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A Proof-theoretical Analysis of Semiconstructive Intermediate Theories

Abstract. In the 80's Pierangelo Miglioli, starting from motivations in the framework of Abstract Data Types and Program Synthesis, introduced semiconstructive theories, a family of "large subsystems" of classical theories that guarantee the computability of functions and predicates represented by suitable formulas. In general, the above computability results are guaranteed by algorithms based on a recursive enumeration of the theorems of the whole system. In this paper we present a family of semiconstructive systems, we call *uniformly semiconstructive*, that provide computational procedures only involving formulas with bounded complexity. We present several examples of uniformly semiconstructive systems containing Harrop theories, induction principles and some well-known predicate intermediate principles. Among these, we give an account of semiconstructive and uniformly semiconstructive systems which lie between Intuitionistic and Classical Arithmetic and we discuss their constructive incompatibility.

Keywords: intermediate semiconstructive systems, information extraction

1. Introduction

The notion of *uniformly semiconstructive system* has been recently developed by the authors together with Pierangelo Miglioli [9, 12]; it is a refinement of the notion of *semiconstructive system*, which arises from motivations in the areas of Abstract Data Types Specification [20, 21] and Program Synthesis [2, 21]. In order to outline the formal framework, let $\mathbf{T}\oplus\mathbf{L}$ be a logical system, where $\mathbf{T}$ is a first order theory (the mathematical part) and $\mathbf{L}$ is a superintuitionistic logic (the deductive apparatus). $\mathbf{T}\oplus\mathbf{L}$ is *semiconstructive* if it satisfies the *weak disjunction property* (if a closed formula $A\vee B \in \mathbf{T}\oplus\mathbf{L}$ then either A or B belongs to the corresponding classical theory $\mathbf{T}\oplus\mathbf{Cl}$) and the *weak explicit definability property* (if a closed formula $\exists x A(x) \in \mathbf{T}\oplus\mathbf{L}$ then $A(t)$ belongs to $\mathbf{T}\oplus\mathbf{Cl}$ for some closed term t). We remark that the above notion is weaker than the usual one of *constructive system* based on the *full disjunction property* (if a closed formula $A\vee B \in \mathbf{T}\oplus\mathbf{L}$ then either $A \in \mathbf{T}\oplus\mathbf{L}$ or $B \in \mathbf{T}\oplus\mathbf{L}$) and the *full explicit definability property* (if a closed formula $\exists x A(x) \in \mathbf{T}\oplus\mathbf{L}$ then $A(t) \in \mathbf{T}\oplus\mathbf{L}$ for some closed term t).

Constructive and semiconstructive systems have been studied in the framework of Abstract Data Type Specification for their relevance in connection with a classical semantics for Abstract Data Types, the so called *isoinitial semantics* [4, 19, 21]. In this context, semiconstructive systems

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can be seen as "large subsystems" of classical theories which guarantee the computability of functions and predicates represented by suitable formulas of the system. To exemplify, let $\mathbf{T}$ be a recursively axiomatizable theory with an isoinitial (classical) model, and let us suppose that the deductive system $\mathbf{T} \oplus \mathbf{L}$ is semiconstructive. In this hypothesis, if a formula of the kind $\forall x \exists ! y A(x, y)$ has a proof π in $\mathbf{T} \oplus \mathbf{L}$, then it defines a computable function f_A (namely, a recursive function of the isoinitial model). As a matter of fact, given a closed term s, the formula $\exists ! y A(s, y)$ is provable in $\mathbf{T} \oplus \mathbf{L}$ as well, hence there exists *one* (by the weak explicit definability property) and *only one* (by the classical meaning of $\exists ! y A(s, y)$) closed term t such that $A(s, t)$ is provable in the classical extension $\mathbf{T} \oplus \mathbf{Cl}$; such a term t can be understood as the value of $f_A(s)$ (see [20, 21]). One can compute $f_A(s)$ by recursively enumerating the theorems of $\mathbf{T} \oplus \mathbf{Cl}$, until the formula $A(s, t)$ is generated (note that, if $\mathbf{T} \oplus \mathbf{L}$ is constructive, the formula $A(s, t)$ is already provable in $\mathbf{T} \oplus \mathbf{L}$).

We notice that the above algorithm completely disregards the *information content* of the proof π of $\forall x \exists ! y A(x, y)$: indeed, a proof of $A(s, t)$ is searched through the whole system $\mathbf{T} \oplus \mathbf{Cl}$ and π has only the role of guaranteeing that the search will eventually halt with a correct answer t. Thus, the above algorithm is intrinsically *non-uniform* in the sense that there is no relation between the *logical complexity* of π, that is the complexity of the formulas occurring in π, and the complexity of the formulas generated by the enumeration procedure. In contrast, an algorithm using the information content of π should be *uniform*, namely, it should generate formulas which have proofs with bounded logical complexity. Indeed, it should generate formulas which have proofs "implicitly contained" in π and, for every reasonable characterization of such proofs, their logical complexity should be bounded by a constant k_π depending on π. The uniformity requirement is met by the traditional methods to extract information from constructive proofs, e.g., the methods based on Normalization, Recursive Realizability or Cut-elimination, as well as the Collection Method introduced by P. Miglioli and M. Ornaghi in [22, 23, 24]. A further justification of the above notion of uniformity, interesting for program extraction from constructive proofs, lies in the fact that the computation of a correct program P can be simulated by the (uniform) Collection Method applied to a suitable transformation of a correctness proof of P (for a complete discussion see [23, 24]).

One could argue that constructive systems are more appropriate to extract information from proofs than the semiconstructive ones, since the weak disjunction property and the weak explicit definability property refer to Classical Logic, but, as shown in [12], there exist intrinsically non-uniform constructive systems. Indeed, in [12] the authors exhibit a constructive system $\mathbf{T} \oplus \mathbf{L}$ and a formula $\forall x \exists! y A(x, y)$ provable in it, with the following property: for every natural number n, there are two closed terms s_n, t_n such that $A(s_n, t_n)$ is provable, but no proof of it has logical complexity less than n. This entails that, even in presence of full disjunction property and full explicit definability property, the search of the term t_n may involve pieces of information that cannot be directly extracted from a single proof or even from a set of proofs with bounded logical complexity. In particular, traditional extraction mechanisms as Normalization and Cut-elimination, as well as the Collection Method, cannot work because they are uniform. By the above discussion, the properties that characterize constructive and semiconstructive systems are not sufficient to support information extraction from proofs. To this aim, a deeper proof-theoretical analysis is needed. In previous works [9, 12] a proof-theoretical study of *uniform* constructive systems has been developed applying ideas coming from the Collection Method. Here we extend this study to semiconstructive systems, introducing a notion of *uniformly semiconstructive system*. In such a system a proof π of $\forall x \exists! y A(x, y)$ contains enough information to compute f_A. The formula $A(s, t)$ can be searched within a calculus, we call *extraction calculus*, which is built up exploiting the information contained in π; in particular, proofs of extraction calculi have a *bounded* logical complexity (where the bound essentially depends on π).

In this paper we use extraction calculi as a "general-purpose" tool to study the constructive properties of intermediate systems from a purely logical viewpoint. In our proof-theoretical analysis we only take into account the logical complexity of proofs, while disregarding the complexity of terms. The latter might heavily affect the computational complexity of the extraction procedures (see, e.g., [5]); on the other hand, as discussed in [1, 8], for some classes of theories "goal oriented" extraction calculi can be defined, which lead to efficient proof search strategies. Finally, we point out that the traditional techniques used to define extraction methods for wide families of constructive systems (based on Normalization Theorems and Recursive Realizability Interpretations [27, 28]) seem hardly applicable in the general framework of semiconstructive theories.

In the following sections we recall the preliminary notions and we introduce our information extraction mechanism. Then, we provide several examples of uniformly semiconstructive systems containing *Harrop theories*, *induction principles* and some well-known intermediate principles, such as *Kuroda principle*, *Kreisel-Putnam principle*, *Markov principle* and a weak version of *Grzegorczyk principle*. Finally, we concentrate on some intermediate systems which lie between Intuitionistic Arithmetic and Classical Arithmetic; we analyze their constructive properties and the constructive incompatibilities which arise from their combinations.

2. Preliminaries

Let $\mathcal{A}$ be an extra-logical alphabet. The set of *terms* and the set of (first-order) *formulas* of the language $\mathcal{L}_{\mathcal{A}}$ are built up in the usual way, starting from $\mathcal{A}$, a denumerable set of individual variables, the binary relation symbol $=$, and the logical constants $\bot$, $\wedge$, $\vee$, $\rightarrow$, $\forall$, $\exists$ (we consider $\neg A$ as an abbreviation for $A \rightarrow \bot$). The notion of *substitution* is the usual one; we call *closed substitution* any substitution associating a closed term with every individual variable. Given a formula A, a *closed instance* of A is any formula obtained by applying a closed substitution to A. The *degree* $\mathrm{dg}(A)$ of a formula A is inductively defined as: $\mathrm{dg}(A) = 1$ if A is atomic; $\mathrm{dg}(A) = \max\{\mathrm{dg}(B), \mathrm{dg}(C)\} + 1$ if A is $B \wedge C$, $B \vee C$ or $B \rightarrow C$; $\mathrm{dg}(A) = \mathrm{dg}(B) + 1$ if A is either $\neg B$, $\exists x B(x)$ or $\forall x B(x)$. The *degree of a finite set of formulas* Γ is $\max\{\mathrm{dg}(A) : A \in \Gamma\}$.

Int and **Cl** denote respectively the set of intuitionistically and classically valid formulas of the pure first-order language with identity $\mathcal{L}$. A *(first-order) intermediate logic* is any set of formulas **L** such that: $\mathbf{Int} \subseteq \mathbf{L} \subseteq \mathbf{Cl}$ and **L** is closed under *modus ponens*, *generalization* and predicate substitution (see, e.g., [26]). Passing from the pure first-order language $\mathcal{L}$ to $\mathcal{L}_{\mathcal{A}}$ (where the relation declarations of $\mathcal{A}$ are seen as *constant relation declarations*, and hence predicate substitutions are not allowed), $\mathbf{Int}_{\mathcal{A}}$ and $\mathbf{Cl}_{\mathcal{A}}$ denote the subsets of $\mathcal{L}_{\mathcal{A}}$ obtained by correctly substituting the predicate variables with formulas of $\mathcal{L}_{\mathcal{A}}$ in the formulas of **Int** and **Cl** respectively.

Let **T** be an $\mathcal{A}$*-theory*, that is a recursively enumerable set of closed formulas of $\mathcal{L}_{\mathcal{A}}$. Given a set of formulas Γ, $\mathbf{T} \oplus \Gamma$ is the smallest set of formulas of $\mathcal{L}_{\mathcal{A}}$ containing **T** and Γ and closed under modus ponens and generalization. We denote with (A) the axiom schema generated by A and with $\mathbf{T} \oplus (A)$ the set of formulas $\mathbf{T} \oplus \Gamma$, where Γ is the set of all the instances of (A) in $\mathcal{L}_{\mathcal{A}}$. We simply write $\mathbf{T} \oplus \mathbf{Int}$ and $\mathbf{T} \oplus \mathbf{Cl}$ for $\mathbf{T} \oplus \mathbf{Int}_{\mathcal{A}}$ and $\mathbf{T} \oplus \mathbf{Cl}_{\mathcal{A}}$ respectively.

An $\mathcal{A}$-theory **T** is *consistent* if it is classically consistent, that is $\mathbf{T} \oplus \mathbf{Cl} \neq \mathcal{L}_{\mathcal{A}}$. In this paper we are mainly concerned with *Harrop theories*, that is theories only consisting of Harrop formulas. A *Harrop formula* is either an atomic or a negated formula, or a formula of the kind $A \wedge B$, $C \rightarrow A$, $\forall x A$ where A and B are Harrop formulas and C is any formula.

