EIGENVALUES AND EQUIVALENT TRANSFORMATION OF A TRIGONOMETRIC MATRIX ASSOCIATED WITH FILTER DESIGN

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Abstract. The $N \times N$ trigonometric matrix $P(\omega)$ whose entries are $P(\omega)(i,j) = \frac{1}{2}(i+j-2)\cos(i-j)\omega$ appears in connection with the design of finite impulse response (FIR) digital filters with real coefficients. We prove several results about its eigenvalues; in particular, assuming $N \geq 4$ we prove that $P(\omega)$ has one positive and one negative eigenvalue when $\frac{\omega}{\pi}$ is an integer, while it has two positive and two negative eigenvalues when $\frac{\omega}{\pi}$ is not an integer. We also show that for $\frac{\omega}{\pi}$ not being an integer and a sufficiently large N, the two positive eigenvalues converge to $\alpha_+ N^2$ and the two negative eigenvalues to $\alpha_- N^2$, where $\alpha_\pm = (1 \pm 2/\sqrt{3})/8$. Furthermore, an equivalent transformation diagonalizing $P(\omega)$ is described.

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1. Introduction

Trigonometric matrices are widely used in various applications, such as image processing [3], communication systems [7], filter design [6, 8, 9, 11], etc. In filter design, trigonometric matrices arise in the formulation of certain design problems, such as the design of finite impulse response (FIR) filters with low group delays and arbitrarily prescribed magnitude [6, 8, 9, 11]. In the design of FIR filters with complex coefficients [8, 11], an eigenvalue problem of trigonometric matrices associated with the reduction of the group delay of an FIR filter was posed in [8] and investigated in [10]. In the design of FIR filters with real coefficients, the group delay of an FIR filter to be designed is also associated with a trigonometric matrix [6, 9]. Hence, it is of interest, both mathematically and practically, to investigate the eigenvalue problem of the trigonometric matrix associated with an FIR filter having real coefficients.

To formulate the problem and provide some relevant background, let

$$H(z) := \sum_{n=0}^{N-1} h(n) z^{-n}$$

be the transfer function of an FIR filter of length N and with real coefficients. Note that $H(z)$ is the z-transform of the unit impulse response of the filter $h(n)$. The frequency response $H(\omega)$, phase response $\phi(\omega)$ and group delay $\tau(\omega)$ of the filter $H(z)$ are given by

$$H(\omega) = \sum_{n=0}^{N-1} h(n) e^{-j\omega n} = \mathbf{h}_x^\mathsf{T}(\mathbf{c}(\omega) + j\mathbf{s}(\omega)),$$

$$\phi(\omega) := \tan^{-1}\left(\frac{\mathbf{h}_x^\mathsf{T}\mathbf{s}(\omega)}{\mathbf{h}_x^\mathsf{T}\mathbf{c}(\omega)}\right) \pm \pi, \qquad \tau(\omega) := -\frac{d}{d\omega}\phi(\omega) = -\frac{d}{d\omega}\tan^{-1}\left(\frac{\mathbf{h}_x^\mathsf{T}\mathbf{s}(\omega)}{\mathbf{h}_x^\mathsf{T}\mathbf{c}(\omega)}\right),$$

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respectively, where $\omega \in \mathbb{R}$ is the digital frequency variable and

$$\mathbf{h}_x := \begin{bmatrix} h(0) & h(1) & \dots & h(N-1) \end{bmatrix}^{\mathsf{T}},$$
$$\mathbf{c}(\omega) := \begin{bmatrix} 1 & \cos\omega & \dots & \cos(N-1)\omega \end{bmatrix}^{\mathsf{T}},$$
$$\mathbf{s}(\omega) := \begin{bmatrix} 0 & -\sin\omega & \dots & -\sin(N-1)\omega \end{bmatrix}^{\mathsf{T}}.$$

Let

$$\tilde{\mathbf{c}}(\omega) := \frac{d\mathbf{c}(\omega)}{d\omega} = \begin{bmatrix} 0 & -\sin\omega & \dots & -(N-1)\sin(N-1)\omega \end{bmatrix}^{\mathsf{T}},$$
$$\tilde{\mathbf{s}}(\omega) := \frac{d\mathbf{s}(\omega)}{d\omega} = \begin{bmatrix} 0 & -\cos\omega & \dots & -(N-1)\cos(N-1)\omega \end{bmatrix}^{\mathsf{T}}.$$

With simple manipulations, we arrive at the following analytic expression for the group delay

$$\tau(\omega) = \frac{\mathbf{h}_x^{\mathsf{T}} P_1(\omega)\mathbf{h}_x}{|H(\omega)|^2},$$

where

$$P_1(\omega) := \mathbf{s}(\omega)\tilde{\mathbf{c}}(\omega)^{\mathsf{T}} - \tilde{\mathbf{s}}(\omega)\mathbf{c}(\omega)^{\mathsf{T}}.$$

The above derivation follows easily from [8] by restricting the discussion in [8] to the case with real filter coefficients only. It could also be found in [9] but with slightly different notation. For band-selective filters, it may be assumed that $|H(\omega)| \approx 1$ in the passbands. Furthermore, when using the semidefinite programming (SDP) approach [8] or the second-order cone programming (SOCP) approach [11], $P_1(\omega)$ is required to be symmetric, which could be done by introducing a new symmetric matrix $P := \frac{1}{2}(P_1(\omega) + P_1^T(\omega))$. Hence, the group delay of the filter in the passbands is approximately given by

$$\tau(\omega) \approx \mathbf{h}_x^{\mathsf{T}} P(\omega)\mathbf{h}_x$$

where $P(\omega)$ is of dimension $N \times N$ and is expressed as

$$P(\omega)(i,j) := \frac{1}{2}(i+j-2)\cos(i-j)\omega. \tag{1}$$

In order to design FIR filters with reduced group delays, i.e., to minimize $\tau(\omega)$ in the passbands, it is important to understand the structure and eigenvalues of $P(\omega)$. In particular, in the case that $P(\omega)$ is not a positive definite matrix, it is required that the positive eigenvalues of $P(\omega)$ are sufficiently larger than the absolute values of the negative eigenvalues for the optimization techniques adopted in [8, 11] to be effective. In [10], the eigenvalue problem related to FIR filters with complex coefficients was discussed. Here we focus on the same eigenvalue problem but for FIR filters with real coefficients. Although the $P(\omega)$ matrix here already appears as one of the block sub-matrices of the matrix in [10], their eigenvalues are quite different. Specifically, while the eigenvalues of the matrix in [10] are independent of ω, those of $P(\omega)$ depend on ω in a quite peculiar way, as we will show. In fact, we prove that for $N \geq 4$, $P(\omega)$ has one positive and one negative eigenvalue when $\frac{\omega}{\pi}$ is an integer, and two positive and two negative eigenvalues when $\frac{\omega}{\pi}$ is not an integer. We also give an asymptotic property of the eigenvalues of $P(\omega)$ by showing that for $\frac{\omega}{\pi}$ not being an integer and large enough N, the two positive eigenvalues are close to $\alpha_+ N^2$

