocks, the structural parameters are always identified unless the unconditional covariance matrices of the reduced form are proportional. He also discusses identification conditions under more general conditions, such as more than two regimes, when common unobservable shocks exist, and situations in which the nature of the heteroskedasticity is misspecified. However, Rigobon (2003) does not provide necessary and sufficient conditions for identification for systems of $n > 2$ equations.

Lanne and Lütkepohl (2008) extend Rigobon's (2003) approach to the context of SVARs and propose a test of overidentifying restrictions within the SVAR framework of Bernanke and Mihov (1998). These authors exploit the knowledge of the break date T_B, the condition $\Sigma_{\varepsilon,1} \neq \Sigma_{\varepsilon,2}$, and the factorization in Eq (8) to identify the structural parameters, see our Proposition 1; Ehrmann *et al.* (2011) follow a similar approach. Closely related works are Lanne *et al.* (2010), who generalize the idea of Lanne and Lütkepohl

[8] Keating (2004) proposes a VAR-based approach for testing block recursive economic theories by exploiting structural breaks.

(2008) to the case in which the changes in the unconditional VAR covariance matrix are not known *a priori* but are governed by an underlying Markov-switching process, and Lanne and Lütkepohl (2010) who model the VAR disturbances as a mixture of normal distributions and exploit their non-normality to identify the shocks. All these contributions share the assumption that the structural parameters remain constant across the detected heteroskedasticity regimes.

Our paper is similar to the above mentioned works as concerns the idea that the heteroskedasticity found in the data can be used constructively to identify the SVAR. However, it departs substantially from the current literature on the volatility-driven approach to the identification of SVARs as concerns the assumption that the structural parameters do not change across heteroskedasticity regimes. In our setup, the occurrence of break(s) in the unconditional VAR covariance matrix is automatically associated with change(s) in the structural parameters and the identification of the shocks is addressed simultaneously across heteroskedasticity regimes. The statistical information provided by the detected change in the VAR unconditional covariance matrix is not sufficient alone to identify the SVAR and must be mixed up with theory-driven information. Our Proposition 1 (Proposition 2, Appendix C) shows that the proper combination of these two sources of identification results in novel necessary and sufficient conditions and enlarges the identification possibility typically available to practitioners.

5 A small monetary policy SVAR

In this section we apply the identification strategy and identification rules derived in Section 3 by estimating a small monetary policy SVAR on U.S. quarterly data. The idea is to exploit the change in volatility occurred in several macroeconomic time series in the switch from the 'Great Inflation' to the 'Great Moderation' period, documented in e.g. McConnell and Perez-Quiros (2002), Boivin and Giannoni (2006) and Lanne and Lütkepohl (2008), among many others.

We focus on a SVAR based on the vector $Z_t{:=}(\tilde{y}_t, \pi_t, R_t)'$ ($n{:=}3$), where $\tilde{y}_t$ is a measure of the output gap, π_t the inflation rate and R_t a nominal policy interest rate. We deal with quarterly data, sample 1954.q3-2008.q3 (including initial values). Our measure of real activity, $\tilde{y}_t$, is the Congressional Budget Office (CBO) output gap, constructed as percentage log-deviations of real GDP with respect to CBO potential output. The measure of inflation, π_t, is the annualized quarter-on-quarter GDP deflator inflation rate, while the policy instrument, R_t, is the Federal funds rate (average of monthly observations). The data were collected from the website of the Federal Reserve Bank of St. Louis. We refer to Giordani (2004) for detailed explanations of why the output gap should be preferred to real (possibly detrended) output in this setup.

The reduced form VAR is a system with six lags (k:=6) and a constant. The VAR lag order k:=6 was obtained by combining LR-type reduction tests with standard information criteria. Henceforth we denote the reference reduced form VAR with k:=6 lags with the acronym 'VAR(6)'.

In line with the empirical literature on the 'Great Moderation', we divide the postwar period 1954.q3-2008.q3 into two sub-periods: the 'Pre-Volcker' period, 1954.q3-1979.q2, and the 'post-Volcker' period, 1979.q3-2008.q3, and test wether the covariance matrix of the VAR(6) changes in these two sub-periods. In our notation, T_B:=1979.q2. We are aware that many other choices for T_B are equally possible: an alternative would be to start the second period 1984.q1 as in e.g. McConnell and Perez-Quiros (2002) but we follow Boivin and Giannoni (2006) who also identify the quarter 1979.q2 as the date which separates two regimes of U.S. monetary policy.[9] Further, our statistical tests, presented below, show that the two chosen periods can be regarded as two regimes characterized by different volatility.

Table 1 reports the estimated covariance matrices of the VAR(6) and some (multivariate) residual diagnostic tests relative to the whole period and the two sub-periods, respectively. Although the chosen lag order is sufficient to obtain uncorrelated residuals, the vector test for the joint normality of the disturbances signals strong departures from the hypothesis of Gaussian distributions, especially on the whole sample and on the 'post-Volcker' sample. In this exercise we take for granted that Assumption 2 holds.[10]

The first hypothesis to test is that a break occurred at T_B:=1979.q2 in the reduced form coefficients θ:=$(\pi', \sigma_+')'$ of the VAR(6), in particular in the covariance matrix σ_+. We first use a standard Chow-type (quasi-)LR test for the null H_0: θ_1=θ_2:=θ against the alternative H_1: $\theta_1 \neq \theta_2$, where p:=dim(θ):=63. The results in Table 1 suggest that H_0 is strongly rejected because the (quasi-)LR test is equal to LR:=-2[719.98 - (322.86+488.96)]:=183.68 and has a p-value of 0.000 (taken from the $\chi^2(63)$ distribution). Successively we test the null $H_0^{(1)}$:σ_{+1}:=vech($\Sigma_{\varepsilon,1}$)=σ_{+2}:=vech($\Sigma_{\varepsilon,2}$)=σ_+ vs $H_1^{(1)}$:$\sigma_{+1} \neq \sigma_{+2}$ under the maintained assumption π_1=π_2=π. Also in this case the computed (quasi-)LR test leads us to sharply reject the null $H_0^{(1)}$. Similar outcomes are obtained by applying conventional 'small sample' corrections. We thus conclude that a statistically significant break in the reduced form parameters occurred at time T_B:=1979.q2, similarly to what discovered by Boivin and Giannoni (2006) by following a different approach.

[9] Unfortunately, the sub-period 1979.q3-1984.q1 is not long enough to give us the possibility of considering two break dates and three potentially heteroskedastic regimes in the empirical analysis.

[10] We treat the VAR(6) as a stationary system and ignore the non-stationarity issue because it is not central to our scopes and exposition. Moreover, the evidence in favor of unit roots (cointegration) in the VAR(6) is not clear-cut in both sub-periods. Results are available upon request.

Having established that the 'Pre-Volcker' and the 'post-Volcker' periods are characterized by different covariance matrices, $\Sigma_{\varepsilon,1} \neq \Sigma_{\varepsilon,2}$ (other than $\Pi_1 \neq \Pi_2$), next we proceed with the identification of the structural shocks. The most popular identification strategy for monetary policy shocks is the 'Cholesky' scheme. Such scheme orders R_t after π_t and $\tilde{y}_t$ (or after $\tilde{y}_t$ and π_t) on the basis of the observation that (i) monetary authorities contemporaneously react to macroeconomic indicators; (ii) inflation and output are affected by policy shocks with lags.[11] Given that ordering, the 'Cholesky' scheme is obtained as a special case of the system in Eq.s (15)-(16) by specifying C and Q lower triangular; in addition to the 'Cholesky' structure, we consider other possible identification schemes summarized in Table 2. Each model is denoted with the symbol $\mathcal{M}_i$, $i = 1, ..., 4$. In each model, given the vector of structural shocks e_t:=$(e_t^{\tilde{y}},\ e_t^{\pi},\ e_t^{R})'$, we call e_t^R the 'monetary policy shock', $e_t^{\tilde{y}}$ the 'output shock' and e_t^{π} the 'inflation shock'.

