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# Second–order mollified derivatives and optimization\*

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## Abstract

The class of strongly semicontinuous functions is considered. For these functions the notion of mollified derivatives, introduced by Ermoliev, Norkin and Wets, is extended to the second order. By means of a generalized Taylor’s formula, second order necessary and sufficient conditions are proved for both unconstrained and constrained optimization.

**Keywords:** Mollifiers, Optimization, Smooth approximations, Strong semicontinuity .

## 1 Introduction

In this paper we extend to the second-order the approach introduced by Ermoliev, Norkin and Wets [8] to define generalized derivatives even for discontinuous functions, which often arise in applications (see [8] for references). To deal with such applications a number of approaches have been proposed to develop a subdifferential calculus for nonsmooth and even discontinuous functions. Among the many possibilities, let us remember the notions due to Aubin [2], Clarke [5], Ioffe [13], Michel and Penot [20], Rockafellar [21], in the context of Variational Analysis. The previous approaches are based on the introduction of first-order generalized derivatives. Extensions to higher-order derivatives have been provided for instance by Hiriart-Hurruty, Strodiot and Hien Nguyen [12], Jeyakumar and Luc [14], Klatte and Tammer [15], Michel and Penot [19], Yang and Jeyakumar [26] , Yang [27]. Most of these higher-order approaches assume that the functions involved are of class  $C^{1,1}$ , that is once differentiable with locally Lipschitz gradient, or at least of class  $C^1$ . Anyway, another possibility, concerning the differentiation of nonsmooth functions dates back to the 30’s and is related to the names of Sobolev [25], who introduced the concept of “weak derivative” and later of Schwartz [24] who generalized Sobolev’s approach with the “theory of distributions”. These techniques are

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widely used in the theory of partial differential equations, in Mathematical Physics and in related problems, but they have not been applied to deal with optimization problems involving nonsmooth functions, until the work of Ermoliev, Norkin and Wets.

The tools which allow to link the “modern” and the “ancient” approaches to Non-smooth Analysis are those of “mollifier” and of “mollified functions”. More specifically, the approach followed by Ermoliev, Norkin and Wets appeals to some of the results of the theory of distributions. They associate with a point  $x \in \mathbb{R}^n$  a family of mollifiers (density functions) whose support tends toward  $x$  and converges to the Dirac function. Given such a family, say  $\{\psi_\epsilon, \epsilon > 0\}$ , one can define a family of mollified functions associated to a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  as the convolution of  $f$  and  $\psi_\epsilon$  (mollified functions will be denoted by  $f_\epsilon$ ). Hence a mollified function can be viewed as an averaged function. The mollified functions possess the same regularity of the mollifiers  $\psi_\epsilon$  and hence, if they are at least of class  $C^2$ , one can define first and second-order generalized derivatives as the cluster points of all possible values of first and second-order derivatives of  $f_\epsilon$ . For more details one can see [8].

In this paper, section 2 recalls the notions of mollifier, of epi-convergence of a sequence of functions and some definitions introduced in [8]. Section 3 is devoted to the introduction of second-order derivatives by means of mollified functions; sections 4 and 5 deal, respectively, with second-order necessary and sufficient optimality conditions for unconstrained and constrained problems.

## 2 Preliminaries

To follow the approach presented in [8] we first need to introduce the notion of mollifier (see e.g. [4]):

**Definition 1** *A sequence of mollifiers is any sequence of functions  $\psi := \{\psi_\epsilon\} : \mathbb{R}^m \rightarrow \mathbb{R}_+$ ,  $\epsilon \downarrow 0$ , such that:*

$$i) \text{ supp } \psi_\epsilon := \{x \in \mathbb{R}^n \mid \psi_\epsilon(x) > 0\} \subseteq \rho_\epsilon \text{cl}\mathcal{B}, \rho_\epsilon \downarrow 0,$$

$$ii) \int_{\mathbb{R}^n} \psi_\epsilon(x) dx = 1,$$

where  $\mathcal{B}$  is the unit ball in  $\mathbb{R}^n$  and  $\text{cl}X$  means the closure of the set  $X$ .

Although in the sequel we may consider general families of mollifiers, some examples may be useful:

**Example 1** *Let  $\epsilon$  be a positive number.*

*i) The functions:*

$$\psi_\epsilon(x) = \begin{cases} \frac{1}{\epsilon^m}, & \max_{1, \dots, m} |x_i| \leq \frac{\epsilon}{2} \\ 0, & \text{otherwise} \end{cases}$$

*are called Steklov mollifiers.*

ii) The functions:

$$\psi_\epsilon(x) = \begin{cases} \frac{C}{\epsilon^m} \exp\left(\frac{\epsilon^2}{\|x\|^2 - \epsilon^2}\right), & \text{if } \|x\| < \epsilon \\ 0, & \text{if } \|x\| \geq \epsilon \end{cases}$$

with  $C \in \mathbb{R}$  such that  $\int_{\mathbb{R}^m} \psi_\epsilon(x) dx = 1$ , are called standard mollifiers.

It is easy to check that both the previous families of functions are of class  $\mathcal{C}^\infty$ .

**Definition 2** ([4]) Given a locally integrable function  $f : \mathbb{R}^m \rightarrow \mathbb{R}$  and a sequence of bounded mollifiers, define the functions  $f_\epsilon(x)$  through the convolution:

$$f_\epsilon(x) := \int_{\mathbb{R}^m} f(x - z) \psi_\epsilon(z) dz.$$

The sequence  $f_\epsilon(x)$  is said a sequence of mollified functions.

In the following all the functions considered will be assumed to be locally integrable.

**Remark 1** There is no loss of generality in considering  $f : \mathbb{R}^m \rightarrow \mathbb{R}$ . The results in this paper remain true also if  $f$  is defined on an open subset of  $\mathbb{R}^m$ .

Some properties of the mollified functions can be considered classical:

**Theorem 1** ([4]) Let  $f \in \mathcal{C}(\mathbb{R}^m)$ . Then  $f_\epsilon$  converges continuously to  $f$ , i.e.  $f_\epsilon(x_\epsilon) \rightarrow f(x)$  for all  $x_\epsilon \rightarrow x$ . In fact  $f_\epsilon$  converges uniformly to  $f$  on every compact subset of  $\mathbb{R}^m$  as  $\epsilon \downarrow 0$ .

The previous convergence property can be generalized.

**Definition 3** ([1], [23]) A sequence of functions  $\{f_n : \mathbb{R}^m \rightarrow \mathbb{R}\}$  epi-converges to  $f : \mathbb{R}^m \rightarrow \mathbb{R}$  at  $x$ , if:

i)  $\liminf_{n \rightarrow +\infty} f_n(x_n) \geq f(x)$  for all  $x_n \rightarrow x$ ;

ii)  $\lim_{n \rightarrow +\infty} f_n(x_n) = f(x)$  for some sequence  $x_n \rightarrow x$ .