and the two negative eigenvalues to $\alpha_- N^2$, where $\alpha_\pm = (1 \pm 2/\sqrt{3})/8$. We prove also a result on an equivalent transformation of $P(\omega)$ into a diagonal matrix.

Before ending this section, we list the notation we use in the paper:

- $\mathbf{0}_{m,l}$, $\mathbf{I}_n$: the $m \times l$ zero and the $n \times n$ identity matrices;
- $\|x\|$: the minimal distance of x to $\mathbb{Z}$, i.e. $\min\{|x-n| \colon n \in \mathbb{Z}\}$;
- $[n]_a$, $\left\{ {m \atop n} \right\}$: the falling factorial symbol and the Stirling number of the second kind (see [1, Ch. III]);
- $\delta(l)$: the discrete function whose value at $l = 0$ is one, 0 otherwise;
- $\overline{O}(f(x))$: a function $g(x)$ satisfying the inequality $|g(x)| \leq |f(x)|$.

2. Main results

In this section, we first present new results on the eigenvalues and an equivalent transformation of $P(\omega)$ in (1) for any N, then another result on the eigenvalues of $P(\omega)$ for a sufficiently large N.

Theorem 1. *For every $N \geq 4$, we have*

1): *When $\frac{\omega}{\pi}$ is an integer $P(\omega)$ has one positive eigenvalue $\tilde{\lambda}_+$ and one negative eigenvalue $\tilde{\lambda}_-$ whose values are $\frac{N}{4}\big(N - 1 \pm \sqrt{\frac{4N^2-6N+2}{3}}\big)$; the other eigenvalues are zero.*

2): *When $\frac{\omega}{\pi}$ is not an integer $P(\omega)$ has two positive eigenvalues $\lambda_{+,1}(\omega)$, $\lambda_{+,2}(\omega)$ and two negative eigenvalues $\lambda_{-,1}(\omega)$, $\lambda_{-,2}(\omega)$; the other eigenvalues are zero.*

We have not been able to discover the general analytic form of a trigonometric matrix $A(\omega)$ such that $A(\omega)P(\omega)A^{-1}(\omega)$ is diagonal, but we have found a matrix $A(\omega)$ such that $A(\omega)P(\omega)A^{\mathsf{T}}(\omega)$ is diagonal (see Thm. 2 here below). This suffices to prove the second part of Theorem 1 as a consequence of the Sylvester's law of inertia for symmetric matrices.

Theorem 2. *For $N \geq 4$, there exists a trigonometric matrix $A(\omega)$ with $\det A(\omega) = -1$, such that*

$$A(\omega)P(\omega)A^{\mathsf{T}}(\omega) = D(\omega) \tag{2}$$

where $D(\omega) := \mathrm{diag}\{1, -1, \sin^4\omega, -\sin^4\omega, 0, \dots, 0\}$ has dimension N.

Proof. We prove that a suitable matrix $A(\omega)$ is given as

$$A(\omega) := \begin{bmatrix} A_4(\omega) & \mathbf{0}_{4,N-4} \\ \multicolumn{2}{c}{F(\omega)} \end{bmatrix} \tag{3}$$

where

$$A_4(\omega) := \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{9}{4} & -\frac{3}{2}\cos\omega & \frac{1}{4} & 0 \\ \frac{9}{4}\cos^2\omega - \frac{7}{4} & -\frac{3}{2}\cos^3\omega + \frac{1}{2}\cos\omega & \frac{1}{4}\cos^2\omega + \frac{1}{4} & 0 \\ -\frac{3}{2}\cos\omega & 2\cos^2\omega + 2 & -\frac{7}{2}\cos\omega & 1 \end{bmatrix}$$

$$F(\omega) := \begin{bmatrix} 1 & u & v & u & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & u & v & u & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & u & v & u & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 1 & u & v & u & 1 \end{bmatrix} \quad \text{with} \quad \begin{array}{l} u := -4\cos\omega \\ v := 4\cos^2\omega + 2 \end{array}.$$

Note that $F(\omega)$ is an $(N-4) \times N$ matrix where each row is obtained by circularly shifting its preceding row to the right by one position.

To simplify the notation, in the following we omit the argument ω and use P_N, A_N and D_N to denote the $N \times N$ matrices $P(\omega)$, $A(\omega)$ and $D(\omega)$, respectively. Note that P_N, A_N and D_N can be obtained from P_{N+1}, A_{N+1} and D_{N+1} by deleting the $(N{+}1)$-th row and column. The proof is carried out by induction in a manner similar to what was done in [10].

Step I: When $N = 4$, $F(\omega)$ and $\mathbf{0}_{4,N-4}$ are zero dimensional and must be suppressed in (3). Thus, A is just A_4. It is then straightforward to verify that $A_4 P_4 A_4^\mathsf{T} = D_4$ and $\det A_4 = -1$. Hence, the theorem is true for $N = 4$.