Model $\mathcal{M}_1$ maintains that the 'Cholesky scheme' holds both in the 'Pre-Volcker' and 'post-Volcker' periods, hence it is based on the idea that the identification structure of the shocks is the same across the two regimes but the magnitude of the response on impact of the variables to the shocks changes across regimes. In principle, model $\mathcal{M}_1$ could also be analyzed by the identification Strategy A, i.e. by two separate SVARs, one for the pre-change period and the other for the post-change period. However, in that case it is not possible to evaluate whether the patterns of impulse responses is statistically significant in the two regimes, see e.g. Boivin and Giannoni (2006). On the other hand, with the proposed identification approach, the detection of significant elements of the Q matrix can be associated with the idea that the response on impact of the variables to the shocks is different across periods, see Eq.s (33)-(34).

Model $\mathcal{M}_2$ can be interpreted as restricted overidentifying counterpart of model $\mathcal{M}_1$ and can be tested empirically. Model $\mathcal{M}_3$ implies three testable overidentifying restrictions and maintains that only the parameters associated with the policy rule change across the two regimes. Finally, the exact identified model $\mathcal{M}_4$ is interesting for two opposite reasons. On the one hand, it can be regarded as an 'agnostic' identification scheme which maximizes the role attached to the data and annihilates the information provided by the theory in the identification of the shocks. On the other hand, model $\mathcal{M}_4$ can be related, with qualifications, to the discussion in the Example $3a$ about the consistency of SVAR analysis and small-scale DSGE modeling.

The four models $\mathcal{M}_i$, $i = 1, 2, 3, 4$ pass the local 'identification check' provided by the algorithm reported in sub-section 3.3, see the last column of Table 2. This simply

[11] The appealing feature of such an identification scheme is that it does not require the researcher to take a position on the identification of other shocks; we refer to Christiano *et al.* (1999) for an extensive discussion on this issue and to Castelnuovo (2011) for a recent investigation.

means that there are points in the admissible parameter space that are locally identified but nothing can be said as to whether or not the four model are identified globally. The (Q)ML estimates of the structural parameters of the four models are summarized in Table 3, which also reports the likelihood associated with each model and a (quasi-)LR test for the specification when the SVAR features overidentifying restrictions.

Studies based on SVARs typically find that U.S. monetary policy shocks, defined as unexpected exogenous changes in R_t, have had a much smaller impact on output gap and inflation since the beginning of the 1980s: overall, the results in Table 3 seem to confirm such evidence. In addition, we detect significant changes in the structural parameters in the move from the 'Pre-Volcker' to the 'post-Volcker' period, and many specified elements of the Q matrices are found highly significant in all estimated SVARs. Interestingly, the sharp statistical rejection of model $\mathcal{M}_3$ in Table 3 indicates that the change in the structural parameters has not to be solely ascribed to changes in the monetary policy rule pursued by the Fed but to other parameters as well.

Figure 1 displays for both samples the impulse response functions relative to a monetary policy shock e_t^R and the associated 90% (asymptotic) confidence interval over an horizon of 40 periods. The reference model is $\mathcal{M}_2$ which other than being not rejected by the data, is the estimated SVAR with associated higher likelihood. The pattern of the two sets of impulse responses reveals the change in monetary policy conduct. The key result from the comparison between the 'Pre-Volcker' period (left column) and the 'post-Volcker' period (right column) in Figure 1 is that the effect of a monetary policy shock was stronger before the 1980s. In addition, we can claim that the response on impact of the variables to the shocks is different in the two periods.

Figure 2 displays for both samples the response of the Federal funds rate to the shocks $e_t^{\tilde{y}}$ and e_t^{π}, respectively. While the sensitiveness of the short term nominal interest rate to the two shocks seems to be weak before the 1980s, the Fed's responsiveness to these two shocks is clear cut in the 'post-Volcker' period. According to a large (but debated) strand of the literature, this evidence reflects the switch to a more aggressive 'active' policy intended to rule out the possibility of sunspot fluctuations induced by self-fulfilling expectations, see e.g. Clarida *et al.* (2000).

A remarkable fact that emerges from the impulse responses in Figure 1 and Figure 2, is the absence of the 'price puzzle' on the 'post-Volcker' period. This evidence, which is also documented in e.g. Barth and Ramey (2001), Hanson (2004), Boivin and Giannoni (2006) and Castelnuovo and Surico (2010), supports the view that the 'price puzzle' phenomenon is much more evident in situations in which the central bank responds weakly to inflationary dynamics.

Finally, as concerns model $\mathcal{M}_4$, one indication stemming from Table 3 is that the estimated small monetary policy SVAR is potentially consistent with what would be the solution of a small DSGE model for Z_t:=$(\tilde{y}_t, \pi_t, R_t)'$ of the type discussed in e.g. Lubik and Schorfheide (2004), Boivin and Giannoni (2006) anf Carlstrom *et al.* (2009) if a break in the structural parameters is admitted at time T_B:=1979.q2.

6 Conclusions

A recent stand of the literature makes use of the heteroskedasticity found in the data to identify SVARs. The maintained assumption is that the structural parameters remain constant when the unconditional VAR covariance matrix changes. In this paper we relax this assumption and derive the identification rules that hold when the reduced form covariance matrix changes along with the structural parameters at known dates.

We derive a necessary and sufficient rank condition for local identification and show that the combination of heteroskedasticity and theory-driven restrictions opens interesting possibility for practitioners, unexplored so far.

We illustrate the usefulness of our approach by focusing on a small-scale monetary policy SVAR model estimated on U.S. quarterly data. In order to identify the shocks, we exploit basic theory-driven restrictions and the change in volatility occurred in a VAR system for output gap, inflation and the policy nominal interest rate in the switch from the 'Great Inflation' to the 'Great Moderation' period. Overall, our results are not at odds with the view that monetary policy has become more effective at stabilizing the economy after the 1980s.

A Appendix: Nonstationary cointegrated variables

In this Appendix we show how the reference SVAR model with breaks at known dates discussed in the paper can be adapted to the case of nonstationary cointegrated variables.

When the companion matrix $\mathring{A}$ associated with the VAR in Eq. (1) is such that $\lambda_{\max}(\mathring{A})$=1, three additional restrictions must be considered: (i) it is assumed that the VAR system features exactly $0 < n - r \leq n$, $0 < r < n$, eigenvalues with real part equal to one and imaginary part equal to zero (unit-roots at zero frequency), which amounts to the existence of the r-dimensional asymptotically stationary vector $\beta' Z_t$ and Vector Equilibrium Correction (VEqC) representation

$$\Delta Z_t = \alpha\beta' Z_{t-1} + \Gamma_1 \Delta Z_{t-1} \ldots + \Gamma_{k-1}\Delta Z_{t-k+1} + \Psi D_t + \varepsilon_t \quad , \quad t = 1, ..., T \tag{35}$$

where α is a $n \times r$ matrix, $\alpha\beta' := -(I_n - A_1 - \cdots - A_k)$, $\Gamma_i := \sum_{j=i+1}^{k} A_j$, $i = 1, ..., k-1$, see Johansen (1996); (ii) it is assumed that the $n \times r$ matrix β is constrained such that $\beta := \beta_I$, where $\beta_I' Z_t$ is an identified system of simultaneous equations involving the levels of Z_t that collects the r theoretically meaningful stationary relations of the system, and β_I embodies at least $\ell \geq r^2$ *linear* identifying restrictions (including normalization) and fulfills the necessary and sufficient identification restrictions discussed in Johansen (1995); (iii) defined the matrices $\Lambda := [\alpha\beta_I', \ \Gamma_1, ..., \Gamma_{k-1}]$ and $\Upsilon := [\alpha, \ \Gamma_1, ..., \Gamma_{k-1}]$, it is assumed that the autoregressive parameters $A := [A_1, ..., A_k]$ of the VAR system are restricted such that $A := \tilde{A}$, where

$$\tilde{A} := \Lambda F + H \quad , \ \Lambda := \Upsilon \Theta_{\beta_I} \ \ , \Theta_{\beta_I} := \begin{bmatrix} \beta_I' & 0_{r \times n(k-1)} \\ 0_{n(k-1) \times n} & I_{n(k-1)} \end{bmatrix} \tag{36}$$

$$F := \begin{bmatrix} -I_n & 0_{n\times n} & \cdots & 0_{n\times n} & 0_{n\times n} \\ I_n & -I_n & \cdots & 0_{n\times n} & 0_{n\times n} \\ \vdots & \ldots & \ddots & \vdots & \vdots \\ 0_{n\times n} & 0_{n\times n} & \cdots & -I_n & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & \cdots & I_n & -I_n \end{bmatrix}, \ H := [I_n, \, 0_{n\times n} \ , \ \cdots, \, 0_{n\times n} \ ].$$

Eq. (36) maps the parameters of the cointegrated VEqC system to the parameters of the VAR in levels. When in the reduced form VAR $\lambda_{\max}(\mathring{A}) = 1$ and the conditions (i)-(iii) hold, the Π matrix in Eq. (3) is assumed to be restricted as $\Pi := (\tilde{A}, \Psi)$.