The sequence  $\{f_n\}$  epi-converges to  $f$  if this holds for all  $x \in \mathbb{R}^m$ , in which case we write  $f = e - \lim f_n$ .

**Remark 2** It can be easily checked that when  $f$  is the epi-limit of some sequence  $f_n$  then  $f$  is lower semicontinuous. Moreover if  $f_n$  converges continuously, then also epi-converges.

**Definition 4** ([8]) A function  $f : \mathbb{R}^m \rightarrow \mathbb{R}$  is said strongly lower semicontinuous (s.l.s.c.) at  $x$  if it is lower semicontinuous at  $x$  and there exists a sequence  $x_n \rightarrow x$  with  $f$  continuous at  $x_n$  (for all  $n$ ) such that  $f(x_n) \rightarrow f(x)$ . The function  $f$  is strongly lower semicontinuous if this holds at all  $x$ .

The function  $f$  is said strongly upper semicontinuous (s.u.s.c.) at  $x$  if it is upper semicontinuous at  $x$  and there exists a sequence  $x_n \rightarrow x$  with  $f$  continuous at  $x_n$  (for all  $n$ ) such that  $f(x_n) \rightarrow f(x)$ . The function  $f$  is strongly lower semicontinuous if this holds at all  $x$ .

**Proposition 1** If  $f : \mathbb{R}^m \rightarrow \mathbb{R}$  is s.l.s.c., then  $-f$  is s.u.s.c. .

*Proof:* It follows directly from the definitions. □

**Theorem 2** ([8]) *Let  $\varepsilon_n \downarrow 0$ . For any s.l.s.c. function  $f : \mathbb{R}^m \rightarrow \mathbb{R}$ , and any associated sequence  $f_{\varepsilon_n}$  of mollified functions we have  $f = e - \lim f_{\varepsilon_n}$ .*

**Remark 3** It can be seen that, according to Remark 2, Theorem 1 follows from Theorem 2.

**Theorem 3** *Let  $\varepsilon_n \downarrow 0$ . For any s.u.s.c. function  $f : \mathbb{R}^m \rightarrow \mathbb{R}$  and any associated sequence  $f_{\varepsilon_n}$  of mollified functions, we have for any  $x \in \mathbb{R}^m$ :*

- i)  $\limsup_{n \rightarrow +\infty} f_{\varepsilon_n}(x_n) \leq f(x)$  for any sequence  $x_n \rightarrow x$ ;
- ii)  $\lim_{n \rightarrow +\infty} f_{\varepsilon_n}(x_n) = f(x)$  for some sequence  $x_n \rightarrow x$ .

*Proof:* Since  $f$  is s.u.s.c., we have  $-f$  s.l.s.c. and thus Theorem 2 applies:

- i) for any sequence  $x_n \rightarrow x$ ,  $\liminf_{n \rightarrow +\infty} (-f_{\varepsilon_n}(x_n)) \geq -f(x)$ , which implies:

$$\limsup_{n \rightarrow +\infty} f_{\varepsilon_n}(x_n) = -\liminf_{n \rightarrow +\infty} (-f_{\varepsilon_n}(x_n)) \leq f(x);$$

- ii) for some sequence  $x_n \rightarrow x$ ,  $\lim_{n \rightarrow +\infty} (-f_{\varepsilon_n}(x_n)) = -f(x)$ , from which we conclude:

$$\lim_{n \rightarrow +\infty} f_{\varepsilon_n}(x_n) = f(x).$$

□

The following Proposition plays a crucial role in the sequel.

**Proposition 2** ([24, 25]) *Whenever the mollifiers  $\psi_\varepsilon$  are of class  $\mathcal{C}^k$ , so are the associated mollified functions  $f_\varepsilon$ .*

By means of mollified functions it is possible to define generalized directional derivatives for a nonsmooth function  $f$ , which, under suitable regularity of  $f$ , coincide with Clarke's generalized derivative. Such an approach has been deepened by several authors (see e.g. [7, 8]) in the first-order case.

**Definition 5** ([8]) *Let  $f : \mathbb{R}^m \rightarrow \mathbb{R}$ ,  $\varepsilon_n \downarrow 0$  as  $n \rightarrow +\infty$  and consider the sequence  $\{f_{\varepsilon_n}\}$  of mollified functions with associated mollifiers  $\psi_{\varepsilon_n} \in \mathcal{C}^1$ . The upper mollified derivative of  $f$  at  $x$  in the direction  $d \in \mathbb{R}^m$ , with respect to (w.r.t.) the mollifiers sequence  $\psi_{\varepsilon_n}$  is defined as:*

$$\overline{\mathcal{D}}_\psi f(x, d) := \sup_{x_n \rightarrow x} \limsup_{n \rightarrow +\infty} \nabla f_{\varepsilon_n}(x_n)^\top d.$$

Similarly, we might introduce the following:

**Definition 6** *Let  $f : \mathbb{R}^m \rightarrow \mathbb{R}$ ,  $\varepsilon_n \downarrow 0$  as  $n \rightarrow +\infty$  and consider the sequence  $\{f_{\varepsilon_n}\}$  of mollified functions with associated mollifiers  $\psi_{\varepsilon_n} \in \mathcal{C}^1$ . The lower mollified derivative of  $f$  at  $x$  in the direction  $d \in \mathbb{R}^m$ , w.r.t. the mollifiers sequence  $\psi_{\varepsilon_n}$  is defined as:*

$$\underline{\mathcal{D}}_\psi f(x, d) := \inf_{x_n \rightarrow x} \liminf_{n \rightarrow +\infty} \nabla f_{\varepsilon_n}(x_n)^\top d.$$

In [8] it has been defined also a generalized gradient w.r.t. the mollifiers sequence  $\psi_{\epsilon_n}$ , in the following way:

$$\partial_\psi f(x) := \{L := \text{Lim sup}_{n \rightarrow +\infty} \nabla f_{\epsilon_n}(x_n), x_n \rightarrow x\}$$

i.e. the set of cluster points of all possible sequences  $\{\nabla f_{\epsilon_n}(x_n)\}$  such that  $x_n \rightarrow x$ . Clearly (see e.g. [8]) for the above mentioned upper mollified derivative it holds:

$$\begin{aligned} \overline{\mathcal{D}}_\psi f(x; d) &\geq \sup_{L \in \partial_\psi f(x)} L^\top d, \\ \underline{\mathcal{D}}_\psi f(x; d) &\leq \inf_{L \in \partial_\psi f(x)} L^\top d. \end{aligned}$$

This generalized gradient has been used in [7] and [8] to prove first-order necessary optimality conditions for nonsmooth optimization. The equivalence with the well-known notions of Nonsmooth Analysis is contained in the following proposition:

**Proposition 3 ([8])** *Let  $f : \mathbb{R}^m \rightarrow \mathbb{R}$  be locally Lipschitz at  $x$ ; then  $\partial_\psi f(x)$  coincides with Clarke's generalized gradient and  $\overline{\mathcal{D}}_\psi f(x, d)$  coincides with Clarke's generalized derivative [5].*

**Remark 4** From the previous proposition and the well-known properties of Clarke's generalized gradient, we deduce that, if  $f$  and  $\psi_\varepsilon \in \mathcal{C}^1$ , then  $\partial_\psi f(x) = \nabla f(x)$ .

