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On the Motive of a Bundle

MAT/03 - Geometria

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Chapter 1

Introduction

One of the most outstanding advances in algebraic geometry in the past few decades has been achieved by V. Voevodsky (Fields Medal, 2002) developing a universal cohomology theory for algebraic varieties: among the consequences of his work are the solutions of the Milnor and Bloch-Kato Conjectures. The existence of a universal cohomology theory was already envisioned by A. Grothendieck in his famous letter to J.-P. Serre in 1964. In his theory, Voevodsky defines several triangulated categories over a perfect field k, namely

$$DM_{gm}^{eff}(k), DM_{-}^{eff}(k), DM_{-,et}^{eff}(k), DM_h(k)_{\mathbb{Q}}.$$

These are, essentially, categories of bounded complexes of sheaves with "homotopy invariant" homology.

The following sequence of functors between the category of effective Chow motives, introduced by Alexander Grothendieck, and these other categories is well known:

$$M^{eff}(k) \to DM_{gm}^{eff}(k) \to DM_{-}^{eff}(k) \to DM_{-,et}^{eff}(k) \to DM_h(k).$$

Let X be a variety over a perfect field k. Suppose that X admits a resolution of singularities $\tilde{X} \to X$ and further assume that there is a tower

$$\begin{array}{c} \tilde{X}_n := \tilde{X} \\ \downarrow {\scriptstyle p_{n-1}} \\ \tilde{X}_{n-1} \\ \vdots \\ \downarrow \\ \tilde{X}_0, \end{array} \tag{1.0.1}$$

where $\tilde{X}_i \to \tilde{X}_{i-1}$'s are fibrations with proper fibre F_i. Indeed such resolutions appear in various contexts, which we address later. In the first part of the thesis we study the motive associated to such X in the Voevodsky motivic categories. Then we implement this to study motives associated with Schubert varieties inside twisted affine flag varieties.

In the first glance, to relate motives of X and $\tilde{X}$, one needs a motivic version of the glorious achievement of the theory of perverse sheaves, i.e. the *Decomposition Theorem* of Beilinson, Bernstein, Deligne (and Gabber) [BBD]. Although it may look optimistic to state the decomposition theorem for motives, nevertheless L. Migliorini and M. A. de Cataldo have established this theory for semismall maps. Note that in [CH] A. Corti and M. Hanamura have shown that the general case is implied by conjectures of Grothendieck and Murre.

On the other hand to compute the motive associated to $\tilde{X}$ we introduce a motivic version of the Leray-Hirsch theorem. Recall that the topological Leray-Hirsch theorem determines the cohomology of a fibre bundle in terms of cohomologies of its base and fibre. The truth is that this naive version of *Leray-Hirsch theorem* does not even hold for the Fulton chow groups. One way to tackle the problem in the algebraic set up is to impose some stronger conditions on the fibre F. For instance one should assume that the fibre

C1) admits cell decomposition, i.e. admits a filtration

$$\emptyset = F_{-1} \subset F_0 \subset \ldots \subset F_n = F$$

by its closed subschemes, such that $F_i \setminus F_{i-1}$ is isomorphic to some affine space, and

C2) satisfies Poincaré duality (this is automatic when $\mathrm{char}\, k = 0$).

Let **Ab** be the category of abelian groups. Let us recall that there is a fully faithful tensor triangulated functor $i : D^b_f(\mathbf{Ab}) \to DM^{eff}_{gm}(k)$, see proposition 4.3.1 for the definitions.

A fundamental result of V. Voevodsky shows that the higher chow groups of a quasi projective variety X coincide certain motivic cohomologies of X. Implementing this result A. Huber and B. Kahn compute the motive of a pure Tate variety in terms of its fundamental invariants, see proposition 4.3.11. Regarding this we prove a motivic version of Leray-Hirsch theorem for cellular fibrations.

Theorem 1.0.1. *Let X be a smooth irreducible variety over a field k of characteristic* 0*. Let Γ be a variety over k and $\pi : \Gamma \to X$ be a proper smooth locally trivial (for Zariski topology) fibration with fiber F. Furthermore assume that F is cellular. Then the motive, $M_{gm}(\Gamma)$, associated to Γ in $DM^{eff}_{gm}(k)$ decomposes as follows:*

$$M_{gm}(\Gamma) \cong \coprod_{p \geqslant 0} CH_p(F) \otimes M_{gm}(X)(p)[2p].$$

Now we may formulate the following theorem:

Theorem 1.0.2. *Let $\tilde{X} \to X$ be a semi-small resolution as in (1.0.1). Let $\{X_\alpha\}$ be a set of connected relevant stratum. Furthermore assume that all fibres F_i satisfy conditions C1 and C2. Then for any α the motive $M(\bar{X}_\alpha)$ (associated to the closure of X_α), is a summand of*

$$\bigotimes_{i=0}^{n-1} \left(\coprod_{p \geqslant 0} CH_p(F_i) \otimes \mathbb{Z}(p)[2p] \right) \otimes M_{gm}(\tilde{X}_0).$$

Note however that we prove a slightly more general theorem, namely we consider the case that $\tilde{X} \to X$ is an alteration in the sense of de Jong, see definition 5.3.1 and theorem 5.3.2.
We then use the above approach to study motives of Schubert varieties in *twisted affine flag varieties.* These varieties first introduced by G. Pappas and M. Rapoport in [PR] and later received much attention from other mathematicians due to their significance in the theory of local models for shimura varieties, see [PRS], [Ha] and [Ri].

The second part of the thesis is devoted to analyse the motive of G-bundles, where G is a reductive algebraic group. In fact we develop a theory

which relates motives of G-bundles with the motives of cellular fibrations. Along the way, we establish a method to filter motives of G-bundles in terms of the faces of weight polytope (associated to certain representation of G) and incidence relation among them, see 7.3.
To build up this relation between motive of G-bundles and motive of cellular fibrations (or even pure Tate fibrations), we use the theory of wonderful compactification of reductive algebraic groups, which has introduced by De Concini and Procesi. Their method produces a smooth canonic compactification $\overline{G}$ of a reductive algebraic group G of adjoint type. Note that in [CP] they only study the case that the group G is defined over $\mathbb{C}$. Although most of the theory carries over for any algebraically closed field of arbitrary characteristic, there are some subtleties which occur in positive characteristic which we mention later.
As a feature of this compactification there is a natural $G \times G$-action on $\overline{G}$, and the arrangement of the orbits can be explained by the associated weight polytope $\mathcal{P}$, see section 6.2.

To establish a reasonable frame work to study cohomology of such compactifications in positive characteristic we proposed the notion of *motivic relatively cellular*. Notice that this notion is slightly weaker than the geometric notion of relatively cellular introduced by Chernousov, Gille, Merkurjev [CGM] and also Karpenko [Kar]. However we show that the similar decomposition principal holds for such motives, see section4.5.

D. Timashev has studied equivariant compactification of reductive groups in [Tim]. In particular he shows that there is a one-to-one correspondence between the $G \times G$-orbits of $\overline{G}$ and the orbits of the action of the Weyl group on the faces of the polytope $\mathcal{P}$, which preserves the incidence relation among orbits. Using this result we observe that when $\operatorname{char} k = 0$ (resp. char > 0) $D_{\mathcal{F}}$ is cellular (resp. motivic cellular), where $D_{\mathcal{F}}$ is the closure of the orbit in $\overline{G}$ corresponding to the face $\mathcal{F}$ of $\mathcal{P}$.

Theorem 1.0.3. *Let G be a reductive algebraic group over k and $X \in Ob(Sm_k)$ be irreducible. Let $\mathcal{G}$ be a G-bundle over X. Then $M_{gm}(\mathcal{G})$ is geometrically mixed Tate in either of the following cases:*

a) $\operatorname{char} k = 0$*, X is geometrically mixed Tate and $\mathcal{G}$ is locally trivial for the Zariski topology on X.*

b) X is a geometrically cellular variety.

To prove this theorem we first introduce the notion of the *configuration of mixed Tate varieties* and prove that the motive associated to such configuration is mixed Tate. Then we use the wonderful compactification and weight polytope combinatorics to conclude, see chapter 6 and section 7.2.

As an immediate consequence of the above theorem we observe that the motive of a split reductive group is mixed Tate. This has already shown by Biglari, A. Huber and B. Kahn using the theory of slice filtration, see [Big] and [HK]. The proof given by B. Kahn and A. Huber in fact relies on a result which produces a filtration on the motive of a split torus bundle $\mathcal{T}$ in $DM_{gm}^{eff}(k)$.

Our method also leads to a filtration on the motive of G-bundles over any base scheme. In this filtration, applying the motivic version of Lray-Hirsch theorem, the motives which filter $M(\mathcal{G})$ can be computed recursively in terms of the motive of base and $G \times G$-orbit closures.
When the base scheme is a smooth projective curve and G has connected center, using Drinfeld-Simpson theorem 3.2.11 we produce a simple (i.e. could be determined by the cocharacter group of a torus) filtration for $M(\mathcal{G})$.

Let us go briefly through the content of the thesis:

In the next chapter we give an overview of the theory of motives.
In the third chapter we present some materials from the theory of reductive algebraic groups, root systems, G-bundles and etc., which we commonly use in the rest of the thesis. Further we state two strong results due to Raghunathan and Drinfeld-Simpson, about the triviality of certain G-bundles.

In the chapter 4 we have intended to approach to a motivic version of Leray-Hirsch theorem, which we prove in the last section, in a conceptual way. We also introduce the notion of motivic relatively cellular and prove the corresponding decomposition result, see 4.5.3. We present a discussion about the theory of slice filtration because of its contribution with the filtration on the motive of a torus bundle.

In the first section of chapter 5 we recall the "decomposition theorem" of Beilinson, Bernstein, Deligne and Gabber. Then we state a motivic version of the theorem due to Migliorini and deCataldo, cf. [CM] and Corti and Hanamura cf. [CH]. Afterwards, using this theory and the results from previous chapters we study the motive of a variety X that admits certain type of resolution of singularities, see section 5.3. Finally, as an application, we study the motive of "Schubert varieties" in a twisted affine flag variety, see section 5.4.

In the chapter 6 we introduce the notion of "mixed Tate configuration" and show that the motive associated to such configuration of varieties is mixed Tate. Then in the section 6.2 we relate this notion with the geometry of $G \times G$-orbits of the wonderful compactification of a reductive group G of adjoint type.

In the last chapter we study the motive of G-bundles over a base scheme X. We first recall the slice filtration on the motive of a torus bundle introduced by A. Huber and B. Kahn in [HK]. Then introducing a different approach, we treat the more general case of "G-bundles". Our method is based on geometric observations and the weight polytope combinatorics of the wonderful compactification of G.

1.1 Notation and Conventions

Throughout this thesis we assume that k is a perfect field (unless otherwise stated). Fix a separable closure $\overline{k}$ of k.
We denote by Sch_k (resp. Sm_k, resp. $PropSch_k$, resp. $Smproj_k$) the category of schemes (resp. smooth schemes, resp. proper schemes, resp. smooth projective) of finite type over k.
For a morphism $f : X \to Y$ and a given point $y \in Y$, we denote by X_y the fibre over y, i.e. $X_y := X \times_Y \operatorname{Spec} \kappa(y)$, where $\kappa(y)$ is the residue field $\mathcal{O}_{Y,y}/\mathfrak{m}_y$.

For X in $Ob(Sch_k)$, let $CH_i(X)$ and $CH^i(X)$ denote Fulton's i-th Chow groups and let $CH_*(X) := \oplus_i CH_i(X)$ (resp. $CH^*(X) := \oplus_i CH^i(X)$), see section 2.1.

We denote by Sch_k^{fr} (resp. Sm_k^{fr}) the full subcategory of Sch_k (resp. Sm_k) consisting of those $X \in Ob(Sch_k)$ (resp. $X \in Ob(Sm_k)$) that $CH_*(X)$ is free of finite rank.

Remark 1.1.1. The category Sm_k^{fr} need not be a tensor category. Even after passing to the coefficients in $\mathbb{Q}$, it is not obvious to the authors that whether the full subcategory of Sm_k consisting of objects X with $rk_{\mathbb{Q}} K_0(X) < \infty$ is a tensor category or not. However if one assumes the Bass conjecture, then this is a trivial consequence.

CAUTION: Throughout this thesis we either assume that k admits resolution of singularities or coefficients in $\mathbb{Q}$.

Chapter 2

Motives

In this chapter we briefly present some preliminaries about the intersection theory and the theory of Motives which we widely use in the rest of the thesis.

2.1 Fulton Chow groups

Definition 2.1.1. For $X \in Sch_k$, let $z_r(X)$ be the free abelian group generated by integral closed subschemes of X of dimension r over k. Every element $Z = \sum_i n_i Z_i$ of the graded group $z_*(X) := \oplus_r z_r(X)$ is called an algebraic cycle.

To each closed subscheme $Z \subset X$ of pure dimension n, with (reduced) irreducible components $Z_1, ..., Z_r$ one may associates the following cycle in $z_*(X)$:

$$\mid Z \mid := \sum_{i=1}^{r} [\ell_{\mathcal{O}_{X,Z_i}}(\mathcal{O}_{Z,Z_i})].Z_i.$$

Here $\ell_{\mathcal{O}_{X,Z_i}}(\mathcal{O}_{Z,Z_i})$ is the length of $\mathcal{O}_{Z,Z_i}$ as an $\mathcal{O}_{X,Z_i}$-module. Conversely we define the support of a cycle $Z := \sum n_i Z_i$, with $n_i \neq 0$ for all i, to be $Supp(Z) := \cup_i Z_i$.
Let $X \in Ob(Sch_k)$. For a Cartier divisor D on X and a closed integral subscheme Z of X which is not contained in the support of D, define the intersection product $D \cdot Z$ as the support of their scheme theoretic intersection $|D \times_X Z|$. One could extend the intersection product by linearity to get an operation:

$$D \cdot - : z_n(X)_D \to z_{n-1}(D)$$

where $z_n(X)_D$ is generated by those n-dimensional closed and integral subschemes of X which are not contained in the support of D.

Recall that two cycles $z, z' \in z_d(X)$ are rationally equivalent if there is a cycle $Z \in z_{d+1}(X \times \mathbb{A}^1)_{X\times 0+X\times 1}$ such that $z - z' = (X \times 0 - X \times 1) \cdot Z$ and we write $z \sim_r z'$.

Definition 2.1.2. Define the d-th chow group $CH_d(X)$ of $X \in Ob(Sch_k)$, to be the group of algebraic cycles of dimension d modulo rational equivalence.

Suppose $\dim X = n$, we set $z^d(X) := z_{n-d}(X)$ and $CH^d(X) := CH_{n-d}(X)$.

Let us remind some basic properties of the chow groups.
i)Projective push-forward: Let $X, Y \in Ob(Sch_k)$, $f : X \to Y$ a projective morphism and $Z \subseteq X$ an integral closed subscheme of dimension n. Then $f(Z) \subseteq Y$ is a closed integral subscheme with $dim_k f(Z) \leq n$. Moreover $k(Z)/k(f(Z))$ is a finite field extension if and only if $\dim_k f(Z) = n$. We define the push-forward $f_*(Z)$ of Z as follows:

$$f_*(Z) := \begin{cases} 0 & dim_k f(Z) < n, \\ (deg_{k(f(Z))} k(Z)) \cdot f(Z) & dim_k f(Z) = n \;, \end{cases}$$

By linearity, this extends to the following morphism:

$$f_* : z_n(X) \to z_n(Y)$$

Note that push-forward is functorial (i.e. $(gf)_* = g_* f_*$), and descends to a morphism:

$$f_* : CH_n(X) \to CH_n(Y)$$

ii) Pull back: Let $f : X \to Y$ be a morphism in Sch_k where Y is smooth over k. For simplicity suppose Y and X are integral. Take an integral subscheme $W \subset Y$ of codimension n. One says that W is in *good position* for f if each irreducible component Z of $f^{-1}(W)$ has codimension n as a subscheme of X. Define the partially defined pull back morphism:

$$\begin{array}{rcc} z^n(Y)_f & \xrightarrow{f^*} & z^n(X) \\ W & \mapsto & \sum_Z m(W, Z; f) Z \end{array}$$

Here $z^n(Y)_f$ is the subgroup of $z^n(Y)$ generated by all cycles W in good position for f and the sum is taken over all irreducible components Z of $f^{-1}(W)$. The intersection multiplicity $m(W, Z; f)$ is given by Serre's formula:

$$m(W, Z; f) := \sum_{i \geq 0} (-1)^i \ell_{\mathcal{O}_{X,Z}} (Tor_i^{\mathcal{O}_{Y,W}} (\mathcal{O}_W, \mathcal{O}_{X,Z}))$$

Note that the assumption that Y is smooth, ensures that the above sum is finite. Finally f^* descends to a well-defined pull back for chow groups:

$$f^* : CH^n(Y) \to CH^n(X).$$

This is actually one of the main features of the definition of rational equivalence. Note that for a morphism $f : X \to Y$ in Sm_k, f^* is a ring homomorphism. Moreover if both Y and X are smooth, this pull back morphism is functorial, i.e. $(fg)^* = g^* f^*$. One can define pull back morphism in general case by Fulton's method [Fu] or the K-theoretic approach of Quillen-Grayson [Gr].

Remark 2.1.3. Consider the Cartesian diagram:

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g'}\downarrow & & {\scriptstyle g}\downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

where f and g are morphisms in Sm_k, f is projective and $X' := X \times_Y Y'$ is smooth of dimension $\dim X + \dim Y' - \dim Y$. Then $g^* f_* = f'_* g'^*$.

iii)Products: Let $W \subseteq X$ and $W' \subseteq Y$ be integral subschemes of dimensions n and m respectively. Define $|W \times_k W'|$ to be the external product of W and W'. Extending by linearity, one can define the external product

$$* : z_n(X) \otimes z_m(Y) \to z_{n+m}(X \times Y),$$

which is commutative, associative and unital. This product descends to an external product

$$* : CH_n(X) \otimes CH_m(Y) \to CH_{n+m}(X \times Y).$$

Let $\delta : X \to X \times X$ be the diagonal morphism and assume that X is smooth. When X=Y, composing the external product with δ^* gives the cup product:

$$\cup_X : CH^n(X) \otimes CH^m(X) \to CH^{n+m}(X), \quad a \cup_X b := \delta^*(a * b)$$

which makes $CH^*(X)$ into a commutative graded ring.

Remark 2.1.4. *(projection formula)* For a projective morphism $f : Y \to X$ in Sm_k, $a \in CH^n(X)$ and $b \in CH^m(Y)$ one has:

$$f_*(f^*(a) \cup_Y b) = a \cup_X f_*(b),$$

see [Fu, chapter 4].

2.2 Chow motives

A cycle of dimension $n := dimX$ on $X \times Y$ is called a correspondence from X to Y. Let X, Y and Z be smooth and projective schemes, we can compose the correspondences $V \in CH_{dimX}(X \times Y)$ and $W \in CH_{dimY}(Y \times Z)$in the following way

$$W \circ V := p_{XZ*}(p^*_{XY}(V) \cdot p^*_{YZ}(W)).$$

Remark 2.2.1. Let $SmProj_k$ be the category of smooth projective k-schemes. For $f : X \to Y$ a morphism in $SmProj_k$, we have the class $[\Gamma_f] \in CH_{dimX}(X \times Y)$ of the graph of f. One easily checks that for morphisms $f : X \to Y$, $g : Y \to Z$, we have $[\Gamma_g] \circ [\Gamma_f] = [\Gamma_{gf}]$.

Definition 2.2.2. Form the category of Chow correspondences $Cor_{CH}(k)$ consisting of the objects $[X]$ for $X \in SmProj_k$, and morphisms

$$Hom_{Cor_{CH}(k)}([X], [Y]) := \bigoplus_{i=1}^{n} CH_{dimX_i}(X_i \times Y)$$

where $X_1, ..., X_r$ are irreducible components of X. The composition of correspondences is as above.

Let us briefly recall the categorical notion of the pseudo-abelian envelope of an additive category $\mathcal{C}$. We call that $\mathcal{C}$ is pseudo-abelian if for each idempotent endomorphism $P : A \to A$, $P^2 = P$, there exist objects A_0 and A_1 of the category $\mathcal{C}$ and an isomorphism $Q : A \to A_0 \oplus A_1$ such that

$$Q \circ P \circ Q^{-1} = 0_{A_0} \oplus id_{A_1}.$$

Given an arbitrary additive category $\mathcal{C}$, there is an additive functor from $\mathcal{C}$ to a pseudo-abelian category $Split(\mathcal{C})$ like $\imath : \mathcal{C} \to Split(\mathcal{C})$, which is universal for additive functors of $\mathcal{C}$ to pseudo-abelian categories. The category $Split(\mathcal{C})$ is constructed as follows:
The objects of $Split(\mathcal{C})$ are pairs (A, P) with $A \in \mathcal{C}$ and $P : A \to A$ an idempotent endomorphism. The morphisms are given by

$$Hom_{Split(\mathcal{C})}((A, P), (A', P')) := \{P' \circ f \circ P | f \in Hom_{\mathcal{C}}(A, A')\}$$

with composition $(P'' \circ g \circ P') \circ (P' \circ f \circ P) := P'' \circ (g \circ P' \circ f) \circ P$. The functor $\imath$ is given by $\imath(A) := (A, id)$. For a projector $P : A \to A$, the identity map on A gives the following isomorphism in $Split(\mathcal{C})$

$$(A, id) \cong (A, 1 - P) \oplus (A, P).$$

This is then clear that $Split(\mathcal{C})$ is pseudo-abelian.

Definition 2.2.3. The category $M^{eff}(k)$ of *effective homological Chow motives over* k is the pseudo-abelian envelope $Split(Cor_{CH}(k))$ of $Cor_{CH}(k)$. We denote the object $([X], id)$ of $M^{eff}(k)$ by $M_{CH}(X)$.