Under the restrictions (i)-(iii), if the number of unit roots $n - r$ is known (or the cointegration rank r is known) and the matrix Θ_{β_I} in Eq. (36) is treated as known, the parameters $\Pi := (\tilde{A}, \Psi)$, Σ_ε and $\Phi_h := J'(\mathring{A}^r)^h J$, $h = 1, 2, ...$, can be estimated consistently and the corresponding ML estimators are asymptotically Gaussian. Thus, given the mapping $\varepsilon_t := Ce_t$, impulse response analysis can be carried out as in e.g. Lütkepohl and Reimers (1992), Amisano and Giannini (1997, Ch. 6), Phillips (1998) and Vlaar (2004).[12] Instead we refer to Gonzalo and Ng (2001) and Pagan and Pesaran (2008) for examples in which the identification analysis of nonstationary SVARs is explicitly based on permanent/transitory decompositions of the shocks.

In the presence of structural breaks occuring at known dates, the matrices Π and Σ_ε obey Eq.s (9)-(11). In this case, our setup is consistent with Hansen (2002). To see this,

[12] Intuitively, if the model is correctly specified, the super-consistency result (Johansen, 1996) ensures that β_I can be replaced in empirical analyses in Eq. (36) with its ML estimator $\hat{\beta}_I$ and treated as the 'true' β_I without consequences on the asymptotic distributions of the ML estimators of the remaining parameters.

consider the case of a single break and the following generalization of system (35):

$$\Delta Z_t = \alpha(t)\beta(t)' Z_{t-1} + \Gamma_1(t)\Delta Z_{t-1} + \ldots + \Gamma_{k-1}(t)\Delta Z_{t-k+1} + \Psi(t)D_t + \varepsilon_t$$

where

$$\alpha(t)\beta(t)' := \alpha_1\beta_1' \times 1\,(t \leq T_B) + \alpha_2\beta_2' \times 1\,(t > T_B) \quad , \ \ t = 1, ..., T$$

$$\Gamma_j(t) := \Gamma_{j,1} \times 1\,(t \leq T_B) + \Gamma_{j,2} \times 1\,(t > T_B) \quad , \ \ t = 1, ..., T \ , \ j = 1, ..., k-1$$

$$\Psi(t) := \Psi_1 \times 1\,(t \leq T_B) + \Psi_2 \times 1\,(t > T_B) \quad , \ \ t = 1, ..., T$$

and

$$\Sigma_\varepsilon(t) := \Sigma_{\varepsilon,1} \times 1\,(t \leq T_B) + \Sigma_{\varepsilon,2} \times 1\,(t > T_B) \quad , \ \ t = 1, ..., T.$$

The matrices α_i, β_i, $\Gamma_{j,i}$ and Ψ_i, $i = 1, 2$, are of conformable dimensions. The counterpart of Eq. (36) is given by

$$\tilde{A}(t) := \Lambda(t)F + H \ \ , \ \ \Lambda(t) := \Upsilon(t)\Theta_{\beta_{I(t)}} \quad , \ \ t = 1, ..., T$$

where

$$\Upsilon(t) := \Upsilon_1 \times 1\,(t \leq T_B) + \Upsilon_2 \times 1\,(t > T_B)$$

$$\Theta_{\beta_{I(t)}} := \begin{bmatrix} \beta_{I,1}' & 0_{r\times n(k-1)} \\ 0_{n(k-1)\times n} & I_{n(k-1)} \end{bmatrix} \times 1\,(t \leq T_B) + \begin{bmatrix} \beta_{I,2}' & 0_{r\times n(k-1)} \\ 0_{n(k-1)\times n} & I_{n(k-1)} \end{bmatrix} \times 1\,(t > T_B)$$

and

$$\Upsilon_i := [\alpha_i, \Gamma_{1,i}, ..., \Gamma_{k-1,i}] \ \ , \ i = 1, 2$$

and for $i = 1, 2$, $\beta_{I,i}$ are restricted as in the point (ii) above.

B Appendix: Estimation issues

In this Appendix we sketch the estimation issues related to the SVAR model introduced in Section 3. Under Assumptions 1-2 and adding the condition that the distribution of the disturbances id Gaussian within each regime, the likelihood-based estimation and inference in the SVAR with known break(s) is of standard type and no new result is needed. Here we simply focus on the likelihood function of the SVAR, its score and information matrix. In the presence of departures from normality, all the estimators derived below have to be interpreted as quasi-ML (QML) estimators. We consider, for simplicity, the case of a single break; the extension to the general case is straightforward. Further details about estimation and testing issues may be found in e.g. Amisano and Giannini (1997).

The whole vector of reduced form parameters of the VAR system in Eq.s (9)-(11) is $\theta{:=}(\theta_1', \theta_2')'$ and its unrestricted VAR log-likelihood function can be split as

$$\log L_T(\theta_1, \theta_2){:=}\text{const}{+}l_{T_B}(\theta_1) + l_{T-(T_B+1)}(\theta_2)$$

where

$$\begin{aligned} l_{T_B}(\theta_1){:=} \quad & -\frac{T_B}{2}\log(\det(\Sigma_{\varepsilon,1})) - \frac{1}{2}\sum_{t=1}^{T_B}(Z_t - \Pi_1 W_t)'\Sigma_{\varepsilon,1}^{-1}(Z_t - \Pi_1 W_t) \\ := \quad & -\frac{T_B}{2}\log(\det(\Sigma_{\varepsilon,1})) - \frac{1}{2}tr\left[\Sigma_{\varepsilon,1}^{-1}(Z_{T_B} - W_{T_B}\Pi_1')'(Z_{T_B} - W_{T_B}\Pi_1')\right] \end{aligned}$$

and

$$\begin{aligned} l_{T-(T_B+1)}(\theta_2){:=} \quad & -\frac{T-(T_B+1)}{2}\log(\det(\Sigma_{\varepsilon,2})) - \frac{1}{2}\sum_{t=T_B+1}^{T-(T_B+1)}(Z_t - \Pi_2 W_t)'\Sigma_{\varepsilon,2}^{-1}(Z_t - \Pi_2 W_t) \\ := \quad & -\frac{T-T_B-1}{2}\log(\det(\Sigma_{\varepsilon,2})) - \frac{1}{2}tr\left[\Sigma_{\varepsilon,2}^{-1}(Z_{T-T_B-1} - W_{T-T_B-1}\Pi_2')'(Z_{T-T_B-1} - W_{T-T_B-1} \right. \end{aligned}$$

where tr is the trace operator, $Z_{T_B}{:=}[Z_1, ..., Z_{T_B}]'$, $W_{T_B}{:=}[W_1, ..., W_{T_B-1}]'$, $Z_{T-T_B-1}{:=}\left[Z_{T-(T_B+1)}, ..., Z_T\right]'$ and $W_{T-T_B-1}{:=}\left[W_{T-(T_B+1)}, ..., W_T\right]'$.

We define the vectors $\alpha := (vec\,(C)', vec(\,Q)')'$ and $\varphi := (\gamma', q')'$, where the latter is the vector of (free) structural parameters. α and φ are linked by the mapping in Eq. (19) (where the S_I matrix may be zero or not, depending on the nature of the restrictions at hand), which we compact in the expression

$$\alpha{:=}G\varphi + g. \tag{37}$$

In Eq. (37), G is a $n^2 \times (a_C + a_Q)$ selection matrix which has full column-rank, and g is a $n^2 \times 1$ known vector. It is convenient to derive the score and information matrix of the SVAR with respect to α, and then use Eq. (37) to obtain equivalent counterparts for the parameters of interest in φ.