DEFINITION 2.1. (**T**-system) Let **T** be an $\mathcal{A}$-theory. An *(intermediate)* **T**-*system* is any set $\mathbf{S} \subseteq \mathcal{L}_{\mathcal{A}}$ such that $\mathbf{T} \oplus \mathbf{Int} \subseteq \mathbf{S} \subseteq \mathbf{T} \oplus \mathbf{Cl}$ and **S** is closed under modus ponens and generalization.

Given $\Gamma, \Delta \subseteq \mathcal{L}_{\mathcal{A}}$, we say that Γ is *constructive in* Δ iff the *weak disjunction property* (wDP) and the *weak explicit definability property* (wED) hold:

(wDP) If $A \vee B \in \Gamma$ and $A \vee B$ is a closed formula, then either $A \in \Delta$ or $B \in \Delta$.

(wED) If $\exists x A(x) \in \Gamma$ and $\exists x A(x)$ is a closed formula, then $A(t/x) \in \Delta$ for some closed term t of the language.

DEFINITION 2.2. (Semiconstructive **T**-system) Let **S** be a **T**-system. **S** is *semiconstructive* iff **S** is constructive in $\mathbf{S} \oplus \mathbf{Cl}$.

We remark that the notion formalized by the above definition is weaker than the usual notion of *constructive* **T**-*system* based on the *full disjunction property* (if $A \vee B \in \mathbf{S}$ and $A \vee B$ is a closed formula, then either $A \in \mathbf{S}$ or $B \in \mathbf{S}$) and the *full explicit definability* property (if $\exists x A(x) \in \mathbf{S}$ and $\exists x A(x)$ is a closed formula, then $A(t/x) \in \mathbf{S}$ for some closed term t of the language). Indeed, every constructive **T**-system is semiconstructive; on the other hand, as we show in §5, there exist recursively axiomatizable semiconstructive **T**-systems that cannot be extended in recursively enumerable **T**-systems satisfying the full disjunction property and the full explicit definability property.

In the following we deal with natural deduction calculi in sequent-style presentation (see [30]) obtained by extending the calculus $\mathcal{ND}_{\mathbf{Int}}$ for first-order Intuitionistic Logic of Table 1. The calculus for first-order Classical Logic $\mathcal{ND}_{\mathbf{Cl}}$ is obtained by replacing the rule $\bot_{\mathrm{Int}}$ of the calculus $\mathcal{ND}_{\mathbf{Int}}$ with the rule:

$$\frac{\Gamma, \neg A \vdash \bot}{\Gamma \vdash A}\,\bot_{\mathrm{Cl}}$$

Let $\mathcal{ND}$ be one of the natural deduction calculi defined in this paper and let **T** be a recursive $\mathcal{A}$-theory; we denote with $\mathcal{ND}(\mathbf{T})$ the calculus obtained by adding to $\mathcal{ND}$, for every $A \in \mathbf{T}$, the *axiom-rule*:

$$\frac{}{\vdash A}\,\mathbf{T}$$

Hereafter we assume the usual conventions on *proper parameters* and *free variables* of the natural deduction rules stated in [29, 30], in such a way to guarantee that the tree-structure $\theta\pi$ obtained by replacing some of the free variables of a proof π with terms is a well-defined proof.

$$\frac{}{A \vdash A}\,\mathrm{Id} \qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash A}\,\bot_{\mathbf{Int}}$$ where A is an atomic formula.

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B}\,\mathrm{I}\wedge \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\,\mathrm{E}\wedge \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}\,\mathrm{E}\wedge$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\,\mathrm{I}\vee \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\,\mathrm{I}\vee \qquad \frac{\Gamma \vdash A \vee B \quad \Delta, A \vdash C \quad \Theta, B \vdash C}{\Gamma, \Delta, \Theta \vdash C}\,\mathrm{E}\vee$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\,\mathrm{I}\to \qquad \frac{\Gamma \vdash A \quad \Delta \vdash A \to B}{\Gamma, \Delta \vdash B}\,\mathrm{E}\to$$

$$\frac{\Gamma \vdash A(y/x)}{\Gamma \vdash \forall x A(x)}\,\mathrm{I}\forall$$ where y does not occur free in Γ or $\forall x A(x)$.

$$\frac{\Gamma \vdash \forall x A(x)}{\Gamma \vdash A(t/x)}\,\mathrm{E}\forall$$

$$\frac{\Gamma \vdash A(t/x)}{\Gamma \vdash \exists x A(x)}\,\mathrm{I}\exists \qquad \frac{\Gamma \vdash \exists x A(x) \quad \Delta, A(y/x) \vdash C}{\Gamma, \Delta \vdash C}\,\mathrm{E}\exists$$ where y does not occur free in Δ, $\exists x A(x)$ or C.

$$\frac{}{\Gamma \vdash x = x}\,\mathrm{id}_1 \qquad \frac{\Gamma \vdash A(t/x) \quad \Delta \vdash t = t'}{\Gamma, \Delta \vdash A(t'/x)}\,\mathrm{id}_2$$ where $A(x)$ is an atomic formula.

Table 1. The natural deduction calculus $\mathcal{ND}_{\mathbf{Int}}$ for Intuitionistic Logic

3. The extraction calculus

Our method to extract information from proofs is independent of any specific notion of calculus, therefore we base it on an abstract notion of calculus. First of all a *proof* over a language $\mathcal{L}_{\mathcal{A}}$ is any finite object π with associated a finite set of formulas $\mathrm{Wffs}(\pi)$ of $\mathcal{L}_{\mathcal{A}}$, we call the *formulas occurring in* π, and a sequent $\mathrm{Seq}(\pi) = \Gamma \vdash \Delta$, we call the *sequent proved by* π; we assume that $\Delta \neq \emptyset$ and $\Gamma \cup \Delta \subseteq \mathrm{Wffs}(\pi)$. Intuitively Γ is the set of *assumptions* of π and Δ the set of *consequences* of π. The compact notation $\pi : \Gamma \vdash \Delta$ is used to indicate that $\mathrm{Seq}(\pi) = \Gamma \vdash \Delta$. With $\mathrm{dg}(\pi)$ we denote the *degree of* π defined as $\max\{\mathrm{dg}(A) : A \in \mathrm{Wffs}(\pi)\}$.

A *calculus* over $\mathcal{L}_{\mathcal{A}}$ is a pair $\mathbf{C} = (C, [\cdot])$, where C is a recursive set of proofs over $\mathcal{L}_{\mathcal{A}}$ and $[\cdot] : C \to 2^C$ is a recursive map associating with every proof of C the set of its subproofs. We require $[\cdot]$ to have the following natural properties: $\pi \in [\pi]$; for every $\pi' \in [\pi]$, $[\pi'] \subseteq [\pi]$; for every $\pi' \in [\pi]$,

$\mathrm{dg}(\pi') \leq \mathrm{dg}(\pi)$. Note that any recursive $\Pi \subseteq C$ determines a calculus, namely the calculus $(\Pi, [\cdot]_{\restriction\Pi})$, where $[\cdot]_{\restriction\Pi}$ is the restriction of $[\cdot]$ to Π. In the following, to simplify the notation, we identify a calculus $\mathbf{C}$ with the set of its proofs.

We remark that the above definition does not refer to any particular inference system, but all the usual inference systems are calculi according to our definition; for instance, one can easily check that $\mathcal{ND}_{\mathbf{Int}}$ and $\mathcal{ND}_{\mathbf{Cl}}$ meet the above definition.

Given $\Pi \subseteq \mathbf{C}$, $[\Pi]$ is the *closure under subproofs* of Π in the calculus $\mathbf{C}$, namely:

$$[\Pi] = \{\pi' : \text{ there exists } \pi \in \Pi \text{ such that } \pi' \in [\pi]\}$$

Finally, we associate with each $\Pi \subseteq \mathbf{C}$ the following attributes: $\mathrm{Seq}(\Pi) = \{\mathrm{Seq}(\pi) : \pi \in \Pi\}$; $\mathrm{dg}(\Pi)$ is the *degree* of Π, i.e., $\mathrm{dg}(\Pi) = \infty$ if Π contains proofs that exceed any complexity bound, and $\mathrm{dg}(\Pi) = \max\{\mathrm{dg}(\pi) : \pi \in \Pi\}$ otherwise; $\mathrm{Theo}(\Pi) = \{A : \vdash A \in \mathrm{Seq}(\Pi)\}$ is the set of *theorems proved in* Π.

DEFINITION 3.1. Given two calculi $\mathbf{C}$ and $\mathbf{C}'$ over the same language, $\mathbf{C}$ *is constructive in* $\mathbf{C}'$ iff $\mathrm{Theo}(\mathbf{C})$ is constructive in $\mathrm{Theo}(\mathbf{C}')$.

Now, we briefly introduce our mechanism to extract information from proofs of an arbitrary calculus; for a complete discussion we refer the reader to [7, 12]. Given a language $\mathcal{L}_{\mathcal{A}}$, let $\Xi_{\mathcal{A}}$ be the set of all the sequents over $\mathcal{L}_{\mathcal{A}}$ and let $\Xi^*_{\mathcal{A}}$ be the set of all the finite sequences of sequents in $\Xi_{\mathcal{A}}$; a *generalized rule (over $\mathcal{L}_{\mathcal{A}}$)* is a relation $\mathcal{R} \subseteq \Xi^*_{\mathcal{A}} \times \Xi_{\mathcal{A}}$. We denote with ϵ the empty sequence of sequents and we write $\sigma \in \mathcal{R}(\sigma^*)$ as a shorthand for $(\sigma^*, \sigma) \in \mathcal{R}$. Let Π be a set of proofs of a calculus $\mathbf{C}$; Π is *$\mathcal{R}$-closed* if, for every $\pi_1 : \sigma_1, \ldots, \pi_n : \sigma_n \in \Pi$, if $\sigma \in \mathcal{R}(\sigma_1; \ldots; \sigma_n)$, then there exists a proof $\pi : \sigma \in \Pi$. Moreover, given a function $\phi : \mathbf{N} \to \mathbf{N}$, Π is *uniformly $\mathcal{R}$-closed w.r.t. ϕ* if, for every $\pi_1 : \sigma_1, \ldots, \pi_n : \sigma_n \in \Pi$, if $\sigma \in \mathcal{R}(\sigma_1; \ldots; \sigma_n)$, then there exists a proof $\pi : \sigma \in \Pi$ with

$$\mathrm{dg}(\pi) \leq \max\{\mathrm{dg}(\pi_1), \ldots, \mathrm{dg}(\pi_n), \phi(\mathrm{dg}(\sigma_1)), \ldots, \phi(\mathrm{dg}(\sigma_n)), \phi(\mathrm{dg}(\sigma))\}$$

We say that Π is *uniformly $\mathcal{R}$-closed* if there exists a $\phi : \mathbf{N} \to \mathbf{N}$ such that Π is *uniformly $\mathcal{R}$-closed w.r.t. ϕ*. Finally, $\mathcal{R}$ is *non-increasing* iff:

(i) There exists a positive integer k such that, for every $\sigma \in \mathcal{R}(\epsilon)$, $\mathrm{dg}(\sigma) \leq k$.

(ii) For every nonempty sequence $\sigma_1; \ldots; \sigma_n$, if $\sigma \in \mathcal{R}(\sigma_1; \ldots; \sigma_n)$, then

$$\mathrm{dg}(\sigma) \leq \max\{\mathrm{dg}(\sigma_1), \ldots, \mathrm{dg}(\sigma_n)\} .$$

Examples of generalized rules are:

- *Substitution rule* $\textsc{Subst}$: its domain is the set of all the sequents and, for every substitution θ, $\theta\Gamma \vdash \theta\Delta \in \textsc{Subst}(\Gamma \vdash \Delta)$.
- *Cut rule* $\textsc{Cut}$: its domain contains all the sequences of sequents of the kind $\Gamma_1 \vdash H; \Gamma_2, H \vdash A$ and $\Gamma_1, \Gamma_2 \vdash A \in \textsc{Cut}(\Gamma_1 \vdash H; \Gamma_2, H \vdash A)$.