Step II: For $N \geq 4$, partition A_{N+1} and P_{N+1} as

$$A_{N+1} = \begin{bmatrix} A_N & \mathbf{0}_{N,1} \\ A_{1,N} & 1 \end{bmatrix} \qquad\qquad P_{N+1} = \begin{bmatrix} P_N & P_{N,1} \\ P_{N,1}^\mathsf{T} & N \end{bmatrix}$$

where

$$\begin{aligned} A_{1,N} &:= \begin{bmatrix} 0 & \cdots & 0 & 1 & u & v & u \end{bmatrix}, \\ P_{N,1} &:= \frac{1}{2} \begin{bmatrix} N\cos N\omega & (N+1)\cos(N-1)\omega & \cdots & (2N-1)\cos\omega \end{bmatrix}^\mathsf{T}. \end{aligned}$$

Hence

$$\begin{aligned} A_{N+1} P_{N+1} A_{N+1}^\mathsf{T} &= \begin{bmatrix} A_N & \mathbf{0}_{N,1} \\ A_{1,N} & 1 \end{bmatrix} \begin{bmatrix} P_N & P_{N,1} \\ P_{N,1}^\mathsf{T} & N \end{bmatrix} \begin{bmatrix} A_N^\mathsf{T} & A_{1,N}^\mathsf{T} \\ \mathbf{0}_{1,N} & 1 \end{bmatrix} \\ &= \begin{bmatrix} A_N P_N A_N^\mathsf{T} & A_N(P_N A_{1,N}^\mathsf{T} + P_{N,1}) \\ (A_{1,N} P_N + P_{N,1}^\mathsf{T}) A_N^\mathsf{T} & A_{1,N}(P_N A_{1,N}^\mathsf{T} + P_{N,1}) + P_{N,1}^\mathsf{T} A_{1,N}^\mathsf{T} + N \end{bmatrix}. \end{aligned}$$

Under the inductive assumption of $A_N P_N A_N^\mathsf{T} = D_N$, the problem of proving $A_{N+1} P_{N+1} A_{N+1}^\mathsf{T} = D_{N+1}$ is reduced to that of proving

$$A_N(P_N A_{1,N}^\mathsf{T} + P_{N,1}) = \mathbf{0}_{N,1} \tag{4}$$

$$A_{1,N}(P_N A_{1,N}^\mathsf{T} + P_{N,1}) + P_{N,1}^\mathsf{T} A_{1,N}^\mathsf{T} + N = 0. \tag{5}$$

From (3) and the structure of $F(\omega)$, it is obvious that $\det A_N = \det A_4 = -1 \neq 0$. Thus, A_N is invertible and Equations (4- 5) can be further simplified to

$$P_N A_{1,N}^\mathsf{T} + P_{N,1} = \mathbf{0}_{N,1} \tag{6}$$

$$P_{N,1}^\mathsf{T} A_{1,N}^\mathsf{T} + N = 0. \tag{7}$$

Conditions (6-7) can be checked elementarily since only the last four entries of $A_{1,N}^\mathsf{T}$ are nonzero; we leave to the reader the necessary computations. $\square$

We are now in a position to prove Theorem 1.

Proof of Theorem 1.
1) As $P(\omega)$ is periodic with a 2π period, it suffices to consider $P(0)$ and $P(\pi)$. The claim for $P(0)$ has been proved in [10]. The claim for $P(\pi)$ easily follows from this, since $P(\pi) = WP(0)W = WP(0)W^{-1}$, where $W = \mathrm{diag}\{1, -1, 1, -1, \ldots\}$ has dimension N and $\det W = 1$.
2) According to the Sylvester's law of inertia (see [4, Ch. X, Sec. 2], [5, Ch. VIII, Sec. 6]), symmetric matrices B and C have the same number of positive/negative/zero eigenvalues, whenever $C = ABA^{\mathsf{T}}$ for any invertible matrix A. By (2), The matrices $P(\omega)$ and $D(\omega)$ satisfy this condition. Therefore the second claim of the theorem follows by noticing that when $\frac{\omega_0}{\pi}$ is not an integer, $D(\omega_0)$ has exactly two positive eigenvalues, two negative eigenvalues and an $(N-4)$-dimensional kernel. $\square$

Theorem 1 states that when $\frac{\omega}{\pi}$ is not an integer $P(\omega)$ has two positive and two negative eigenvalues, but it does not tell what these four non-zero eigenvalues look like. This is somewhat unsatisfactory since in the filter design problem discussed in [8, 11], it is required that the positive eigenvalues of $P(\omega)$ must be sufficiently larger than the absolute values of the negative eigenvalues, as already mentioned in the Introduction. To investigate further properties of the four non-zero eigenvalues of $P(\omega)$, we numerically evaluate them for $N = 4, 10, 50, 200$ with $\omega \in [0, 2\pi]$ in a step of $2\pi/100$ and depict the results in Figure 1.