The log-likelihood function concentrated with respect to the autoregressive parameters

Π_1 and Π_2 can be written as:

$$\begin{aligned}\log L_T(\alpha) := \quad & \text{const} - \frac{T_B}{2}\log(\det(C)^2) - \frac{T-T_B}{2}\log(\det(C+Q)^2) \\ & -\frac{T_B}{2} tr\left(C^{-1\prime}C^{-1}\hat{\Sigma}_{\varepsilon,1}\right) - \frac{T-T_B}{2} tr\left[(C+Q)^{-1\prime}(C+Q)^{-1}\hat{\Sigma}_{\varepsilon,2}\right]\end{aligned}$$

where

$$\hat{\Sigma}_{\varepsilon,1} := \frac{1}{T_B}(Z_{T_B} - W_{T_B}\hat{\Pi}_1')'(Z_{T_B} - W_{T_B}\hat{\Pi}_1')$$

$$\hat{\Sigma}_{\varepsilon,2} := \frac{1}{T-T_B-1}(Z_{T-T_B-1} - W_{T-T_B-1}\hat{\Pi}_2')'(Z_{T-T_B-1} - W_{T-T_B-1}\hat{\Pi}_2').$$

By applying standard derivative rules, it is seen that the score of the SVAR is given by

$$\begin{aligned} s_T(\alpha) := \quad & \frac{\partial \log L_T(\alpha)}{\partial \alpha} \\ := \quad & \left(\begin{array}{c} vec\left[-T_B C'^{-1} - (T-T_B-1)(C+Q)'^{-1} + T_B\left(C'^{-1}C^{-1}\hat{\Sigma}_{\varepsilon,1}C'^{-1}\right) + \right. \\ \left. + (T-T_B-1)(C+Q)'^{-1}(C+Q)^{-1}\hat{\Sigma}_{\varepsilon,2}(C+Q)'^{-1}\right] \\ \\ vec\left[-(T-T_B-1)(C+Q)'^{-1} + (T-T_B-1)(C+Q)'^{-1}(C+Q)^{-1}\hat{\Sigma}_{\varepsilon,2}(C+Q)'^{-1}\right] \end{array}\right)\end{aligned}$$

while, by computing second derivatives, the information matrix is given by

$$\mathcal{I}_T(\alpha) := E\left(-\frac{\partial^2 \log L_T(\alpha)}{\partial\alpha\partial\alpha'}\right) := \left[\begin{array}{cc} \mathcal{I}_T^{11}(\alpha) & \mathcal{I}_T^{12}(\alpha) \\ \mathcal{I}_T^{12}(\alpha) & \mathcal{I}_T^{22}(\alpha)\end{array}\right]$$

where

$$\begin{aligned} \mathcal{I}_T^{11}(\alpha) := \quad & 2T_B\left(I_n \otimes C^{-1\prime}\right)N_n\left(I_n \otimes C^{-1}\right) + \\ & +2(T-T_B-1)\left[I_n \otimes (C+Q)^{-1\prime}\right]N_n\left[I_n \otimes (C+Q)^{-1}\right]; \\ \mathcal{I}_T^{12}(\alpha) := \quad & 2(T-T_B-1)\left[I_n \otimes (C+Q)^{-1\prime}\right]N_n\left[I_n \otimes (C+Q)^{-1}\right]; \\ \mathcal{I}_T^{22}(\alpha) := \quad & 2(T-T_B-1)\left[I_n \otimes (C+Q)^{-1\prime}\right]N_n\left[I_n \otimes (C+Q)^{-1}\right]. \end{aligned}$$

It follows that $s_T(\varphi) := G' s_T(\alpha)$ and $\mathcal{I}_T(\varphi) := G'\mathcal{I}_T(\alpha)G$. The form of $s_T(\alpha)$ and $\mathcal{I}_T(\alpha)$ in the equations above suggest that the log-likelihood of the SVAR can conveniently be maximized by the *score algorithm* (e.g. Fletcher, 1987).

Under Assumptions 1-2, LR tests for the overidentifying restrictions are asymptotically $\chi^2(n(n+1) - (a_C + a_Q))$-distributed (see Proposition 2).

C Appendix: Multiple breaks

The analysis developed in the paper is based on the assumption that there exists a single break in the reduced form coefficients and two heteroskedasticity regimes upon which the SVAR can be identified. In this Appendix we extend our results to the case in which the number of breaks is $s \geq 2$ and there are $s+1$ heteroskedasticity regimes in the data.

The VAR reduced form parameters are allowed to change at the break points $T_{B1}, \ldots, T_{Bs}$, where $1 < T_{B1}, \ldots, < T_{Bs} < T$. We conventionally assume that T_{B_0}:=1 and T_{Bs+1}:=T, and consider the following assumption that generalizes Assumption 1 in the text.

Assumption 3 The break points $1 < T_{B_1} < \ldots < T_{Bs} < T$, are known, $T_{B_1} \geq f$, $T_{Bi} - T_{Bi-1} \geq f$, $i = 2, \ldots, s+1$, and the reduced form unconditional covariance matrix $\sigma_+(t)$:=$vech(\Sigma(t))$ is given by

$$\sigma_+(t) := \sum_{i=1}^{s+1} \sigma_{+i} \times 1\left(T_{Bi-1} < t \leq T_{Bi}\right) \quad , \quad t = 1, ..., T \tag{38}$$

where

$$\sigma_{+i} := vech(\Sigma_{\varepsilon,i}) \neq \sigma_{+j} := vech(\Sigma_{\varepsilon,j}) \quad \forall\ i \neq j.$$

Similarly to the case of a single break, the matrix that links the structural shocks and reduced form disturbances is given by

$$C(t) := C + \sum_{i=1}^{s} Q_i \times 1\left(T_{Bi-1} < t \leq T_{Bi}\right) \quad , \quad t = 1, ..., T \tag{39}$$

where Q_j, $j = 1, ..., s$ are $n \times n$ matrices. In Eq. (39), the matrix C contains the structural parameters before any break occurs while the matrices Q_j, $j = 1, ..., s$, describe how the identification structure changes across heteroskedasticity regimes.

The mapping between reduced form and structural parameters is given by

$$\Sigma_{\varepsilon,1} := CC' \quad , \quad \Sigma_{\varepsilon,i} := \left(C + Q_{i-1}\right)\left(C + Q_{i-1}\right)' \quad , \quad i = 2, ..., s+1$$

and the linear restrictions on C and Q_j, $j = 1, ..., s$, are expressed in the form

$$vec(C) := S_C \gamma + \sum_{i=1}^{s} S_{I_i} q_i + s_C$$

$$vec(Q_j) := S_{Q_i} q_i + \sum_{i \neq j} S_{IQ_i} q_i + s_{Q_j} \quad , \ j = 1, ..., s$$

and are compacted in the expression

$$\begin{pmatrix} vec(C) \\ vec(Q_1) \\ \vdots \\ vec(Q_s) \end{pmatrix} := \begin{bmatrix} S_C & S_{I_1} & S_{I_2} & \cdots & S_{I_s} \\ & S_{Q_1} & S_{IQ_2} & \cdots & S_{IQ_s} \\ & & \ddots & \vdots & \vdots \\ & & & & S_{Q_s} \end{bmatrix} \begin{pmatrix} \gamma \\ q_1 \\ q_2 \\ \vdots \\ q_s \end{pmatrix} + \begin{pmatrix} s_C \\ s_{Q_1} \\ \vdots \\ s_{Q_s} \end{pmatrix}. \tag{40}$$

In Eq. (40), the matrices S_{I_i}, $i = 1, ..., s$ and S_{IQ_j}, $j \neq i$ can be possibly zero and accommodate the possibility of cross-regime restrictions; as usual, γ is the vector that collects the free elements of the matrix C, q_i is the a_{Q_i}-dimensional vector containing the free elements of the matrix Q_i, $i = 1, ..., s$, and S_C, S_{Q_i} and s_{Q_i} have obvious interpretation. To simplify notation, we denote the 'big' $(s+1)n^2 \times (a_C + a_{Q_1} + \ldots + a_{Q_s})$ matrix in Eq. (40) with S^*.

The next proposition generalizes Proposition 1 to the case of $s \geq 2$ breaks.