Properties of these generalized derivatives and their applications to optimization problems are investigated in [7, 8]. By the way, for the aim of our paper, we will need to point out the following proposition (contained in [8]) of which we give an alternative proof.

**Proposition 4** *Let  $f : \mathbb{R}^m \rightarrow \mathbb{R}$  and  $x \in \mathbb{R}^m$ . Then:*

- i)  $\overline{\mathcal{D}}_\psi f(\cdot; d)$  is upper semicontinuous (u.s.c.) at  $x$  for all  $d \in \mathbb{R}^m$ ;*
- ii)  $\underline{\mathcal{D}}_\psi f(\cdot; d)$  is lower semicontinuous (l.s.c.) at  $x$  for all  $d \in \mathbb{R}^m$ .*

*Proof:* We can prove only *i*), since *ii*) follows with the same reasoning. Assume  $d \in \mathbb{R}^m$  is fixed. First we note that the upper semicontinuity is obvious if  $\overline{\mathcal{D}}_\psi f(x; d) = +\infty$ . Otherwise, for all  $K > \overline{\mathcal{D}}_\psi f(x; d)$ , there exists a neighborhood  $U(x)$  and an integer  $n_0$  so that:

$$\nabla f_{\epsilon_n}(x')^\top d < K, \quad \forall n > n_0, \quad \forall x' \in U(x).$$

Therefore, for each  $x' \in U(x)$ , we have:

$$\overline{\mathcal{D}}_\psi f(x'; d) = \sup_{x_n \rightarrow x'} \limsup_{n \rightarrow +\infty} \nabla f_{\epsilon_n}(x_n)^\top d \leq K,$$

which shows that  $\overline{\mathcal{D}}_\psi f(\cdot; d)$  is u.s.c. indeed. □

Furthermore, we point out the following property:

**Proposition 5**  $\overline{\mathcal{D}}_\psi f(x; \cdot)$  and  $\underline{\mathcal{D}}_\psi f(x; \cdot)$  are positively homogeneous functions. Furthermore, if  $\overline{\mathcal{D}}_\psi f(x; \cdot)$  ( $\underline{\mathcal{D}}_\psi f(x; \cdot)$  respectively) is finite then it is subadditive (resp. superadditive) and hence convex (resp. concave) as a function of the direction  $d$ .

*Proof:* The positive homogeneity is trivial. Concerning the second part of the Theorem, we have,  $\forall d_1, d_2 \in \mathbb{R}^m$ :

$$\begin{aligned} \overline{\mathcal{D}}_\psi f(x; d_1 + d_2) &= \sup_{x_n \rightarrow x} \limsup_{n \rightarrow +\infty} \nabla f(x_n)^\top (d_1 + d_2) \leq \\ &\leq \sup_{x_n \rightarrow x} \limsup_{n \rightarrow +\infty} \nabla f(x_n)^\top d_1 + \sup_{x_n \rightarrow x} \limsup_{n \rightarrow +\infty} \nabla f(x_n)^\top d_2 = \\ &= \overline{\mathcal{D}}_\psi f(x; d_1) + \overline{\mathcal{D}}_\psi f(x; d_2), \end{aligned}$$

and hence  $\overline{\mathcal{D}}_\psi f(x; \cdot)$  is subadditive. Convexity follows considering positive homogeneity and subadditivity. The proof for  $\underline{\mathcal{D}}_\psi f(x; \cdot)$  is analogous.  $\square$

### 3 Second-order mollified derivatives

As suggested in [8], by requiring some more regularity of the mollifiers, it is possible to construct also second-order generalized derivatives.

**Definition 7** Let  $f : \mathbb{R}^m \rightarrow \mathbb{R}$ ,  $\epsilon_n \downarrow 0$  and consider the sequence of mollified functions  $\{f_{\epsilon_n}\}$ , obtained from a family of mollifiers  $\psi_{\epsilon_n} \in \mathcal{C}^2$ . We define the second-order upper mollified derivative of  $f$  at  $x$  in the directions  $d$  and  $v \in \mathbb{R}^m$ , w.r.t. to the mollifiers sequence  $\{\psi_{\epsilon_n}\}$ , as:

$$\overline{\mathcal{D}}_\psi^2 f(x; d, v) := \sup_{x_n \rightarrow x} \limsup_{n \rightarrow +\infty} d^\top H f_{\epsilon_n}(x_n) v,$$

where  $H f_{\epsilon_n}(x)$  is the Hessian matrix of the function  $f_{\epsilon_n} \in \mathcal{C}^2$  at the point  $x$ .

**Definition 8** Let  $f : \mathbb{R}^m \rightarrow \mathbb{R}$ ,  $\epsilon_n \downarrow 0$  and consider the sequence of mollified functions  $\{f_{\epsilon_n}\}$ , obtained from a family of mollifiers  $\psi_{\epsilon_n} \in \mathcal{C}^2$ . We define the second-order lower mollified derivative of  $f$  at  $x$  in the directions  $d$  and  $v \in \mathbb{R}^m$ , w.r.t. the mollifiers sequence  $\{\psi_{\epsilon_n}\}$ , as:

$$\underline{\mathcal{D}}_\psi^2 f(x; d, v) := \inf_{x_n \rightarrow x} \liminf_{n \rightarrow +\infty} d^\top H f_{\epsilon_n}(x_n) v.$$

**Proposition 6** Let  $f : \mathbb{R}^m \rightarrow \mathbb{R}$  and  $x \in \mathbb{R}^m$ .

i) If  $\lambda > 0$ , then:

$$\begin{aligned} \overline{\mathcal{D}}_\psi^2 \lambda f(x; d) &= \lambda \overline{\mathcal{D}}_\psi^2 f(x; d); \\ \underline{\mathcal{D}}_\psi^2 \lambda f(x; d) &= \lambda \underline{\mathcal{D}}_\psi^2 f(x; d). \end{aligned}$$

Moreover, if  $\lambda < 0$  we get:

$$\overline{\mathcal{D}}_\psi^2 \lambda f(x; d) = \lambda \underline{\mathcal{D}}_\psi^2 f(x; d).$$