It is easy to check that $\mathcal{ND}_{\mathbf{Int}}$ and $\mathcal{ND}_{\mathbf{Cl}}$ are uniformly $\textsc{Subst}$-closed and uniformly $\textsc{Cut}$-closed w.r.t. a linear function.

In the following we use generalized rules to extract information from proofs of a given calculus, to this aim we introduce a calculus having generalized rules as inference rules.

DEFINITION 3.2. Let $\mathcal{R}$ be a generalized rule over $\mathcal{L}_{\mathcal{A}}$ and let Σ be any set of sequents over the same language. $\mathbb{D}(\mathcal{R}, \Sigma)$ is the set of proof-trees inductively defined as follows:

(i) If $\sigma \in \Sigma$ or $\sigma \in \mathcal{R}(\epsilon)$, then $\tau \equiv \sigma$ is a proof-tree of $\mathbb{D}(\mathcal{R}, \Sigma)$ with root σ and $\mathrm{depth}(\tau) = 1$.

(ii) If $\tau_1 : \sigma_1, \ldots, \tau_n : \sigma_n$ are proof-trees of $\mathbb{D}(\mathcal{R}, \Sigma)$ (where σ_i is the root of τ_i) then, for every $\sigma \in \mathcal{R}(\sigma_1; \ldots; \sigma_n)$, the proof-tree

$$\tau \equiv \frac{\tau_1 : \sigma_1 \ \ldots \ \tau_n : \sigma_n}{\sigma} \mathcal{R}$$

with root σ belongs to $\mathbb{D}(\mathcal{R}, \Sigma)$ and

$$\mathrm{depth}(\tau) = \max\{\mathrm{depth}(\tau_1), \ldots, \mathrm{depth}(\tau_n)\} + 1 .$$

We remark that, if both $\mathcal{R}$ and Σ are recursive, then $\mathbb{D}(\mathcal{R}, \Sigma)$ is a calculus, where we consider the obvious subproof map $[\cdot]$ determined by the inductive definition of the proofs of $\mathbb{D}(\mathcal{R}, \Sigma)$.

DEFINITION 3.3. (E-rule and extraction calculus) Let $\mathbf{C}$ be a calculus.

(i) An *extraction rule for* $\mathbf{C}$ (*e-rule* for short) is a recursive non-increasing generalized rule $\mathcal{R}$ such that $\mathbf{C}$ is uniformly $\mathcal{R}$-closed.

(ii) Given an extraction rule $\mathcal{R}$ for $\mathbf{C}$ and a recursive $\Pi \subseteq \mathbf{C}$, the *extraction calculus for* Π is the calculus $\mathbb{D}(\mathcal{R}, \mathrm{Seq}([\Pi]))$.

We remark that, if Π is a recursive set of proofs with bounded logical complexity, then the extraction calculus $\mathbb{D}(\mathcal{R}, \mathrm{Seq}([\Pi]))$ has a bounded logical complexity, as stated by the following result:

PROPOSITION 3.4. *Let* Π *be a recursive set of proofs of a calculus* $\mathbf{C}$ *and let* $\mathcal{R}$ *be an e-rule for* $\mathbf{C}$. *If, for every* $\sigma \in \mathcal{R}(\epsilon)$, $\mathrm{dg}(\sigma) \leq k$, *then, for every* $\tau \in \mathbb{D}(\mathcal{R}, \mathrm{Seq}([\Pi]))$, $\mathrm{dg}(\tau) \leq \max\{k, \mathrm{dg}(\Pi)\}$.

Now, we can introduce our main definition:

DEFINITION 3.5. Let $\mathbf{C}$ and $\mathbf{C}'$ be calculi over the same language $\mathcal{L}_{\mathcal{A}}$. $\mathbf{C}$ is *uniformly constructive* in $\mathbf{C}'$ iff:

(i) $\mathrm{Seq}(\mathbf{C}) \subseteq \mathrm{Seq}(\mathbf{C}')$;

(ii) There exists an e-rule $\mathcal{R}$ for $\mathbf{C}'$ such that, for every recursive subset Π of $\mathbf{C}$, $\mathrm{Theo}([\Pi])$ is constructive in $\mathrm{Theo}(\mathbb{D}(\mathcal{R}, \mathrm{Seq}([\Pi])))$.

Clearly, if $\mathbf{C}$ is uniformly constructive in $\mathbf{C}'$, then any calculus $\mathbf{C}''$ contained in $\mathbf{C}$ (where, according to the previous conventions, $\mathbf{C}''$ is identified by a recursive subset of $\mathbf{C}$) is uniformly constructive in $\mathbf{C}'$ as well. We stress that the notion of uniform constructivity explained in Definition 3.5 actually implies the notion of constructivity of Definition 3.1. As a matter of fact, let $\mathbf{C}$ be uniformly constructive in $\mathbf{C}'$. If $A \vee B \in \mathrm{Theo}(\mathbf{C})$, there exists a proof $\pi : \vdash A \vee B \in \mathbf{C}$. Let us consider the calculus $\mathbb{D}(\mathcal{R}, \mathrm{Seq}([\pi]))$; by Point (ii) of Definition 3.5, $\mathbb{D}(\mathcal{R}, \mathrm{Seq}([\pi]))$ either contains a proof of $\vdash A$ or a proof of $\vdash B$. Moreover, by Point (i) of Definition 3.5 and by the fact that $\mathbf{C}'$ is $\mathcal{R}$-closed, any theorem of $\mathbb{D}(\mathcal{R}, \mathrm{Seq}([\pi]))$ can be proved in $\mathbf{C}'$; therefore, either $A \in \mathrm{Theo}(\mathbf{C}')$ or $B \in \mathrm{Theo}(\mathbf{C}')$, and this proves (wDP) (the proof of (wED) is similar). On the other hand, to find such a proof we do not need to consider the whole calculus $\mathbf{C}'$, but we can search the calculus $\mathbb{D}(\mathcal{R}, \mathrm{Seq}([\pi]))$ which, by Proposition 3.4, has a bounded logical complexity. We remark that the extraction calculus is uniform only with respect to the degree of the formulas and it does not fix any bound on the complexity of the terms involved in the extraction procedure. In contrast, the complexity of the terms must be considered to study the computational complexity of the extraction procedure (see, e.g., [5]). Since in this paper we are mainly interested to use extraction calculi as a tool to study the uniform constructivity properties of calculi, we disregard issues related to the computational complexity. We also point out that the extraction calculi presented in this paper use extraction rules that are not suitable to get efficient proof search in $\mathbb{D}(\mathcal{R}, \mathrm{Seq}([\pi]))$. However, as discussed in [1, 8], for some classes of theories goal-oriented extraction calculi can be defined.

Let $\mathbf{T}$ be an $\mathcal{A}$-theory, let $\mathbf{S}$ be a $\mathbf{T}$-system and let $\mathbf{C}$ be a calculus over $\mathcal{L}_{\mathcal{A}}$. We say that $\mathbf{C}$ *generates* $\mathbf{S}$ if $\mathbf{S} = \mathrm{Theo}(\mathbf{C})$.

DEFINITION 3.6. (Uniformly semicostructive $\mathbf{T}$-system) A $\mathbf{T}$-system $\mathbf{S}$ is *uniformly semiconstructive* iff there exist two calculi $\mathbf{C}$ and $\mathbf{C}'$ such that:

(i) $\mathbf{C}$ generates $\mathbf{S}$;

(ii) $\mathbf{C}'$ generates $\mathbf{S} \oplus \mathbf{Cl}$;

(iii) $\mathbf{C}$ is uniformly constructive in $\mathbf{C}'$.

For a detailed discussion on the properties of semiconstructive systems and on the stronger notion of uniform constructivity we refer the reader to [9, 12]. To conclude this section, we remark that the notions of semiconstructivity and uniform semiconstructivity do not coincide; indeed in [12] it is shown that there exists a $\mathbf{T}$-system which is semiconstructive but not uniformly semiconstructive.

4. Cover Set Induction

In this paper we restrict our attention to $\mathbf{T}$-systems defined starting from a recursive Harrop theory $\mathbf{T}$ and containing a schema of Cover Set Induction. Harrop theories have been deeply investigated in the framework of constructive and semiconstructive formal systems [21, 29] and seems to be the widest family of theories for which a general result of uniform semiconstructivity can be given. As for the Cover Set Induction, it provides a general schema of induction that can be specialized to obtain specific induction principles in several theories formalizing Abstract Data Types; see [21] for some examples of application of this schema. Here we define the notion of cover set, the Cover Set Induction rule and the conditions needed to treat such a rule in the context of uniform semiconstructivity.

Given a set $\mathcal{C}$ of terms of $\mathcal{L}_{\mathcal{A}}$ containing at least one closed term, we denote with $\mathrm{Cterm}_{\mathcal{C}}$ the union of the sets of closed terms $\mathcal{C}_i$ inductively defined as follows:

- $\mathcal{C}_0$ is the set of closed terms in $\mathcal{C}$;

- $\mathcal{C}_{i+1} = \mathcal{C}_i \cup \{t(s_1,\dots,s_n) \mid t(x_1,\dots,x_n) \in \mathcal{C}$ and $s_1 \in \mathcal{C}_i,\dots,s_n \in \mathcal{C}_i\}$.

DEFINITION 4.1. Let $\mathbf{T}$ be an $\mathcal{A}$-theory and let $\mathcal{C}$ be a finite set of terms of $\mathcal{L}_\mathcal{A}$ containing at least one closed term. $\mathcal{C}$ *is a cover set for* $\mathbf{T}$ iff:

(i) No term of $\mathcal{C}$ is a variable;

(ii) For every closed term t of $\mathcal{L}_\mathcal{A}$, there is a term $t' \in \mathrm{Cterm}_\mathcal{C}$ such that $t = t' \in \mathbf{T} \oplus \mathbf{Int}$.

Let $\mathcal{C} = \{t_1,\dots,t_n\}$ be a cover set for an $\mathcal{A}$-theory $\mathbf{T}$. By Definition 4.1 we know that any closed term of $\mathcal{L}_\mathcal{A}$ is equivalent to a closed term that can be built in finitely many steps starting from the terms in $\mathcal{C}$. This justifies the introduction of the *Cover Set Induction* rule $\mathcal{C}$-Ind, where the proof of a formula $\forall x A(x)$ is accomplished by induction on the "structural complexity" of terms; the base cases are represented by the closed terms of $\mathcal{C}$, whereas the inductive steps are represented by the open terms of $\mathcal{C}$. We can formalize $\mathcal{C}$-Ind as follows:

$$\frac{\Gamma_1,\Delta_1 \vdash A(t_1) \quad \dots \quad \Gamma_n,\Delta_n \vdash A(t_n)}{\Gamma_1,\dots,\Gamma_n \vdash A(x)}\ \mathcal{C}\text{-Ind}$$

where, for $1 \leq i \leq n$:

(1) If t_i is a closed term then Δ_i is empty;

(2) If t_i is of the kind $s(y_1^i,\dots,y_{k_i}^i)$, then $\Delta_i = \{A(y_1^i),\dots,A(y_{k_i}^i)\}$.