Proposition 2 [Identification of $C, Q_1, \ldots, Q_s$] Assume that the data generating process belongs to the class of SVARs whose reduced form parameters are given by

$$\theta(t) := \begin{pmatrix} \pi(t) \\ \sigma_+(t) \end{pmatrix} \quad , \quad t = 1, ..., T$$

and $\sigma_+(t)$ is given in Assumption 3, Eq. (38). Given the structural specification in Eq. (39), assume that the matrices C, Q_j, $j = 1, ..., s$ are subject to the restrictions in Eq. (40) and let γ_0 and $q_{0,i}$ be the vectors containing the 'true' values of γ and q_j, $j = 1, \ldots, s$.

Then necessary and sufficient condition for the SVAR to be locally identified is that the $\frac{1}{2}(s+1)n^2 \times (a_C + a_{Q_1} + \ldots + a_{Q_s})$ matrix

$$(I_{s+1} \otimes D_n^+) \begin{bmatrix} (C \otimes I_n) & & & \\ (C+Q_1) \otimes I_n & (C+Q_1) \otimes I_n & & \\ \vdots & \vdots & \ddots & \\ (C+Q_s) \otimes I_n & & & (C+Q_s) \otimes I_n \end{bmatrix} S^* \tag{41}$$

has full-column rank evaluated at C_0, $Q_{0,i}$, $i = 1, ..., s$; necessary order condition is

$$(a_C + a_{Q_1} + \ldots + a_{Q_s}) \leq (s+1)\, n(n+1)/2.$$

Proof: Appendix D.

When in Proposition 2 s:=1, the matrix in Eq. (41) collapses to the matrix in Eq. (29) and the order condition collapse to the one in Eq. (28). All remarks made for Proposition 1 can be extended to Proposition 2.

The next example consider a SVAR with s:=2 breaks.

Example 4 [Two breaks and three heteroskedasticity regimes] Consider a SVAR with three variables (n:=3) and the case s:=2, where T_{B_1} and T_{B_2} are the two break dates in which the VAR covariance matrix changes ($\Sigma_{\varepsilon,1} \neq \Sigma_{\varepsilon,2}$, $\Sigma_{\varepsilon,2} \neq \Sigma_{\varepsilon,3}$). In this situation, a possible identification structure which is consistent with the requirements of Proposition 2 is e.g. given by

$$\underset{\varepsilon_t}{\underbrace{\begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{pmatrix}}} := \left\{ \underset{C}{\underbrace{\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}}} + \begin{bmatrix} q_{1,11} & 0 & 0 \\ q_{1,21} & q_{1,22} & 0 \\ q_{1,31} & q_{1,32} & q_{1,33} \end{bmatrix} \times 1\left(T_{B_1} < t \leq T_{B_2}\right) \right.$$

$$\left. + \begin{bmatrix} q_{2,11} & 0 & 0 \\ 0 & q_{2,22} & 0 \\ 0 & 0 & q_{2,33} \end{bmatrix} \times 1\left(T_{B_2} < t \leq T\right) \right\} \underset{e_t}{\underbrace{\begin{pmatrix} e_{1t} \\ e_{2t} \\ e_{3t} \end{pmatrix}}}$$

where e_t is, as usual, the vector of structural shocks.

If the specified matrices C and Q_j, $j = 1, ..., s$ meet the requirements of Proposition 2, the (population) orthogonalized impulse responses are given by

$$\Xi_{1,h} := \left[\psi_{1,lm,h}\right] := J'(\mathring{A}_1)^h JC \ , \quad h = 0, 1, 2, ... \quad \text{'pre-change' regime}$$

$$\Xi_{i,h} := \left[\psi_{i,lm,h}\right] := J'(\mathring{A}_i)^h J(C+Q_{i-1}) \ , \quad h = 0, 1, 2, ..., \ i = 2, ..., s+1 \quad \text{'post-change' regimes}$$

where

$$\mathring{A}_i := \begin{bmatrix} & A_i & \\ I_{n(k-1)} & , & 0_{n(k-1)\times n} \end{bmatrix} \quad , \ i = 1, ..., s+1$$

and the matrices A_is may be restricted as explained in Appendix A in case of nonstationary cointegrated variables. The coefficient $\psi_{i,lm,h}$ is the response of variable l to a one-time impulse in variable m, h periods before, in the volatility regime i.

D Appendix: Proofs

Proof of Proposition 1. (a) Given the vectors $\alpha{:=}(vec(C)', vec(Q)')'$ and $\varphi{:=}(\gamma', q')'$, define the $n(n+1)$-dimensional vector $\sigma_+^*{:=}(\sigma_{+1}', \sigma_{+2}')$. The mapping between the reduced form and the structural parameters is given by the first two sets of equations of system (23)-(26), which we re-write in the form

$$\begin{aligned}\sigma_{+1} &= vech(CC')\\ \sigma_{+2} &= vech[(C+Q)(C+Q)']\end{aligned}$$

where C and Q depend on γ and q, respectively. By Assumption 1, $\sigma_{+1} \neq \sigma_{+2}$ implies $Q \neq 0_{n\times n}$. Necessary and sufficient condition for local identification is the non-singularity of the $n(n+1)\times(a_C + a_Q)$ Jacobian matrix $\frac{\partial \sigma_+^*}{\partial \varphi'}$ evaluated at C_0 and Q_0 (Rothemberg, 1971). By the chain rule we have

$$\frac{\partial \sigma_+^*}{\partial \varphi'}{:=}\frac{\partial \sigma_+^*}{\partial \alpha'} \times \frac{\partial \alpha}{\partial \varphi'}$$

where

$$\underset{n(n+1)\times 2n^2}{\frac{\partial \sigma_+^*}{\partial \alpha'}{:=}\begin{bmatrix} \frac{\partial \sigma_{+1}}{\partial vec(C)'} & \frac{\partial \sigma_{+1}}{\partial vec(Q)'} \\ \frac{\partial \sigma_{+2}}{\partial vec(C)'} & \frac{\partial \sigma_{+2}}{\partial vec(Q)'}\end{bmatrix}}\ ,\ \underset{2n^2\times(a_C+a_Q)}{\frac{\partial \alpha}{\partial \varphi'}{:=}S{:=}\begin{bmatrix} S_C & S_I \\ 0_{n^2\times a_C} & S_Q\end{bmatrix}}.$$

By using the properties of the matrices K_n, N_n and D_n^+ (Magnus and Neudecker, 2007) and applying standard derivative rules, the four blocks of the matrix $\frac{\partial \sigma_+^*}{\partial \alpha'}$ are given by:

$$\frac{\partial \sigma_{+1}}{\partial vec(C)'}{:=}\frac{\partial vech(CC')}{\partial vec(C)'}{:=}D_n^+\frac{\partial vec(CC')}{\partial vec(C)'}{:=}2\ D_n^+N_n(C\otimes I_n){:=}2\ D_n^+(C\otimes I_n);$$

$$\frac{\partial \sigma_{+1}}{\partial vec(Q)'}{:=}0_{\frac{1}{2}n(n+1)\times n^2};$$

$$\frac{\partial \sigma_{+2}}{\partial vec(C)'}{:=}\frac{\partial vech[(CC'+CQ'+QC'+QQ']}{\partial vec(C)'}\quad{:=}\frac{\partial vech(CC')}{\partial vec(C)'}+\frac{\partial vech(CQ')}{\partial vec(C)'}+\frac{\partial vech(QC')}{\partial vec(C)'}$$

$${:=}2\ D_n^+(C\otimes I_n)+D_n^+(Q\otimes I_n)+D_n^+(I_n\otimes Q)K_n\quad{:=}2\ D_n^+(C\otimes I_n)+2D_n^+(Q\otimes I_n);$$

$$\frac{\partial \sigma_{+2}}{\partial vec(Q)'}{:=}\frac{\partial vech[(CC'+CQ'+QC'+QQ']}{\partial vec(Q)'}\quad{:=}2\ D_n^+(C\otimes I_n)+2D_n^+(Q\otimes I_n).$$