Remark 2.2.4. If one take the correspondence group $\oplus_{i=1}^{n} CH^{dimX_i}(X_i \times Y)$ as the morphisms from X to Y instead of $\oplus_{i=1}^{n} CH_{dimX_i}(X_i \times Y)$, the analogous construction leads to the category of *effective cohomological motives* $M_{eff}(k)$, and the functor:

$$M^{CH} : Sm_k^{op} \to M_{eff}(k).$$

The category $M^{eff}(k)$ inherits the additive structure of $Cor(k)$. Moreover, it is a tensor category with respect to the tensor product:

$$(X, P) \otimes (Y, P') = (X \times_{\operatorname{Spec} k} Y, P \times P').$$

Here $X,\ Y \in Cor_{CH}(k)$ and $P,\ P'$ are idempotent endomorphisms on X and Y respectively. Since $X \times_{\operatorname{Spec} k} \operatorname{Spec} k \cong X$, the motive $M_{CH}(\operatorname{Spec} k)$ is the neutral element for tensor multiplication.

Definition 2.2.5. Let x be a k-point of $\mathbb{P}_k^1$, and P the class of $\mathbb{P}_k^1 \times x$ in $CH_1(\mathbb{P}_k^1 \times \mathbb{P}_k^1)$. One can see $(\mathbb{P}_k^1, P) \in M^{eff}(k)$ does not depend on the choice of x. We call this motive the *Tate motive* and will denote it by $\mathbb{L}$. One has $M_{CH}(\mathbb{P}_k^1) \cong \mathbb{L} \oplus M_{CH}(\operatorname{Spec} k)$.

We denote by $\mathbb{L}^n$ the n-th tensor power of $\mathbb{L}$. For $M \in M^{eff}(k)$, set $M(n) := M \otimes \mathbb{L}^n$

Let $X,\ Y \in SmProj_k$. Let $X_1, ..., X_n$ be the irreducible components of X.For non-negative integers $r,\ t$ we have

$$Hom_{M^{eff}}(M_{CH}(X)(r), M_{CH}(Y)(t)) = \bigoplus_{i=1}^{n} CH_{dimX_i + r - t}(X_i \times Y). \quad (2.2.1)$$

2.3 Geometric Motives

As we have seen in the previous section, one can associates an object of the category of effective chow motives to a smooth projective scheme X. Moreover recall that the morphisms in this category are given by rational equivalence classes of cycles. One might naturally look for a way to associate a motive to quasi-projective schemes, while in this case chow ring is incapable to keep the records of all morphisms (e.g. $CH_*(\mathbb{A}^n) = \mathbb{Z}$). In this section we briefly explain the V. Voevodsky's method, which in particular

serves this desire.
To introduce the Voevodsky's effective geometric motives, we need to define *finite correspondences*.

Definition 2.3.1. Let $X, Y \in Ob(Sch_k)$. The group $c(X, Y)$ is the subgroup of $z(X \times_k Y)$ generated by those integral closed subschemes W of $X \times_k Y$ that satisfy the following:

i) The projection $p_1 : W \to X$ is finite,

ii) $p_1(W)$ is an irreducible component of X.

The elements of $c(X, Y)$ are called *finite correspondences* from X to Y.

Definition 2.3.2. We define the category of correspondences $Cor(k)$, as the category whose objects are the smooth schemes over k and morphisms as follows:

$$Hom_{Cor(k)}(X, Y) := c(X, Y).$$

We write $[X]$ to denote the scheme X as an object of $Cor(k)$. We denote by f_* the morphism in $c(X, Y)$ associated with the graph Γ_f of $f : X \to Y$. This establishes the functor $Sch_k \to Cor(k)$.
We write $Cor(X, Y)$ to denote $Hom_{Cor(k)}(X, Y)$.

Definition 2.3.3. Consider the homotopy category $\mathcal{H}^b(Cor(k))$ of bounded complexes over $Cor(k)$. Assume that T is the minimal thick subcategory of $\mathcal{H}^b(Cor(k))$ which contains all the classes of complexes of the following two forms:

1) $p_* : [X \times \mathbb{A}^1] \to [X]$, for every $X \in Sm_k$ (p is the projection morphism).

2) $[U \cap V] \xrightarrow{[j_U] \oplus [j_V]} [U] \oplus [V] \xrightarrow{[i_U] \oplus (-[i_V])} [X]$, for any $X \in Sm_k$ and an open covering $X = U \cup V$. Here the morphisms j_U, j_V, i_U, i_V are the obvious open embeddings.

Definition 2.3.4. The category $DM^{eff}_{gm}(k)$ of *effective geometric motives* is the pseudo-abelian envelope of the localization of $\mathcal{H}^b(Cor(k))$ (as a triangulated tensor category) with respect to T. Let M_{gm} denote the obvious functor

$$Sm_k \to DM^{eff}_{gm}(k).$$

We denote the unit object $M_{gm}(\operatorname{Spec} k)$ of the tensor structure by $\mathbb{Z}$. For any $X \in Ob(Sm_k)$, consider the morphism $M_{gm}(X) \to \mathbb{Z}$ induced by the morphism $X \to \operatorname{Spec} k$. Thus we get the following canonical distinguished triangle

$$\tilde{M}_{gm}(X) \to M_{gm}(X) \to \mathbb{Z} \to \tilde{M}_{gm}(X)[1]$$

where $\tilde{M}_{gm}(X)$ is the *reduced motive* of X represented in $\mathcal{H}^b(Cor(k))$ by the complex $[X] \to [\operatorname{Spec} k]$.

Definition 2.3.5. Define the Tate object $\mathbb{Z}(1)$ of $DM_{gm}^{eff}(k)$ as $\tilde{M}_{gm}(\mathbb{P}^1)[-2]$. Further let $\mathbb{Z}(n)$ be the n-th tensor power of $\mathbb{Z}(1)$. For any A in $DM_{gm}^{eff}(k)$, denote by $A(n)$ the object $A \otimes \mathbb{Z}(n)$.

Definition 2.3.6. The triangulated category $DM_{gm}(k)$ of geometric motives over k is the category obtained from $DM_{gm}^{eff}(k)$ by inverting $\mathbb{Z}(1)$. In other words it has objects of the form $A(n)$ for A in $DM_{gm}^{eff}(k)$ and $n \in \mathbb{Z}$, together with morphisms

$$Hom_{DM_{gm}(k)}(A(n), B(m)) := \varinjlim_{N} Hom_{DM_{gm}^{eff}(k)}(A \otimes \mathbb{Z}(n+N), B \otimes \mathbb{Z}(m+N)).$$

Remark 2.3.7. There is functor $\imath : DM_{gm}^{eff}(k) \to DM_{gm}(k)$, defined by sending A to $A(0)$. For the morphisms use the canonical embedding

$$Hom_{DM_{gm}^{eff}(k)}(A, B) \to \varinjlim_{N} Hom_{DM_{gm}^{eff}(k)}(A \otimes \mathbb{Z}(N), B \otimes \mathbb{Z}(N)).$$

Note that for $n \geq 0$, we have the evident map $\imath(A \otimes \mathbb{Z}(n)) \to A(n)$, which is an isomorphism.

Theorem 2.3.8. *The functor $\imath : DM_{gm}^{eff}(k) \to DM_{gm}(k)$ is a fully faithful embedding.*

Proof. See [VSF, Chapter 5, Theorem 4.3.1]. □

2.4 The category of motivic complexes

Definition 2.4.1. A *presheaf* $\mathcal{P}$ on a small category $\mathcal{C}$ with values in a category $\mathcal{A}$ is a functor

$$\mathcal{P} : \mathcal{C}^{op} \to \mathcal{A}.$$

The category $PreShv^{\mathcal{A}}(\mathcal{C})$ of $\mathcal{A}$-valued presheaves on $\mathcal{C}$, is a category with above functors as its objects and natural transformations of such functors as its morphisms. For a morphism $f : A \to B$ in $\mathcal{C}$ we will denote the morphism $P(f) : P(B) \to P(A)$ by f^*.

Remark 2.4.2. i) If $\mathcal{A}$ is an abelian category then $PreShv^{\mathcal{A}}(\mathcal{C})$ is also an abelian category. The kernel and cokernel for $f : \mathcal{F} \to \mathcal{G}$ are defined as follows:

$$ker(f)(x) = ker(f(x) : \mathcal{F}(x) \to \mathcal{G}(x))$$

$$coker(f)(x) = coker(f(x) : \mathcal{F}(x) \to \mathcal{G}(x))$$

ii) The category $PreShv^{\mathbf{Ab}}(\mathcal{C})$ of presheaves with values in the category of abelian groups, has enough injectives, see Grothendieck's Tohoko article [Gro].

Notation: Let τ be a Grothendieck pre-topology on the category $\mathcal{C}$ and let $X \in Ob(\mathcal{C})$. We denote by $Cov_\tau(X)$ the set of covering families of X.

Definition 2.4.3. Let $\mathcal{P}$ be a presheaf of abelian groups on $\mathcal{C}$. For every covering family $\{f_i : U_i \to X\} \in Cov_\tau(X)$ (for a pre-topology τ), Let $p_{1,ij} : U_i \times_X U_j \to U_i$ and $p_{2,ij} : U_i \times_X U_j \to U_j$ be the obvious projections. We say $\mathcal{P}$ is a sheaf for τ, if for any covering family the following sequence is exact:

$$0 \to \mathcal{P}(X) \xrightarrow{(f_i^*)} \prod_i \mathcal{P}(U_i) \xrightarrow{(p_{1,ij}^* - p_{2,ij}^*)} \prod_{i,j} \mathcal{P}(U_i \times_X U_j).$$

Definition 2.4.4. Let X be a k-scheme of finite type. A *Nisnevich cover* $\mathcal{U} \to X$ is an etale morphism of finite type such that for each finitely generated separable field extension F of k, the map on F-valued points $\mathcal{U}(F) \to X(F)$ is surjective.

Considering Nisnevich covers as the covering families, one get the *small Nisnevich site* X_{Nis}, on X. One can define the *big Nisnevich site* in the similar way, take the underlying category to be Sm_k and for any X in Sm_k define the covering families of X as that of X_{Nis}.

Let $Sh^{Nis}(X)$ denote the category of Nisnevich sheaves of abelian groups on X and $Sh^{Nis}(k)$ denote the category of Nisnevich sheaves of abelian groups on Sm_k. For a presheaf $\mathcal{F}$ on X_{Nis} or Sm_k, we let $\mathcal{F}_{Nis}$ denote the associated sheaf.

Definition 2.4.5. Let $\mathcal{F}$ and $\mathcal{G}$ be presheaves of abelian groups. Define their tensor product $\mathcal{F} \otimes \mathcal{G}$, as the presheaf which sends X to

$$(\mathcal{F} \otimes \mathcal{G})(X) := \mathcal{F}(X) \otimes_{\mathbb{Z}} \mathcal{G}(X)$$

One also defines the internal Hom, $\mathcal{H}om(\mathcal{F}, \mathcal{G})$, as follows

$$\mathcal{H}om(\mathcal{F}, \mathcal{G})(X) := Hom_{PreShv^{\mathbf{Ab}}(Sm_k)}(\mathcal{F} \otimes \mathbb{Z}(X), \mathcal{G})$$

where $\mathbb{Z}(X)$ denotes the presheaf of abelian groups on Sm_k freely generated by $Hom_{Sch_k}(-, X)$.

The category $Sh^{Nis}(Sm_k)$ is a tensor category with tensor given by the shififying the presheaf obtained by the above tensor product. Moreover the internal Hom is given by

$$\mathcal{H}om(\mathcal{F}, \mathcal{G})(X) := Hom_{Sh^{Nis}(Sm_k)}(\mathcal{F} \otimes \mathbb{Z}_{Nis}(X), \mathcal{G}).$$

Definition 2.4.6.

i) The category $PST(k)$ of presheaves with transfer is the category of additive presheaves of abelian groups on $Cor(k)$, i.e. the category of additive functors $\mathcal{F} : Cor(k)^{op} \to \mathbf{Ab}$.

ii) The category $Sh^{Nis}(Cor(k))$ of Nisnevich sheaves with transfer on Sm_k is the full subcategory of $PST(k)$ whose objects are those presheaves $\mathcal{F}$ such that, for each $X \in ob(Sm_k)$, the restriction of $\mathcal{F}$ to X_{Nis} is a sheaf.

Remark 2.4.7. In fact a presheaf with transfer $\mathcal{F}$ is a presheaf on Sm_k together with transfer maps $Tr(a) : \mathcal{F}(Y) \to \mathcal{F}(X)$ for every $a \in Cor(X, Y)$ which satisfies:

i) $Tr(\Gamma_f) = f^*$.

ii) $Tr(a \circ b) = Tr(b) \circ Tr(a)$.

iii) $Tr(a \pm b) = Tr(a) \pm Tr(b)$

Definition 2.4.8. For every $X \in Ob(Sch_k)$, one associates the Nisnevich sheaf with transfers $L(X)$ defined by

$$L(X) : Y \mapsto Cor(Y, X).$$

Remark 2.4.9. The sheaf $L(X)$ is freely generated by the representable sheaf of sets $Hom(-,X)$.
In particular, we have the canonical isomorphism

$$Hom_{Sh^{Nis}(Cor(k))}(L(X),\mathcal{F}) \cong \mathcal{F}(X).$$

In fact this is the special case of the canonical isomorphism

$$Ext^n_{Sh^{Nis}(Cor(k))}(L(X),\mathcal{F}) \cong H^n(X_{Nis},\mathcal{F}).$$

Remark 2.4.10. Let $X \in Ob(Sch_k)$. In the definition 2.3.1 of finite correspondences if we replace the condition of finiteness of the projection $p_1 : W \to X$, with quasi finiteness, then in a similar way as the above definition of $L(X)$, we can define a Nisnevich sheaf with transfer which we denote by $L^c(X)$.

The presheaves $L(X)$ (resp. $L^c(X)$) are covariantly functorial with respect to X (resp. with respect to proper morphisms on X). We have:

$$L(-) : Sck/k \to PreShv^{\mathbf{Ab}}(Cor(k))$$

$$L^c(-) : Sck^{prop}/k \to PreShv^{\mathbf{Ab}}(Cor(k))$$

Remark 2.4.11. One can define a tensor structure on $Sh^{Nis}(Cor(k))$ and $PST(k)$. There is also an internal Hom in $Sh^{Nis}(Cor(k))$ and $PST(k)$. See [Le, Section 4.2].

Definition 2.4.12. Let $\mathcal{F}$ be an object of $PreShv^{\mathbf{Ab}}(Sm_k)$. We say that $\mathcal{F}$ is *homotopy invariant* if for any smooth scheme X over k the morphism

$$p^* : \mathcal{F}(X) \to \mathcal{F}(X \times \mathbb{A}^1),$$

is an isomorphism, where p is the projection to the first factor. We say $\mathcal{F}$ is *strictly homotopy invariant* if for any $q \geq 0$ the cohomology presheaf $X \mapsto H^q(X_{Nis},\mathcal{F}_{Nis})$ is homotopy invariant.

Lemma 2.4.13. *Let $X \in Ob(Sm_k)$, S be a finite set of its points and $j_U : U \to X$ an open subscheme. Then there exists an open neighbourhood $j_V : V \to X$ of S in X and a finite correspondence $a \in Cor(V,U)$ such that for every homotopy invariant presheaf with transfer $\mathcal{F}$, one has*

$$Tr(a) \circ j_U^* = j_V^*.$$

Proof. cf. [VSF, Chapter 3, Lemma 4.17]. □

Definition 2.4.14. The category $DM_{-}^{eff}(k)$ is the full subcategory of the derived category $D^{-}(Sh^{Nis}(Cor(k)))$ consisting of complexes whose cohomology sheaves are homotopy invariant.

Proposition 2.4.15. *$DM_{-}^{eff}(k)$ is a triangulated subcategory of the category $D^{-}(Sh^{Nis}(Cor(k)))$.*

Proof. cf. [VSF, Chapter 5, Section 3.1]. □

2.5 Embedding theorem I

Let X be a scheme of finite type over a field k and $r \geqslant 0$ an integer. Define $z_{equi}(X, r)$ to be the presheaf on the category of smooth schemes, which sends a smooth scheme Y to the free abelian group generated by closed and integral equidimensional subschemes of $X \times Y$ of relative dimension r over Y. It is easy to show that $z_{equi}(X, r)$ is a sheaf in the Nisnevich topology and that it has a canonical structure of presheaf with transfers.
One can easily verify that any morphism $f : Y' \to Y$ induces a unique homomorphism $z(f) : z_{equi}(X, r)(Y) \to z_{equi}(X, r)(Y')$ such that the following holds:

1) For $Y'' \xrightarrow{g} Y' \xrightarrow{f} Y$ one has $z(fg) = z(g)z(f)$.

2) For a dominant morphism $f : Y' \to Y$ one has

$$z(f)(\sum n_i Z_i) = \sum n_i \mid Z_i \times_Y Y' \mid$$

   where $\mid Z_i \times_Y Y' \mid$ is the associated cycle to $Z_i \times_Y Y'$ in $z_*(X \times Y')$

3) For a morphism $f : Y' \to Y$ and a closed subscheme Z of $X \times Y$ which is flat and equidimensional of relative dimension r over Y, we have $z(f)(|Z|) = |Z \times_Y Y'|$. Here $|Z|$ (resp. $|Z \times_Y Y'|$) is the cycle associated to Z (resp. $Z \times_Y Y'$) in $X \times Y$ (resp. $X \times Y'$))

Let X be a scheme of finite type over k, V a smooth scheme and U a smooth euidimensional scheme of finite type. Obviously any elements of $z_{equi}(X,r)(U \times V)$ belongs to $z_{equi}(X \times U, r + dimU)(V)$. Moreover for any morphism $f : V' \to V$ of smooth schemes over k, one can verify the following equality of cycles on $X \times U \times V'$:

$$z(f)(\mathcal{Z}) = z(Id_U \times f)(\mathcal{Z}). \tag{2.5.2}$$

See [VSF, Chapter2, Theorem 3.7.3].
Let $X \in Ob(Sch_k)$ and $U, V \in Ob(Sm_k)$. We let $z_{equi}(U,X,r)(V)$ denote the group $z_{equi}(X,r)(U \times V)$. In fact $z_{equi}(U,X,r)$ is a presheaf on smooth schemes which is contravariantly functorial with respect to U. The equation (2.5.2) asserts that there is a canonical embedding of presheaves

$$D : z_{equi}(U,X,r) \to z_{equi}(X \times U, r + dimU),$$

which is consistent with covariant (contravariant) functoriality of both of the above presheaves with respect to proper (resp. flat equidimensional) morphisms $X \to X'$.

Set $\Delta^n := \mathrm{Spec}(k[x_0, ..., x_n]/\sum x_i = 1)$. As in topological situation one can define boundary and degeneracy morphisms:

$$\partial_i^n : \Delta^{n-1} \to \Delta^n$$

$$\sigma_i^n : \Delta^{n+1} \to \Delta^n$$

and then get the cosimplicial object $\Delta^\bullet(\Delta^n, \partial_i^n, \sigma_i^n)$ in the category Sm_k of smoot schemes.
Let $\mathcal{F}$ be a presheaf of abelian groups on Sch_k. For every integer n, one defines the presheaf $C_n(\mathcal{F})$ by sending $X \in Ob(Sch_k)$ to:

$$C_n(\mathcal{F})(X) := \mathcal{F}(\Delta^n \times X).$$

One can form a complex $C_*(\mathcal{F})$ of presheaves such that the differential is given by alternating sums of homomorphisms $\mathcal{F}(\partial_i^n \times Id_X)$. The complex $C_*(\mathcal{F})$ is called the *singular simplicial complex of* $\mathcal{F}$. Denote the cohomology presheaves of $C_*(\mathcal{F})$ by:

$$\underline{h}_i(\mathcal{F}) = H^{-i}(C_*(\mathcal{F}))$$

We will see below that $\underline{h}_i(\mathcal{F})$ are homotopy invariant.

Lemma 2.5.1. *Let $\mathcal{F}$ be a presheaf on Sch_k (resp. Sm_k). For any $X \in Ob(Sch_k)$ (resp. $X \in Ob(Sm_k)$) and any integer $i \in \mathbb{Z}$, the projection $X \times \mathbb{A}^1 \to X$ induces an isomorphism:*

$$\underline{h}_i(\mathcal{F})(X) \to \underline{h}_i(\mathcal{F})(X \times \mathbb{A}^1)$$

Proof. Let $i_0, i_1 : X \to X \times \mathbb{A}^1$ be the closed embeddings $Id_X \times \{0\}$ and $Id_X \times \{1\}$ respectively. First we will show that these two morphisms induce homotopic morphisms of complexes of abelian groups

$$i_0^*, i_1^* : C_*(\mathcal{F})(X \times \mathbb{A}^1) \to C_*(\mathcal{F})(X).$$

Define a homomorphism

$$s_n := \sum_{i=0}^{n} (-1)^i (Id_X \times \psi_i)^* : \mathcal{F}(X \times \mathbb{A}^1 \times \Delta^n) \to \mathcal{F}(X \times \Delta^{n+1})$$

where $\psi_i : \Delta^{n+1} \to \Delta^n \times \mathbb{A}^1$ is a linear isomorphism such that:

$$\psi_i(\nu_j) = \begin{cases} \nu_j \times 0 & \text{for } j \leq i \ , \\ \nu_{j-1} \times 1 & \text{for } j > i \ , \end{cases}$$

Here $\nu_j = (0, ..., 1, .., 0)$ is the j-th vertex of Δ^{n+1} (resp. Δ^n). One can show easily that $sd + ds = i_1^* - i_0^*$. So $i_0^*, i_1^* : C_*(\mathcal{F})(X \times \mathbb{A}^1) \to C_*(\mathcal{F})(X)$ are homotopic and therefore

$$i_0^* : \underline{h}_i(\mathcal{F})(X \times \mathbb{A}^1) \to \underline{h}_i(\mathcal{F})(X)$$

and

$$i_1^* : \underline{h}_i(\mathcal{F})(X \times \mathbb{A}^1) \to \underline{h}_i(\mathcal{F})(X)$$

coincide.
Using multiplication morphism $\mu : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$, we define the following morphisms:

$$I_0 : X \times \mathbb{A}^1 \xrightarrow{i_0} (X \times \mathbb{A}^1) \times \mathbb{A}^1 \xrightarrow{Id_X \times \mu} X \times \mathbb{A}^1$$

$$I_1 : X \times \mathbb{A}^1 \xrightarrow{i_1} (X \times \mathbb{A}^1) \times \mathbb{A}^1 \xrightarrow{Id_X \times \mu} X \times \mathbb{A}^1$$

By the above i_0^* and i_1^* coincide and therefore

$$I_0^*, I_1^* : \underline{h}_i(\mathcal{F})(X \times \mathbb{A}^1) \to \underline{h}_i(\mathcal{F})(X \times \mathbb{A}^1)$$

coincide. But I_1 is in fact the identity morphism and I_0 is equal to the following composition:

$$f : X \times \mathbb{A}^1 \xrightarrow{pr_1} X \xrightarrow{i_0} X \times \mathbb{A}^1$$

Therefore the induced homomorphism $f^* : \underline{h}_i(\mathcal{F})(X \times \mathbb{A}^1) \to \underline{h}_i(\mathcal{F})(X \times \mathbb{A}^1)$ is identity. So we conclude. $\square$

-Let us recall the following facts
Let

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

be a short exact sequence of presheaves of abelian groups on either Sch_k or Sm_k. Then

$$0 \to C_*(\mathcal{F}) \to C_*(\mathcal{G}) \to C_*(\mathcal{H}) \to 0$$

is exact sequence of complexes of presheaves. Therefore we get the following long exact sequence:

$$... \to \underline{h}_i(\mathcal{F}) \to \underline{h}_i(\mathcal{G}) \to \underline{h}_i(\mathcal{H}) \to \underline{h}_{i-1}(\mathcal{F}) \to ...$$

Let k be any field. Here we list up a sort of interesting features of bivariant cycle cohomology, which surprisingly unifies Borel-Moore homology, Milnor K-theory and étale cohomology. See [VSF, Chapter 4, pages 152-153]

- Let Y be any smooth scheme over k. For any scheme of finite type X over k, $\underline{h}_i(z_{equi}(X,r))(Y)$ is contravariantly functorial with respect to U.