- ii) The maps  $(d, v) \rightarrow \overline{\mathcal{D}}_\psi^2 f(x; d, v)$  and  $(d, v) \rightarrow \underline{\mathcal{D}}_\psi^2 f(x; d, v)$  are symmetric (that is  $\overline{\mathcal{D}}_\psi^2 f(x; d, v) = \overline{\mathcal{D}}_\psi^2 f(x; v, d)$  and  $\underline{\mathcal{D}}_\psi^2 f(x; d, v) = \underline{\mathcal{D}}_\psi^2 f(x; v, d)$ ).
- iii) The functions  $\overline{\mathcal{D}}_\psi^2 f(x; d, \cdot)$  and  $\underline{\mathcal{D}}_\psi^2 f(x; d, \cdot)$  are positively homogeneous, for any fixed  $d \in \mathbb{R}^m$ .
- iv) If  $\overline{\mathcal{D}}_\psi^2 f(x; \cdot, \cdot)$  ( $\underline{\mathcal{D}}_\psi^2 f(x; \cdot, \cdot)$  resp.) is finite, then it is sublinear (superlinear).
- v)  $\overline{\mathcal{D}}_\psi^2 f(x; d, -v) = -\underline{\mathcal{D}}_\psi^2 f(x; d, v)$ .
- vi)  $\overline{\mathcal{D}}_\psi^2 f(\cdot; d, v)$  is upper semicontinuous (u.s.c.) at  $x$  for every  $d, v \in \mathbb{R}^m$ .
- vii)  $\underline{\mathcal{D}}_\psi^2 f(\cdot; d, v)$  is lower semicontinuous (l.s.c.) at  $x$  for every  $d, v \in \mathbb{R}^m$ .

*Proof:* i), ii) and iii) are obvious from the definitions. The proof of iv) is similar to that of Proposition 5.

To prove v), observe that we have:

$$\begin{aligned}
\overline{\mathcal{D}}_\psi^2 f(x; d, -v) &= \sup_{x_n \rightarrow x} \limsup_{n \rightarrow +\infty} -d^\top Hf_{\epsilon_n}(x_n)v = \\
&= \sup_{x_n \rightarrow x} -\liminf_{n \rightarrow +\infty} d^\top Hf_{\epsilon_n}(x_n)v = \\
&= -\inf_{x_n \rightarrow x} \liminf_{n \rightarrow +\infty} d^\top Hf_{\epsilon_n}(x_n)v = \\
&= -\underline{\mathcal{D}}_\psi^2 f(x; d, v).
\end{aligned}$$

The proves of vi) and vii) are analogous to that of Proposition 4. □

In the following we will set for simplicity:

$$\overline{\mathcal{D}}_\psi^2 f(x; d) := \overline{\mathcal{D}}_\psi^2 f(x; d, d)$$

and:

$$\underline{\mathcal{D}}_\psi^2 f(x; d) := \underline{\mathcal{D}}_\psi^2 f(x; d, d).$$

**Remark 5** Clearly the previous derivatives may be infinity. A sufficient condition for these derivatives to be finite is to require  $f \in \mathcal{C}^{1,1}$  (that is once differentiable with locally Lipschitz partial derivatives). In fact, in this case the second-order mollified derivatives can be viewed as first-order mollified derivatives of a locally Lipschitz function and thus Proposition 3 applies.

**Remark 6** It is important to underline that the second-order mollified derivatives are dependent on the specific family of mollifiers which we choose and also on the sequence  $\epsilon_n$ . Practically by changing one of these choices we might obtain a different result for  $\overline{\mathcal{D}}_\psi^2 f(x; d, v)$ . However, the results which follow hold true for any mollifiers sequence (provided they are at least of class  $\mathcal{C}^2$ ) and any choice of  $\epsilon_n$ . Moreover, by Proposition 4.10 in [8], we have that, if  $f \in \mathcal{C}^1$ , then for any choice of the sequence of mollifiers and of  $\epsilon_n$ ,  $\overline{\mathcal{D}}_\psi^2 f(x; d, v)$  coincides with:

$$\limsup_{x' \rightarrow x, t \downarrow 0} \frac{\nabla f(x' + td)^\top v - \nabla f(x')^\top v}{t}.$$

So, when  $f$  is of class  $\mathcal{C}^{1,1}$  at  $x$ ,  $\underline{\mathcal{D}}_\psi^2 f(x; d, v)$  coincides with the derivative introduced in [12] (that is second-order generalized derivative in Clarke's sense). Hence when  $f \in \mathcal{C}^{1,1}$  it is possible to recover results presented in [12] from those which follow.

**Remark 7** One of the main advantages of considering mollified derivatives is that we need not to go through a first-order approximation to get the second-order derivative. Practically we derive both first and second-order generalized derivatives as the limit of two independent well defined sequences of “numbers”.

Using these notions of derivatives, we shall introduce a Taylor’s formula for strongly semicontinuous functions:

**Theorem 4 (Lagrange theorem and Taylor’s formula)** *Let  $f : \mathbb{R}^m \rightarrow \mathbb{R}$  be a s.l.s.c. (resp. s.u.s.c.) function and let  $\epsilon_n \downarrow 0$ ,  $t > 0$ ,  $d$  and  $x \in \mathbb{R}^m$ .*

*i) If  $\psi_{\epsilon_n} \in \mathcal{C}^1$  is a sequence of mollifiers, there exists a point  $\xi \in [x, x + td]$  such that:*

$$\begin{aligned} f(x + td) - f(x) &\leq t\overline{\mathcal{D}}_{\psi}f(\xi; d) \\ (f(x + td) - f(x)) &\geq t\underline{\mathcal{D}}_{\psi}f(\xi; d) \end{aligned} .$$

*ii) If  $\psi_{\epsilon_n} \in \mathcal{C}^2$  is a sequence of mollifiers, there exists  $\xi \in [x, x + td]$  such that:*

$$\begin{aligned} f(x + td) - f(x) &\leq t\overline{\mathcal{D}}_{\psi}f(x; d) + \frac{t^2}{2}\overline{\mathcal{D}}_{\psi}^2f(\xi; d), \\ (f(x + td) - f(x)) &\geq t\underline{\mathcal{D}}_{\psi}f(x; d) + \frac{t^2}{2}\underline{\mathcal{D}}_{\psi}^2f(\xi; d) \end{aligned}$$

*assuming that the righthand sides are well defined, i.e. it does not happen the expression  $+\infty - \infty$ .*

*Proof:* We prove only the second part. The proof of the first part is similar.