It turns out that

$$\frac{\partial\sigma_{+}^{*}}{\partial\varphi'}{:=}\begin{bmatrix} 2\ D_{n}^{+}(C\otimes I_{n}) & 0_{\frac{1}{2}n(n+1)\times n^{2}} \\ 2\ D_{n}^{+}(C\otimes I_{n})+2D_{n}^{+}(Q\otimes I_{n}) & 2\ D_{n}^{+}(C\otimes I_{n})+2D_{n}^{+}(Q\otimes I_{n}) \end{bmatrix} S$$

$$:=2\underset{n(n+1)\times 2n^{2}}{\begin{bmatrix} D_{n}^{+} & 0_{\frac{1}{2}n(n+1)\times n^{2}} \\ 0_{n^{2}\times 0_{\frac{1}{2}n(n+1)\times n^{2}}} & D_{n}^{+} \end{bmatrix}}\underset{2n^{2}\times 2n^{2}}{\begin{bmatrix} (C\otimes I) & 0_{n^{2}\times n^{2}} \\ (C+Q)\otimes I & (C+Q)\otimes I \end{bmatrix}}\underset{2n^{2}\times(a_{C}+a_{Q})}{\begin{bmatrix} S_{C} & S_{I} \\ 0_{n^{2}\times a_{C}} & S_{Q} \end{bmatrix}}.$$

Aside from the multiplicative scalar 2, the matrix above is the same as the matrix in Eq. (27). The necessary order condition is obviously given by $(a_C+a_Q)\leq n\,(n+1)$. (b) If $C{:=}\tilde{C}$ is an identified version of C and is kept fixed, $S_I{:=}0_{n^2\times a_Q}$ and the actual mapping between reduced form and free structural parameters is given by

$$\sigma_{+2}=vech[(\tilde{C}+Q)(\tilde{C}+Q)'].$$

The computation of the $\frac{1}{2}n\,(n+1)\times a_Q$ Jacobian $\frac{\partial\sigma_{+2}}{\partial q'}$ delivers the matrix in Eq. (29) of the text, and the necessary order condition is $a_Q\leq\frac{1}{2}n\,(n+1)\,.$■

Proof of Proposition 2 The proof is a straightforward generalization of the proof of Proposition 1. It suffices to define the vectors $\alpha^{*}{:=}(vec(C)',vec(Q_1)',...,vec(Q_s)')'$ and $\varphi^{*}{:=}(\gamma',q_1',...q_s')'$, the $\frac{1}{2}n(n{+}1)(s{+}1)$-dimensional vector $\sigma_{+}^{*}{:=}(\sigma_{+1}',\sigma_{+2}',...,\sigma_{+(s+1)}')$ and observe that the mapping between reduced form and structural parameters is given by

$$\begin{aligned}\sigma_{+1} &= vech(CC')\\ \sigma_{+j} &= vech[(C+Q_j)(C+Q_j)'] \quad , \qquad j{=}1,...,s\end{aligned}$$

whereas the derivative $\frac{\partial\alpha^{*}}{\partial\varphi^{*\prime}}$ corresponds to the S^{*} matrix implied by Eq. (40). By using the properties of the matrices K_n, N_n and D_n^{+} and standard derivative rules, it is seen that the Jacobian $\frac{\partial\sigma_{+}^{*}}{\partial\varphi^{*\prime}}$ corresponds to the matrix in Eq. (41) multiplied by 2. The necessary order condition holds trivially.■

References

Amisano, G. and Giannini, C. (1997), *Topics in Structural VAR Econometrics*, 2nd edn, Springer, Berlin.

Barth, M.J. and Ramey, V.A. (2001), The cost channel of monetary policy transmission, *NBER Macroeconomics Annual*, 199-239.

Bernanke, B.S. and Mihov, I. (1998), Measuring monetary policy, *Quarterly Journal of Economics* 113, 869-902.

Blanchard, O. and Perotti, R. (2002), An empirical characterization of the dynamic effects of changes in government spending and taxes on output, *Quarterly Journal of Economics* 117, 1329-1368.

Blanchard, O. and Quah, D. (1989) The dynamic effects of aggregate demand and supply disturbances, *American Economic Review* 79, 655-673.

Boivin, J. and Giannoni, M.P. (2006), Has monetary policy become more effective? *Review of Economics and Statistics* 88, 445-462.

Caporale, G.M., Cipollini, A. and Demetriades, P. (2005), Monetary policy and the exchange rate during the Asian crisis: Identification through heteroskedasticity, *Journal of International Money and Finance* 24, pages 39-53.

Caporale, G.M., Cipollini, A. and Spagnolo, N. (2005), Testing for contagion: A conditional correlation analysis, *Journal of Empirical Finance* 12, 476-489.

Carlstrom, C.T., Fuerst, T.S. and Paustian, M. (2009), Monetary policy shocks, Cholesky identification, and DNK models, *Journal of Monetary Economics* 56, 1014-1021.

Castelnuovo, E. (2011), In Cholesky-VARs we trust? An empirical investigation for the U.S., Working Paper, University of Padova.

Castelnuovo, E. and Surico, P. (2010), Monetary policy, inflation expectations and the price puzzle, *Economic Journal* 120, 1262-1283.

Christiano, L., Eichenbaum, J.M. and Evans, C (1999), Monetary policy shocks: What have we learned and to what end?, in Taylor, J.B., Woodford, M. (Eds.), *Handbook of Macroeconomics*, Elsevier Science, 65–148.

Clarida, R.J., Galì, J. and Gertler, M. (2000), Monetary policy rules and macroeconomic stability: evidence and some theory, *Quarterly Journal of Economics* 115, 147-180.

Dungey, M. and Martin, V.L. (2001), Contagion across financial markets: An empirical assessment, Australian National University mimeograph.

Ehrmann, M., Fratzscher, M. and Rigobon, R. (2011), Stocks, bonds, money markets and exchange rates: Measuring international financial transmission, *Journal of Applied Econometrics* 26, 948–974.

Fletcher, R., (1987), *Practical methods of optimization*, Wiley-Interscience, New York.

Fry, R., and Pagan, A. (2011), Sign restrictions in Structural Vector Autoregressions: A critical review, *Journal of Economic Literature* 49, 938-60.

Giannini, C. (1992), *Topics in Structural VAR Econometrics*, Springer, Berlin.

Giordani, P. (2004), An alternative explanation of the price puzzle, *Journal of Monetary Economics* 51, 1271-1296.

Gonzalo, J. and Ng., S. (2001), A systematic framework for analyzing the dynamic effects of permanent and transitory shocks, *Journal of Economic Dynamics and Control* 25, 1527-1546.

Hamilton, J.D. (1994), *Time Series Analysis*, Princeton University Press, Princeton.

Hansen, P. R. (2003), Structural changes in the cointegrated vector autoregressive model, *Journal of Econometrics* 114, 261-295.

Hanson, M.S. (2004), The price puzzle reconsidered, *Journal of Monetary Economics* 51, 1385-413.

Horn, R.A. and Johnson, C. R. (1985), *Matrix analysis*, Cambridge, Cambridge University Press.

Iskrev, N. (2010), Local identification in DSGE models, *Journal of Monetary Economics* 57, 189-202.

Johansen, S. (1995), Identifying restrictions of linear equations with applications to simultaneous equations and cointegration, *Journal of Econometrics* 69, 111–132.

Johansen, S. (1996). *Likelihood-based inference in cointegrated vector auto-regressive models*, Oxford University Press, Oxford.

Keating, J. (2004), Testing block recursive economic theories by looking for structural breaks, Working Paper.

Kilian, L. (2012), *Structural vector autoregressions*, prepared for the Handbook of Research Methods and Applications in Empirical Macroeconomics, Hashimzade, N., and M. Thornton (eds.), Camberley, UK: Edward Elgar, 2012.

King, M., Sentana, E. and Wadhwani, S. (1994), Volatility and links between national stock markets, *Econometrica* 62, 901-933.

Klein, R. and Vella, F. (2010), Estimating a class of triangular simultaneous equations models without exclusion restrictions, *Journal of Econometrics* 154, 154-164.

Lanne, M. and Lütkepohl, H. (2008), Identifying monetary policy shocks via changes in volatility, *Journal of Money, Credit and Banking* 40, 1131-1149.

Lanne, M. and Lütkepohl, H. (2010), Structural vector autoregressions with non normal residuals, *Journal of Business and Economic Statistics* 28, 159-168.

Lanne, M., Lütkepohl, H. and Maciejowska, K. (2010), Structural vector autoregressions with Markov switching, *Journal of Economic Dynamics and Control* 34, 121-131.