The formulas in Δ_i are the induction hypotheses, the variables $y_1^i,\dots,y_{k_i}^i$ are called *proper parameters* of the Cover Set Induction rule and must not occur free in $\Gamma_1,\dots,\Gamma_n, A(x)$. We extend to the rule $\mathcal{C}$-Ind the usual conditions on proper parameters made in [30]. Given a recursive theory $\mathbf{T}$ with cover set $\mathcal{C}$ and a natural deduction calculus $\mathcal{ND}(\mathbf{T})$, we denote with $\mathcal{ND}^{\mathcal{C}\text{-Ind}}(\mathbf{T})$ the calculus obtained by adding to $\mathcal{ND}(\mathbf{T})$ the instance of the rule $\mathcal{C}$-Ind corresponding to the cover set $\mathcal{C}$. Obviously the $\mathcal{C}$-Ind rule can also be formalized as the following axiom schema:

$$(\mathcal{C}\text{-Ind}) \equiv (\bigwedge \Delta_1 \rightarrow A(t_1)) \wedge \dots \wedge (\bigwedge \Delta_n \rightarrow A(t_n)) \rightarrow \forall x A(x)$$

As an example, the induction principle for Arithmetic is a specialization of Cover Set Induction. Indeed, let $\mathcal{L}_{\mathbf{Ar}}$ be the language based on the alphabet of Arithmetic $\{0, s, +, *\}$ and let $\mathbf{Ar}$ be the Harrop theory consisting of the usual axioms:

$$\begin{array}{ll} \forall x \neg(0 = s(x)) & \forall x \forall y (s(x) = s(y) \rightarrow x = y) \\ \forall x (x + 0 = x) & \forall x \forall y (x + s(y) = s(x + y)) \\ \forall x (x * 0 = 0) & \forall x \forall y (x * s(y) = x * y + x) \end{array}$$

It is easy to check that $\mathcal{C} = \{0, s(x)\}$ is a cover set for **Ar**; $\mathrm{Cterm}_{\mathcal{C}}$ contains the usual canonical terms of Arithmetic of the form $s^n(0)$ and Condition (ii) of Definition 4.1 can be easily proved by induction on the structure of t. The associated rule $\mathcal{C}$-Ind coincides with the induction rule for Arithmetic:

$$\frac{\Gamma_1 \vdash A(0) \quad \Gamma_2, A(y) \vdash A(s(y))}{\Gamma_1, \Gamma_2 \vdash A(x)}\ \mathcal{C}\text{-Ind}$$

Therefore, $\mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\mathbf{Int}}(\mathbf{Ar})$ generates Intuitionistic Arithmetic $\mathbf{HA} = \mathbf{Ar} \oplus (\mathcal{C}\text{-Ind}) \oplus \mathbf{Int}$, while $\mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\mathbf{Cl}}(\mathbf{Ar})$ generates Classical Arithmetic $\mathbf{PA} = \mathbf{Ar} \oplus (\mathcal{C}\text{-Ind}) \oplus \mathbf{Cl}$. We remark that in general a theory admits several cover sets; e.g., $\{0, s(0), x + y\}$ and $\{0, s(x), x + y, x * y\}$ are cover sets for **Ar** as well, and correspond to different formalizations of the induction principle for Arithmetic.

In our approach it is important to determine cover sets where canonical terms can be computed by e-rules. This is formalized as follows:

DEFINITION 4.2. Let **T** be a recursive $\mathcal{A}$-theory. **T** *has an adequate cover set* $\mathcal{C}$ iff $\mathcal{C}$ is a cover set for **T** and there exists a generalized rule $\mathrm{Cov}_{\mathcal{C}}$ (we call the *e-rule related to* $\mathcal{C}$) such that:

(i) $\mathrm{Cov}_{\mathcal{C}}$ is an e-rule for $\mathcal{ND}_{\mathbf{Int}}(\mathbf{T})$;

(ii) For every closed term $t \in \mathcal{L}_{\mathcal{A}}$ and for every $t' \in \mathrm{Cterm}_{\mathcal{C}}$, $t = t' \in \mathbf{T}\oplus\mathbf{Int}$ iff $\vdash t = t'$ is provable in $\mathbb{D}(\mathrm{Cov}_{\mathcal{C}}, \emptyset)$.

We remark that, by Proposition 3.4 and Condition (ii), the degree of the proofs of $\mathbb{D}(\mathrm{Cov}_{\mathcal{C}}, \emptyset)$ only depends on the e-rule $\mathrm{Cov}_{\mathcal{C}}$.

It is easy to check that $\mathcal{C} = \{0, s(x)\}$ is an adequate cover set for **Ar**, where the related e-rule $\mathrm{Cov}_{\{0,s(x)\}}$ consists of the union of the generalized rules SUBST, ID_1 and ID_2 of Table 2 and of the generalized rules SUM and PROD below:

$$\begin{array}{ll} \vdash x + 0 = x \in \mathrm{Sum}(\epsilon) & \vdash x * 0 = 0 \in \mathrm{Prod}(\epsilon) \\ \vdash x + s(y) = s(x + y) \in \mathrm{Sum}(\epsilon) & \vdash x * s(y) = x * y + x \in \mathrm{Prod}(\epsilon) \end{array}$$

In the following sections we prove the uniform semiconstructivity of some families of **T**-systems, with **T** a Harrop theory, involving Cover Set Induction and some intermediate principles already investigated in the literature on constructive and intermediate systems.

5. The family of uniformly semiconstructive systems $\mathrm{KWD}^{\mathcal{C}}(\mathbf{T})$

Let $\mathbf{T}$ be a recursive Harrop theory admitting an adequate cover set $\mathcal{C}$; we denote with $\mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\mathrm{KWD}}(\mathbf{T})$ the calculus obtained by adding to $\mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\mathbf{Int}}(\mathbf{T})$ the following rules:

$$\frac{\Gamma \vdash \forall x \neg\neg A(x)}{\Gamma \vdash \neg\neg \forall x A(x)}\,\mathrm{Kur} \qquad \frac{}{\vdash \exists x A(x) \vee \forall x(A(x) \rightarrow B \vee \neg B)}\,\mathrm{DT}$$

$$\frac{\Gamma_1 \vdash \forall x \neg\neg A(x) \quad \Gamma_2 \vdash \forall x(A(x) \vee B)}{\Gamma_1, \Gamma_2 \vdash \forall x A(x) \vee B}\,\mathrm{wGrz} \quad \text{where } x \text{ is not free in } B$$

These rules correspond to the intermediate principles:

$$\begin{aligned}
(\mathrm{Kur}) &\equiv \forall x \neg\neg A(x) \rightarrow \neg\neg \forall x A(x) \\
(\mathrm{wGrz}) &\equiv \forall x \neg\neg A(x) \wedge \forall x(A(x) \vee B) \rightarrow \forall x A(x) \vee B \quad (x \text{ not free in } B) \\
(\mathrm{DT}) &\equiv \exists x A(x) \vee \forall x(A(x) \rightarrow B \vee \neg B)
\end{aligned}$$

As far as we know, the principles (wGrz) and (DT) have never been investigated in the context of constructive systems. (wGrz) is a weakened form of the *Grzegorczyk Principle* $\forall x(A(x) \vee B) \rightarrow \forall x A(x) \vee B$ whose addition to Intuitionistic Logic gives rise to the intermediate constructive logic of *constant domains* [15]. As for (DT), it is a predicate extension of the propositional principle characterizing the intermediate propositional logic whose frames have depth at most 2 (see [6]). On the other hand, *Kuroda Principle* (Kur) has been deeply investigated in the literature on constructive systems [13, 29]. Moreover, it has been considered in the context of Abstract Data Types specification based on *isoinitial (classical) semantics* for the role it plays with respect to Classical Logic (see [19, 21]). Indeed, a theory $\mathbf{T}$ is classically consistent iff $\mathbf{T} \oplus \mathbf{L}$ is consistent for every intermediate predicate logic $\mathbf{L}$ including Kuroda Principle [13]. We remark that, as a consequence of this feature (see [29]), the $\mathbf{T}$-systems including Kuroda Principle are not in the scope of those recursive realizability interpretations, like Kleene's 1945-realizability [16], which can be used to get consistency proofs for "anticlassical" systems (e.g., Intuitionistic Arithmetic enriched by Church's Thesis [29]).

Here we prove that, for every recursive Harrop theory $\mathbf{T}$ with an adequate cover set $\mathcal{C}$, the calculus $\mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\mathrm{KWD}}(\mathbf{T})$ is uniformly constructive in $\mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\mathbf{Cl}}(\mathbf{T})$. To this aim we introduce the following notion of *evaluation*:

DEFINITION 5.1. (Evaluation) Let Π be a set of proofs and let A be a formula of $\mathcal{L}_{\mathcal{A}}$. A *is evaluated in* Π, and we write $\Pi \rhd A$, iff the following conditions hold:

(i) There is a proof $\pi : \vdash A \in \Pi$;

(ii) For every closed instance θA of A, one of the following conditions holds:

  (a) θA is atomic or negated;
  (b) $\theta A \equiv B \wedge C$ and $\Pi \rhd B$ and $\Pi \rhd C$;
  (c) $\theta A \equiv B \vee C$ and either $\Pi \rhd B$ or $\Pi \rhd C$;
  (d) $\theta A \equiv B \rightarrow C$ and, if $\Pi \rhd B$ then $\Pi \rhd C$;
  (e) $\theta A \equiv \exists x B(x)$ and $\Pi \rhd B(t/x)$ for some closed term t of $\mathcal{L}_{\mathcal{A}}$;
  (f) $\theta A \equiv \forall x B(x)$ and $\Pi \rhd B(t/x)$ for every closed term t of $\mathcal{L}_{\mathcal{A}}$.

A set Γ of formulas is evaluated in a set of proofs Π, and we write $\Pi \rhd \Gamma$, if $\Pi \rhd A$ for every formula $A \in \Gamma$.

LEMMA 5.2. *Let Π be a set of proofs over $\mathcal{L}_{\mathcal{A}}$ such that Π is* ID_2*-closed (see Table 2). If t and t' are closed terms of $\mathcal{L}_{\mathcal{A}}$ such that $\Pi \rhd t = t'$ then, for every formula $A(x)$ of $\mathcal{L}_{\mathcal{A}}$, $\Pi \rhd A(t)$ iff $\Pi \rhd A(t')$.*

PROOF. Since Π is ID_2-closed, $\vdash A(t) \in \text{Seq}(\Pi)$ iff $\vdash A(t') \in \text{Seq}(\Pi)$. The proof of Point (ii) of Definition 5.1 easily follows by induction on the structure of $A(x)$. ∎

LEMMA 5.3. *Let Π be a set of proofs over $\mathcal{L}_{\mathcal{A}}$ such that Π is closed w.r.t.* SUBST, RE$\wedge$, RE$\forall$, RMP *(see Table 2). For every Harrop formula H, if $\vdash H \in \text{Seq}(\Pi)$ then $\Pi \rhd H$.*

PROOF. We proceed by induction on the degree of the Harrop formula H. If H is atomic or negated the assertion immediately follows. If $H \equiv A \wedge B$ or $H \equiv \forall x A(x)$ the assertion follows by the closure of Π w.r.t. the rules RE$\wedge$ and RE$\forall$ and by the induction hypothesis. Let $H \equiv A \rightarrow B$, let θ be a closed substitution and suppose that $\Pi \rhd \theta A$. Since Π is SUBST-closed, $\vdash \theta(A \rightarrow B) \in \text{Seq}(\Pi)$ and, by the fact that $\vdash \theta A \in \text{Seq}(\Pi)$ and Π is RMP-closed, it follows that $\vdash \theta B \in \text{Seq}(\Pi)$. Since θB is a Harrop formula, by the induction hypotheses we get $\Pi \rhd \theta B$. ∎

$\theta\Gamma \vdash \theta A \in \textsc{Subst}(\Gamma \vdash A)$	$\Gamma \vdash A_i \in \textsc{Re}\wedge(\Gamma \vdash A_1 \wedge A_2),\ i \in \{1,2\}$
$\Gamma_1, \Gamma_2 \vdash A \in \textsc{Cut}(\Gamma_1 \vdash H; \Gamma_2, H \vdash A)$	$\Gamma \vdash A(x) \in \textsc{Re}\forall(\Gamma \vdash \forall x A(x))$
$\vdash x = x \in \textsc{Id}_1(\epsilon)$	$\Gamma, \Delta \vdash B \in \textsc{Rmp}(\Gamma \vdash A \to B; \Delta \vdash A)$
$\Gamma, \Delta \vdash A(t') \in \textsc{Id}_2(\Gamma \vdash A(t); \Delta \vdash t = t')$	$\Gamma \vdash \forall x A(x) \in \textsc{Rcl}(\Gamma \vdash \forall x \neg\neg A(x))$

Table 2. The generalized rule RHR

$\Gamma \vdash \exists x A(x) \in \textsc{Rdt}_1(\Gamma \vdash A(t); \vdash \exists x A(x) \vee \forall x(A(x) \to B \vee \neg B))$
$\vdash \forall x(A(x) \to B \vee \neg B) \in \textsc{Rdt}_2(\vdash \exists x A(x) \vee \forall x(A(x) \to B \vee \neg B))$

Table 3. The generalized rule RDT

Now, let R_1 be the generalized rule consisting of the union of the generalized rules of Tables 2 and 3 and of $\textsc{Cov}_{\mathcal{C}}$ (the e-rule related to $\mathcal{C}$). It is easy to check that R_1 is an e-rule for $\mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\mathrm{CI}}(\mathbf{T})$. For a recursive $\Pi \subseteq \mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\mathrm{KWD}}(\mathbf{T})$, let us denote with $\mathbb{D}_1(\Pi)$ the extraction calculus $\mathbb{D}(\mathrm{R}_1, \mathrm{Seq}([\Pi]))$.