- Let Y be any smooth scheme over k. For any scheme of finite type X over k, $\underline{h}_i(z_{equi}(X,r))(Y)$ is covariantly functorial with respect to proper morphisms in X.

- Let Y be any smooth scheme over k. For any scheme of finite type X over k, $\underline{h}_i(z_{equi}(X,r))(Y)$ is contravariantly functorial with respect to flat equidimensional morphisms in X.

- The groups $\underline{h}_n(z_{equi}(\mathbb{A}^n, 0))(\operatorname{Spec} k)$ are isomorphic to the Milnor K-groups $K_n^M(k)$.

- If k is an algebraically closed field which admits resolution of singularities, X is a smooth scheme over k of dimension m, and $n \neq 0$ is an integer prime to $char(k)$, the groups
$$\underline{h}_i(z_{equi}(X,0) \otimes \mathbb{Z}/n\mathbb{Z})(\mathrm{Spec}\, k)$$
are isomorphic to the etale cohomology groups $H^{2m-i}_{et}(X, \mathbb{Z}/n\mathbb{Z})$.

- For any scheme X over k the group $\underline{h}_0(z_{equi}(X,r))(\mathrm{Spec}\, k)$ is canonically isomorphic to the group $CH_r(X)$ of cycles of dimension r on X modulo rational equivalences.

- For any equidimensional affine scheme X over k and any $r \geq 0$ there are canonical isomorphisms
$$\underline{h}_i(z_{equi}(X,r))(\mathrm{Spec}\, k) \to CH^{dimX-r}(X,i)$$
where the groups on the right hand side are higher chow groups of X.

- Assume that k is of characteristic 0 and X is a normal equidimensional scheme of pure dimension n or that k is a perfect field and that X is a normal affine scheme of pure dimension n. Then the groups $\underline{h}_i(z_{equi}(X, n-1))(\mathrm{Spec}\, k)$ are of the form:
$$\underline{h}_i(z_{equi}(X,n-1))(\mathrm{Spec}\, k) = \begin{cases} CH_{n-1}(X) & \text{for } i = 0 \ , \\ \mathcal{O}^*(X) & \text{for } i = 1 \ , \\ 0 & \text{for } i \neq 0, 1, \end{cases}$$

Theorem 2.5.2. *Let X, Y be smooth projective equidimensional schemes over a field k. Then the embedding of presheaves*
$$\mathcal{D} : z_{equi}(Y, X, r) \to z_{equi}(X \times Y, r + dimY)$$
induces isomorphisms
$$\underline{h}_i(z_{equi}(Y, X, r)) \to \underline{h}_i(z_{equi}(X \times Y, r + dimY))$$
for all $i \in \mathbb{Z}$.

Proof. See [VSF, Chapter 4, Theorem.7.1]. □

Remark 2.5.3. In the above theorem, when k admits resolution of singularities then one can remove the hypotheses that X and Y are projective and smooth, cf. [VSF, Chapter 4].

Theorem 2.5.4. *There exists a functor* $M^{eff}(k) \to DM_{gm}^{eff}(k)$ *such that*

$$\begin{array}{ccc} SmProj_k & \longrightarrow & Sm_k \\ {\scriptstyle M_{CH}(-)}\downarrow & & {\scriptstyle M_{gm}(-)}\downarrow \\ M^{eff}(k) & \longrightarrow & DM_{gm}^{eff}(k) \end{array}$$

commutes.

Proof. It is enough to show that for $X, Y \in Smproj_k$, there is a canonical homomorphism:

$$CH_{dimX}(X \times Y) \to Hom_{DM_{gm}^{eff}}(M_{gm}(X), M_{gm}(Y))$$

There exists a canonical morphism

$$f : c(X,Y) \to Hom_{DM_{gm}^{eff}}(M_{gm}(X), M_{gm}(Y)).$$

Define the morphism

$$g : c(X \times \mathbb{A}^1, Y) \to c(X,Y)$$

which sends a correspondence $s \in c(X \times \mathbb{A}^1, Y)$ to $s|_{X\times\{0\}\times Y} - s|_{X\times\{1\}\times Y}$. One can easily verify that these two morphisms form a complex (i.e. $f \circ g = 0$). To complete the proof it only remains to show that $\operatorname{coker} g$ is isomorphic to $CH_{dimX}(X \times Y)$. Consider the isomorphism on cohomology presheaves for $i = 0$ in theorem 2.5.2. By taking the sections over $\operatorname{Spec} k$, this isomorphism gives the following isomorphism of groups:

$$\underline{h}_0(z_{equi}(Y, X, r)) \xrightarrow{\sim} \underline{h}_0(z_{equi}(X \times Y.r + dimY))$$

By the properties we mentioned for presheaves $\underline{h}_i(z_{equi}(X, r))$ and definition of $\underline{h}_0(z_{equi}(Y, X, r))$ we get the desired isomorphism:

$$\operatorname{coker} g \xrightarrow{\sim} CH_{dimX}(X \times Y).$$

□

2.6 Embedding theorem II

In this section we state a theorem of Voevodsky which shows that there is a natural full embedding of tensor triangulated categories from $DM_{gm}^{eff}(k)$ to the category $DM_{-}^{eff}(k)$.
In this section we will assume that k is perfect.

Remark 2.6.1. As we have seen in 2.5.1, $\underline{h}_i(\mathcal{F})$ are homotopy invariant. Therefore by theorem [VSF, Chapter 3, Theorem 4.27 and Theorem 5.7], the associated Nisnevich sheaves $\underline{h}_i^{Nis}(\mathcal{F})$ are strictly homotopy invariant for every presheaf with transfer $\mathcal{F}$. Therefore there is a functor

$$C_* : PST(k) \to DM_{-}^{eff}(k),$$

sending $\mathcal{F}$ to $C_*(\mathcal{F})$ which factors through the canonical functor

$$PST(k) \to Sh^{Nis}(Cor(k)).$$

We will denote the functor $Sh^{Nis}(Cor(k)) \to DM_{-}^{eff}(k)$ with the same notation C_*.

Remark 2.6.2. The functor C_* from the category of Nisnevich sheaves with transfer on Sm_k to $DM_{-}^{eff}(k)$ can be extended to a functor:

$$\mathbf{R}C : D^{-}(Sh^{Nis}(Cor(k))) \to DM_{-}^{eff}(k)$$

which is left adjoint to the natural embedding. The above functor identifies $DM_{-}^{eff}(k)$ with localization of $D^{-}(Sh^{Nis}(Cor(k)))$ with respect to the localizing subcategory generated by complexes of the form

$$L(X \times \mathbb{A}^1) \xrightarrow{p_1} L(X), \quad X \in Sm_k,$$

where p_1 is induced by the first projection, see [VSF, Chapter 5, Proposition 3.2.3].

Consider the functor

$$L : Cor(k) \to Sh^{Nis}(Cor(k)),$$

sending X to the representable sheaf $L(X)$. One can extend this to the homotopy category of bounded complexes:

$$L : \mathcal{H}^b(Cor(k)) \to D^{-}(Sh^{Nis}(Cor(k)))$$

Now we state the following important embedding theorem, due to V. Voevodsky, which enables one to apply the machinery of sheaves to the category $DM_{gm}^{eff}(k)$ of effective geometric motives.

Theorem 2.6.3. *Let k be perfect. There is a commutative diagram of exact tensor functors:*

$$\begin{array}{ccc} \mathcal{H}^b(Cor(k)) & \xrightarrow{L} & D^-(Sh^{Nis}(Cor(k))) \\ \downarrow & & \downarrow \boldsymbol{R}C \\ DM_{gm}^{eff}(k) & \xrightarrow{\imath} & DM_-^{eff}(k) \end{array}$$

such that

i) $\imath$ *is a full embedding with dense image, and*

ii) $\boldsymbol{R}C(L(X)) \cong C_*(X)$.

Proof. See [VSF, Chapter 5, Theorem 3.2.6]. □

2.7 Motives with compact support

Recall that as we have seen in the previous section, see remarks 2.4.10 and 2.6.1, one has the following two functors:

$$\begin{aligned} \underline{C}_*(-) : Sch_k &\to DM_-^{eff}(k) \\ X &\mapsto C_*(L(X)) \end{aligned}$$

and

$$\begin{aligned} \underline{C}_*^c(-) : Sch_k^{prop} &\to DM_-^{eff}(k) \\ X &\mapsto C_*(L^c(X)). \end{aligned}$$

Remark 2.7.1. If k admits resolution of singularities, then one can show that the above functors factor through the canonical embedding

$$DM_{gm}^{eff}(k) \to DM_-^{eff}(k).$$

This gives the functors

$$M_{gm}(-) : Sch_k \to DM_{gm}^{eff}(k),$$

sending X to the motive of X and

$$M_{gm}^c(-) : Sch_k^{prop} \to DM_{gm}^{eff}(k),$$

sending X to the motive with compact support of X. Note that the first functor extends the functor M_{gm} which we have introduced in definition 2.3.4. Let us list up some well-known properties of the above functor in the following theorem

Theorem 2.7.2. *The functor $M_{gm}(-)$ has the following properties:*

i. *(Kunneth formula) For every $X,\ Y \in Sch_k$ there is a canonical isomorphism $M_{gm}(X \times Y) = M_{gm}(X) \otimes M_{gm}(Y)$.*

ii. *(Homotopy invariance) For $X \in Sch_k$ the morphism*

$$M_{gm}(X \times \mathbb{A}^1) \to M_{gm}(X)$$

induced by the first projection, is an isomorphism.

iii. *(Mayer-Vietoris axiom) For an open covering $X = U \cup V$ of X in Sch_k we have a canonical distinguished triangle:*

$$M_{gm}(U \cap V) \to M_{gm}(U) \oplus M_{gm}(V) \to M_{gm}(X) \to M_{gm}(U \cap V)[1].$$

iv. *(Blow-up triangle) Let $X \in Sch_k$ and $Z \subset X$ its closed subscheme. Denote by $p_Z : X_Z \to X$ the blow-up of Z in X. Then one has a canonical distinguished triangle:*

$$M_{gm}(p_Z^{-1}(Z)) \to M_{gm}(X_Z) \oplus M_{gm}(Z) \to M_{gm}(X) \to M_{gm}(p_Z^{-1}(Z))[1].$$

v. *(Motives of blow-ups) Let k be perfect field. Let $X \in Sm_k$ and Z its closed subscheme which is smooth and everywhere of codimension d. Then there is a canonical isomorphism:*

$$M_{gm}(X_Z) = M_{gm}(X) \oplus (\oplus_{n=1}^{d-1} M_{gm}(Z)(n)[2n])$$

vi. *(Gysin distinguished triangle) Considering the assumptions in the previous part one has a canonical distinguished triangle:*

$$M_{gm}(X - Z) \to M_{gm}(X) \to M_{gm}(Z)(d)[2d] \to M_{gm}(X - Z)[1].$$

Proof. See [VSF, Chapter 5]. □

Theorem 2.7.3. *The functor $M^c_{gm}(-)$ has the following properties:*

i. For a proper scheme X over k there is a canonical isomorphism:

$$M^c_{gm}(X) = M_{gm}(X).$$

ii. Let Z be a closed subscheme of a scheme X of finite type. There is a canonical distinguished triangle:

$$M^c_{gm}(Z) \to M^c_{gm}(X) \to M^c_{gm}(U := X - Z) \to M^c_{gm}(Z)[1].$$

iii. Let $f : X \to Y$ be a flat equidimensional morphism of schemes of finite type with relative dimension n. There is a canonical morphism:

$$M^c_{gm}(Y)(n)[2n] \to M^c_{gm}(X).$$

iv. For any scheme of finite type X over k, we have a canonical isomorphism:

$$M^c_{gm}(X \times \mathbb{A}^1) = M^c_{gm}(X)(1)[2]$$

Proof. See [VSF, Chapter 5]. □

Duality Let k be a field which admits resolution of singularities. Then $DM_{gm}(k)$ is a rigid tensor triangulated category.

(a) For any pair of objects A, B in $DM_{gm}(k)$ there exists an internal Hom-object $\mathcal{H}om_{DM_{gm}(k)}(A, B)$. We set $A^\vee := \mathcal{H}om_{DM_{gm}(k)}(A, \mathbb{Z})$ to be the dual of A.

(b) For any object A in $DM_{gm}(k)$ the canonical morphism $A \to (A^\vee)^\vee$ is an isomorphism.

(c) For any pair of objects A, B in $DM_{gm}(k)$ there are canonical isomorphisms
i)$\mathcal{H}om_{DM_{gm}}(A, B) = A^\vee \otimes B$,

ii)$(A \otimes B)^\vee = A^\vee \otimes B^\vee$.

For any smooth equidimensional scheme X of dimension n over k there is a canonical isomorphism:

$$M_{gm}(X)^\vee = M^c_{gm}(X)(-n)[-2n].$$

Let X be a smooth equidimensional scheme of dimension n over k and Z be a closed subscheme of X. Applying duality to the localization sequence for M^c_{gm} we get the following generalized Gysin distinguished triangle:

$$M_{gm}(X-Z) \to M_{gm}(X) \to M^c_{gm}(Z)^\vee(n)[2n] \to M_{gm}(X-Z)[1].$$

Let us finish this section by recalling the following two important results of V. Voevodsky:

Theorem 2.7.4. *Let X be a quasi-projective equidimensional scheme over k of dimension n. Then for all $i, j \in \mathbb{Z}$ there are canonical isomorphisms:*

$$CH^{n-i}(X, j-2i) \cong \begin{cases} Hom_{DM^{eff}_-}(\mathbb{Z}(i)[j], \underline{C}^c_*(X)) & \textit{for } i \geq 0\ , \\ Hom_{DM^{eff}_-}(\mathbb{Z}, \underline{C}^c_*(X)(-i)[-j]) & \textit{for } i \leq 0\ , \end{cases}$$

which commute with the boundary maps in the localization long exact sequences.

Proof. See [VSF, Chapter 5, Proposition 4.2.9]. □

Theorem 2.7.5. *Let $X \in Sm_k$. There is a canonical isomorphism:*

$$CH^i(X) \simeq Hom_{DM^{eff}_{gm}}(M_{gm}(X), \mathbb{Z}(i)[2i])$$

Proof. See [VSF, Chapter 5, corollary 4.2.5]. □

2.8 Borel-Moore motivic homology

Theorem 2.8.1. *Assume k admits resolution of singularities. For any closed subscheme $Z \xrightarrow{i} X$ with complement $U \xrightarrow{j} X$, there is a distinguished triangle:*

$$M^c_{gm}(Z) \xrightarrow{i_*} M^c_{gm}(X) \xrightarrow{j^*} M^c_{gm}(U) \to M^c_{gm}(Z)[1]$$

Proof. See [MVW, Theorem 16.15]. □

Definition 2.8.2. Let X be any scheme of finite type over k and $i \geq 0$. Define the *motivic cohomology with compact supports* of X with coefficients in a ring R to be

$$H^{n,i}_c(X, R) = Hom_{DM^{eff}_-}(M^c_{gm}(X), R(i)[n]).$$

Dually define the *(Borel-Moore) motivic homology with compact supports* of X in the following way

$$H^{BM}_{n,i}(X, R) = Hom_{DM^{eff}_-}(R(i)[n], M^c_{gm}(X))$$

Apply Hom to the triangle in theorem 2.8.1 to get the long exact localization sequences for motivic cohomology and homology with compact support:

$$H^{n,i}_c(U, \mathbb{Z}) \to H^{n,i}_c(X, \mathbb{Z}) \to H^{n,i}_c(Z, \mathbb{Z}) \to H^{n+1,i}_c(U, \mathbb{Z}),$$

$$H^{BM}_{n,i}(Z, \mathbb{Z}) \to H^{BM}_{n,i}(X, \mathbb{Z}) \to H^{BM}_{n,i}(U, \mathbb{Z}) \to H^{BM}_{n-1,i}(Z, \mathbb{Z}).$$

Let us finally mention that the higher chow groups of $X \in Sch_k$ can be realized as the motivic cohomology of X (and as the Borel-Moore homology of X).

Theorem 2.8.3. *Let X be a smooth separated scheme over a perfect field k, then for all n and $i \geq 0$ there is a natural isomorphism:*

$$H^{n,i}(X, \mathbb{Z}) \xrightarrow{\sim} CH^i(X, 2i - n),$$

where $CH^i(X, n)$ denotes the higher chow groups of X.

Proof. See [MVW, Theorem 19.1]. □

Theorem 2.8.4. *Assume that k admits resolution of singularities. Let X be a quasi-projective equidimensional scheme over k of dimension d. Then for every positive $i \leq d$ and n there is a canonical isomorphism:*

$$CH^{d-i}(X, n) \cong H^{BM}_{2i+n,i}(X, \mathbb{Z}) = Hom(\mathbb{Z}(i)[2i + n], M^c(X)).$$

Proof. See [MVW, Proposition 19.18]. □

Chapter 3

Linear Algebraic Groups and G-Bundles

3.1 Reductive groups and root systems

Throughout this section we denote by G a linear algebraic group over a field k (which is not necessarily perfect). We will use G° to denote its identity component.

Definition 3.1.1. An algebraic group T over k is called a torus of rank r if it becomes isomorphic to a product of multiplicative group, $\mathbb{G}^r_{m,\bar{k}}$, over $\overline{k}$. If moreover T is isomorphic to $\mathbb{G}^r_{m,k}$ then T is called a k-split torus.

Definition 3.1.2. A group G is called solvable if it has a subnormal series whose factor groups are all abelian. One can equivalently say that the derived series

$$\cdots \subseteq DDDG \subseteq DDG \subseteq DG$$

terminates in e.

Definition 3.1.3. A Borel subgroup of G is a closed connected solvable subgroup B with largest possible dimension.

Definition 3.1.4. The radical $R(G)$ of an algebraic group G is the identity component of the maximal normal solvable subgroup of G. The group of unipotent elements of $R(G)$ is called unipotent radical of G and will be denoted by $R_u(G)$.

Definition 3.1.5. A linear algebraic group G is called semi-simple if the radical of the identity component of $G_{\bar{k}}$ is trivial. Equivalently, a semisimple linear algebraic group has no non-trivial connected, normal, abelian subgroup.

Definition 3.1.6. A reductive group is a smooth affine algebraic group such that the unipotent radical of $G_{\bar{k}}$ is trivial.

Lemma 3.1.7. *For a reductive algebraic group G the radical $R(G)$ of G is a torus. Furthermore this equals to the identity component, $Z(G)^\circ$, of the center of G.*

Proof. [Hu, Section 19.5] □

Let G be an algebraic group defined over k and L a field extension of k in $\overline{k}$. We call a morphism of algebraic L-groups $\chi : G_L \to \mathbb{G}_{m,\overline{k}}$ a *character* of G defined over L. We denote the abelian group consisting of all characters of G (with the natural product) defined over L by $X^*(G)_L$, when the field L coincide with $\overline{k}$ we simply drop the subscript L and call it the character group. Dually, $X_*(G)_L$ (resp. $X_*(G)$) denotes the abelian group consisting of all *cocharacters* $\lambda : \mathbb{G}_{m,L} \to G_L$ defined over L (resp. $\overline{k}$). We have a natural pairing

$$(-,-) : X_*(G) \times X^*(G) \to \mathbb{Z}.$$

Note that a torus T is a split torus over k if and only if $X_*(T) = X_*(T)_k$. In this case the above pairing is precisely a perfect pairing.

Let G be a reductive group over k. Let S denotes a maximal k-split torus of G. Let $Z_G(S)$ (resp. $N_G(S)$) be the centralizer (resp. normalizer) of S in G.
We will denote the derivation of G by G^{der}. Define $S^{der} := (S \cap G^{der})^\circ$.

Definition 3.1.8. The *Weyl goup* $W := W(G,S)$ is defined to be the finite k-group $N_G(S)/Z_G(S)$.

Let $Lie(G)$ be the tangent space to G at the identity element $e \in G$. For a morphism $G \xrightarrow{f} G'$ let $d_e(f) : Lie(G) \to Lie(G')$ denotes the associated morphism on tangent spaces. The adjoint representation of a linear algebraic group G is the linear representation Ad of G in $Lie(G)$ induced by adjunction. More precisely the representation Ad is given by the mapping which sends each $g \in G$ to the differential $Ad(g) := d_e(Int(g))$ of the inner automorphism $Int(g) : x \mapsto gxg^{-1}$.
Let Z be the center of G, and $G_{ad} = G/Z$. This is a semi-simple k-group of adjoint type, (i.e. the adjoint representation is faithful). We set $S_{ad} := S/S \cap Z$.