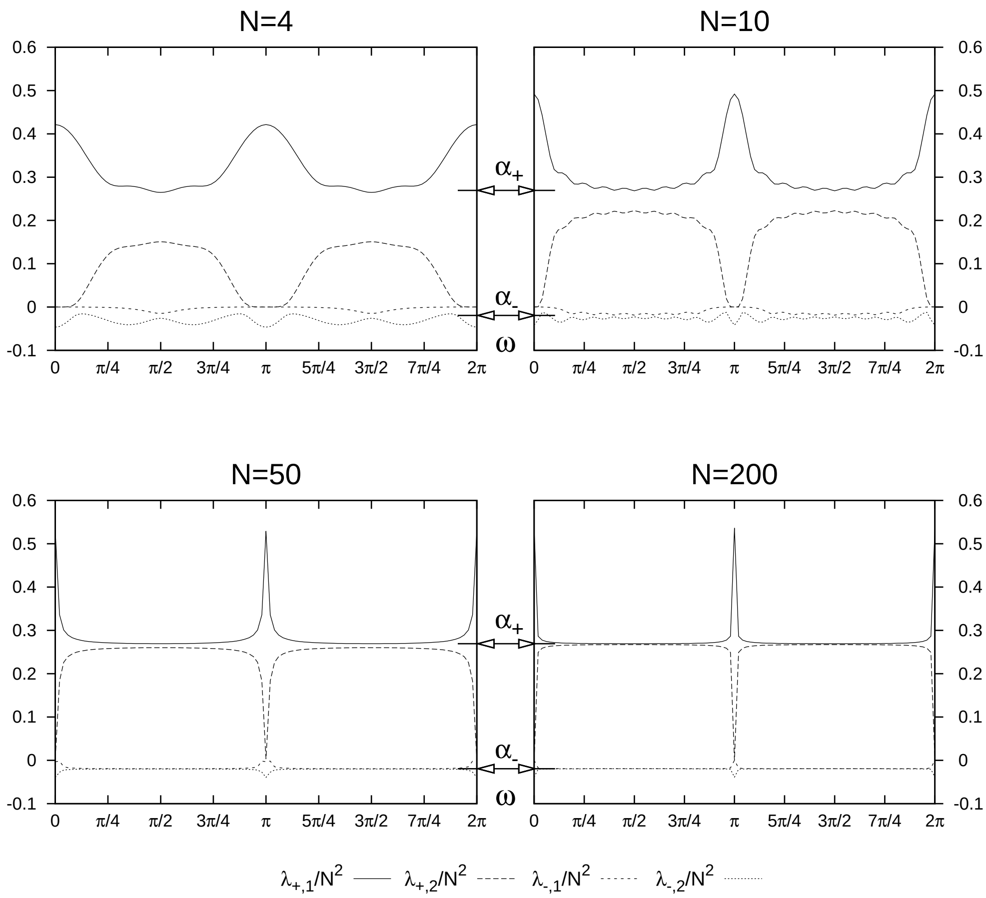

FIGURE 1. Eigenvalues of $P(\omega)$ for $N = 4, 10, 50, 200$, normalized to N^2.

The figure shows that the two positive eigenvalues are quite close to each other and similarly for the two negative eigenvalues when $N = 50$; for higher values of N this fact is even more evident and for $N = 200$ they are almost identical. This asymptotic property of the eigenvalues is stated in the next theorem.

Theorem 3. *When $\frac{\omega}{\pi}$ is not an integer, the nonzero eigenvalues of $P(\omega)$ satisfy the inequalities*

$$\left|\lambda_{+,1,2} - \alpha_+ N^2\right| \leq \sqrt{\frac{1.05}{\|\omega/\pi\|}}\, N^{3/2}, \tag{8}$$

$$\left|\lambda_{-,1,2} - \alpha_- N^2\right| \leq \sqrt{\frac{0.61}{\|\omega/\pi\|}}\, N^{3/2}, \tag{9}$$

whenever $\|\omega/\pi\|N \geq 41$, and where $\alpha_\pm := (1 \pm 2/\sqrt{3})/8$.

The proof of Theorem 3 requires the following lemmas.

Lemma 1. *Let $a, b > 0$, then*

$$\frac{a^a b^b}{(a+b)^{a+b}} \leq \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+1)}.$$

Proof. Let

$$f(a,b) := \frac{\Gamma(b+1)}{\Gamma(a+b+1)} \frac{(a+b)^{a+b}}{b^b}.$$

This map can be extended as a continuous map in $b = 0$ with $f(a,0) = a^a/\Gamma(a+1)$ for every $a > 0$. Therefore the proposed inequality can be stated as $f(a,b) \geq f(a,0)$ and can be proved by proving that the partial derivative with respect to b of $f(a,b)$ is nonnegative. The values of $f(a,b)$ are positive. Therefore the sign of $\partial_b f(a,b)$ coincides with that one of $\partial_b \log f(a,b)$, which is

$$\frac{\Gamma'(b+1)}{\Gamma(b+1)} - \frac{\Gamma'(a+b+1)}{\Gamma(a+b+1)} + \log(a+b) - \log b.$$

This function is equal to zero for $a = 0$. Hence, in order to prove that it is nonnegative for every $a, b > 0$, it is sufficient to prove that its partial derivative with respect to a is nonnegative. Using the representation $-\Gamma'(x)/\Gamma(x) = \gamma + \sum_{k=1}^{\infty}(\frac{1}{x+k-1} - \frac{1}{k})$ (see [2, Thm. 1.2.5]), this derivative can be written as

$$\partial_a \partial_b \log f(a,b) = \frac{1}{a+b} - \sum_{k=1}^{\infty} \frac{1}{(a+b+k)^2}.$$

Let c be a positive constant; then, adding the inequalities $\frac{1}{c+k-1} - \frac{1}{c+k} \geq \frac{1}{(c+k)^2}$ for $k = 1, 2, \ldots$, we see that $\frac{1}{c} \geq \sum_{k=1}^{\infty} \frac{1}{(c+k)^2}$, thus proving that $\partial_a \partial_b \log f(a,b) \geq 0$ for $a, b > 0$. □

Lemma 2. *Let $\Delta_d(N)$ be the set $\{\mathbf{n} \in \mathbb{N}^d \colon n_1 + \cdots + n_d \leq N\}$. Let $a_1, \ldots, a_d \geq 0$ and let $\mathbf{a} := a_1 + \cdots + a_d$. Then we have*

$$\Big|\sum_{\mathbf{n} \in \Delta_{d-1}(N)} n_1^{a_1} \cdots n_{d-1}^{a_{d-1}} (N - n_1 - \cdots - n_{d-1})^{a_d}\Big| \leq \frac{\prod_{m=1}^{d} \Gamma(a_m + 1)}{\Gamma(\mathbf{a} + d)}\, N^{\mathbf{a}} \sum_{u=0}^{d-1} \binom{d-1}{u} [\mathbf{a} + d - 1]_u N^{d-1-u}.$$

Since $[\mathbf{a}+d-1]_u \le (\mathbf{a}+d-1)^u$, *we deduce that*

$$\Big|\sum_{\mathbf{n}\in\Delta_{d-1}(N)} n_1^{a_1}\cdots n_{d-1}^{a_{d-1}}(N-n_1-\cdots-n_{d-1})^{a_d}\Big| \le \frac{\prod_{m=1}^d \Gamma(a_m+1)}{\Gamma(\mathbf{a}+d)}\, N^{\mathbf{a}}(N+\mathbf{a}+d-1)^{d-1}.$$

This result is essentially optimal under hypotheses as general as those ones assumed here. In fact, the inequality holds as equality when $d=2$ and $a_1=a_2=0$, and as asymptotic equality when N increases for every fixed set of exponents a_j and every dimension. On the other hand, for fixed N and nonzero exponents, tighter bounds are possible for the coefficients of the non-maximal powers of N, but at the cost of a greater complexity of the result.