For any  $x_n \rightarrow x$ , we can easily write Taylor’s formula for each mollified function:

$$f_{\epsilon_n}(x_n + td) - f_{\epsilon_n}(x_n) = t\nabla f_{\epsilon_n}(x_n)^{\top}d + \frac{t^2}{2}d^{\top}Hf_{\epsilon_n}(\xi_n)d$$

where  $\xi_n \in (x_n, x_n + td)$ . Without loss of generality, we can think that  $\xi_n \rightarrow \xi \in [x, x + td]$ . Now, we consider the lim sup as  $n \rightarrow +\infty$  and the definition of  $\overline{\mathcal{D}}_{\psi}f(x; d)$  and  $\overline{\mathcal{D}}_{\psi}^2f(x; d)$  to get:

$$\begin{aligned} \limsup_{n \rightarrow +\infty} f_{\epsilon_n}(x_n + td) - \limsup_{n \rightarrow +\infty} f_{\epsilon_n}(x_n) &\leq \limsup_{n \rightarrow +\infty} [f_{\epsilon_n}(x_n + td) - f_{\epsilon_n}(x_n)] \leq \\ &\leq t \limsup_{n \rightarrow +\infty} \nabla f_{\epsilon_n}(x_n)^{\top}d + \frac{t^2}{2} \limsup_{n \rightarrow +\infty} d^{\top}Hf_{\epsilon_n}(\xi_n)d \leq t\overline{\mathcal{D}}_{\psi}f(x; d) + \frac{t^2}{2}\overline{\mathcal{D}}_{\psi}^2f(\xi; d). \end{aligned}$$

By the strong lower semicontinuity assumed on  $f$ , there exists a sequence  $y_n \rightarrow x$  such that:

$$\lim_{n \rightarrow +\infty} f_{\epsilon_n}(y_n) = f(x).$$

Thus, recalling Theorem 2, we have, for this sequence:

$$\begin{aligned} \limsup_{n \rightarrow +\infty} f_{\epsilon_n}(y_n + td) - \limsup_{n \rightarrow +\infty} f_{\epsilon_n}(y_n) &= \limsup_{n \rightarrow +\infty} f_{\epsilon_n}(y_n + td) - \lim_{n \rightarrow +\infty} f_{\epsilon_n}(y_n) \geq \\ \liminf_{n \rightarrow +\infty} f_{\epsilon_n}(y_n + td) - \lim_{n \rightarrow +\infty} f_{\epsilon_n}(y_n) &\geq f(x + td) - f(x), \end{aligned}$$

from which the thesis follows.

The other formula follows in a similar way, recalling Theorem 3 instead of Theorem 2.  $\square$

It should be clear that, for both semicontinuity of the generalized derivatives and Taylor’s formula, we need some conditions to avoid “triviality” of the derivatives, such as  $f$  locally Lipschitz so that, as already seen, the first-order mollified derivative is finite, since it coincides with Clarke’s derivative.

## 4 Unconstrained Optimization

In this section we wish to give second-order necessary and sufficient conditions for unconstrained optimization problems of the form:

$$P_1) \quad \min_{x \in \Omega} f(x)$$

where  $f : \mathbb{R}^m \rightarrow \mathbb{R}$  and  $\Omega \subseteq \mathbb{R}^m$  is an open set.

Some first-order necessary optimality conditions have already been studied in [8], under very weak hypotheses on  $f$ .

**Theorem 5 ([8])** *Let  $f : \mathbb{R}^m \rightarrow \mathbb{R}$  be s.l.s.c. and assume that  $x_0$  is a local solution of problem  $P_1$ ). Then for any sequence of mollifiers, we have:*

$$0 \in \partial_\psi f(x_0).$$

**Theorem 6** *Let  $x_0$  be a local solution of problem  $P_1$ ) with  $f : \mathbb{R}^m \rightarrow \mathbb{R}$  s.l.s.c.. Then the following conditions hold:*

- i)  $\overline{\mathcal{D}}_\psi f(x_0; d) \geq 0, \quad \forall d \in \mathbb{R}^m .$
- ii)  $\overline{\mathcal{D}}_\psi^2 f(x_0; d) \geq 0, \quad \forall d \in \mathbb{R}^m \quad \text{such that} \quad \overline{\mathcal{D}}_\psi f(x_0; d) = 0 .$

*Proof:*

- i) By Theorem 5 we know  $0 \in \partial_\psi f(x_0)$ . Thus, by definition:

$$\overline{\mathcal{D}}_\psi f(x_0; d) \geq 0.$$

- ii) Let  $d$  be such that  $\overline{\mathcal{D}}_\psi f(x_0; d) = 0$  and apply Taylor's formula to get:

$$f(x_0 + td) - f(x_0) \leq \frac{t^2}{2} \overline{\mathcal{D}}_\psi^2 f(\xi; d),$$

for  $t > 0$ ,  $d \in \mathbb{R}^m$  and  $\xi \in [x_0, x_0 + td]$ . For  $t$  "small enough", since  $x_0$  is a local minimizer we obtain  $f(x_0 + td) - f(x_0) \geq 0$  and hence, using the upper semicontinuity of  $\overline{\mathcal{D}}_\psi^2 f(\cdot; d)$ :

$$0 \leq \limsup_{t \downarrow 0} 2 \frac{f(x_0 + td) - f(x_0)}{t^2} \leq \limsup_{t \downarrow 0} \overline{\mathcal{D}}_\psi^2 f(\xi; d) \leq$$

$$\limsup_{x \rightarrow x_0} \overline{\mathcal{D}}_\psi^2 f(x; d) \leq \overline{\mathcal{D}}_\psi^2 f(x_0; d),$$

so that condition ii) is proved. □

**Theorem 7** Let  $f : \mathbb{R}^m \rightarrow \mathbb{R}$  be s.u.s.c.. Moreover, assume that at  $x_0$  the following conditions hold for any  $d \in S^1$  (the unit sphere in  $\mathbb{R}^m$ ):

i)  $\underline{\mathcal{D}}_\psi f(x_0; d) > 0$  implies that there exist a real number  $\alpha(d) > 0$  and a neighborhood of the direction  $d$ ,  $U(d)$  such that:

$$\underline{\mathcal{D}}_\psi f(x_0 + td'; d') > 0, \quad \forall t \in (0, \alpha(d)), \quad \forall d' \in U(d).$$

ii)  $\underline{\mathcal{D}}_\psi f(x_0; d) = 0$  implies that there exist a real number  $\alpha(d) > 0$  and a neighborhood of  $d$ ,  $U(d)$ , such that  $\underline{\mathcal{D}}_\psi f(x_0; d') \geq 0$  and  $\underline{\mathcal{D}}_\psi^2 f(x_0 + td'; d') > 0$ ,  $\forall t \in (0, \alpha(d))$  and  $\forall d' \in U(d)$ .

Then  $x_0$  is a (strict) local solution of  $P_1$ .