Let $\varrho : G \to Gl(V)$ be a representation of G in a vector space V. The induced action of S on V decomposes the representation to its weight spaces

$$V = \bigoplus_{\chi \in X^*(S)} V_\chi,$$

here $V_\chi = \{\nu \in V; \varrho(s)(\nu) = \chi(s)\nu \forall s \in S\}$. Those χ for which $V_\chi \neq 0$ are called weights of ϱ.
Now setting $\varrho := Ad$ we obtain

$$Lie(G) = \bigoplus_{\chi \in X^*(S)} Lie(G)_\chi,$$

Definition 3.1.9. The set $\Phi(G, S)$ of non-zero characters such that the weight space $Lie(G)_\chi$ is non-trivial is called the set of roots of G with respect to S.

Let us rewrite the above decomposition in the following way

$$Lie(G) = Lie(Z_G(S)) \oplus \bigoplus_{\alpha \in \Phi(G,S)} Lie(G)_\alpha.$$

This is often called the root decomposition of $Lie(G)$.

Let V be a finite dimensional $\mathbb{R}$-vector space, and $V^\vee = Hom(V, \mathbb{R})$ be its dual space. An element $\omega \in GL(V)$ is called a reflection if $\omega^2 = 1$ and the fixed point set of ω is a hyperplane $H \subset V$. Let $\alpha \in V$ be a nonzero vector satisfying $\omega(\alpha) = -\alpha$. Then define $\alpha^\vee \in V^\vee$ to be the unique vector such that $\alpha^\vee \mid_H = 0$ and $(\alpha, \alpha^\vee) = 2$. Then $\omega(\beta) = \beta - (\beta, \alpha^\vee)\alpha$.

Definition 3.1.10. An abstract root system is a pair (V, Φ) consisting of a finite dimensional real vector space V and a finite subset Φ of vectors in V , satisfying:

(a) Φ does not contain 0, and spans V .

(b) For each α there is a reflection ω_α with respect to α which preserves Φ, and $\omega_\alpha(\alpha) = -\alpha$.

(c) For any $\alpha, \beta \in \Phi$ one has $(\alpha, \beta^\vee) \in \mathbb{Z}$. Here $\beta^\vee$ is called the coroot corresponding to β.

An abstract root system Φ is called reduced if for any $\alpha \in \Phi$ and any $t \in \mathbb{R}$ with $|t| > 1$, $\frac{1}{t}\alpha$ is not in Φ.

An isomorphism from a root system $(V;\Phi)$ to another one (V',Φ') is a linear isomorphism $f: V \to V'$ which sends Φ to Φ'.

Definition 3.1.11. The group of automorphisms of a root system is denoted by $\mathrm{Aut}(\Phi)$. The subgroup of $\mathrm{Aut}(\Phi)$ generated by the reflections w_a, $a \in \Phi$ is called the *abstract Weyl group* of the root system. This is a finite group, and is denoted by W_Φ.

Definition 3.1.12. A subset $\Delta \subseteq \Phi$ is called a system of simple (fundamental) roots for Φ if

(a) Δ form a basis for V,

(b) any root in Φ can be represented as a sum $\sum_{\alpha\in\Delta} m_\alpha \alpha$ with m_α being integral coefficients, all non-positive or all non-negative.

We will denote by Φ^+ the set of roots which are decomposed into a linear combination of elements of Δ with non-negative coefficients. The elements of Φ^+ are called the positive roots with respect to Δ.

An element $\omega \in W_\Phi$ can be represented as a product $\omega = \omega_1 \dots \omega_n$ such that ω_i is a reflection for every i. Assigning the number of factors of the shortest expression of ω defines a length function $\ell : W_\Phi \to \mathbb{Z}_{\geq 0}$.

Let I be a subset of Φ. We let Δ_I denote the subset of Δ which is generated by I. Let W_I be the subgroup of W_Φ generated by all reflections corresponding to the elements of I. One can observe that Φ_I is a root system and the associated Weyl group W_{Φ_I} is W_I.

Remark 3.1.13. The natural action of W on $X^*(S)$ leaves $\Phi(G,S)$ stable, and thus induces homomorphism $W \to \mathrm{Aut}(\Phi(G,S))$.

Proposition 3.1.14. *Let G be a reductive group over k and S be a maximal split k-torus. Let $\Phi(G,S)$ be the set of roots of G with respect to S. It maps bijectively to its image under the projection $X^*(S) \to X^*(S^{der})$, and we denote this image still by $\Phi(G,S)$. Then*

(a) $(V := X^*(S^{der}) \otimes \mathbb{R}, \Phi(G,S))$ is an abstract root system,

(b) For any $a \in \Phi(G,S)$ let

$$\mathfrak{g}_a := \bigoplus_{\alpha\in\langle a\rangle} Lie(G)_\alpha$$

where $\langle a\rangle$ denote the subset of $\Phi(G,S)$ consisting of positive multiples of a, i.e. a and possibly $2a$ ($\frac{1}{2}a$). Then there is a unique closed unipotent k-subgroup U_a of G, which is normalized by $Z_G(S)$ with $\mathfrak{g}_a = Lie(U_a)$. As a k-variety U_a is isomorphic to an affine k-space.

(c) *The natural homomorphism $W(k) \to Aut(\Phi(G,S))$ induces an isomorphism between the Weyl group and the abstract Weyl group W_Φ.*

(d) *If S is a maximal torus of G, so that G is k-split, then the root system $(V,\Phi(G,S))$ is reduced. Furthermore, $\dim_k \mathfrak{g}_a = 1$ for any $a \in \Phi(G,S)$, and $U_a = \mathbb{G}_a$ and the morphism induced by the multiplication map*

$$\mathbb{G}_a \times \cdots \times \mathbb{G}_a \tilde{\to} \prod_{\alpha\in\Phi} U_\alpha \to U$$

is an isomorphism. Here U denotes the unipotent part of Borel subgroup B.

Proof. See [Zh, theorem 1.1]. □

3.2 G-bundles and the induced representation

In this section we fix a split reductive group G over a field k and a Borel subgroup B which contains a maximal k-split torus T.

Definition 3.2.1. A principal G-bundle $\mathcal{G}$ over X is a family $\mathcal{G} \to X$ together with an action $* : \mathcal{G} \times G \to \mathcal{G}$ such that the morphism

$$\mathcal{G} \times G \to \mathcal{G} \times_X \mathcal{G},$$

which sends (x,g) to $(x, x*g)$, is an isomorphism.

Remark 3.2.2. One can equivalently say that $\mathcal{G}$ is a G-bundle over X if it is locally trivial for the flat topology on X, i.e. there is a flat covering $\{U_i \to X\}$ such that the restriction $\mathcal{G}_{U_i}$ is isomorphic with trivial G-bundle $G \times U_i$ over U_i.

Remark 3.2.3. Let G be an affine group over X. There is a canonical bijection between the isomorphism classes of G-bundles and the elements of the pointed set $H^1(X_{fl}, G)$, where $H^1(X_{fl}, G)$ denotes the first chech cohomology of X with coefficients in G with respect to the flat site X_{fl} on X.

Theorem 3.2.4. *(Hilbert 90)The canonical maps*

$$H^1(X_{Zar}, Gl_n) \to H^1(X_{\acute{e}t}, Gl_n) \to H^1(X_{fl}, Gl_n)$$

are isomorphisms.

Proof. See [Mi, Chapter 3, Proposition 4.9]. □

Definition 3.2.5. Let Y be a variety with left G-action. To a G-bundle $\mathcal{G}$ on X one associates a fibration $\mathcal{G} \times^G Y$ with fibre Y over X, defined by the following quotient

$$\mathcal{G} \times Y / \sim,$$

where $(x, y) \sim (x * g, g^{-1}.y)$ for every $g \in G$.

Remark 3.2.6. Note that if $\mathcal{G}$ is locally trivial for étale (resp. Zariski) topology then so is the fibration $\mathcal{G} \times^G Y$.

Remark 3.2.7. Let $\mathscr{H}^1(G, X)$ denotes the category whose objects are G-bundles over X and whose morphisms are the natural isomorphisms of G-bundles. Let $Vect^n(X)$ denotes the category of vector bundles on X of rank n. Finally $\mathrm{Rep}_n(G)$ denotes the category of n-dimensional representations of G. We have the following functor

$$\mathscr{H}^1(G, X) \times \mathrm{Rep}_n(G) \to Vect^n(X)$$

$$(\mathcal{G}, V) \mapsto \mathcal{G}_V := \mathcal{G} \times^G V.$$

For a given subgroup H of G one may regard G as a principal H-bundle over G/H via the usual projection. Now for a representation $\rho : H \to Gl(V)$ we may assign a vector bundle $\mathcal{E}_V := G \times^H V$ over G/H. Let $\mathcal{O}_{\mathcal{E}_V}$ denotes the associated sheaf of sections. Note that the set of global sections of $\mathcal{O}_{\mathcal{E}_V}$ is in bijection with the set of H-morphisms from G to V

$$Mor_H(G, V) = \{f : G \to V; f(gh) = \rho(h^{-1})(f(g))\forall g \in G, h \in H\}.$$

By the construction, the vector bundle $\mathcal{E}_V$ is equipped with a natural transitive left G-action (this is why it is sometimes called homogeneous vector bundle). Hence we obtain a representation of G on the space of global sections of $\mathcal{E}_\mathcal{V}$, which we denote it by $\mathrm{Ind}_H^G(\rho)$. More explicitly for $f \in Mor_H(G, V)$ and $g \in G$

$$(\mathrm{Ind}_H^G(\rho)(g)f)(x) = f(g^{-1}x).$$

The above construction can be formulate in the following way. There is a functor

$$Ind_H^G(-) : \mathrm{Rep}(H) \to \mathrm{Rep}(G),$$

which is a left adjoint to the obvious restriction functor

$$\mathrm{Rep}(G) \to \mathrm{Rep}(H).$$

Definition 3.2.8. Let $\overline{B} \subseteq G$ be the Borel subgroup opposite to B. Let λ be a dominant weight of G. Note that by the decomposition $B = TU$ where U is the unipotent radical of G, one may view λ as a linear representation of a Borel subgroup, such that U lies in the kernel. The Weyl module V_λ with highest weight λ is defined to be the dual space $V_\lambda := \mathrm{Ind}_{\overline{B}}^G(\lambda)^\vee$ of the induced module $\mathrm{Ind}_{\overline{B}}^G(\lambda)$.

3.2.1 Discussion about triviality of G-bundles

In this subsection we state two strong results about the triviality of G-bundles. We will make use of these theorems in the last chapter.

Affine spaces are topologically contractible, so they admit no non-trivial topological vector bundle. Jean-Pierre Serre, in his 1955 paper "Faisceaux algbriques cohrents", mentioned that the equivalent algebraic question for vector bundles (and equivalently GL_n-bundles) is not known.
In the commutative algebra setting this question can be formulated as whether exist projective modules over polynomial ring $k[x_1, ..., x_n]$ of finite type which are not free. Quillen and Suslin independently proved that GL_r-bundles (equivalently, vector bundles of rank r) over $\mathbb{A}^n$ are trivial.

The similar triviality question for G-bundles were considered by Raghunathan. Through his deep considerations he proves

Theorem 3.2.9. *let k be an algebraically closed field. Then G-bundles on $\mathbb{A}^n$ are trivial.*

Proof. cf. [Ra2]. □

Remark 3.2.10. However when k is not algebraically closed, there can be reductive groups G such that the principal G-bundles over $\mathbb{A}^2$ are not isomorphic to pull backs of G-bundles over $\operatorname{Spec} k$ under the structure morphism. (Note that in general G-bundles over $\operatorname{Spec} k$ may not be trivial (unlike in the case $G = GL_r$)).

Let us also state the following theorem of Drinfeld and Simpson.

Theorem 3.2.11. *Suppose G is semi-simple. Let C be a smooth projective curve. Let p be a closed point of C and $\dot{C}$ denote the complement of p in C. Let R be a k-algebra and $\mathcal{G}$ be a G-bundle over C_R. Then the restriction of $\mathcal{G}$ to $\dot{C}_R$ is trivial, locally for the $fppf$-topology over* $\operatorname{Spec} R$. *If $\operatorname{char} k$ does not divide the order of $\pi_1(G)$, then this is even true locally for the étale topology over* $\operatorname{Spec} R$.

Proof. cf. [DS]. □

Chapter 4

Motivic Leray-Hirsch Theorem

4.1 Chow motive of relatively cellular varieties

Lemma 4.1.1. *Let $f : N \to P$ be a morphism in $M^{eff}(k)$. The following are equivalent:*

i) f is an isomorphism.

ii) For every $Y \in SmProj_k$,

$$Hom_{M^{eff}(k)}(\mathbb{L}^n, M_{CH}(Y) \otimes N) \xrightarrow{\mathcal{H}om(\mathbb{L}^n, M_{CH}(Y)\otimes -)} Hom_{M^{eff}(k)}(\mathbb{L}^n, M_{CH}(Y) \otimes P)$$

is an isomorphism for every n.

iii)For every $Y \in SmProj_k$,

$$Hom_{M^{eff}(k)}(M_{CH}(Y) \otimes P, \mathbb{L}^n) \xrightarrow{\mathcal{H}om(M_{CH}(Y)\otimes -, \mathbb{L}^n)} Hom_{M^{eff}(k)}(M_{CH}(Y) \otimes N, \mathbb{L}^n)$$

is an isomorphism for every n.

Proof. See [EKM, Sections 63,64]. □

Theorem 4.1.2. *Let $E \to X$ be a vector bundle of rank r over a smooth projective scheme X. Then $M_{CH}(\mathbb{P}(E))$ is naturally isomorphic to the motive $\coprod_{i=0}^{r-1} M_{CH}(X)(i)$.*

Proof. See [EKM, Theorem 63.10]. □

Definition 4.1.3. A scheme $X \in Ob(Sch_k)$ is called *relatively cellular* if it admits a filtration by its closed subschemes:

$$\emptyset = X_{-1} \subset X_0 \subset ... \subset X_n = X$$

where $p_i : U_i := X_i \setminus X_{i-1} \to Y_i$ is an affine bundle of rank d_i and Y_i is a smooth complete scheme. Each U_i is called a cell of X and Y_i is the base of the cell U_i.

By an affine bundle of rank d we mean a flat morphism $f : X \to Y$ such that $f^{-1}(y)$ is isomorphic to the affine space $\mathbb{A}^d_{k(y)}$, for any point $y \in Y$.

Keep the notation of the definition 4.1.3. Consider Γ_{p_i} (i.e. the graph of p_i) as a subscheme of $U_i \times Y_i$ and let $c_i \in CH(X_i \times Y_i)$ be the class of its closure in $X_i \times Y_i$. We have $c_0 = \Gamma_{p_0}$. View c_i as a correspondence of degree 0 from X_i to Y_i. For the closed embedding $f_i : X_i \to X$, the correspondence $f_i \circ c_i^t \in CH(Y_i \times X)$ is well-defined and of degree d_i for all $i \geq 0$.
Define homomorphisms:

$$\pi_i : CH_{*-d_i}(Z \times Y_i) \to CH_*(Z \times X)$$

$\pi_i = (1_Z \times f_i \circ c_i^t)_*$ for $0 \leq i \leq n$.

Theorem 4.1.4. *Let $X \in SmProj_k$ admits a cell decomposition as in definition 4.1.3. Then the sequence of correspondences $f_i \circ c_i^t$ induce a morphism:*

$$\coprod_{i=0}^{n} M_{CH}(Y_i)(d_i) \to M_{CH}(X)$$

which is an isomorphism in $M^{eff}(k)$.

Proof. It is a consequence of lemma 4.1.1 and the following proposition. □

Proposition 4.1.5. *Let X be a cellular scheme with a filtration as in definition 4.1.3. Then for every $Z \in Sch_k$, the morphism*

$$\sum_{i=0}^{n} \pi_i : \coprod_{i=0}^{n} CH_{*-d_i}(Z \times Y_i) \to CH_*(Z \times X)$$

is an isomorphism.

Proof. See [EKM, Theorem 66.2]. □

4.2 Leray-Hirsch Theorem for Chow

In this section we first remind the Leray-Hirsch theorem from Algebraic Topology. Let $\pi : \Gamma \to X$ be a fibre bundle with fibre F. This theorem gives the description of the singular cohomology of Γ as $H^*(X) - module$ according to singular cohomology of the fibre F.

Theorem 4.2.1. *Let $\pi : \Gamma \to X$ be a fibre bundle with fibre F. Assume that for each $p \in \mathbb{Z}$, the p-th singular cohomology $H^p(F) = H^p(F, \mathbb{Q})$ has finite dimension m_p. Further assume that there exists classes $c_{1,p}, c_{2,p}, ..., c_{m_p,p}$ in $H^p(\Gamma)$ which restrict on each fibre F to a basis for the p-th singular cohomology $H^p(F)$. Then the following morphism is an isomorphism of $H^*(X) - modules$:*

$$H^*(F) \otimes H^*(X) \longrightarrow H^*(\Gamma)$$

$$\sum_{i,j,k} a_{i,j,k} \iota^*(c_{i,j}) \otimes b_k \mapsto \sum_{i,j,k} a_{i,j,k} c_{i,j} \wedge \pi^*(b_k),$$

where $\iota : F \hookrightarrow \Gamma$ is an inclusion of a fibre and $\{b_k\}$ is a basis for $H^*(X)$, and thus, induces a basis $\{\iota^*(c_{i,j}) \otimes b_k\}$ for $H^*(F) \otimes H^*(X)$.

Proof. See [Hat, Theorem 4D.1]. □

Let us now move slightly toward the algebraic situation, namely we are going to state the theorem for the Chow ring of a locally trivial fibre bundle. Unfortunately at least the naive version of Kunneth formula does not hold for the Chow functor. Thus the Leray-Hirsch Theorem is not true for every locally trivial fibre bundle as well. In fact to get a reasonable set up one has to impose some extra assumptions on the fibre. Let's now discuss one of the possible conditions on the fibre which fulfils our desire.
Consider a smooth proper morphism $f : \Gamma \to X$ of relative dimension d, which is locally trivial in the Zariski topology, with fibre F. As in topology, assume that there are elements in the chow ring of Γ which restrict to a basis for the chow ring of the fibre. We would like to conclude that these elements give a basis for $CH^*(\Gamma)$ over $CH^*(X)$, at least for non-singular X. In order to state a version of Leray-Hirsch theorem for chow groups, one needs to impose the following conditions on F.
a) F satisfies Poincaré duality.
Recall that one says F satisfies *Poincaré duality* if:

i) The degree map from $CH_0(F)$ to $\mathbb{Z}$ is an isomorphism, and

ii) The intersection pairings

$$CH^i(F) \otimes CH^{d-i}(F) \to CH^d(F) \approxeq CH_0(F) \approxeq \mathbb{Z}$$

are perfect pairings for all i.

b) F is a cellular variety.

Recall that:

Definition 4.2.2. We say that a scheme X over a field k is *cellular*, or admits a *cell decomposition*, if there exists a filtration of X by closed subschemes:

$$X_0 = \emptyset \subset X_1 \subset .... \subset X_n = X$$

such that for every $1 \leq i \leq n$, we have $U_i = X_i - X_{i-1} = \bigcup_{j=1}^{n_i} U_{i,j}$, where each $U_{i,j}$ is isomorphic to some affine space.

Before stating the Leray-Hirsch theorem for Chow groups, due to Colino and Fulton [CF], let us mention that if the restriction map from $CH^*(\Gamma)$ to $CH^*(\Gamma_x)$ is surjective for a point $x \in X$ in each irreducible component of X, e.g. generic point, then this will be true for every fibre.

Proposition 4.2.3. *Let $f : \Gamma \to X$ be a proper smooth morphism of relative dimension d, which is locally trivial in the Zariski topology, with fibre F. Let F admits cell decomposition and satisfies Poincare duality. Let $\gamma_1, ...\gamma_m$ be homogeneous elements of $CH^*(\Gamma)$ whose restrictions to fibres form a basis over $\mathbb{Z}$. Then every element in $CH_*(\Gamma)$ has a unique expression of the form:*

$$\sum_{i=1}^{m} \gamma_i \cap f^*\alpha_i, \quad \alpha_i \in CH_*(X)$$

In other words the following homomorphism is an isomorphism:

$$\begin{array}{ccc} \oplus_{i=1}^m CH_*(X) & \xrightarrow{\Phi} & CH_*(\Gamma) \\ \oplus\alpha_i & \mapsto & \sum \gamma_i \cap f^*\alpha_i \end{array}$$

For non-singular X, it means that γ_is form a free basis for $CH^*(\Gamma)$ as a $CH^*(X)$-module.

Proof. We prove by induction on $\dim X$. We may assume that X is irreducible with function field K. Let $\Gamma_K \cong F \otimes_k K$ denote the generic fibre and $\varrho : CH_*(\Gamma) \to CH_*(\Gamma_K)$ denote the restriction morphism. Consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker\varphi & \longrightarrow & \bigoplus_{i=1}^m CH_*(X) & \stackrel{\varphi}{\longrightarrow} & CH_*(\Gamma_K) & \longrightarrow & 0 \\ & & \downarrow & & {\scriptstyle\Phi}\downarrow & & \| & & \\ 0 & \longrightarrow & \ker\varrho & \longrightarrow & CH_*(\Gamma) & \stackrel{\varrho}{\longrightarrow} & CH_*(\Gamma_K) & \longrightarrow & 0 \end{array}$$

Let α be an element in $\ker\varrho$. There is an open subscheme $U_\alpha \subset X$ such that the restriction of α to $CH_*(\Gamma_{U_\alpha})$ vanishes. Set $Z_\alpha := X - U_\alpha$. Consider the following diagram

$$\begin{array}{ccccccc} \bigoplus_{i=1}^m CH_*(Z_\alpha) & \longrightarrow & \bigoplus_{i=1}^m CH_*(X) & \longrightarrow & \bigoplus_{i=1}^m CH_*(U_\alpha) & \longrightarrow & 0 \\ \downarrow & & {\scriptstyle\Phi}\downarrow & & \downarrow & & \\ CH_*(f^{-1}(Z_\alpha)) & \longrightarrow & CH_*(\Gamma) & \longrightarrow & CH_*(f^{-1}(U_\alpha)) & \longrightarrow & 0. \end{array}$$

Since α maps to zero in $CH_*(f^{-1}(U_\alpha))$ and the left vertical arrow is an isomorphism by induction hypothesis, we see that the element $\alpha \in CH_*(\Gamma)$ has a preimage under Φ. By diagram chase this element lies in $\ker\varphi$. This shows that $\ker\varphi \to \ker\varrho$ is surjective. Thus Φ is surjective by five lemma.