Proof. The proof is by induction on d. For $d=2$ the claim states that

$$\sum_{n=0}^N n^a(N-n)^b \le \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)}N^{a+b}(N+a+b+1) \tag{10}$$

for every $a,b\ge 0$ and for every N. The inequality is evident if $a=b=0$. Hence we can further assume that $a+b>0$. Under this hypothesis the function $x\to x^a(N-x)^b$ has a unique maximum at $N^*:=aN/(a+b)$. Splitting the domain of the sum in integers $n<N^*$ and $n\in[N^*,N]$, and using the comparison of the sum and integral in each domain we have that

$$\sum_{n=0}^N n^a(N-n)^b \le \int_0^N x^a(N-x)^b\,dx + N^{*a}(N-N^*)^b.$$

We get the claim firstly by substituting $x\to Nx$ in the integral and N^* with $aN/(a+b)$, then recalling that $\int_0^1 x^{a-1}(1-x)^{b-1}dx=\Gamma(a)\Gamma(b)/\Gamma(a+b)$ (see. [2, Thms. 1.1.4 and 1.8.1]) and using the inequality in Lemma 1 to compare the second term to the first one.
For $d>2$, the claim follows splitting the sum as

$$\sum_{(n_1,\dots,n_{d-1})\in\Delta_{d-1}(N)}\cdots = \sum_{n_1=0}^N n_1^{a_1}\Big[\sum_{(n_2,\dots,n_{d-1})\in\Delta_{d-2}(N-n_1)} n_2^{a_2}\cdots n_{d-1}^{a_{d-1}}((N-n_1)-\cdots-n_{d-1})^{a_d}\Big],$$

using the inductive hypothesis to bound the inner sum and (10) to bound the remaining sum. □

Lemma 3. *Let* $\omega\ne r\pi$ $(r\in\mathbb{Z})$ *and let* $h\in\mathbb{N}$. *Then*

$$\Big|\sum_{n=0}^N n^h\cos(\phi+2n\omega)\Big| \le \frac{1}{4\|\omega/\pi\|}\sum_{k=0}^h \left\{ {h \atop k} \right\} \sum_{l=0}^k \frac{k!}{l!}(1+\delta(l))\frac{(N+1)^l}{(4\|\omega/\pi\|)^{k-l}},$$

where ϕ *is an arbitrary function which is independent of* n.

Proof. The elementary identity

$$\sum_{n=0}^N [n]_k\, z^n = z^k\Big(\frac{d}{dz}\Big)^k\frac{z^{N+1}-1}{z-1} = z^k\sum_{l=0}^k\binom{k}{l}(z^{N+1}-1)^{(l)}\Big(\frac{1}{z-1}\Big)^{(k-l)}$$

implies that

$$\Big|\sum_{n=0}^N [n]_k\, e^{2in\omega}\Big| \le \sum_{l=0}^k\frac{k!}{l!}(1+\delta(l))\frac{(N+1)^l}{|e^{2i\omega}-1|^{k-l+1}}.$$

The result follows by the lower bound $|e^{2i\omega}-1| = 2|\sin\omega| \geq 4\|\omega/\pi\|$ and the identity $\sum_{k=0}^{h} \left\{ {h \atop k} \right\} [n]_k = n^h$ (See [1, Prop. 3.24]). $\square$

Lemma 4. *Let $\omega \neq r\pi$ $(r \in \mathbb{Z})$, $a_1, \ldots, a_d \in \mathbb{N}$ and let ϕ be an arbitrary function independent of n_d. Suppose that $4\|\omega/\pi\|N \geq c$ for a fixed parameter $c > 0$, independent of ω. Then*

$$\Big| \sum_{\mathbf{n}\in\Delta_d(N)} n_1^{a_1} \cdots n_d^{a_d} \cos(\phi + 2n_d\omega) \Big| \leq I(a_1, \ldots, a_d) \frac{(N+1)^{\mathbf{a}}(N+\mathbf{a}+d)^{d-1}}{\|\omega/\pi\|},$$

where $\mathbf{a}$ is defined in Lemma 2 and

$$I(a_1,\ldots,a_d) := \frac{1}{4} \sum_{k=0}^{a_d} \left\{ {a_d \atop k} \right\} \sum_{l=0}^{k} (1+\delta(l)) \frac{2^{a_d-k}}{c^{a_d-l}} \frac{k! \prod_{m=1}^{d-1} \Gamma(a_m+1)}{\Gamma(a_1+\cdots+a_{d-1}+l+d)}.$$

Proof. This is a consequence of Lemmas 2-3 and of the hypothesis $4\|\omega/\pi\|N > c$ which implies that $(4\|\omega/\pi\|)^{-1} \leq N/c$ and that $N \geq c/2$. $\square$

We are now in a position to prove the last theorem.

Proof of Theorem 3. Using an explicit form of the characteristic equation given in [4, Ch. 3, Sec. 7], we have

$$\det(\lambda \mathbf{I}_N - P(\omega)) = \lambda^4 - S_1(\omega)\lambda^3 + S_2(\omega)\lambda^2 - S_3(\omega)\lambda + S_4(\omega) = 0, \tag{11}$$

where $S_j(\omega)$ $(j = 1, \ldots, 4)$ is the sum of the principal minors of order j of $P(\omega)$. Let Q_4 be the 4×4 symmetric matrix

$$Q_4 := \frac{1}{2} \begin{bmatrix} q_{11} & q_{12} & q_{13} & q_{14} \\ q_{12} & q_{22} & q_{23} & q_{24} \\ q_{13} & q_{23} & q_{33} & q_{34} \\ q_{14} & q_{24} & q_{34} & q_{44} \end{bmatrix},$$

where

$q_{11} := 2n_1$ $\quad$ $q_{13} := (2n_1+n_2+n_3)\cos(n_2+n_3)\omega$
$q_{22} := 2n_1+2n_2$ $\quad$ $q_{14} := (2n_1+n_2+n_3+n_4)\cos(n_2+n_3+n_4)\omega$
$q_{33} := 2n_1+2n_2+2n_3$ $\quad$ $q_{23} := (2n_1+2n_2+n_3)\cos n_3\omega$
$q_{44} := 2n_1+2n_2+2n_3+2n_4$ $\quad$ $q_{24} := (2n_1+2n_2+n_3+n_4)\cos(n_3+n_4)\omega$
$q_{12} := (2n_1+n_2)\cos n_2\omega$ $\quad$ $q_{34} := (2n_1+2n_2+2n_3+n_4)\cos n_4\omega$