*Proof:* By contradiction, assume that there exists  $x_n$  such that:  $f(x_n) - f(x_0) \leq 0$ . It can be easily written  $x_n = x_0 + t_n d_n$ ,  $d_n \in S^1$ ,  $d_n \rightarrow d \in S^1$ ,  $t_n \downarrow 0$ .

i) If  $\underline{\mathcal{D}}_\psi f(x_0; d) > 0$ , then:

$$0 \geq f(x_0 + t_n d_n) - f(x_0) \geq t_n \underline{\mathcal{D}}_\psi f(\xi_n; d_n),$$

where  $\xi_n \in [x_0, x_0 + t_n d_n]$ . Which contradict the hypothesis.

ii) If  $\underline{\mathcal{D}}_\psi f(x_0; d) = 0$ , then we have the following contradiction:

$$0 \geq f(x_0 + t_n d_n) - f(x_0) \geq t_n \underline{\mathcal{D}}_\psi f(x_0; d_n) + \frac{t_n^2}{2} \underline{\mathcal{D}}_\psi^2 f(\xi_n; d_n) > 0,$$

for  $n$  sufficiently large.

□

## 5 Constrained Optimization

In this section we give second-order necessary and sufficient optimality conditions for constrained optimization problems. We begin considering the following problem:

$$P_2) \quad \min_{x \in K} f(x)$$

where  $K \subseteq \mathbb{R}^m$ .

**Definition 9** The cone of feasible directions of the set  $K$  at  $x$  is given by:

$$F(K, x) := \{d \in \mathbb{R}^m \mid \exists \alpha > 0 : \forall t \in [0, \alpha], x + td \in K\}.$$

**Definition 10** The set:

$$T(K, x_0) := \{d \in \mathbb{R}^m \mid \exists \{d_n\} \rightarrow d, \exists \{t_n\} \downarrow 0 : x_0 + t_n d_n \in K\}$$

is called the Bouligand tangent cone to the set  $K$  at  $x$ .

Clearly we have the following inclusion:

$$F(K, x) \subseteq T(K, x).$$

**Theorem 8** *Let  $f : \mathbb{R}^m \rightarrow \mathbb{R}$  be s.l.s.c.. If  $x_0 \in K$  is a local solution of problem  $P_2$ ), then:*

- i)  $\overline{\mathcal{D}}_\psi f(x_0; d) \geq 0, \forall d \in F(K, x_0),$*
- ii)  $\overline{\mathcal{D}}_\psi^2 f(x_0; d) \geq 0, \forall d \in F(K, x_0)$  such that  $\overline{\mathcal{D}}_\psi f(x_0; d) = 0.$*

*Proof:* Let  $x_0$  be a local minimizer of  $f$  over  $K$ . For  $t$  “sufficiently small” and  $d \in F(K, x_0)$ , we have:

$$0 \leq \frac{f(x_0 + td) - f(x_0)}{t} \leq \overline{\mathcal{D}}_\psi f(\xi; d),$$

for some  $\xi \in [x_0, x_0 + td]$ . Taking lim sup as  $t \downarrow 0$  of both members and recalling the upper semicontinuity of  $\overline{\mathcal{D}}_\psi f(\cdot; d)$ , we obtain:

$$0 \leq \limsup_{t \downarrow 0} \overline{\mathcal{D}}_\psi f(\xi; d) \leq \overline{\mathcal{D}}_\psi f(x_0; d).$$

To prove condition *ii)*, let  $d \in F(K, x_0)$  be such that  $\overline{\mathcal{D}}_\psi f(x_0; d) = 0$ . Using Theorem 4, we have, for  $t > 0$  “sufficiently small”:

$$0 \leq f(x_0 + td) - f(x_0) \leq +\frac{t^2}{2} \overline{\mathcal{D}}_\psi^2 f(\xi; d),$$

where  $\xi \in [x_0, x_0 + td]$ . Dividing by  $t^2$ , taking lim sup for  $t \downarrow 0$  and using the upper semicontinuity of  $\overline{\mathcal{D}}_\psi^2 f(\cdot; d)$ , the thesis follows.  $\square$

**Theorem 9** *Let  $f : \mathbb{R}^m \rightarrow \mathbb{R}$  be s.u.s.c.,  $x_0 \in K$  and assume that  $\forall d \in T(K, x_0) \cap S^1$  one of the following conditions holds:*

- i) if  $\underline{\mathcal{D}}_\psi f(x_0; d) > 0$ , then there exist a real number  $\alpha(d) > 0$  and a neighborhood of the direction  $d$ , say  $U(d)$  such that:*

$$\underline{\mathcal{D}}_\psi f(x_0 + td'; d') > 0, \quad \forall t \in (0, \alpha(d)), \quad \forall d' \in U(d) \cap S^1;$$

- ii) if  $\underline{\mathcal{D}}_\psi f(x_0; d) = 0$ , then there exist a real number  $\alpha(d) > 0$  and a neighborhood of the direction  $d$ , say  $U(d)$ , such that, for each  $t \in (0, \alpha(d))$  and for each  $d' \in U(d) \cap S^1$  we have  $\underline{\mathcal{D}}_\psi f(x_0; d') \geq 0$  and  $\underline{\mathcal{D}}_\psi^2 f(x_0 + td'; d') > 0.$*

*Then  $x_0$  is a local solution of problem  $P_2$ ).*

*Proof:* Ab assurdo, let assume there exists a feasible sequence  $\{x_n\} \rightarrow x_0$  such that  $f(x_n) < f(x_0)$ . It can be easily written, without loss of generality  $x_n = x_0 + t_n d_n$ ,  $d_n \in S^1$ ,  $d_n \rightarrow d \in S^1$ ,  $t_n \downarrow 0$ , and hence  $d \in T(K, x_0)$ .

i) If  $\underline{\mathcal{D}}_\psi f(x_0; d) > 0$ , then, as  $n \rightarrow +\infty$ :

$$0 \geq f(x_0 + t_n d_n) - f(x_0) \geq t_n \underline{\mathcal{D}}_\psi f(\xi_n; d_n),$$

with  $\xi_n \in [x_0, x_0 + t_n d_n]$ , which is trivially a contradiction.

ii) If  $\underline{\mathcal{D}}_\psi f(x_0; d) = 0$ , then:

$$0 \geq f(x_0 + t_n d_n) - f(x_0) \geq t_n \underline{\mathcal{D}}_\psi f(x_0; d_n) + \frac{t_n^2}{2} \underline{\mathcal{D}}_\psi^2 f(\xi_n; d_n),$$

with  $\xi_n \in [x_0, x_0 + t_n d_n]$ , which is again a contradiction.

□

Now we deal with the following constrained optimization problem:

$$\begin{aligned} P_3) \quad & \min f(x) \\ & \text{s.t.} \\ & g_i(x) \leq 0, \quad i = 1, \dots, r \end{aligned}$$

where  $f, g_i : \mathbb{R}^m \rightarrow \mathbb{R}$ . We will define the set of active constraints at a point  $x_0$  as the index set  $I(x_0) : \{i = 1, \dots, r : g_i(x_0) = 0\}$  and the feasible set as  $\Gamma := \{x \in \mathbb{R}^m : g_i(x) \leq 0, i = 1, \dots, r\}$ .