Lewbel, A. (2012), Using heteroskedasticity to identify and estimate mismeasured and endogenous regressor models, *Journal of Business and Economic Statistics* 30, 67-80.

Lubik, T. A., Schorfheide, F. (2004), Testing for indeterminacy: An application to U.S. monetary policy, *American Economic Review* 94, 190-217.

Lucchetti, R. (2006), Identification of covariance structures, *Econometric Theory* 22, 235-257.

Lütkepohl, H. (2005), *New introduction to multiple time series*, Springer, Berlin Heidelberg.

Lütkepohl, H. and Reimers, H.E. (1992), Impulse response analysis of cointegrated systems with an application to German Money demand, *Journal of Economic Dynamics and Control* 16, 53-78.

Magnus, J.R. and Neudecker, H. (2007), *Matrix differential calculus with applications in statistics and econometrics*, John Wiley and Sons.

Mavroeidis, S. and Magnusson, L. (2010), Identification using stability restrictions, Working Paper, Tulane University.

McConnell, M.M. and Perez-Quiros, G. (2000), Output fluctuations in the United States: what has changed since the early 1980's? *American Economic Review* 90, 1464-1476.

Normandin, M. and Phaneuf, B. (2004), Monetary policy shocks: Testing identification conditions under time-varying conditional volatility, *Journal of Monetary Economics* 51, 1217-43.

Pagan, A.R. and Pesaran, M.H. (2008), Econometric analysis of structural systems with permanent and transitory shocks, *Journal of Economic Dynamics and Control* 32, 3376-3395.

Phillips, P.C.B. (1998), Impulse response and forecast error variance asymptotics in nonstationary VARs, *Journal of Econometrics* 83, 21–56.

Rigobon, R. (2003), Identification through heteroskedasticity, *Review of Economics and Statistics* 85, 777-792.

Rigobon, R. and Sack B. (2003), Measuring the reaction of monetary policy to the stock market, *Quarterly Journal of Economics* 118, 639-669.

Rigobon, R. and Sack B. (2004), The impact of monetary policy on asset prices, *Journal of Monetary Economics* 51, 1553-75.

Rothenberg, T.J. (1971), Identification in parametric models, *Econometrica* 39, 577-591.

Rubio-Ramirez, J.F., Waggoner, D. and Zha, T. (2010), Structural Vector Autoregressions: Theory of identification and algorithms for inference, *Review of Economic Studies* 77, 665-696.

Sentana, E. (1992), Identification of multivariate conditionally heteroskedastic factor models, LSE-FMG Discussion Paper No. 139.

Sentana, E. and Fiorentini, G. (2001), Identification, estimation and testing of conditionally heteroskedastic factor models, *Journal of Econometrics* 102, 143-164.

Vlaar, P.J.G. (2004), On the asymptotic distribution of impulse response functions with long-run restrictions, *Econometric Theory* 20, 891-903.

TABLE 1

Estimated covariance matrices and diagnostic tests from the VAR(6). Break date T_B:=1979.q2

Overall period	1954.q3-2008.q3 (T=211)	dim(θ):=63
LM_{AR5}:= 1.064 [0.363]	$\hat{\Sigma}_{\varepsilon}:=\begin{bmatrix} 0.762 & -0.012 & 0.037 \\ & 0.273 & 0.007 \\ & & 0.191 \end{bmatrix}$	log-Likelihood:=719.98
JB_N:=186.27 [0.000]		
'Pre-Volcker' period	1954.q3-1979.q2 (T=94)	dim(θ_1):=63
LM_{AR5}:= 0.838 [0.753]	$\hat{\Sigma}_{\varepsilon,1}:=\begin{bmatrix} 0.989 & -0.021 & 0.036 \\ & 0.314 & 0.007 \\ & & 0.152 \end{bmatrix}$	log-Likelihood:=322.86
JB_N:= 14.525 [0.0234]		
'post-Volcker' period	1979.q3-2008.q3 (T=117)	dim(θ_2):=63
LM_{AR5}:= 1.69 [0.010]	$\hat{\Sigma}_{\varepsilon,2}:=\begin{bmatrix} 0.562 & 0.010 & 0.045 \\ & 0.204 & 0.007 \\ & & 0.194 \end{bmatrix}$	log-Likelihood:=488.96
JB_N:= 50.91 [0.000]		

NOTES: LM_{AR5} is the Lagrange Multiplier vector test for the absence of residuals autocorrelation against the alternative of autocorrelated VAR disturbances up to lag 5; JB_N is the Jarque-Bera multivariate test for Gaussian disturbances. Number in brackets are p-values.

TABLE 2

Four models, known break date T_B			Z_t:=$(\tilde{y}_t, \pi_t, R_t)'$, e_t:=$(e_t^{\tilde{y}}, e_t^{\pi}, e_t^{R})'$	
Model	C	Q	# over-identif. $n(n+1)-(a_C+a_Q)$	rank check M:=5000 draws from uniform dist.
$\mathcal{M}_1$	$\begin{bmatrix} * & & \\ * & * & \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * & & \\ * & * & \\ * & * & * \end{bmatrix}$	exact ident.	passed
$\mathcal{M}_2$	$\begin{bmatrix} * & & \\ & * & \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * & & \\ & * & \\ * & & * \end{bmatrix}$	3	passed
$\mathcal{M}_3$	$\begin{bmatrix} * & & \\ * & * & \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} & & \\ & & \\ * & * & * \end{bmatrix}$	3	passed
$\mathcal{M}_4$	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * & & \\ & * & \\ & & * \end{bmatrix}$	exact ident.	passed

NOTES: Asterisks denote generic coefficients while blank entries correspond to zeros. ‘The ‘rank check’ column has been obtained by the algorithm sketched in Sub-section 3.3 by considering M:=5000 draws from the uniform distribution and the interval $\mathcal{U}$:=[-1.5, 1.5] for all non-zero parameters entering C and Q, see also footnote 8.

TABLE 3

Estimated SVARs with known break date T_B:=1979.q3 on US quarterly data

Model: $C(t){:=}C + Q \times 1\,(t > T_B)\ ,\ \ t = 1, ..., T$

C | Q | $(C+Q),\ t = T_B, ..., T$

$\mathcal{M}_1$

$$\begin{bmatrix} \underset{(0.064)}{0.883} & & \\ \underset{(0.028)}{-0.019} & \underset{(0.020)}{0.281} & \\ \underset{(0.014)}{0.032} & \underset{(0.013)}{0.021} & \underset{(0.009)}{0.130} \end{bmatrix} \quad \begin{bmatrix} \underset{(0.073)}{-0.369} & & \\ \underset{(0.034)}{0.036} & \underset{(0.024)}{0.094} & \\ \underset{(0.021)}{0.041} & \underset{(0.020)}{-0.006} & \underset{(0.014)}{0.029} \end{bmatrix} \quad \begin{bmatrix} 0.514 & & \\ 0.017 & 0.375 & \\ 0.073 & 0.015 & 0.159 \end{bmatrix}$$

Log-Likelihood = 811.81   exact identification

$\mathcal{M}_2$

$$\begin{bmatrix} \underset{(0.064)}{0.883} & & \\ & \underset{(0.021)}{0.281} & \\ \underset{(0.014)}{0.034} & \underset{(0.010)}{0.024} & \underset{(0.009)}{0.130} \end{bmatrix} \quad \begin{bmatrix} \underset{(0.073)}{-0.369} & & \\ & \underset{(0.024)}{-0.094} & \\ \underset{(0.021)}{0.037} & & \underset{(0.014)}{-0.290} \end{bmatrix} \quad \begin{bmatrix} 0.514 & & \\ & 0.187 & \\ 0.071 & 0.024 & 0.160 \end{bmatrix}$$

Log-Likelihood =811.09   LR test = $\underset{[0.70]}{1.44}$

$\mathcal{M}_3$

$$\begin{bmatrix} \underset{(0.034)}{0.703} & & \\ \underset{(0.016)}{-0.004} & \underset{(0.011)}{0.233} & \\ \underset{(0.011)}{0.027} & \underset{(0.011)}{0.018} & \underset{(0.010)}{0.130} \end{bmatrix} \quad \begin{bmatrix} & & \\ & & \\ \underset{(0.023)}{0.070} & \underset{(0.022)}{0.016} & \underset{(0.014)}{0.029} \\ & & \end{bmatrix} \quad \begin{bmatrix} 0.703 & & \\ -0.004 & 0.233 & \\ 0.096 & 0.034 & 0.160 \end{bmatrix}$$