Each S_j can be computed as the determinant of the principal and upper minor of order j of Q_4 summed over every combination of nonnegative indexes $n_1, \ldots, n_j$ such that $n_1 + \cdots + n_j$ is strictly lower than N and each n_j but n_1 is strictly positive. Thus for example

$$S_1 = \frac{1}{2}\sum_{n_1=0}^{N-1} q_{11}, \qquad S_2 = \frac{1}{2^2}\sum_{n_1=0}^{N-2}\sum_{n_2=1}^{N-1-n_1} \det \begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix}.$$

Moreover, from the definition of Q_4 it is clear that S_j can be written as

$$\sum_{[abc]} \sum_{\substack{\mathbf{n}\in\Delta_j(N-1) \\ n_k>0\ \forall k\neq 1}} P^{(j)}_{[abc]} \cos(an_2 + bn_3 + cn_4)\omega$$

where each $P^{(j)}_{[abc]}$ is an homogeneous polynomial of degree j in the $n_1, \ldots, n_4$ indeterminates for a suitable set of multi-integers $[abc]$. The computation of $P^{(j)}_{[abc]}$ is a bit tedious, the final result is collected in Table 1.

TABLE 1. Polynomials $P^{(j)}_{[abc]}$

$P^{(1)}_{[\ldots]}$	
[000]	n_1
$P^{(2)}_{[\ldots]}$	
[000]	$\frac{1}{4}(2n_1^2 + 2n_1n_2 - \frac{1}{2}n_2^2)$
[200]	$\frac{1}{8}(2n_1 + n_2)^2$
$P^{(3)}_{[\ldots]}$	
[000]	$\frac{1}{8}(-(n_1 + n_2)n_2^2 - (n_1n_2 + \frac{3}{2}n_2^2)n_3 - (n_1 + \frac{1}{2}n_2)n_3^2)$
[200]	$\frac{1}{16}((2n_1 + n_2 + n_3)n_2 + 2n_1n_3)n_3$
[020]	$\frac{1}{16}(2n_1 + 2n_2 + n_3)(n_2 + n_3)n_2$
[220]	$-\frac{1}{16}(2n_1 + n_2 + n_3)n_2n_3$
$P^{(4)}_{[\ldots]}$	
[000]	$\frac{1}{16}((\frac{1}{4}n_3^2 + \frac{1}{4}n_3n_4 + \frac{1}{4}n_4^2)n_2^2 + (\frac{1}{2}n_3^2 + \frac{3}{4}n_3n_4 + \frac{1}{4}n_4^2)n_2n_3 + (\frac{1}{4}n_3^2 + \frac{1}{2}n_3n_4 + \frac{1}{4}n_4^2)n_3^2)$
[200]	$\frac{1}{16}((-\frac{1}{4}n_3^2 - \frac{1}{4}n_3n_4)n_2^2 + (-\frac{1}{2}n_3^3 - \frac{3}{4}n_3^2n_4 - \frac{1}{4}n_3n_4^2)n_2 - \frac{1}{4}n_3^4 - \frac{1}{2}n_3^3n_4 - \frac{1}{4}n_3^2n_4^2)$
[020]	$\frac{1}{16}(-8n_1^4 + (-24n_2 + (-16n_3 - 8n_4))n_1^3 + (-24n_2^2 + (-32n_3 - 16n_4)n_2 + (-8n_3^2 - 8n_3n_4 - 2n_4^2))n_1^2 + (-8n_2^3 + (-16n_3 - 8n_4)n_2^2 + (-8n_3^2 - 8n_3n_4 - 2n_4^2)n_2)n_1 + ((-\frac{1}{4}n_3n_4 - \frac{1}{4}n_4^2)n_2^2 + (-\frac{1}{4}n_3^2n_4 - \frac{1}{4}n_3n_4^2)n_2))$
[002]	$\frac{1}{16}(8n_1^4 + (24n_2 + 16n_3 + 8n_4)n_1^3 + (24n_2^2 + (32n_3 + 16n_4)n_2 + (8n_3^2 + 8n_4n_3 + 2n_4^2))n_1^2 + (8n_2^3 + (16n_3 + 8n_4)n_2^2 + (8n_3^2 + 8n_4n_3 + 2n_4^2)n_2)n_1 + ((-\frac{1}{4}n_3^2 - \frac{1}{4}n_4n_3)n_2^2 + (-\frac{1}{2}n_3^3 - \frac{3}{4}n_4n_3^2 - \frac{1}{4}n_4^2n_3)n_2 + (-\frac{1}{4}n_3^4 - \frac{1}{2}n_4n_3^3 - \frac{1}{4}n_4^2n_3^2)))$
[202]	$\frac{1}{16}((\frac{1}{8}n_3^2 + \frac{1}{4}n_4n_3 + \frac{1}{8}n_4^2)n_2^2 + (\frac{1}{4}n_3^3 + \frac{1}{2}n_4n_3^2 + \frac{1}{4}n_4^2n_3)n_2 + \frac{1}{8}n_3^4 + \frac{1}{4}n_4n_3^3 + \frac{1}{8}n_4^2n_3^2)$
[20-2]	$\frac{1}{16}(\frac{1}{8}n_3^2n_2^2 + (\frac{1}{4}n_3^3 + \frac{1}{4}n_4n_3^2)n_2 + \frac{1}{8}n_3^4 + \frac{1}{4}n_4n_3^3 + \frac{1}{8}n_4^2n_3^2)$
[242]	$\frac{1}{128}n_2^2n_4^2$
[220]	$\frac{1}{64}n_2n_3n_4(n_2 + n_3 + n_4)$
[022]	$\frac{1}{64}n_2n_3n_4(n_2 + n_3 + n_4)$
[222]	$\frac{1}{16}(2n_1^4 + (6n_2 + (4n_3 + 2n_4))n_1^3 + (13/2n_2^2 + (9n_3 + 4n_4)n_2 + (3n_3^2 + 3n_4n_3 + \frac{1}{2}n_4^2))n_1^2 + (3n_2^3 + (\frac{13}{2}n_3 + \frac{5}{2}n_4)n_2^2 + (\frac{9}{2}n_3^2 + 4n_4n_3 + \frac{1}{2}n_4^2)n_2 + (n_3^3 + \frac{3}{2}n_4n_3^2 + \frac{1}{2}n_4^2n_3))n_1 + (\frac{1}{2}n_2^4 + (3/2n_3 + \frac{1}{2}n_4)n_2^3 + (\frac{13}{8}n_3^2 + n_4n_3 - \frac{1}{8}n_4^2)n_2^2 + (\frac{3}{4}n_3^3 + \frac{3}{4}n_4n_3^2)n_2 + (\frac{1}{8}n_3^4 + \frac{1}{4}n_4n_3^3 + \frac{1}{8}n_4^2n_3^2)))$