LEMMA 5.4. *Let* $\mathbf{T}$ *be a recursive Harrop* $\mathcal{A}$*-theory with an adequate cover set* $\mathcal{C}$ *and let* Π *be any recursive set of proofs of* $\mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\mathrm{KWD}}(\mathbf{T})$*. For every proof* $\pi : \Gamma \vdash A$ *belonging to the closure under substitution of* $[\Pi]$*, if* $\mathbb{D}_1(\Pi) \rhd \Gamma$ *then* $\mathbb{D}_1(\Pi) \rhd A$*.*

PROOF. Since π belongs to the closure under substitution of $[\Pi]$, there exist $\pi' : \Gamma' \vdash A' \in [\Pi]$ and a substitution θ such that $\theta\Gamma' \vdash \theta A' \equiv \Gamma \vdash A$. Thus, by definition, $\mathbb{D}_1(\Pi)$ contains a proof of the sequent $\Gamma' \vdash A'$ and hence, since $\mathbb{D}_1(\Pi)$ is $\textsc{Subst}$-closed, there exists a proof $\tau : \Gamma \vdash A$ in $\mathbb{D}_1(\Pi)$. Moreover, since $\Gamma = \{B_1, \ldots, B_n\}$ is evaluated in $\mathbb{D}_1(\Pi)$, there exist proofs $\tau_1 : \vdash B_1, \ldots, \tau_n : \vdash B_n$ in $\mathbb{D}_1(\Pi)$. Since $\mathbb{D}_1(\Pi)$ is $\textsc{Cut}$-closed and contains the proofs $\tau, \tau_1, \ldots \tau_n$, it also contains a proof $\tau^* : \vdash A$. This proves Point (i) of Definition 5.1; to prove Point (ii) we proceed by induction on depth(π).

Basis: If depth(π) $= 0$, the only rule applied in π is either an assumption introduction, an axiom-rule or an instance of the rule DT. In the first case $\Gamma = \{A\}$ and hence A is trivially evaluated in $\mathbb{D}_1(\Pi)$. In the second case $\Gamma = \emptyset$ and A is a Harrop formula; since $\vdash A \in \text{Seq}([\Pi])$, it is provable in $\mathbb{D}_1(\Pi)$ and, by Lemma 5.3, $\mathbb{D}_1(\Pi) \rhd A$. Finally, let us consider the case where the only rule applied in π is DT; in this case Γ is empty and $A \equiv \exists x B(x) \vee \forall x(B(x) \to C \vee \neg C)$. Let θ be a closed substitution and let us suppose that $\theta \exists x B(x)$ is not evaluated in $\mathbb{D}_1(\Pi)$; we prove that $\mathbb{D}_1(\Pi) \rhd \theta \forall x(B(x) \to C \vee \neg C)$. Since $\mathbb{D}_1(\Pi)$ contains the e-rules Subst and Rdt_2, it contains a proof of $\vdash \theta \forall x(B(x) \to C \vee \neg C)$. Let t be any closed term of $\mathcal{L}_{\mathcal{A}}$; since $\mathbb{D}_1(\Pi)$ contains the rules Re$\forall$ and Subst, $\mathbb{D}_1(\Pi)$ contains a proof of $\vdash \theta(B(t) \to C \vee \neg C)$. Moreover, $\theta B(t)$ is not evaluated in $\mathbb{D}_1(\Pi)$ (otherwise, by the presence of Rdt_1 in $\mathbb{D}_1(\Pi)$, $\theta \exists x B(x)$ would be evaluated in $\mathbb{D}_1(\Pi)$), thus $\mathbb{D}_1(\Pi) \rhd \theta(B(t) \to C \vee \neg C)$, and this concludes the proof.

Step: Let us suppose that $\text{depth}(\pi) = h + 1$. The proof goes on by cases according to the last rule applied in π; here we only discuss some representative cases. Notice that the case corresponding to the Kur rule is trivial since A is a negated formula.

Disjunction Elimination.

$$\pi : \Gamma \vdash A \equiv \frac{\pi_0 : \Gamma_0 \vdash B_1 \vee B_2 \qquad \pi_1 : \Gamma_1, B_1 \vdash A \qquad \pi_2 : \Gamma_2, B_2 \vdash A}{\Gamma_0, \Gamma_1, \Gamma_2 \vdash A}\,\text{E}\vee$$

Let θ be a closed substitution. Since $\mathbb{D}_1(\Pi) \rhd \theta\Gamma_0$, $\theta\pi_0$ belongs to the closure under substitution of $[\Pi]$, and $\text{depth}(\pi_0) \leq h$, we get, by induction hypothesis, that $\theta B_1 \vee \theta B_2$ is evaluated in $\mathbb{D}_1(\Pi)$. Thus, there exists $i \in \{1, 2\}$ such that $\mathbb{D}_1(\Pi) \rhd \theta B_i$ and, since $\theta\pi_i : \theta\Gamma_i, \theta B_i \vdash \theta A$ belongs to the closure under substitution of $[\Pi]$, we get, by induction hypothesis, that $\mathbb{D}_1(\Pi) \rhd \theta A$.

Generalized induction rule.

$$\pi : \Gamma \vdash A \equiv \frac{\pi_1 : \Gamma_1, \Delta_1 \vdash B(p_1) \quad \ldots \quad \pi_n : \Gamma_n, \Delta_n \vdash B(p_n)}{\Gamma_1, \ldots, \Gamma_n \vdash B(x)}\,\mathcal{C}\text{-Ind}$$

Firstly we prove that, for every $i \geq 0$ and every $t \in \mathcal{C}_i$ (where $\mathcal{C}_i$ is defined as in §4), $\mathbb{D}_1(\Pi) \rhd B(t)$. We proceed by a secondary induction on i (which corresponds to an induction on the structural complexity of t). Let $i \geq 0$ and let $t \in \mathcal{C}_i$. Since $\vdash B(x)$ is provable in $\mathbb{D}_1(\Pi)$, by applying the Subst rule we get a proof of $\vdash B(t)$. Let θ be any closed substitution, we show that $\mathbb{D}_1(\Pi) \rhd \theta B(t)$. If $i = 0$ then $t \in \mathcal{C}$, hence, by definition of the rule $\mathcal{C}$-Ind, π must have a subproof $\pi' : \Gamma \vdash B(t)$; thus the assertion immediately follows from the principal induction hypothesis applied to the proof $\theta\pi'$. Now, let us suppose that the assertion holds for i and let $t(s_1, \ldots, s_n) \in \mathcal{C}_{i+1}$. By the definition of the rule $\mathcal{C}$-Ind, π contains a subproof $\pi' : \Gamma, B(p_1), \ldots, B(p_n) \vdash B(t(p_1, \ldots, p_n))$. Let us consider the proof

$\theta[s_1/p_1, \ldots, s_n/p_n]\pi'$; by the convention on the proper parameters, this is a proof of the sequent $\theta\Gamma, \theta B(s_1), \ldots, \theta B(s_n) \vdash \theta B(t(s_1, \ldots, s_n))$ and it belongs to the closure under substitution of $[\Pi]$. By the secondary induction hypothesis the premises of this proof are evaluated in $\mathbb{D}_1(\Pi)$, thus, by the main induction hypothesis, $\mathbb{D}_1(\Pi) \rhd \theta B(t(s_1, \ldots, s_n))$ and this concludes the above claim. It remains to prove that, for every closed substitution θ, $\mathbb{D}_1(\Pi) \rhd \theta B(x)$. Let $\theta(x) = t$; since R_1 contains $\mathrm{Cov}_{\mathcal{C}}$, $\mathbb{D}_1(\Pi)$ contains a proof of $\vdash t = t'$ with $t' \in \mathrm{Cterm}_{\mathcal{C}}$. Since $t' \in \mathcal{C}_i$ for some i, by the above proof $\mathbb{D}_1(\Pi) \rhd B(t')$; by Lemma 5.2 it follows $\mathbb{D}_1(\Pi) \rhd B(t)$, which implies $\mathbb{D}_1(\Pi) \rhd \theta B(x)$.

Rule wGrz.

$$\pi : \Gamma \vdash A \equiv \cfrac{\pi_1 : \Gamma_1 \vdash \forall x \neg\neg B(x) \qquad \pi_2 : \Gamma_2 \vdash \forall x (B(x) \vee C)}{\Gamma_1, \Gamma_2 \vdash \forall x B(x) \vee C}\,\mathrm{wGrz}$$

Let θ be a closed substitution; we must prove that one between the formulas $\theta\forall x B(x)$ and θC is evaluated in $\mathbb{D}_1(\Pi)$. Let us suppose that θC is not evaluated in $\mathbb{D}_1(\Pi)$. Since, by induction hypothesis on $\theta\pi_2$, $\mathbb{D}_1(\Pi) \rhd \theta\forall x(B(x) \vee C)$, we deduce that, for every closed term t of $\mathcal{L}_A$, $\mathbb{D}_1(\Pi) \rhd \theta B(t/x)$. To prove that $\mathbb{D}_1(\Pi) \rhd \theta\forall x B(x)$, we only need to show that $\theta\forall x B(x)$ is provable in $\mathbb{D}_1(\Pi)$. By induction hypothesis on the proof $\theta\pi_1$, the sequent $\vdash \theta\forall x \neg\neg B(x)$ is provable in $\mathbb{D}_1(\Pi)$; since the latter set of proofs is RCL-closed, we get that also the sequent $\vdash \theta\forall x B(x)$ is provable in $\mathbb{D}_1(\Pi)$. ■

COROLLARY 5.5. *Let* $\mathbf{T}$ *be a recursive Harrop theory with an adequate cover set* $\mathcal{C}$ *and let* Π *be any recursive set of proofs of* $\mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\mathrm{KWD}}(\mathbf{T})$. *Then* $\mathrm{Theo}([\Pi])$ *is constructive in* $\mathrm{Theo}(\mathbb{D}_1(\Pi))$.

PROOF. If $A \vee B$ is a closed formula in $\mathrm{Theo}([\Pi])$, then there exists a proof $\pi : \vdash A \vee B$ in the closure under substitution of $[\Pi]$. Since the empty set of premises is evaluated in $\mathbb{D}_1(\Pi)$, by Lemma 5.4 $\mathbb{D}_1(\Pi) \rhd A \vee B$. Thus, at least one of the formulas A and B is evaluated in $\mathbb{D}_1(\Pi)$; by Definition 5.1, this means that one of the sequents $\vdash A$ and $\vdash B$ is provable in $\mathbb{D}_1(\Pi)$. The proof of the (wED) property is similar. ■

Since $\mathrm{Seq}(\mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\mathrm{KWD}}(\mathbf{T})) \subseteq \mathrm{Seq}(\mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\mathrm{Cl}}(\mathbf{T}))$ and R_1 is an e-rule for the calculus $\mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\mathrm{Cl}}(\mathbf{T})$, by the previous corollary we get:

THEOREM 5.6. *Let* $\mathbf{T}$ *be a recursive Harrop theory with an adequate cover set* $\mathcal{C}$. *Then* $\mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\mathrm{KWD}}(\mathbf{T})$ *is uniformly constructive in* $\mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\mathrm{Cl}}(\mathbf{T})$.