It remains to show that $\ker\Phi = 0$. Let $\hbar \in CH^d(F)$ be the generator corresponding to $1 \in \mathbb{Z}$. Let us relabel the homogeneous elements γ_i which lie in $CH^j(\Gamma)$ by double subscript γ_i^j. Since F satisfies the Poincaré duality we may choose elements $\vartheta_i^j \in CH^{d-j}(\Gamma)$ whose restrictions to fibres give the dual basis of the restrictions of γ_i^j. Now if $\sum_{ij} \gamma_i^j \cap f^*\alpha_{ij} = 0$ and p be the maximum number for which $\alpha_{qp} \neq 0$ for some q, then:

$$f_*(\vartheta_q^p(\sum_{ij} \gamma_i^j \cap f^*\alpha_{ij})) = 0$$

But one can show:

$$f_*(\vartheta_q^p\gamma_i^j \cap f^*\alpha_{ij}) = \begin{cases} \alpha_{ij} & \text{if } i = q \textit{ and } j = p\ , \\ 0 & \text{otherwise.} \end{cases}$$

For the first case, note that by the definition of dual basis $\vartheta_q^p\gamma_q^p$ is an element of $CH^d(\Gamma)$ such that its restriction to a fibre is $\hbar$. It is enough to show the equality in the case $\alpha_{qp} = [X]$ ($[X]$ is the class associated to X in chow group). But $f_*(\vartheta_q^p\gamma_i^j \cap f^*[X]) = n'[X]$ for some integer n', and by restricting to a fibre we conclude $n' = 1$. We prove the second as follows:

i) If $j < p$, by a similar proof as above one can show $f_*(\vartheta_q^p \gamma_i^j \cap f^* \alpha_{ij})$ vanishes.

ii) If $j = p$, obviously $f_*(\vartheta_q^p \gamma_i^p \cap f^* \alpha_{ip}) = 0$ for $i \neq q$.

Therefore we conclude. □

In section 4.6 we prove a variant of the Leray-Hirsch theorem for Voevodsky motives.

4.3 Slice filtration

In this section we either assume $char k = 0$ or coefficients in $\mathbb{Q}$.

Recall that the Fulton chow groups of X can be derived from the motive $M_{gm}^c(X)$ associated to X, see theorem 2.7.4. Conversely one might ask the following:

Question A: Suppose that the chow groups $CH^*(X)$ of an irreducible proper scheme X are given. Then under which conditions on X we can recover $M_{gm}^c(X)$?

The following discussion is an attempt to answer this question.

Let **Ab** denote the category of abelian groups and $D^-(\mathbf{Ab})$ its bounded above derived category. Let $D^b(\mathbf{Ab})$ be the bounded derive category of **Ab** and $D_f^b(\mathbf{Ab})$ its full subcategory consisting of objects with finitely generated cohomology groups. $D_f^b(\mathbf{Ab})$ is equivalent to the bounded derived category of finitely generated abelian groups. The category $D_f^b(\mathbf{Ab})$ is a rigid tensor triangulated category.

Proposition 4.3.1. *There exists a unique triangulated functor*

$$\imath : D_f^b(\boldsymbol{Ab}) \to DM_{gm}^{eff}(k)$$

sending $\mathbb{Z}$ to $\mathbb{Z}(0)$. It is fully faithful and respects the tensor structures. Its essential image is the thick tensor subcategory of $DM_{gm}^{eff}(k)$ generated by $\mathbb{Z}(0)$.

Proof. see [HK, Proposition 4.5.]. □

We are going to define the slice filtration and the fundamental invariants associated to a motive $M \in M_-^{eff}(k)$, introduced in [HK].

For $n \geq 0$, let $DM_-^{eff}(k)(n)$ be the full subcategory of $DM_-^{eff}(k)$ whose set of objects is $\{M(n) | M \in DM_-^{eff}(k)\}$. Define the triangulated functor

$$\begin{array}{rcl} DM_-^{eff}(k) & \xrightarrow{\nu^{\geqslant n}} & DM_-^{eff}(k)(n) \\ M & \mapsto & \mathcal{H}om(\mathbb{Z}(n), M)(n) \end{array}$$

where $\mathcal{H}om$ is the internal Hom.

Theorem 4.3.2. *The functor $\nu^{\geqslant n}$ is right adjoint to the inclusion*

$$DM_-^{eff}(k)(n) \hookrightarrow DM_-^{eff}(k).$$

Proof. See [HK, Proposition 1.1]. □

By the above adjunction, for any $n \geq 0$ we get the morphism

$$a^n : \nu^{\geqslant n} M \to M$$

induced by the identity map of $\mathcal{H}om(\mathbb{Z}(n), M)$. Moreover for any $n > 0$, there is a morphism $f^n : \nu^{\geqslant n} M \to \nu^{\geqslant n-1} M$ such that $a^{n-1} \circ f^n = a^n$. In fact by adjunction we get the morphism

$$\mathcal{H}om(\mathbb{Z}(n), M)(1) \to \mathcal{H}om(\mathbb{Z}(n-1), M)$$

and then tensoring with $\mathbb{Z}(n-1)$ gives f^n.

Definition 4.3.3. Define $\nu_{\leqslant n} DM_-^{eff}(k)$ to be the full subcategory of the category $DM_-^{eff}(k)$, consisting of those objects on which $\nu^{\geqslant n+1}$ vanishes.

Proposition 4.3.4. *Let $M \in M_-^{eff}(k)$.*

i) Let $\nu_{<n} M = \nu_{\leqslant n-1} M$ be an object which fits in an exact triangle:

$$\nu^{\geqslant n} M \xrightarrow{a^n} M \to \nu_{<n} M \to \nu^{\geqslant n} M[1]$$

This object is uniquely defined up to unique isomorphism. For every $n \geq 0$, $\nu_{<n}$ defines a triangulated endofunctor of $DM_-^{eff}(k)$. The natural transformations $a_n : Id \to \nu_{<n}$ factor canonically through natural transformations $f_n : \nu_{<n+1} \to \nu_{<n}$.

ii) $\nu_{\leqslant n}$ *is left adjoint to the inclusion* $\nu_{\leqslant n} DM_{-}^{eff}(k) \to DM_{-}^{eff}(k)$.

iii) *Let* $\nu_n M$ *be a object fitting in an exact triangle:*

$$\nu_n M \to \nu_{<n+1} M \xrightarrow{f_n} \nu_{<n} M \to \nu_n M[1]$$

This object is uniquely defined up to unique isomorphism. The functor ν_n *defines another triangulated endofunctor of* $DM_{-}^{eff}(k)$. *There is also a functorial exact triangle:*

$$\nu^{\geqslant n+1} M \xrightarrow{f^n} \nu^{\geqslant n} M \to \nu_n M$$

iv) *For any* $M \in DM_{-}^{eff}(k)$, *one writes canonically*

$$\nu_n M = c_n(M)(n)[2n]$$

and the $c_n s$ *also define triangulated endofunctors of* $DM_{-}^{eff}(k)$. *One has the identities*

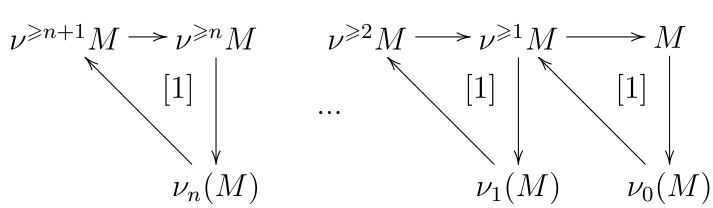
$$\nu^{\geqslant n}(M(1)) = (\nu^{\geqslant n-1} M)(1),$$

$$c_n(M(1)[2]) = c_{n-1}(M).$$

Proof. See [HK, Corollary 1.4] □

Definition 4.3.5. The $c_n(M)$ are called the *fundamental invariants* of M. For a variety X we abbreviate $c_n(X) := c_n(M(X))$.

Consider the following diagram of distinguished triangles:



Note that $\nu_n M = c_n(M)(n)[2n]$. This diagram illustrates that how one can implement fundamental invariants associated to M, in order to analyse the motive M whenever we have vanishing $\nu^{\geqslant N} M = 0$ for N large enough.

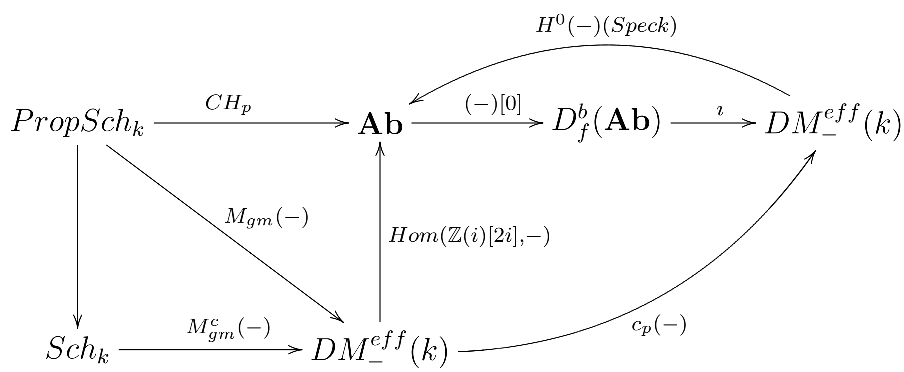
B. Kahn and A. Huber have shown in [HK, Theorem 2.2] that For a variety X the complex $c_n M^c_{gm}(X)$ is concentrated in non-positive degrees and moreover

$$H^0(c_n(M^c_{gm}(X))) = \underline{CH}_n(X),$$

where the values of the Nisnevich sheaf with transfers $\underline{CH}_n(X)$ are given by

$$\underline{CH}_n(X)(U) = CH_n(X_{k(U)}).$$

Let us summarize the above discussion in following diagram which gives a panoramic view of the theory:



The above insight motivates the following question:

Question B: Let M be a motive in $DM^{eff}_-(k)$. Suppose that the fundamental invariants of M are given. Then in general what one can say about M? More precisely, is there any non trivial subcategory $\mathcal{C}$ of $DM^{eff}_-(k)$, for which we can retrieve every M in $\mathcal{C}$ from its fundamental invariants (and what is the largest possible $\mathcal{C}$)?

The first part of the above question is kind of tautological question regarding the theory of slice filtration, see the filtration in the previous page. In the sequel we shall discuss some partial answers to the second part of the question concerning the easiest cases. These cases were studied by B. Kahn and Huber in [HK].

Let us define the following subcategories of Voevodsky's motivic categories

Definition 4.3.6. An object of DM_{gm} is called pure Tate motive if it is a (finite) direct sum of copies of $\mathbb{Z}(p)[2p]$ for $p \in \mathbb{Z}$.

Definition 4.3.7. The thick subcategory of $DM_{gm}^{eff}(k)$, generated by $\mathbb{Z}(0)$ and $\mathbb{Z}(1)$ is called *the category of mixed Tate motives* and we denote it by $TDM_{gm}^{eff}(k)$. Any object of $TDM_{gm}^{eff}(k)$ is called a *mixed Tate motive.* A motive M is *geometrically mixed Tate* if it becomes mixed Tate over $\overline{k}$. Similarly we define the category $TDM_{-}^{eff}(k)$.

Proposition 4.3.8. *An object $M \in DM_{gm}^{eff}(k)$ is geometrically mixed Tate if and only if there is a finite separable extension E of k such that the restriction of M to $DM_{gm}^{eff}(E)$ is mixed Tate.*

Proof. c.f. [HK, Proposition 5.3]. □

Remark 4.3.9. For a cellular variety X one can show that Chow functor satisfies Kunneth formula, i.e.

$$\bigoplus_{i+j=p} CH^i(X) \otimes CH^j(Y) \to CH^p(X \times Y)$$

is an isomorphism, see 4.2.3.
Analogously one can show that a more general fact holds for $c_p(-)$, namely assume that M and N are in $DM_{-}^{eff}(k)$ and further assume M is mixed Tate, then we have the following Kunneth isomorphism:

$$\bigoplus_{i+j=p} c_i(N) \otimes c_j(M) \to c_p(N \otimes M),$$

see [HK, Lemma 4.8].

The following proposition shows that having the fundamental invariants of the motive M one can verify whether M is in $TDM_{gm}^{eff}(k)$ ($TDM_{-}^{eff}(k)$) or not.

Proposition 4.3.10. *A motive M in $DM_{gm}^{eff}(k)$ is in $TDM_{gm}^{eff}(k)$ if and only if $c_n(M)$ lies in $D_f^b(\boldsymbol{Ab})$ for all n and $c_n(M) = 0$ for n large enough. If M is in $TDM_{-}^{eff}(k)$, then $c_n(M)$ lies in $D^-(\boldsymbol{Ab})$.*

Proof. See [HK, Proposition 4.6]. □

Finally let us mention the following observation of A. Huber and B. Kahn. This in fact gives positive answer to the question B for some specific cases, e.g. when $M(X)$ is pure Tate.

Proposition 4.3.11. *Let X be a smooth variety over k. If $M_{gm}(X)$ be a pure Tate motive then there is a natural isomorphism:*

$$M_{gm}(X) \cong \bigoplus_p c_p(X)(p)[2p].$$

Here $c_p(X) = CH^p(X)^\vee[0]$, i.e. the image of the $\mathbb{Z}$-dual of $CH^p(X)$ under $\imath$.

Proof. There is a natural isomorphism:

$$CH^p(X) \cong Hom(M(X), \mathbb{Z}(p)[2p]),$$

see 2.7.5. Since $M(X)$ is pure Tate this is a free group of finite type and therefore we get a canonical morphism $M(X) \otimes CH^p(X) \to \mathbb{Z}(p)[2p]$ and consequently:

$$\varphi : M(X) \to \bigoplus_p CH^p(X)^\vee(p)[2p].$$

Note that for any q we have the following isomorphism:

$$Hom(\bigoplus_p CH^p(X)^\vee(p)[2p], \mathbb{Z}(q)[2q]) \xrightarrow{\sim} Hom(M(X), \mathbb{Z}(q)[2q]).$$

Therefore by Yoneda lemma φ is an isomorphism and we conclude. □

4.4 Motives of cellular varieties

Proposition 4.4.1. *Let X be a cellular variety over k. Then we have the following cases:*

i) If $char k = 0$, then $M^c_{gm}(X)$ is a pure Tate motive and in particular

$$M^c_{gm}(X) = \bigoplus_p c_p(M^c_{gm}(X))(p)[2p],$$

with $c_p(M^c_{gm}(X)) = CH_p(X)[0]$.

ii) If X is smooth, then $M_{gm}(X)$ is a pure Tate motive and in particular

$$M_{gm}(X) = \bigoplus_p c_p(M_{gm}(X))(p)[2p],$$

with $c_p(M_{gm}(X)) = CH^p(X)^\vee[0]$.

Proof. The proof is by induction on the length of the filtration of X as a cellular vriety, using gysin triangle and proposition 4.3.11. For details see [HK, Proposition 4.11]. □

Corollary 4.4.2. *Let X be as above. Assume further that it is equidimentional and smooth. Then there is a natural isomorphism in $DM_{gm}^{eff}(k)$:*

$$\coprod_{p\geqslant 0} CH^p(X)^\vee \otimes \mathbb{Z}(p)[2p] \to M_{gm}(X),$$

where $CH^p(X)^\vee$ denotes the dual $\mathbb{Z}$-module.

Proof. cf. [Kah, Corollary 3.5]. □

The most famous examples of cellular varieties are in fact generalized flag varieties. In the following subsection we state an easy consequence of the above proposition applied to this particular example.

4.4.1 Motive of Schubert varieties inside generalized flag varieties

In this section we fix a split reductive group G over a perfect field k and a maximal k-split torus T.

Proposition 4.4.3. *Let B be any Borel subgroup of G. Then G/B is a projective variety, and all other Borel subgroups of G are conjugate to B.*

Proof. See [Hu, Section 21.3] □

Remark 4.4.4. By the above proposition one can view the homogeneous variety G/B as parameter space for Borel subgroups of G.

Definition 4.4.5. A closed subgroup P of G is called parabolic if G/P is projective.

Proposition 4.4.6. *A closed subgroup of G is parabolic if and only if it contains a Borel subgroup. In particular, a connected subgroup H of G is a Borel subgroup if and only if H is solvable and G/H is projective. Indeed B is the subgroup such that G/B is the largest homogeneous space for G having the structure of a projective variety.*

Proof. See [Hu, Section 21.3] □

Theorem 4.4.7. *Let G be a reductive group and let P be a k-parabolic subgroup of G. We have the following statements*

(a) The fibration $G \to G/P$ is locally trivial for the Zariski topology.

(b) The variety G/P is k-rational.

Proof. cf. [BoT2]. □

Remark 4.4.8. Fix a Borel subgroup B containing T. A parabolic subgroup P of G is called standard, if it contains B. Since all Borel subgroups of G are conjugate, any parabolic subgroup is conjugate to a unique standard parabolic subgroup. All standard parabolic subgroups of G are of the form $P_I := BW_IB$ for a subset $I \subseteq \Delta$.

-*Bruhat Decomposition* Let P, Q be two standard parabolic subgoups of G corresponding to the subsets I, J of Δ. Then one has the following Bruhat decomposition

$$G = \bigsqcup_{\tilde{\omega}} P\tilde{\omega}Q,$$

here $\tilde{\omega}$ is in the set of representatives of double quotient $W_I \backslash W/W_J$.

In particular from the above lemma we see that $G = \bigsqcup_{\tilde{\omega} \in W^I} B\tilde{\omega}P$, where W^I is a set of representatives for W/W_I with minimal length.
Let $X_\omega \subseteq G/P$ denote the image of the orbit of ω under the left B-action. We call the closure $\overline{X}_\omega$ of X_ω the closed Schubert variety in the (generalized) flag variety G/P.

We have the following well-known theorem:

Theorem 4.4.9. *Let G be a split reductive group over k. Let X denote the projective homogeneous variety G/P. We have the following statements*

(a) $X_\omega \cong \mathbb{A}^{\ell(\omega)}$,

(b) $\overline{X}_\omega \smallsetminus X_\omega = \bigsqcup_{\ell(\omega') < \ell(\omega)} X_{\omega'}$.

This result is due to Kock [Ko]. He treats a slightly more general case. Namely, he considers G which is not necessarily k-split, but only k-isotropic, i.e. G possesses a non-trivial k-split subtorus, and rather obtains a relative cellular filtration (see definition 4.1.3) with Y_is are projective homogeneous varieties.

Corollary 4.4.10. *Let X be as above. Let $\{\omega_{\ell(\omega)}\}$ be a set of representatives of W_I, ordered by the length function ℓ. The filtration*

$$\emptyset = \overline{X}_{\ell(\omega_0)} \subset \overline{X}_{\ell(\omega_1)} \subset \cdots \subset \overline{X}_{\ell(\omega_n)} = X$$

is a cell decomposition for X.

Proposition 4.4.11. *Let G be a split reductive algebraic group, and let P be a parabolic subgroup of G which is conjugate with a standard parabolic subgroup P_I. Then there is an isomorphism*

$$M_{gm}(G/P) \cong \coprod_{w \in W^I} \mathbb{Z}(l(\omega))[2l(w)].$$

In particular the generalized flag variety G/P is pure Tate.

Proof. As we have seen above decomposition $G = \coprod_{w \in W^I} BwP$ induces a cell decomposition $G/P \cong G/P_I = \coprod_{w \in W^I} X_w$, where $X_w \cong \mathbb{A}^{l(w)}$. Then $CH_*(G/P)$ is generated by the cycles $[\overline{X_w}]$ and thus we may conclude by 4.4.2. □

Let us finish this subsection by recalling the following result of B.Kock about Poincaré duality for projective homogeneous varieties:

Remark 4.4.12. Let P_I be a standard parabolic subgroup of G associated to $I \subseteq \Delta$. Let w_I denotes the longest element in W_I. As we observed in corollary 4.4.10 the Schubert varieties $\overline{X}_w$ for $w \in W^I$ produce a cell decomposition for the projective homogenoues variety G/P. In particular they form a basis for the chow ring $CH_*(G/P)$. In [Ko], B. Kock establishes the Poincaré duality for G/P. He gives a concrete formula for the product of two basis elements in $CH^*(G/P)$. More explicitly the product of two basis elements $[\overline{X}_w] \in CH_*(G/P)$ and $[\overline{X}_{w'}]$ in $CH^*(G/P)$ is given by the following formula

$$[\overline{X}_w].[\overline{X}_{w'}] = \delta_{w, w_\Delta w' w_I}[pt],$$

where $[pt]$ denotes the class of a point in $CH^*(G/P)$.

4.5 Motivic relatively cellular

First we introduce the notion of motivic relatively cellular. Notice that this notion is slightly weaker than the geometric notion of relatively cellular introduced by Chernousov, Gille, Merkurjev [CGM] and also Karpenko [Kar].

Definition 4.5.1. A scheme $X \in Ob(Sch_k)$ is called *motivic relatively cellular with respect to* the functor $M^c_{gm}(-)$ if it admits a filtration by its closed subschemes:

$$\emptyset = X_{-1} \subset X_0 \subset ... \subset X_n = X$$

together with flat equidimensional morphisms $p_i : U_i := X_i \smallsetminus X_{i-1} \to Y_i$ of relative dimension d_i, such that the induced morphisms

$$p_i^* : M^c_{gm}(Y_i)(d_i)[2d_i] \to M^c_{gm}(U_i)$$

are isomorphisms in $DM^{eff}_{gm}(k)$. Here Y_i is smooth proper scheme for all $1 \leq i \leq n$.