The main contribution to S_j comes from

$$\sum_{\substack{\mathbf{n}\in\Delta_j(N-1) \\ n_k>0\ \forall k\neq 1}} P^{(j)}_{[000]}$$

which produces the polynomials

$j = 1$	$\frac{1}{2}N^2 - \frac{1}{2}N$
$j = 2$	$\frac{5}{96}N^4 - \frac{5}{24}N^3 + \frac{19}{96}N^2 - \frac{1}{24}N$
$j = 3$	$\frac{-1}{384}N^6 + \frac{1}{128}N^5 - \frac{1}{384}N^4 - \frac{1}{128}N^3 + \frac{1}{192}N^2$
$j = 4$	$\frac{1}{36864}N^8 - \frac{1}{7680}N^7 + \frac{7}{92160}N^6 + \frac{1}{3072}N^5 - \frac{11}{36864}N^4 - \frac{1}{5120}N^3 + \frac{1}{5120}N^2$

In each polynomial we retain only the main term and we estimate the contribution of the remaining ones. Since for $x \in [0,1]$ we have

$$\begin{aligned}
&-\tfrac{5}{24}+\tfrac{19}{96}x-\tfrac{1}{24}x^2 \in (\tfrac{-5}{24},0)\\
&|\tfrac{1}{128}-\tfrac{1}{384}x-\tfrac{1}{128}x^2+\tfrac{1}{192}x^3| \leq \tfrac{1}{128}\\
&-\tfrac{1}{7680}+\tfrac{7}{92160}x+\tfrac{1}{3072}x^2-\tfrac{11}{36864}x^3-\tfrac{1}{5120}x^4+\tfrac{1}{5120}x^5 \in (\tfrac{-1}{7680},0),
\end{aligned}$$

these contributions can be estimated as

$j=1$	$\frac{1}{2}N^2+\overline{O}(\frac{1}{2}N)$	$j=2$	$\frac{5}{96}N^4+\eta N^3$ with $\eta\in(\frac{-5}{24},0)$
$j=3$	$\frac{-1}{384}N^6+\overline{O}(\frac{1}{128}N^5)$	$j=4$	$\frac{1}{36864}N^8+\xi N^7$ with $\xi\in(\frac{-1}{7680},0)$

The other sum contributing to S_j is

$$\sum_{[abc]\neq[000]}\ \sum_{\substack{\mathbf{n}\in\Delta_j(N-1)\\ n_k>0\ \forall k\neq 1}} P^{(j)}_{[abc]}\cos(an_2+bn_3+cn_4)\omega$$

and here each inner term can be estimated using the explicit representations of $P^{(j)}_{[abc]}$ contained in Table 1 and Lemma 4 with $c=164$ (since Theorem 3 assumes $\|\omega/\pi\|N\geq 41$). After some computations we get the following equalities:

$$\begin{aligned}
S_1 &= \Big(\tfrac{1}{2}+\overline{O}\big(\tfrac{1/2}{N}\big)\Big)N^2\\
S_2 &= \Big(\tfrac{5}{96}+\overline{O}\big(\eta+\tfrac{18677/161376}{\|\omega/\pi\|}(1+\tfrac{3}{N})\big)\tfrac{1}{N}\Big)N^4\\
S_3 &= \Big(-\tfrac{1}{384}+\overline{O}\big(\tfrac{1}{128}+\tfrac{13655/1291008}{\|\omega/\pi\|}(1+\tfrac{5}{N})^2\big)\tfrac{1}{N}\Big)N^6\\
S_4 &= \Big(\tfrac{1}{36864}+\overline{O}\big(\xi+\tfrac{39558023933/3645909872640}{\|\omega/\pi\|}(1+\tfrac{7}{N})^3\big)\tfrac{1}{N}\Big)N^8.
\end{aligned}$$

The constant η in S_2 is negative and, in absolute value, smaller than $\frac{18677/161376}{\|\omega/\pi\|}$ for every ω. Their values are comparable in size when $\|\omega/\pi\|$ is close to $1/2$, therefore in this case their sum shows a considerable cancellation. However, this effect disappears when $\|\omega/\pi\|$ is close to zero: since this is the most delicate part of the range for ω, there is essentially no convenience in keeping η, and we bound the $\overline{O}(\cdot)$ term in S_2 simply with the greater $\overline{O}\big(\frac{18677/161376}{\|\omega/\pi\|}(1+\frac{1}{N})^3\big)$. An analogous remark applies to the S_4 term. Summing up, for $N\geq 82$ (another consequence of the assumption $\|\omega/\pi\|N\geq 41$) we deduce that

$$\begin{aligned}
S_1 &= \Big(\tfrac{1}{2}+\overline{O}\big(\tfrac{0.25}{\|\omega/\pi\|N}\big)\Big)N^2 & S_3 &= \Big(-\tfrac{1}{384}+\overline{O}\big(\tfrac{0.01582}{\|\omega/\pi\|N}\big)\Big)N^6\\
S_2 &= \Big(\tfrac{5}{96}+\overline{O}\big(\tfrac{0.11998}{\|\omega/\pi\|N}\big)\Big)N^4 & S_4 &= \Big(\tfrac{1}{36864}+\overline{O}\big(\tfrac{0.01388}{\|\omega/\pi\|N}\big)\Big)N^8.
\end{aligned}$$

Substituting these relations into (11) and simplifying, we have

$$\begin{aligned}
\det(\lambda\mathbf{I}_N-P(\omega)) = \lambda^4 &- \Big(\frac{1}{2}+\overline{O}\big(\tfrac{0.25}{\|\omega/\pi\|N}\big)\Big)N^2\lambda^3+\Big(\frac{5}{96}+\overline{O}\big(\tfrac{0.11998}{\|\omega/\pi\|N}\big)\Big)N^4\lambda^2\\
&+\Big(\frac{1}{384}+\overline{O}\big(\tfrac{0.01582}{\|\omega/\pi\|N}\big)\Big)N^6\lambda+\Big(\frac{1}{36864}+\overline{O}\big(\tfrac{0.01388}{\|\omega/\pi\|N}\big)\Big)N^8.
\end{aligned}$$