Concerning this problem, we will first investigate first-order conditions expressed by means of mollified derivatives.

**Lemma 1 (Generalized Abadie Lemma)** *Let  $f$  and  $g_i, i \in I(x_0)$  be s.l.s.c.,  $g_i, i \notin I(x_0)$  be u.s.c. and assume that  $x_0 \in \Gamma$  is a local solution of problem  $P_3$ ).*

*Then  $\nexists d \in \mathbb{R}^m$  such that:*

$$\begin{cases} \overline{\mathcal{D}}_\psi f(x_0; d) < 0 \\ \overline{\mathcal{D}}_\psi g_i(x_0; d) < 0, \quad i \in I(x_0) \end{cases}$$

*Proof:* Since  $x_0$  is a local solution of  $P_3$ ), we can easily check that,  $\forall d \in \mathbb{R}^m$ ,  $\nexists \alpha(d) > 0$  such that  $\forall t \in (0, \alpha(d))$ :

$$\begin{cases} \overline{\mathcal{D}}_\psi f(x_0 + td; d) < 0 \\ \overline{\mathcal{D}}_\psi g_i(x_0 + td; d) < 0, \quad i \in I(x_0) \end{cases}$$

Infact, if for some  $d$  such an  $\alpha(d)$  would exist, from Theorem 4 we would get,  $\forall t \in (0, \alpha(d))$ :

$$f(x_0 + td) < f(x_0)$$

and

$$g_i(x_0 + td) < 0, \quad i \in I(x_0).$$

Since  $g_i, i \notin I(x_0)$  are u.s.c. we obtain also, for  $t$  “small enough”,  $g_i(x_0 + td) < 0, i \notin I(x_0)$ . This fact contradicts that  $x_0 \in \Gamma$  is a local solution of  $P_3$ ).

Hence, for any fixed  $d \in \mathbb{R}^m$  one can find a sequence  $t_n \downarrow 0$  such that for all  $n$  it holds or  $\overline{\mathcal{D}}_\psi f(x_0 + t_n d; d) \geq 0$  either  $\overline{\mathcal{D}}_\psi g_{\bar{i}}(x_0 + t_n d; d) \geq 0$ , for some fixed  $\bar{i} \in I(x_0)$ . Recalling that the first-order upper mollified derivative is u.s.c. we obtain that either  $\overline{\mathcal{D}}_\psi f(x_0; d) \geq 0$  or  $\overline{\mathcal{D}}_\psi g_{\bar{i}}(x_0; d) \geq 0$  and hence we get the thesis. □

**Theorem 10 (Generalized F. John Conditions)** *Let  $f, g_i, i \in I(x_0)$  be s.l.s.c. and  $g_i, i \notin I(x_0)$  be u.s.c.. Assume that  $x_0 \in \Gamma$  is a local solution of problem  $P_3$  and that  $\overline{\mathcal{D}}_\psi f(x_0; \cdot)$  and  $\overline{\mathcal{D}}_\psi g_i(x_0; \cdot), i \in I(x_0)$  are finite. Then there exist scalars  $\tau \geq 0, \lambda_i \geq 0, i \in I(x_0)$ , not all zero, such that:*

$$\tau \overline{\mathcal{D}}_\psi f(x_0; d) + \sum_{i \in I(x_0)} \lambda_i \overline{\mathcal{D}}_\psi g_i(x_0; d) \geq 0, \quad \forall d \in \mathbb{R}^m. \quad (1)$$

*Proof:* From the previous Lemma we know that the system:

$$\begin{cases} \overline{\mathcal{D}}_\psi f(x_0; d) < 0 \\ \overline{\mathcal{D}}_\psi g_i(x_0; d) < 0 \quad i \in I(x_0) \end{cases}$$

has no solution. Since the first-order upper mollified derivatives are convex (Proposition 5), from a well known Theorem of the alternative ([3] Theorem 7.1.2), we obtain the thesis.  $\square$

**Remark 8** *Of course a relevant question is which conditions would ensure  $\tau > 0$  (or equivalently  $\tau = 1$ ) in formula (1). It can be easily seen that this is the case if the following generalized Slater-type constraint qualification condition holds:*

$$\exists \bar{d} \in \mathbb{R}^m \quad \text{such that} \quad \overline{\mathcal{D}}_\psi g_i(x_0; \bar{d}) < 0, \quad i \in I(x_0).$$

Now we prove necessary and sufficient second-order optimality conditions for problem  $P_3$ .

**Theorem 11** *Let  $f, g_i, i \in I(x_0)$  be s.l.s.c.,  $g_i, i \notin I(x_0)$  be u.s.c. and assume that  $x_0 \in \Gamma$  is a local solution of problem  $P_3$ . Moreover assume that  $\overline{\mathcal{D}}_\psi f(x_0; \cdot)$  and  $\overline{\mathcal{D}}_\psi g_i(x_0; \cdot), i \in I(x_0)$  are finite.*

*Then, if  $\tau \geq 0, \lambda_i \geq 0, i \in I(x_0)$  satisfy (1), the following condition holds:*

$$\begin{aligned} \text{if } d \in F(\Gamma(\lambda), x_0) \text{ is such that } \tau \overline{\mathcal{D}}_\psi f(x_0; d) + \sum_{i \in I(x_0)} \lambda_i \overline{\mathcal{D}}_\psi g_i(x_0; d) = 0 \\ \text{then} \quad \tau \overline{\mathcal{D}}_\psi^2 f(x_0; d) + \sum_{i \in I(x_0)} \lambda_i \overline{\mathcal{D}}_\psi^2 g_i(x_0; d) \geq 0 \end{aligned} \quad (2)$$

where  $\Gamma(\lambda) = \{x \in \Gamma \mid \sum_{i \in I(x_0)} \lambda_i g_i(x) = 0\}$ .

*Proof:* Let  $d \in F(\Gamma(\lambda), x_0)$  be such that  $\tau \overline{\mathcal{D}}_\psi f(x_0; d) + \sum_{i \in I(x_0)} \lambda_i \overline{\mathcal{D}}_\psi g_i(x_0; d) = 0$  and observe that, since  $\overline{\mathcal{D}}_\psi f(x_0; \cdot)$  and  $\overline{\mathcal{D}}_\psi g_i(x_0; \cdot), i \in I(x_0)$  are finite, we can write, for  $t > 0$ :

$$\begin{aligned} f(x_0 + td) - f(x_0) &\leq t \overline{\mathcal{D}}_\psi f(x_0; d) + \frac{t^2}{2} \overline{\mathcal{D}}_\psi^2 f(\xi; d) \\ g_i(x_0 + td) - g_i(x_0) &\leq t \overline{\mathcal{D}}_\psi g_i(x_0; d) + \frac{t^2}{2} \overline{\mathcal{D}}_\psi^2 g_i(\xi_i; d), \quad i \in I(x_0) \end{aligned}$$

where  $\xi, \xi_i \in [x_0, x_0 + td]$ .