Remark 4.5.2. Note that X is cellular if p_i is affine bundle and $Y_i = \operatorname{Spec} k$, for $0 \leq i \leq n$.

Proposition 4.5.3. *Suppose k admits resolution of singularities. Assume that $X \in \mathcal{O}b(Sch_k)$ is equidimensional of dimension n, which admits a filtration as in the definition 4.5.1. Then we have the following decomposition*

$$M^c(X) = \bigoplus_i M^c(Y_i)(d_i)[2d_i].$$

Proof. We prove by induction on $\dim X$. Consider the following distinguished triangle

$$M^c_{gm}(X_{j-1}) \to M^c_{gm}(X_j) \xrightarrow{g_j} M^c_{gm}(U_j) \to M^c_{gm}(X_{j-1})[1].$$

Take the closure of the graph of $p_j : U_j \to Y_j$ in $X_j \times Y_j$. This defines a cycle in $CH_{dim\ X_j}(X_j \times Y_j)$ and since Y_j is smooth this gives a morphism $\gamma_j : M^c_{gm}(Y_j)(d_j)[2d_j] \to M^c_{gm}(X_j)$ by [VSF, Chap. 5, Thm. 4.2.2.3) and Prop. 4.2.3] such that $g_j \circ \gamma_j = p_j^*$. Thus the above distinguished triangle splits. The theorem now follows from induction hypothesis. □

Corollary 4.5.4. *Keep the notation and assumptions of the above proposition. The we have:*

$$H^{BM}_{n,i}(X, R) = \bigoplus_i H^{BM}_{n-2d_i,i-d_i}(Y_i, R)$$

In particular if $Y_i \in Ob(Sm^{fr}/k)$ for every i then $X \in Ob(Sm^{fr}/k)$.

Proof. Apply $Hom(R(i)[n], -)$ to the decomposition we obtained in the above proposition. □

Remark 4.5.5. Note that one can define a variant of the definition 4.5.1 with respect to the functor $M_{gm}(-)$. In this case one has to replace p_i^* by p_{i_*} and it is not necessary to assume that p_is are flat. With this definition it is not hard to see that a variant of the proposition 4.5.3 holds after imposing some additional conditions. Indeed to apply Gysin triangle we have to assume that all X_is which appear in the filtration of X are smooth. Note that in this case we don't need to assume k admits resolution of singularities. The proof goes similar to the proof of proposition 4.5.3. Similarly one could deduce an analogous corollary concerning motivic cohomology sequence rather than the Borel-Moore one.

Remark 4.5.6. Assume that X is a motivic relatively cellular scheme, such that Y_i is pure Tate for every $1 \leq i \leq n$. Then using noetherian induction and gysin triangle one can show that X is pure Tate.

4.6 Motivic Leray-Hirsch theorem

In this section we prove a motivic version of the Leray-Hirsch theorem in $DM_{gm}^{eff}(k)$ which gives the decomposition of the motive of a fibre bundle subject to certain assumptions.

Theorem 4.6.1. *Let X be a smooth irreducible variety over a field k of characteristic 0. Let $\pi : \Gamma \to X$ be a proper smooth locally trivial (for Zariski topology) fibration with fiber F. Furthermore assume that F is cellular. Then one has an isomorphism in $DM_{gm}^{eff}(k)$*

$$M_{gm}(\Gamma) \cong \coprod_{p \geqslant 0} CH_p(F) \otimes M_{gm}(X)(p)[2p].$$

Proof. Take a set of homogeneous elements $\{\zeta_{i,p}\}_{i,p}$ of $CH^*(\Gamma)$ such that for any p the restrictions of $\{\zeta_{i,p}\}_i$ to any fiber $\Gamma_x \cong F$ form a basis for $CH^p(\Gamma_x)$. Notice that since X is irreducible, it is enough that the restrictions of the ζ_is generate $CH_*(\Gamma_x)$ for the fiber Γ_x over a particular x.

By the theorem 2.7.5, for each i, $\zeta_{i,p}$ defines a morphism

$$M_{gm}(\Gamma) \to \mathbb{Z}(p)[2p].$$

Summing up all these morphisms and taking dual, by Poincaré duality we get the following morphism

$$\varphi : M_{gm}(\Gamma) \to \bigoplus_p CH_p(F) \otimes \mathbb{Z}(p)[2p].$$

Composing the morphism

$$M_{gm}(\Delta) : M_{gm}(\Gamma) \to M_{gm}(\Gamma \times \Gamma) \cong M_{gm}(\Gamma) \otimes M_{gm}(\Gamma)$$

which is induced by the diagonal map $\Delta : \Gamma \times \Gamma \to \Gamma$, with $M_{gm}(\pi) \otimes \varphi$ we obtain a morphism $M_{gm}(\Gamma) \to \bigoplus_p CH^p(F) \otimes M_{gm}(X)(p)[2p]$. Now take a covering $\{U_i\}$ of X that trivializes Γ. The restriction of this global morphism to U_j is induced by the restriction of ζ_is to U_j. The same holds over intersections, i.e. these morphisms fit together when we pass to the intersections $U_j \cap U_k$. Thus by Mayer-Vietoris we may reduce to the case that Γ is a trivial fibration $X \times_k F$. This precisely follows from Kunneth formula 2.7.3 and corollary 4.4.2.

□

We finish this section by reproving the V. Voevodsky's projective bundle theorem [MVW, Theorem 15.12].

Corollary 4.6.2. *Let X be a smooth scheme over k and $\mathcal{E}$ be a vector bundle over X. Denote by $p : \mathbb{P}(\mathcal{E}) \to X$ the projective bundle over X associated with $\mathcal{E}$. Then one has a canonical isomorphism in $DM_{gm}^{eff}(k)$ of the form:*

$$M_{gm}(\mathbb{P}(\mathcal{E})) = \oplus_{n=0}^{dim\mathcal{E}-1} M_{gm}(X)(n)[2n]$$

Proof. This follows from the above theorem, corollary 4.4.2 and the fact that $CH^*(\mathbb{P}^n) \cong \mathbb{Z}[\zeta]/\langle\zeta^{n+1}\rangle$, where ζ corresponds to the class of hyperplane in $\mathbb{P}^n$. □

Chapter 5

Resolution by a Tower variety

In the first section of this chapter we recall the "decomposition theorem" of Beilinson, Bernstein, Deligne and Gabber. Then we state a motivic version of the theorem due to Migliorini and de Cataldo, cf. [CM] and Corti and Hanamura cf. [CH]. Using this theory and the results from previous chapters we study the motive of a variety X that admits certain type of resolution of singularities, see section 5.3. Finally, as an application, we study the motive of Schubert varieties in a twisted affine flag variety, see section 5.4.

5.1 The Decomposition theorem of BBDG

In this section, for the sake of completeness, we briefly recall the glorious achievement of the theory of perverse sheaves,"The Decomposition Theorem", of Beilinson, Bernstein, Deligne and Gabber [BBD]. However to avoid lots of complications arising in this theory, we stick to the case of semi-small maps. For those readers who are not familiar with these notions, this hopefully will not make a major issue, since in the rest of the thesis we only implement the motivic version of the decomposition theorem, thanks to the works of Migliorini and deCataldo, cf. [CM] and Corti and Hanamura cf. [CH].

For a proper morphism $f : X \to Y$ we assume that there exist a finite algebraic stratification $Y = \bigsqcup_{b \in B} Y_b$ which satisfies the following conditions

(a) every space Y_b is a locally closed and smooth subvariety of Y and

(b) the induced maps $f|_{f^{-1}(Y_b)} : f^{-1}(Y_b) \to Y_b$ are locally trivial fibrations over Y_b.

Note that the existence of such stratification in characteristic zero is guaranteed, see [GM, p.43].

Here we recall the definition of semismall map.

Definition 5.1.1. A morphism $f : X \to Y$ is called semi-small if all irreducible components of $X \times_Y X$ have dimension less than or equal to $\dim X$. Furthermore we say that f is small if in addition the irreducible components of dimension $n := \dim X$ of $X \times_Y X$ dominate Y.

Proposition 5.1.2. *Let $f' : X' \to Y$ and $f : X \to Y$ be proper maps, semismall over their images. Then*

$$\dim X' \times_Y X \leq 1/2(\dim X + \dim X').$$

Proof. easy. □

Remark 5.1.3. Note that the semi-smallness of f is equivalent to the following condition:

For a positive integer $\delta \in \mathbb{N}$ let $Y^\delta := \{y \in Y | dim X_y = \delta\}$, then

$$2\delta \leq dimY - dimY^\delta.$$

Furthermore f is small if the above inequality is strict for any $\delta > 0$.

Remark 5.1.4. A semismall map $f : X \to Y$ is generically finite. Indeed, since $dimY^\delta < dimY$ for $\delta > 0$, therefore Y^0 is the open dense subvariety. Hence f is generically finite and $\dim Y = \dim X$.

Definition 5.1.5. Let

$$\mathcal{A} := \{a \in B; 2 \dim f^{-1}(y) = \dim Y - \dim Y_a, y \in Y_a\}.$$

This set is called the set of relevant strata for f.

Remark 5.1.6. In general we may have different stratifications for f, however the set of subvarieties $\{\overline{Y_a}\}_{a \in \mathcal{A}}$ is uniquely determined by f.

Remark 5.1.7. Note that a small map has only one relevant stratum, i.e. the dense one.

Let us back to the Decomposition theorem. Borho and MacPherson first observed that in fact a simplified version of the Decomposition theorem can be established for semi-small maps of manifolds, see [BM]. Also in the proof given by Migliorini and de Cataldo they show that the decomposition theorem can be deduced from the non-degeneracy of certain bilinear forms. In addition, as the crucial part of their approach, they deduce the general case by reducing to the semismall case. This is done by induction on the "defect of semi-smallness" (this notion actually measures that how far away a map is from being semi-small) and by taking hyperplane sections to reduce the defect of semi-smallness.

A subset $U \subseteq Z$ is called constructible if it is obtained from a finite sequence of unions, intersections, or complements of algebraic subvarieties of Z.
A local system on Z is a locally constant sheaf on Z with finite-dimensional stalks. Note that a local system on Z corresponds to a finite-dimensional representation of the fundamental group of Z.

We say that a complex $(K^{\cdot}, \delta^{\cdot})$ of sheaves on Z has constructible cohomology sheaves if there exist a decomposition $Z := \sqcup_{\alpha} Z_{\alpha}$ into finitely many constructible subsets such that for each i, the cohomology sheaves $H^i(K^{\cdot}) := Ker(\delta^i)/Im(\delta^{i-1})$ are locally constant along each Z_{α} with finite-dimensional stalks.

Bounded complexes with constructible cohomology sheaves are called *constructible complexes*. The *constructible bounded derived category* $D^b_{cc}(Z)$ is the full subcategory of the bounded derived category of sheaves $D^b(Z)$ whose objects are the constructible complexes.

Let L be a local system on a Zariski dense open subset of the regular locus Z_{reg}. The intersection complex $IC_Z(L)$ is a complex of sheaves on Z, which extends the complex $L[dimZ]$ on Z_{reg} and is determined, up to unique isomorphism in $D^b_{cc}(Z)$, by certain support and co-support conditions, see [GM].

If $L = \mathbb{Q}_{Z_{reg}}$ then we simply denote it by IC_Z.

For a morphism $f : X \to Z$ and a bounded complex of sheaves $\mathcal{F}^{\cdot}$ on X, let $Rf_*\mathcal{F}^{\cdot}$ denotes the usual (derived) direct image of $\mathcal{F}^{\cdot}$.

Theorem 5.1.8. *Let $f : X \to Y$ be a proper, semismall algebraic map of algebraic varieties. Suppose that X is smooth or with rational singularities.*

Set $n := \dim X$. Then there is a canonical isomorphism

$$Rf_*\mathbb{Q}_X[n] \cong_{D^b_{cc}(Y)} \bigoplus_{a\in\mathcal{A}} IC_{\overline{Y_a}}(L_a)$$

Here L_a is the semisimple local system on Y_a given by the monodromy action on the maximal dimensional irreducible components of the fibers of f over Y_a.

5.2 The Motivic Decomposition Theorem

In this section, let X be a variety which is locally in the étale topology isomorphic to the quotient of a nonsingular variety X' by the action of a finite group Γ of automorphisms of X'. We call such X, *a* $\mathbb{Q}$*-variety* and denote by $\mathbb{Q} - var$ the full subcategory whose objects are $\mathbb{Q}$-varieties.

Let us introduce the following category:

The category $\mathcal{C}_{S,\mathbb{Q}}$ has proper maps $f : X \to S$ from a $\mathbb{Q}$-variety X as its objects. Now we want to introduce its morphisms.

Consider the object $Y \to S$ where Y is an equi-dimensional variety. Set

$$Hom_{\mathcal{C}_{S,\mathbb{Q}}}(X \to S, Y \to S) := CH_{dimY}(X \times_S Y).$$

By additivity the definition extends to the case that Y is not equi-dimensional.

To define the composition of the morphisms one needs to introduce the refined gysin map.

Refined gysin map:

Consider the following fibre product diagram:

$$\begin{array}{ccc} X' := X \times_Y Y' & \longrightarrow & X \\ \big\downarrow & & \big\downarrow \iota \\ Y' & \xrightarrow{f} & Y \end{array}$$

where ι is a regular embedding of codimension d with normal bundle $\mathcal{N}_X Y$ and Y' is a pure l-dimensional variety. We want to define the refined gysin homomorphism:

$$\iota^! : CH_*(Y') \to CH_{*-d}(X').$$

Note that it is enough to construct $\iota^!([Y'])$ (for a general $\alpha \in CH_*(Y')$, replace Y' with support of α and map the resulting class to $CH_*(Y')$).

The map $X' \to Y'$ is a closed embedding and the normal cone $\mathcal{C}_{X'}Y'$ is a pure l-dimensional subscheme of the pull back of $N_X Y$ to X'; its cycle class is therefore equivalent to the flat pull back of a unique cycle class $\iota^!([Y']) \in CH_{l-d}(X')$.

Now let $\delta : Y \to Y \times Y$ denotes the diagonal map. Consider the following cartesian diagram

$$\begin{array}{ccc} X \times_S Y \times_S Z & \longrightarrow & X \times_S Y \times Y \times_S Z \\ \big\downarrow & & \big\downarrow \\ Y & \xrightarrow{\delta} & Y \times Y. \end{array}$$

Now for given

$$\varphi \in CH_{dimY}(X \times_S Y) = Hom_{\mathcal{C}_{S,\mathbb{Q}}}(X \to S, Y \to S)$$

and

$$\psi \in CH_{\dim Z}(Y \times_S Z) = Hom_{\mathcal{C}_{S,\mathbb{Q}}}(Y \to S, Z \to S),$$

define $\psi \circ \varphi := p_{XZ_*}\delta^!(\varphi \times \psi)$, where $p_{XZ} : X \times_S Y \times_S Z \to X \times_S Z$ is the projection morphism.

Note that one can define the Tate object in $\mathcal{C}_{S,\mathbb{Q}}$ in a usual way. This leads to the following definition.

Definition 5.2.1. The category $CHM_{S,\mathbb{Q}}$ of pure Chow motives over S is defined by adding the Tate twist and taking pseudo-abelian envelope of $\mathcal{C}_{S,\mathbb{Q}}$.

Note that the category $CHM_{\operatorname{Spec} k,\mathbb{Q}}$ coincides with the classical category of rational Chow motives over k.

We represent an element of $CHM_{S,\mathbb{Q}}$ by $(f : X \to S, P)(r)$ where $P \in CH_{\dim X}(X \times_S X)$ is a projector, i.e. $P \circ P = P$ as correspondences, and r is an integer. If $r = 0$, then we drop it from the notation. Furthermore if P is the diagonal morphism Δ_X, then we simply write $f : X \to S$. We assign to a $\mathbb{Q}$-variety X over S a *pure chow motive* in the obvious way.

Let us list up some facts about this category:

- If S is a $\mathbb{Q}$-variety, then

$$Hom_{CHM_{S,\mathbb{Q}}}((X \to S)(i), (id : S \to S)(j)) = CH_{\dim S+i-j}(X).$$

- Any proper map $S \to S'$ induces a functor

$$CHM_{S,\mathbb{Q}} \to CHM_{S',\mathbb{Q}},$$

  in particular:

- if S is proper, then there is a natural map

$$CHM_{S,\mathbb{Q}} \to M^{eff}(k) \otimes \mathbb{Q}$$

  to the category of rational Chow motives.

- The category of relative chow motives $CHM_{S,\mathbb{Q}}$ admits a topological realization in $D^b_{cc}(S)$, indeed Corti and Hanamura in [CH] and de Cataldo and Migliorini in [CM] observed that the natural assignment

$$\begin{array}{rcl} CHM_{S,\mathbb{Q}} & \to & D^b_{cc}(S) \\ (f : X \to S)(r) & \mapsto & Rf_*\mathbb{Q}_X[2r] \end{array}$$

  is "functorial". To see this first notice that by lemma 2.21 and lemma 2.23 of [CH] we have

$$\mathrm{Hom}_{D^b_{cc}(S)}(Rf_*\mathbb{Q}_X[i], Rg_*\mathbb{Q}_Y[j]) \cong H^{BM}_{2dimY+i-j}(X \times_S Y).$$

  Then the cycle map $CH_{\dim Y+i-j}(X \times_S Y) \to H^{BM}_{2(\dim Y+i-j)}(X \times_S Y)$ induces

$$Hom_{CHM_{S,\mathbb{Q}}}((X \to S)(i), (Y \to S)(j)) \longrightarrow$$
$$Hom_{D^b_{cc}(S)}(Rf_*\mathbb{Q}_X[2i], Rg_*\mathbb{Q}_Y[2j]).$$

  This morphism establishes the functoriality.

Remark 5.2.2. Let $X \xrightarrow{f} S$ and $Y \xrightarrow{g} S$ be proper maps of $\mathbb{Q}$-varieties, semismall on their images and suppose that $t := (1/2)(dimY - dimX)$ is an integer. Then the map:

$$Hom_{CHM_{S,\mathbb{Q}}}(X \xrightarrow{f} S, (Y \xrightarrow{g} S)(t)) \cong CH_{(1/2)(\dim X+\dim Y)}(X \times_S Y)$$
$$\to Hom_{D^b_{cc}(S)}(Rf_*\mathbb{Q}_X[dimX], Rg_*\mathbb{Q}_Y[dimY]) \cong H^{BM}_{\dim X+\dim Y}(X \times_S Y)$$

is an isomorphism, see lemma 2.3.7 [CM].

We are going to state the main theorem of de Cataldo and Migliorini in [CM]. Let X be a $\mathbb{Q}-variety$ of dimension n and $f : X \to Y$ be a proper , semismall and surjective map. Let $\mathcal{A}$ be the set of relevant strata for f. For $a \in \mathcal{A}$, let Y_a denote the corresponding relevant stratum. Let $t_a := \frac{dimY_a - dimX}{2}$, $y_a \in Y_a$ and E_a be the $\pi_1(Y_a, y_a) - set$ given by the monodromy action of $\pi_1(Y_a, y_a)$ on the maximal dimensional irreducible components of $f^{-1}(y_a)$. Let $\nu_a : Z_a \to Y_a$ be the étale covering associated with $\pi_1(Y_a, y_a) - set\ E_a$. We let $Z_{a,i}$ denote the connected components of Z_a. Let $\overline{Z_{a,i}}$ be a $\mathbb{Q}-variety$ containing $Z_{a,i}$ as a Zariski-dense open subset and such that there is a proper surjective map $\overline{\nu_{a,i}} : \overline{Z_{a,i}} \to \overline{Y_a}$ extending $\nu_{a,i} := \nu_a|_{Z_{a,i}}$ (Note that $\overline{Y_a}$ is the closure of Y_a in Y). Set

$$\overline{\nu_a} := \coprod_i \overline{\nu_{a,i}} : \overline{Z_a} = \coprod_i \overline{Z_{a,i}} \to \overline{Y_a}.$$

We denote the composition $\overline{Z_a} \xrightarrow{\overline{\nu_a}} \overline{Y_a} \hookrightarrow Y$ by the same symbol $\overline{\nu_a}$. Consider the decomposition

$$Rf_*\mathbb{Q}_X[n] \cong_{D^b_{cc}(Y)} \bigoplus_{a\in\mathcal{A}} IC_{\overline{Y_a}}(L_a),$$

see theorem 5.1.8 , and let $P_a^{top} \in End_{D^b_{cc}}(Rf_*\mathbb{Q}_X[dimX])$ be the corresponding projectors. By remark 5.2.2 they corresponds to the projectors

$$P_a \in End_{CHM_{Y,\mathbb{Q}}}(X \xrightarrow{f} Y).$$

Denote by IH_f the projector corresponding to the open stratum of Y. Note that if f is generically one-to-one, IH_f^{top} gives the projection on the intersection cohomology complex of Y.

The following theorem is the main theorem of Migliorini and de Cataldo in [CM].

Theorem 5.2.3. *Keep the above notations, the following holds:*

a) There is an isomorphism in $CHM_{Y,\mathbb{Q}}$

$$(X \xrightarrow{f} Y) \cong \bigoplus_{a\in\mathcal{A}} (X \xrightarrow{f} Y, P_a).$$

b) If the maps $\overline{\nu_{a,i}} : \overline{Z_{a,i}} \to \overline{Y_a}$ are semi-small, then there is an isomorphism in $CHM_{Y,\mathbb{Q}}$

$$(X \xrightarrow{f} Y, IH_f) \cong \bigoplus_{a\in\mathcal{A}} (\overline{Z_a} \xrightarrow{\overline{\nu_a}} Y, IH_{\overline{\nu_a}})(t_a).$$

c) Assume that $\overline{\nu_{a,i}} : \overline{Z_{a,i}} \to \overline{Y_a}$ *are small. Then*

$$(X \xrightarrow{f} Y) \cong \bigoplus_{a\in\mathcal{A}} (\overline{Z_a} \xrightarrow{\overline{\nu_a}} \overline{Y_a})(t_a).$$

Note that in [CM], de Cataldo and Migliorini also construct a correspondence $\overline{\Gamma} \subseteq \coprod_a \overline{Z_a} \times X$ which induces the following decomposition:

$$\overline{\Gamma} : \bigoplus_{a\in\mathcal{A}} [\overline{Z_a}](t_a) \simeq [X].$$

5.3 Resolution by a Tower Variety

Recall that in [dJ], de Jong introduced the notion of the alteration of a variety X. However this notion is slightly weaker than resolution of singularities but made him able to prove that any variety has an alteration which is regular.