Letting $y = \lambda N^{-2}$, the characteristic equation $\det(\lambda \mathbf{I}_N - P(\omega)) = 0$ becomes for y

$$y^4 - \Big(\frac{1}{2} + \overline{O}\big(\tfrac{0.25}{\|\omega/\pi\|N}\big)\Big) y^3 + \Big(\frac{5}{96} + \overline{O}\big(\tfrac{0.11998}{\|\omega/\pi\|N}\big)\Big) y^2 \\ + \Big(\frac{1}{384} + \overline{O}\big(\tfrac{0.01582}{\|\omega/\pi\|N}\big)\Big) y + \Big(\frac{1}{36864} + \overline{O}\big(\tfrac{0.01388}{\|\omega/\pi\|N}\big)\Big) = 0.$$

Let $q_N(y)$ denote the polynomial appearing to the left hand side of the previous equation, and let $q_\infty(y)$ be that one we obtain setting $N \to \infty$, so that

$$q_\infty(y) := y^4 - \frac{1}{2} y^3 + \frac{5}{96} y^2 + \frac{1}{384} y + \frac{1}{36864}.$$

Then

$$|q_N(y) - q_\infty(y)| \le \frac{|0.01388 + 0.01582\, y + 0.11998\, y^2 + 0.25\, y^3|}{\|\omega/\pi\|N}. \tag{12}$$

The polynomial $q_\infty(y)$ factorizes as $(y - \alpha_+)^2 (y - \alpha_-)^2$. Moreover, we have the elementary inequality

$$|0.01388 + 0.01582\, y + 0.11998\, y^2 + 0.25\, y^3| < 1.05\, |y - \alpha_-|^2 \tag{13}$$

for every complex y satisfying $|y - \alpha_+| \le 0.16004$. Under the hypothesis $\|\omega/\pi\|N \ge 41$ we have $(1.05/\|\omega/\pi\|N)^{1/2} < 0.16004$, so that from (12-13) and the factorization of q_∞ we get

$$|q_N(y) - q_\infty(y)| < |q_\infty(y)| \qquad \forall\, y \in \mathbb{C}\colon |y - \alpha_+| = \sqrt{\frac{1.05}{\|\omega/\pi\|N}}.$$

By the Rouché's Theorem we can conclude that for those N the polynomial $q_N(y)$ has in the disk $|y - \alpha_+| \le (1.05/(\|\omega/\pi\|N))^{1/2}$ as many roots as $q_\infty(y)$, which are exactly two if N is large enough. This proves the claim for the positive eigenvalues as $\lambda = yN^2$. The second claim for the negative eigenvalues is proved with an analogous argument. □

Theorem 3 assures that for $\frac{\omega}{\pi}$ being not an integer, the two positive eigenvalues of $P(\omega)$ approach $\alpha_+ N^2$ asymptotically and similarly for the two negative eigenvalues approaching $\alpha_- N^2$. As a result, for a sufficiently large N and for $\frac{\omega}{\pi}$ being not an integer, the ratio of the positive eigenvalues to the absolute values of the negative eigenvalues is approximated by $\alpha_+ N^2 / |\alpha_- N^2| \approx 14$, which is sufficiently large to ensure the optimization techniques in [8, 11] to work well when adopted for the design of real FIR filters.

We admit that the error bounds given in Theorem 3 are not tight, particularly for ω far away from the central frequency $\pi/2$. For example, when $\omega = 0.1\pi$, Theorem 3 requires the minimal N to be 410 and the corresponding errors bound (the right hand side of (8)) for the positive eigenvalues is about 26900, while the actual numerical errors (the left hand side of (8)) are only about 405 and 213, respectively, because in this case the two positive eigenvalues are about 45488 and 44870, respectively, while $\alpha_+ 410^2 \approx 45275$. Furthermore, for $\omega = 0.1\pi$ and $N = 50$, the two positive eigenvalues are about 624 and 699, and the actual numerical errors (the left hand side of (8)) are about 50 and 26, respectively, and $\alpha_+ 50^2 \approx 673$. Hence, the maximum relative error for the positive eigenvalues is about 7.4%. Similarly, for the same $\omega = 0.1\pi$ and $N = 50$, the two negative eigenvalues are about -45.8 and -51.7, and the actual numerical errors (the left hand side of (9)) are about 2.5 and 3.4, respectively, and $\alpha_- 50^2 \approx -48.3$. Hence, the maximum relative error for the negative eigenvalues is about 7.1%. The above numerical errors lead to the difference between the approximate ratio of the positive eigenvalues to the absolute values of the negative eigenvalues, $673/48.3 \approx 14$, and the actual ratio of the smaller positive eigenvalue to the absolute values of the smaller negative eigenvalue, $624/51.7 \approx 12.1$. However, both ratios are still large enough for ensuring the filter design techniques adopted in [8, 11] to perform well. To reduce the error bounds further, some of the previous inequalities could be improved. For example we could use the full strength of Lemma 2, and the fact that Lemma 3 holds with $[N+1]_l$ in place of $(N+1)^l$; also the contributions to S_j coming from the whole main terms could be retained. In this way we can prove Theorem 3 under the weaker hypothesis $\|\omega/\pi\|N \geq 35$ and with slightly smaller constants in the error bounds. In our opinion, such a small improvement is not worth the more complicated formulas we need to prove it. A stronger improvement would certainly follow if we could take account of the fact that in several polynomials $P^{(j)}_{[abc]}$ there are more than one oscillating cosine, so that some of these polynomials should show extra cancellation (at least when there are no "1 to 1 resonances" between the frequencies), and that the contributes of different $P^{(j)}_{[abc]}$ polynomials have different sign. However, at this moment we do not see an easy way to exploit these cancellations. In conclusion, we hope that the results presented in this paper provide the theoretical support for adopting the optimization techniques in [8, 11] to the design of FIR filters of real coefficients and would also motivate further study in reducing the error bounds in estimating the asymptotical eigenvalues of the trigonometric matrix significantly under a much weaker hypothesis.

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