Hence we have:

$$\tau f(x_0 + td) + \sum_{i \in I(x_0)} \lambda_i g_i(x_0 + td) - \tau f(x_0) - \sum_{i \in I(x_0)} \lambda_i g_i(x_0) \leq$$

$$\leq \frac{t^2}{2} [\tau \bar{\mathcal{D}}_\psi^2 f(\xi; d) + \sum_{i \in I(x_0)} \lambda_i \bar{\mathcal{D}}_\psi^2 g_i(\xi_i; d)].$$

For  $t$  “small enough”, the lefthandside is nonnegative and hence, using the upper semicontinuity of second-order mollified derivatives:

$$\begin{aligned} 0 &\leq \limsup_{t \downarrow 0} [\tau \bar{\mathcal{D}}_\psi^2 f(\xi; d) + \sum_{i \in I(x_0)} \lambda_i \bar{\mathcal{D}}_\psi^2 g_i(\xi_i; d)] \leq \\ &\leq \tau \limsup_{t \downarrow 0} \bar{\mathcal{D}}_\psi^2 f(\xi; d) + \sum_{i \in I(x_0)} \lambda_i \limsup_{t \downarrow 0} \bar{\mathcal{D}}_\psi^2 g_i(\xi_i; d) \leq \\ &\leq \tau \bar{\mathcal{D}}_\psi^2 f(x_0; d) + \sum_{i \in I(x_0)} \lambda_i \bar{\mathcal{D}}_\psi^2 g_i(x_0; d), \end{aligned}$$

and so we get the thesis.  $\square$

**Theorem 12** *Let  $f, g_i, i \in I(x_0)$  be s.u.s.c. and  $x_0 \in \Gamma$ . Moreover, assume there exist scalars  $\lambda_i \geq 0, i \in I(x_0)$  such that  $\forall d \in T(\Gamma, x_0) \cap S^1$  one of the following conditions holds:*

i) *If  $\underline{\mathcal{D}}_\psi f(x_0; d) + \sum_{i \in I(x_0)} \lambda_i \underline{\mathcal{D}}_\psi g_i(x_0; d) > 0$ , then there exist a real  $\alpha(d) > 0$  and a neighborhood of the direction  $d, U(d)$ , so that:*

$$\underline{\mathcal{D}}_\psi f(x_0 + td'; d') + \sum_{i \in I(x_0)} \lambda_i \underline{\mathcal{D}}_\psi g_i(x_0 + td'; d') > 0 \quad \forall t \in (0, \alpha(d)), \forall d' \in U(d).$$

ii) *If  $\underline{\mathcal{D}}_\psi f(x_0; d) + \sum_{i \in I(x_0)} \lambda_i \underline{\mathcal{D}}_\psi g_i(x_0; d) = 0$ , then there exist a real  $\alpha(d) > 0$  and a neighborhood of the direction  $d, U(d)$ , so that:*

$$\underline{\mathcal{D}}_\psi^2 f(x_0 + td'; d') + \sum_{i \in I(x_0)} \lambda_i \underline{\mathcal{D}}_\psi^2 g_i(x_0 + td'; d') > 0 \quad \forall t \in (0, \alpha(d)), \forall d' \in U(d).$$

Then  $x_0$  is a (strict) local solution of  $P_3$ ).

*Proof:* By contradiction assume there exists a feasible sequence  $\{x_n\} \rightarrow x_0$  so that  $f(x_n) - f(x_0) \leq 0$ . We shall write  $x_n = x_0 + t_n d_n$  for some  $d_n \rightarrow d \in T(\Gamma, x_0) \cap S^1$ .

i) If  $\underline{\mathcal{D}}_\psi f(x_0; d) + \sum_{i \in I(x_0)} \lambda_i \underline{\mathcal{D}}_\psi g_i(x_0; d) > 0$ , then we would have:

$$\begin{aligned} f(x_0 + t_n d_n) - f(x_0) &\geq t_n \underline{\mathcal{D}}_\psi f(\xi_n; d_n) \\ \text{and } g_i(x_0 + t_n d_n) - g_i(x_0) &\geq t_n \underline{\mathcal{D}}_\psi g_i(\xi_n^i; d_n), \quad i \in I(x_0) \end{aligned}$$

where  $\xi_n, \xi_n^i \in [x_0, x_0 + t_n d_n]$ . Using multipliers  $\lambda_i$ , we get:

$$\begin{aligned} 0 &\geq f(x_0 + t_n d_n) + \sum_{i \in I(x_0)} \lambda_i g_i(x_0 + t_n d_n) - f(x_0) - \sum_{i \in I(x_0)} \lambda_i g_i(x_0) \geq \\ &\geq t_n \underline{\mathcal{D}}_\psi f(\xi_n; d_n) + t_n \sum_{i \in I(x_0)} \lambda_i \underline{\mathcal{D}}_\psi g_i(\xi_n^i; d_n) \end{aligned}$$

which contradict the hypothesis for  $n$  large enough.

ii) If  $\underline{\mathcal{D}}_\psi f(x_0; d) + \sum_{i \in I(x_0)} \lambda_i \underline{\mathcal{D}}_\psi g_i(x_0; d) = 0$ , then we shall write:

$$f(x_0 + t_n d_n) - f(x_0) \geq t_n \underline{\mathcal{D}}_\psi f(x_0; d_n) + \frac{t_n^2}{2} \underline{\mathcal{D}}_\psi^2 f(\xi_n; d_n)$$

$$\text{and } g_i(x_0 + t_n d_n) - g_i(x_0) \geq t_n \underline{\mathcal{D}}_\psi g_i(x_0; d_n) + \frac{t_n^2}{2} \underline{\mathcal{D}}_\psi^2 g_i(\xi_n^i; d_n), \quad i \in I(x_0)$$

where  $\xi_n, \xi_n^i \in [x_0, x_0 + t_n d_n]$ . Using multipliers  $\lambda_i$  and the assumption, we get:

$$0 \geq f(x_0 + t_n d_n) + \sum_{i \in I(x_0)} \lambda_i g_i(x_0 + t_n d_n) - f(x_0) - \sum_{i \in I(x_0)} \lambda_i g_i(x_0) \geq$$

$$\geq \frac{t_n^2}{2} \underline{\mathcal{D}}_\psi^2 f(\xi_n; d_n) + \frac{t_n^2}{2} \sum_{i \in I(x_0)} \lambda_i \underline{\mathcal{D}}_\psi^2 g_i(\xi_n^i; d_n)$$

which contradict again the hypothesis for  $n$  large enough. □

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