Definition 5.3.1. Let X be a variety over a field k. An alteration of X is a proper morphism which is generically finite

In the sequel of the section we study motives of varieties admitting certain type of alterations.

Let $\tilde{X}$ be a variety over a perfect field k. Suppose that $\tilde{X}$ sits in the tower

$$\begin{array}{c} \tilde{X}_n := \tilde{X} \\ {\scriptstyle p_{n-1}}\downarrow \\ \tilde{X}_{n-1} \\ \vdots \\ \downarrow \\ \tilde{X}_0 \end{array}$$

where $\tilde{X}_i \to \tilde{X}_{i-1}$ is a proper smooth locally trivial fibration with fibre F_i, such that F_i is cellular and satisfies Poincaré duality. We call such a variety a tower variety over $\tilde{X}_0$.

In the rest of this section we will assume that $\tilde{X}_0$ is smooth.

Theorem 5.3.2. *Let $f : \tilde{X} \to X$ be a surjective semismall morphism. Assume that it is an alteration in the sense of de Jong. Further assume that $\tilde{X}$ is a tower variety over smooth proper scheme $\tilde{X}_0$ as above. Then the motive associated to X, $M(X)$, is a summand of*

$$\bigotimes_{i=0}^{n-1}\left(\coprod_{p\geqslant 0} CH_p(F_i)\otimes \mathbb{Z}(p)[2p]\right)\otimes M(\tilde{X}_0).$$

Proof. The morphism $f : \tilde{X} \to X$ can be factored, using the Stein factorization, as $\tilde{X} \xrightarrow{g} X' \xrightarrow{h} X$, where $g : \tilde{X} \to X'$ is a semi-small birational morphism and $h : X' \to X$ is a finite morphism. Assume that $h : X' \to X$ is of degree d. Therefore this morphism induces a morphism $\varphi : M(X) \to M(X')$ such that the composition $M(h)\circ\varphi$ is multiplication by d, being the coefficients in $\mathbb{Q}$, $M(X)$ is a direct summand of $M(X')$.

Now by theorems 2.5.4 and 5.2.3, $M(X')$ is a direct summand of $M(\tilde{X})$. On the other hand, since F_is are cellular and satisfy Poincaré duality, by theorem 4.6.1 the motive $M(\tilde{X})$ is of the form

$$\bigotimes_{i=0}^{n-1}\left(\coprod_{p\geqslant 0} CH_p(F_i)\otimes \mathbb{Z}(p)[2p]\right)\otimes M(\tilde{X}_0).$$

□

Corollary 5.3.3. *Let $\tilde{X} \to X$ be a semi-small resolution of singularities and assume that $\tilde{X}$ is a tower variety over smooth proper scheme $\tilde{X}_0$. Let $\{X_\alpha\}$ be a set of connected relevant strata. Then for each α the motive $M(\overline{X}_\alpha)$ (associated to the closure of X_α), appears as a direct summand of*

$$\bigotimes_{i=0}^{n-1}\left(\coprod_{p\geqslant 0} CH_p(F_i)\otimes \mathbb{Z}(p)[2p]\right)\otimes M(\tilde{X}_0).$$

Remark 5.3.4. Assuming conjectures of Grothendieck and Murre, see [CH, Paragraph 3.6], Corti and Hanamura prove that the decomposition theorem holds in $CHM_{S,\mathbb{Q}}$. In this case we may drop the semismallness condition from the hypotheses of the above corollary.

Since the category of mixed Tate motives is thick, the above theorem has the following obvious consequence:

Corollary 5.3.5. *Keep the assumptions of the above theorem. Assume that the motive associated with $M(\tilde{X}_0)$ is mixed Tate (resp. geometrically mixed Tate). Then $M(X)$ is also mixed Tate (resp. geometrically mixed Tate).*

5.4 Motive of Schubert varieties in Affine Flag varieties

In section 4.4.1 we saw that the motive associated with a Schubert variety in a (finite dimensional) generalized flag variety G/P has a relatively easy description. More precisely for a split reductive group G over a perfect field k, one can construct an explicit filtration by closed sub-Schubert varieties by a Schubert variety $\overline{X_\omega}$, furthermore this filtration form a cellular decomposition for $\overline{X_\omega}$. In particular one can give a concrete description of the motive of such varieties in the terms of their fundamental invariants.
In the current section we first recall the definition of Schubert varieties S_w in twisted affine flag varieties, introduced by Pappas and Rapoport, see [PR]. Note that computing the motive $M(S_w)$ is slightly more complicated than that of Schubert varieties in a generalized flag variety. Namely it turns out that the filtration obtained by the closed Schubert varieties is not in general a topological cell decomposition, see [Ri, Remark 2.6].

At the end of this section, as an application of the results of previous section, we"relate" the motive associated to an affine Schubert variety S_w to the motive of projective homogeneous varieties, which are well understood (as we mentiond above).

To define a Schubert variety in a twisted affine flag variety we need to recall some basic facts from Bruhat's theory of buildings. Since giving a detailed explanation about buildings will stray us away from the purpose of the thesis, we would prefer to state some general facts and avoid a detailed explanation of the theory.

We refer the reader to [AB] for a detailed explanation about Coxeter complexes, buildings and etc.

Recall that to a Coxeter group W, one associates a simplicial complex $\Sigma = \Sigma(W, S)$, called the *Coxeter complex*.
A *building* is a simplicial complex Δ that can be expressed as the union of subcomplexes Σ (called *apartments*) subject to the following axioms:

B0 Each apartment Σ is a Coxeter complex.

B1 For any two simplices $A, B \in \Delta$, there is an apartment Σ containing both of them.

B2 If Σ and Σ' are two apartment containing A and B, then there is an isomorphism $\Sigma \to \Sigma'$ fixing A and B pointwise.
We say that a map fixes a simplex A *pointwise* if it fixes every vertex of A.

Let G be a connected reductive linear algebraic group over the local field $K := k((z))$ of Laurent series with algebraically closed residue field k and ring of integers $\mathcal{O}_K = k[\![z]\!]$.

To a connected reductive group G over a local field K, Bruhat and Tits associate a building $\mathcal{B}(G)$. Moreover any maximal split torus S defines an apartment $\mathcal{A} := \mathcal{A}(G, S)$ which is called the reduced apartment of G associated with S.

For any $F \in \mathcal{A}(G, S)$, there exists a unique connected $\mathcal{O}_K$-model $\mathcal{P}_F$ of G, cf. [BT].

Definition 5.4.1. Let C be a maximal dimensional simplex (alcove) in $\mathcal{A}(G, S)$. The group scheme $\mathcal{P}_C$ is called the *Iwahori* group scheme of G associated to (G, S, C), we denote it by B. For a general facet F, $\mathcal{P}_F$ is called the *parahoric* group scheme of G associated to (G, S, F).

Fix a maximal K-split torus S. Let $T := Z_G(S)$ be the centralizer of S (a maximal torus), and let $N := N_G(S)$ be the normalizer of S. Denote by $W_0 = N(K)/T(K)$ the relative Weyl group of G with respect to S.

The *Iwahori-Weyl group* $\tilde{W}(G, S)$ of G with respect to S is

$$\tilde{W} = \tilde{W}(G, S) := N(K)/\mathcal{T}_0(\mathcal{O}_K)$$

where $\mathcal{T}_0$ is the connected Néron model of T over $\mathcal{O}_K$.
In a similar manner as in the section 3.1, one can observe that $\tilde{W}$ is endowed with a length function ℓ and a (partial) order $\leq$.

As we mentioned before to a facet F in $\mathcal{A}$ one associates a parahoric subgroup $\mathcal{P}_F$. The subgroup $\tilde{W}_F$ of the Iwahori-Weyl group $\tilde{W}$ corresponding to F is

$$\tilde{W}_F := N(K) \cap \mathcal{P}_F/\mathcal{T}^0(\mathcal{O}_K)$$

Proposition 5.4.2. *Let B be the Iwahori subgroup of $G(K)$ associated to an alcove contained in the apartment associated to the maximal split torus S. Then $G(K) = BN(K)B$ and the map which sends BnB to $\overline{n} \in \tilde{W}$ induces a bijection (8.2)*

$$B\backslash G(K)/B \tilde{\to} \tilde{W}.$$

Let F, F' be two facets contained in the apartment corresponding to S, then there is a bijection

$$\tilde{W}_{F'}\backslash\tilde{W}/\tilde{W}_F \tilde{\to} \mathcal{P}_{F'}\backslash G(K)/\mathcal{P}_F.$$

Proof. cf. [PR, Proposition 8.1]. □

Let X be a scheme over K. Define the functor

$$LX : Aff/k \to \mathcal{S}ets$$

$$\operatorname{Spec} R \mapsto X(R((z))).$$

from the category of affine schemes over k to the category of sets. Likewise, for any scheme $\mathcal{X}$ over $\mathcal{O}_K$ define the functor

$$L^+\mathcal{X} : Aff/k \to \mathcal{S}ets$$

$$\operatorname{Spec} R \mapsto \mathcal{X}(R[\![z]\!]).$$

Let G be as above and $\mathcal{P}$ be a parahoric group. The functors LG and $L^+\mathcal{P}$ give rise to sheaves in the fpqc-topology on the category of affine k-schemes.

Recall that a k-space is called an ind-scheme if it is the inductive limit of the functors associated to a directed family of k-schemes. An ind-group scheme is an ind-scheme which is a group object in the category of ind-schemes.

Proposition 5.4.3. *If $\mathcal{X}$ is an affine scheme of finite type over $\mathcal{O}_K$ (resp. K) then $L^+\mathcal{X}$ (resp. LX) is represented by an affine scheme (resp. ind-scheme).*

Proof. cf. [PR, Section 1] □

Definition 5.4.4. Let G (resp. $\mathcal{P}$) be a linear algebraic group over K (resp. $\mathcal{O}_K$). The group of loops (resp. positive loops) associated to G is the ind-group scheme (resp. k-scheme) LG (resp. $L^+\mathcal{P}$) over $\operatorname{Spec} k$.

Definition 5.4.5. The twisted affine flag variety of G corresponding to $F \in \mathcal{A}(G, S)$ is the $fppf$-quotient sheaf

$$\mathcal{F}\ell_F := \mathcal{F}\ell_{\mathcal{P}_F} = LG/L^+\mathcal{P}_F.$$

Theorem 5.4.6. *Let $\mathcal{P}_F$ be an affine group scheme smooth and of finite type over $\mathcal{O}_K$ corresponding to $F \in \mathcal{A}(G,S)$. Then the $fppf$-sheaf $\mathcal{F}\ell_F$ is represented by an ind-k-scheme of ind-finite type over k. The quotient morphism $LG \to \mathcal{F}\ell_F$ admits sections locally for the étale topology.*

Proof. cf. [PR, Theorem 1.4]. □

Definition 5.4.7. Let $w \in \tilde{W} = \tilde{W}(G,S)$ be an element of the Iwahori-Weyl group. Let $\mathcal{P}$ and $\mathcal{P}'$ be parahoric groups associated to facets F and F'. The $(\mathcal{P}',\mathcal{P})$-Schubert cell $C_w = C_w(\mathcal{P}',\mathcal{P})$ is the reduced subscheme of $L^+\mathcal{P}'$-orbit of w in $\mathcal{F}\ell_{\mathcal{P}}$ under the left action of $L^+\mathcal{P}'$. The $(\mathcal{P}',\mathcal{P})$-Schubert variety $S_w = S_w(\mathcal{P}',\mathcal{P})$ is the reduced scheme with underlying set, the Zariski closure of C_w.

Theorem 5.4.8. *Suppose that G splits over a tamely ramified extension of K. Furthermore assume that the characteristic of k does not divide the order of the fundamental group of the derived group $\pi_1(G^{der})$. Then for any $w \in \tilde{W}$ the Schubert variety S_w is normal, Frobenius-split (when $\mathrm{char} k > 0$) and has only rational singularities.*

Proof. cf. [PR, Theorem 8.4]. □

Proposition 5.4.9. *Let $w \in \tilde{W}_{F'} \backslash \tilde{W} / \tilde{W}_F$. Let $\mathcal{P}_F$ and $\mathcal{P}_{F'}$ be as 5.4.2, then*

(a) The Schubert variety $S_w = S_w(\mathcal{P}_{F'},\mathcal{P}_F)$ is set theoretically the disjoint union of locally closed subvarieties

$$S_w = \bigsqcup_{w' \leq w} C_{w'}(\mathcal{P}_{F'},\mathcal{P}_F)$$

where w' runs over the cosets of the double quotient $\tilde{W}_{F'} \backslash \tilde{W} / \tilde{W}_F$

(b) $\dim S_w = \ell(w)$.

Proof. cf. [Ri, Proposition 2.8] □

Note that in our terminology the word "cell" is somewhat misleading. In fact unlike the situation in subsection 4.4.1 the $(\mathcal{P}', \mathcal{P})$-Schubert cells $C_w = C_w(\mathcal{P}', \mathcal{P})$ need not to be a topological cell unless the case that the parahoric group $\mathcal{P}'$ equals to an Iwahori-group [Ri, Remark 2.6].
Below we recall Bott-Samelson-Demazure resolution for an affine Schubert variety S_ω, as it is constructed in [Ri].

Definition 5.4.10. A sequence of parahoric group schemes $\mathcal{P}_1, \ldots, \mathcal{P}_n$ and $\mathcal{Q}_1, \ldots, \mathcal{Q}_n$ with

(i) $L^+\mathcal{Q}_i \subset L^+\mathcal{P}_i \cap L^+\mathcal{P}_{i+1}$ for $i = 1, \ldots, n-1$,

(ii) $L^+\mathcal{Q}_n \subset L^+\mathcal{P}_n \cap L^+\mathcal{P}_F$

is called resolutive with respect to S_ω, if the morphism given by multiplication

$$L^+\mathcal{P}_1 \times^{L^+\mathcal{Q}_1} \cdots \times^{L^+\mathcal{Q}_{n-1}} L^+\mathcal{P}_n / L^+\mathcal{Q}_n \to \mathcal{F}\ell_F$$

factors through S_ω and induces a birational morphism

$$m : L^+\mathcal{P}_1 \times^{L^+\mathcal{Q}_1} \cdots \times^{L^+\mathcal{Q}_{n-1}} L^+\mathcal{P}_n / L^+\mathcal{Q}_n \to S_\omega.$$

The morphism m is $L^+\mathcal{P}_1$-equivariant, and hence $L^+\mathcal{P}_1$ must stabilize the Schubert variety S_ω.

We call $\Omega(S_\omega) := L^+\mathcal{P}_1 \times^{L^+\mathcal{Q}_1} \cdots \times^{L^+\mathcal{Q}_{n-1}} L^+\mathcal{P}_n / L^+\mathcal{Q}_n$, the *Bott-Samelson-Demazure variety* corresponding to S_ω.

Theorem 5.4.11. *Let F' be another facet in the Bruhat-Tits building of G and let $S_\omega(\mathcal{P}_{F'}, \mathcal{P}_F)$ be any $(\mathcal{P}_{F'}, \mathcal{P}_F)$-Schubert variety in $\mathcal{F}_{\ell_F}$. Then there exists a resolutive sequence of parahoric group schemes for $S_\omega(\mathcal{P}_{F'}, \mathcal{P}_F)$ such that the corresponding birational morphism is $L^+\mathcal{P}_{F'}$-equivariant.*

Proof. See [Ri, Corollary 3.5]. □

By the above theorem we may take a resolutive sequence for S_ω (as in the definition 5.4.10). Let $\overline{P}_i^{red}$ denotes the maximal reductive quotient of the special fibre of $\mathcal{P}_i$. Consider the special fibre of the morphism $\mathcal{Q}_i \hookrightarrow \mathcal{P}_i$. Then the image $\overline{Q}_i$ in $\overline{P}_i^{red}$ is a parabolic subgroup and the reduction mod $\mathfrak{m}_K$ gives an isomorphism

$$\frac{L^+\mathcal{P}_i}{L^+\mathcal{Q}_i} \cong \frac{\overline{P}_i^{red}}{\overline{Q}_i} \cong F_i,$$

with F_i a projective homogeneous variety. Hence the Bott-Samelson-Demazure variety is a tower variety with fibres $\{F_i\}$, which are projective varieties and in particular they admits cell decomposition, see section 4.4.1 .
By remark 4.4.12 a fibre F_i satisfies Poincaré duality and therefore we may apply Leray-Hirsch theorem to obtain

$$M(\Omega(S_w)) = \bigotimes_{i=1}^{n} M(F_i).$$

Now the following corollary follows from corollary 5.3.3.

Corollary 5.4.12. *Keep the above notation. Assume φ is semismall. Then $M(S(F', F))$ is a summand of $\bigotimes_{i=1}^{n} M(F_i)$, where F_i is the projective homogeneous variety $\overline{P}_i^{red}/\overline{Q}_i$.*

Remark 5.4.13. Note that in certain cases (e.g. the case of minuscule Schubert varieties), the semismallness of such resolution (or a minimal model for that) is known, compare [Pe].

Chapter 6

Mixed Tate Configuration

In this chapter we introduce the notion of "mixed Tate configuration" and show that the motive associated to such configuration of varieties is mixed Tate. Then in the section 6.2 we relate this notion with the geometry of $G \times G$-orbits of the wonderful compactification of a reductive group G of adjoint type.

6.1 Mixed Tate configuration

Definition 6.1.1. Let $X \in \mathcal{O}b(Sch_k)$. We say that X is *mixed Tate* if the associated motive $M^c_{gm}(X)$ is an object of the subcategory of mixed Tate motives $TDM^{eff}_{gm}(k)$. Let $\{X_i\}_{i=1}^n$ be the set of irreducible components of X. We call X a *configuration of mixed Tate varieties* if

i) X_i is mixed Tate for $1 \leq i \leq n$, and

ii) Union of the elements of any arbitrary subset of $\{X_{ij} := X_i \cap X_j\}_{i \neq j}$ is a configuration of mixed Tate varieties or is empty.

Lemma 6.1.2. *The motive of every configuration of mixed Tate varieties is mixed Tate.*

Proof. We prove by induction on r, the dimension of mixed Tate configuration. The statement is obvious for $r = 0$. Suppose that the lemma holds for all mixed Tate configurations of dimension $r < m$. Let $X = X_1 \cup \cdots \cup X_n$ be a configuration of mixed Tate varieties of dimension m, where X_is are its irreducible components. For inclusion $\bigcup_{i\neq j} X_{ij} \subset \bigcup_{i=1}^{n} X_i$, we have the following induced localization distinguished triangle:

$$M_{gm}^c(\bigcup_{i\neq j} X_{ij}) \to M_{gm}^c(X_1 \cup ... \cup X_n) \to M_{gm}^c(\bigcup_{i=1}^{n} X_i \smallsetminus \bigcup_{i\neq j} X_{ij}) \to M_{gm}^c(\bigcup_{i\neq j} X_{ij})[1].$$

By the induction assumption, $M_{gm}^c(\bigcup_{i\neq j} X_{ij})$ is mixed Tate.
On the other hand we have:

$$M_{gm}^c(\bigcup_{i=1}^{n}(X_i \smallsetminus \bigcup_{i\neq j} X_{ij})) = \bigoplus_{i=1}^{n} M_{gm}^c(X_i \smallsetminus \bigcup_{i\neq j} X_{ij}).$$

It only remains to show that for every i, $M_{gm}^c(X_i \smallsetminus \bigcup_{i\neq j} X_{ij})$ is mixed Tate. To see this, for a given i consider the following distinguished triangle:

$$M_{gm}^c(\bigcup_{j\neq i} X_{ij}) \to M_{gm}^c(X_i) \to M_{gm}^c(X_i \smallsetminus \bigcup_{j\neq i} X_{ij}) \to M_{gm}^c(\bigcup_{j\neq i} X_{ij})[1].$$

Notice that $M_{gm}^c(\bigcup_{i\neq j} X_{ij})$ is mixed Tate by induction hypothesis. □

6.2 Wonderful Compactification

Let k be a perfect field. Let G be a split connected reductive group over k. Fix a maximal split torus T in G.

In this section we first recall the definition of wonderful compactification of G, and then will study some of its properties.

In [CP], De Concini and Procesi have introduced the wonderful compactification of a symmetric space. In particular their method produces a smooth canonic compactification $\overline{G}$ of a semisimple algebraic group G of adjoint type. Note that in [CP] they study only the case that the group G is defined over $\mathbb{C}$. Most of the theory carries over for any algebraically closed field of arbitrary characteristic. However there are some subtleties in positive characteristic which we mention later.
As a feature of this compactification there is a natural $G \times G$-action on $\overline{G}$, and the arrangement of the orbits can be explained by the associated weight polytope. Let us briefly recall some facts about the construction of $\overline{G}$ and the geometry of its $G \times G$-orbits and their closure.

Let $\rho_\lambda : G \to GL(V_\lambda)$ be an irreducible faithful representation of G with strictly dominant highest weight λ. We define the compactification X_λ of G as follows

$$X_\lambda = \overline{\mathbb{P}(\rho_\lambda(G))},$$

where the closure is taken inside $\mathbb{P}(End(V_\lambda))$.
It is verified in [CP] that when G is of adjoint type, X_λ is smooth and independent of the choice of the highest weight. This compactification is called *wonderful compactification* and we denote it by $\overline{G}$.

Consider all weights of the representation ρ_λ with respect to the maximal torus T and take their convex hull $\mathcal{P}$ in $X^*(T) \otimes \mathbb{R}$. Then it is easy to see that $\mathcal{P}$ is a polytope invariant under the action of the Weyl group of G. It is called the weight polytope of the representation ρ_λ. Fix a system of fundamental roots Δ.

The following proposition explains the geometry of the wonderful compactification and the closures of its $G \times G$-orbits. Furthermore it provides an effective method to compute their cohomologies.