Log-Likelihood =787.48   LR test = $\underset{[0.00]}{48.66}$

$\mathcal{M}_4$

$$\begin{bmatrix} \underset{(0.065)}{0.876} & \underset{(0.055)}{-0.045} & \underset{(0.054)}{-0.105} \\ \underset{(0.022)}{-0.020} & \underset{(0.030)}{0.255} & \underset{(0.038)}{-0.115} \\ \underset{(0.012)}{0.049} & \underset{(0.027)}{0.072} & \underset{(0.018)}{0.105} \end{bmatrix} \quad \begin{bmatrix} \underset{(0.073)}{-0.374} & & \\ & \underset{(0.052)}{-0.402} & \\ & & \underset{(0.032)}{-0.26} \end{bmatrix} \quad \begin{bmatrix} 0.502 & 0.045 & 0.105 \\ 0.017 & 0.147 & 0.115 \\ 0.073 & -0.072 & 0.155 \end{bmatrix}$$

Log-Likelihood =811.81   exact identification

NOTES: Standard errors in parenthesis, p-values by squared brackets. The columns of the matrix $(C+Q)$ have been normalized such that the elements on the main diagonal are positive. Empty entries correspond to zeros.

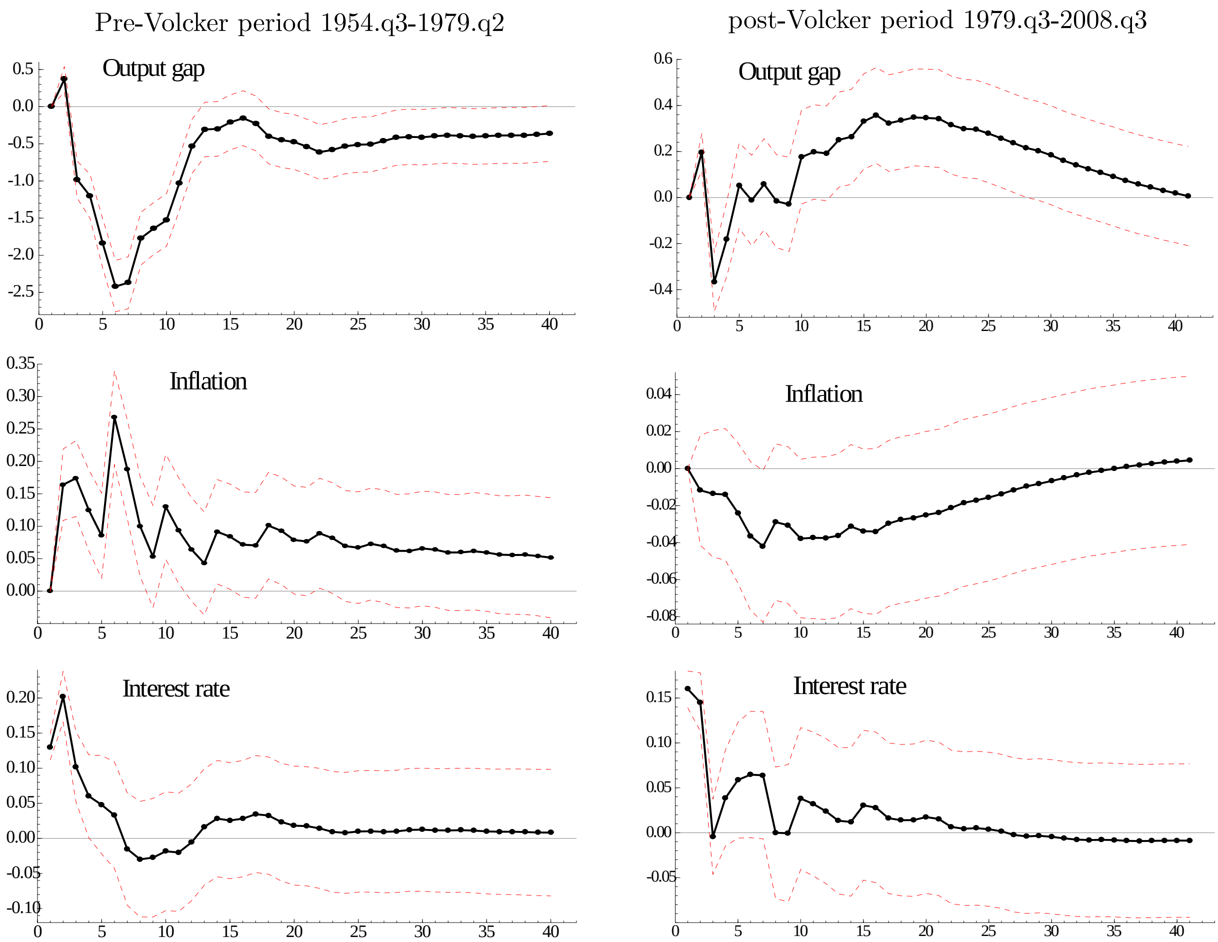

Figure 1. Impulse response functions to a same-size monetary policy shock e_t^R with 90% confidence bands based on the SVAR model $\mathcal{M}_2$ in tables 2-3.

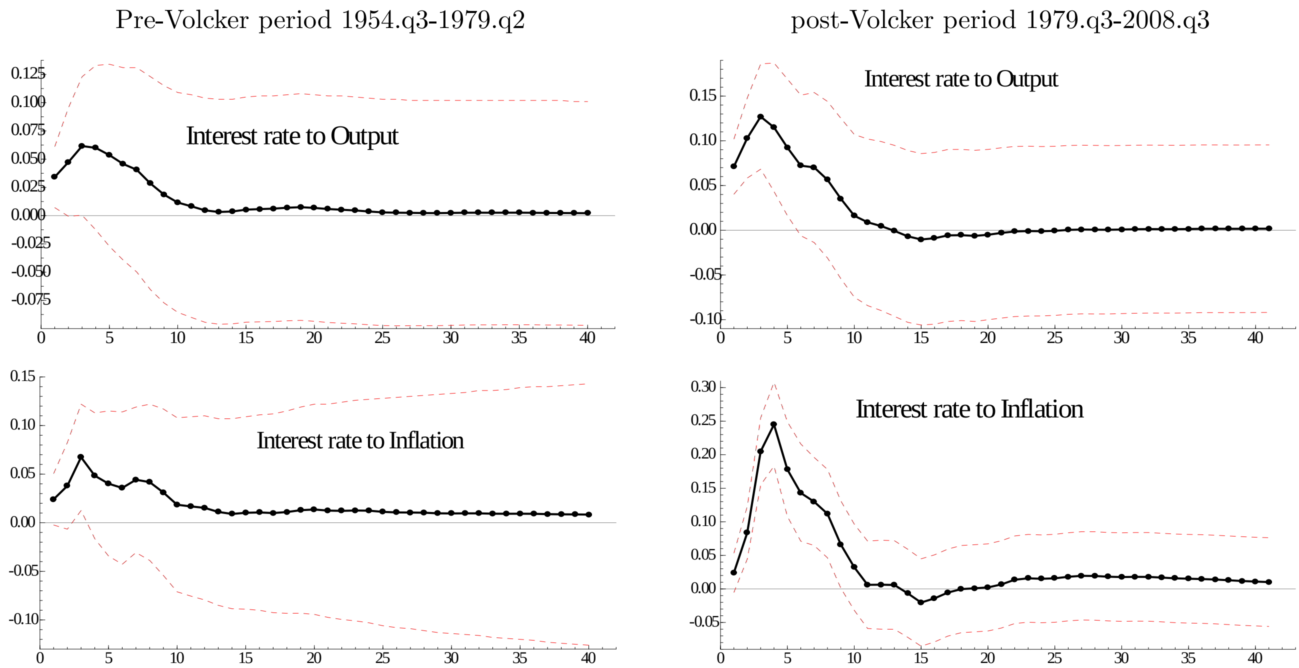

Figure 2. Impulse responses of R_t to same-size output and inflation shocks $e_t^{\tilde{y}}$ and e_t^{π} with 90% confidence bands based on the SVAR model $\mathcal{M}_2$ of tables 2-3.