Let us define:

$$\mathrm{KWD}^{\mathcal{C}}(\mathbf{T}) = \mathbf{T} \oplus (\mathcal{C}\text{-Ind}) \oplus (\mathrm{Kur}) \oplus (\mathrm{wGrz}) \oplus (\mathrm{DT}) \oplus \mathbf{Int}$$

Since the calculus $\mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\mathrm{KWD}}(\mathbf{T})$ generates $\mathrm{KWD}^{\mathcal{C}}(\mathbf{T})$ and $\mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\mathbf{Cl}}(\mathbf{T})$ generates $\mathrm{KWD}^{\mathcal{C}}(\mathbf{T}) \oplus \mathbf{Cl}$ (which coincides with $\mathbf{T} \oplus (\mathcal{C}\text{-Ind}) \oplus \mathbf{Cl}$), by Theorem 5.6 we conclude:

THEOREM 5.7. *Let* $\mathbf{T}$ *be a recursive Harrop theory with an adequate cover set* $\mathcal{C}$*. Then* $\mathrm{KWD}^{\mathcal{C}}(\mathbf{T})$ *is uniformly semiconstructive.*

We remark that Theorem 5.6 implies that *every* subcalculus of $\mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\mathrm{KWD}}(\mathbf{T})$ is uniformly constructive in $\mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\mathbf{Cl}}(\mathbf{T})$; in some cases such calculi are even constructive (or uniformly constructive, see [9, 12]). On the other hand, as we prove in the rest of this section, the rules wGrz and DT do not have a constructive meaning and may give rise to uniformly semiconstructive systems which are not constructive.

To this aim, let $\mathbf{Ar}$ be the Harrop theory of Arithmetic and let $\mathbf{HA} = \mathbf{Ar} \oplus (\mathcal{C}\text{-Ind}) \oplus \mathbf{Int}$ be the system of Intuitionistic Arithmetic (see §4). The $\mathbf{HA}$-system $\mathbf{HA}^{+} = \mathbf{HA} \oplus (\mathrm{wGrz})$ contained in $\mathrm{KWD}^{\mathcal{C}}(\mathbf{Ar})$ is uniformly semiconstructive; as a matter of fact, $\mathbf{HA}^{+}$ is generated by the calculus $\mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\mathrm{WGRZ}}(\mathbf{Ar})$ obtained by adding the rule wGrz to $\mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\mathbf{Int}}(\mathbf{Ar})$ and, by Theorem 5.6, $\mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\mathrm{WGRZ}}(\mathbf{Ar})$ is uniformly constructive in $\mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\mathbf{Cl}}(\mathbf{Ar})$. On the other hand, $\mathbf{HA}^{+}$ cannot be extended in a $\mathbf{HA}$-system satisfying the *full* disjunction property and the *full* explicit definability property, as proved in the next theorem.

THEOREM 5.8. *Let* $\mathbf{T}$ *be a consistent theory such that* $\mathbf{HA} \subseteq \mathbf{T}$*. There exists no consistent and recursively axiomatizable constructive* $\mathbf{T}$*-system* $\mathbf{S}$ *such that* $\mathbf{HA}^{+} \subseteq \mathbf{S}$.

PROOF. Let $\mathbf{S}$ be a constructive recursively axiomatizable $\mathbf{T}$-system including $\mathbf{HA}^{+}$ (with $\mathbf{HA} \subseteq \mathbf{T}$). Let $p(x)$ be a unary recursively enumerable but not recursive predicate. By Kleene Normal Form Theorem for recursively enumerable relations (see [25]), there exists a primitive recursive binary predicate $q(x,y)$ such that $p(x) \leftrightarrow \exists y q(x,y)$. Now, by well-known representability results, there exists a formula $A(x,y)$ with two free variables of the language of Arithmetic which strongly represents the predicate $q(x,y)$ in $\mathbf{HA}$; this means that $\forall x \forall y(A(x,y) \vee \neg A(x,y)) \in \mathbf{HA}$, and, for every $a,b \in \mathbf{N}$, denoting with $\widetilde{a}$ and $\widetilde{b}$ the corresponding numerals, if $q(a,b)$ is true then $A(\widetilde{a},\widetilde{b}) \in \mathbf{HA}$, if $q(a,b)$ is false then $\neg A(\widetilde{a},\widetilde{b}) \in \mathbf{HA}$. Moreover, let G be a closed formula of the language of Arithmetic such that $G \notin \mathbf{S}$ and $\neg G \notin \mathbf{S}$ (such a formula exists by the intuitionistic version of Gödel Incompleteness Theorem [29]). Let us consider the formula $H(x) \equiv \exists y A(x,y) \vee \forall y(\neg A(x,y) \vee (G \vee \neg G))$. It is easy to check that $H(x)$ belongs to $\mathbf{HA}^+$ (in fact both $\forall z(\exists y A(x,y) \vee (\neg A(x,z) \vee (G \vee \neg G)))$ and $\forall z \neg\neg(\neg A(x,z) \vee (G \vee \neg G))$ belong to $\mathbf{HA}$, and $\mathbf{HA}^+$ contains (wGrz)). We show that one can effectively decide whether $p(k)$ holds or not, for every natural number k. Indeed, since $\mathbf{S}$ is constructive and recursively axiomatizable, there is a terminating effective procedure which, for every input $k \in \mathbf{N}$, outputs either a proof of $\exists y A(\tilde{k}, y)$ or a proof of $\forall y(\neg A(\tilde{k}, y) \vee (G \vee \neg G))$. If $\exists y A(\tilde{k}, y) \in \mathbf{S}$, then, by the constructivity of $\mathbf{S}$, $A(\tilde{k}, \tilde{b}) \in \mathbf{S}$ for some numeral $\tilde{b}$; this implies that $\exists y q(k,y)$ holds. On the other hand, if $\forall y(\neg A(\tilde{k}, y) \vee (G \vee \neg G)) \in \mathbf{S}$, since, by the hypotheses on the formula G, $G \vee \neg G \notin \mathbf{S}$, we deduce that $\neg A(\tilde{k}, \tilde{b}) \in \mathbf{S}$, for every $b \in \mathbf{N}$; this implies that $\exists y q(k,y)$ does not hold. Since $p(x) \leftrightarrow \exists y q(x,y)$, we have that $p(x)$ is recursive, contrary to the assumptions. ■

The above proof essentially depends on the presence of the axiom schema (wGrz), but a similar result can be given for other $\mathbf{HA}$-systems contained in $\mathrm{KWD}^{\mathcal{C}}(\mathbf{Ar})$. Let us consider the $\mathbf{HA}$-system $\mathbf{HA}^{++} = \mathbf{HA} \oplus (\mathrm{DT})$, which is generated by the calculus $\mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\mathrm{DT}}(\mathbf{Ar})$ obtained by adding the rule DT to $\mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\mathbf{Int}}(\mathbf{Ar})$. By Theorem 5.6, $\mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\mathrm{DT}}(\mathbf{Ar})$ is uniformly constructive in $\mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\mathbf{Cl}}(\mathbf{Ar})$, hence $\mathbf{HA}^{++}$ is uniformly semiconstructive; on the other hand:

THEOREM 5.9. *Let* $\mathbf{T}$ *be a consistent theory such that* $\mathbf{HA} \subseteq \mathbf{T}$. *There exists no consistent and recursively axiomatizable constructive* $\mathbf{T}$*-system* $\mathbf{S}$ *such that* $\mathbf{HA}^{++} \subseteq \mathbf{S}$.

PROOF. Let $\mathbf{S}$ be a recursively axiomatizable and constructive $\mathbf{T}$-system including $\mathbf{HA}^{++}$ (with $\mathbf{HA} \subseteq \mathbf{T}$). We show that, for every closed formula A, one can decide whether $A \in \mathbf{S}$ or not. Indeed, let G be a closed formula of the language of Arithmetic such that $G \notin \mathbf{S}$ and $\neg G \notin \mathbf{S}$. Since $\mathbf{S}$ is constructive and recursively axiomatizable and, for every closed formula A, $A \vee (A \to G \vee \neg G) \in \mathbf{HA}^{++}$, there is a terminating effective procedure which, taking any closed formula A of the language of Arithmetic as an input, outputs either a proof of A or a proof of $A \to G \vee \neg G$. If $A \to G \vee \neg G \in \mathbf{S}$, by the choice of G and the constructivity of $\mathbf{S}$, $A \notin \mathbf{S}$. Hence, the set of theorems of $\mathbf{S}$ is recursive, contrary to the intuitionistic version of Church's Theorem. ■

6. Adding the Markov Principle

Another well-known intermediate principle that has been deeply studied in the context of constructive mathematics [29] and in the context of program synthesis [31] is *Markov Principle* (Mk) $\equiv \forall x(A(x) \vee \neg A(x)) \wedge \neg\neg\exists x A(x) \rightarrow \exists x A(x)$. (Mk) gives rise to the following rule:

$$\frac{\Gamma_1 \vdash \neg\neg\exists x A(x) \quad \Gamma_2 \vdash \forall x(A(x) \vee \neg A(x))}{\Gamma_1, \Gamma_2 \vdash \exists x A(x)}\text{Mk}$$

Let $\mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\text{MKWD}}(\mathbf{T})$ be the calculus obtained by adding Mk to $\mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\text{KWD}}(\mathbf{T})$.

The proof of the semiconstructivity result for $\mathbf{T}$-systems containing the Markov Principle requires $\mathbf{T}$ to satisfy the following semantical condition, not needed to treat the rules of the above section. Namely:

DEFINITION 6.1. Given an $\mathcal{A}$-theory $\mathbf{T}$ with an adequate cover set $\mathcal{C}$, we say that $\mathbf{T}$ *has a reachable model w.r.t.* $\mathcal{C}$ iff there exists a classical model $\mathfrak{M}$ of $\mathbf{T} \oplus (\mathcal{C}\text{-Ind})$ such that $\mathfrak{M}$ is *reachable*, that is every element of the domain of $\mathfrak{M}$ is denoted by a closed term of $\mathcal{L}_{\mathcal{A}}$.

On the other hand no special e-rule is needed to treat Markov Rule, hence we use the same extraction calculus introduced in the previous section.

LEMMA 6.2. *Let* $\mathbf{T}$ *be a recursive Harrop* $\mathcal{A}$*-theory with an adequate cover set* $\mathcal{C}$ *such that* $\mathbf{T}$ *has a reachable model w.r.t.* $\mathcal{C}$*. Let* Π *be any recursive set of proofs of* $\mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\text{MKWD}}(\mathbf{T})$*. For every* $\pi : \Gamma \vdash A$ *belonging to the closure under substitution of* $[\Pi]$*, if* $\mathbb{D}_1(\Pi) \rhd \Gamma$ *then* $\mathbb{D}_1(\Pi) \rhd A$*.*

PROOF. The proof is analogous to the one of Lemma 5.4; we only have to analyze the case corresponding to the rule Mk.