Proposition 6.2.1. *Keep the above notation, we have the following statements:*

a) There is a one-to-one correspondence between the $G \times G$-orbits of $\overline{G}$ and the orbits of the action of the Weyl group on the faces of the polytope $\mathcal{P}$, which preserves the incidence relation among orbits (i.e. consider the faces $\mathcal{F}_1 \subseteq \mathcal{F}_2$ of the polytope $\mathcal{P}$, the orbit corresponds to the face $\mathcal{F}_1$ is contained in the closure of the orbit which corresponds to $\mathcal{F}_2$).

b) Let $I \subset \Delta$ and $\mathcal{F} = \mathcal{F}_I$ the associated face of $\mathcal{P}$. Let $D_\mathcal{F}$ be the closure of the orbit corresponding to the face $\mathcal{F}$. Then $D_\mathcal{F} = \sqcup_{\alpha \in W \times W} C_{\mathcal{F},\alpha}$, such that for each $\alpha := (u, v)$ there is a bijective morphism

$$\mathbb{A}^{n_{\mathcal{F},\alpha}} \to C_{\mathcal{F},\alpha},$$

where $n_{\mathcal{F},\alpha} = l(w_0) - l(u) + \mid I \cap I_u \mid + l(v)$ and w_0 denotes the longest element of the Weyl group. In particular when $\mathrm{char}\, k = 0$ (resp. $\mathrm{char} > 0$) $D_\mathcal{F}$ is cellular (resp. motivic cellular).

c) $\overline{G} \smallsetminus G$ is a normal crossing divisor, and its irreducible components form a mixed Tate configuration.

Proof. For the proof of a) we refer to [Tim, Prop.8]. The existence of the bijective morphism in part b) is the main result of Renner in [Re]. The fact that $D_{\mathcal{F}}$ is cellular in characteristic zero follows from Zariski main theorem. In positive characteristic this follows from the fact that any universal topological homeomorphism induces isomorphism of the associated h-sheaves, see [Vo2, Prop. 3.2.5]. Finally c) follows from a), b) and remark 4.5.6. □

Chapter 7

Filtration on the motives of G-bundles

Let k be a perfect field and G a split reductive group over k. In this chapter we study the motive of G-bundles over a base scheme X. We first recall the slice filtration on the motive of a torus bundle introduced by A. Huber and B. Kahn in [HK]. Then introducing a different approach, we treat the more general case of G-bundles. Our method is essentially based on geometric observations and the weight polytope combinatorics of the wonderful compactification of G.

7.1 Slice filtration and motive of torus bundles

Let T be a split torus of dimension r and $\mathcal{T}$ a principal T-bundle over a smooth variety Y over a field k. Let $X^*(T) := Hom(T, \mathbb{G}_m)$ denotes the character group of T. We claim that there is a canonical map

$$X^*(T) \otimes M(Y) \xrightarrow{e_T} M(Y)(1)[2].$$

Let $\chi \in X^*(T)$. The induced action of T on $\mathbb{G}_m$ gives us the $\mathbb{G}_m$-bundle $\mathcal{G}_\chi := \mathcal{T} \times^T \mathbb{G}_m$ over Y. We denote by $\mathcal{L}_\chi$ the line bundle associated to $\mathcal{G}_\chi$. Let i_0 be its zero section. A. Huber and B. Kahn defined the motivic Euler class $e(\mathcal{L}_\chi)$ to be the composition of the gysin morphism for i_0 with the isomorphism induced by homotopy invariance

$$e(\mathcal{L}_\chi) : M(Y) \cong M(\mathcal{L}_\chi) \xrightarrow{i_0^*} M(Y)(1)[2].$$

And thus we get the desired canonical map e_T.

Remark 7.1.1. There is a generalization of motivic Euler class for vector bundles. Let $\mathcal{E} \to Y$ be a vector bundle with zero section i_0. The *motivic Euler class* is the composition of the gysin morphism for i_0 with the isomorphism induced by homotopy invariance,

$$e(\mathcal{E}) : M(Y) \cong M(\mathcal{E}) \xrightarrow{i_0^*} M(Y)(r)[2r]$$

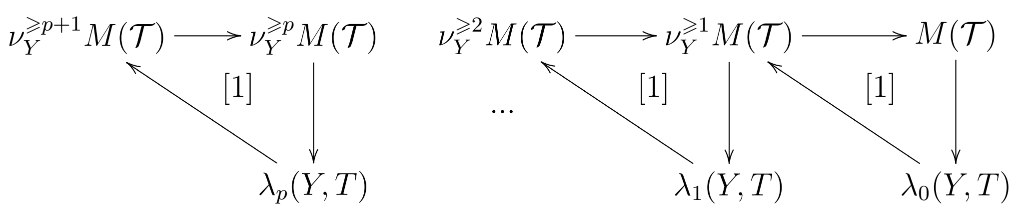
Let $X_*(T) := Hom(\mathbb{G}_m, T)$ be the $\mathbb{Z}$-dual of $X^*(T)$.

Definition 7.1.2. Let $d^0 : M(Y) \to M(Y)(1)[2] \otimes X_*(T)$ be the dual of the map e_T. Let

$$d^p : M(Y)(p)[2p] \otimes \Lambda^p(X_*(T)) \to M(Y)(p+1)[2p+2] \otimes \Lambda^{p+1}(X_*(T))$$

be its extension to the exterior powers (induced by the algebra structure of $\Lambda^*(X_*(T))$).

Theorem 7.1.3. *Consicer the following diagram of distinguished triangles in $DM_{gm}^{eff}(k)$,*



with $M(\mathcal{T}) \cong \nu_Y^{\geqslant 0}M(\mathcal{T})$, $0 = \nu_Y^{\geqslant r+1}M(\mathcal{T})$. *Here*

$$\lambda_p(Y,T) := M(Y)(p)[p] \otimes \Lambda^p(X^*(T))$$

for $0 \leq p \leq r$. *The induced map*

$$M(Y)(p)[p] \otimes \Lambda^p(X^*(T)) \to \nu_Y^{\geqslant p+1}M(\mathcal{T})[1] \to M(Y)(p+1)[p+2]\Lambda^{p+1}(X^*(T))$$

equals $d^p[-p]$, *where* d^p *is as in definition 7.1.2 We call* $\nu_Y^{\geqslant p}M(\mathcal{T})$ *the relative slice filtration of* $\mathcal{T}$ *over* Y.

Proof. For the construction see [HK]. □

7.2 Motive of G-bundles

Let G be a (split) reductive group. Let $X \in Ob(Sch_k)$ be a mixed Tate variety (i.e. $M(X)$ is mixed Tate). In the following theorem we will verify that, with some extra assumptions, the motive associated to a G-bundle over X lies in the category of mixed Tate motives (resp. geometrically mixed Tate motives).

Theorem 7.2.1. *Let G be a connected reductive group over k and $X \in Ob(Sm_k)$ be irreducible. Let $\mathcal{G}$ be a G-bundle over X. Then $M_{gm}(\mathcal{G})$ is geometrically mixed Tate in either of the following cases:*

a) $\operatorname{char} k = 0$*, X is geometrically mixed Tate and $\mathcal{G}$ is locally trivial for the Zariski topology on X.*

b) X is a geometrically cellular variety.

Proof. a) We may assume that the base field k is algebrically closed. Let us first assume that G is a semisimple group of adjoint type. Then G admits a wonderful compactification $\overline{G}$ which is smooth. By construction, there is a $(G \times G)$-action on $\overline{G}$. Let G acts on $\overline{G}$ via the first factor and consider the $\overline{G}$-fibration $\overline{\mathcal{G}} := \mathcal{G} \times^G \overline{G}$ over X. Clearly we have the open immersion $\mathcal{G} \hookrightarrow \overline{\mathcal{G}}$ of varieties over X. So we get the following generalized Gysin distinguished triangle:

$$M_{gm}(\mathcal{G}) \to M_{gm}(\overline{\mathcal{G}}) \to M^c_{gm}(\overline{\mathcal{G}} \setminus \mathcal{G})^*(n)[2n] \to M_{gm}(\mathcal{G})[1]$$

where $n := dim\overline{\mathcal{G}}$.
By proposition 6.2.1, $\overline{G}$ admits a cell decomposition. Therefore by theorem 4.6.1, $M^c_{gm}(\overline{\mathcal{G}})$ is mixed Tate. So to prove the theorem it is enough to show $M^c_{gm}(\overline{\mathcal{G}} \setminus \mathcal{G})$ is mixed Tate.
Let's now look at the geometry of the closures of $(G \times G)$-orbits. As it is mentioned in proposition 6.2.1 a), these orbit closures could be indexed by a subset of faces of Weyl chamber in such a way that the incidence relation between faces gets preserved. Note that by proposition 6.2.1 b) the closure of these orbits admit cell decomposition. Thus by theorem 4.6.1 the irreducible components of $\overline{\mathcal{G}} \setminus \mathcal{G}$ form a mixed Tate configuration. Now lemma 6.1.2 implies that $M^c_{gm}(\overline{\mathcal{G}} - \mathcal{G})$ is mixed Tate.
Now assume that G is a reductive algebraic group and let $\mathcal{G}$ be a G-bundle. We may assume that $Z(G)$ is connected. Note that since G is reductive $Z := Z(G)$ is a torus. Let $\mathcal{G}'$ be the associated G_{ad}-bundle. By the above

statements we know that $M(\mathcal{G}')$ is mixed Tate. Notice that any torus bundle is locally trivial for the Zariski topology by the theorem Hilbert90. Take a toric compactification $\overline{Z}$ of Z and embed $\mathcal{G}$ into $\mathcal{Z} := \mathcal{G}\times^{Z}\overline{Z}$, which is a toric fibration over $\mathcal{G}'$. Now the irreducible components of the complement of $\mathcal{G}$ in $\mathcal{Z}$ are obviously toric fibrations over $\mathcal{G}'$. Since fibers are toric and $M(\mathcal{G}')$ is mixed Tate therefore by theorem 4.6.1 we argue that these irreducible components form a mixed Tate configuration and we may argue as above. To prove part b) we first show that the motive associated with a connected reductive group is geometrically mixed Tate.

Lemma 7.2.2. *Let G be a connected reductive group over k, then the motive associated to G is geometrically mixed Tate. Furthermore if G is a split reductive group then $M_{gm}(G)$ is mixed Tate.*

Proof. We may assume that k is algebraically closed. Let T be a maximal split torus in G of rank r. We consider the projection $\pi : G \to G/T$ and view G as a T-bundle over G/T. Consider $\mathbb{P}^r_k$ as the obvious compactification of T and let $\overline{\mathcal{T}} := G \times^T \mathbb{P}^r_k$ be the associated projective bundle over G/T. By projective bundle formula

$$M_{gm}(\overline{\mathcal{T}}) = M_{gm}(\mathbb{P}^r_k) \otimes M_{gm}(G/T)$$

(see [MVW, Theorem 15.12].) On the other hand $B = T \ltimes U$, where B is a Borel subgroup of G containing T and U is the unipotent part of B. Notice that as a variety U is isomorphic to an affine space over k. Since the fibration $G \to G/B$ is the composition of $G \to G/T$ and U-fibration $G/T \to G/B$, we deduce by corollary 4.4.10 and theorem4.6.1 that $M_{gm}(\overline{\mathcal{T}})$ is pure Tate. Finally one can apply the same trick as in part a) to conclude $M_{gm}(G)$ is mixed Tate. The second part of the lemma is similar,only we don't require to pass to an algebraic closure. □

proof of part b) We may assume that k is algebraically closed. Let

$$\emptyset = X_{-1} \subset X_0 \subset ... \subset X_n = X$$

be a cell decomposition for X where $U_i := X_i \smallsetminus X_{i-1}$ is isomorphic to $\mathbb{A}^{d_i}_k$. We prove by induction on n. Consider the following generalized gysin distinguished triangle:

$$M_{gm}(\mathcal{G}|_{U_n}) \to M_{gm}(\mathcal{G}) \to M_{gm}(\mathcal{G}|_{X_{n-1}})$$

By Raghunathan's result the restriction of $\mathcal{G}$ to U_n is trivial bundle over U_n and therefore is mixed Tate. Also $M_{gm}(\mathcal{G}|X_{n-1})$ is mixed Tate by induction hypothesis. So we conclude.

□

Let us now explain some application of the above theorem.

Recall that the Voevodsky's theory of motives over perfect field k can be established for the schemes over more general base scheme S. We refer the reader to the article [Vo1] of Voevodsky or to [CD] of Cisinski and Deglise. They construct the triangulated category of mixed motives $DM(S)$. Here S is any locally noetherian scheme of finite dimension. This category is constructed from the category of Nisnevich sheaves with transfers over X. For the details about the category of mixed Tate motives over a number ring we refer to [Sc].

Corollary 7.2.3. *Let $\mathcal{A}$ be a henselian discrete valuation ring with perfect field of fractions K and perfect residue field k of characteristic $p > 0$. Let G be a connected reductive algebraic group over K and P be a parahoric group associated to a facet $\underline{a}$ of the building $\mathcal{B}_G$. Suppose that the maximal torus of G splits over an unramified extension of K. Then the motive $M(P)$ becomes mixed Tate over an étale covering of $\mathcal{A}$.*

Proof. We consider the six functors introduced in [CD, section 1], for the situation that:

$$i : \operatorname{Spec} k \to \operatorname{Spec} \mathcal{A},$$

$$j : \operatorname{Spec} K \to \operatorname{Spec} \mathcal{A}.$$

Then we have the following distinguished triangle:

$$j_! j^* \to id \to i_* i^*$$

in $DM_{gm}(\mathcal{A})$. The generic fiber P_K of P is G itself, hence by lemma 7.2.2 we know that $M_{gm}(P_K)$ is geometrically mixed Tate. On the other hand since G splits over a tamely ramified extension of K, we may argue by [Mc] that the special fiber P_k has a Levi subgroup $\mathcal{L}$, in particular $\mathcal{L} \times R_u(P_k) \to P_k$ is an isomorphism. Notice that the unipotent radical $R_u(P_k)$ is isomorphic to an affine space. Choose a finite extension of k that splits $\mathcal{L}$. Since motive of the levi component becomes mixed Tate over this extension, thus also the motive associated to the special fiber P_k. Since k is perfect and $\mathcal{A}$ is henselian, this extension gives an étale cover of $\mathcal{A}$. Now the assertion follows from the localization property, see [CD] or [Sc], and the above distinguished triangle.

□

7.3 Filtration associated to the Weight Polytopes

In section 7.2 we studied the motive of a G-bundle over base scheme X, when the associated motive $M(X)$ is geometrically mixed Tate. In the sequel we produce a filtration for the motive of G-bundles over any base scheme X in Sch_k in terms of weight polytope and the incidencerelation amont its faces.

Let $\mathcal{G}$ be a G-bundle over X, where G is a linear algebraic group. Let $\overline{G}$ be a compactification of G. Suppose that $D := \overline{G} \setminus G$ form a mixed Tate configuration $D = \cup_{i=1}^{m} D_i$, such that $D^J := \cap_{i\in J} D_i$ is either irreducible or empty for any $J \subseteq \{1, \ldots, m\}$. Assume that there exist a polytope whose faces correspond to those subsets $J \subseteq \{1, ..., m\}$ such that D^J is non empty. Let $\mathcal{P}$ be the dual of this polytope. For each face $\mathcal{F}$ of $\mathcal{P}$, let $D_{\mathcal{F}}$ denote the associated subvariety of D regarding the above correspondence. For each $1 \leq r \leq m$, let Q_r be the set consisting of all faces in $\mathcal{P}$ of codimension r. Set $\partial\mathcal{F} := \{\mathcal{F} \cap \mathcal{T} | \mathcal{T} \in Q_1\} - \{\mathcal{F}\}$.

Let $\overline{\mathcal{G}}$ denote the compactification $\mathcal{G} \times^G \overline{G}$ of $\mathcal{G}$ and let $\mathcal{D}_{\mathcal{F}}$ be the associated $D_{\mathcal{F}}$-fibration over X. We may form the following nested filtration on $M^c_{gm}(\mathcal{G})$ by the distinguished triangles, indexed by codimension r and faces $\mathcal{F} \in Q_r$

$$M^c_{gm}(\overline{\mathcal{G}} \setminus \mathcal{G}) \to M^c_{gm}(\overline{\mathcal{G}}) \to M^c_{gm}(\mathcal{G})$$

$$\vdots$$

$$M^c_{gm}(\bigcup_{\mathcal{F}\in Q_{r+1}} \mathcal{D}_{\mathcal{F}}) \to M^c_{gm}(\bigcup_{\mathcal{F}\in Q_r} \mathcal{D}_{\mathcal{F}}) \to \oplus_{\mathcal{F}\in Q_r} M^c_{gm}(\mathcal{D}_{\mathcal{F}} \setminus \bigcup_{\mathcal{F}'\in\partial\mathcal{F}} \mathcal{D}_{\mathcal{F}'}),$$
$$M^c_{gm}(\bigcup_{\mathcal{F}'\in\partial\mathcal{F}} \mathcal{D}_{\mathcal{F}'})) \to M^c_{gm}(\mathcal{D}_{\mathcal{F}}) \to M^c_{gm}(\mathcal{D}_{\mathcal{F}} \setminus \bigcup_{\mathcal{F}'\in\partial\mathcal{F}} \mathcal{D}_{\mathcal{F}'}), \tag{7.3.1}$$

where for each $\mathcal{F} \in Q_r$ the last triangle is the first line of a nested triangle obtained by replacing $\mathcal{P}$ by $\mathcal{F}$.

Note that this filtration is particularly interesting when $\mathcal{D}_{\mathcal{F}}$s are cellular fibrations. In this situation we may apply theorem 4.6.1 to compute $M^c_{gm}(\mathcal{D}_{\mathcal{F}})$. Let us recall two of such cases.

Example 7.3.1. Let T be a split torus of rank n and let $\mathcal{T}$ be a T-bundle over X. Consider a torus embedding of T into the projective space $\mathbb{P}^n$ associated with the standard n-simplex Δ^n. So we put $P := \Delta^n$ in the above filtration. Note that in this case for each face $\mathcal{F} \in \Delta^n$, $\mathcal{D}_{\mathcal{F}}$ is a projective bundle and hence one may use the projective bundle formula [MVW, Thm 15.11] to compute $M^c_{gm}(\mathcal{D}_{\mathcal{F}})$. In particular when $M^c_{gm}(X)$ is mixed Tate, one may prove recursively that $M^c_{gm}(\mathcal{T})$ is mixed Tate, compare section 7.1.

Example 7.3.2. Let G be a semi-simple group of adjoint type and $\overline{G}$ its wonderful compactification. In this case the polytope P is the Wyle chamber. Recall that for every face $\mathcal{F}$ of P, $D_{\mathcal{F}}$ admits a cell decomposition. Let us mention that for any regular compactification (see [Br] for details)of G and any vertex $\mathcal{F}$, $D_{\mathcal{F}}$ is isomorphic to $G/B \times G/B$ and in particular $\mathcal{D}_{\mathcal{F}}$ is a cellular fiberation.

The Case of 1-motives:
In practice for some applications it might happen that the motive of the base variety X is far from being mixed Tate. Already it can happen for the case of 1-motives. Recall that the motive $M(C)$ of a curve C decomposes in $DM^{eff}_{gm}(k) \otimes \mathbb{Q}$ as follows

$$M(C) = M_0(C) \oplus M_1(C) \oplus M_2(C), \tag{7.3.2}$$

where $M_i(C) := TotLiAlb^{\mathbb{Q}}(C)[i]$. For the definition of $LiAlb^{\mathbb{Q}}(C)$ and detailed explanation of the theory we refer to section 3.12 of [BK].

In the sequel we explain the special case when X is a relative curve.

Example 7.3.3. Let G be a reductive group over k. Let $\mathcal{G}$ be a G-bundle over C. Let C be a smooth projective curve over a field k. Fix a closed point p of C and set $\dot{C} := C \setminus \{p\}$. Assume that char k does not divide the order of $\pi_1(\mathbb{C})$ therefore by the well-known theorem of Drinfeld and Simpson 3.2.11 we may take a finite extension of k which simultaneously trivializes the restriction of $\mathcal{G}_s$ over $\dot{C}$ and the fiber over p. Therefore we obtain the following distinguished triangle

$$M(G_s \times \dot{C}_{k'}) \to M(\mathcal{G}_{s,k'}) \to M(G_s \times k')(n)[2n],$$

and by the Kunneth theorem

$$M(G_s) \otimes M(\dot{C}_{k'}) \to M(\mathcal{G}_{s,k'}) \to M(G_s \times k')(n)[2n].$$

Let us assume that G has a connected center. Since G is reductive $Z := Z(G)^\circ$ is a split torus. We may now apply either of the filtrations 7.3.1 or 7.1.3 to the torus bundle $\mathcal{G} \to \mathcal{G}_s$. For instance from the latter filtration we get the following

i) A filtration $\{\varphi_i : M_i \to M_{i-1}\}_{i \in \mathbb{N}}$ where $M_i := \nu^{\geq i}_{\mathcal{G}_{s,k'}} M(\mathcal{G}_{k'})$. In particular $M_0 = M(\mathcal{G}_{k'})$ and $M_r = 0$ for $r > \operatorname{rk} Z(G)^\circ$

ii) The following sort of distinguished triangles

$$M(\dot{C}_{k'}) \otimes M(G_s) \to M(\mathcal{G}_{s,k'}) \to M(G_s) \otimes M(k')(n)[2n]$$

$$M_{i+1} \to M_i \to M(\mathcal{G}_{s,k'})(i)[i] \otimes F_i,$$

where F_i be the i-th vedge power of the group $\Xi := Hom(\mathbb{G}_m, Z)$.

Note that $\nu^{\geq r}_{\mathcal{G}_{s,S'}} M(\mathcal{G}_{k'}) = 0$ for $r > rkZ(G)^\circ$.

Remark 7.3.4. In the above example one may consider a G-bundle $\mathcal{G}_{C_S}$ over the relative curve $C \times_k S$ over a scheme S. Then the above distinguished triangles lie over an étale cover $S' \to S$. Note that our discussion in particular shows that the class $[\mathcal{G}] - [G \times_S C_S]$ lies in the kernel of the natural morphism $K_0(Var_S) \to K_0(Var_{S'})$ of the Grothendieck K-rings.

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