Markov Rule.

$$\pi : \Gamma \vdash A \equiv \frac{\pi_1 : \Gamma_1 \vdash \neg\neg\exists x B(x) \quad \pi_2 : \Gamma_2 \vdash \forall x(B(x) \vee \neg B(x))}{\Gamma_1, \Gamma_2 \vdash \exists x B(x)}\text{Mk}$$

Let us consider a closed substitution θ. By induction hypothesis on the proof $\theta\pi_2$, $\mathbb{D}_1(\Pi) \rhd \theta\forall x(B(x) \vee \neg B(x))$. Hence, for every closed term t of $\mathcal{L}_{\mathcal{A}}$, $\mathbb{D}_1(\Pi) \rhd \theta B(t/x) \vee \neg\theta B(t/x)$. Let us suppose that, for every closed term t of $\mathcal{L}_{\mathcal{A}}$, $\mathbb{D}_1(\Pi) \rhd \neg\theta B(t/x)$; then, for every closed term t of $\mathcal{L}_{\mathcal{A}}$, $\vdash \theta\neg B(t/x)$ is provable in $\mathbb{D}_1(\Pi)$. By induction hypothesis on the proof $\theta\pi_1$, we also have $\mathbb{D}_1(\Pi) \rhd \theta\neg\neg\exists x B(x)$. Since $\text{Seq}(\mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\text{MKWD}}(\mathbf{T})) \subseteq \text{Seq}(\mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\mathbf{Cl}}(\mathbf{T}))$ and R_1 is an e-rule for $\mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\mathbf{Cl}}(\mathbf{T})$, every formula provable in $\mathbb{D}_1(\Pi)$ belongs to $\mathbf{T} \oplus (\mathcal{C}\text{-Ind}) \oplus \mathbf{Cl}$. It follows that the formulas $\theta\neg B(t/x)$, for every closed term t of $\mathcal{L}_{\mathcal{A}}$, and $\theta\exists x B(x)$ belong to $\mathbf{T} \oplus (\mathcal{C}\text{-Ind}) \oplus \mathbf{Cl}$; this contradicts the hypothesis that $\mathbf{T} \oplus (\mathcal{C}\text{-Ind})$ has a reachable model. Hence there exists a closed term t of $\mathcal{L}_{\mathcal{A}}$ such that $\mathbb{D}_1(\Pi) \rhd \theta B(t/x)$, and $\mathbb{D}_1(\Pi) \rhd \exists x B(x)$. ■

From the previous lemma, reasoning as in the proof of Corollary 5.5, we get:

COROLLARY 6.3. *Let* $\mathbf{T}$ *be a recursive Harrop theory with an adequate cover set* $\mathcal{C}$ *such that* $\mathbf{T}$ *has a reachable model w.r.t.* $\mathcal{C}$. *Let* Π *be any recursive set of proofs of* $\mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\mathrm{MKWD}}(\mathbf{T})$. *Then* $\mathrm{Theo}([\Pi])$ *is constructive in* $\mathrm{Theo}(\mathbb{D}_1(\Pi))$.

Since R_1 is an e-rule for the calculus $\mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\mathbf{Cl}}(\mathbf{T})$ and $\mathrm{Seq}(\mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\mathrm{MKWD}}(\mathbf{T})) \subseteq \mathrm{Seq}(\mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\mathbf{Cl}}(\mathbf{T}))$, by the previous corollary we get:

THEOREM 6.4. *Let* $\mathbf{T}$ *be a recursive Harrop theory with an adequate cover set* $\mathcal{C}$ *such that* $\mathbf{T}$ *has a reachable model w.r.t.* $\mathcal{C}$. *Then* $\mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\mathrm{MKWD}}(\mathbf{T})$ *is uniformly constructive in* $\mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\mathbf{Cl}}(\mathbf{T})$.

Finally, for a Harrop theory $\mathbf{T}$ with an adequate cover set $\mathcal{C}$ let:

$$\mathrm{MKWD}^{\mathcal{C}}(\mathbf{T}) = \mathrm{KWD}^{\mathcal{C}}(\mathbf{T}) \oplus (\mathrm{Mk})$$

Since $\mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\mathrm{MKWD}}(\mathbf{T})$ generates $\mathrm{MKWD}^{\mathcal{C}}(\mathbf{T})$, by the above discussion we can conclude:

THEOREM 6.5. *Let* $\mathbf{T}$ *be a recursive Harrop theory with an adequate cover set* $\mathcal{C}$ *such that* $\mathbf{T}$ *has a reachable model w.r.t.* $\mathcal{C}$. *Then* $\mathrm{MKWD}^{\mathcal{C}}(\mathbf{T})$ *is uniformly semiconstructive.*

7. The family of uniformly semiconstructive systems $\mathrm{KKW}^{\mathcal{C}}(\mathbf{T})$

In this section we investigate the uniform semiconstructivity of a family of $\mathbf{T}$-systems including the principles:

$$(\mathrm{KP}_\vee) \;\equiv\; (\neg A \to B \vee C) \to (\neg A \to B) \vee (\neg A \to C)$$
$$(\mathrm{KP}_\exists) \;\equiv\; (\neg A \to \exists x B(x)) \to \exists x(\neg A \to B(x)) \quad \text{(where } x \text{ is not free in } A\text{)}$$

$(\mathrm{KP}_\vee)$ is the *Kreisel-Putnam Principle*, a principle well-known in the literature on propositional intermediate logics [17], while $(\mathrm{KP}_\exists)$, also known in the area of constructivism as (IP) [29], naturally completes the meaning of $(\mathrm{KP}_\vee)$ at the predicate level, where existential formulas can be seen as infinitary disjunctions. A maximal constructive intermediate predicate logic including $(\mathrm{KP}_\vee)$ and $(\mathrm{KP}_\exists)$ is studied in [3].

The above principles can be expressed by the following rules:

$$\frac{\Gamma, \neg A \vdash B \vee C}{\Gamma \vdash (\neg A \to B) \vee (\neg A \to C)}\text{KP}_\vee \qquad \frac{\Gamma, \neg A \vdash \exists x B(x)}{\Gamma \vdash \exists x(\neg A \to B(x))}\text{KP}_\exists \quad \text{where } x \text{ is not free in } A$$

Given a recursive Harrop theory with an adequate cover set $\mathcal{C}$, we denote with $\mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\text{KKW}}(\mathbf{T})$ the calculus obtained by adding to the calculus $\mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\mathbf{Int}}(\mathbf{T})$ of §4 the rules wGrz and Kur of §5 and the rules $\text{KP}_\vee$ and $\text{KP}_\exists$. We prove that $\mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\text{KKW}}(\mathbf{T})$ is uniformly constructive in the calculus $\mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\mathbf{Cl}}(\mathbf{T})$. To this aim, we introduce a new notion of evaluation:

DEFINITION 7.1. (Neg-evaluation) Let Π be a set of proofs over $\mathcal{L}_\mathcal{A}$, and let Neg and A be a set of closed negated formulas and a formula in the language $\mathcal{L}_\mathcal{A}$ respectively. A is Neg-*evaluated in* Π ($\Pi \rhd_{\text{Neg}} A$) iff the following conditions hold:

(i) Either $A \in \text{Neg}$ or there exists a proof $\pi : \Phi \vdash A \in \Pi$ with $\Phi \subseteq \text{Neg}$;

(ii) For every closed instance θA of A, one of the following conditions holds:

  (a) θA is atomic or negated;

  (b) $\theta A \equiv B \wedge C$, and both $\Pi \rhd_{\text{Neg}} B$ and $\Pi \rhd_{\text{Neg}} C$;

  (c) $\theta A \equiv B \vee C$, and either $\Pi \rhd_{\text{Neg}} B$ or $\Pi \rhd_{\text{Neg}} C$;

  (d) $\theta A \equiv B \to C$, and, for every set Neg′ of closed negated formulas of $\mathcal{L}_\mathcal{A}$ such that $\text{Neg}' \supseteq \text{Neg}$, if $\Pi \rhd_{\text{Neg}'} B$ then $\Pi \rhd_{\text{Neg}'} C$;

  (e) $\theta A \equiv \exists x B(x)$, and $\Pi \rhd_{\text{Neg}} B(t/x)$ for some closed term t of $\mathcal{L}_\mathcal{A}$;

  (f) $\theta A \equiv \forall x B(x)$, and, for every closed term t of $\mathcal{L}_\mathcal{A}$, $\Pi \rhd_{\text{Neg}} B(t/x)$.

A set Γ of formulas is Neg-evaluated in a set of proofs Π, written $\Pi \rhd_{\text{Neg}} \Gamma$, if $\Pi \rhd_{\text{Neg}} A$ for every formula $A \in \Gamma$. It is easy to prove the following properties:

LEMMA 7.2. *Let* Neg *be a set of closed negated formulas, let* Π *be a set of proofs and let* A *be a formula.*

(1) If $\Pi \rhd_{\text{Neg}} A$, *then* $\Pi \rhd_{\text{Neg}'} A$ *for every set of closed negated formulas* Neg′ *including* Neg.

(2) If Π *is closed w.r.t. the generalized rule* CUT, $\neg H$ *is a closed formula,* $\Pi \rhd_{\text{Neg}} \neg H$ *and* $\Pi \rhd_{\text{Neg} \cup \{\neg H\}} A$, *then* $\Pi \rhd_{\text{Neg}} A$.

The following results can be easily proved along the lines of the analogous lemmas of §5.

LEMMA 7.3. *Let* Π *be a set of proofs over* $\mathcal{L}_\mathcal{A}$ *such that* Π *is* ID_2*-closed and let* Neg *be a set of closed negated formulas of* $\mathcal{L}_\mathcal{A}$*. If* t *and* t' *are closed terms of* $\mathcal{L}_\mathcal{A}$ *such that* $\Pi \rhd_{\text{Neg}} t = t'$ *then, for every formula* $A(x)$ *of* $\mathcal{L}_\mathcal{A}$*,* $\Pi \rhd_{\text{Neg}} A(t)$ *iff* $\Pi \rhd_{\text{Neg}} A(t')$.

LEMMA 7.4. *Let* Π *be a set of proofs over* $\mathcal{L}_\mathcal{A}$ *such that* Π *is closed w.r.t.* SUBST, RE$\wedge$, RE$\forall$, RMP *(see Table 2). Let* Neg *be a set of closed negated formulas of* $\mathcal{L}_\mathcal{A}$*. For every Harrop formula* H *and for every* $\Phi \subseteq \text{Neg}$*, if* $\Phi \vdash H \in \text{Seq}(\Pi)$*, then* $\Pi \rhd_{\text{Neg}} H$.

Let R_2 be the generalized rule consisting of the union of $\text{COV}_\mathcal{C}$ (the e-rule related to $\mathcal{C}$), the generalized rules of Table 2, and the following generalized rules:

$$\Gamma, \Delta \vdash \neg A \rightarrow B \in \text{RKP}\vee_1(\Gamma \vdash B; \Delta \vdash (\neg A \rightarrow B) \vee (\neg A \rightarrow C)) \text{ with } \neg A \notin \Gamma$$
$$\Gamma, \Delta \vdash \neg A \rightarrow B \in \text{RKP}\vee_1(\Gamma, \neg A \vdash B; \Delta \vdash (\neg A \rightarrow B) \vee (\neg A \rightarrow C))$$
$$\Gamma, \Delta \vdash \neg A \rightarrow C \in \text{RKP}\vee_2(\Gamma \vdash C; \Delta \vdash (\neg A \rightarrow B) \vee (\neg A \rightarrow C)) \text{ with } \neg A \notin \Gamma$$
$$\Gamma, \Delta \vdash \neg A \rightarrow C \in \text{RKP}\vee_2(\Gamma, \neg A \vdash C; \Delta \vdash (\neg A \rightarrow B) \vee (\neg A \rightarrow C))$$
$$\neg B \vdash \neg B \in \text{RKP}\vee_3(\Delta \vdash (\neg A \rightarrow \neg B) \vee (\neg A \rightarrow C))$$
$$\neg C \vdash \neg C \in \text{RKP}\vee_4(\Delta \vdash (\neg A \rightarrow B) \vee (\neg A \rightarrow \neg C))$$

$$\Gamma, \Delta \vdash \neg A \rightarrow B(t) \in \text{RKP}\exists_1(\Gamma \vdash B(t); \Delta \vdash \exists x(\neg A \rightarrow B(x))) \quad \text{with } \neg A \notin \Gamma$$
$$\Gamma, \Delta \vdash \neg A \rightarrow B(t) \in \text{RKP}\exists_1(\Gamma, \neg A \vdash B(t); \Delta \vdash \exists x(\neg A \rightarrow B(x)))$$
$$\neg B(t) \vdash \neg B(t) \in \text{RKP}\exists_2(\Delta \vdash \exists x(\neg A \rightarrow \neg B(x)))$$

It is easy to check that R_2 is an e-rule for $\mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\mathbf{Cl}}(\mathbf{T})$. We denote with $\mathbb{D}_2(\Pi)$ the extraction calculus $\mathbb{D}(\text{R}_2, \text{Seq}([\Pi]))$.

LEMMA 7.5. *Let* $\mathbf{T}$ *be a recursive Harrop theory with an adequate cover set* $\mathcal{C}$*. Let* Π *be any recursive set of proofs of* $\mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\text{KKW}}(\mathbf{T})$ *and let* Neg *be a set of closed negated formulas of* $\mathcal{L}_\mathcal{A}$*. For every proof* $\pi : \Gamma \vdash A$ *belonging to the closure under substitution of* $[\Pi]$*, if* $\mathbb{D}_2(\Pi) \rhd_{\text{Neg}} \Gamma$ *then* $\mathbb{D}_2(\Pi) \rhd_{\text{Neg}} A$.

PROOF. Since π belongs to the closure under substitution of $[\Pi]$, there exist a proof $\pi' : \Gamma' \vdash A' \in [\Pi]$ and a substitution θ such that $\theta\Gamma' \vdash \theta A' \equiv \Gamma \vdash A$. Thus, by definition, $\mathbb{D}_2(\Pi)$ contains a proof of the sequent $\Gamma' \vdash A'$ and, since it contains the SUBST rule, it also contains a proof $\tau : \Gamma \vdash A$. Let $\Gamma = \Phi_0 \cup \{H_1, \dots, H_n\}$, where $\Phi_0 = \Gamma \cap \text{Neg}$. Since Γ is Neg-evaluated in $\mathbb{D}_2(\Pi)$, $\mathbb{D}_2(\Pi)$ contains the proofs $\tau_1 : \Phi_1 \vdash H_1$, $\dots$, $\tau_n : \Phi_n \vdash H_n$ with $\Phi_1 \cup \dots \cup \Phi_n \subseteq \text{Neg}$. Let $\Phi^* = \Phi_0 \cup \Phi_1 \cup \dots \cup \Phi_n$; by repeatedly applying the CUT rule to the proofs $\tau, \tau_1, \dots, \tau_n$, we can construct in $\mathbb{D}_2(\Pi)$ a proof $\tau^* : \Phi^* \vdash A$.

The proof of Point (ii) goes on by induction on depth(π). If depth(π) = 0, the only rule applied in π is either an assumption introduction or an axiom introduction. In the former case the proof is trivial, in the latter case we apply Lemma 7.4. To prove the induction step, we proceed by cases according to the last rule applied in π. The proof is similar to the one of Lemma 5.4. Here we only treat the representative case of the $\mathrm{KP}_\vee$ rule, the case of the $\mathrm{KP}_\exists$ rule being similar.

Rule $\mathrm{KP}_\vee$.

$$\pi : \Gamma \vdash A \equiv \frac{\pi_1 : \Gamma, \neg B \vdash C \vee D}{\Gamma \vdash (\neg B \to C) \vee (\neg B \to D)}\,\mathrm{KP}_\vee$$

We must prove that, for every closed substitution θ, one of the formulas $\theta(\neg B \to C)$ and $\theta(\neg B \to D)$ is Neg-evaluated in $\mathbb{D}_2(\Pi)$. Since Γ is Neg-evaluated in $\mathbb{D}_2(\Pi)$, we have that $\theta\Gamma \cup \{\theta\neg B\}$ is $\mathrm{Neg} \cup \{\theta\neg B\}$-evaluated in $\mathbb{D}_2(\Pi)$. Then, by the induction hypothesis on the proof $\theta\pi_1$, either θC or θD is $\mathrm{Neg} \cup \{\theta\neg B\}$-evaluated in $\mathbb{D}_2(\Pi)$. Let us assume that θC is the evaluated formula; this implies that there exists a proof $\tau : \Phi \vdash \theta C \in \mathbb{D}_2(\Pi)$, with $\Phi \subseteq \mathrm{Neg} \cup \{\theta\neg B\}$. We remark that, if $\theta C \in \mathrm{Neg} \cup \{\theta\neg B\}$, we can construct τ starting from the proof $\tau^* : \Phi^* \vdash A$ defined in the proof of Point (i) as follows:

$$\frac{\dfrac{\tau^* : \Phi^* \vdash (\neg B \to C) \vee (\neg B \to D)}{\theta\Phi^* \vdash \theta(\neg B \to C) \vee \theta(\neg B \to D)}\,\textsc{Subst}}{\theta C \vdash \theta C}\,\textsc{Rkpv}_3$$

Hence, the proof

$$\frac{\tau : \Phi \vdash \theta C \qquad \dfrac{\tau^* : \Phi^* \vdash (\neg B \to C) \vee (\neg B \to D)}{\theta\Phi^* \vdash \theta(\neg B \to C) \vee \theta(\neg B \to D)}\,\textsc{Subst}}{\Phi \setminus \{\theta\neg B\},\ \theta\Phi^* \vdash \theta(\neg B \to C)}\,\textsc{Rkpv}_1$$

belongs to $\mathbb{D}_2(\Pi)$, and this proves Point (i) of Definition 7.1 for $\theta(\neg B \to C)$. To prove Point (ii) for this formula, let us suppose that $\theta\neg B$ is Neg'-evaluated in $\mathbb{D}_2(\Pi)$, with $\mathrm{Neg} \subseteq \mathrm{Neg}'$. We already know that θC is $\mathrm{Neg} \cup \{\theta\neg B\}$-evaluated in $\mathbb{D}_2(\Pi)$, and hence, by Point (1) of Lemma 7.2, it is also $\mathrm{Neg}' \cup \{\theta\neg B\}$-evaluated in $\mathbb{D}_2(\Pi)$. Since $\theta\neg B$ is Neg'-evaluated in $\mathbb{D}_2(\Pi)$, by Point (2) of Lemma 7.2 it follows that θC is Neg'-evaluated in $\mathbb{D}_2(\Pi)$, and this concludes the proof. ∎

By the previous lemma, we get as in Corollary 5.5:

COROLLARY 7.6. *Let* $\mathbf{T}$ *be a recursive Harrop theory with an adequate cover set* $\mathcal{C}$ *and let* Π *be any recursive set of proofs of* $\mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\text{KKW}}(\mathbf{T})$. *Then* $\text{Theo}([\Pi])$ *is constructive in* $\text{Theo}(\mathbb{D}_2(\Pi))$.

Since R_2 is an e-rule for the calculus $\mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\mathbf{Cl}}(\mathbf{T})$ and $\text{Seq}(\mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\text{KKW}}(\mathbf{T})) \subseteq \text{Seq}(\mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\mathbf{Cl}}(\mathbf{T}))$, as a consequence of the previous corollary we get:

THEOREM 7.7. *Let* $\mathbf{T}$ *be a recursive Harrop theory with an adequate cover set* $\mathcal{C}$. *Then* $\mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\text{KKW}}(\mathbf{T})$ *is uniformly constructive in* $\mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\mathbf{Cl}}(\mathbf{T})$.

Given a Harrop theory $\mathbf{T}$ with an adequate cover set $\mathcal{C}$, let:

$$\text{KKW}^{\mathcal{C}}(\mathbf{T}) = \mathbf{T} \oplus (\mathcal{C}\text{-Ind}) \oplus (\text{Kur}) \oplus (\text{wGrz}) \oplus (\text{KP}_\vee) \oplus (\text{KP}_\exists) \oplus \mathbf{Int}$$

Since $\mathcal{ND}^{\mathcal{C}\text{-Ind}}_{\text{KKW}}(\mathbf{T})$ generates $\text{KKW}^{\mathcal{C}}(\mathbf{T})$, by the above theorem we conclude:

THEOREM 7.8. *Let* $\mathbf{T}$ *be a recursive Harrop theory with an adequate cover set* $\mathcal{C}$. *Then* $\text{KKW}^{\mathcal{C}}(\mathbf{T})$ *is uniformly semiconstructive.*

8. Incompatibility

In the previous sections we have discussed the semiconstructivity of some families of intermediate systems. In particular, given a recursive Harrop theory $\mathbf{T}$ with an adequate cover set $\mathcal{C}$, the following systems have been proved to be uniformly semiconstructive:

- $\mathbf{S}_1 = \mathbf{T} \oplus \mathbf{Int} \oplus (\text{wGrz}) \oplus (\text{Kur}) \oplus (\mathcal{C}\text{-Ind})$ (see §5).
- $\mathbf{S}_1 \oplus (\text{DT}) \oplus (\text{Mk})$ and its subsystems, where the presence of (Mk) requires the additional semantical condition of Definition 6.1 (see §6).
- $\mathbf{S}_1 \oplus (\text{KP}_\exists) \oplus (\text{KP}_\vee)$ and its subsystems (see §7).

One may wonder whether we can combine (Mk) and $(\text{KP}_\vee)$ and $(\text{KP}_\exists)$ or (DT) and $(\text{KP}_\vee)$ and $(\text{KP}_\exists)$ within a semiconstructive system. Unfortunately, the general tools used so far are not useful to this purpose. Indeed, one can realize that the semantical condition on the reachable model used to treat Markov rule and the notion of Neg-evaluation used to treat $\text{KP}_\vee$ and $\text{KP}_\exists$ rules cannot be put together. On the other hand, we can assure that no general result of uniform semiconstructivity can be proved for the above systems due to the following incompatibility results:

THEOREM 8.1. *There exists no semiconstructive* **HA**-*system* **S** *containing the following combinations of principles:*

(1) (Mk) and ($KP_\vee$);
(2) (Mk) and ($KP_\exists$);
(3) (DT) and ($KP_\vee$);
(4) (DT) and ($KP_\exists$).

PROOF. We prove the illustrative example of Case 1. By Gödel-Rosser-Mostowski-Kripke-Myhill Theorem [29] there exist two formulas $A(x)$ and $B(x)$ such that:

(i) $\forall x(A(x) \vee \neg A(x)) \in \mathbf{HA}$ and $\forall x(B(x) \vee \neg B(x)) \in \mathbf{HA}$;

(ii) $\exists x A(x) \rightarrow \exists x B(x) \notin \mathbf{PA}$ and $\exists x B(x) \rightarrow \exists x A(x) \notin \mathbf{PA}$.

We proceed indirectly. Let us suppose that there exists a semiconstructive **HA**-system **S** containing (Mk) and ($KP_\vee$). Since $\mathbf{HA} \subseteq \mathbf{S}$, by (i) it follows that $\forall x((A(x) \vee B(x)) \vee \neg(A(x) \vee B(x))) \in \mathbf{S}$. Since **S** contains (Mk), we get $\neg\neg\exists x(A(x) \vee B(x)) \rightarrow \exists x(A(x) \vee B(x)) \in \mathbf{S}$ which implies $\neg\neg\exists x(A(x) \vee B(x)) \rightarrow \exists x A(x) \vee \exists x B(x) \in \mathbf{S}$; since **S** also contains ($KP_\vee$), it follows that $(\neg\neg\exists x(A(x) \vee B(x)) \rightarrow \exists x A(x)) \vee (\neg\neg\exists x(A(x) \vee B(x)) \rightarrow \exists x B(x))$ belongs to **S**. By the semiconstructivity of **S** we get $\neg\neg\exists x(A(x) \vee B(x)) \rightarrow \exists x A(x) \in \mathbf{PA}$ or $\neg\neg\exists x(A(x) \vee B(x)) \rightarrow \exists x B(x) \in \mathbf{PA}$. This implies that either $\exists x B(x) \rightarrow \exists x A(x) \in \mathbf{PA}$ or $\exists x A(x) \rightarrow \exists x B(x) \in \mathbf{PA}$, in contradiction with (ii). ■

We remark that in Case (2) of the above theorem **S** coincides with **PA**. As for Case (4), we can even strengthen the result. Indeed, let us consider the intermediate principle (St) $\equiv ((\neg\neg A \rightarrow A) \rightarrow A \vee \neg A) \rightarrow \neg A \vee \neg\neg A$, also known as *Scott Principle* (see [10]). (St) has been widely studied by people working in intermediate propositional logics, see, e.g., [6, 10, 11, 14, 18]. We can prove that *there exists no semiconstructive* **HA**-*system* **S** *containing* (St) *and* ($KP_\exists$); since (St) is provable in $\mathbf{Int} \oplus (KP_\exists)$, this implies Case (4) of the previous theorem.

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